Improved Convergence of Landau-de Gennes Minimizers in the Vanishing Elasticity Limit

Wei Wang

School of Mathematical Sciences,
Peking University

Joint work with Haotong Fu and Huaijie Wang

July 21, 2025

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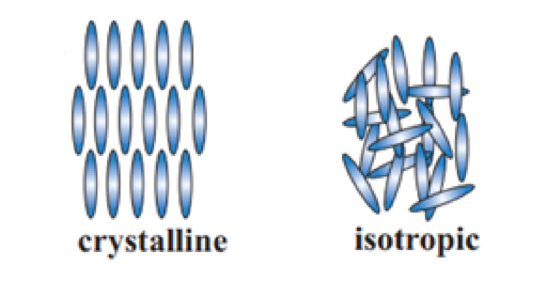
Liquid Crystals

Liquid crystals (LCs) are anisotropic fluids, where the anisotropy arises from the directional nature of the molecular geometry, physical, or chemical properties.

LCs are mesophases that exist between anisotropic crystalline and isotropic liquid phases.

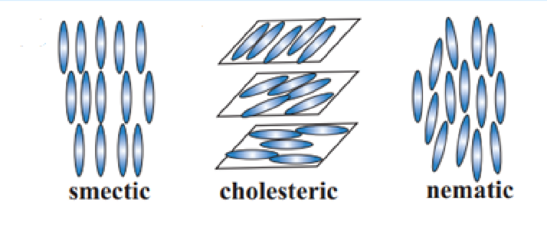


Order v.s. Disorder

Three Phases of LCs

There are three major classes of LCs.

1. **Nematic**: There is a long-range orientational order, that is, the molecules almost align parallel to each other, but no long-range correlation to the molecular center of mass positions.
2. **Cholesteric**: On a larger scale, the director of cholesteric molecules follows a helix with a spatial period.
3. **Smectic**: It has one degree of translational ordering, resulting in a layered structure.



Different LCs' Models

There are different mathematical models to characterize LCs.

1. **Vector model**: Assume that there exists a locally preferred direction $\mathbf{n}(x) \in \mathbb{S}^2$ for the alignment of LC molecules at each material point x.
   - Advantage: Simple but works well in many cases.
   - Drawback: It does not respect the head-to-tail symmetry of rod-like molecular $-\mathbf{n} \sim \mathbf{n}$, leading to an incorrect description of some systems, especially when defects are present.
2. **Molecular model**: The alignment behavior is described by an orientational distribution function $f(x, \mathbf{m})$, representing the number density of molecules with orientation $\mathbf{m} \in \mathbb{S}^2$ at material point x.
   - Advantage: Provide a more accurate description.
   - Drawback: The computational cost is usually prohibitive.
3. **Q-tensor model**: It uses a traceless symmetric 3×3-matrix $\mathbf{Q}(x)$ to describe the alignment of LC molecules at the position x.
   - Advantage: It does not assume that the molecular alignment has a preferred direction and thus can describe the biaxiality
   - Drawback: The analysis is also complicated.

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$\mathbf{Q}$-tensor Model (Landau-de Gennes Model)

From a viewpoint of molecular model, we understand $\mathbf{Q}(x)$ as

$$\mathbf{Q}(x) = \int_{\mathbb{S}^2} \left(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\mathbf{I} \right) f(x, \mathbf{m}) \mathrm{d}\mathbf{m} \in \mathbb{S}_0,$$

where $\mathbb{S}_0 := \{\mathbf{Q} \in \mathbb{M}^{3\times 3} : \mathbf{Q}^{\mathrm{T}} = \mathbf{Q},\ \operatorname{tr} \mathbf{Q} = 0\}$. Let $\mathbf{Q} \in \mathbb{S}_0$ and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of $\mathbf{Q}$. Then

$$\mathbf{Q} = \lambda_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \lambda_3 \mathbf{n}_3 \otimes \mathbf{n}_3,$$

where $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \in \mathbb{S}^2$ with $\mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij}\,(1 \le i, j \le 3)$. Rewrite $\mathbf{Q}$ as

$$\mathbf{Q} = s\left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I} \right) + r\left(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\mathbf{I} \right), \quad s, r \in \mathbb{R}$$

where $\mathbf{n}, \mathbf{m}$ are eigenvectors of $\mathbf{Q}$ with $\mathbf{n} \cdot \mathbf{m} = 0$.

1. **Isotropic**: If $s = r = 0$, namely, $\mathbf{Q} = \mathbf{O}$.
2. **Biaxial**: If s and r are different and non-zero.
3. **Uniaxial**: If $s = r \neq 0$ or $s = 0, r \neq 0$ or $s \neq 0, r = 0$. In particular,

$$\mathbf{Q} = \tilde{s}\left(\widetilde{\mathbf{n}} \otimes \widetilde{\mathbf{n}} - \frac{1}{3}\mathbf{I} \right), \quad \widetilde{\mathbf{n}} \in \mathbb{S}^2.$$

The central object in this model is the free energy functional

$$F(\mathbf{Q},\Omega) := \int_\Omega (f_e(\mathbf{Q}) + f_b(\mathbf{Q}))\mathrm{d}x, \quad \Omega \subset \mathbb{R}^d \ (d = 2, 3),$$

- $f_e(\mathbf{Q})$: The elastic energy density. We express

$$f_e(\mathbf{Q}) = \frac{L_1}{2}|\nabla \mathbf{Q}|^2 + \frac{L_2}{2}\partial_j \mathbf{Q}_{ij}\partial_k \mathbf{Q}_{ik} + \frac{L_3}{2}\partial_k \mathbf{Q}_{ij}\partial_j \mathbf{Q}_{ik}, \quad L_i \geq 0 \ (i = 1, 2, 3).$$

- $f_b(\mathbf{Q})$: The bulk potential density. The simplest form is

$$f_b(\mathbf{Q}) = -\frac{a}{2}\operatorname{tr}\mathbf{Q}^2 - \frac{b}{3}\operatorname{tr}\mathbf{Q}^3 + \frac{c}{4}(\operatorname{tr}\mathbf{Q}^2)^2,$$

  where $a \geq 0, b, c > 0$ are material constants.

Stable equilibrium configurations of the liquid crystalline system with a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ correspond to local minimizers of this functional.

Vanishing Elasticity Limit: Background

Main reductive assumption: Let $L_2 = L_3 = 0$ for simplicity.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Consider the Landau-de Gennes energy

$$E_\varepsilon(\mathbf{Q}, \Omega) := \int_\Omega e_\varepsilon(\mathbf{Q})\mathrm{d}x, \quad e_\varepsilon(\mathbf{Q}) := \frac{1}{2}|\nabla \mathbf{Q}|^2 + \frac{1}{\varepsilon^2} f(\mathbf{Q}), \tag{LdG}$$

where

$$f(\mathbf{Q}) = k - \frac{a}{2}\operatorname{tr}\mathbf{Q}^2 - \frac{b}{3}\operatorname{tr}\mathbf{Q}^3 + \frac{c}{4}(\operatorname{tr}\mathbf{Q}^2)^2, \quad \mathbf{Q} \in \mathbb{S}_0.$$

Here, k is constant s.t. $\inf_{\mathbf{Q}\in\mathbb{S}_0} f(\mathbf{Q}) = 0$.

The observation in physics: Elastic constants are so small compared to the bulk constants.

Question (Vanishing elasticity limit)

The asymptotic behavior of minimizers (or critical points) to (LdG) as $\varepsilon \to 0^+$.

Limiting Functional

Define the vacuum manifold as

$$\mathcal{N} := \left\{ s_* \left(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathbf{I} \right) : \mathbf{n} \in \mathbb{S}^2 \right\} = f^{-1}(0),$$

where

$$s_* := s_*(a, b, c) = \frac{b + \sqrt{b^2 + 24ac}}{4c}.$$

Letting $\varepsilon \to 0^+$, the term $\frac{1}{\varepsilon^2} f(\mathbf{Q})$ in (LdG) forces the minimizers to take the value in $\mathcal{N}$.

The limiting energy functional is

$$E(\mathbf{Q}, \Omega) := \int_\Omega |\nabla \mathbf{Q}|^2 \mathrm{d}x, \quad \mathbf{Q} \in H^1(\Omega, \mathcal{N}). \tag{Dir}$$

Vanishing elasticity limit: Previous Results

1. Majumdar-Zarnescu (2010) proved: Assume $\Omega \in C^\infty$, $\mathbf{Q}_b \in C^\infty(\partial\Omega, \mathcal{N})$, and $\{\mathbf{Q}_\varepsilon\}$ are minimizers of (LdG) with $\mathbf{Q}_\varepsilon|_{\partial\Omega} = \mathbf{Q}_b$. Then, $\exists \varepsilon_i \to 0^+$, s.t.
   - $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ in H^1; $\mathbf{Q}_0$: a minimizer of (Dir) with $\mathbf{Q}_0|_{\partial\Omega} = \mathbf{Q}_b$.
   - $\forall K \subset\subset \Omega \backslash \operatorname{sing}(\mathbf{Q}_0)$, $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ uniformly in K.
2. Nguyen-Zarnescu (2013) showed that $\forall j \in \mathbb{Z}_+$ and $\forall K \subset\subset \Omega \backslash \operatorname{sing}(\mathbf{Q}_0)$, $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ in $C^j(K)$
3. Canevari (2017) studied the asymptotic behavior of $\mathbf{Q}_\varepsilon$ with line defects, and W.-Zhang (2024) extended his results to include sextic bulk energy potential.
4. Contreras-Lamy (2022) and Feng-Hong (2022) generalized results by Nguyen-Zarnescu to the cases with non-zero L_2 and L_3.
5. Wang-Wang-Zhang (2017) and Huang-Lin (2022) analyzed the gradient flow setting of $\mathbf{Q}$-tensor model.

Vanishing Elasticity Limit: Main Results

Theorem (Fu-Wang-W. 2025)

Suppose $\{\mathbf{Q}_\varepsilon\}_{\varepsilon\in(0,1)} \subset H^1_{\mathrm{loc}}(\Omega, \mathbb{S}_0)$ *are local minimizers of* (LdG) *s.t.*

$$\sup_{\varepsilon\in(0,1)} \left(E_\varepsilon(\mathbf{Q}_\varepsilon, \Omega) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)}\right) \leq M.$$

Then, $\exists \mathbf{Q}_0 \in H^1(\Omega, \mathcal{N})$*, a local minimizer of* (Dir) *s.t.* $\exists \varepsilon_i \to 0^+$, $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ *strongly in* $H^1_{\mathrm{loc}}(\Omega, \mathbb{S}_0)$*, and the following properties hold.*

1. $\forall p \in (1, +\infty)$ *and* $K \subset\subset \Omega$,

$$\lim_{i\to+\infty} \|\mathbf{Q}_{\varepsilon_i} - \mathbf{Q}_0\|_{L^p(K)} = 0.$$

2. $\forall K \subset\subset \Omega$,

$$\int_K \frac{1}{\varepsilon_i^2} f(\mathbf{Q}_{\varepsilon_i}) \mathrm{d}x \leq C\varepsilon_i,$$

where $C = C(a, b, c, K, M) > 0$.

Remarks on Main Results

1. Our arguments also show that $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ strongly in $W^{1,p}_{\mathrm{loc}}$ $\forall 1 < p < 3$.
2. We indeed prove that

$$\int_K \frac{1}{\varepsilon^2} f(\mathbf{Q}_\varepsilon)\mathrm{d}x \le C(a,b,c,K)\varepsilon, \quad K \subset\subset \Omega.$$

3. Our convergence results in the main theorem are already optimal.
4. The condition

$$\sup_{\varepsilon\in(0,1)} \left(E_\varepsilon(\mathbf{Q}_\varepsilon,\Omega) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(\Omega)}\right) \le M.$$

   is satisfied, e.g. if $\Omega \in C^\infty$ and $\mathbf{Q}_\varepsilon|_{\partial\Omega} = \mathbf{Q}_b \in C^\infty(\partial\Omega,\mathcal{N})$.

5. Our method applies to other settings, such as the generalized Ginzburg-Landau model (Monteil-Rodiac-Schaftingen, 2021) and torus-like solutions of (LdG) as in (Dipasquale-Millot-Pisante, 2021).

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Problem Setting

$\{\mathbf{Q}_\varepsilon\} \subset H^1(B_1, \mathbb{S}_0)$ are minimizers of (LdG) with

$$\mathbf{Q}|_{\partial B_1} = \mathbf{Q}_\mathrm{b} := s_* \left(x \otimes x - \frac{1}{3}\mathbf{I} \right).$$

$\exists \varepsilon_i \to 0^+$ s.t.

$$\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0 = s_* \left(\mathbf{n}_0 \otimes \mathbf{n}_0 - \frac{1}{3}\mathbf{I} \right) \quad \text{in } H^1(B_1, \mathbb{S}_0),$$

where $\mathbf{n}_0$ is a minimizer of

$$\inf_{\mathbf{n}=x \text{ on } \partial B_1} \int_{B_1} |\nabla \mathbf{n}|^2 \mathrm{d}x.$$

Brézis-Coron-Lieb (1986) proved that $\mathbf{n}_0(x) = \frac{x}{|x|}$. Then

$$\mathbf{Q}_0 = s_* \left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\mathbf{I} \right)$$

is the so-called hedgehog solution.

Sharpness of the Main Theorem

L^p **convergence**:
Since $\partial B_1 \in C^\infty$ and $\mathrm{sing}(\mathbf{Q}_0) = \{0\}$, we have

$$\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0 \text{ uniformly in } B_1 \backslash B_r, \quad \forall r \in (0,1).$$

Note that $\mathbf{Q}_0$ is not smooth, showing that the L^p convergence is sharp. Indeed, if

$$\lim_{\varepsilon \to 0^+} \|\mathbf{Q}_\varepsilon - \mathbf{Q}_0\|_{L^\infty(B_{\frac{1}{2}})} = 0,$$

then $\mathbf{Q}_0$ is continuous at $0 \Rightarrow \mathbf{Q}_0 \in C^\infty(B_{\frac{1}{2}})$, a contradiction.

The convergence rate $\int \varepsilon^{-2} f(\mathbf{Q}_\varepsilon) \mathrm{d}x$:
The sharpness follows from the following proposition.

Proposition

For $\mathbf{Q}_{\varepsilon_i}$ given as above, there is $C = C(a,b,c) > 0$ s.t. for $i \gg 1$,

$$\int_{B_{\frac{3}{4}}} f(\mathbf{Q}_{\varepsilon_i}) \mathrm{d}x \geq \frac{\varepsilon_i^3}{C}.$$

Proof of the Example's Properties

Lemma

$\exists \eta = \eta(a,b,c) > 0$ s.t. $\{\mathbf{Q} \in \mathbb{S}_0 : f(\mathbf{Q}) < \eta\} \subset \{\mathbf{Q} \in \mathbb{S}_0 : \lambda_1(\mathbf{Q}) > \lambda_2(\mathbf{Q})\}$, where $\lambda_1(\mathbf{Q}) \geq \lambda_2(\mathbf{Q}) \geq \lambda_3(\mathbf{Q})$ are three eigenvalues of $\mathbf{Q}$ in order.

Assume that $\lambda_1(\mathbf{Q}) = \lambda_2(\mathbf{Q}) = \lambda$. We have $\lambda_3(\mathbf{Q}) = -2\lambda$. As a result,

$$f(\mathbf{Q}) = k - 3a\lambda^2 + 2b\lambda^3 + 9c\lambda^4 := g(\lambda).$$

By calculations, $k = \frac{s_*^2}{27}(9a + 2bs_* - 3cs_*^2)$. Indeed, $g(\lambda)$ achieves the minimum at $\lambda_* = \frac{-b+\sqrt{b^2+24ac}}{12c}$. Moreover, $g(\lambda_*) > 0$. Choosing $\eta \in (0, g(\lambda_*))$, we obtain the desired property.

Lemma

$\nexists \mathbf{Q} \in C(\overline{B}_1, \mathcal{N})$ s.t. $\mathbf{Q}|_{\partial B_1}(x) = s_* \left(x \otimes x - \frac{1}{3}\mathbf{I}\right)$.

The existence of such a $\mathbf{Q}$ contradicts the property that there is no retraction from B_1 to $\partial B_1 = \mathbb{S}^2$.

Claim: $\exists y_i \in B_{\frac{1}{2}}$ s.t. $f(\mathbf{Q}_{\varepsilon_i}(y_i)) > \eta(a,b,c) > 0$. If $\forall y \in B_1$, $f(\mathbf{Q}_{\varepsilon_i})(y) \ll 1$, then $\forall y \in \overline{B}_1$, $\lambda_1(\mathbf{Q}_{\varepsilon_i}) > \lambda_2(\mathbf{Q}_{\varepsilon_2})$. Thus, $\exists$ C^1 nearest point projection

$$\Pi : \{\mathbf{Q} \in \mathbb{S}_0 : \lambda_1(\mathbf{Q}) > \lambda_2(\mathbf{Q})\} \to \mathcal{N}.$$

Hence $\Pi \circ \mathbf{Q}_{\varepsilon_i} \in C(\overline{B}_1, \mathcal{N})$ and $\Pi \circ \mathbf{Q}_{\varepsilon}|_{\partial B_1} = s_* \left(x \otimes x - \frac{1}{3}\mathbf{I}\right)$, a contradiction. Recalling that $\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0$ uniformly in $B_1 \backslash B_{\frac{1}{2}}$, for $i \gg 1$, we have $y_i \in B_{\frac{1}{2}}$.

By elliptic estimates,

$$\|\nabla \mathbf{Q}_{\varepsilon_i}\|_{L^\infty(B_{\frac{3}{4}})} \leq \frac{C(a,b,c)}{\varepsilon_i}.$$

Then, $\exists \delta = \delta(a,b,c) \in (0, \frac{1}{4})$ s.t.

$$\inf_{y \in B_{\delta\varepsilon_i}(y_i)} f(\mathbf{Q}_{\varepsilon_i}(y)) > \frac{\eta}{2}.$$

If $\varepsilon_i \in (0, \frac{1}{2})$, then $B_{\delta\varepsilon_i}(y_i) \subset B_{\frac{3}{4}}$ and

$$\int_{B_{\frac{3}{4}}} f(\mathbf{Q}_{\varepsilon_i}) \geq \int_{B_{\delta\varepsilon_i}(y_i)} f(\mathbf{Q}_{\varepsilon_i}) \geq \frac{\varepsilon_i^3}{C(a,b,c)}.$$

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Tool I: Monotonicity

Definition

Let $\phi \in C^\infty([0,+\infty),\mathbb{R}_{\geq 0})$ s.t. the following properties hold.

1. $\operatorname{supp}\phi \subset [0,10)$.
2. $\forall t \in [0,+\infty)$, $\phi(t) \geq 0$ and $|\phi'(t)| \leq 100$.
3. $-2 \leq \phi'(t) \leq -1$ $\forall t \in [0,8]$.
4. $\forall t \in \mathbb{R}_+$, $\phi'(t) \leq 0$.

Let $\mathbf{Q} \in H^1(\Omega,\mathbb{S}_0)$, $x \in \Omega$, and $0 < r < \frac{1}{10}\operatorname{dist}(x,\partial\Omega)$. Define

$$\Theta_r^\phi(\mathbf{Q},x) := \frac{1}{r}\int e_\varepsilon(\mathbf{Q})\phi\left(\frac{|y-x|^2}{r^2}\right)\mathrm{d}y.$$

Proposition (Monotonicity)

Assume that $\mathbf{Q}_\varepsilon : \Omega \to \mathbb{S}_0$ *is a smooth critical point of* (LdG). *Let* $x \in \Omega$ *and* $0 < r < R < \frac{1}{10}\operatorname{dist}(x, \partial\Omega)$. *Then*

$$\Theta_R^\phi(\mathbf{Q}_\varepsilon, x) - \Theta_r^\phi(\mathbf{Q}_\varepsilon, x)$$
$$= \int_r^R \left[-\frac{2}{\rho^2} \int \left| \frac{y-x}{\rho} \cdot \nabla \mathbf{Q}_\varepsilon \right| \phi' \left(\frac{|y-x|^2}{\rho^2} \right) + \frac{2}{\varepsilon^2 \rho^2} \int f(\mathbf{Q}_\varepsilon) \phi \left(\frac{|y-x|^2}{\rho^2} \right) \right] \mathrm{d}\rho.$$

The proof follows from the stress identity

$$\partial_j (e_\varepsilon(\mathbf{Q}_\varepsilon)\delta_{ij} - \partial_i \mathbf{Q}_\varepsilon : \partial_j \mathbf{Q}_\varepsilon) = 0, \quad \forall i \in \{1, 2, 3\}.$$

Tool II: Compactness

Proposition (H^1-compactness)

$\{\mathbf{Q}_\varepsilon\}_{\varepsilon\in(0,1)} \subset H^1_{\mathrm{loc}}(\Omega, \mathbb{S}_0)$*: a sequence of local minimizers of* (LdG) *s.t.*

$$\sup_{\varepsilon\in(0,1)} \left(E_\varepsilon(\mathbf{Q}_\varepsilon, K) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(K)}\right) < +\infty, \quad \forall K \subset\subset \Omega.$$

Then, $\exists \varepsilon_i \to 0^+$ *s.t.*

$$\begin{aligned} &\mathbf{Q}_{\varepsilon_i} \to \mathbf{Q}_0 \text{ strongly in } H^1_{\mathrm{loc}}(\Omega, \mathbb{S}_0), \\ &\frac{1}{\varepsilon_i^2} f(\mathbf{Q}_{\varepsilon_i}) \to 0 \text{ strongly in } L^1_{\mathrm{loc}}(\Omega), \end{aligned}$$

where $\mathbf{Q}_0 \in H^1_{\mathrm{loc}}(\Omega, \mathcal{N})$ *is a local minimizer of* (Dir).

The proof follows from the application of a comparison map using interpolation results developed by Luckhaus (1988).

Tool III: Regularity and Partial Regularity

Lemma (A priori estimate)

$\mathbf{Q}_\varepsilon \in H^1(B_{2r}(x), \mathbb{S}_0)$: *a critical point of* (LdG) *with* $\|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{2r}(x))} \le M$. *Then,* $\mathbf{Q}_\varepsilon \in C^\infty$ *and* $\|\nabla \mathbf{Q}_\varepsilon\|_{L^\infty(B_r(x))} \le C(a,b,c,M)(\varepsilon^{-1} + r^{-1})$.

Lemma (Partial regularity I)

$\mathbf{Q}_\varepsilon \in H^1(B_{4r}(x), \mathbb{S}_0)$: *a local minimizer of* (LdG) *with* $\|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{4r}(x))} \le M$. $\forall \delta > 0$, $\exists (\eta, \Lambda) = (\eta, \Lambda)(a,b,c,\delta,M) > 0$ *s.t. if* $r \in (\Lambda\varepsilon, 1)$ *and*

$$r^{-1} E_\varepsilon(\mathbf{Q}_\varepsilon, B_{2r}(x)) < \eta \quad \Rightarrow \quad \|\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})\|_{L^\infty(B_r(x))} < \delta.$$

Lemma (Partial regularity II)

Assume as (Partial regularity I). $\exists (\eta, \Lambda) = (\eta, \Lambda)(a,b,c,M) > 0$ *s.t. if* $r \in (\Lambda\varepsilon, 1)$,

$$r^{-1} E_\varepsilon(\mathbf{Q}_\varepsilon, B_{2r}(x)) < \eta \quad \Rightarrow \quad r^2 \|e_\varepsilon(\mathbf{Q}_\varepsilon)\|_{L^\infty(B_r(x))} \le C_0(a,b,c,M,\eta),$$

with $\lim_{\eta \to 0^+} C_0(a,b,c,M,\eta) = 0$.

Characterization of the Bad Behavior

We first introduce the regular scale to give a quantitative characterization of regularity.

Definition (Regular scale)

Let $\varepsilon \in (0,1)$. $\mathbf{Q}_\varepsilon \in H^1(B_2, \mathbb{S}_0)$: a local minimizer of (LdG). For $x \in B_1$, we define

$$r(\mathbf{Q}_\varepsilon, x) := \sup\{0 \le r \le 1 : r^2 \|e_\varepsilon(\mathbf{Q}_\varepsilon)\|_{L^\infty(B_r(x))} \le 1\}.$$

For a minimizer $\mathbf{Q}_\varepsilon$ of (LdG) in $\Omega \subset \mathbb{R}^3$, the partial regularity lemmas motivate us that the "bad points" are those where $e_\varepsilon(\mathbf{Q}_\varepsilon)$ and $\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})$ are large. To this reason, for $\mathbf{Q}_\varepsilon$ as in (12), we define the collection of *bad points* with $\delta, r > 0$ being parameters as

$$\operatorname{Bad}(\mathbf{Q}_\varepsilon; r, \delta) := \{y \in \Omega : r(\mathbf{Q}_\varepsilon, y) < r\} \cup \{y \in \Omega : \operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N}) > \delta\}.$$

Crucial Lemma: Dichotomy

Proposition (Dichotomy result)

$\beta \in (0, \frac{1}{2})$, $\delta > 0$. $\mathbf{Q}_\varepsilon \in H^1(B_{40}, \mathbb{S}_0)$: *a local minimizer of* (LdG) *with*

$$E_\varepsilon(\mathbf{Q}_\varepsilon, B_{40}) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{40})} \leq M.$$

$\exists(\eta, \eta', \Lambda) = (\eta, \eta', \Lambda)(a, b, \beta, c, \delta, M) > 0$ *s.t. if for some* $y \in B_{2r}(x)$,

$$\Theta_r^\phi(\mathbf{Q}_\varepsilon, y) - \Theta_{\frac{r}{2}}^\phi(\mathbf{Q}_\varepsilon, y) < \eta,$$

and $r \in (\Lambda\varepsilon, 1)$, *then*

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta' r, \delta) \cap B_r(x) \subset B_{2\beta r}(y).$$

In other words, if $r \in (\Lambda\varepsilon, 1)$, either

$$\Theta_{\frac{r}{2}}^\phi(\mathbf{Q}_\varepsilon, y) < \Theta_r^\phi(\mathbf{Q}_\varepsilon, y) - \eta$$

or

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta' r, \delta) \cap B_r(x) \subset B_{2\beta r}(y).$$

Key Observation: High Symmetry $\Rightarrow$ Regularity

Lemma (Symmetry implies regularity)

$\varepsilon \in (0,1)$. $\mathbf{Q}_\varepsilon \in H^1(B_{10r}(x), \mathbb{S}_0)$: *a local minimizer of* (LdG) *with*

$$r^{-1} E_\varepsilon(\mathbf{Q}_\varepsilon, B_{10r}(x)) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{10r}(x))} \le M.$$

$\forall \delta > 0$, $\exists (\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$, *s.t. if* $r \in (\Lambda\varepsilon, 1)$,

$$\Theta_r^\phi(\mathbf{Q}_\varepsilon, x) - \Theta_{\frac{r}{2}}^\phi(\mathbf{Q}_\varepsilon, x) < \eta, \quad \inf_{v \in \mathbb{S}^2} \frac{1}{r} \int_{B_r(x)} |v \cdot \nabla \mathbf{Q}_\varepsilon|^2 < \eta,$$

then

$$\|\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})\|_{L^\infty\left(B_{\frac{r}{2}}(x)\right)} < \delta \quad \textit{and} \quad r(\mathbf{Q}_\varepsilon, x) \ge \frac{r}{2}.$$

Moreover, $x \notin \operatorname{Bad}\left(\mathbf{Q}_\varepsilon; \frac{r}{2}, \delta\right)$.

Pinching of density+Approximated 1-invariance$\Rightarrow$Regularity

Proof of the Key Observation

If not, $\exists \mathbf{Q}_{\varepsilon_i}$, with $\frac{\varepsilon_i}{r_i} \to 0^+$,

$$\Theta^{\phi}_{r_i}(\mathbf{Q}_{\varepsilon_i}, x_i) - \Theta^{\phi}_{\frac{r_i}{2}}(\mathbf{Q}_{\varepsilon_i}, x_i) < i^{-1}, \quad \inf_{v \in \mathbb{S}^2} \frac{1}{r_i} \int_{B_{r_i}(x)} |v \cdot \nabla \mathbf{Q}_{\varepsilon_i}|^2 < i^{-1}, \qquad (1)$$

but $r(\mathbf{Q}_{\varepsilon_i}, x_i) < \frac{r_i}{2}$ or $\mathrm{dist}(\mathbf{Q}_{\varepsilon_i}, \mathcal{N}) \geq \delta > 0$ in $B_{\frac{r_i}{2}}(x_i)$. Let

$$\widetilde{\mathbf{Q}}_{\widetilde{\varepsilon}_i}(y) := \mathbf{Q}_{\varepsilon_i}(x_i + r_i y).$$

Up to a subsequence,

$$\widetilde{\mathbf{Q}}_{\widetilde{\varepsilon}_i} \to \widetilde{\mathbf{Q}}_0 \in \mathcal{N} \text{ strongly in } H^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{S}_0),$$

where $\widetilde{\mathbf{Q}}_0$ is a local minimizer of (Dir).

$$(1) \Rightarrow \text{homogeneity} + 1\text{-invariance} \Rightarrow \widetilde{\mathbf{Q}}_0 \text{ is a constant.}$$

The result follows from the partial regularity.

A Corollary of the Key Observation

Corollary (Smaller ball implies pinching)

$\varepsilon \in (0,1)$, $\delta > 0$. $\mathbf{Q}_\varepsilon \in H^1(B_{40}, \mathbb{S}_0)$: *a local minimizer of* (LdG) *with*

$$E_\varepsilon(\mathbf{Q}_\varepsilon, B_{40}) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{40})} \le M.$$

$\exists(\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$, *s.t. if*

$$\inf_{v \in \mathbb{S}^2} \frac{1}{r} \int_{B_r(x)} |v \cdot \nabla u|^2 < \eta,$$

then the following holds. $\forall y \in B_{\frac{r}{2}}(x)$, $\exists r_y \in [\eta^{\frac{1}{2}} r, 1]$ *s.t. if* $r \in (\Lambda\varepsilon, 1)$, *then*

$$\|\operatorname{dist}(\mathbf{Q}_\varepsilon, \mathcal{N})\|_{L^\infty\left(B_{\frac{r_y}{2}}(y)\right)} < \delta \quad \textit{and} \quad r(\mathbf{Q}_\varepsilon, y) \ge \frac{r_y}{2}.$$

The proof is due to the key observation and dyadic decompositions of the radius.

Proof of the Dichotomy Lemma

If $z \in B_r(x) \backslash B_{2\beta r}(y)$, we choose $\sigma = \sigma(\beta) > 0$ s.t.

$$B_{\sigma r}(z) \subset B_{4r}(y) \cap (B_r(x) \backslash B_{2\beta r}(y)).$$

Monotonicity implies

$$\int_{B_{4r}(y)} |(\zeta - y) \cdot \nabla \mathbf{Q}_\varepsilon|^2 \mathrm{d}\zeta \leq C\eta r^3. \tag{2}$$

Note $|z - y| \geq 2\beta r$. As a result,

$$\begin{aligned}
&\int_{B_{\sigma r}(z)} \left| \frac{z-y}{|z-y|} \cdot \nabla \mathbf{Q}_\varepsilon \right|^2 \\
&\leq \frac{C}{r^2} \left(\int_{B_{\sigma r}(z)} |(\zeta - y) \cdot \nabla \mathbf{Q}_\varepsilon|^2 \mathrm{d}\zeta + \int_{B_{\sigma r}(z)} |(\zeta - z) \cdot \nabla \mathbf{Q}_\varepsilon|^2 \mathrm{d}\zeta \right) \\
&\overset{(2)}{\leq} C\eta r + 2\sigma^2 r \left(\frac{1}{r} \int_{B_{\sigma r}} |\nabla \mathbf{Q}_\varepsilon|^2 \right) \\
&\overset{\text{Monotonicity}}{\leq} C(\beta, M)(\eta + 2\sigma^2) r,
\end{aligned}$$

Choosing a sufficiently small

$$(\eta, \Lambda^{-1}, \sigma) = (\eta, \Lambda^{-1}, \sigma)(a, b, \beta, c, \delta, M) > 0,$$

we apply the corollary of our key observation to deduce that if $r \in (\Lambda\varepsilon, 1)$, then $\exists \eta' = \eta'(a, b, \beta, c, \delta, M) > 0$ s.t.

$$r(\mathbf{Q}_\varepsilon, z) \geq \eta' r \quad \text{and} \quad \mathrm{dist}(\mathbf{Q}_\varepsilon(z), \mathcal{N}) < \delta,$$

completing the proof.

Covering Arguments

Lemma (Main covering)

Let $\delta > 0$, $\varepsilon \in (0,1)$, $0 < r < R \leq 1$, *and* $x_0 \in B_2$. $\mathbf{Q}_\varepsilon \in H^1(B_{40}, \mathbb{S}_0)$*: a local minimizer of* (LdG) *with* $E_\varepsilon(\mathbf{Q}_\varepsilon, B_{40}) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{40})} \leq M$.
$\exists(\eta, \Lambda) = (\eta, \Lambda)(a, b, c, \delta, M) > 0$ *s.t. if* $r \in (\Lambda\varepsilon, 1)$*, the following hold.*
$\exists B_{2r_x}(x) \subset B_{2R}(x_0)$ *with* $r_x \geq r$ *s.t.*

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R(x_0) \subset B_{r_x}(x).$$

Moreover, either $r_x = r$ *or* $\sup_{y \in B_{2r_x}(x)} \Theta^\phi_{\frac{r_x}{20}}(\mathbf{Q}_\varepsilon, y) \leq \sup_{y \in B_{2R}(x_0)} \Theta^\phi_R(\mathbf{Q}_\varepsilon, y) - \eta$.

By further covering, there is a collection of balls $\{B_{r_y}(y)\}_{y \in \mathcal{D}}$ with $\inf_{y \in \mathcal{D}} r_y \geq r$,

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R(x_0) \subset \bigcup_{y \in r_y} B_{r_y}(y)$$

such that either $r_y = r$ or $\sup_{\zeta \in B_{2r_y}(y)} \Theta^\phi_{r_y}(\mathbf{Q}_\varepsilon, \zeta) \leq \sup_{z \in B_{2R}(x_0)} \Theta^\phi_R(\mathbf{Q}_\varepsilon, z) - \eta$.
Moreover, $\#\mathcal{D} \leq C$, where $C > 0$ is an absolute constant.

Lemma (Final covering)

$\exists(\eta,\Lambda)=(\eta,\Lambda)(a,b,c,\delta,M)>0$ s.t. if $r\in(\Lambda\varepsilon,1)$, then $\exists\{x_i\}_{i=1}^N\subset B_R(x_0)$, satisfying

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon;\eta r,\delta)\cap B_R(x_0)\subset\bigcup_{i=1}^N B_r(x_i),$$

where $N=N(a,b,c,\delta,M)\in\mathbb{Z}_+$. In particular,

$$\mathcal{L}^3(B_r(\mathrm{Bad}(\mathbf{Q}_\varepsilon;\eta r,\delta)\cap B_1))\leq C(a,b,c,\delta,M)r^3.$$

The proof follows from the iterative applications of the main covering lemma.

Proof of Main Covering Lemma

Up to a translation, let $x_0 = 0$. For $x \in B_R$ and $0 < \rho < R$, define

$$F_\eta(\mathbf{Q}_\varepsilon; x, \rho) := \left\{ y \in B_{2\rho}(x) : \Theta^{\phi}_{\frac{\rho}{20}}(\mathbf{Q}_\varepsilon, y) > E - \eta \right\}, \quad E := \sup_{y \in B_{2R}} \Theta^{\phi}_{R}(\mathbf{Q}_\varepsilon, y).$$

$\exists \ell \in \mathbb{Z}_+$ s.t. $2^{-\ell} R \le r < 2^{-\ell+1} R$. If $F_\eta(\mathbf{Q}_\varepsilon; 0, R) = \emptyset$, then B_R is as desired. WLOG, assume $F_\eta(\mathbf{Q}_\varepsilon; 0, R) \neq \emptyset$. For $i = 1$, by the dichotomy result, for sufficiently small

$$(\eta, \Lambda^{-1}) = (\eta, \Lambda^{-1})(a, b, c, \delta, M, \rho) > 0,$$

$\exists x_1 \in F_\eta(\mathbf{Q}_\varepsilon; 0, R)$ s.t.

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R \subset B_{\frac{R}{10}}(x_1) \cap B_R.$$

Choose $r_1 = \frac{R}{2}$. If $F_\eta(\mathbf{Q}_\varepsilon; x_1, r_1) = \emptyset$, then the ball $B_{r_1}(x_1)$ is what we need. On the other hand, we proceed to obtain $B_{r_2}(x_2)$ by using the dichotomy result again such that $r_2 = \frac{R}{2^2}$ and

$$\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta r, \delta) \cap B_R \subset B_{\frac{1}{2} \cdot \frac{R}{10}}(x_2) \cap B_R.$$

Repeating the procedure, we will stop at the desired stage.

Completing the Proof

Regarding the L^p convergence, we show that $\forall K \subset\subset \Omega$,

$$\sup_{\varepsilon\in(0,1)} \|\nabla \mathbf{Q}_\varepsilon\|_{L^{3,\infty}(K)} \leq C(a,b,c,K,M).$$

where $L^{3,\infty}$ is the Lorentz space and

$$\|\nabla \mathbf{Q}_\varepsilon\|_{L^{3,\infty}(K)} := \sup_{t>0}[\mathcal{L}^3(\{x\in K : |\nabla \mathbf{Q}_\varepsilon| > t\})]^{\frac{1}{3}}.$$

The convergence property in $L^p(\Omega, \mathbb{S}_0)$ follows directly from the interpolation and the Sobolev embedding theorem. WLOG, $K = B_{\frac{1}{2}}$, $\Omega = B_{40}$, and

$$E_\varepsilon(\mathbf{Q}_\varepsilon, B_{40}) + \|\mathbf{Q}_\varepsilon\|_{L^\infty(B_{40})} \leq M.$$

By the final covering, for $r \in (\Lambda\varepsilon, 1)$,

$$\mathcal{L}^3(B_r((\{y : r(\mathbf{Q}_\varepsilon, y) < \eta r\}) \cap B_1)) \leq C(a,b,c,M)r^3,$$

where $(\eta,\Lambda) = (\eta,\Lambda)(a,b,c,M) > 0$. If $0 < r \leq \Lambda\varepsilon$, A priori estimate implies that $\forall y \in B_1$,

$$r|\nabla \mathbf{Q}_\varepsilon(y)| \leq C\left(\frac{r}{\varepsilon} + 1\right) \leq C(\Lambda + 1).$$

As a result, $r(\mathbf{Q}_\varepsilon, \cdot) > c_0 r$ in $B_{\frac{3}{4}}$, where $c_0 = c_0(a, b, c, M) > 0$. Choose $\eta \in (0, c_0)$, $\forall r \in (0, 1)$,

$$\mathcal{L}^3(B_r((\{y : r(\mathbf{Q}_\varepsilon, y) < \eta r\}) \cap B_1)) \leq C(a, b, c, M) r^3,$$

Letting $r = t^{-1}$, when $t > 0$, we have

$$\mathcal{L}^3(\{y \in B_1 : |\nabla \mathbf{Q}_\varepsilon(y)| > t\}) \leq C(a, b, c, M) t^{-3}.$$

Fix $0 < \nu < 1$. $\forall \varepsilon > 0$ with $\Lambda\varepsilon < 1$, $\exists n(\varepsilon) \in \mathbb{Z}_+$ s.t. $\nu^{n(\varepsilon)-1} \in [\Lambda\varepsilon, \nu^{-1}\Lambda\varepsilon]$. Then

$$\begin{aligned}\int_{B_1} f(\mathbf{Q}_\varepsilon)\mathrm{d}x \leq & \int_{B_{\Lambda\varepsilon}(\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta\Lambda\varepsilon, \delta) \cap B_1)} f(\mathbf{Q}_\varepsilon)\mathrm{d}x \\ & + \sum_{j=0}^{n(\varepsilon)-2} \int_{\mathcal{A}_j} f(\mathbf{Q}_\varepsilon)\mathrm{d}x + \int_{B_1 \setminus B_1(\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta, \delta))} f(\mathbf{Q}_\varepsilon)\mathrm{d}x\end{aligned}$$

where

$$\mathcal{A}_j := B_{\nu^j}(\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta\nu^j, \delta) \cap B_1) \setminus B_{\nu^{j+1}}(\mathrm{Bad}(\mathbf{Q}_\varepsilon; \eta\nu^{j+1}, \delta) \cap B_1).$$

Nguyen-Zarnescu (2013) showed that

$$f(\mathbf{Q}_\varepsilon) \leq C(a,b,c,M)\varepsilon^4\nu^{-4(j+1)}$$

in $\mathcal{A}_j$. Then

$$\int_{B_1} f_\varepsilon(\mathbf{Q}_\varepsilon)\mathrm{d}x \leq C\left(\varepsilon^3 + \sum_{j=0}^{n(\varepsilon)-2}\varepsilon^4\nu^{-(j+1)} + \varepsilon^4\right) \leq C\varepsilon^3,$$

completing the proof.

Thank you for listening!