Semilinear elliptic equation with singular nonlinearity: Regularity and Singularity

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December 21, 2024

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Equations and Backgrounds

The semilinear elliptic equation with singular nonlinearity is given by

$$\Delta u = u^{-p} + f, \ \ u \geq 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \ \ n \geq 2, \ \ p \geq 0, \tag{SN}$$

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Some mathematical and physical backgrounds are as follows.

1. $p = 0$: $\Delta u = \chi_{\{u>0\}}$, obstacle problem, free boundary problem.
2. $p > 0$:
    - $p > 1$: Thin film theory.
    - $p = 2$: Simplified micro-electromechanical system (MEMS).
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Example

For $p = 2$, $n = 1$ and $f \equiv 0$, an easy derivation of the model is the dynamical analysis of two homopolar charges under the Coulomb force.

Problems Setting: Motivation

Consider the regularity theory for solutions of (SN) with $p > 0$.

If $u \in C^0$ and $f \in C^\infty$, then

$$x \in \{u > 0\} \;\Rightarrow\; \exists r > 0, \text{ s.t. } \inf_{B_r(x)} u > 0 \;\Rightarrow\; u \in C^\infty(B_r(x)).$$

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How large are rupture sets for solutions of (SN) with $p > 0$? What assumptions will we have to obtain such results?

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For $0 < p < 1$, solutions behave like those for case $p = 0$. There are almost complete results even for two phases (Tavares and Terracini, JMPA 2019). We mainly consider the case $p > 1$, which is very different from $0 < p < 1$.

Problems Setting: Different Solutions

For (SN) with $p > 1$, some definitions are as below.

1. **Weak solution:** $u \in (H^1_{\text{loc}} \cap L^{-p}_{\text{loc}})(\Omega)$, (SN) is satisfied in the sense of distribution.

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3. **Stationary solution:** weak solution+stationary condition, namely, $\forall Y \in C_0^\infty(\Omega, \mathbb{R}^n)$,

$$\int_\Omega \left[\left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} \right) \operatorname{div} Y - DY(\nabla u, \nabla u) - f(Y \cdot \nabla u) \right] = 0. \quad \text{(SC)}$$

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The corresponding functional of (SN) is

$$\mathcal{F}_f(u, \Omega) := \int_\Omega \left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} + fu \right).$$

Then

$$\text{(SC)} \Leftrightarrow \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}_f(u(\cdot + tY(\cdot)), \Omega) = 0.$$

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3. **Stationary solution:**
   - $d_u \leq n - 2 + \frac{4}{p+1}$ (Guo-Wei, MM 2006).
   - $d_u \leq n - 2$ (Dávila-Wang-Wei, AIHP 2016).

   The best estimate for d_u is achieved for stationary solutions!

Remarks on Results by Dávila-Wang-Wei

Theorem (Dávila-Wang-Wei, AIHP 2016)

Assume that $u \in C^{0,\frac{2}{p+1}}_{\mathrm{loc}}$ is a stationary solution of $\Delta u = u^{-p}$. Then $\{u = 0\}$ is a relatively closed set with $d_u \leq n - 2$. In particular, if $n = 2$, then $\{u = 0\}$ is discrete.

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Some remarks are as follows.

1. The assumption of $\frac{2}{p+1}$-Hölder continuity is optimal and corresponds to the $C^{1,1}_{\mathrm{loc}}$ regularity in the obstacle problem.
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2. It is an a priori result. It is still unknown if a stationary solution must be $\frac{2}{p+1}$-Hölder continuous.
3. The result is sharp. If $n = 2$,

$$u(x) = u(|x|) := \left(\frac{2}{p+1}\right)^{-\frac{2}{p+1}} |x|^{\frac{2}{p+1}}$$

is a stationary solution for $\Delta u = u^{-p}$ and $\{u = 0\} = \{0\}$.

Motivation of Our Works

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Before we present the main results, we introduce the concept of rectifiability. Rectifiable sets can be viewed as manifolds for analyst.

Definition (Rectifiability)

Let $N \in \mathbb{Z}_+$ and $k \in \mathbb{Z} \cap [1, N]$. We call a set $M \subset \mathbb{R}^N$ as k-rectifiable if

$$M \subset M_0 \cup \bigcup_{i \in \mathbb{Z}_+} f_i(\mathbb{R}^k),$$

where $\mathcal{H}^k(M_0) = 0$, and $f_i : \mathbb{R}^k \to \mathbb{R}^N$ is a Lipschitz map $\forall i \in \mathbb{Z}_+$.

Remark: Here, "Lipschitz" can be replaced by "C^1."

Main Results

Theorem (Wang-Zhang, arXiv: 2411.16048)

$\frac{1}{2}+\frac{1}{2p}<\frac{q}{n}$. $u\in C^{0,\frac{2}{p+1}}_{\mathrm{loc}}(B_4)$ *is a stationary solution of* (SN) *with* $f\in L^q_{\mathrm{loc}}(B_4)$, *satisfying* $\|u\|_{L^1(B_2)}+\|f\|_{L^q(B_2)}\le\Lambda$. *The following properties hold.*

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1. $\exists\varepsilon, C=\varepsilon, C(\Lambda,n,p,q)>0$, *s.t.*

$$\mathcal{L}^n(B_r(\{u<\varepsilon r^{\frac{2}{p+1}}\}\cap B_1))\le Cr^2,\quad 0<r<1.$$

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$$\mathcal{L}^n(B_r(\{u<\varepsilon r^{\frac{2}{p+1}}\}\cap B_1))\le Cr^2,\quad 0<r<1.$$

2. *If* $f\in W^{j-1,\infty}_{\mathrm{loc}}(B_4)$ *with* $j\in\mathbb{Z}_+$ *and* $\|f\|_{W^{j-1,\infty}(B_2)}\le\Lambda'$, *then*

$$\sup\left\{\lambda>0:\lambda^{\frac{2(p+1)}{j(p+1)-2}}\mathcal{L}^n(\{x\in B_1:|D^j u(x)|>\lambda\})\right\}\le C',$$

where $C'=C'(\Lambda,\Lambda',j,n,p,q)>0$.

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3. $\{u=0\}$ *is* $(n-2)$*-rectifiable, and for* $n=2$, $\{u=0\}$ *is a discrete set.*

Some Remarks on Our Results

1. We estimate the $(n-2)$-dimensional Minkowski r-content for $\{u < \varepsilon r^{\frac{2}{p+1}}\}$. In particular,

$$\dim_{\mathrm{Min}}(\{u=0\} \cap B_1) \leq n-2.$$

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2. The $L^{\frac{2(p+1)}{j(p+1)-2},\infty}$ and $(n-2)$-rectifiability of $\{u=0\}$ are both sharp.

3. By standard interpolation,
$$D^j u \in L^{\frac{2(p+1)}{j(p+1)-2}-}(B_1), \quad j \in \mathbb{Z}_+,$$
i.e. $\forall 0 < s < \frac{2(p+1)}{j(p+1)-2}$, $\exists C = C(\Lambda, \Lambda', j, n, p, q, s) > 0$, s.t.
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For $j=1$, $u \in W^{1,\frac{2(p+1)}{p-1}}(B_1)$, improving the H^1 regularity.

4. In fact, any k-stratum of $\{u=0\}$ is k-rectifiable with $k \in \mathbb{Z} \cap [0, n-2]$. Here, the stratification is based on the tangent function.

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Terminologies of Harmonic Maps

$\Omega \subset \mathbb{R}^n$: bounded domain. $\mathcal{N} \hookrightarrow \mathbb{R}^d$ smooth, compact manifold, embedded into $\mathbb{R}^d$. Harmonic map: the critical point of the variational problem

$$\mathcal{E}(\Phi, \Omega) := \int_\Omega |\nabla \Phi|^2, \quad \Phi = (\Phi^1, \Phi^2, ..., \Phi^d) \in H^1(\Omega, \mathcal{N}).$$

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$\Phi \in H^1(\Omega, \mathcal{N})$ is a local minimizer of if $\forall B_r(x) \subset\subset \Omega$, and $\Psi \in H^1(B_r(x), \mathcal{N})$ with $\Phi = \Psi$ on $\partial B_r(x)$,

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Φ is a weakly harmonic map if $\forall \varphi = (\varphi^i)_{i=1}^d \in C_0^\infty(\Omega, \mathbb{R}^d)$,

$$\int_\Omega (\nabla \Phi \cdot \nabla \varphi - A(\Phi)(\nabla \Phi, \nabla \Phi) \cdot \varphi) = 0,$$

where $A(y)(\cdot, \cdot) : T\mathcal{N} \times T\mathcal{N} \to (T\mathcal{N})^\perp$ is the second fundamental form of $\mathcal{N}$

Definition (Stationary harmonic map)

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$$\int_\Omega (|\nabla\Phi|^2 \operatorname{div} Y - 2DY(\nabla\Phi, \nabla\Phi)) = 0,$$

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For a harmonic map $\Phi \in H^1(\Omega, \mathcal{N})$, the singular set is

$$S(\Phi) = \{x \in \Omega : \forall r > 0,\ \Phi \text{ is not continuous in } B_r(x)\}.$$

Results on $S(\Phi)$ for Harmonic Maps

Let $d_\Phi := \dim_{\mathcal{H}}(S(\Phi))$.

1. **Estimates of the Hausdorff dimension:**
   - **Local minimizer**: $d_\Phi \leq n - 3$ (Schoen and Uhlenbeck, JDG 1982).
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2. **Rectifiability:**
   - **Local minimizer**: For real and analytic $\mathcal{N}$, $S(\Phi)$ is $(n-3)$-rectifiable and the k-stratum is k-rectifiable (Simon, CVPDE 1995).
   - **Stationary harmonic map**:
     - The concentration set is $(n-2)$-rectifiable (Lin, AM 1999).
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The results in (Dávila-Wang-Wei, AIHP 2016) follows from similar arguments in (Schoen and Uhlenbeck, JDG 1982), applying Federer's dimensional reduction!

Difficulties in the Proof of Rectifiability

Assumption (Simplified model)

$u \in C^{0,\frac{2}{p+1}}_{\mathrm{loc}}(B_2)$ *is a stationary solution of* $\Delta u = u^{-p}$ *in* B_2 *with* $p > 1$.

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For $B_r(x) \subset B_2$, the widely used density is

$$\theta(u; x, r) := r^{\frac{2(p-1)}{p+1} - n} \int_{B_r(x)} \left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} \right) - \frac{r^{-\frac{4}{p+1} - n}}{p+1} \int_{\partial B_r(x)} u^2.$$

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Moreover, it satisfies the monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\theta(u; x, r) = r^{-\frac{4}{p+1} - n} \int_{\partial B_r(x)} \left| (y - x) \cdot \nabla u - \frac{2u}{p+1} \right|^2 \mathrm{d}\mathcal{H}^{n-1}(y) \geq 0.$$

$$x \in \{u > 0\} \ \Leftrightarrow \ \lim_{r \to 0^+} \theta(u; x, r) = -\infty.$$

$\theta(u; x, r)$ can be negative!

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$$\theta(u;x,r) := r^{\frac{2(p-1)}{p+1}-n} \int_{B_r(x)} \left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} \right) - \frac{r^{-\frac{4}{p+1}-n}}{p+1} \int_{\partial B_r(x)} u^2.$$

Moreover, it satisfies the monotonicity formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\theta(u;x,r) = r^{-\frac{4}{p+1}-n} \int_{\partial B_r(x)} \left| (y-x)\cdot \nabla u - \frac{2u}{p+1} \right|^2 \mathrm{d}\mathcal{H}^{n-1}(y) \geq 0.$$

$$x \in \{u > 0\} \;\Leftrightarrow\; \lim_{r\to 0^+} \theta(u;x,r) = -\infty.$$

$\theta(u;x,r)$ can be negative!

Previous arguments in harmonic maps require the nonnegativity of the density!

Blow up Analysis I

Classical blow-ups. (Dávila-Wang-Wei, AIHP 2016) For $x \in B_2$,

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For $x \in \{u > 0\}$, the blow-ups do not have a limit!

Stratification I

Definition (k-symmetric function)

$k \in \mathbb{Z} \cap [0, n]$. $h \in C^{0,\frac{2}{p+1}}_{\text{loc}}(\mathbb{R}^n))$ is k-symmetric at $x \in \mathbb{R}^n$ wrt $V \subset \mathbb{R}^n$ with $\dim V = k$ if h is $\frac{2}{p+1}$-homogeneous at x and invariant with respect to V. If $x = 0$, we call that h is k-symmetric.

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$\forall k \in \mathbb{Z} \cap [0, n-1]$, define the k-stratum of u by

$$S^k_{(\text{I})}(u) := \{x \in \{u = 0\} : \text{no tangent function } v \text{ of } u \text{ at } x \text{ is } (k+1)\text{-symmetric}\}.$$

As a result,

$$S^0_{(\text{I})}(u) \subset S^1_{(\text{I})}(u) \subset S^2_{(\text{I})}(u) \subset ... \subset S^{n-1}_{(\text{I})}(u) = \{u = 0\}.$$

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Indeed,

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Rectifiability: Idea

After (Naber and Valtorta, AM 2017), in (A. Naber and D. Valtorta, MZ 2018), there are simplified arguments only requiring the boundedness of the density.

For u satisfying the assumption of the simplified model,

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$$\theta_r(u; x, r) \text{ is locally bounded in } \{u = 0\}.$$

Idea: Restrict the analysis on the rupture set and apply methods by Naber and Valtorta in MZ 2018.

Rectifiability: Modified Densities

Following (Hirsch, Stuvard, and Valtorta, TAMS 2019), we also modify the density $\theta_r(u; x, r)$ by

$$\vartheta(u; x, r) = r^{\frac{2(p-1)}{p+1} - n} \int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} \right) \phi_{x,r} + \frac{2r^{-\frac{4}{p+1} - n}}{p+1} \int_{\partial B_r(x)} u^2 \dot{\phi}_{x,r}.$$

Definition

Let $\phi : [0, +\infty) \to [0, +\infty)$.

- $\operatorname{supp} \phi \subset [0, 10)$ and ϕ is decreasing.
- $\phi(t) \geq 0$, and $|\phi'(t)| \leq 100$, $\forall t \in [0, +\infty)$.
- $-2 \leq \phi'(t) \leq -1$, $\forall t \in [0, 8]$.

For $x \in \mathbb{R}^n$,

$$\phi_{x,r}(y) := \phi\left(\frac{|y-x|^2}{r^2}\right), \quad \dot{\phi}_{x,r}(y) := \phi'\left(\frac{|y-x|^2}{r^2}\right).$$

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Such a "modification" can avoid the application of unique continuation.

Difficulties in the Proof of Regularity Improvement

To enhance the regularity, we have to obtain.

$$\mathcal{L}^n(B_r(\{u < \varepsilon r^{\frac{2}{p+1}}\} \cap B_1)) \leq C\left([u]_{C^{0,\frac{2}{p+1}}}, n, p\right) r^2.$$

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Difficulty: The density does not have a uniform bound in $\{u < \varepsilon r^{\frac{2}{p+1}}\}$.

For u and $B_r(x) \subset \{u > 0\}$, we have the nondegeneracy

$$\sup_{B_r(x)} u \geq C'\left([u]_{C^{0,\frac{2}{p+1}}}, n, p\right) r^{\frac{2}{p+1}}.$$

We need the one with "inf" replacing "sup"!

Blow up Analysis II and Stratification II

Refined blow-ups. For $x \in B_2$, let

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Alternative Result: Intuition

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1. $0 \leq u(x) \ll r^{\frac{2}{p+1}}$:
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2. $u(x) \gtrsim r^{\frac{2}{p+1}}$:
   We use the standard regularity theory for elliptic equations to find a small ball $B_{\delta r}(x)$ with $\delta \ll 1$ s.t. u exhibits nice properties.

Quantitative Stratification: Definitions

Definition (Quantitative symmetry)

$\varepsilon > 0$, and $k \in \mathbb{Z} \cap [0, n]$. u is (k, ε)-symmetric in $B_r(x) \subset\subset B_2$, if there is a k-symmetric function $h \in C^{0,\frac{2}{p+1}}_{\mathrm{loc}}(\mathbb{R}^n)$ s.t $\|\widetilde{u}_{x,r} - h\|_{L^\infty(B_1)} < \varepsilon$.

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$\varepsilon > 0$, $k \in \mathbb{Z} \cap [0, n-1]$, and $0 < r < 1$, the k-th (ε, r)-stratification of u, denoted by $S^k_{\varepsilon,r}(u)$, is

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In fact,

$$S^k_{(\mathrm{II})}(u) \cap B_1 = \bigcup_{\varepsilon>0} S^k_\varepsilon(u) = \bigcup_{\varepsilon>0} \bigcap_{0<r<1} S^k_{\varepsilon,r}(u).$$

Quantitative Stratification: Main Theorems

Theorem (Main)

$\varepsilon > 0$ and $k \in \mathbb{Z} \cap [0, n-2]$. Assume that $[u]_{C^{0,\frac{2}{p+1}}(\overline{B}_2)} \leq \Lambda$. If $0 < r < 1$, then

$$\mathcal{L}^n(B_r(S^k_{\varepsilon,r}(u))) \leq C(\varepsilon, \Lambda, n, p) r^{n-k}.$$

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1. This theorem is sharp, giving paralleled results as harmonic maps.
2. The result implies the Ahlfors regularity of $S^k_\varepsilon(u)$ and then the rectifiability.
3. The proof of

$$\mathcal{L}^n(B_r(S^k_{\varepsilon,r}(u) \cap \{u = 0\})) \leq C(\varepsilon, \Lambda, n, p) r^{n-k}$$

is much easier and enough to obtain the rectifiability of $S^k(u)$. However, it is not sufficient to obtain the improvement of regularity.

Quantitative Stratification: Lemmas I

By simple compactness arguments, we have:

Lemma

$[u]_{C^{0,\frac{2}{p+1}}(\overline{B}_2)} \leq \Lambda$. $\exists \varepsilon = \varepsilon(\Lambda, n, p) > 0$ *s.t.* $\forall 0 < r < 1$

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By the estimate of $S^k_{\varepsilon,r}(u)$, this lemma implies that

$$\begin{aligned}
&\mathcal{L}^n(\{x \in B_1 : u(x) < \varepsilon r^{\frac{2}{p+1}}\}) \\
&\leq \mathcal{L}^n(B_r(S^{n-2}_{\varepsilon,r}(u))) \\
&\leq C(\varepsilon, \Lambda, n, p) r^{n-(n-2)} \\
&\leq C(\varepsilon, \Lambda, n, p) r^2.
\end{aligned}$$

Quantitative Stratification: Lemma II

The key step in obtaining the main theorem is the following lemma. The rest of the proof is an application of arguments by Naber and Valtorta through lengthy and tricky covering methods.

Lemma

$k \in \mathbb{Z} \cap [0, n-2]$, $0 < s \leq 1$, and $x \in \mathbb{R}^n$. $[u]_{C^{0,\frac{2}{p+1}}(\overline{B}_{15s}(x))} \leq \Lambda$. $\forall \varepsilon > 0$, there exist $\delta, \delta' = \delta, \delta'(\varepsilon, \Lambda, n, p) > 0$ s.t. if

$$\inf_{V \subset \mathbb{R}^n,\ \dim V = k+1} \left(s^{\frac{2(p-1)}{p+1} - n} \int_{B_s(x)} |V \cdot \nabla u|^2 \right) < \delta,$$

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1. For harmonic maps, the proof of a similar result is simple.
2. If we use $u_{x,r}$ to define quantitative stratification, we cannot obtain such a result without assuming that $u(x) = 0$.

Quantitative Stratification: Alternative Lemma

Lemma (Alternative lemma)

$k \in \mathbb{Z} \cap [0, n-2]$, $0 < r \le 1$, and $x \in \mathbb{R}^n$. $[u]_{C^{0,\frac{2}{p+1}}(\overline{B}_{15r}(x))} \le \Lambda$. $\forall \varepsilon > 0$, there are $\delta = \delta(\varepsilon, \Lambda, n, p) > 0$ and $\gamma = \gamma(n,p) > 0$ s.t. if

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then either $u(x) \ge \delta^\gamma r^{\frac{2}{p+1}}$, or $\exists r_x \in [\delta^\gamma r, r]$ s.t.

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2. By this lemma, if $u(x) \ge \delta^\gamma r^{\frac{2}{p+1}}$, we can use the regularity theory of elliptic equations to obtain that in a smaller ball with radius which is comparable to r_x, where u is $(k+1, \varepsilon)$-symmetric in it.

Thank you for listening!