Improved Convergence of Landau-de Gennes Minimizers in the Vanishing Elasticity Limit

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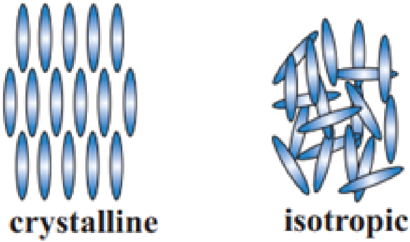
Liquid Crystals

Liquid crystals (LCs) are anisotropic fluids. The anisotropy arises from the directional nature of the molecular geometry, physical, or chemical properties.

LCs are between anisotropic crystalline and isotropic liquid phases.



Order v.s. Disorder

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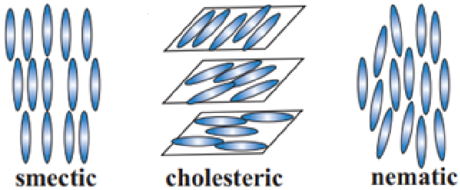

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Define $\mathbb{S}_0 := \{\mathbf{Q} \in \mathbb{M}^{3\times 3} : \mathbf{Q}^{\mathrm{T}} = \mathbf{Q},\ \operatorname{tr}\mathbf{Q} = 0\}$.
Let $\mathbf{Q} \in \mathbb{S}_0$ and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $\mathbf{Q}$. Then

$$\mathbf{Q} = \lambda_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \lambda_3 \mathbf{n}_3 \otimes \mathbf{n}_3,$$

where $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \in \mathbb{S}^2$ with $\mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij}\,(1 \le i, j \le 3)$. Rewrite $\mathbf{Q}$ as

$$\mathbf{Q} = s\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I}\right) + r\left(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\mathbf{I}\right), \quad s, r \in \mathbb{R},$$

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3. **Uniaxial**: If $s = r \neq 0$ or $s = 0, r \neq 0$ or $s \neq 0, r = 0$. In particular,

$$\mathbf{Q} = t\left(\mathbf{u} \otimes \mathbf{u} - \frac{1}{3}\mathbf{I}\right), \quad \mathbf{u} \in \mathbb{S}^2.$$

The free energy functional is

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- $f_e(\mathbf{Q})$: The elastic energy density. For $L_i \geq 0$ with $(i = 1, 2, 3)$,

$$f_e(\mathbf{Q}) = \frac{L_1}{2}|\nabla\mathbf{Q}|^2 + \frac{L_2}{2}\partial_j\mathbf{Q}_{ij}\partial_k\mathbf{Q}_{ik} + \frac{L_3}{2}\partial_k\mathbf{Q}_{ij}\partial_j\mathbf{Q}_{ik}, \quad .$$

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Stable equilibrium configurations of the liquid crystalline system with a domain $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ correspond to local minimizers of $F(\cdot,\Omega)$.

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where

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Question (Vanishing elasticity limit)

The asymptotic behavior of minimizers (or critical points) to (LdG) as $\varepsilon \to 0^+$.

Limiting Functional

The vacuum manifold is

$$\mathcal{N} := \left\{ s_* \left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} \mathbf{I} \right) : \mathbf{n} \in \mathbb{S}^2 \right\} = f^{-1}(0),$$

where

$$s_* := s_*(a, b, c) = \frac{b + \sqrt{b^2 + 24ac}}{4c}.$$

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$$E(\mathbf{Q}, \Omega) := \frac{1}{2} \int_\Omega |\nabla \mathbf{Q}|^2 \mathrm{d}x, \quad \mathbf{Q} \in H^1(\Omega, \mathcal{N}). \tag{Dir}$$

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4. Contreras-Lamy (2022) and Feng-Hong (2022) generalized results by Nguyen-Zarnescu to the cases with non-zero L_2 and L_3.

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4. Contreras-Lamy (2022) and Feng-Hong (2022) generalized results by Nguyen-Zarnescu to the cases with non-zero L_2 and L_3.
5. Wang-Wang-Zhang (2017) and Huang-Lin (2022) analyzed the gradient flow settings of $\mathbf{Q}$-tensor model.

Vanishing Elasticity Limit: Main Results

Theorem (Fu-Wang-W. 2025)

Suppose $\{\mathbf{Q}_\varepsilon\}_{\varepsilon\in(0,1)} \subset H^1_{\mathrm{loc}}(\Omega,\mathbb{S}_0)$ *are local minimizers of* (LdG) *s.t.*

$$E_\varepsilon(\mathbf{Q}_\varepsilon,\Omega) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)} \le M, \quad \forall \varepsilon \in (0,1).$$

Then, $\exists \mathbf{Q}_0 \in H^1(\Omega,\mathcal{N})$, *a local minimizer of* (Dir) *s.t.* $\exists \varepsilon_i \to 0^+$, $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ *strongly in* $H^1_{\mathrm{loc}}(\Omega,\mathbb{S}_0)$, *and the following hold.*

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1. $\forall p \in (1, +\infty)$ *and* $\forall K \subset\subset \Omega$,

$$\|\mathbf{Q}_{\varepsilon_i} - \mathbf{Q}_0\|_{L^p(K)} \to 0, \quad \text{as} \quad i \to +\infty.$$

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2. $\forall K \subset\subset \Omega$, $\exists C = C(a,b,c,K,M) > 0$, *s.t.*

$$\int_K \frac{1}{\varepsilon_i^2} f(\mathbf{Q}_{\varepsilon_i}) \mathrm{d}x \leq C\varepsilon_i.$$

Remarks on Main Results

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is satisfied, e.g. if $\Omega \in C^\infty$ and $\mathbf{Q}_\varepsilon|_{\partial\Omega} = \mathbf{Q}_b \in C^\infty(\partial\Omega, \mathcal{N})$.

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5. Our method applies to other settings, such as the generalized Ginzburg-Landau model (Monteil-Rodiac-Schaftingen, 2021) and torus-like solutions of (LdG) as in (Dipasquale-Millot-Pisante, 2021).

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Problem Setting

$\{\mathbf{Q}_\varepsilon\} \subset H^1(B_1, \mathbb{S}_0)$ are minimizers of (LdG) with

$$\mathbf{Q}|_{\partial B_1} = \mathbf{Q}_\mathrm{b} := s_* \left(x \otimes x - \frac{1}{3}\mathbf{I} \right).$$

$\exists \varepsilon_i \to 0^+$ s.t.

$$\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0 = s_* \left(\mathbf{n}_0 \otimes \mathbf{n}_0 - \frac{1}{3}\mathbf{I} \right) \quad \text{in } H^1(B_1, \mathbb{S}_0),$$

where $\mathbf{n}_0$ is a minimizer of

$$\inf_{\mathbf{n}=x \text{ on } \partial B_1} \int_{B_1} |\nabla \mathbf{n}|^2 \mathrm{d}x.$$

Brézis-Coron-Lieb (1986): $\mathbf{n}_0(x) = \frac{x}{|x|}$. Then

$$\mathbf{Q}_0 = s_* \left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\mathbf{I} \right)$$

is the so-called hedgehog solution.

Sharpness of the Main Theorem

L^p-**convergence**:

Since $\partial B_1 \in C^\infty$ and $\mathrm{sing}(\mathbf{Q}_0) = \{0\}$, we have

$$\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0 \text{ uniformly in } B_1 \backslash B_r, \quad \forall r \in (0,1).$$

Note that $\mathbf{Q}_0$ is not smooth, showing that the L^p-convergence is sharp. Indeed, if

$$\lim_{\varepsilon \to 0^+} \|\mathbf{Q}_\varepsilon - \mathbf{Q}_0\|_{L^\infty(B_{1/2})} = 0,$$

then $\mathbf{Q}_0$ is continuous at $0 \Rightarrow \mathbf{Q}_0 \in C^\infty(B_{1/2}, \mathcal{N})$, a contradiction!

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The convergence rate $\int \varepsilon^{-2} f(\mathbf{Q}_\varepsilon) \mathrm{d}x$:

The sharpness follows from the following proposition.

Proposition

For $\mathbf{Q}_{\varepsilon_i}$ given as above, $\exists C = C(a,b,c) > 0$ s.t. for $i \gg 1$,

$$\int_{B_{3/4}} f(\mathbf{Q}_{\varepsilon_i}) \mathrm{d}x \geq \frac{\varepsilon_i^3}{C}.$$

Proof of the Example's Properties

Lemma

$\exists \eta = \eta(a,b,c) > 0$ s.t. $\mathbb{S}_0 \cap \{f(\mathbf{Q}) < \eta\} \subset \mathbb{S}_0 \cap \{\lambda_1(\mathbf{Q}) > \lambda_2(\mathbf{Q})\}$, where $\lambda_1(\mathbf{Q}) \geq \lambda_2(\mathbf{Q}) \geq \lambda_3(\mathbf{Q})$ are three eigenvalues of $\mathbf{Q}$ in order.

Assume $\lambda_1(\mathbf{Q}) = \lambda_2(\mathbf{Q}) = \lambda$. Then $\lambda_3(\mathbf{Q}) = -2\lambda$ and

$$f(\mathbf{Q}) = k - 3a\lambda^2 + 2b\lambda^3 + 9c\lambda^4 := g(\lambda).$$

In particular, $k = \frac{s_*^2}{27}(9a + 2bs_* - 3cs_*^2)$. Indeed, $g(\lambda)$ gets the minimum at $\lambda_* = \frac{-b+\sqrt{b^2+24ac}}{12c}$ and $g(\lambda_*) > 0$. Let $\eta \in (0, g(\lambda_*)) \Rightarrow$ done!

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Lemma

$\nexists \mathbf{Q} \in C(\overline{B}_1, \mathcal{N})$ s.t. $\mathbf{Q}|_{\partial B_1}(x) = s_* \left(x \otimes x - \frac{1}{3}\mathbf{I} \right)$.

The existence of such a $\mathbf{Q}$ contradicts the property that there is no retraction from B_1 to $\partial B_1 = \mathbb{S}^2$.

Claim: $\exists y_i \in B_{1/2}$ s.t. $f(\mathbf{Q}_{\varepsilon_i}(y_i)) > \eta(a,b,c) > 0$.
Pf: If $\forall y \in B_1$, $f(\mathbf{Q}_{\varepsilon_i})(y) \ll 1$, then $\forall y \in \overline{B}_1$, $\lambda_1(\mathbf{Q}_{\varepsilon_i}) > \lambda_2(\mathbf{Q}_{\varepsilon_i})$. Thus, $\exists$ C^1-nearest point projection

$$\Pi : \{\mathbf{Q} \in \mathbb{S}_0 : \lambda_1(\mathbf{Q}) > \lambda_2(\mathbf{Q})\} \to \mathcal{N}.$$

Hence $\Pi \circ \mathbf{Q}_{\varepsilon_i} \in C(\overline{B}_1, \mathcal{N})$ and

$$\Pi \circ \mathbf{Q}_{\varepsilon}|_{\partial B_1} = s_* \left(x \otimes x - \frac{1}{3}\mathbf{I} \right),$$

a contradiction. Since $\mathbf{Q}_{\varepsilon_i} \overset{\text{uniform}}{\to} \mathbf{Q}_0$ in $B_1 \backslash B_{1/2}$, for $i \gg 1$, $y_i \in B_{1/2}$.

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$$\Pi \circ \mathbf{Q}_\varepsilon|_{\partial B_1} = s_* \left(x \otimes x - \frac{1}{3}\mathbf{I} \right),$$

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By elliptic estimates,

$$\|\nabla \mathbf{Q}_{\varepsilon_i}\|_{L^\infty(B_{3/4})} \leq C(a,b,c)\varepsilon_i^{-1}.$$

Then, $\exists \delta = \delta(a,b,c) \in (0, \frac{1}{4})$ s.t.

$$\inf_{y \in B_{\delta\varepsilon_i}(y_i)} f(\mathbf{Q}_{\varepsilon_i}(y)) > \eta/2.$$

If $\varepsilon_i \in (0, \frac{1}{2})$, then $B_{\delta\varepsilon_i}(y_i) \subset B_{3/4}$ and

$$\int_{B_{3/4}} f(\mathbf{Q}_{\varepsilon_i}) \geq \int_{B_{\delta\varepsilon_i}(y_i)} f(\mathbf{Q}_{\varepsilon_i}) \geq \frac{\varepsilon_i^3}{C(a,b,c)}.$$

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Tool I: Monotonicity

Definition

Let $\phi \in C^\infty([0,+\infty), \mathbb{R}_{\geq 0})$ s.t. the following properties hold.

1. $\operatorname{supp} \phi \subset [0, 10)$.
2. $\forall t \in [0,+\infty)$, $\phi(t) \geq 0$ and $|\phi'(t)| \leq 100$.
3. $-2 \leq \phi'(t) \leq -1 \ \forall t \in [0, 8]$.
4. $\forall t \in \mathbb{R}_+$, $\phi'(t) \leq 0$.

Let $\mathbf{Q} \in H^1(\Omega, \mathbb{S}_0)$, $x \in \Omega$, and $0 < r < \frac{1}{10}\operatorname{dist}(x, \partial\Omega)$. Define

$$\Theta_r^\phi(\mathbf{Q}, x) := \frac{1}{r} \int e_\varepsilon(\mathbf{Q}) \phi\left(\frac{|y-x|^2}{r^2}\right) \mathrm{d}y.$$

Proposition (Monotonicity)

Assume that $\mathbf{Q}_\varepsilon : \Omega \to \mathbb{S}_0$ *is a smooth critical point of* (LdG). *Let* $x \in \Omega$ *and* $0 < r < R < \frac{1}{10}\,\mathrm{dist}(x, \partial\Omega)$. *Then*

$$\begin{aligned}
&\Theta_R^\phi(\mathbf{Q}_\varepsilon, x) - \Theta_r^\phi(\mathbf{Q}_\varepsilon, x) \\
&= \int_r^R \left[-\frac{2}{\rho^2} \int \left| \frac{y-x}{\rho} \cdot \nabla \mathbf{Q}_\varepsilon \right|^2 \phi' \left(\frac{|y-x|^2}{\rho^2} \right) \mathrm{d}y \right] \mathrm{d}\rho \\
&\quad + \int_r^R \left[\frac{2}{\varepsilon^2 \rho^2} \int f(\mathbf{Q}_\varepsilon) \phi \left(\frac{|y-x|^2}{\rho^2} \right) \mathrm{d}y \right] \mathrm{d}\rho.
\end{aligned}$$

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The proof follows from the stress identity:

$$\partial_j (e_\varepsilon(\mathbf{Q}_\varepsilon)\delta_{ij} - \partial_i \mathbf{Q}_\varepsilon : \partial_j \mathbf{Q}_\varepsilon) = 0, \quad \forall i \in \{1, 2, 3\}.$$

Tool II: Compactness

Proposition (H^1-compactness)

$\{\mathbf{Q}_\varepsilon\}_{\varepsilon\in(0,1)} \subset H^1_{\mathrm{loc}}(\Omega, \mathbb{S}_0)$: *a sequence of local minimizers of* (LdG) *s.t.*

$$\sup_{\varepsilon\in(0,1)} \Big(E_\varepsilon(\mathbf{Q}_\varepsilon, K) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(K)} \Big) < +\infty, \quad \forall K \subset\subset \Omega.$$

Then, $\exists \varepsilon_i \to 0^+$ *s.t.*

$$\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0 \textit{ strongly in } H^1_{\mathrm{loc}}(\Omega, \mathbb{S}_0),$$
$$\frac{1}{\varepsilon_i^2} f(\mathbf{Q}_{\varepsilon_i}) \to 0 \textit{ strongly in } L^1_{\mathrm{loc}}(\Omega),$$

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The proof follows from constructing appropriate comparison maps using interpolation results developed by Luckhaus (1988).

Tool III: Regularity and Partial Regularity

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Lemma (A priori estimate)

$\mathbf{Q}_\varepsilon \in H^1(B_{2r}, \mathbb{S}_0)$: *a critical point of* (LdG) *with* $\|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{2r})} \le M$.
Then, $\mathbf{Q}_\varepsilon \in C^\infty$ *and* $\|\nabla \mathbf{Q}_\varepsilon\|_{L^\infty(B_r)} \le C(a, b, c, M)(\varepsilon^{-1} + r^{-1})$.

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Lemma (Partial regularity I)

$\mathbf{Q}_\varepsilon \in H^1(B_{4r}, \mathbb{S}_0)$: *a local minimizer of* (LdG) *with* $\|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{4r})} \leq M$.
$\forall \delta > 0$, $\exists (\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$ *s.t. if* $r \in (\Lambda\varepsilon, 1)$, *then*
$E_\varepsilon(\mathbf{Q}_\varepsilon, B_{2r}) < \eta r \Rightarrow \|\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})\|_{L^\infty(B_r)} < \delta$.

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$E_\varepsilon(\mathbf{Q}_\varepsilon, B_{2r}) < \eta r \Rightarrow \|\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})\|_{L^\infty(B_r)} < \delta$.

Lemma (Partial regularity II)

Assume as (Partial regularity I). $\exists (\eta, \Lambda) = (\eta, \Lambda)(a, b, c, M) > 0$ *s.t. if*
$r \in (\Lambda\varepsilon, 1)$, *then* $E_\varepsilon(\mathbf{Q}_\varepsilon, B_{2r}) < \eta r \Rightarrow r\|\nabla \mathbf{Q}_\varepsilon\|_{L^\infty(B_r)} \leq C_0(a, b, c, M, \eta)$,
with $\lim_{\eta \to 0^+} C_0(a, b, c, M, \eta) = 0$.

Characterization of the Bad Behavior

We first introduce the regular scale to give a quantitative characterization of regularity.

Definition (Regular scale)

Let $\varepsilon \in (0,1)$. $\mathbf{Q}_\varepsilon \in H^1(B_2, \mathbb{S}_0)$: a local minimizer of (LdG). For $x \in B_1$, we define

$$r(\mathbf{Q}_\varepsilon, x) := \sup\{0 \leq r \leq 1 : r\|\nabla \mathbf{Q}_\varepsilon\|_{L^\infty(B_r(x))} \leq 1\}.$$

For a minimizer $\mathbf{Q}_\varepsilon$ of (LdG) in $\Omega \subset \mathbb{R}^3$, the partial regularity lemmas motivate us that the "bad points" are those where

$$|\nabla \mathbf{Q}_\varepsilon| \gg 1 \quad \text{and} \quad \operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N}) \gtrsim 1.$$

For this reason, for $\mathbf{Q}_\varepsilon$ as in (12), we define the collection of *bad points* with $\delta, r > 0$ being parameters as

$$\operatorname{Bad}(\mathbf{Q}_\varepsilon; r, \delta) := \{y \in \Omega : r(\mathbf{Q}_\varepsilon, y) < r \text{ or } \operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N}) > \delta\}.$$

Crucial Lemma: Dichotomy

Proposition (Dichotomy result)

$\beta \in (0, \frac{1}{2})$, $\delta > 0$, *and* $x \in B_2$. $\mathbf{Q}_\varepsilon \in H^1(B_{40}, \mathbb{S}_0)$*: a local minimizer of* (LdG) *with*

$$E_\varepsilon(\mathbf{Q}_\varepsilon, B_{40}) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{40})} \leq M.$$

$\exists(\eta, \eta', \Lambda) = (\eta, \eta', \Lambda)(a, b, \beta, c, \delta, M) > 0$ *s.t. if for some* $y \in B_{2r}(x)$,

$$\Theta_r^\phi(\mathbf{Q}_\varepsilon, y) - \Theta_{r/2}^\phi(\mathbf{Q}_\varepsilon, y) < \eta,$$

and $r \in (\Lambda\varepsilon, 1)$*, then* $\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta' r, \delta) \cap B_r(x) \subset B_{2\beta r}(y)$.

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$\exists(\eta, \eta', \Lambda) = (\eta, \eta', \Lambda)(a, b, \beta, c, \delta, M) > 0$ s.t. if for some $y \in B_{2r}(x)$,

$$\Theta_r^\phi(\mathbf{Q}_\varepsilon, y) - \Theta_{r/2}^\phi(\mathbf{Q}_\varepsilon, y) < \eta,$$

and $r \in (\Lambda\varepsilon, 1)$, then $\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta' r, \delta) \cap B_r(x) \subset B_{2\beta r}(y)$.

In other words, if $r \in (\Lambda\varepsilon, 1)$, either

$$\Theta_{r/2}^\phi(\mathbf{Q}_\varepsilon, y) < \Theta_r^\phi(\mathbf{Q}_\varepsilon, y) - \eta$$

or $\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta' r, \delta) \cap B_r(x) \subset B_{2\beta r}(y)$.

Key Observation: High Symmetry $\Rightarrow$ Regularity

Lemma (Symmetry implies regularity)

$\varepsilon \in (0,1)$. $\mathbf{Q}_\varepsilon \in H^1(B_{10r}(x), \mathbb{S}_0)$: *a local minimizer of* (LdG) *with*

$$r^{-1} E_\varepsilon(\mathbf{Q}_\varepsilon, B_{10r}(x)) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{10r}(x))} \le M.$$

$\forall \delta > 0$, $\exists (\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$, *s.t. if* $r \in (\Lambda\varepsilon, 1)$,

$$\Theta_r^\phi(\mathbf{Q}_\varepsilon, x) - \Theta_{r/2}^\phi(\mathbf{Q}_\varepsilon, x) < \eta, \quad \inf_{v \in \mathbb{S}^2} \frac{1}{r} \int_{B_r(x)} |v \cdot \nabla \mathbf{Q}_\varepsilon|^2 < \eta,$$

then

$$\|\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})\|_{L^\infty(B_{r/2}(x))} < \delta \quad \textit{and} \quad r(\mathbf{Q}_\varepsilon, x) \ge \frac{r}{2}.$$

Moreover, $x \notin \operatorname{Bad}\left(\mathbf{Q}_\varepsilon; \frac{r}{2}, \delta\right)$.

Key Observation: High Symmetry $\Rightarrow$ Regularity

Lemma (Symmetry implies regularity)

$\varepsilon \in (0,1)$. *$\mathbf{Q}_\varepsilon \in H^1(B_{10r}(x), \mathbb{S}_0)$: a* local minimizer *of* (LdG) *with*

$$r^{-1}E_\varepsilon(\mathbf{Q}_\varepsilon, B_{10r}(x)) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{10r}(x))} \leq M.$$

$\forall \delta > 0$, $\exists(\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$, *s.t. if* $r \in (\Lambda\varepsilon, 1)$,

$$\Theta_r^\phi(\mathbf{Q}_\varepsilon, x) - \Theta_{r/2}^\phi(\mathbf{Q}_\varepsilon, x) < \eta, \quad \inf_{v \in \mathbb{S}^2} \frac{1}{r}\int_{B_r(x)} |v \cdot \nabla \mathbf{Q}_\varepsilon|^2 < \eta,$$

then

$$\|\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})\|_{L^\infty(B_{r/2}(x))} < \delta \quad \textit{and} \quad r(\mathbf{Q}_\varepsilon, x) \geq \frac{r}{2}.$$

Moreover, $x \notin \operatorname{Bad}\left(\mathbf{Q}_\varepsilon; \frac{r}{2}, \delta\right)$.

Pinching of density+Approximated 1-invariance$\Rightarrow$Regularity

Proof of the Key Observation

If not, $\exists \mathbf{Q}_{\varepsilon_i}$, with $\frac{\varepsilon_i}{r_i} \to 0^+$,

$$\Theta^{\phi}_{r_i}(\mathbf{Q}_{\varepsilon_i}, x_i) - \Theta^{\phi}_{r_i/2}(\mathbf{Q}_{\varepsilon_i}, x_i) < i^{-1}, \quad \inf_{v \in \mathbb{S}^2} \frac{1}{r_i} \int_{B_{r_i}(x)} |v \cdot \nabla \mathbf{Q}_{\varepsilon_i}|^2 < i^{-1}, \quad (1)$$

but $r(\mathbf{Q}_{\varepsilon_i}, x_i) < \frac{r_i}{2}$ or $\mathrm{dist}(\mathbf{Q}_{\varepsilon_i}, \mathcal{N}) \geq \delta > 0$ in $B_{r_i/2}(x_i)$. Let

$$\widetilde{\mathbf{Q}}_{\widetilde{\varepsilon}_i}(y) := \mathbf{Q}_{\varepsilon_i}(x_i + r_i y).$$

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$$\widetilde{\mathbf{Q}}_{\widetilde{\varepsilon}_i}(y) := \mathbf{Q}_{\varepsilon_i}(x_i + r_i y).$$

Up to a subsequence,

$$\widetilde{\mathbf{Q}}_{\widetilde{\varepsilon}_i} \to \widetilde{\mathbf{Q}}_0 \in \mathcal{N} \text{ strongly in } H^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{S}_0),$$

where $\widetilde{\mathbf{Q}}_0$ is a local minimizer of (Dir).

$$(1) \Rightarrow \text{homogeneity} + \text{1-invariance} \Rightarrow \widetilde{\mathbf{Q}}_0 \text{ is a constant.}$$

The result follows from the partial regularity.

A Corollary of the Key Observation

Corollary (Smaller ball implies pinching)

$\varepsilon \in (0,1)$, $\delta > 0$. $\mathbf{Q}_\varepsilon \in H^1(B_{40}, \mathbb{S}_0)$: *a local minimizer of* (LdG) *with*

$$E_\varepsilon(\mathbf{Q}_\varepsilon, B_{40}) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{40})} \le M.$$

$\exists(\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$, *s.t. if* $x \in B_2$ *and*

$$\inf_{v \in \mathbb{S}^2} \frac{1}{r} \int_{B_r(x)} |v \cdot \nabla u|^2 < \eta,$$

then the following holds. $\forall y \in B_{r/2}(x)$, $\exists r_y \in [\eta^{\frac{1}{2}} r, 1]$ *s.t. if* $r \in (\Lambda\varepsilon, 1)$, *then*

$$\|\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})\|_{L^\infty(B_{r_y/2}(y))} < \delta \quad \textit{and} \quad r(\mathbf{Q}_\varepsilon, y) \ge \frac{r_y}{2}.$$

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The proof is due to the key observation and dyadic decompositions of the radius.

Proof of the Dichotomy Lemma

Proof of the Dichotomy Lemma

If $z \in B_r(x) \backslash B_{2\beta r}(y)$, we choose $\sigma = \sigma(\beta) > 0$ s.t.

$$B_{\sigma r}(z) \subset B_{4r}(y) \cap (B_r(x) \backslash B_{2\beta r}(y)).$$

Monotonicity implies

$$\int_{B_{4r}(y)} |(\zeta - y) \cdot \nabla \mathbf{Q}_\varepsilon|^2 \mathrm{d}\zeta \leq C\eta r^3. \tag{2}$$

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Note $|z - y| \geq 2\beta r$. As a result,

$$\begin{aligned}
&\int_{B_{\sigma r}(z)} \left| \frac{z-y}{|z-y|} \cdot \nabla \mathbf{Q}_\varepsilon \right|^2 \\
&\leq \frac{C}{r^2} \left(\int_{B_{\sigma r}(z)} |(\zeta - y) \cdot \nabla \mathbf{Q}_\varepsilon|^2 \mathrm{d}\zeta + \int_{B_{\sigma r}(z)} |(\zeta - z) \cdot \nabla \mathbf{Q}_\varepsilon|^2 \mathrm{d}\zeta \right) \\
&\overset{(2)}{\leq} C\eta r + 2\sigma^2 r \left(\frac{1}{r} \int_{B_{\sigma r}(z)} |\nabla \mathbf{Q}_\varepsilon|^2 \right) \\
&\overset{\text{Monotonicity}}{\leq} C(\beta, M)(\eta + 2\sigma^2) r,
\end{aligned}$$

Choosing a sufficiently small

$$(\eta, \Lambda^{-1}, \sigma) = (\eta, \Lambda^{-1}, \sigma)(a, b, \beta, c, \delta, M) > 0,$$

we apply the corollary of our key observation to deduce that if $r \in (\Lambda\varepsilon, 1)$, then $\exists \eta' = \eta'(a, b, \beta, c, \delta, M) > 0$ s.t.

$$r(\mathbf{Q}_\varepsilon, z) \geq \eta' r \quad \text{and} \quad \mathrm{dist}(\mathbf{Q}_\varepsilon(z), \mathcal{N}) < \delta,$$

completing the proof.

Covering Arguments

Lemma (Main covering)

Let $\delta > 0$, $\varepsilon \in (0,1)$, $0 < r < R \leq 1$, *and* $x_0 \in B_2$. $\mathbf{Q}_\varepsilon \in H^1(B_{40}, \mathbb{S}_0)$*: a local minimizer of* (LdG) *with* $E_\varepsilon(\mathbf{Q}_\varepsilon, B_{40}) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{40})} \leq M$.
$\exists(\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$ *s.t. if* $r \in (\Lambda\varepsilon, 1)$*, then the following hold.* $\exists B_{2r_x}(x) \subset B_{2R}(x_0)$ *with* $r_x \geq r$ *s.t.*

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R(x_0) \subset B_{r_x}(x).$$

Moreover, either $r_x = r$ *or*

$$\sup_{y \in B_{2r_x}(x)} \Theta^\phi_{\frac{r_x}{20}}(\mathbf{Q}_\varepsilon, y) \leq \sup_{y \in B_{2R}(x_0)} \Theta^\phi_R(\mathbf{Q}_\varepsilon, y) - \eta.$$

By further covering, $\exists\{B_{r_y}(y)\}_{y\in\mathcal{D}}$ with

$$\inf_{y\in\mathcal{D}} r_y \geq r,$$

and

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon;\eta r,\delta)\cap B_R(x_0)\subset\bigcup_{y\in r_y} B_{r_y}(y)$$

s.t. either $r_y = r$ or

$$\sup_{\zeta\in B_{2r_y}(y)}\Theta^\phi_{r_y}(\mathbf{Q}_\varepsilon,\zeta)\leq\sup_{z\in B_{2R}(x_0)}\Theta^\phi_R(\mathbf{Q}_\varepsilon,z)-\eta.$$

Moreover, $\#\mathcal{D}\leq C$, where $C>0$ is an absolute constant.

Lemma (Final covering)

Assume as (Main covering). $\exists(\eta,\Lambda)=(\eta,\Lambda)(a,b,c,\delta,M)>0$ *s.t. if* $r\in(\Lambda\varepsilon,1)$, *then* $\exists\{x_i\}_{i=1}^N\subset B_R(x_0)$, *satisfying*

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon;\eta r,\delta)\cap B_R(x_0)\subset\bigcup_{i=1}^N B_r(x_i),$$

where $N=N(a,b,c,\delta,M)\in\mathbb{Z}_+$. *In particular,*

$$\mathcal{L}^3(B_r(\mathrm{Bad}(\mathbf{Q}_\varepsilon;\eta r,\delta)\cap B_1))\le C(a,b,c,\delta,M)r^3.$$

Lemma (Final covering)

Assume as (Main covering). $\exists(\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$ *s.t. if* $r \in (\Lambda\varepsilon, 1)$*, then* $\exists\{x_i\}_{i=1}^N \subset B_R(x_0)$*, satisfying*

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R(x_0) \subset \bigcup_{i=1}^N B_r(x_i),$$

where $N = N(a, b, c, \delta, M) \in \mathbb{Z}_+$*. In particular,*

$$\mathcal{L}^3(B_r(\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_1)) \leq C(a, b, c, \delta, M) r^3.$$

The proof follows from the iterative applications of the main covering lemma.

Proof of Main Covering Lemma

Up to a translation, let $x_0 = 0$. For $x \in B_R$ and $0 < \rho < R$, define

$$F_\eta(\mathbf{Q}_\varepsilon; x, \rho) := \left\{\Theta^\phi_{\rho/20}(\mathbf{Q}_\varepsilon, \cdot) > E - \eta\right\} \cap B_{2\rho}(x), \; E := \sup_{y \in B_{2R}} \Theta^\phi_R(\mathbf{Q}_\varepsilon, y).$$

$\exists \ell \in \mathbb{Z}_+$ s.t. $2^{-\ell}R \leq r < 2^{-\ell+1}R$. If $F_\eta(\mathbf{Q}_\varepsilon; 0, R) = \emptyset$, then B_R is as desired. WLOG, assume $F_\eta(\mathbf{Q}_\varepsilon; 0, R) \neq \emptyset$.

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$$(\eta, \Lambda^{-1}) = (\eta, \Lambda^{-1})(a, b, c, \delta, M, \rho) > 0,$$

$\exists x_1 \in F_\eta(\mathbf{Q}_\varepsilon; 0, R)$ s.t.

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R \subset B_{R/10}(x_1) \cap B_R.$$

Choose $r_1 = \frac{R}{2}$. If $F_\eta(\mathbf{Q}_\varepsilon; x_1, r_1) = \emptyset$, then the ball $B_{r_1}(x_1)$ is what we need.

Proof of Main Covering Lemma

Up to a translation, let $x_0 = 0$. For $x \in B_R$ and $0 < \rho < R$, define

$$F_\eta(\mathbf{Q}_\varepsilon; x, \rho) := \left\{\Theta^\phi_{\rho/20}(\mathbf{Q}_\varepsilon, \cdot) > E - \eta\right\} \cap B_{2\rho}(x), \ E := \sup_{y \in B_{2R}} \Theta^\phi_R(\mathbf{Q}_\varepsilon, y).$$

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$$(\eta, \Lambda^{-1}) = (\eta, \Lambda^{-1})(a, b, c, \delta, M, \rho) > 0,$$

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$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R \subset B_{R/10}(x_1) \cap B_R.$$

Choose $r_1 = \frac{R}{2}$. If $F_\eta(\mathbf{Q}_\varepsilon; x_1, r_1) = \emptyset$, then the ball $B_{r_1}(x_1)$ is what we need. On the other hand, we proceed to obtain $B_{r_2}(x_2)$ by using the dichotomy result again such that $r_2 = 2^{-2}R$ and

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R \subset B_{R/20}(x_2) \cap B_R.$$

Repeating the procedure, we will stop at the desired stage.

Completing the Proof

Regarding the L^p-convergence, we show that $\forall K \subset\subset \Omega$,

$$\sup_{\varepsilon \in (0,1)} \|\nabla \mathbf{Q}_\varepsilon\|_{L^{3,\infty}(K)} \leq C(a, b, c, K, M).$$

where $L^{3,\infty}$ is the Lorentz space and

$$\|\nabla \mathbf{Q}_\varepsilon\|_{L^{3,\infty}(K)} := \sup_{t>0} [\mathcal{L}^3(\{x \in K : |\nabla \mathbf{Q}_\varepsilon| > t\})]^{\frac{1}{3}}.$$

The result follows from the interpolation and the Sobolev embedding theorem.

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The result follows from the interpolation and the Sobolev embedding theorem. WLOG, $K = B_{1/2}$, $\Omega = B_{40}$, and

$$E_\varepsilon(\mathbf{Q}_\varepsilon, B_{40}) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{40})} \leq M.$$

By the final covering, for $r \in (\Lambda\varepsilon, 1)$,

$$\mathcal{L}^3(B_r((\{y : r(\mathbf{Q}_\varepsilon, y) < \eta r\}) \cap B_1)) \leq C(a,b,c,M)r^3,$$

where $(\eta,\Lambda) = (\eta,\Lambda)(a,b,c,M) > 0$. If $0 < r \leq \Lambda\varepsilon$, A priori estimate implies that $\forall y \in B_1$,

$$r|\nabla \mathbf{Q}_\varepsilon(y)| \leq C(\varepsilon^{-1}r + 1) \leq C(\Lambda + 1).$$

As a result, $r(\mathbf{Q}_\varepsilon, \cdot) > c_0 r$ in $B_{3/4}$, where $c_0 = c_0(a, b, c, M) > 0$. Choose $\eta \in (0, c_0)$, $\forall r \in (0, 1)$,

$$\mathcal{L}^3(B_r((\{y : r(\mathbf{Q}_\varepsilon, y) < \eta r\}) \cap B_1)) \leq C(a, b, c, M) r^3,$$

Letting $r = t^{-1}$, when $t > 0$, we have

$$\mathcal{L}^3(\{y \in B_1 : |\nabla \mathbf{Q}_\varepsilon(y)| > t\}) \leq C(a, b, c, M) t^{-3}.$$

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$$\mathcal{L}^3(\{y \in B_1 : |\nabla \mathbf{Q}_\varepsilon(y)| > t\}) \leq C(a,b,c,M) t^{-3}.$$

Fix $0 < \nu < 1$. $\forall \varepsilon > 0$ with $\Lambda\varepsilon < 1$, $\exists n(\varepsilon) \in \mathbb{Z}_+$ s.t. $\nu^{n(\varepsilon)-1} \in [\Lambda\varepsilon, \nu^{-1}\Lambda\varepsilon]$. Then

$$\int_{B_1} f(\mathbf{Q}_\varepsilon)\mathrm{d}x \leq \int_{B_{\Lambda\varepsilon}(\mathrm{Bad}(\mathbf{Q}_\varepsilon;\eta\Lambda\varepsilon,\delta)\cap B_1)} f(\mathbf{Q}_\varepsilon)\mathrm{d}x + \sum_{j=0}^{n(\varepsilon)-2} \int_{\mathcal{A}_j} f(\mathbf{Q}_\varepsilon)\mathrm{d}x + \int_{B_1 \setminus B_1(\mathrm{Bad}(\mathbf{Q}_\varepsilon;\eta,\delta))} f(\mathbf{Q}_\varepsilon)\mathrm{d}x$$

where

$$\mathcal{A}_j := B_{\nu^j}(\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta\nu^j, \delta) \cap B_1) \setminus B_{\nu^{j+1}}(\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta\nu^{j+1}, \delta) \cap B_1).$$

Nguyen-Zarnescu (2013) showed that

$$f(\mathbf{Q}_\varepsilon) \leq C(a,b,c,M)\varepsilon^4 \nu^{-4(j+1)}$$

in $\mathcal{A}_j$. Then

$$\int_{B_1} f_\varepsilon(\mathbf{Q}_\varepsilon)\mathrm{d}x \leq C\left(\varepsilon^3 + \sum_{j=0}^{n(\varepsilon)-2} \varepsilon^4 \nu^{-(j+1)} + \varepsilon^4\right) \leq C\varepsilon^3,$$

completing the proof.

Thank you for listening!