SOME QUESTIONS OF HARMONIC ANALYSIS

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Abstract. This is a compilation of questions on harmonic analysis. The majority of these questions are sourced from the lecture notes on harmonic analysis delivered by Dongyi Wei, a highly talented professor from the School of Mathematical Science at Peking University. Due to his exceptional abilities, the problems in this course are quite challenging, which served as the motivation behind creating this note. Additionally, this note will also include some fundamental questions from GTM249 and GTM250. Our intention is to provide useful study materials for the qualifying exam of this course at Peking University.

Contents

1. Fourier series and Fourier transform

1. Let $f \in BV(\mathbb{T})$, show that $\widehat{f}(k) = O(\frac{1}{|k|})$.

Proof. We will show a more precise result, i.e. $|\widehat{f}(k)| \leq \frac{\mathrm{Var(f)}}{2\pi|k|}$ for $k \neq 0$. By using integration by parts and the observation that f is of period 1, we can get that

$$\widehat{f}(k) = \int_{\mathbb{T}} f(x)e^{-2\pi ikx}dx = -\int_{\mathbb{T}} \frac{e^{-2\pi ikx}}{2\pi ik}df.$$

By the property of $BV(\mathbb{T})$, the norm of the measure df is exactly $\mathrm{Var}(f)$ and then $|\widehat{f}(k)| \leq \frac{\mathrm{Var}(f)}{2\pi|k|}$, which completes the proof. $\square$

Remark 1.1. Indeed, we have an additional method to establish its validity, considering the arguments presented in the proof of the Riemann-Lebesgue lemma. Specifically, given that function f is defined on the interval $\mathbb{T}$, we have

$$\widehat{f}(k) = \int_0^1 (-1)^j f(x)e^{-2\pi ik\left(x+\frac{j}{2k}\right)}dx = \int_0^1 (-1)^j f\left(x - \frac{j}{2k}\right)e^{-2\pi ikx}dx.$$

Summing the index j up from 0 to $k-1$, we can obtain

$$|2k\widehat{f}(k)| \leq \int_0^1 \sum_{j=0}^{k-1} \left| f\left(x - \frac{2j}{2k}\right) - f\left(x - \frac{2j+1}{2k}\right)\right| dx \leq \mathrm{Var}(f),$$

which implies that $\|\widehat{f}(k)\| \leq \frac{\mathrm{Var}(f)}{2|k|}$. From this observation, we can see that the obtained bound in this case is comparatively weaker than the one obtained through the first method.

2. Let $f \in L^1(\mathbb{T})$, $\sigma_N(f)$ is the Fejér sum corresponding to f. Assume that x_0 is a Lebesgue point of f. Show that $\lim_{N\to\infty} \sigma_N f(x_0) = f(x_0)$.

Proof. Through translations, we can establish the assumption that $x_0 = 0$. Additionally, by modifying $f(x)$ to $f(x) - f(0)$, we can further assume that $f(0) = 0$. In order to streamline the notation within the proof, we denote

$$I_0 = [-1, 1],\ I_k = [-2^{k-1}, 2^k] \cup [2^{k-1}, 2^k],\ k \in \mathbb{Z}_+ \text{ and } J_k = [-2^k, 2^k],\ k \in \mathbb{Z}_+.$$

Now, by using the property that $\sigma_N f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x-t) F_N(t) dt$ and the estimate for the Fejér kernel $F_N(t)$:

$$F_N(t) = \frac{1}{N+1} \frac{\sin^2(\pi(N+1)t)}{\sin^2(\pi t)} \leq \frac{\pi^2}{2} \frac{N+1}{1+(N+1)^2 t^2},$$

we have

$$\begin{aligned} |\sigma_N f(0)| &\leq C \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(N+1)|f(t)|}{1+(N+1)^2 t^2} dt = C \int_{-(N+1)/2}^{(N+1)/2} \frac{1}{1+t^2} \left| f\left(\frac{t}{N+1}\right) \right| dt \\ &\leq C \int_{\mathbb{R}} \frac{1}{1+t^2} \left| f\left(\frac{t}{N+1}\right) \right| dt \leq C \sum_{k=0}^{\infty} \int_{I_k} \frac{1}{1+t^2} \left| f\left(\frac{t}{N+1}\right) \right| dt \\ &\leq C \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \int_{J_k} \left| f\left(\frac{t}{N+1}\right) \right| dt. \end{aligned}$$

Let us consider the sequence

$$\{a_N\}_{N=1}^{\infty},\ \text{where } a_{N,k} = \frac{1}{2^{2k}} \int_{J_k} \left| f\left(\frac{t}{N+1}\right) \right| dt.$$

By change of variables and applying the assume that 0 is the Lebesgue point, it can be obtained that

$$a_{N,k} = \frac{1}{2^{k-1}} \frac{1}{2 \cdot 2^k (N+1)^{-1}} \int_{-2^k (N+1)^{-1}}^{2^k (N+1)^{-1}} |f(t)| dt \to 0$$

when $N \to \infty$ for fixed k. Then there exists $\delta > 0$ such that if $2^k (N+1)^{-1} < \delta$, it follows that $|a_{N,k}| \leq \frac{1}{2^{k-1}}$. If $\delta < 2^k (N+1)^{-1} \leq 1$, we can deduce that

$$|a_{N,k}| \leq \frac{1}{2^{k-2}} \frac{1}{2\delta} \int_0^1 |f(t)| dt = \frac{4}{2^k \delta} \|f\|_1.$$

If $1 < 2^k (N+1)^{-1}$, by utilizing the periodicity of f, it can be obtained that

$$|a_{N,k}| \leq \frac{4}{2^{k-1}} \frac{1}{2 \cdot 2^k (N+1)^{-1}} \cdot \frac{2^k}{N+1} \int_0^1 |f(t)| dt \leq \frac{8}{2^{k-1}} \|f\|_1.$$

Now, combining all the discussions above, we have $|a_{N,k}| \leq \frac{C}{2^k} \|f\|_1$. Then the result $\sigma_N f(0) \to 0$ when $N \to \infty$ follows directly from dominant convergence theorem. □

3. Let $P_N(x)$ be an triangular polynomial defined on $\mathbb{T}$ with order N. Show that

$$\|P_N'\|_\infty \leq \pi N(N+1) \|P_N\|_\infty.$$

Proof. By the assumption of P_N, we can get that

$$P_N(x) = \sum_{k=0}^{N} a_k e^{2\pi i k x},\ P_N'(x) = 2\pi i \sum_{k=0}^{N} k a_k e^{2\pi i k x}.$$

Let $Q(x) = 2\pi i \sum_{\ell=0}^{N} \ell e^{2\pi i \ell x}$, one has

$$\begin{aligned} P_N * Q(x) &= \int_0^1 P_N(x-y) Q(y) dy = \int_0^1 \left(\sum_{j=0}^{N} a_j e^{2\pi i j(x-y)} \right) \cdot \left(2\pi i \sum_{\ell=0}^{N} \ell e^{2\pi i \ell y} \right) dy \\ &= 2\pi i \sum_{j=0}^{N} \sum_{\ell=0}^{N} \ell a_j a_\ell e^{2\pi i j t} \int_0^1 e^{2\pi i (\ell - j) y} dy = 2\pi i \sum_{j=0}^{N} j a_j e^{2\pi i j t} \delta_{j\ell} = P_N'(t). \end{aligned}$$

Now by using Young inequality, we can obtain

$$\|P_N'\|_\infty = \|P_N * Q\|_\infty \le \|P_N\|_\infty \|Q\|_1 \le \pi N(N+1)\|P_N\|_\infty,$$

which is exactly the result. $\square$

Remark 1.2. In fact, we can further show that $\|P_N'\| \le 4\pi N\|P_N\|_\infty$. To prove this, we firstly claim that $P_N'(x)$ can be represented by

$$\frac{P_N'(x)}{2\pi i N} = \left((e^{-2\pi i N(\cdot)}P_N) * F_{N-1}\right)(x)e^{2\pi i N x} - \left((e^{2\pi i N(\cdot)}P_N) * F_{N-1}\right)(x)e^{-2\pi i N x}.$$

If this claim is true, the improvement for the results can be easily got that

$$\left\|\frac{P'}{2\pi i N}\right\|_\infty \le \|e^{-2\pi i N(\cdot)}P_N\|_\infty \|F_{N-1}\|_1 + \|e^{2\pi i N(\cdot)}P_N\|_\infty \|F_{N-1}\|_1 \le 2\|P_N\|_\infty,$$

which completes the proof. Now we will show the claim. By the linearity, we only need to verify the result for $P_N(x) = ae^{2\pi i k x}$. Indeed,

$$\int_0^1 e^{-2\pi i N(x-y)} \cdot ae^{2\pi i k(x-y)} F_{N-1}(y) e^{2\pi i N x} dy = \begin{cases} \frac{ka}{N} e^{2\pi i k x} & \text{if } k > 0, \\ 0 & \text{if } k \le 0, \end{cases}$$

$$\int_0^1 e^{2\pi i N(x-y)} \cdot ae^{2\pi i k(x-y)} F_{N-1}(y) e^{-2\pi i N x} dy = \begin{cases} \frac{ka}{N} e^{2\pi i k x} & \text{if } k < 0, \\ 0 & \text{if } k \ge 0. \end{cases}$$

4. Assume that $f \in L^2(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$. Show that $\widehat{f} \in L^1(\mathbb{R})$.

Proof. Since $f \in L^2(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$, we have $\widehat{f} \in L^2(\mathbb{R})$ and $\xi\widehat{f} \in L^2(\mathbb{R})$. Then, by employing Hölder inequality and Cauchy inequality, we obtain the following expression

$$\begin{aligned} \int_{\mathbb{R}} |\widehat{f}(\xi)| d\xi &= \int_{|\xi|\ge 1} |\widehat{f}(\xi)| d\xi + \int_{|\xi|<1} |\widehat{f}(\xi)| d\xi \\ &\le C\left(\int_{|\xi|<1} |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} + \left(\int_{|\xi|\ge 1} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{|\xi|\ge 1} \frac{1}{|\xi|^2} d\xi\right)^{\frac{1}{2}} < \infty, \end{aligned}$$

which comepletes the proof. $\square$

5. Assume that $f \in L^1(\mathbb{R})$ and $f' \in L^1(\mathbb{R})$. Show that $\widehat{f} \in L^2(\mathbb{R})$.

Proof. Since $f, f' \in L^1(\mathbb{R})$, we get $\widehat{f}, \xi\widehat{f} \in L^\infty$. We can thus assume that $|\widehat{f}| \le M$ and $|\xi||\widehat{f}| \le M$ for some positive constant $M > 0$. Consequently, we can deduce that

$$\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi = \int_{|\xi|\ge 1} |\widehat{f}(\xi)|^2 d\xi + \int_{|\xi|<1} |\widehat{f}(\xi)|^2 d\xi \le \int_{|\xi|<1} M d\xi + \int_{|\xi|\ge 1} \frac{M}{|\xi|^2} d\xi < \infty,$$

which completes the proof. $\square$

6. Let $f \in L^1(\mathbb{R}^n)$. Assume that f is continuous at 0 and $\widehat{f} \ge 0$. Show that $\widehat{f} \in L^1(\mathbb{R}^n)$.

Proof. Let $g(x) = e^{-\pi|x|^2}$ and $g_\lambda(x) = \lambda^{-n} g(\lambda^{-1}x)$. Since $f \in L^1(\mathbb{R}^n)$, then $f * g_\lambda(x)$ is well defined. Indeed, we get

$$\|f * g_\lambda(x)\|_1 \le \|f\|_1 \|g_\lambda\|_1 = \|f\|_1 \|g\|_1.$$

Now we have $\widehat{f * g_\lambda}(\xi) = \widehat{f}(\xi) g(\lambda\xi)$. Also, by $f \in L^1(\mathbb{R}^n)$, we get $\widehat{f} \in L^\infty(\mathbb{R}^n)$ and $\widehat{f} g(\lambda \cdot) \in L^1(\mathbb{R}^n)$. Then it follows that

$$f * g_\lambda(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) g(\lambda\xi) e^{2\pi i x\cdot\xi} d\xi.$$

Considering $x = 0$ and $\lambda \to 0$, we observe that the left-hand side tends to $f(0)$ while the right-hand side tends to $\int_{\mathbb{R}^n} \widehat{f}(\xi) d\xi$. This implies that $\widehat{f} \in L^1(\mathbb{R}^n)$. In the convergence of the left-hand side, we rely on the fact that g_λ converges to the Dirac function. Meanwhile, for the convergence of the right-hand side, we utilize the assumption that $\widehat{f} \geq 0$ and apply the monotone convergence theorem for integrals. $\square$

7. Let $|f(x)| \leq C(1+|x|)^{-1-\delta}$, $|\widehat{f}(\xi)| \leq C(1+|\xi|)^{-1-\delta}$, $f, \widehat{f} \in C(\mathbb{R})$, where the constants $\delta, C > 0$. Show the Poisson formula

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{2\pi ikx}.$$

Proof. Let $g(x) = \sum_{k\in\mathbb{Z}} f(x+k)$. Given that $|f(x)| \leq C(1+|x|)^{-1-\delta}$, we can observe that $g \in C(\mathbb{T})$, as demonstrated below:

$$|g(x)| \leq \sum_{k\in\mathbb{Z}} |f(x+k)| \leq \sum_{k\in\mathbb{Z}} C(1+||x|-|k||)^{-1-\delta} < \infty, \quad \forall x \in [0,1].$$

Therefore, g is well-defined and belongs to $C(\mathbb{T})$. Furthermore, we can establish that:

$$\int_0^1 |g(x)|dx \leq \sum_{k\in\mathbb{Z}} \int_0^1 |f(x+k)|dx = \sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(x)|dx = \int_{\mathbb{R}} |f(x)|dx.$$

Based on the previous discussion, we can define the Fourier series on $[0,1]$ as follows:

$$\widehat{g}(n) = \sum_{k\in\mathbb{Z}} \int_0^1 f(x+k)e^{-2\pi inx}dx = \sum_{k\in\mathbb{Z}} \int_k^{k+1} f(x)e^{-2\pi inx}dx = \int_{\mathbb{R}} f(x)e^{-2\pi inx}dx = \widehat{f}(n).$$

Let $S_N(x) = \sum_{|k|\leq N} \widehat{f}(k)e^{2\pi ikx}$. Since $|\widehat{f}(\xi)| \leq C(1+|\xi|)^{-1-\delta}$, by utilizing the Weierstrass principle, we observe that $S_N(x)$ converges uniformly to the function $S(x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx}$, which is continuous. Consequently, we can conclude that $\widehat{(g-S)}(k) = 0$ for any $k \in \mathbb{Z}$. By employing the Riemann-Lebesgue lemma, we deduce that $g = S$ almost everywhere. Since both g and S are continuous, we can further establish that $g = S$ everywhere in $[0,1]$, thereby completing the proof. $\square$

8. Let $1 \leq p, q \leq \infty$, show that if there exists $C > 0$ such that $\|\widehat{f}\|_q \leq C\|f\|_p$ for any $f \in S(\mathbb{R}^n)$, then $q = p'$ and $1 \leq p \leq 2$.

Proof. For $f \in S(\mathbb{R}^n)$, let us define $f_\lambda(x) = \lambda^{-n} f(\lambda^{-1}x)$. It follows that

$$\|f_\lambda\|_p = \lambda^{-n}\lambda^{\frac{n}{p}}\|f\|_p, \ \|\widehat{f_\lambda}\|_q = \lambda^{-\frac{n}{q}}\|\widehat{f}\|_q.$$

Using the assumption $\|\widehat{f_\lambda}\|_q \leq C\|f_\lambda\|_p$ we can consider the cases $\lambda \to 0$ and $\lambda \to \infty$, and deduce that $\lambda^{-n(1-\frac{1}{p})} = \lambda^{-\frac{n}{q}}$, which implies $q = \frac{p}{p-1}$. Now we need to prove that $1 \leq p \leq 2$. Choosing $f(x) = e^{-\pi\lambda^2|x|^2}$ with $\operatorname{Re}\lambda^2 > 0$. Assume that $\lambda^2 = a + bi$, we have $|\lambda| = (a^2+b^2)^{\frac{1}{4}}$. By simple calculations, it can be obtained that $\widehat{f}(\xi) = \lambda^{-n}e^{-\pi\lambda^{-2}|x|^2}$. Now we can get that

$$|f(x)| = e^{-\pi a|x|^2}, \ |\widehat{f}(\xi)| = \frac{1}{(a^2+b^2)^{\frac{n}{4}}} e^{-\frac{a\pi|\xi|^2}{a^2+b^2}}.$$

Then, by employing the assumption and computations, it follows that

$$\|f\|_p = Ca^{-\frac{n}{2p}}, \ \|\widehat{f}\|_q = \frac{C}{(a^2+b^2)^{\frac{n}{4}}}\left(\frac{a}{a^2+b^2}\right)^{-\frac{n}{2q}}$$

and

$$\frac{1}{(a^2+b^2)^{\frac{n}{4}}}\left(\frac{a}{a^2+b^2}\right)^{-\frac{n}{2q}} \leq Ca^{-\frac{n}{2p}}.$$

Letting $b \to \infty$, we see that $\frac{n}{4} - \frac{n}{2q} \geq 0$, i.e. $q \geq 2$ and then $1 \leq p \leq 2$. $\square$

2. Hardy-Littlewood maximal function

1. Let $\varphi \in L^1 \cap C(\mathbb{R}^n)$ and $\psi(x) = \sup_{|y| \geq |x|} |\varphi(y)|$. Assume that $\psi \in L^1(\mathbb{R}^n)$. Let

$$F^*(x) = \sup_{t>0} \sup_{|x-y|<t} |\varphi_t * f(y)|.$$

Show that $F^*(x) \leq CMf(x)$.

Proof. For $t > 0$ and $|x - y| < t$, we have the inequality

$$F^*(x) \leq \int_{B(x,16t)} |\psi_t(y-z)||f(z)|dz + \sum_{k=4}^{\infty} \int_{B(x,2^{k+1}t)\setminus B(x,2^k t)} |\psi_t(y-z)||f(z)|dz.$$

First, let us consider the integral over $B(x, 16t)$:

$$\int_{B(x,16t)} |\psi_t(y-z)||f(z)|dz \leq \frac{C}{(16t)^n} \int_{B(x,16t)} |f(z)|dz \leq CMf(x),$$

where C is a constant and $Mf(x)$ denotes the Hardy-Littlewood maximal function of f.

By assumptions, we set $\psi(x) = \rho(|x|)$ with $\psi \in L^1(\mathbb{R}^n)$. Next, we examine the sum over k:

$$\sum_{k\in\mathbb{Z}} \rho(2^{k+1})(2^k)^n \leq C \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \rho(r) r^{n-1} dr \leq \sum_{k\in\mathbb{Z}} \int_{2^k<|x|\leq 2^{k+1}} |\psi(x)|dx < \infty.$$

Now, for $2^k t < |z - x| < 2^{k+1}t$ with $k \geq 4$, we have $2^{k-1}t < |z - y| < 2^{k+2}t$, and

$$|\psi_t(y-z)| \leq C\rho(2^{k-1})(2^k)^n.$$

Using this estimate, we can deduce that

$$\sum_{k=4}^{\infty} \int_{B(x,2^{k+1}t)\setminus B(x,2^k t)} |\psi_t(y-z)||f(z)|dz$$
$$\leq \sum_{k=4} C\rho(2^{k-1})(2^k)^n \left(\frac{1}{(2^k t)^n} \int_{B(x,2^k t)} |f(z)|dz \right) \leq CMf(x).$$

Combining the two estimates, we obtain the result $F^*(x) \leq CMf(x)$. This completes the proof. □

2. Show that if $0 < p < r < q \leq \infty$, then $L^{p,\infty} \cap L^{q,\infty} \subset L^r$.

Proof. We will divide the proof into two cases.

Case 1. Assume $0 < p < r < q < \infty$. In this case, we have $a_f(\lambda) \leq \frac{\|f\|^p_{p,\infty}}{\lambda^p}$ and $a_f(\lambda) \leq \frac{\|f\|^q_{q,\infty}}{\lambda^q}$, where $a_f(\lambda) = |\{x \in \mathbb{R}^n : |f| > \lambda\}|$. We can express the L^r norm of f as follows:

$$\|f\|^r_r = \int_0^\infty r\lambda^{r-1} a_f(\lambda)d\lambda = \int_0^A r\lambda^{r-1} a_f(\lambda)d\lambda + \int_A^\infty r\lambda^{r-1} a_f(\lambda)d\lambda,$$

where A is a positive constant to be determined.

Using the above inequalities for $a_f(\lambda)$, we can estimate the first integral as:

$$\int_0^A r\lambda^{r-1} a_f(\lambda)d\lambda \leq \int_0^A r\lambda^{r-p-1} \|f\|^p_{p,\infty} d\lambda = C\|f\|^p_{p,\infty} \int_0^A r\lambda^{r-p-1} d\lambda = CA^{r-p}\|f\|^p_{p,\infty}.$$

For the second integral, we can use the inequality $a_f(\lambda) \leq \frac{\|f\|^q_{q,\infty}}{\lambda^q}$ to obtain:

$$\int_A^\infty r\lambda^{r-1} a_f(\lambda)d\lambda \leq \int_A^\infty r\lambda^{r-q-1} \|f\|^q_{q,\infty} d\lambda = C\|f\|^q_{q,\infty} \int_A^\infty r\lambda^{r-q-1} d\lambda = CA^{r-q}\|f\|^q_{q,\infty}.$$

Combining these estimates, we have:

$$\|f\|^r_r \leq \int_0^A r\lambda^{r-p-1}\|f\|^p_{p,\infty} d\lambda + \int_A^\infty r\lambda^{r-q-1}\|f\|^q_{q,\infty} d\lambda \leq CA^{r-p}\|f\|^p_{p,\infty} + CA^{r-q}\|f\|^q_{q,\infty}.$$

Now, if we choose A such that $A^{r-p}\|f\|_{p,\infty}^p = A^{r-q}\|f\|_{q,\infty}^q$, we can simplify the inequality to obtain:

$$\|f\|_r \le C\|f\|_{p,\infty}^{\frac{(r-p)q}{(q-p)r}}\|f\|_{q,\infty}^{\frac{(q-r)p}{(q-p)r}}.$$

This completes the proof for Case 1.

Case 2. $0<p<r<q=\infty$. Let $A=2\|f\|_\infty$, we get $\int_A^\infty r\lambda^{r-1}a_f(\lambda)d\lambda=0$. Then it can be deduced that

$$\begin{aligned}\|f\|_r^r &= \int_0^\infty r\lambda^{r-1}a_f(\lambda)d\lambda = \int_0^A r\lambda^{r-1}a_f(\lambda)d\lambda + \int_A^\infty r\lambda^{r-1}a_f(\lambda)d\lambda \\ &\le \int_0^A r\lambda^{r-p-1}\|f\|_{p,\infty}^p d\lambda \le C\|f\|_\infty^{r-p}\|f\|_{p,\infty}^p,\end{aligned}$$

which means that the inequality for the first case is still true. □

3. Let $a_k>0$, show that $\|\sum_k f_k\|_{1,\infty} \le (1+\sum_k a_k)\sum_k \|f_k\|_{1,\infty}\ln(1+a_k^{-1})$.

Proof. For any $\lambda>0$ and $0<\varepsilon<1$, let $g_k=\min\{|f_k|-\lambda a_k, \lambda(1+\varepsilon)\}$, we have

$$\left|\left\{x : \left|\sum_{k=1}^\infty f_k\right| > \left(1+\sum_{k=1}^\infty a_k\right)\lambda\right\}\right| \le \left|\left\{x : \sum_{k=1}^\infty (|f_k|-\lambda a_k)_+ > \lambda\right\}\right| \le \frac{1}{\lambda}\left\|\sum_{k=1}^\infty g_k\right\|_{1,\infty} \le \frac{1}{\lambda}\sum_{k=1}^\infty \|g_k\|_1.$$

By using $\|g_k\|_1 = \int_0^\infty |\{g_k>\mu\}|d\mu$ and the fact that $0\le g_k\le \lambda(1+\varepsilon)$, we have

$$\begin{aligned}\|g_k\|_1 &= \int_0^{(1+\varepsilon)\lambda} |\{x : g_k>\mu\}|d\mu \le \int_0^{(1+\varepsilon)\lambda} \frac{\|f_k\|_{1,\infty}}{\mu+\lambda a_k}d\mu \\ &= \ln\left(\frac{\lambda a_k + \lambda(1+\varepsilon)}{\lambda a_k}\right)\|f_k\|_{1,\infty} = \|f_k\|_{1,\infty}\ln(1+(1+\varepsilon)a_k^{-1}).\end{aligned}$$

Then we can obtain

$$\left|\left\{x : \left|\sum_{k=1}^\infty f_k\right| > \left(1+\sum_{k=1}^\infty a_k\right)\lambda\right\}\right| \le \frac{1}{\lambda}\sum_{k=1}^\infty \|f_k\|_{1,\infty}\ln(1+(1+\varepsilon)a_k^{-1}),$$

which implies the result if we let $\varepsilon\to 0^+$. □

4. Let $1<p\le 2$. Show that

$$\|\widehat{f}\|_{L^p(\mathbb{R}^n,|x|^{-n(2-p)}dx)} \le C\|f\|_{L^p(\mathbb{R}^n,dx)}.$$

Proof. Let $d\mu=|x|^{-2n}dx$ and $Tf(x)=\widehat{f}(x)|x|^n$. To establish the desired inequality, it suffices to prove that

$$\|Tf\|_{L^p(\mathbb{R}^n,d\mu)} \le C\|f\|_{L^p(\mathbb{R}^n,dx)},$$

where C is a constant. The case $p=2$ can be obtained using the Plancherel identity. Thus, it remains to prove that

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^n,d\mu)} \le C\|f\|_{L^1(\mathbb{R}^n,dx)}$$

together with the use of Marcinkiewicz interpolation. First, note that

$$\mu(\{x : |Tf(x)|>\lambda\}) = \mu(\{x : |\widehat{f}(x)||x|^n>\lambda\}) \le \mu(\{x : |x|^{-n} < \lambda^{-1}\|f\|_1\}) \le \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n,dx)},$$

where we utilized the fact that $\|\widehat{f}\|_{L^\infty(\mathbb{R}^n,dx)} \le \|f\|_{L^1(\mathbb{R}^n,dx)}$ and

$$\mu(\{x : |x|^{-n}>t\}) = \int_{|x|>t^{\frac{1}{n}}} \frac{1}{|x|^{2n}}dx \le Ct,\ t>0.$$

By establishing above inequalities, the proof is completed. □

5. Let $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Show that $Mf(x)<\infty$ a.e. or $Mf(x)=\infty$ a.e.

Proof. If there exists a measurable set $E\subset\mathbb{R}^n$ such that $0<|E|<\infty$ and $Mf(x)=\infty$ in E, we aim to prove that $Mf(x)=\infty$ almost everywhere in $\mathbb{R}^n$. For $x\in E$, we consider three cases:

- There exists $r_k\to 0$, such that $\frac{1}{\omega_n r_k^n}\int_{B(x,r_k)}|f(y)|dy\to\infty$ when $k\to\infty$.

- There exists $r_k \to r$ with $r > 0$, such that $\frac{1}{\omega_n r_k^n} \int_{B(x,r_k)} |f(y)|dy \to \infty$ when $k \to \infty$.
- There exists $r_k \to \infty$, such that $\frac{1}{\omega_n r_k^n} \int_{B(x,r_k)} |f(y)|dy \to \infty$ when $k \to \infty$.

We denote $E = E_1 \cup E_2 \cup E_3$ corresponding to the above three cases. Note that $|E_1| = 0$ since $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Hence, we have

$$\frac{1}{\omega_n r_k^n} \int_{B(x,r_k)} |f(y)|dy \to |f(x)| < \infty$$

almost everywhere in $\mathbb{R}^n$. If $E_3 \neq \emptyset$, let $y \in \mathbb{R}^n$, $N > 0$, and $x \in E_3$. Then, there exists $r > |x-y|$ such that $\int_{B(x,r)} |f(z)|dz > N\omega_n r^n$. By simple observation, we have

$$\int_{B(y,2r)} |f(z)|dz \geq \int_{B(x,r)} |f(z)|dz > N\omega_n r^n,$$

which implies $Mf(y) = \infty$ by the arbitrariness of N. The same arguments can be applied if $E_2 \neq \emptyset$. In fact, for any $x \in E_2$, we find that $\int_{B(x,r)} |f(y)|dy > M$ for any $M > 0$, contradicting the assumption that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. □

6. Let $p \in (1, \infty)$. Show that $\|Mf\|_{p,\infty} \leq C\|f\|_{p,\infty}$.

Proof. Firstly, we claim that for any $\lambda > 0$,

$$|\{x : Mf > 2\lambda\}| \leq \frac{C}{\lambda} \int_{\{|f|>\lambda\}} |f(x)|dx.$$

Assuming this claim, we can proceed to show that for any $\lambda > 0$,

$$\begin{aligned}
\lambda^p|\{x : Mf > 2\lambda\}| &\leq C\lambda^{p-1} \int_{\{|f|>\lambda\}} |f(x)|dx = C\lambda^{p-1} \int_0^\infty |\{x : |f| > \lambda, |f| > \mu\}|d\mu \\
&= C\lambda^{p-1} \int_0^\lambda |\{x : |f| > \lambda\}|d\mu + C\lambda^{p-1} \int_\lambda^\infty |\{x : |f| > \mu\}|d\mu \\
&\leq C\lambda^p|\{x : |f| > \lambda\}| + C\lambda^{p-1} \int_\lambda^\infty \frac{\|f\|_{p,\infty}^p}{\mu^p} d\mu \leq C\lambda^p|\{x : |f| > \lambda\}| \leq C\|f\|_{p,\infty}^p.
\end{aligned}$$

This implies that $\|Mf\|_{p,\infty} \leq C\|f\|_{p,\infty}$. Now let us prove the claim. Let $f = f_1 + f_2$ such that

$$f_1 = f\chi_{\{|f|>\lambda\}},\ f_2 = f\chi_{\{|f|<\lambda\}}.$$

Using the weak $(1,1)$ property for the maximal operator, we have

$$\begin{aligned}
|\{x : Mf > 2\lambda\}| &\leq |\{x : Mf_1 > \lambda\}| + |\{x : Mf_2 > \lambda\}| = |\{x : Mf_1 > \lambda\}| \\
&\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f_1(x)|dx = \frac{C}{\lambda} \int_{\{|f|>\lambda\}} |f(x)|dx,
\end{aligned}$$

which proves the claim. □

Remark 2.1. Since for any $\lambda > 0$, the set $\{x : Mf(x) > \lambda\}$ is open, it can be obtained by modify the proof of result that M is weak $(1,1)$ a little to obtain that

$$|\{x : Mf(x) > \lambda\}| \leq \frac{C}{\lambda} \int_{\{Mf>\lambda\}} |f(x)|dx.$$

By using this and arguments above, one can also show that M is bounded in $L^{p,\infty}$ for $1 < p < \infty$.

Remark 2.2. In fact, we can prove a stronger result. If T is weak (p,p) for any $1 < p < \infty$, then for any $1 < r < \infty$, we have $\|Tf\|_{r,\infty} \leq C\|f\|_{r,\infty}$. Let us choose $p < r < q$ and $\lambda > 0$. We decompose $f = f_1 + f_2$ such that $f_1 = f\chi_{\{|f|>\lambda\}}$. Then

$$|\{x : |Tf| > \lambda\}| \leq \left|\left\{x : |Tf_1| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x : |Tf_2| > \frac{\lambda}{2}\right\}\right| \leq C\left(\frac{\|f_1\|_p^p}{\lambda^p} + \frac{\|f_2\|_q^q}{\lambda^q}\right).$$

To proceed, we note that

$$\|f_1\|_p^p = \int_0^\infty p\mu^{p-1}|\{|f_1| > \mu\}|d\mu \le \int_\lambda^\infty p\mu^{p-1}\frac{\|f_1\|_{r,\infty}^r}{\mu^r}d\mu \le C\lambda^{p-r}\|f\|_{r,\infty}^r,$$
$$\|f_2\|_q^q = \int_0^\infty q\mu^{q-1}|\{|f_1| > \mu\}|d\mu \le \int_0^\lambda q\mu^{q-1}\frac{\|f_2\|_{r,\infty}^r}{\mu^r}d\mu \le C\lambda^{q-r}\|f\|_{r,\infty}^r.$$

By using these estimates, we conclude that $\|Tf\|_{r,\infty} \le C\|f\|_{r,\infty}$, which is the desired result.

7. Assume that the sequence of operators $\{T_k\}$ is uniformly weak $(1,1)$, i.e. there exists $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |T_k f(x)| > \lambda\}| \le C\lambda^{-1}\|f\|_1, \quad \lambda > 0.$$

Let $\{C_k\}$ is a sequence of positive number such that $\sum_{k=1}^\infty C_k^{\frac{1}{2}} < \infty$ and define $Tf(x) = \sum_{k=1}^\infty C_k T_k f(x)$. Show that T is weak $(1,1)$.

Proof. Let $\alpha = \sum_{k=1}^\infty C_k^{\frac{1}{2}}$. Firstly, we note that

$$|\{x : |Tf(x)| > \lambda\}| \le \left|\left\{x : \sum_{k=1}^\infty |T_k f(x)| > \sum_{k=1}^\infty C_k^{\frac{1}{2}}\lambda\alpha^{-1}\right\}\right| \le \sum_{k=1}^\infty \left|\left\{x : C_k|T_k f(x)| > C_k^{\frac{1}{2}}\alpha^{-1}\right\}\right|.$$

In view of the assumption that $\{T_k\}$ is uniformly weak $(1,1)$, it follows that

$$\left|\left\{x : C_k|T_k f(x)| > C_k^{\frac{1}{2}}\lambda\alpha^{-1}\right\}\right| \le \frac{C\alpha C_k^{\frac{1}{2}}}{\lambda}.$$

This, together with the calculations above, implies that T is weak $(1,1)$. □

8. Let $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a bounded operator and $\|Tf\|_2 = A\|f\|_2$. Assume that $\|Tf\|_p \le C\|f\|_p$ for any $f \in (L^2 \cap L^p)(\mathbb{R}^n)$ with $p \ge 1$. Show that $\|f\|_{p'} \le CA^{-2}\|Tf\|_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Since $\|Tf\|_2 = A\|f\|_2$ for any $f \in L^2(\mathbb{R}^n)$, we see that $A^{-1}T$ is an operator preserving the inner product of $L^1(\mathbb{R}^n)$, i.e. $(Tf, Tg)_{L^2(\mathbb{R}^n)} = A^2(f, g)_{L^2(\mathbb{R}^n)}$. Then the result follows directly by using basic duality arguments. Indeed, for any $g \in (L^2 \cap L^p)(\mathbb{R}^n)$, we have

$$\left|\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx\right| = A^{-2}\left|\int_{\mathbb{R}^n} Tf(x)\overline{Tg(x)}dx\right| \le A^{-2}\|Tf\|_{p'}\|Tg\|_p \le C\|Tf\|_{p'}\|g\|_p.$$

Then the desired result is true by duality. □

9. Assume that $Q \subset \mathbb{R}^n$ is a cube and $\operatorname{supp}(f) \subset Q$ with $f \in L^1(Q)$. Show that $Mf \in L^1(Q)$ is and only if $|f|\ln^+|f| \in L^1(Q)$, where $\ln^+ t = \max\{\ln t, 0\}$.

Proof. Without loss of generality, we can assume that $f \ge 0$. Firstly, suppose that $f\ln^+ f \in L^1(Q)$. By the arguments in the proof of problem 6 of this section, we have

$$|\{x : Mf > \lambda\}| \le \frac{C}{\lambda}\int_{\{f > \frac{\lambda}{2}\}} f(x)dx.$$

Then it follows by the usage of distributional function that

$$\|Mf\|_{L^1(Q)} = \int_0^\infty |\{x \in Q : Mf > \lambda\}|d\lambda \le |Q| + \int_1^\infty \frac{C}{\lambda}\int_{\{f > \frac{\lambda}{2}\}} f(x)dxd\lambda.$$

Applying Fubini theorem, we can easily obtain that $\|Mf\|_1 \le |Q| + C\|f\ln^+ f\|_1$. On the other hand, we assume that $Mf \in L^1(Q)$, then $E_d f \in L^1(Q)$, where $E_d f$ is the dyadic maximal function. On the other hand, we have to prove the following inequality.

$$|\{x : Mf > \lambda\}| \ge \frac{C}{\lambda}\int_{\{f > \lambda\}} f(x)dx$$

for any $\lambda > 0$. Applying Calderón-Zygumund decomposition to f at the level λ, we can obtain a sequence of disjoint dyadic cubes $\{Q_k\}$ such that $\lambda < \fint_{Q_k} f \le 2^n\lambda$ and $0 < f \le \lambda$ in $\mathbb{R}^n \setminus \cup_{k\ge1} Q_k$. This implies that for any $x \in \cup_{k\ge1} Q_k$, $Mf(x) \ge \fint_{Q_k} f > \lambda$ and then

$$\begin{aligned}|\{x : Mf > \lambda\}| &\ge \left|\bigcup_{k\ge1} Q_k\right| = \sum_{k=1}^{\infty} |Q_k| \ge \sum_{k=1}^{\infty} \frac{1}{2^n\lambda} \int_{Q_k} f(x)dx \\ &= \frac{1}{2^n\lambda} \int_{\cup_{k\ge1} Q_k} f(x)dx \ge \frac{C}{\lambda} \int_{\{f>\lambda\}} f(x)dx.\end{aligned}$$

By using almost the same arguments above, we can obtain that $f \ln^+ f \in L^1(Q)$. $\square$

10. Show that a sublinear operator T is weak $(1,1)$ and strong (∞,∞) if and only if there exists $C_1, C_2 > 0$ such that for any $f \in (L^1 + L^\infty)(\mathbb{R}^n)$,

$$|\{x : |Tf(x)| > \lambda\}| \le \frac{C_1}{\lambda} \int_{\frac{\lambda}{C_2}}^{\infty} |\{x : |f(x)| > t\}| dt.$$

Proof. If T is weak $(1,1)$ and strong (∞,∞), for $f \in (L^1 + L^\infty)(\mathbb{R}^n)$, we can decompose f as $f = f_1 + f_2$ such that $f_1 = f\chi_{\{|f|>\mu\}}$ and obtain

$$\|Tf_1\|_{1,\infty} \le A\|f_1\|_1, \ \|Tf_2\|_\infty \le B\|f_2\|_\infty \le B\mu,$$

where A and B are the boundedness of T with respect to weak $(1,1)$ and strong (∞,∞). By simple observation, it follows that

$$|\{x : |Tf| > \lambda\}| \le \left|\left\{x : |Tf_1| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x : |Tf_2| > \frac{\lambda}{2}\right\}\right|$$

If we choose μ such that $B\mu = \frac{\lambda}{4}$, it can be obtained that $\left|\left\{x : |Tf_2| > \frac{\lambda}{2}\right\}\right| = 0$ and then

$$|\{x : |Tf| > \lambda\}| \le \frac{2A}{\lambda} \int_{\mathbb{R}^n} |f_1(x)| dx = \frac{2A}{\lambda} \int_{\frac{\lambda}{4B}}^{\infty} |\{x : |f(x)| > t\}| dt.$$

This directly implies the result by choosing $C_1 = A$ and $C_2 = 4B$. On the other hand, one only needs to choose $f \in L^1(\mathbb{R}^n)$ and $f \in L^\infty(\mathbb{R}^n)$ in the formula. $\square$

11.(A. Kolmogorov) Let S be a sublinear operator that maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ with norm B. Suppose that $f \in L^1(\mathbb{R}^n)$. Prove that for any set A of finite Lebesgue measure and for all $0 < q < 1$ we have

$$\int_A |S(f)(x)|^q dx \le (1-q)^{-1} B^q |A|^{1-q} \|f\|_1^q.$$

Proof. By utilizing the distributional function, we get

$$\begin{aligned}\int_A |S(f)(x)|^q dx &= \int_0^\infty q\lambda^{q-1} |\{x \in A : |S(f)(x)| > \lambda\}| d\lambda \\ &= \left(\int_0^\mu + \int_\mu^\infty\right) q\lambda^{q-1} |\{x \in A : |S(f)(x)| > \lambda\}| d\lambda \\ &\le \int_0^\mu q\lambda^{q-1}|A| d\lambda + \int_\mu^\infty Bq\lambda^{q-2}\|f\|_1 d\lambda \le |A|\mu^q + \frac{Bq}{1-q}\mu^{q-1}\|f\|_1.\end{aligned}$$

Choosing μ such that $|A|\mu^q = B\mu^{q-1}\|f\|_1$, the result follows directly. $\square$

3. Hilbert transform

1. Calculate the Hilbert transform of $\chi_{[a,b]}$, the characteristic function on $[a,b]$.

Proof. By the definition of Hilbert transform, we have

$$Hf(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} dy = \frac{1}{\pi} \ln \left| \frac{x-a}{x-b} \right|.$$

To get the above formula, one need to consider cases that $x \in (a,b)$, $x \in (b,\infty)$ and $x \in (-\infty, a)$. □

2. Let $A = \cup_{i=1}^{N} [a_i, b_i]$, where $b_j < a_{j+1}$. Show that

$$|\{x \in \mathbb{R} : |H\chi_A(x)| > \lambda\}| = \frac{2|A|}{\sinh(\pi\lambda)}.$$

Proof. Since $\chi_A = \sum_{i=1}^{N} \chi_{[a_i,b_i]}$, we can express the Hilbert transform of χ_A as follows:

$$H\chi_A = \frac{1}{\pi} \sum_{i=1}^{N} \ln \left| \frac{x-a_i}{x-b_i} \right|.$$

Since for $x \to a_j^{+,-}$, $H\chi_A \to -\infty$ and for $x \to b_j^{+,-}$, $H\chi_A \to +\infty$, then the equation $H\chi_A(x) = \lambda$ has exactly one root ρ_j in each interval (a_j, b_j) and exactly one root in each interval (b_j, a_{j+1}). Due to this observation, it follows that

$$|\{x \in \mathbb{R} : H(\chi_A)(x) > \lambda\}| = \sum_{j=1}^{N} r_j - \sum_{j=1}^{N} \rho_j.$$

To complete the proof, it remains to calculate $\sum_{j=1}^{N} \rho_j$ and $\sum_{j=1}^{N} r_j$. Firstly, we note that ρ_j satisfies

$$\prod_{k=1}^{n} (\rho_j - a_k) = e^{\pi\lambda} \prod_{k=1}^{n} (\rho_j - b_k).$$

By applying Vieta theorem, we can get that

$$\sum_{j=1}^{N} \rho_j = \frac{1}{1+e^{\pi\lambda}} \left(\sum_{j=1}^{N} a_j + e^{\pi\lambda} \sum_{j=1}^{N} b_j \right).$$

By using almost the same computations,

$$\sum_{j=1}^{N} r_j = \frac{1}{1-e^{\pi\lambda}} \left(\sum_{j=1}^{N} a_j - e^{\pi\lambda} \sum_{j=1}^{N} b_j \right).$$

Then $\sum_{j=1}^{N} r_j - \sum_{j=1}^{N} \rho_j = \frac{|A|}{\sinh(\pi\lambda)}$. Repeating the arguments above, we see that

$$|\{x \in \mathbb{R} : H(\chi_A)(x) < -\lambda\}| = \frac{|A|}{\sinh(\pi\lambda)}$$

as well, so the result is true. □

3. Let $f(x) = \frac{1}{1+x^2}$, $g(x) = \frac{x}{(1+x^2)^2}$, calculate the Fourier and Hilbert transform of f and g.

Proof. To calculate $\widehat{f}(\xi) = \int_{\mathbb{R}} \frac{e^{2\pi i x\xi}}{1+x^2} dx$, we have to consider two cases. If $\xi > 0$, we get

$$\widehat{f}(\xi) = 2\pi i \cdot \frac{e^{2\pi i \cdot i\xi}}{2i} = \pi e^{-2\pi\xi}.$$

For $\xi \leq 0$, we have $\widehat{f}(\xi) = \pi e^{2\pi\xi}$. Combined calculations above, we have $\widehat{f}(\xi) = \pi e^{-\pi|\xi|}$. By this, we can get

$$\widehat{g}(\xi) = -\frac{\widehat{f'}}{2}(\xi) = 2\pi i\xi \cdot \left(-\frac{1}{2}\right) \cdot \pi e^{-2\pi|\xi|} = -\pi^2 i\xi e^{-2\pi|\xi|}.$$

To give the Hilbert transform of $f(x)$, we note that

$$Hf(x) = \int_{\mathbb{R}} \pi i \operatorname{sgn}(\xi) e^{2\pi|\xi|+2\pi i x\xi} d\xi = \frac{x}{1+x^2}.$$

For $Hg(x)$, we have

$$Hg(x) = \int_{\mathbb{R}} -i \operatorname{sgn}(\xi)(-\pi^2 i\xi e^{-2\pi|\xi|}) e^{2\pi i x\xi} d\xi = \frac{1-x^2}{2(1+x^2)^2},$$

which completes the calculations. $\square$

4. Let $f \in S(\mathbb{R})$. Show the identity of Hilbert transform:

$$(\widehat{Hf} * \widehat{Hf})(\xi) = (\widehat{f} * \widehat{f})(\xi) - 2i \operatorname{sgn}(\widehat{f} * \widehat{Hf})(\xi).$$

Proof. By the definition of Hilbert transform, we have

$$\begin{aligned}(\widehat{Hf} * \widehat{Hf})(\xi) &= \int_{\mathbb{R}} -i \operatorname{sgn}(\xi-\eta)\widehat{f}(\xi-\eta)(-i\operatorname{sgn}(\eta)\widehat{f}(\eta))d\eta \\ &= -\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)\operatorname{sgn}(\eta)\widehat{f}(\xi-\eta)\widehat{f}(\eta)d\eta = \left(\int_{\xi}^{\infty} - \int_0^{\xi} + \int_{-\infty}^{0}\right)\widehat{f}(\xi-\eta)\widehat{f}(\eta)d\eta.\end{aligned}$$

On the other hand,

$$(\widehat{f} * \widehat{f})(\xi) = \left(\int_{-\infty}^{\xi} + \int_{\xi}^{\infty}\right)\widehat{f}(\xi-\eta)\widehat{f}(\eta)d\eta$$

and

$$-2i\operatorname{sgn}(\xi)(\widehat{f} * \widehat{Hf})(\xi) = -2i\int_{-\infty}^{\infty}(-i\operatorname{sgn}(\xi-\eta))\widehat{f}(\xi-\eta)\widehat{f}(\eta)d\eta = \left(2\int_{\xi}^{\infty} - 2\int_{-\infty}^{\xi}\right)\widehat{f}(\xi-\eta)\widehat{f}(\eta)d\eta.$$

Now it can be observed that

$$\begin{aligned}(\widehat{f} * \widehat{f})(\xi) - 2i\operatorname{sgn}(\xi)(\widehat{f} * \widehat{Hf})(\xi) &= \left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi} + 2\int_{\xi}^{\infty}\right)\widehat{f}(\xi-\eta)\widehat{f}(\eta)d\eta \\ = \left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi} + 2\int_{-\infty}^{0}\right)\widehat{f}(\xi-\eta)\widehat{f}(\eta)d\eta &= \left(\int_{\xi}^{\infty} + \int_{-\infty}^{0} - \int_{0}^{\xi}\right)\widehat{f}(\xi-\eta)\widehat{f}(\eta)d\eta = (\widehat{Hf} * \widehat{Hf})(\xi),\end{aligned}$$

where for the third inequality, we have used change of variables $\xi - \eta \mapsto \eta$. $\square$

Remark 3.1. Consider the analytic function $F_f(z)$ defined on the upper half-plane by

$$F_f(z) = \frac{i}{\pi}\int_{\mathbb{R}} \frac{f(t)}{z-t}dt = \frac{1}{\pi}\int_{\mathbb{R}} \frac{yf(t)}{(x-t)^2+y^2}dt + i\left(\frac{1}{\pi}\int_{\mathbb{R}} \frac{(x-t)f(t)}{(x-t)^2+y^2}dt\right).$$

If we fix x and let $y \to 0^+$, we observe that $F_f(x+iy) \to f(x) + Hf(x)$. Intuitively, this implies that the Hilbert transform transfers the function f to the imaginary part of $F_f(x)$ for $x \in \mathbb{R}$. Using the identity $(u+iv)^2 = u^2 - v^2 + 2iuv$, we can see that $H(f - H^2 f) = 2fHf$, which precisely matches the result.

5. Let $\varphi \in S(\mathbb{R})$. Show that

$$\lim_{N\to\infty} \text{p. v.} \int_{\mathbb{R}} \frac{e^{2\pi i N x}}{x}\varphi(x)dx = \varphi(0)\pi i.$$

Proof. By utilizing change or variables, it can be got that

$$\begin{aligned}\text{p. v.} \int_{\mathbb{R}} \frac{e^{2\pi i N x}}{x}\varphi(x)dx &= \int_{\mathbb{R}} \frac{e^{2\pi i N x}}{x}(\varphi(x)-\varphi(0))dx + \text{p. v.}\int_{\mathbb{R}} \frac{\cos x}{x}\varphi(0)dx + i\int_{\mathbb{R}} \frac{\sin x}{x}\varphi(0)dx \\ &= \int_{\mathbb{R}} \frac{e^{2\pi i N x}}{x}(\varphi(x)-\varphi(0))dx + i\int_{\mathbb{R}} \frac{\sin x}{x}\varphi(0)dx.\end{aligned}$$

Since $\int_{\mathbb{R}} \frac{\sin x}{x} dx = \pi$, we only need to show that

$$\int_{\mathbb{R}} \frac{e^{2\pi i N x}}{x} (\varphi(x) - \varphi(0)) dx \to 0$$

when $N \to 0$. For any $\varepsilon > 0$, there exist $A > 0$ such that

$$\left| \int_{|x|>A} \frac{e^{2\pi i N x}}{x} (\varphi(x) - \varphi(0)) dx \right| \leq \int_{|x|>A} \frac{|\varphi(x)|}{|x|} dx + \left| \int_{|x|>A} \frac{\varphi(0) \sin x}{x} dx \right| < \varepsilon$$

where we have used the assumption that $\varphi \in S(\mathbb{R})$. For the term $\int_{|x|>A} \frac{e^{2\pi i N x}}{x} (\varphi(x) - \varphi(0)) dx$, it tends to 0 by Riemann-Lebesgue lemma. $\square$

6. Let $f \in S(\mathbb{R})$. Show that $Hf \in L^1(\mathbb{R})$ if and only if $\int_{\mathbb{R}} f(x) dx = 0$.

Proof. We firstly claim that if $\int_{\mathbb{R}} f(x) dx = 0$, then $H(xf) = xH(f)$. This is because

$$\begin{aligned} H(xf) &= \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{(x-y) f(x-y)}{y} dy \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \frac{x f(x-y)}{y} dy - \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|y-x|>\varepsilon} f(y) dy = xH(f). \end{aligned}$$

Then, we can see that

$$\begin{aligned} \int_{\mathbb{R}} |Hf(x)| dx &= \int_{|x|\leq 1} |Hf(x)| dx + \int_{|x|>1} |Hf(x)| dx \leq C \|Hf\|_2 + \int_{|y|>1} \frac{|H(xf)(y)|}{|y|} dy \\ &\leq C \|Hf\|_2 + \left(\int_{|y|>1} \frac{1}{|y|^2} dy \right)^{\frac{1}{2}} \left(\int_{|y|>1} |H(xf)(y)|^2 dy \right)^{\frac{1}{2}}, \end{aligned}$$

which implies that $Hf \in L^1(\mathbb{R})$. On the other hand, since $Hf \in L^1(\mathbb{R})$, we see that $\widehat{H}f(\xi) = -i \operatorname{sgn}(\xi) \widehat{f}(\xi)$ is continuous, which shows that $\widehat{f}(0) = 0$ and then $\int_{\mathbb{R}} f(x) dx = 0$. $\square$

7. Let $1 \leq q < p \leq \infty$ and T is strong (p, q). Assume that T is commutative to the translation operator. Prove that $Tf = 0$, for any $f \in S(\mathbb{R}^n)$.

Proof. Let τ_h such that $\tau_h(f)(x) = f(x+h)$. Then we have

$$\|\tau_h T(f) + Tf\|_q = \|T(\tau_h f + f)\|_p \leq \|T\|_{p\to q} \|\tau_h f + f\|_p.$$

Letting $h \to \infty$, it follows that

$$2^{\frac{1}{q}} \|Tf\|_q \leq 2^{\frac{1}{p}} \|T\|_{p\to q} \|f\|_p,$$

where we used $\lim_{h\to\infty} \|g + \tau_h g\|_r = 2^{\frac{1}{r}} \|g\|_r$. This implies that $\|T\|_{p\to q} = 0$ and then $Tf = 0$ a.e. $\square$

4. Singular integral

1. Let $\Omega \in L^1(S^{n-1})$ and $\int_{S^{n-1}} \Omega(u) d\sigma(u) = 0$. Show that if $Tf = f * \mathrm{p.v.} \frac{\Omega(x')}{|x|^n}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $p = q$ or $T = 0$, where $x' = \frac{x}{|x|}$.

Proof. Let $f \in S(\mathbb{R}^n)$, we can set $f_\lambda(x) = f(\lambda x)$. Then it follows from simple calculations that

$$Tf_\lambda(x) = \lim_{\varepsilon\to 0^+} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} f(\lambda(x-y)) dy = \lim_{\varepsilon\to 0^+} \int_{|z|>\lambda\varepsilon} \frac{\Omega(z')}{|z|^n} f(\lambda x - z) dz = Tf(\lambda x).$$

Applying this, one can deduce that $\|Tf_\lambda\|_q = \lambda^{-\frac{n}{q}} \|Tf\|_q$. Combined with $\|f_\lambda\|_p = \lambda^{-\frac{n}{p}} \|f\|_p$ and, we obtain that

$$\|Tf\|_q \leq \lambda^{n\left(\frac{1}{q} - \frac{1}{p}\right)} \|f\|_p.$$

If $p \neq q$, there are two cases.

- If $q > p$, choose $\lambda \to 0$ and then $\|Tf\|_q = 0$ for any $f \in L^p(\mathbb{R}^n)$, i.e. $T = 0$.
- If $q > p$, choose $\lambda \to \infty$ and then $\|Tf\|_q = 0$ for any $f \in L^p(\mathbb{R}^n)$, i.e. $T = 0$.

By the discussions above, we can complete the proof. □

2. Let $\Omega \in L^1(S^{n-1})$ such that $\int_{S^{n-1}} \Omega(u)\,\mathrm{sgn}(u\cdot\xi)d\sigma(u) = 0$ for any $\xi \in \mathbb{R}^n$. Show that Ω is an even function.

Proof. By using the theorem for the multiplier of p. v. $\frac{\Omega(x')}{|x|^m}$, it can be got that

$$\mathcal{F}\left(\mathrm{p.\,v.}\,\frac{\Omega(x')}{|x|^n}\right) = m(\xi) = \int_{S^{n-1}} \Omega(u)\left(\ln\left(\frac{1}{|u\cdot\xi'|}\right) - i\frac{\pi}{2}\,\mathrm{sgn}(u\cdot\xi')\right)d\sigma(u).$$

To show the result, we can decompose $\Omega(u) = \Omega_o(u) + \Omega_e(u)$, where

$$\Omega_o(u) = \frac{\Omega(u)-\Omega(-u)}{2},\ \Omega_e(u) = \frac{\Omega(u)+\Omega(-u)}{2}.$$

In view of this notations,

$$\mathcal{F}\left(\mathrm{p.\,v.}\,\frac{\Omega_o(x')}{|x|^n}\right) = \int_{S^{n-1}} \Omega(u)\,\mathrm{sgn}(u\cdot\xi')d\sigma(u) = 0.$$

This implies that $\Omega_o(u) \equiv 0$ and then Ω is an even function. □

3. Define the fractional integral operator I_α as

$$I_\alpha f(x) = \pi^{\alpha-\frac{n}{2}}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma(\frac{\alpha}{2})}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy \text{ or } \widehat{I_\alpha f}(\xi) = |\xi|^{-\alpha}\widehat{f}(\xi)$$

for $f \in S(\mathbb{R}^n)$. Show that for $1 \le p < \frac{n}{\alpha}$,

$$|I_\alpha f(x)| \le C\|f\|_p^{\frac{\alpha p}{n}} Mf(x)^{1-\frac{\alpha p}{n}}.$$

Proof. We have the decomposition as follows.

$$I_\alpha f(x) = C\int_{|x|\le A}\frac{f(y)}{|x-y|^{n-\alpha}}dy + C\int_{|x|>A}\frac{f(y)}{|x-y|^{n-\alpha}}dy = \mathrm{I} + \mathrm{II}.$$

For the estimate of I, by using the approximation of identity, it follows that

$$\mathrm{I} \le C\left(\int_0^A \rho^{n-1}\rho^{-n+\alpha}d\rho\right)Mf(x) = CA^\alpha Mf(x).$$

Regarding the second term II, employing the Hölder inequality, we obtain

$$\mathrm{II} \le C\left(\int_{|x|>A}\frac{1}{|x-y|^{(n-\alpha)p'}}dy\right)^{\frac{1}{p'}}\|f\|_p \le C\left(\int_A^\infty \rho^{n-1-(n-\alpha)p'}d\rho\right)^{\frac{1}{p'}}\|f\|_p \le CA^{\alpha-\frac{n}{p}}\|f\|_p,$$

where $\frac{1}{p}+\frac{1}{p'} = 1$. Now choosing A such that $A^\alpha Mf(x) = A^{\alpha-\frac{n}{p}}\|f\|_p$, we can deduce that

$$|I_\alpha f(x)| \le C\|f\|_p^{\frac{\alpha p}{n}} Mf(x)^{1-\frac{\alpha p}{n}}$$

as desired. □

4. Show that I_α is strong (p,q) $(1<p<\frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p}-\frac{\alpha}{n})$ and weak $(1, \frac{n}{n-\alpha})$.

Proof. By applying $|I_\alpha f(x)| \le C\|f\|_p^{\frac{\alpha p}{n}} Mf(x)^{1-\frac{\alpha p}{n}}$, it follows that

$$\left(\int_{\mathbb{R}^n}|I_\alpha f(x)|^{\frac{pn}{n-\alpha p}}dx\right)^{\frac{n-\alpha p}{pn}} \le C\|f\|_p^{\frac{\alpha p}{n}}\left(\int_{\mathbb{R}^n}Mf(x)^p dx\right)^{\frac{n-\alpha p}{pn}} \le C\|f\|_p$$

for $1<p<\frac{n}{\alpha}$, where we have used the property that M is strong (p,p) for $p>1$. Now, considering $p=1$, we can deduce

$$|\{x\in\mathbb{R}^n : |I_\alpha f(x)| > \lambda\}| = \left|\left\{x\in\mathbb{R}^n : Mf(x) > \left(\frac{\lambda}{\|f\|_1^{\frac{\alpha}{n}}}\right)^{\frac{n}{n-\alpha}}\right\}\right| \le C\left(\frac{\|f\|_1}{\lambda}\right)^{\frac{n}{n-\alpha}},$$

where we have used the fact that M is weak $(1,1)$. □

5. Calculate the Riesz transform of Poisson kernel denoted by $Q_t^{(j)}(x) = R_j(P_t)(x)$.

Proof. By the definition of Riesz transform and Poisson formula, we get

$$Q_t^{(j)}(x) = \int_{\mathbb{R}^n} -i\frac{\xi_j}{|\xi|}e^{-2\pi t|\xi|}e^{2\pi i x\cdot\xi}d\xi.$$

Then it can be deduced from simple calculations that

$$\partial_t Q_t^{(j)}(x) = \int_{\mathbb{R}^n} 2\pi i\xi_j e^{-2\pi t|\xi|}e^{2\pi i x\cdot\xi}d\xi = \partial_j P_t(x).$$

Since $P_t(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}}$, we get

$$\partial_t Q_t^{(j)}(x) = \partial_j P_t(x) = -\frac{(n+1)\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\frac{x_j t}{(t^2+|x|^2)^{\frac{n+1}{2}}}.$$

Therefore

$$Q_t^{(j)}(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\frac{x_j}{(t^2+|x|^2)^{\frac{n+1}{2}}}, \quad \forall t > 0$$

as desired. ☐

6. Find $\Gamma \in S'(\mathbb{R}^n)$ such that $\Delta\Gamma = \delta$, where Δ is the Laplace operator and δ is the Dirac function with respect to 0.

Proof. For the simplicity, we only consider $n \geq 2$ and the case $n = 1$ can be analyze in the same way. Since the solutions of $\Delta\Gamma = 0$ are harmonic polynomials, we only need to find a particular solution for $\Delta\Gamma = \delta$. We claim that $\Delta\Gamma = \delta$ if

$$\Gamma_0(x) = \begin{cases} \text{p. v.} -\dfrac{1}{(n-2)\alpha(n)}\dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\ \text{p. v.}\ \dfrac{1}{2\pi}\ln|x| & \text{if } n = 2, \end{cases}$$

where $\alpha(n) = \mathcal{H}^{n-1}(\partial B^n(0,1))$ and $B^n(0,1)$ is the n dimensional unit ball. Indeed, for $\varphi \in S(\mathbb{R}^n)$, we have

$$\begin{aligned}\lim_{\varepsilon\to0^+}\int_{|x|>\varepsilon}\frac{1}{|x|^{n-2}}\Delta\varphi dx &= \lim_{\varepsilon\to0^+}\left(-\int_{|x|=\varepsilon}\frac{1}{|x|^{n-2}}\frac{x}{\varepsilon}\cdot D\varphi(x)d\mathcal{H}^{n-1} + \int_{|x|=\varepsilon}\varphi\frac{x}{\varepsilon}\cdot D\left(\frac{1}{|x|^{n-2}}\right)d\mathcal{H}^{n-1}\right) \\ &= (n-2)\alpha(n)\varphi(0),\end{aligned}$$

where we have used the fact that $\Delta(\frac{1}{|x|^{n-2}}) = 0$ in $\{|x| > \varepsilon\}$ and

$$\int_{|x|=\varepsilon}\varphi\frac{x}{\varepsilon}\cdot D\left(\frac{1}{|x|^{n-2}}\right)dx = -\frac{1}{\varepsilon^{n-1}}\int_{|x|=\varepsilon}\varphi(x)d\mathcal{H}^{n-1} \to -\varphi(0)$$

$$\left|\int_{|x|=\varepsilon}\frac{1}{|x|^{n-2}}\frac{x}{\varepsilon}\cdot D\varphi d\mathcal{H}^{n-1}\right| \leq \frac{1}{\varepsilon^{n-2}}\int_{|x|=\varepsilon}|D\varphi(x)|d\mathcal{H}^{n-1} \to 0.$$

The case for $n = 2$ can be proved in the same manner. Then we can conclude that

$$\Gamma = \Gamma_0 + \text{harmonic polynomials}$$

as desired. ☐

7. Let $P_k(x)$ $(k \geq 1)$ be a k-homogeneous polynomial. Show that

(1) $(P_k(x)e^{-\pi|x|^2})\widehat{\ }(\xi) = i^{-k}P_k(\xi)e^{-\pi|\xi|^2}$.
(2) $\int_{S^{n-1}} P_k(x')d\sigma(x') = 0$.
(3) $\mathcal{F}\left(\text{p. v.}\frac{P_k(x)}{|x|^{n+k}}\right) = i^{-k}\frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{k}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)}\frac{P_k(\xi)}{|\xi|^k}$.

Proof. Firstly, we observe that (2) directly follows from the formula for the mean value and the fact that $P(0)=0$. Before proving (1), we can assume its truth and show (3). In fact, we have

$$|x|^{-n-k} = \frac{\pi^{\frac{n+k}{2}}}{\Gamma\left(\frac{n+k}{2}\right)} \int_0^\infty t^{\frac{n+k}{2}-1} e^{-\pi t|x|^2} dt$$

and

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{P_k(x)}{|x|^{n+k}} \widehat{\phi}(x) dx &= \lim_{\varepsilon\to 0^+} \frac{\pi^{\frac{n+k}{2}}}{\Gamma\left(\frac{n+k}{2}\right)} \int_0^\infty \int_{|x|>\varepsilon} t^{\frac{n+k}{2}-1} e^{-\pi t|x|^2} \widehat{\phi}(x) P_k(x) dx dt \\
&= \lim_{\varepsilon\to 0^+} \frac{\pi^{\frac{n+k}{2}}}{\Gamma\left(\frac{n+k}{2}\right)} \int_{\mathbb{R}^n} \int_0^\infty \int_{|x|>\varepsilon} t^{\frac{n+k}{2}-1} e^{-\pi t|x|^2} \phi(y) e^{-2\pi i x\cdot y} P_k(x) dx dt dy \\
&= \frac{\pi^{\frac{n+k}{2}}}{\Gamma\left(\frac{n+k}{2}\right)} \int_{\mathbb{R}^n} \int_0^\infty \left(\lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} e^{-\pi t|x|^2} e^{-2\pi i x\cdot y} P_k(x) dx \right) t^{\frac{n+k}{2}-1} \phi(y) dt dy
\end{aligned}$$

where we have used the dominated convergence theorem. Using (1), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} e^{-\pi t|x|^2} e^{-2\pi i x\cdot y} P_k(x) dx &= \int_{\mathbb{R}^n} e^{-\pi t|x|^2} e^{-2\pi i x\cdot y} P_k(x) dx \\
&= \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\pi|x|^2} e^{-2\pi i x\cdot \frac{y}{\sqrt{t}}} P_k\left(\frac{x}{\sqrt{t}}\right) dx = i^k t^{-k-\frac{n}{2}} P_k(y) e^{-\pi\frac{|y|^2}{t}}.
\end{aligned}$$

Combining all the above, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{P_k(x)}{|x|^{n+k}} \widehat{\phi}(x) dx &= \frac{\pi^{\frac{n+k}{2}}}{\Gamma\left(\frac{n+k}{2}\right)} \int_0^\infty \int_{\mathbb{R}^n} i^k t^{-\frac{k}{2}-1} P_k(y)\phi(y) e^{-\pi\frac{|y|^2}{t}} dy dt \\
&= \frac{\pi^{\frac{n+k}{2}}}{\Gamma\left(\frac{n+k}{2}\right)} i^k \int_{\mathbb{R}^n} \int_0^\infty s^{\frac{k}{2}-1} e^{-\pi s|y|^2} P_k(y)\phi(y) dt dy = \frac{\pi^{\frac{n+k}{2}}}{\Gamma\left(\frac{n+k}{2}\right)} i^k \frac{\Gamma\left(\frac{k}{2}\right)}{\pi^{\frac{k}{2}}} \int_{\mathbb{R}^n} \frac{P_k(y)}{|y|^k} \phi(y) dy,
\end{aligned}$$

which implies the result. Now it remains to show (1). Indeed

$$\begin{aligned}
e^{\pi|\xi|^2} \int_{\mathbb{R}^n} P_k(x) e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi} dx &= \int_{\mathbb{R}^n} P_k(x) e^{-\pi\sum_{j=1}^n (x_j - i\xi_j)^2} dx = \int_{\mathbb{R}^n} P_k(z+i\xi) e^{-\pi|z|^2} dz \\
&= \int_0^\infty \int_{\{|z|=\rho\}} P_k(z+i\xi) e^{-\pi|z|^2} dS d\rho = \int_0^\infty \int_{\{|z|=\rho\}} P_k(z+i\xi) e^{-\pi\rho^2} dS d\rho.
\end{aligned}$$

In view of the property that $P_k(z+i\xi)$ is harmonic and the mean value formula, there is

$$e^{\pi|\xi|^2} \int_{\mathbb{R}^n} P_k(x) e^{-\pi|x|^2} e^{-2\pi i x\cdot\xi} dx = \int_0^\infty P_k(i\xi) e^{-\pi\rho^2} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \rho^{n-1} d\rho = i^k P_k(i\xi),$$

where for the second equality, we have used the property of Gamma function. □

8. Assume that T is a convention operator and bounded in $L^2(\mathbb{R}^n)$. Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f(x)dx = 0$. Show that if Tf is integrable, then $\int_{\mathbb{R}^n} Tf(x)dx = 0$.

Proof. Let $K \in S'(\mathbb{R}^n)$, be given. We assume the operator T is defined as follows:

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) f(y) dy.$$

Since T is bounded on $L^2(\mathbb{R}^n)$, we can observe from $\widehat{Tg}(\xi) = \widehat{K}(\xi)\widehat{g}(\xi)$ that $\widehat{K} \in L^\infty$ (due to the property that L^2 multiplier is in L^∞). Considering $f \in L^1(\mathbb{R}^n)$, we have $\widehat{f} \in L^\infty(\mathbb{R}^n)$ and $f \in C_0(\mathbb{R}^n)$. If $Tf \in L^1(\mathbb{R}^n)$, we can also conclude that $\widehat{Tf} \in C_0(\mathbb{R}^n)$. By applying the inverse formula, we obtain

$$\widehat{K}(\xi)\widehat{f}(\xi) = \widehat{Tf}(\xi) = \int_{\mathbb{R}^n} Tf(x) e^{-2\pi i x\cdot\xi} dx.$$

Since $\widehat{K}(\xi)\widehat{f}(\xi)$ is continuous and well-defined pointwise, it can be deduced

$$\int_{\mathbb{R}^n} Tf(x)dx = \widehat{K}(0)\int_{\mathbb{R}^n} f(x)dx = 0,$$

where for the last equality, we have utilized the assumption that the integral of function f is 0. $\square$

9. Let B be a unit ball in $\mathbb{R}^n$, $N > \frac{n}{2}$. Set

$$D_N = \{\varphi \in C_0^\infty(\mathbb{R}^n) : \mathrm{supp}(\varphi) \subset B,\ \|D^\alpha\varphi\|_\infty \leq 1,\ 0 \leq |\alpha| \leq N\}.$$

Let $K \in S'(\mathbb{R}^n)$ such that $K \in L^1_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ and

$$|K(x)| + |x||\nabla K(x)| \leq A_1|x|^{-n},\ x \neq 0.$$

Show that $\widehat{K} \in L^\infty$ if and only if

$$\sup_{\varphi\in D_N, R>0} |K(\varphi^R)| \leq A,\ \varphi^R(x) = \varphi\left(\frac{x}{R}\right).$$

Proof. If $\widehat{K} \in L^\infty$, then for any $\varphi \in D_N$, we can deduce from the equation

$$K(\varphi^R) = (K * \varphi^R)(0) = \int_{\mathbb{R}^n} \widehat{K}(\xi)\widehat{\varphi^R}(\xi)d\xi = \int_{\mathbb{R}^n} \widehat{K}(\xi)R^n\widehat{\varphi}(\xi)d\xi$$

that

$$|K(\varphi^R)| \leq \left|\int_{\mathbb{R}^n} \widehat{K}(\xi)R^n\widehat{\varphi}(\xi)d\xi\right| \leq \|\widehat{K}\|_\infty \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)|d\xi.$$

To show that $\int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)|d\xi$ is uniformly bounded, we decompose the integration as follows:

$$\int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)|d\xi = \int_{|\xi|\leq 1} |\widehat{\varphi}(\xi)|d\xi + \int_{|\xi|>1} |\widehat{\varphi}(\xi)|d\xi.$$

On the one hand, we have

$$\|\widehat{\varphi}\|_\infty \leq \int_{\mathbb{R}^n} |\varphi(x)|dx = \int_B |\varphi(x)|dx \leq |B| \Rightarrow \int_{|\xi|\leq 1} |\widehat{\varphi}(\xi)|d\xi \leq \int_{|\xi|\leq 1} \|\widehat{\varphi}\|_\infty d\xi \leq |B|.$$

On the other hand, since $\|D^\alpha\varphi\|_\infty \leq 1$ for any $0 \leq |\alpha| \leq N$, it can obtained that

$$\|D^\alpha\varphi\|_{L^2(\mathbb{R}^n)} \leq |B|^{\frac{1}{2}}.$$

This implies the existence of a dimensional constant C, such that $\||\xi|^\alpha\widehat{\varphi}\|_{L^2(\mathbb{R}^n)} \leq C$ for any $0 \leq |\alpha| \leq N$. Consequently,

$$\int_{|\xi|>1} |\widehat{\varphi}(\xi)|d\xi \leq \left(\int_{|\xi|>1} |\xi|^{-2N}d\xi\right)^{\frac{1}{2}} \left(\int_{|\xi|>1} |\xi|^{2N}|\widehat{\varphi}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \leq C.$$

Hence, $\sup_{\varphi\in D_N, R>0} |K(\varphi^R)| \leq A$. To show the other side, we note that $|K(x)| \leq A_1|x|^{-n}$ implies that there is a dimensional constant C, such that for any $R > 0$,

$$\int_{R<|x|<2R} |K(x)|dx \leq \int_R^{2R} \frac{A_1}{|x|^n}dx \leq CA_1.$$

Furthermore, $|\nabla K(x)| \leq A_1|x|^{-(n+1)}$ implies that $K(x)$ satisfies the Hörmander condition, i.e. there exists $B > 0$ such that

$$\int_{|x|>2|y|} |K(x-y) - K(x)|dx \leq B.$$

Since K is seen as a tempered distribution in view of the principle integral, so to show that $\widehat{K} \in L^\infty$, it suffices to prove that for any $0 < R_1 < R_2 < \infty$,

$$\left|\int_{R_1<|x|<R_2} K(x)dx\right| \leq C,$$

where C is a constant independent of R_1 and R_2. We will divide the proof into two cases.

Case 1. If $R_2 \le 2R_1$, we have

$$\left| \int_{R_1<|x|<R_2} K(x)dx \right| \le \int_{R_1<|x|<2R_1} |K(x)|dx \le CA_1.$$

Case 2. If $R_2 > 2R_1$, we can choose $\varphi \in D_N$ such that $\varphi \equiv 1$ in $\{|x| \le \frac{1}{2}\}$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} K(x)(\varphi^{2R_2}(x) - \varphi^{2R_1}(x))dx = &\int_{2R_1<|x|<R_2} K(x)dx + \int_{R_1<|x|<2R_1} K(x)(1-\varphi^{2R_1}(x))dx \\ &+ \int_{R_2<|x|<2R_2} K(x)\varphi^{2R_2}(x)dx,\end{aligned}$$

which implies that

$$\left| \int_{2R_1<|x|<R_2} K(x)dx \right| \le 2CA_1 + 2A$$

and then completes the proof in view of the first case. □

10. Assume that $K(x, y)$ satisfies the conditions of standard kernel and φ is a smooth radial function satisfying $\varphi(x) = 0$ if $|x| \le \frac{1}{2}$ and $\varphi(x) = 1$ if $|x| \ge 1$. Let $K_\varepsilon(x, y) = K(x, y)\varphi(\frac{x-y}{\varepsilon})$. Show that $K_\varepsilon(x.y)$ satisfies the conditions of standard kernel uniformly of $\varepsilon > 0$.

Proof. Since K satisfies conditions of standard kernel, we have, for any $x, y \in \mathbb{R}^n$ such that $x \ne y$,

$$\begin{aligned}|K(x,y)| &\le \frac{C}{|x-y|^n}, \\ |K(x,y) - K(x,z)| &\le \frac{C|y-z|^\delta}{|x-y|^{n+\delta}} \text{ if } |x-y| > 2|y-z|, \\ |K(x,y) - K(w,y)| &\le \frac{C|x-w|^\delta}{|x-y|^{n+\delta}} \text{ if } |y-x| > 2|x-w|.\end{aligned}$$

for some $C, \delta > 0$. Firstly we notice that

$$|K_\varepsilon(x,y)| \le |K(x,y)| \left| \varphi\left(\frac{x-y}{\varepsilon}\right) \right| \le \frac{C}{|x-y|^n}.$$

Then we only need to verify the next two properties. For the sake of simplicity, we only show the second property since the proof for the third result is almost the same. Assume that here $|x - y| > 2|y - z|$. At this time, it can be seen that

$$\frac{1}{2}|x-y| \le |x-z| \le \frac{3}{2}|x-y|. \qquad (*)$$

By utilizing triangular inequality, it follows that

$$\begin{aligned}|K_\varepsilon(x,y) - K_\varepsilon(x,z)| &= |K(x,y)\varphi_\varepsilon(x-y) - K(x,z)\varphi_\varepsilon(x-z)| \\ &\le |K(x,y)\varphi_\varepsilon(x-y) - K(x,y)\varphi_\varepsilon(x-z)| + |K(x,y)\varphi_\varepsilon(x-z) - K(x,z)\varphi_\varepsilon(x-z)| \\ &\le |K(x,y)||\varphi_\varepsilon(x-y) - \varphi_\varepsilon(x-z)| + |K(x,y) - K(x,z)||\varphi_\varepsilon(x-z)| \\ &\le \frac{C}{|x-y|^n}|\varphi_\varepsilon(x-y) - \varphi_\varepsilon(x-z)| + \frac{C|y-z|^\delta}{|x-y|^{n+\delta}}.\end{aligned}$$

Then, it remains to show that

$$|\varphi_\varepsilon(x-y) - \varphi_\varepsilon(x-z)| \le \frac{C|y-z|^\delta}{|x-y|^\delta}. \quad (**)$$

By the definition of φ, it can be assumed that $|x - y| > \frac{\varepsilon}{4}$. Indeed, if $|x - y| \le \frac{\varepsilon}{4}$, by employing $(*)$, we can deduce that $|x - y| \le \frac{3}{8}\varepsilon$. Consequently, $\varphi_\varepsilon(x - z) = \varphi_\varepsilon(z - y) = 0$, which directly implies $(**)$. Now it follows from Lagrange mean value theorem that

$$|\varphi_\varepsilon(x-y) - \varphi_\varepsilon(x-z)| \le \frac{C|y-z|}{\varepsilon} \le \frac{C|y-z|}{|x-y|},$$

by which we can obtain $(**)$. $\square$

11. Let T_1 and T_2 are Calderón-Zygmund operator and have the same standard kernel. Show that there exists $a \in L^\infty(\mathbb{R}^n)$ such that $(T_1 - T_2)f(x) = a(x)f(x)$.

Proof. Actually, we will show the following lemma.

Lemma 4.1. *Let T be a bounded operator on $L^2(\mathbb{R}^n)$. For any $f \in L^\infty_c(\mathbb{R}^n)$, $Tf = 0$ a.e. in $x \in \operatorname{supp}(f)$. Then there exists $b \in L^\infty$ such that $Tf(x) = b(x)f(x)$ a.e.*

To prove this lemma, we can decompose $\mathbb{R}^n = \cup_{Q\in\mathcal{Q}_0} Q$ as the disjoint union, where

$$\mathcal{Q}_k = \left\{ \prod_{i=1}^n \left[\frac{a_i}{2^k}, \frac{a_i+1}{2^k}\right] : a_1, a_2, ..., a_n \in \mathbb{Z} \right\}.$$

We claim that $b(x) = \cup_{Q\in\mathcal{Q}_0} T\chi_Q(x)\chi_Q(x) \in L^2_{\text{loc}}(\mathbb{R}^n)$. Indeed, for any $Q \in \mathcal{Q}_k$, $k \geq 0$, there exists a unique $Q' \in \mathcal{Q}_0$ such that $Q \subset Q'$. In this case,

$$T(\chi_{Q'} - \chi_Q)(x) = 0, \quad \forall x \in Q$$

by the assumption. This implies that $T\chi_{Q'}(x) = T\chi_Q(x)$ almost everywhere in Q. Also, we have $T\chi_Q(x) = 0$ a.e. in $\mathbb{R}^n \backslash Q$. Then $T\chi_Q(x) = T\chi_{Q'}(x)\chi_Q(x) = b(x)\chi_Q(x)$ a.e. Define

$$V = \operatorname{span}\{\chi_Q : Q \in \mathcal{Q}_k, \text{ for some } k \geq 0\}.$$

Then we can get that $Tf = bf$ if $f \in V$. For any $Q \in \cup_k \mathcal{Q}_k$, we have

$$\int_Q |b|^2 dx = \|b\chi_Q\|_2^2 = \|T\chi_Q\|_2^2 \leq C\|\chi_Q\|_2^2 = C|Q|.$$

By applying the Lebesgue differentiation theorem, we conclude that $|b|^2 \leq C$ a.e. and then $b \in L^\infty$. With this property and the fact that V is dense in $L^2(\mathbb{R}^n)$, it is not difficult to extend T to the space $L^2(\mathbb{R}^n)$ such that for any $f \in L^2(\mathbb{R}^n)$, $Tf = bf$. $\square$

12. Let T be a Calderón-Zygmund operator with standard kernel $K(x, y)$. Let

$$I_{\varepsilon,N}(x) = \int_{\varepsilon<|x-y|<N} K(x, y)dy.$$

Show that

$$\int_{|x-x_0|<N} |I_{\varepsilon,N}(x)|^2 dx \leq CN^n,$$

where C is independent of N, ε and x_0.

Proof. To prove this, we only need to show that

$$\int_{|x-x_0|<N/2} |I_{\varepsilon,N}(x)|^2 dx \leq CN^n,$$

is uniform with respect to x_0, ε and N, since the ball with radius N can be covered by finite number of balls with radius $\frac{N}{2}$. Set $K_\varepsilon(x, y) = K(x, y)\chi_{\{|x-y|>\varepsilon\}}$. Define

$$T_\varepsilon f(x) = \int_{\mathbb{R}^n} K_\varepsilon(x, y)f(y)dy.$$

Since $K(x, y)$ is a standard kernel, then $\|T_\varepsilon\|_{2\to2}$ is uniformly bounded. Now let

$$\chi^{N,x_0}(x) = \chi_{\{y:|y-x_0|<N\}}(x).$$

It is easy to observe that (here for $E_1, E_2 \subset \mathbb{R}^n$, $E_1 \Delta E_2$ denotes $(E_1\backslash E_2) \cup (E_2\backslash E_1)$)

$$\{y : |x-y| < N\}\Delta\{y : |y-x_0| < N\} \subset \left\{y : \frac{N}{2} < |x-y| < \frac{3N}{2}\right\}$$

when $|x-x_0|<\frac{N}{2}$. In view of this observation, we use the formula

$$I_{\varepsilon,N}(x)-T_\varepsilon(\chi^{N,x_0})(x)=\int_{\varepsilon<|x-y|<N}K(x,y)dy-\int_{|x-y|>\varepsilon,|y-x_0|<N}K(x,y)dy$$

to obtain that

$$|I_{\varepsilon,N}(x)-T_\varepsilon(\chi^{N,x_0})(x)|\le\int_{\frac{N}{2}<|x-y|<\frac{3N}{2}}|K(x,y)|dy\le\int_{N/2}^{3N/2}\int_{|x|=r}\frac{C}{|x|^n}dSdr\le C.$$

Therefore, we obtain

$$\int_{|x-x_0|<\frac{N}{2}}|I_{\varepsilon,N}(x)|^2dx\le CN^n+C\|T_\varepsilon(\chi^{N,x_0})\|_2^2\le CN^n+C\|\chi^{N,x_0}\|_2^2\le CN^n$$

as desired. □

13. Let $K\in S'(\mathbb{R}^n)$ such that $K\in L^1_{\text{loc}}(\mathbb{R}^n)$. Assume that

$$\sup_{a>0}\int_{a<|x|<2a}|K(x)|dx<\infty.$$

If $Tf=K*f$ is bounded on $L^2(\mathbb{R}^n)$, show that

$$\sup_{0<a<b}\left|\int_{a<|x|<b}K(x)dx\right|<\infty.$$

Proof. Consider a radial function $\varphi\in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\varphi)\subset\{|x|\le 2\}$ and $\varphi\equiv 1$ in $\{|x|\le 1\}$. For $R>0$, let $\varphi^R(x)=\varphi(\frac{x}{R})$. We have

$$(K*\varphi^R)(0)=\langle K,\varphi^R\rangle_{S'\times S}=\int_{\mathbb{R}^n}\widehat{K}(\xi)R^n\widehat{\varphi}(R\xi)d\xi.$$

Using this, we can obtain

$$\left|\int_{\mathbb{R}^n}K(x)\varphi^R(x)dx\right|\le\left|\int_{\mathbb{R}^n}\widehat{K}(\xi)R^n\widehat{\varphi}(R\xi)d\xi\right|\le\|\widehat{K}\|_\infty\|\widehat{\varphi}\|_1\le\|T\|_{2\to2}\|\widehat{\varphi}\|_1.$$

Fix $0<R_1<R_2<\infty$, we can divide the proof into two cases.
Case 1. If $2R_1\ge R_2$, we have

$$\left|\int_{R_1<|x|<R_2}K(x)dx\right|\le\int_{R_1<|x|<2R_1}|K(x)|dx\le A=\sup_{a>0}\int_{a<|x|<2a}|K(x)|dx.$$

Case 2. If $2R_1<R_2$, then

$$\begin{aligned}\int_{\mathbb{R}^n}K(x)(\varphi^{R_2}(x)-\varphi^{R_1}(x))dx&=\int_{2R_1<|x|<R_2}K(x)dx+\int_{R_1<|x|<2R_1}K(x)(1-\varphi^{R_1}(x))dx\\&\quad+\int_{R_2<|x|<2R_2}K(x)\varphi^{R_2}(x)dx,\end{aligned}$$

and

$$\left|\int_{2R_1<|x|<R_2}K(x)dx\right|\le A+A+2\|T\|_{2\to2}\|\widehat{\varphi}\|_1,$$

which completes the proof. □

5. Hardy and BMO space

1. Let $\varphi \in S(\mathbb{R}^n)$, $\widehat{\varphi}(0) \neq 0$. Define the maximal function

$$M_\varphi^* f(x) = \sup_{t>0} \sup_{|x-y|<t} |\varphi_t * f(y)|.$$

Show that

$$\|M_\varphi^* f\|_1 \leq C\|f\|_{\mathcal{H}^1_{at}(\mathbb{R}^n)}.$$

Proof. By the definition of $\mathcal{H}^1_{at}$, we only need to show that for any atom a satisfying

$$\mathrm{supp}(a) \subset Q,\ \int_Q a = 0 \text{ and } \|a\|_\infty \leq \frac{1}{|Q|},$$

there exists C, a dimensional constant, such that

$$\|M_\varphi^* a\|_1 \leq C,$$

Using simple translations, we can assume that $Q = Q(0,r)$ and $Q^* = B(0, 4\sqrt{n}r)$. On the one hand, by question 1 in Section 2, we have

$$\int_{Q^*} |M_\varphi^* a| dx \leq C \int_{Q^*} |Ma| dx \leq C|Q^*|^{\frac{1}{2}} \int_{Q^*} |Ma|^2 dx \leq C|Q_*|^{\frac{1}{2}} \|a\|_2^2 \leq C.$$

On the other hand, for fixed $x \in \mathbb{R}^n \backslash Q_*$ and $t > 0$, we have

$$|(\varphi_t * a)(y)| \leq \int_Q \frac{1}{t^n} \left| \varphi\left(\frac{y-z}{t}\right) - \varphi\left(\frac{y}{t}\right)\right| |a(z)| dz \leq \int_Q \frac{|z|}{t^{n+1}} \left|\nabla\varphi\left(\frac{y-\theta(z)z}{t}\right)\right| |a(z)| dz$$

for any $|x-y| < t$, where $\theta(z) \in (0,1)$. We claim that

$$|(\varphi_t * a)(y)| \leq \frac{Cl(Q)}{|x|^{n+1}}.$$

If the claim is true, it follows that

$$\sup_{t>0} \sup_{|x-y|<t} |(\varphi_t * a)(y)| \leq \frac{Cl(Q)}{|x|^{n+1}}$$

for any $x \in \mathbb{R}^n \backslash Q^*$ and then

$$\int_{R^n \backslash Q^*} |M_\varphi^* a| dx \leq \int_{\mathbb{R}^n \backslash Q^*} \frac{Cl(Q)}{|x|^{n+1}} dx \leq C,$$

which completes the proof. To show this claim, we will divide the prood into two cases.
Case 1. If $|x| \leq 100t$, then

$$|(\varphi_t * a)(y)| \leq \int_Q \frac{Cl(Q)}{t^{n+1}} |a(z)| dz \leq \frac{Cl(Q)}{|x|^{n+1}}.$$

Case 2. If $|x| > 100t$, by the assumption that φ is Schwartz function, $|\nabla\varphi(w)| \leq C|w|^{-n-1}$ and then

$$\left|\nabla\varphi\left(\frac{y-\theta(z)z}{t}\right)\right| \leq \frac{Ct^{n+1}}{|y-\theta(z)z|^{n+1}}.$$

At this time $|y| \geq |x| - |y-x| \geq 99t$ and then

$$|y-\theta(z)z| \geq ||y|-|z|| \geq ||x|-|x-y|-|z|| \geq \left|\frac{99}{100}|x| - |z|\right| \geq \frac{99}{100}|x| - l(Q) \geq C|x|,$$

which implies that

$$|(\varphi_t * a)(y)| \leq \int_Q \frac{C|z|}{|y-\theta(z)z|^{n+1}} |a(z)| dz \leq \int_Q \frac{C|z|}{|x|^{n+1}} |a(z)| dz \leq \frac{Cl(Q)}{|x|^{n+1}} \int_Q |a(z)| dz \leq \frac{Cl(Q)}{|x|^{n+1}}.$$

□

2. Show that for fractional integral operator I_α with $0 < \alpha < n$,

$$\|I_\alpha f\|_{\frac{n}{n-\alpha}} \leq C\|f\|_{\mathcal{H}^1_{at}(\mathbb{R}^n)}.$$

Proof. By the definition of $\mathcal{H}^1_{at}$, we only need to show that for any atom a such that

$$\mathrm{supp}(a) \subset Q, \ \int_Q a = 0 \text{ and } \|a\|_\infty \leq \frac{1}{|Q|},$$

there is

$$\|I_\alpha a\|_{\frac{n}{n-\alpha}} \leq C,$$

where C is a dimensional constant. By the property of I_α (see question 3 and 4 of Section 4), I_α is weak $(1, \frac{n}{n-\alpha})$ and

$$|I_\alpha a(x)| \leq C\|a\|_p^{\frac{\alpha p}{n}} (Mf(x))^{1-\frac{\alpha p}{n}} \leq C|Q|^{\frac{\alpha}{n}-1}.$$

Using this, we can choose $Q^* = B(c, 4\sqrt{n}r), Q = Q(c,r)$ and obtain that

$$\begin{aligned}\int_{Q^*} |I_\alpha a|^{\frac{n}{n-\alpha}} dx &= \left(\int_0^A + \int_A^\infty\right) \lambda^{\frac{n}{n-\alpha}-1} |\{x \in Q^* : |I_\alpha a| > \lambda\}| d\lambda \\ &\leq C\int_0^A \lambda^{\frac{n}{n-\alpha}} |Q^*| d\lambda + \int_A^{C|Q|^{\frac{\alpha}{n}-1}} \lambda^{-1} d\lambda \leq A^{\frac{n}{n-\alpha}} |Q^*| + \ln\left(\frac{C|Q|^{\frac{\alpha}{n}-1}}{A}\right).\end{aligned}$$

Let $A = |Q|^{\frac{\alpha}{n}-1}$, we obtain that

$$\int_{Q^*} |I_\alpha a|^{\frac{n}{n-\alpha}} dx \leq C.$$

On the other hand, for any $x \in \mathbb{R}^n \backslash Q^*$ and $y \in Q$ with $|x-c| > 2|y-c|$, we can deduce from Lagrange mean value theorem and Minkowski inequality that

$$\begin{aligned}\left(\int_{\mathbb{R}^n\backslash Q^*} |I_\alpha a|^{\frac{n}{n-\alpha}} dx\right)^{\frac{n-\alpha}{n}} &= \left(\int_{\mathbb{R}^n\backslash Q^*} \int_Q \left|\left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x-c|^{n-\alpha}}\right) a(y)\right|^{\frac{n}{n-\alpha}} dydx\right)^{\frac{n-\alpha}{n}} \\ &\leq \left(\int_{\mathbb{R}^n\backslash Q^*} \int_Q \left|\left|\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x-c|^{n-\alpha}}\right| |a(y)|\right|^{\frac{n}{n-\alpha}} dydx\right)^{\frac{n-\alpha}{n}} \\ &\leq \int_Q \left(\int_{\mathbb{R}^n\backslash Q^*} \left(\frac{|y-c|}{|x-c|^{n-\alpha+1}} |a(y)|\right)^{\frac{n}{n-\alpha}} dx\right)^{\frac{n-\alpha}{n}} dy \leq Cl(Q) \left(l(Q)^{-\frac{\alpha}{n-\alpha}}\right)^{\frac{n-\alpha}{n}} \leq C.\end{aligned}$$

Combined discussions above, we have complete the proof. □

3. Let $f \in \mathcal{H}^1_{at}(\mathbb{R}^n)$. Show that

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(y)|}{|y|^n} dy \leq C\|f\|_{\mathcal{H}^1_{at}(\mathbb{R}^n)}.$$

Proof. Also we only need to show that for any atom a such that

$$\mathrm{supp}(a) \subset Q, \ \int_Q a = 0 \text{ and } \|a\|_\infty \leq \frac{1}{|Q|},$$

there is

$$\int_{\mathbb{R}^n} \frac{\widehat{a}(y)}{|y|^n} dy \leq C,$$

where C depends only on n. Note that for $\tau_h a(x) = a(x+h)$, we have $\widehat{\tau_h a}(\xi) = \widehat{a}(\xi) e^{2\pi i h\xi}$ and then

$$\int_{\mathbb{R}^n} \frac{|\widehat{a}(y)|}{|y|^n} dy = \int_{\mathbb{R}^n} \frac{|\widehat{\tau_h a}(y)|}{|h|^n} dy.$$

This implies it can be assumed that Q is centered at 0. For $a_\lambda(x) = \lambda^{-n} a(\lambda^{-1}x)$, we have $\widehat{a}_\lambda(\xi) = \widehat{a}(\lambda\xi)$ and then

$$\int_{\mathbb{R}^n} \frac{|\widehat{a}(y)|}{|y|^n} dy = \int_{\mathbb{R}^n} \frac{|\widehat{a}_\lambda(y)|}{|y|^n} dy.$$

This means that we can assume that $l(Q)$ is 1. Now without loss of generality, we assume that $Q = Q(0, \frac{1}{2})$. Let $Q^* = B(0, 2\sqrt{n})$, then

$$\int_{Q^*} \frac{|\widehat{a}(y)|}{|y|^n} dy = \int_{Q^*} \frac{1}{|y|^n} \left| \int_Q a(x)(e^{-2\pi i x y} - 1) dx \right| dy \leq \int_{Q^*} \frac{1}{|y|^n} \int_Q |a(x)||x||y| dx dy \leq C.$$

On the other hand, by Cauchy inequality, it follows that

$$\int_{\mathbb{R}^n \setminus Q^*} \frac{|\widehat{a}(y)|}{|y|^n} dy \leq \left(\int_{\mathbb{R}^n \setminus Q^*} |\widehat{a}(y)|^2 dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n \setminus Q^*} \frac{1}{|y|^{2n}} dy \right)^{\frac{1}{2}} \leq C,$$

where we have used Plancherel identity for the second inequality. Then we can complete the proof. $\square$

4. Let $|f(x)| \leq \frac{C}{1+|x|^{n+\varepsilon}}$, $\varepsilon > 0$ and $\int_{\mathbb{R}^n} f(x) dx = 0$. Show that $f \in \mathcal{H}^1_{at}(\mathbb{R}^n)$.

Proof. Let R_k $(k = 1, 2, ..., n)$ be Riesz transform. We will use the norm

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|f\|_1 + \sum_{k=1}^{n} \|R_k f\|_1$$

to verify the result. Firstly, we have $\|f\|_1 < \infty$ by the assumption of f. It remains to show that for any $k = 1, 2, ..., n$, $\|R_k f\|_1 < \infty$. To simplify the notations, we define

$$A(r, R) = \{x \in \mathbb{R}^n : r \leq |x| \leq R\}.$$

Since $\int_{\mathbb{R}^n} f(x) dx = 0$, we get

$$|R_k f(x)| \leq \sum_{j=0}^{\infty} \left| \int_{\mathbb{R}^n} \left(\frac{x_k - y_k}{|x-y|^{n+1}} - \frac{x_k}{|x|^{n+1}} \right) f_j(y) dy \right|,$$

where $f_j = f\chi_{A(2^j, 2^{j+1})}$. Let

$$g_j(x) = \int_{A(2^j, 2^{j+1})} \left(\frac{x_k - y_k}{|x-y|^{n+1}} - \frac{x_k}{|x|^{n+1}} \right) f_j(y) dy.$$

If $|x| > 2 \cdot 2^{j+1} = 2^{j+2}$, we have

$$|g_j(x)| \leq \left| \int_{A(2^j, 2^{j+1})} \frac{C|y|}{|x|^{n+1}} f(y) dy \right| \leq \frac{C}{|x|^{n+1}} \frac{(2^j)^{n+1}}{1 + (2^j)^{n+\varepsilon}} \leq \frac{C(2^j)^{1-\varepsilon}}{|x|^{n+1}}.$$

Then

$$\int_{|x| > 2^{j+2}} |g_j(x)| dx = C \int_{2^{j+2}}^{\infty} \frac{(2^j)^{1-\varepsilon}}{\rho^2} d\rho \leq C(2^j)^{-\varepsilon}.$$

On the other hand for $2 \leq |x| < 2^{j+2}$, one has

$$|g_j(x)| \leq \frac{1}{|x|^{n+1}} \int_{A(2^j, 2^{j+1})} |f(y)| dy + |R_k f_j(x)|.$$

For $1 < p < \infty$, by using the property that R_k is bounded on L^p, we obtain

$$\begin{aligned} \|R_k f_j\|_{L^1(A(2, 2^{j+2}))} &\leq C(2^j)^{n(1-\frac{1}{p})} \|R_k f_j\|_{L^p(\mathbb{R}^n)} \leq C(2^j)^{n(1-\frac{1}{p})} \|f_j\|_{L^p(\mathbb{R}^n)} \\ &= C(2^j)^{n(1-\frac{1}{p})} \|f\|_{L^p(A(2^j, 2^{j+1}))} \leq C(2^j)^{n(1-\frac{1}{p})} \frac{(2^j)^{\frac{n}{p}}}{(2^j)^{n+\varepsilon}} \leq C(2^j)^{-\varepsilon}. \end{aligned}$$

Also, there is

$$\left| \frac{1}{|x|^{n+1}} \int_{A(2^j, 2^{j+1})} |f(y)| dy \right|_{L^1(A(2^j, 2^{j+1}))} \leq C(2^j)^{-\varepsilon} \ln(2^j).$$

Combined all above, we get

$$\|R_k f\|_{L^1(\{|x| \geq 2\})} \leq \sum_{j=0}^{\infty} \|g_j\|_{L^1(\{|x| \geq 2\})} \leq \sum_{j=0}^{\infty} C(2^j)^{-\varepsilon} \ln(2^j) < \infty.$$

Again by the boundedness of R_k on L^p, then $R_k f \in L^1$, which completes the proof. $\square$

Remark 5.1. Here is another method to show that $f \in \mathcal{H}^1_{at}$. For any $k \in \mathbb{Z}_+$, we define

$$a_k = \int_{B_{2^k}} f(x)dx = -\int_{\mathbb{R}^n \setminus B_{2^k}} f(x)dx$$

and

$$g_k(x) = \begin{cases} f(x) - \dfrac{1}{|B_{2^k}|} a_k & \text{if } |x| \leq 2^k, \\ 0 & \text{if } |x| > 2^k. \end{cases}$$

Since $|f(x)| \leq \frac{C}{1+|x|^{n+\varepsilon}}$, it follows that

$$|a_k| \leq \int_{B_{2^k}} |f(x)|dx \leq \int_{B_{2^k}} \frac{C}{|x|^{n+\varepsilon}} dx \leq C_1 2^{-k\varepsilon},$$

where C_1 depends only on C and n. Setting $h_k = g_k - g_{k-1}$ if $k \geq 2$ and $h_1 = g_1$, we can obtain that

$$\int_{B_{2^k}} h_k(x)dx = 0, \ \operatorname{supp}(h_k) \subset B_{2^k}$$

and

$$\|h_k\|_{L^\infty(B_{2^k})} \leq \sup_{2^{k-1} < |x| \leq 2^k} |f(x)| + \frac{|a_{k-1}|}{2^{n(k-1)}} + \frac{|a_{k-1}|}{2^{nk}} \leq \frac{C_2}{|B_{2^k}|} \cdot 2^{-k\varepsilon},$$

where C is dependent only on C and n. Now it can be deduced that $\{\widetilde{h_k}\} = \{C_2^{-1} 2^{k\varepsilon} h_k\}$ is a sequence of atoms and there exists the relation

$$f = \sum_{k=1}^{\infty} h_k = \sum_{k=1}^{\infty} C_2 2^{-k\varepsilon} \widetilde{h}_k,$$

which implies that $f \in \mathcal{H}^1_{at}(\mathbb{R}^n)$.

5. Let $f \in \mathrm{BMO}(\mathbb{R}^n)$. Show that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+\varepsilon}} dx < \infty.$$

Proof. Denote $Q_k = [-2^k, 2^k]$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|f(x)|}{1+|x|^{n+\varepsilon}} dx &\leq \int_{\mathbb{R}^n} \frac{|f(x) - f_{Q_0}|}{1+|x|^{n+\varepsilon}} dx + \int_{\mathbb{R}^n} \frac{|f_{Q_0}|}{1+|x|^{n+\varepsilon}} dx \\ &\leq \sum_{k=0}^{\infty} \int_{Q_{k+1} \setminus Q_k} \frac{|f(x) - f_{Q_0}|}{1+|x|^{n+\varepsilon}} dx + \int_{Q_0} \frac{|f(x) - f_{Q_0}|}{1+|x|^{n+\varepsilon}} dx + \int_{\mathbb{R}^n} \frac{|f_{Q_0}|}{1+|x|^{n+\varepsilon}} dx. \end{aligned}$$

By the assumption on f, then

$$\int_{Q_0} \frac{|f(x) - f_{Q_0}|}{1+|x|^{n+\varepsilon}} dx + \int_{\mathbb{R}^n} \frac{|f_{Q_0}|}{1+|x|^{n+\varepsilon}} dx \leq C.$$

It remains to show that

$$\sum_{k=0}^{\infty} \int_{Q_{k+1} \setminus Q_k} \frac{|f(x) - f_{Q_0}|}{1+|x|^{n+\varepsilon}} dx < \infty.$$

Indeed,

$$\begin{aligned}\sum_{k=0}^{\infty}\int_{Q_{k+1}\setminus Q_k}\frac{|f(x)-f_{Q_0}|}{1+|x|^{n+\varepsilon}}dx &\le \sum_{k=0}^{\infty}\int_{Q_{k+1}\setminus Q_k}\frac{|f(x)-f_{Q_k}|}{1+|x|^{n+\varepsilon}}dx+\sum_{k=0}^{\infty}\int_{Q_{k+1}\setminus Q_k}\frac{|f_{Q_k}-f_{Q_0}|}{1+|x|^{n+\varepsilon}}dx\\ &\le \sum_{k=0}^{\infty}2^{-k\varepsilon}\|f\|_*+\sum_{k=0}^{\infty}\int_{Q_{k+1}\setminus Q_k}\sum_{j=0}^{k}\frac{|f_{Q_{j+1}}-f_{Q_j}|}{1+|x|^{n+\varepsilon}}dx\\ &\le \sum_{k=0}^{\infty}2^{-k\varepsilon}\|f\|_*+\sum_{k=0}^{\infty}k\cdot 2^{-k\varepsilon}\|f\|_*<\infty,\end{aligned}$$

which completes the proof. □

6. Show that for any fractional integral operator I_α with $0<\alpha<n$,

$$\|I_\alpha f\|_p \le C\|f\|_{n/\alpha}.$$

Proof. This is a direct consequence of the duality between BMO space and $\mathcal{H}^1_{at}$ and the results of question 2 in this section. □

7. Prove that $\||f|^\alpha\|_* \le 2\|f\|_*^\alpha$ whenever $0<\alpha\le 1$.

Proof. For any $a\in\mathbb{R}$ and $Q\subset\mathbb{R}^n$ being a cube, we see that

$$\frac{1}{|Q|}\int_Q ||f|^\alpha-|a|^\alpha|dx \le \frac{1}{|Q|}\int_Q ||f|-|a||^\alpha dx \le \left(\frac{1}{|Q|}\int_Q ||f|-|a||dx\right)^\alpha,$$

where for the first inequality, we have used Bernoulli inequality and for the second inequality, we have applied Hölder inequality. Then by utilizing the characterization of the norm $\|\cdot\|_*$, the result follows directly. □

8. Let f be a real-valued BMO function on $\mathbb{R}^n$. Prove that the functions

$$f_{KL}(x)=\begin{cases}K & \text{if } f(x)<K,\\ f(x) & \text{if } K\le f(x)\le L,\\ L & \text{if } f(x)>L,\end{cases}$$

satisfy $\|f_{KL}\|_*\le \frac{9}{4}\|f\|_*$.

Proof. An often useful lemma:

Lemma 5.2. *For any $\alpha\in\mathbb{R}$ we have $\frac{1}{|Q|}\int_Q|f-f_Q|\le\frac{2}{|Q|}\int_Q|f-\alpha|$.*

Proof of this lemma.

$$\begin{aligned}\frac{1}{|Q|}\int_Q|f-f_Q| &\le \frac{1}{|Q|}\int_Q|f_Q-\alpha|+\frac{1}{|Q|}\int_Q|f-\alpha| = |\alpha-f_Q|+\frac{1}{|Q|}\int_Q|f-\alpha|\\ &= |(\alpha-f)_Q|+\frac{1}{|Q|}\int_Q|f-\alpha|\le\frac{2}{|Q|}\int_Q|f-\alpha|.\end{aligned}$$

□

Corollary 5.3. *If $|\phi(t)-\phi(s)|\le|t-s|$ then $||\phi\circ f||_*\le 2||f||_*$.*

Proof of this corollary.

$$\frac{1}{|Q|}\int_Q|\phi\circ f-(\phi\circ f)_Q|\le\frac{2}{|Q|}\int_Q|\phi\circ f-\phi(f_Q)|\le\frac{2}{|Q|}\int_Q|f-f_Q|.$$

□

Then the result follows from the choice of ϕ by

$$\phi(x) = \begin{cases} K & \text{if } x < K, \\ x & \text{if } K \le x \le L, \\ L & \text{if } x > L. \end{cases}$$

This is because $|\phi(s) - \phi(t)| \le |s - t|$ for any $t, s \in \mathbb{R}$. □

Remark 5.4. From the calculations above, we can obtain that for any ϕ being Lipschitz, there is the estimate

$$\|\phi \circ f\|_* \le 2 \operatorname{Lip}(\phi)\|f\|_*, \ \operatorname{Lip}(\phi) = \sup_{x,y\in\mathbb{R}, x\neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Remark 5.5. In this problem, the constant $\frac{9}{4}$ may be confusing since we have already improved this estimate for the constant 2. In fact, the design of this problem is from the observation that

$$\|\max\{f, g\}\|_* \le \frac{3}{2}(\|f\|_* + \|g\|_*), \ \|\min\{f, g\}\|_* \le \frac{3}{2}(\|f\|_* + \|g\|_*).$$

This is obtained by simply use the property that $\||f|\|_* \le 2\|f\|_*$. Now we can prove $\|f_{KL}\|_* \le \frac{9}{4}\|f\|_*$ by noting that $f_{KL}(x) = \max\{\min\{f(x), L\}, K\}$.

6. Paley-Littlewood decomposition

1. Let $\psi \in S(\mathbb{R}^n)$, $\psi(0) = 0$. Define $\widehat{S_j f}(\xi) = \widehat{f}(\xi)\psi(2^{-j}\xi)$. Show that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{\frac{1}{2}} \right\|_1 \le C\|f\|_{\mathcal{H}^1_{at}}.$$

Proof. Let $\Psi \in S(\mathbb{R}^n)$ such that $\widehat{\Psi}(\xi) = \psi(\xi)$. Then we get

$$S_j f(x) = \int_{\mathbb{R}^n} (2^j)^n \Psi(2^j(x - y)) f(y) dy,$$

Set $A = \mathbb{C}$ and $B = l^2$, we can consider the vector-valued operator $\vec{S}$

$$\vec{S} f(x) = (S_j f)_{j=-\infty}^{\infty}(x),$$

where $\vec{S} : m(\mathbb{R}^n, \mathbb{C}) \to m(\mathbb{R}^n, l^2)$. Firstly, by using Plancherel identity, we get

$$\|\|\vec{S}f\|_{l^2}\|_2^2 = \left\| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2 \right)^{\frac{1}{2}} \right\|_2^2 = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |S_j f|^2 = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\psi(2^{-j}\xi)|^2 d\xi.$$

Since $\psi \in S(\mathbb{R}^n)$, we get

$$\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 \le C \sum_{j\in\mathbb{Z}} \min\{|2^{-j}\xi|, |2^{-j}\xi|^{-1}\} \le \left(\sum_{j\ge i} 2^{-2j}|\xi|^2 + \sum_{j<i} 2^{2j}|\xi|^{-2} \right) \le C(2^{-2i}|\xi|^2 + 2^{2i}|\xi|^{-2}).$$

Choosing suitable i, we have

$$\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 \le C$$

and then

$$\|\|\vec{S}f\|_{l^2}\|_2 \le C\|f\|_2.$$

Obviously, we can deduce that the lernel of $\vec{S}$ is

$$\vec{K}(x) = (\Psi_j(x))_{j\in\mathbb{Z}} = (2^{nj}\Psi(2^j x))_{j\in\mathbb{Z}}.$$

Then $\vec{K}(x) : \mathbb{C} \to l^2$ and

$$\int_{|x|>2|y|} \|\vec{K}(x-y) - \vec{K}(x)\|_{l^2} dx \leq C.$$

This is because

$$\begin{aligned}\|\nabla\Psi_j(x)\|_{l^2} \leq \|\nabla\Psi_j(x)\|_{l^1} &= \sum_{j\in\mathbb{Z}} (2^j)^{n+1} |\nabla\Psi(2^j x)| \leq C \sum_{j\in\mathbb{Z}} (2^j)^{n+1} \min\{1, |2^j\xi|^{-n-2}\} \\ &\leq C \sum_{j\leq i} (2^j)^{n+1} + C \sum_{j>i} |x|^{-n-2} 2^{-j} \leq C(2^i)^{n+1} + C|x|^{-n-2}2^{-i}.\end{aligned}$$

Choosing i s.t. $2^{-i} \leq |x| < 2^{1-i}$, we have

$$\|\nabla\Psi(x)\|_{l^2} \leq C|x|^{-n-2}|x| \leq C|x|^{-n-1}.$$

Then it follows from Lagrange mean value theorem that

$$\|\vec{K}(x-y) - \vec{K}(x)\|_{l^2} \leq \frac{C|y|}{|x|^{n+1}}.$$

Then we can see that the kernel $\vec{K}$ satisfies the condition of standard kernel, which implies that

$$\|\|\vec{S}f\|_{l^2}\|_1 \leq C\|f\|_{\mathcal{H}^1_{at}},$$

which is exactly the result. $\square$

2. Let $h \in S(\mathbb{R}^n)$ such that $\text{supp}(\widehat{h}) \subset [-\frac{1}{8}, \frac{1}{8}]^n$. Given a sequence $\{a_j\}$, let

$$f(x) = \sum_{j=1}^{\infty} a_j e^{2\pi i 2^j x} h(x).$$

Show that

$$\|f\|_p \leq C \left(\sum_{j=1}^{\infty} |a_j|^2\right)^{\frac{1}{2}} \|h\|_p, \quad \forall 1 < p < \infty.$$

Proof. We can deduce that

$$\widehat{f}(\xi) = \sum_{j=1}^{\infty} a_j \widehat{h}(\xi - 2^j).$$

Without loss of generality, we can assume that $\text{supp}(h) \subset [0, \frac{1}{4}]^n$ for otherwise we can consider the function $\widetilde{h}(x) = h(x)e^{2\pi i \frac{1}{8} x}$. Let $\Delta_j = Q_{j+1} \backslash Q_j$, where $Q_j = [-2^j, 2^j]^n$, we have

$$\widehat{f}(\xi)\chi_{\Delta_j}(\xi) = \chi_{\Delta_j}(\xi) \left(\sum_{k=1}^{\infty} a_k \widehat{h}(\xi - 2^k)\right) = \begin{cases} a_j \widehat{h}(\xi - 2^j) & \text{if} \quad j \geq 1 \\ 0 & \text{if} \quad j < 1. \end{cases}$$

Define operators S_j by

$$\widehat{S_j f}(\xi) = \chi_{\Delta_j}(\xi)\widehat{f}(\xi).$$

By using Paley-Littlewood theorem, we have

$$\begin{aligned}\|f\|_p \leq C \left\| \left(\sum_{j\in\mathbb{Z}} |S_j f|^2\right)^{\frac{1}{2}} \right\|_p &\leq C \left\| \left(\sum_{j=1}^{\infty} |S_j f|^2\right)^{\frac{1}{2}} \right\|_p \\ = C \left\| \left(\sum_{j=1}^{\infty} |a_j|^2 |h(x)|^2\right)^{\frac{1}{2}} \right\|_p &\leq C \left(\sum_{j=1}^{\infty} |a_j|^2\right)^{\frac{1}{2}} \|h\|_p\end{aligned}$$

as desired. $\square$

3. Show that $e^{\frac{i\xi_j}{|\xi|}}$ is a L^p multiplier with $1 < p < \infty$.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $\mathrm{supp}(\psi) \subset \{\frac{1}{2} \le |\xi| \le 2\}$. Then we can see that for $m(\xi) = e^{\frac{i\xi_j}{|\xi|}}$

$$m(2^k\xi)\psi(\xi) = m(\xi)\psi(\xi).$$

Then by the definition of ψ and m, for some $a > \frac{n}{2}$,

$$\sup_k \|m(2^k\cdot)\psi\|_{H^a} < \infty.$$

In view of Hörmander multiplier theorem, $m \in \mathcal{M}_p$ with $1 < p < \infty$. □

Remark 6.1. Here we can see that all homogeneous functions have similar properties.

4. Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $0 \notin \mathrm{supp}(\zeta)$. Let

$$G(f)(x) = \sup_{N>0} \left| \sum_{j<N} \Delta_j^\zeta f(x) \right|,$$

where $\widehat{\Delta_j^\zeta} f(\xi) = \widehat{f}(\xi)\zeta(2^{-j}\xi)$. Show that

$$\|G(f)\|_p \le C\|f\|_p, \quad \forall 1 < p < \infty.$$

Proof. Choose radial Schwartz function φ such that $\sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) = 1$ in $\mathbb{R}^n \backslash \{0\}$ and $\mathrm{supp}(\widehat{\varphi}) \subset \{\xi : \frac{6}{7} \le |\xi| \le 2\}$. Then there exists C_0, depending only on n, such that $\Delta_k^\varphi \Delta_j^\zeta = 0$ for $|j-k| \ge C_0$. By this, we have

$$\begin{aligned} \sum_{j<N} \Delta_j^\zeta f &= \sum_{k<N+C_0} \Delta_k^\varphi \left(\sum_{j<N} \Delta_j^\zeta f \right) \\ &= \sum_{k<N+C_0} \Delta_k^\varphi \left(\sum_{j\in\mathbb{Z}} \Delta_j^\zeta f \right) - \sum_{k<N+C_0} \Delta_k^\varphi \left(\sum_{j\ge N} \Delta_j^\zeta f \right). \end{aligned}$$

Let $\widehat{\psi}(\xi) = \sum_{j<N+C_0} \widehat{\varphi}(2^{N-j}\xi) = \sum_{j>-C_0} \widehat{\varphi}(2^j\xi)$. It can be verifid that $\psi \in L^1$. Indeed, by the definition of φ, $\mathrm{supp}(\widehat{\psi}) \subset \{\xi : |\xi| \le 2 \cdot 2^{C_0}\}$ and $|\widehat{\psi}(\xi)| \le C$. This implies that $\psi \in L^1(\mathbb{R}^n)$ and $\|\psi\|_1$ is independent of N. By this,

$$\sum_{k<N+C_0} \Delta_k^\varphi \left(\sum_{j\in\mathbb{Z}} \Delta_j^\zeta f \right) = (2^N)^n \psi(2^N\cdot) * \left(\sum_{j\in\mathbb{Z}} \Delta_j^\zeta f \right).$$

In view of the approximation of identity, we have

$$\sup_{N>0} \left| \sum_{k<N+C_0} \Delta_k \left(\sum_{j\in\mathbb{Z}} \Delta_j^\zeta f \right) \right| \le CM \left(\sum_{j\in\mathbb{Z}} \Delta_j^\zeta f \right). \qquad (**)$$

On the other hand, there exists $C_1, C_2 > 0$ depending only on n such that

$$\sum_{k<N+C_0} \Delta_k^\varphi \left(\sum_{j\ge N} \Delta_j^\zeta f \right) = \sum_{N-C_1<k\le N+C_0, N\le j\le N+C_2} \Delta_k^\varphi \Delta_j^\zeta f.$$

Then again by approximation of identity (or by question 5 in this section)

$$\left\| \sum_{k<N+C_0} \Delta_k^\varphi \sum_{j\ge N} \Delta_j^\zeta f \right\|_p \le C \sum_{N-C_1<k\le N+C_0, N\le j\le N+C_2} \left\| \Delta_k^\varphi \Delta_j^\zeta f \right\|_p \le C\|f\|_p.$$

This, together with $(**)$ and the property that M is bounded on L^p, implies that

$$\|G(f)\|_p \leq C\|f\|_p + C\left\|\sum_j \Delta_j^\zeta f\right\|_p.$$

Again, by using question 5 in this section, the proof is completed. $\square$

5. Let $\zeta \in C_0^\infty(\mathbb{R}^n)$, $0 \notin \text{supp}(\zeta)$. Assume that $\{a_j\}$ is a bounded sequence. Show that

$$m(\xi) = \sum_{j\in\mathbb{Z}} a_j \zeta(2^{-j}\xi)$$

is an L^p multiplier with $1 < p < \infty$.

Proof. In view of Hörmander multiplier theorem, we only need to verify that for some $a > \frac{n}{2}$, and a radial function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp}(\psi) \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$, there is

$$\sup_j \|m(2^{-j}\cdot)\psi\|_{H^a} < \infty.$$

Actually, for $\beta = (\beta_1, \beta_2, ..., \beta_n)$ be a multiple index, we get

$$D_\xi^\beta(m(2^k\xi)\psi(\xi)) = \sum_{\gamma\leq\beta}\sum_{j\in\mathbb{Z}} a_j D_\xi^\gamma \zeta(2^{k-j}\xi) D_\xi^{\beta-\gamma}\psi(\xi)(2^{k-j})^{|\gamma|}.$$

By the definition of ψ, we see that if $|\xi| < \frac{1}{2}$ or $|\xi| > 2$, then

$$D_\xi^\beta(m(2^k\xi)\psi(\xi)) = 0.$$

So it can be get that there exists $C_1 > 0$ a dimensional constant such that

$$D_\xi^\beta(m(2^k\xi)\psi(\xi)) = \sum_{\gamma\leq\beta}\sum_{k-C_0<j\leq k+C_0} a_j D_\xi^\gamma \zeta(2^{k-j}\xi) D_\xi^{\beta-\gamma}\psi(\xi)(2^{k-j})^{|\gamma|}.$$

By this, we have

$$\|D_\xi^\beta(m(2^k\xi)\psi(\xi))\|_2 \leq C\sum_{\gamma\leq\beta}\sum_{k-C_0<j\leq k+C_0} \|D_\xi^\gamma \zeta(2^{k-j}\xi) D_\xi^{\beta-\gamma}(\xi)(2^{k-j})^\gamma\|_2 < \infty,$$

as desired. $\square$

Remark 6.2. There is also a simpler method to show that $m(\xi)$ is in $\mathcal{M}_p$. For any multiple index β such that $|\beta| \leq \left[\frac{n}{2}\right] + 1$, we consider

$$D_\xi^\beta m(\xi) = \sum_{j\in\mathbb{Z}} 2^{-j|\beta|} a_j D_\zeta^\beta \zeta(2^{-j}\xi).$$

Since $\zeta \in C_0^\infty(\mathbb{R}^n)$ and $0 \notin \text{supp}(\zeta)$, we can assume that $\text{supp}(\zeta) \subset \{x : 0 < a < |x| < b\}$. This implies that the summation above is actually only for the index j such that $a < 2^{-j}|\xi| < b$. Obviously such j has only finite choices and it follows $2^{-j|\beta|} \leq C|\xi|^{-|\beta|}$. By utilizing these facts, we can easily deduce that

$$|D_\xi^\beta m(\xi)| \leq \left|\sum_{j\in\mathbb{Z}} 2^{-j|\beta|} a_j D_\xi^\beta \zeta(2^{-j}\xi)\right| \leq C|\xi|^{-|\beta|}.$$

By using the well-known corollary of Hörmander multiplier theorem, the result follows directly.

6. Let $1 < p < \infty$, $f \in S(\mathbb{R}^n)$, $L_1 = \partial_1 - \partial_2^2 + \partial_3^4$ and $L_2 = \partial_1 + \partial_2^2 + \partial_3^2$. Show that

$$\|\partial_2\partial_3^2 f\|_p \leq C\|L_1 f\|_p, \ \|\partial_1 f\|_p \leq C\|L_2 f\|_p.$$

Proof. By Fourier transform, we only need to show that

$$m_1(\xi) = \frac{i\xi_1}{i\xi_1 - \xi_2^2 - \xi_3^2},\ m_2(\xi) = \frac{\xi_2\xi_3^2}{i\xi_1 + \xi_2^2 + \xi_3^4}$$

are L^p multipliers. We take m_1 as the example and the proof for m_2 is almost the same. Note that

$$m_1(\xi_1, \xi_2, \xi_3, ..., \xi_n) = m_1(\lambda^2\xi_1, \lambda\xi_2, \lambda\xi_3, \lambda\xi_4, ..., \lambda\xi_n).$$

Then for a multiple index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, it can be got that

$$(D_\xi^\alpha m_1)(\xi) = (D_\xi^\alpha m_1)(\lambda^2\xi_1, \lambda\xi_2, ..., \lambda\xi_n)\lambda^{2\alpha_1+\alpha_2+...+\alpha_n}.$$

For any $\lambda \in \mathbb{R}^n\backslash\{0\}$, we can choose special $\lambda_\xi \in \mathbb{R}$ such that

$$(\lambda^2\xi_1, \lambda\xi_2, ..., \lambda\xi_n) \in S^{n-1}.$$

Then $\lambda^2 \le |\xi_1|^{-1}$, $\lambda \le |\xi|^{-1}$ ... $\lambda \le |\xi_n|^{-1}$ and

$$|D_\xi^\alpha m_1(\xi)| \le \left(\sup_{S^{n-1}} |D_\xi^\alpha m_1|\right) |\xi_1|^{-\alpha_1} ... |\xi_n|^{-\alpha_n},$$

which implies that $m_1 \in \mathcal{M}_p$ with $1 < p < \infty$. □