Stratification for Harmonic Map Flows via Tangent Measures: Rectifiability and Regularity

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Harmonic Map Flows: Overview

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and $\mathcal{N} \hookrightarrow \mathbb{R}^d$ $(d \geq 2)$ be a smooth, compact mfd. For $u \in H^1(\Omega, \mathcal{N})$, the **Dirichlet functional**:

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The harmonic map satisfies $-\Delta u = A(u)(\nabla u, \nabla u)$ in Ω, where $A(\cdot)(\cdot, \cdot) : \mathcal{N} \times T\mathcal{N} \times T\mathcal{N} \to (T\mathcal{N})^\perp$ is the second fundamental form.

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Origin: Introduced by [Eells-Sampson, 1964, *AJM*] to construct harmonic maps in any given free homotopy class.

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- [Wang-Lin, 2010, *C. Ann. Math. Series B*]: Uniqueness under additional regularity conditions.

Suitable Solutions

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3. u satisfies the stationary condition: $\forall \xi \in C_0^\infty(P_1, \mathbb{R}^n)$,
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Note: Smooth solutions are suitable solutions.

Localized energy inequality

By (S1), we have:

Lemma

Let $r > 0$ and $X_0 = (x_0, t_0) \in P_2$ be a point such that $P_r(X_0) \subset P_4$. Assume $u : P_4 \to \mathcal{N}$ is a suitable solution, then

$$\int_{B_r(x_0)\times[s,t]} |\partial_t u|^2 \phi^2 dx d\tau + \int_{B_r(x_0)} |\nabla u(\cdot, t)|^2 \phi^2 dx$$
$$\leq \int_{B_r(x_0)} |\nabla u(\cdot, s)|^2 \phi^2 dx + 4 \int_s^t \int_{B_r(x_0)} |\nabla u|^2 |\nabla \phi|^2 dx d\tau$$

for any $\phi \in C_0^\infty(B_r(x_0))$ and a.e. $t_0 - r^2 < s \leq t < t_0 + r^2$.

The Monotonicity Formula

Combining (S1) and (S2), we obtain:

Proposition

Fix $X_0 = (x_0, t_0) \in P_4$. Assume that $u : P_8 \to \mathcal{N}$ is a suitable solution with $\|\nabla u\|_{L^2(P_8)} \leq \Lambda$. Then, for a.e. $0 < r \leq R < 2$,

$$\Psi(u, X_0, R) - \Psi(u, X_0, r) + C_1(R - r)$$
$$\geq C_2 \int_{t_0 - R^2}^{t_0 - r^2} \left(\int_{B_1(x_0)} \varphi_{x_0}^2 \frac{|(x - x_0) \cdot \nabla u + 2(t - t_0)\partial_t u|^2}{|t_0 - t|} G_{X_0}(x, t)\mathrm{d}x \right) \mathrm{d}t,$$

where

$$\Psi(u, X_0, \rho) := \frac{\rho^2}{2} \int_{S_\rho(X_0)} \varphi_{x_0}^2 |\nabla u(x, t)|^2 G_{X_0}(x, t)\mathrm{d}x,$$

$(C_1, C_2)(\Lambda, n) > 0$, and G is the heat kernel.

Partial Regularity

Definition (Singular set)

The singular set of a suitable weak harmonic map is

$$\operatorname{sing}(u) := \{X \in P_r(X_0) : \forall \rho > 0,\ u \notin C^\infty(P_\rho(X))\}.$$

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Partial regularity:

$$r^{-n} \int_{P_{2r}(X)} |\nabla u|^2 \le \varepsilon_0(n, \mathcal{N}) \implies u \in C^\infty(P_r(X), \mathcal{N}).$$

Partial regularity $\Rightarrow \mathcal{P}^n(\mathrm{sing}(u)) = 0$, where $\mathcal{P}^n$ is the n-dimensional parabolic Hausdorff measure.

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- [Liu, 2003, *ARMA*]: Partial regularity extends to general $\mathcal{N}$.

Blow-up Analysis

Definition

Let $U \subset \mathbb{R}^n \times \mathbb{R}$ be an open set and $\mathcal{M}(U)$ be the space of Radon measures on U. For $X_0 = (x_0, t_0) \in U$ and $r > 0$, let

$$\mathcal{P}_{X_0,r}(X) = (x_0 + rx, t_0 + r^2 t), \quad X = (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

For a suitable solution $u : P_1 \to \mathcal{N}$, $\mu \in \mathcal{M}(P_1)$, $r > 0$, and $X \in P_1$, define

$$T_{X,r}u := u \circ \mathcal{P}_{X_0,r}, \quad T_{X,r}\mu := r^{-n}(\mu \circ \mathcal{P}_{X_0,r}).$$

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Then, $\exists r_i \to 0^+$ s.t.

$$w_i := T_{X,r_i}u \rightharpoonup w \text{ weakly in } H^1_{\mathrm{loc}}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^d),$$
$$\frac{1}{2}|\nabla w_i|^2 \mathrm{d}x\mathrm{d}t \rightharpoonup^* \eta := \frac{1}{2}|\nabla w|^2 \mathrm{d}x\mathrm{d}t + \nu \text{ in } \mathcal{M}(\mathbb{R}^n \times \mathbb{R}).$$

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- w: tangent flow of u at X.
- η: tangent measure of u at X.
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Remark

- Tangent flows, and tangent measures depend on the choice of the sequence $\{r_i\}$.
- The defect measure ν quantifies energy loss in the weak convergence.

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- **Space-time strata**:

$$S^k(u) := \{X \in P_1 : \text{no tang. flow at } X \text{ is space-time } (k+1)\text{-sym}\},$$
$$\Sigma^k(u) := \{X \in P_1 : \text{no tang. meas. at } X \text{ is space-time } (k+1)\text{-sym}\}.$$

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- **Spatial strata** (for $t \in (-1, 1)$):

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Dimension Estimates

Using Federer dimension reduction:

Theorem (B. White, 1997, *JRAM*)

Let $u : P_1 \to \mathcal{N}$ be a suitable solution, then

- $\dim_{\mathcal{P}} S^k(u), \Sigma^k(u) \le k$;
- $\dim_{\mathcal{H}} S^k(u,t), \Sigma^k(u,t) \le k \ \forall t$;
- $\dim_{\mathcal{H}} S^k(u) \cap \{t\}, \Sigma^k(u) \cap \{t\} \le k-2$ *for a.e.* t.

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Question

Can we further characterize the strata and singular sets of suitable solutions?

Main Theorem: Rectifiability and Minkowski Dimension

Theorem (Fu-W.-Wu-Zhang, *arXiv: 2504.14880*)

Let $\Lambda > 0$, $k \in \mathbb{Z} \cap [1, n-1]$, $t \in (-1, 1)$. *Assume* $u : P_1 \to \mathcal{N}$ *is a suitable solution with* $\|\nabla u\|_{L^2(P_1)} \le \Lambda$. *Then*

1. ***Rectifiability***:
   - $\Sigma^k(u, t)$ *is* k*-rectifiable* $\forall k \in \mathbb{Z} \cap [1, n-2]$;
   - $\mathrm{sing}(u) \cap \{t\}$ *is* $(n-2)$*-rectifiable.*
2. ***Minkowski content estimate***:

$$\mathrm{Min}_r^{n-2}(\mathrm{sing}(u) \cap (B_1 \times \{t\})) \le C(\Lambda, n, \mathcal{N}).$$

where $\mathrm{Min}_\rho^k(\cdot)$ *is the Minkowski content.*

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The Minkowski dimension estimate: $\dim_{\mathrm{Min}}(\mathrm{sing}(u) \cap \{t\}) \leq n - 2$

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- Assuming $\partial_t u \equiv 0$, the properties degenerate to those for stationary harmonic maps in [Naber-Valtorta, 2017, *Ann. Math.*].

Extension to Ginzburg-Landau Model

One can extend our results to the Ginzburg-Landau model

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \varepsilon^{-2} f(u_\varepsilon), \quad f(p) = -D_p[\chi(\mathrm{dist}^2(p, \mathcal{N}))],$$

where χ is an appropriate cut-off function.

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- **Concentration set**:

$$\Sigma := \bigcap_{r>0} \left\{ X \in P_1 : \liminf_{i\to+\infty} \left(r^{-n} \int_{P_r(X)} |\nabla u_i|^2 \right) \ge \frac{1}{2}\varepsilon_0(n, \mathcal{N}) \right\}.$$

Extension to Ginzburg-Landau Model

One can extend our results to the Ginzburg-Landau model

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \varepsilon^{-2} f(u_\varepsilon), \quad f(p) = -D_p[\chi(\mathrm{dist}^2(p, \mathcal{N}))],$$

where χ is an appropriate cut-off function.

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- [Lin-Wang, 2002, *AHIP*]: Rectifiability for a.e. t.

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$$\int_{\mathbb{R}^k} |\nabla \psi|^2 \exp\left(-\frac{|x|^2}{4}\right) \mathrm{d}x.$$
- **Note**: If ψ is a quasi-harmonic k-sphere, then
$$u(x,t) = \psi\left(\frac{x}{\sqrt{-t}}\right)$$
solves the harmonic map flow for $t < 0$.

Main Theorem: Regularity Improvements

Theorem (Fu-W.-Wu-Zhang, *arXiv:2504.14880*)

Assume $\mathcal{N}$ admits neither harmonic nor quasi-harmonic 2-spheres and $u : P_2 \to \mathcal{N}$ is a suitable solution. Then

$$\sup_{t\in(-1,1)} \|\nabla u(\cdot, t)\|_{L^{3,\infty}(B_1)} \leq C(\Lambda, n, \mathcal{N}).$$

The Lorentz space $L^{p,\infty}(E)$ for $p \in [1, \infty)$ and measurable $E \subset \mathbb{R}^n$ is

$$\|f\|_{L^{p,\infty}(E)} := \sup_{t>0} t\,[\mathcal{L}^n(\{x \in E : |f(x)| > t\})]^{\frac{1}{p}} < +\infty.$$

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Remark

Since $L^{p,\infty} \subset L^q$ for each $q \in [1,3)$, we have

$$\sup_{t\in(-1,1)} \|\nabla u(\cdot,t)\|_{L^{q}(B_1)} \leq C(\Lambda, n, \mathcal{N}, q) \quad \forall q \in [1,3).$$

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  - *Static case*: [Naber-Valtorta, 2017, *Ann. Math.*] proved regularity for Dirichlet energy minimizers.
  - *Dynamic case*: We use properties of $\mathcal{N}$ for sharp regularity bounds.

Strategies

Key Idea: Quantitative stratification

- [Cheeger-Naber, 2013, *Invent. Math.*]: Gromov-Hausdorff limit.
- [Cheeger-Naber, 2013, *CPAM*]: Harmonic maps and minimal currents.

Definition (Quantitative spatial stratification)

Assume $0 < r < R \le 1$, $k \in \mathbb{Z} \cap [0, n-1]$, and $u : P_4 \to \mathcal{N}$ is a suitable solution. Let

$$\Sigma^k_{\varepsilon;r,R}(u,t) := \{x \in B_2 : u \text{ is not spatially } (k,\varepsilon)\text{-sym in } B_s(x) \text{ at } t, \forall s \in [r,R]\}.$$

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We have $\Sigma^k(u,t) = \cup_{\varepsilon>0} \Sigma^k_{\varepsilon,R}(u,t)$.

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$$\mathcal{L}^n(B_r(\Sigma^k_{\varepsilon;r,R}(u,t)\cap B_1)) \leq Cr^{n-k},$$
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for any $\varepsilon > 0$ and $r \in (0,R)$.
- Rectifiability is preserved under the countable union. Then the rectifiability of $\Sigma^k(u,t)$ reduces to $\Sigma^k_{\varepsilon;R}(u,t)$.
- For the top stratum, through compactness arguments, we always have an ε-room to improve, i.e., for some $\varepsilon > 0$,
$$\Sigma^{n-2}_{\varepsilon;0,\varepsilon}(u,t) = \Sigma^{n-2}(u,t).$$

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  - Apply to some more abstract geometric elements
  - When applied to flows, one may need the uniqueness of the blow-ups ([Huang-Jiang, *arXiv:2406.05877*] and [Fang-Li, *arXiv:2504.09811*]) or more information about the tangent flows [Huang-Jiang, *arXiv:2510.17060*].

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- [Edelen-Engelstein, 2019, *TAMS*].

The Key Covering Result

Theorem (Covering of quantitative strata)

Let $k \in \mathbb{Z} \cap [1, n]$, $R \in (0, \frac{1}{10})$, *and* $(x_0, t_0) \in P_2$. *Assume* $u : P_4 \to \mathcal{N}$ *is a suitable solution with* $\|\nabla u\|_{L^2(P_4)} \leq \Lambda$. *Then* $\exists \eta = \eta(\varepsilon, \Lambda, n, \mathcal{N}) > 0$ *s.t. when* $r \in (0, \eta^2)$, *the following properties hold.*

1. $\exists \{B_{Rr}(y)\}_{y \in \mathcal{C}}$ *s.t.*

$$\Sigma^k_{\varepsilon;\eta Rr,1}(u, t_0) \cap B_r(x_0) \subset \bigcup_{y \in \mathcal{C}} B_r(y).$$

2. *We have* $(\#\mathcal{C})R^k \leq C\varepsilon, \Lambda, n, \mathcal{N})$.

The Reifenberg Theorem

Definition (k-dimensional displacement)

Let $k \in \mathbb{Z} \cap [0, n]$, $0 < r \leq 1$, and $U \subset \mathbb{R}^n$ be a bounded open set. Assume that μ is a finite Radon measure on U, namely, $\mu(U) < +\infty$. For $x_0 \in U$ and $0 < r < \mathrm{dist}(x_0, U)$, we define the k-dimensional displacement as

$$D^k_\mu(x_0, r) := \min_{L \in \mathbb{A}(n,k)} \left(r^{-k-2} \int_{B_r(x_0)} \mathrm{dist}^2(y, L) \mathrm{d}\mu(y) \right).$$

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$$D_\mu^k(x_0, r) := \min_{L \in \mathbb{A}(n,k)} \left(r^{-k-2} \int_{B_r(x_0)} \mathrm{dist}^2(y, L) \mathrm{d}\mu(y) \right).$$

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Remark

$D_\mu^k(x_0, r) = 0$ if and only if $\operatorname{supp} \mu \cap B_r(x_0) \subset L$ for some $L \in \mathbb{A}(n, k)$.

The Reifenberg theorem

Theorem (Naber-Valtorta, 2017, *Ann. Math.*)

Let $k \in \mathbb{Z} \cap [0, n]$ *and* $x_0 \in \mathbb{R}^n$. *Assume* $\{B_{r_y}(y)\}_{y \in \mathcal{D}} \subset B_{2r}(x_0)$ *is a collection of pairwise disjoint balls,* $\mathcal{D} \subset B_r(x_0)$, *and*

$$\mu := \sum_{y \in \mathcal{D}} \omega_k r_y^k \delta_y,$$

where ω_k *denotes the volume of a* k*-dimensional unit ball. There exist* $\delta_{\mathrm{R}}(n), C_{\mathrm{R}}(n) > 0$ *s.t. if*

$$\int_{B_t(x)} \left(\int_0^t D_\mu^k(y, s) \frac{\mathrm{d}s}{s} \right) \mathrm{d}\mu(y) < \delta_{\mathrm{R}} t^k, \quad \forall B_t(x) \subset B_{2r}(x_0),$$

then $\mu(B_r(x_0)) \leq C_{\mathrm{R}} r^k$.

Different forms of similar results: [Azzam-Tolsa, 2015, *GAFA*].

Parabolic L^2-best estimates

Proposition

Assume $u : P_4 \to \mathcal{N}$ *is a suitable solution with* $\|\nabla u\|_{L^2(P_4)} \le \Lambda$. $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, \Lambda, n, \mathcal{N}) > 0$ *s.t. if* u *is spatially* $(0, \delta)$*-sym but not spatially* $(k+1, \varepsilon)$*-sym in* $B_{2r}(x_0)$ *at* t_0*, then*

$$D_\mu^k(x_0, r) \le \frac{C}{r^k} \int_{B_r(x_0)} \left[\mathcal{W}\left(u, (y, t_0), 2r, \frac{r}{2}\right) + r\right] \mathrm{d}\mu(y),$$

where

$$\mathcal{W}\left(u, (y, t_0), 2r, \frac{r}{2}\right) := \Psi(u, (y, t_0), 2r) - \Psi\left(u, (y, t_0), \frac{r}{2}\right)$$

represents the difference of densities in scales $2r$ *and* $\frac{r}{2}$.

This proposition establish the connection between the monotonicity formula and the displacements. It is new and key observation.

Main covering

Lemma (Main covering)

Let $k \in \mathbb{Z} \cap [1, n]$, $R \in (0, \frac{1}{10})$, *and* $(x_0, t_0) \in P_2$. *Let* $u : P_4 \to \mathcal{N}$ *be a suitable solution with* $\|\nabla u\|_{L^2(P_4)} \leq \Lambda$. $\exists \eta = \eta(\varepsilon, \Lambda, n, \mathcal{N}) < \frac{1}{10}$ *s.t. if* $r \in (0, \eta^2)$, *then* $\exists \{B_{r_y}(y)\}_{y \in \mathcal{C}}$ *with* $r_y \in [Rr, \frac{r}{10})$ $\forall y \in \mathcal{C}$, *and*

$$\Sigma^k_{\varepsilon;\eta Rr}(u, t_0) \cap B_r(x_0) \subset \bigcup_{y \in \mathcal{C}} B_{r_y}(y), \quad \mathcal{C} \subset \Sigma^k_{\varepsilon;\eta Rr}(t_0) \cap B_r(x_0).$$

1. $\exists C_{\mathrm{M}} = C_{\mathrm{M}}(\varepsilon, \Lambda, n, \mathcal{N}) > 0$ *s.t.* $\sum_{y \in \mathcal{C}} r_y^k \leq C_{\mathrm{M}} r^k$.
2. *For any* $y \in \mathcal{C}$, *either* $r_y = Rr$ *or*

$$\sup_{z \in B_{2r_y}(y)} \Phi(u, (z, t_0), 2r_y) \leq \sup_{z \in B_{2r}(x_0)} \Phi(u, (z, t_0), 2r) - \frac{\eta}{3}.$$

Thank you for listening!