Semilinear elliptic equation with singular nonlinearity: Regularity and Singularity

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Equations and Backgrounds

The semilinear elliptic equation with **singular nonlinearity** is given by

$$\Delta u = u^{-p} + f,\ \ u \geq 0 \quad \text{in } \Omega \subset \mathbb{R}^n,\ \ n \geq 2,\ \ p \geq 0, \tag{SN}$$

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Some mathematical and physical backgrounds are as follows.

1. $p = 0$: $\Delta u = \chi_{\{u>0\}}$, obstacle problem, free boundary problem.
2. $p > 0$:
    - $p > 1$: Thin film theory.
    - $p = 2$: Simplified micro-electromechanical system (MEMS).
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For $p = 2$, $n = 1$ and $f \equiv 0$, an easy derivation of the model is the dynamical analysis of two homopolar charges under the Coulomb force.

Problems Setting: Motivation

Consider the regularity theory for solutions of (SN) with $p > 0$.

If $u \in C^0$ and $f \in C^\infty$, then

$$x \in \{u > 0\} \ \Rightarrow \ \exists r > 0, \ \text{s.t.} \ \inf_{B_r(x)} u > 0 \ \Rightarrow \ u \in C^\infty(B_r(x)).$$

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For $0 < p < 1$, solutions behave like those for case $p = 0$. There are almost complete results even for two phases with analogous methods (H. Tavares and S. Terracini, JMPA 2019). Thus, we mainly consider the case $p > 1$.

Problems Setting: Different Solutions

For (SN) with $p > 1$, we have the following definitions.

1. **Weak solution:** $0 \leq u \in (H^1_{\text{loc}} \cap L^{-p}_{\text{loc}})(\Omega)$, (SN) is satisfied in the sense of distribution.

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3. **Stationary solution:** weak solution+stationary condition, namely, $\forall Y \in C_0^\infty(\Omega, \mathbb{R}^n)$,

$$\int_\Omega \left[\left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} \right) \operatorname{div} Y - DY(\nabla u, \nabla u) - f(Y \cdot \nabla u) \right] = 0. \tag{SC}$$

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The **corresponding functional** of (SN) is

$$\mathcal{F}_f(u, \Omega) := \int_\Omega \left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} + fu \right).$$

Then

$$\text{(SC)} \Leftrightarrow \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \mathcal{F}_f(u(\cdot + tY(\cdot)), \Omega) = 0.$$

Problems Setting: Present results

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   - $d_u \leq n - 2$ (Dávila-Wang-Wei, AIHP 2016).

Remarks on Results by Dávila-Wang-Wei

Theorem (Dávila-Wang-Wei, AIHP 2016)

Assume that $u \in C^{0,\frac{2}{p+1}}_{\mathrm{loc}}$ *is a stationary solution of* $\Delta u = u^{-p}$*. Then the rupture set* $\{u = 0\}$ *is a relatively closed set with* $d_u \leq n - 2$*. If* $n = 2$*, then* $\{u = 0\}$ *is discrete.*

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Some remarks are as follows.

1. The assumption of $\frac{2}{p+1}$-Hölder continuity is **optimal** and corresponds to the $C^{1,1}_{\mathrm{loc}}$ regularity in the obstacle problem.
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2. It is an a priori result. It is still unknown if a stationary solution must be $\frac{2}{p+1}$-Hölder continuous.
3. The result is **sharp**. If $n = 2$,

$$u(x) = u(|x|) := \left(\frac{2}{p+1}\right)^{-\frac{2}{p+1}} |x|^{\frac{2}{p+1}}$$

is a stationary solution for $\Delta u = u^{-p}$ and $\{u = 0\} = \{0\}$.

Motivation of Our Works

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Before we present the main results, we introduce the concept of rectifiability. Rectifiable sets can be viewed as manifolds for analyst

Definition (Rectifiability)

Let $N \in \mathbb{Z}_+$ and $k \in \mathbb{Z} \cap [1, N]$. We call a set $M \subset \mathbb{R}^N$ as k-rectifiable if

$$M \subset M_0 \cup \bigcup_{i \in \mathbb{Z}_+} f_i(\mathbb{R}^k),$$

where $\mathcal{H}^k(M_0) = 0$, and $f_i : \mathbb{R}^k \to \mathbb{R}^N$ is a Lipschitz map $\forall i \in \mathbb{Z}_+$.

Main Results

Theorem (Wang-Zhang, arXiv: 2411.16048)

$\frac{1}{2} + \frac{1}{2p} < \frac{q}{n}$. *$u \in C^{0,\frac{2}{p+1}}_{\mathrm{loc}}(B_4)$ is a stationary solution of* (SN) *with $f \in L^q_{\mathrm{loc}}(B_4)$, satisfying $\|u\|_{L^1(B_2)} + \|f\|_{L^q(B_2)} \le \Lambda$. The following properties hold.*

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1. $\exists \varepsilon, C = \varepsilon, C(\Lambda, n, p, q) > 0$, *s.t.*

$$\mathcal{L}^n(B_r(\{u < \varepsilon r^{\frac{2}{p+1}}\} \cap B_1)) \le Cr^2, \quad 0 < r < 1.$$

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$$\mathcal{L}^n(B_r(\{u<\varepsilon r^{\frac{2}{p+1}}\}\cap B_1))\le Cr^2,\quad 0<r<1.$$

2. *If $f\in W^{j-1,\infty}_{\mathrm{loc}}(B_4)$ with $j\in\mathbb{Z}_+$ and $\|f\|_{W^{j-1,\infty}(B_2)}\le\Lambda'$, then*

$$\sup\left\{\lambda>0:\lambda^{\frac{2(p+1)}{j(p+1)-2}}\mathcal{L}^n(\{x\in B_1:|D^ju(x)|>\lambda\})\right\}\le C',$$

where $C'=C'(\Lambda,\Lambda',j,n,p,q)>0$.

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3. *$\{u=0\}$ is $(n-2)$-rectifiable, and for $n=2$, $\{u=0\}$ is a discrete set.*

Some Remarks on Our Results

1. We actually estimate the $(n-2)$-dimensional Mikowski r-content for $\{u < \varepsilon r^{\frac{2}{p+1}}\}$. In particular, $\dim_{\mathrm{Min}}(\{u = 0\} \cap B_1) \leq n - 2$.

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2. The $L^{\frac{2(p+1)}{j(p+1)-2},\infty}$ and $(n-2)$-rectifiability of $\{u=0\}$ are both **sharp**.
3. By standard interpolation,

$$D^j u \in L^{\frac{2(p+1)}{j(p+1)-2}-}(B_1), \quad j \in \mathbb{Z}_+,$$

i.e. $\forall 0 < s < \frac{2(p+1)}{j(p+1)-2}$, $\exists C = C(\Lambda, \Lambda', j, n, p, q, s) > 0$, s.t.

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   For $j=1$, $u \in W^{1,\frac{2(p+1)}{p-1}}(B_1)$, improving the H^1 regularity.
4. In fact, any k-stratum of $\{u=0\}$ is k-rectifiable with $k \in \mathbb{Z} \cap [0, n-2]$. Here, the stratification is based on the tangent function.

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Terminologies of Harmonic maps

$\Omega \subset \mathbb{R}^n$: bounded domain. $\mathcal{N} \hookrightarrow \mathbb{R}^d$ real, smooth, compact manifold, embedded into $\mathbb{R}^d$. **Harmonic map**: the critical point of the variational problem

$$\mathcal{E}(\Phi, \Omega) := \int_\Omega |\nabla \Phi|^2, \quad \Phi = (\Phi^1, \Phi^2, ..., \Phi^d) \in H^1(\Omega, \mathcal{N}).$$

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$\Phi \in H^1(\Omega, \mathcal{N})$ is a **local minimizer** of if $\forall B_r(x) \subset\subset \Omega$, and $\Psi \in H^1(B_r(x), \mathcal{N})$ with $\Phi = \Psi$ on $\partial B_r(x)$,

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where $A(y)(\cdot, \cdot) : T\mathcal{N} \times T\mathcal{N} \to (T\mathcal{N})^\perp$ is the second fundamental form of $\mathcal{N}$

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For a harmonic map $\Phi \in H^1(\Omega, \mathcal{N})$, the singular set is

$$\operatorname{sing}(\Phi) = \{x \in \Omega : \forall r > 0,\ \Phi \text{ is not continuous in } B_r(x)\}.$$

Results on sing(Φ) for Harmonic Maps

1. **Estimates of the Hausdorff dimension:**
   - For **local minimizers**, $\dim_{\mathcal{H}}(\text{sing}(\Phi)) \leq n-3$ (R. Schoen and K. Uhlenbeck, JDG 1982);
   - For **stationary harmonic maps**, $\dim_{\mathcal{H}}(\text{sing}(\Phi)) \leq n-2$ (F. Bethuel, MM 1993).

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Remark: The results in (Dávila-Wang-Wei, AIHP 2016) follows from similar arguments in (Schoen and Uhlenbeck, JDG 1982).

Difficulties in the Proof of Rectifiability

Assumption (Simplified model)

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$$\theta(u; x, r) := r^{\frac{2(p-1)}{p+1} - n} \int_{B_r(x)} \left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} \right) - \frac{r^{-\frac{4}{p+1} - n}}{p+1} \int_{\partial B_r(x)} u^2.$$

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Moreover, it satisfies the **monotonicity formula**

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$$x \in \{u > 0\} \iff \lim_{r \to 0^+} \theta(u; x, r) = -\infty.$$

$\theta(u; x, r)$ **can be negative!**

Difficulties in the Proof of Rectifiability

Assumption (Simplified model)

$u \in C^{0,\frac{2}{p+1}}_{\text{loc}}$ is a stationary solution of $\Delta u = u^{-p}$ in B_2 with $p > 1$.

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All the previous arguments in harmonic maps require the nonnegativity of the density!

Blow up analysis I

Classical blow-ups. (Dávila-Wang-Wei, AIHP 2016) For $x \in B_2$,

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For $x \in \{u > 0\}$, the blow-ups do not have a limit!

Stratification I

Definition (k-symmetric function)

$k \in \mathbb{Z} \cap [0, n]$. $h \in C^{0,\frac{2}{p+1}}_{\mathrm{loc}}(\mathbb{R}^n))$ is k-symmetric at $x \in \mathbb{R}^n$ wrt $V \subset \mathbb{R}^n$ with $\dim V = k$ if h is $\frac{2}{p+1}$-homogeneous at x and invariant with respect to V. If $x = 0$, we call that h is k-symmetric.

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$\forall k \in \mathbb{Z} \cap [0, n-1]$, define the k-stratum of u by

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As a result,

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Rectifiability: Idea

After (A. Naber and D. Valtorta, AM 2017), in (A. Naber and D. Valtorta, MZ 2018), the same authors developed simplified arguments only requiring the boundedness of the density.

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For u satisfying the assumption of the simplified model,

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Idea: Restrict the analysis on the rupture set and apply methods by A. Naber and D. Valtorta.

Rectifiability: Modified Densities

Following (J. Hirsch, S. Stuvard, and D. Valtorta, TAMS 2019), we also modify the density $\theta_r(u; x, r)$ by

$$\vartheta(u; x, r) = r^{\frac{2(p-1)}{p+1} - n} \int_{\mathbb{R}^n} \left(\frac{|\nabla u|^2}{2} - \frac{u^{1-p}}{p-1} \right) \phi_{x,r} + \frac{2r^{-\frac{4}{p+1} - n}}{p+1} \int_{\partial B_r(x)} u^2 \dot{\phi}_{x,r}.$$

Definition

Let $\phi : [0, +\infty) \to [0, +\infty)$.

- $\operatorname{supp} \phi \subset [0, 10)$.
- $\phi(t) \geq 0$, and $|\phi'(t)| \leq 100$, $\forall t \in [0, +\infty)$.
- $-2 \leq \phi'(t) \leq -1$, $\forall t \in [0, 8]$.

For $x \in \mathbb{R}^n$,

$$\phi_{x,r}(y) := \phi\left(\frac{|y-x|^2}{r^2} \right), \quad \dot{\phi}_{x,r}(y) := \phi'\left(\frac{|y-x|^2}{r^2} \right).$$

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Such a modification can avoid the application of **unique continuation**.

Difficulties in the Proof of Regularity Improvement

To enhance the regularity, we have to obtain.

$$\mathcal{L}^n(B_r(\{u < \varepsilon r^{\frac{2}{p+1}}\} \cap B_1)) \leq C\left([u]_{C^{0,\frac{2}{p+1}}}, n, p\right) r^2.$$

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Difficulty: The density does not have a uniform bound in $\{u < \varepsilon r^{\frac{2}{p+1}}\}$.

For u and $B_r(x) \subset \{u > 0\}$, we have the **Nondegeneracy**

$$\sup_{B_r(x)} u \geq C'\left([u]_{C^{0,\frac{2}{p+1}}}, n, p\right) r^{\frac{2}{p+1}}.$$

Note that we actually need the one with "inf" replacing "sup".

Blow up analysis II and Stratification II

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$$\widetilde{u}_{x,r}(y) := \frac{u(x+ry)-u(x)}{r^{\frac{2}{p+1}}}.$$

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Alternative Results: Intuition

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2. $u(x) \gtrsim r^{\frac{2}{p+1}}$:
   We use the standard regularity theory for elliptic equations to find a small ball $B_{\delta r}(x)$ with $\delta \ll 1$ s.t. u exhibits nice properties.

Quantitative Stratification: Definitions

Definition (Quantitative symmetry)

$\varepsilon > 0$, and $k \in \mathbb{Z} \cap [0, n]$. u is (k, ε)-symmetric in $B_r(x) \subset\subset B_2$, if there is a k-symmetric function $h \in C^{0,\frac{2}{p+1}}_{\mathrm{loc}}(\mathbb{R}^n)$ s.t $\|\widetilde{u}_{x,r} - h\|_{L^\infty(B_1)} < \varepsilon$.

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$\varepsilon > 0$, $k \in \mathbb{Z} \cap [0, n-1]$, and $0 < r < 1$, the k-th (ε, r)-stratification of u, denoted by $S^k_{\varepsilon,r}(u)$, is

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In fact,

$$S^k_{(\mathrm{II})}(u) \cap B_1 = \bigcup_{\varepsilon > 0} S^k_\varepsilon(u) = \bigcup_{\varepsilon > 0} \bigcap_{0 < r < 1} S^k_{\varepsilon,r}(u).$$

Quantitative Stratification: Main Theorems

Theorem (Main)

$\varepsilon > 0$ and $k \in \mathbb{Z} \cap [0, n-2]$. Assume that $[u]_{C^{0,\frac{2}{p+1}}(\overline{B}_2)} \leq \Lambda$. If $0 < r < 1$, then

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1. This theorem is sharp, giving paralleled results as harmonic maps.
2. The result implies the Ahlfors regularity of $S^k_\varepsilon(u)$ and then the rectifiability.
3. The proof of

$$\mathcal{L}^n(B_r(S^k_{\varepsilon,r}(u) \cap \{u = 0\})) \leq C(\varepsilon, \Lambda, n, p) r^{n-k}$$

is much easier and enough to obtain the rectifiability of $S^k(u)$. However, it is not sufficient to obtain the improvement of regularity.

Quantitative Stratification: Lemmas I

By simple compactness arguments, we have:

Lemma

$[u]_{C^{0,\frac{2}{p+1}}(\overline{B}_2)} \leq \Lambda$. $\exists \varepsilon = \varepsilon(\Lambda, n, p) > 0$ *s.t.* $\forall 0 < r < 1$

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By the estimate of $S^k_{\varepsilon,r}(u)$, this lemma implies that

$$\begin{aligned}\mathcal{L}^n(\{x \in B_1 : u(x) < \varepsilon r^{\frac{2}{p+1}}\}) &\leq \mathcal{L}^n(B_r(S^{n-2}_{\varepsilon,r}(u))) \\ &\leq C(\varepsilon, \Lambda, n, p) r^{n-(n-2)} \\ &\leq C(\varepsilon, \Lambda, n, p) r^2.\end{aligned}$$

Quantitative Stratification: Lemma II

The key step in obtaining our main theorem is the following **alternative lemma**. The rest of the proof is a simple application of arguments by A. Naber and D. Valtorta.

Lemma

$k \in \mathbb{Z} \cap [0, n-2]$, $0 < s \leq 1$, and $x \in \mathbb{R}^n$. $[u]_{C^{0,\frac{2}{p+1}}(\overline{B}_{15s}(x))} \leq \Lambda$. $\forall \varepsilon > 0$, there exist $\delta, \delta' = \delta, \delta'(\varepsilon, \Lambda, n, p) > 0$ s.t. either

$$\inf_{V \subset \mathbb{R}^n,\ \dim V = k+1} \left(s^{\frac{2(p-1)}{p+1} - n} \int_{B_s(x)} |V \cdot \nabla u|^2 \right) < \delta,$$

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1. For harmonic maps, the proof is simple.
2. If we use $u_{x,r}$ to define quantitative stratification, we cannot obtain such a result without assuming that $u(x) = 0$.

Quantitative Stratification: Lemma III

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then either $u(x) \ge \delta^\gamma r^{\frac{2}{p+1}}$, or $\exists r_x \in [\delta^\gamma r, r]$ s.t. u is $(k+1, \varepsilon)$-symmetric in $B_{r_x}(x)$.

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2. By this lemma, if $u(x) \ge \delta^\gamma r^{\frac{2}{p+1}}$, we can use the regularity theory of elliptic equations to obtain that in a smaller ball with radius which is comparable to r_x, u is $(k+1, \varepsilon)$-symmetric.

Thank you for listening!