

Stratification for Harmonic Map Flows via Tangent Measures: Rectifiability and Regularity

Wei Wang

School of Mathematical Sciences

Peking University

Joint work with Haotong Fu, Ke Wu, and Zhifei Zhang

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Harmonic Map Flows: Overview

Let Ω be a bounded domain in \mathbb{R}^n with $n \geq 2$ and $\mathcal{N} \hookrightarrow \mathbb{R}^d$ ($d \geq 2$) be a smooth, compact mfd. For $u \in H^1(\Omega, \mathcal{N})$, the **Dirichlet functional**:

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The harmonic map satisfies $-\Delta u = A(u)(\nabla u, \nabla u)$ in Ω , where $A(\cdot)(\cdot, \cdot) : \mathcal{N} \times T\mathcal{N} \times T\mathcal{N} \rightarrow (T\mathcal{N})^\perp$ is the second fundamental form.

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Origin: Introduced by [Eells-Sampson, 1964, *AJM*] to construct harmonic maps in any given free homotopy class.

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- [Wang-Lin, 2010, *C. Ann. Math. Series B*]: Uniqueness under additional regularity conditions.

Suitable Solutions

Definition

A map $u : P_1 = B_1 \times (-1, 1) \rightarrow \mathcal{N}$ is a suitable solution of the harmonic map flow if $\partial_t u, \nabla u \in L^2(P_1)$ and:

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- 1 u is a weak solution;
- 2 u satisfies the localized energy inequality: $\forall \theta \in C_0^\infty(P_1, \mathbb{R}_{\geq 0})$,

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Note: Smooth solutions are suitable solutions.

Localized energy inequality

By (S1), we have:

Lemma

Let $r > 0$ and $X_0 = (x_0, t_0) \in P_2$ be a point such that $P_r(X_0) \subset P_4$. Assume $u : P_4 \rightarrow \mathcal{N}$ is a suitable solution, then

$$\begin{aligned} & \int_{B_r(x_0) \times [s, t]} |\partial_t u|^2 \phi^2 dx d\tau + \int_{B_r(x_0)} |\nabla u(\cdot, t)|^2 \phi^2 dx \\ & \leq \int_{B_r(x_0)} |\nabla u(\cdot, s)|^2 \phi^2 dx + 4 \int_s^t \int_{B_r(x_0)} |\nabla u|^2 |\nabla \phi|^2 dx d\tau \end{aligned}$$

for any $\phi \in C_0^\infty(B_r(x_0))$ and a.e. $t_0 - r^2 < s \leq t < t_0 + r^2$.

The Monotonicity Formula

Combining (S1) and (S2), we obtain:

Proposition

Fix $X_0 = (x_0, t_0) \in P_4$. Assume that $u : P_8 \rightarrow \mathcal{N}$ is a suitable solution with $\|\nabla u\|_{L^2(P_8)} \leq \Lambda$. Then, for a.e. $0 < r \leq R < 2$,

$$\begin{aligned} & \Psi(u, X_0, R) \exp(C_1(R - r)) - \Psi(u, X_0, r) + C_1(R - r) \\ & \geq C_2 \int_r^R \left(\int_{T_\rho(X_0)} \varphi_{x_0}^2 \frac{|(x - x_0) \cdot \nabla u + 2(t - t_0) \partial_t u|^2}{|t_0 - t|} G_{X_0}(x, t) dx dt \right) \frac{d\rho}{\rho}, \end{aligned}$$

where

$$\Psi(u, X_0, \rho) := \frac{1}{2} \int_{T_\rho(X_0)} \varphi_{x_0}^2(x) |\nabla u(x, t)|^2 G_{X_0}(x, t) dx dt,$$

$(C_1, C_2)(\Lambda, n) > 0$, and G is the heat kernel.

Partial Regularity

Definition (Singular set)

The singular set of a suitable weak harmonic map is

$$\text{sing}(u) := \{X \in P_r(X_0) : \forall \rho > 0, u \notin C^\infty(P_\rho(X))\}.$$

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Partial regularity:

$$r^{-n} \int_{P_{2r}(X)} |\nabla u|^2 \leq \varepsilon_0(n, \mathcal{N}) \implies u \in C^\infty(P_r(X), \mathcal{N}).$$

Partial regularity $\Rightarrow \mathcal{P}^n(\text{sing}(u)) = 0$, where \mathcal{P}^n is the n -dimensional parabolic Hausdorff measure.

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- [Liu, 2003, *ARMA*]: Partial regularity extends to general \mathcal{N} .

Blow-up Analysis

Definition

Let $U \subset \mathbb{R}^n \times \mathbb{R}$ be an open set and $\mathcal{M}(U)$ be the space of Radon measures on U . For $X_0 = (x_0, t_0) \in U$ and $r > 0$, let

$$\mathcal{P}_{X_0, r}(X) = (x_0 + rx, t_0 + r^2 t), \quad X = (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

For a suitable solution $u : P_1 \rightarrow \mathcal{N}$, $\mu \in \mathcal{M}(P_1)$, $r > 0$, and $X \in P_1$, define

$$T_{X, r} u := u \circ \mathcal{P}_{X_0, r}, \quad T_{X, r} \mu := r^{-n} (\mu \circ \mathcal{P}_{X_0, r}).$$

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Then, $\exists r_i \rightarrow 0^+$ s.t.

$$\begin{aligned} w_i &:= T_{X, r_i} u \rightharpoonup w \text{ weakly in } H_{\text{loc}}^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^d), \\ \frac{1}{2} |\nabla w_i|^2 dx dt &\rightharpoonup^* \eta := \frac{1}{2} |\nabla w|^2 dx dt + \nu \text{ in } \mathcal{M}(\mathbb{R}^n \times \mathbb{R}). \end{aligned}$$

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- η : tangent measure of u at X .
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Remark

- Tangent flows, and tangent measures depend on the choice of the sequence $\{r_i\}$.
- The defect measure ν quantifies energy loss in the weak convergence.

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- **Space-time strata:**

$$S^k(u) := \{X \in P_1 : \text{no tang. flow at } X \text{ is space-time } (k+1)\text{-sym}\},$$

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- **Spatial strata** (for $t \in (-1, 1)$):

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Dimension Estimates

Using Federer dimension reduction:

Theorem (B. White, 1997, *JRAM*, *Crella*)

Let $u : P_1 \rightarrow \mathcal{N}$ be a suitable solution, then

- $\dim_{\mathcal{P}} S^k(u), \Sigma^k(u) \leq k$;
- $\dim_{\mathcal{H}} S^k(u, t), \Sigma^k(u, t) \leq k \ \forall t$;
- $\dim_{\mathcal{H}} S^k(u) \cap \{t\}, \Sigma^k(u) \cap \{t\} \leq k - 2$ for a.e. t .

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Question

Can we further characterize the strata and singular sets of suitable solutions?

Main Theorem: Rectifiability and Minkowski Dimension

Theorem (Fu-W.-Wu-Zhang, *arXiv: 2504.14880*)

Let $\Lambda > 0$, $k \in \mathbb{Z} \cap [1, n-1]$, $t \in (-1, 1)$. Assume $u : P_1 \rightarrow \mathcal{N}$ is a suitable solution with $\|\nabla u\|_{L^2(P_1)} \leq \Lambda$. Then

① Rectifiability:

- $\Sigma^k(u, t)$ is k -rectifiable $\forall k \in \mathbb{Z} \cap [1, n-2]$;
- $\text{sing}(u) \cap \{t\}$ is $(n-2)$ -rectifiable.

② Minkowski content estimate:

$$\text{Min}_r^{n-2}(\text{sing}(u) \cap (B_1 \times \{t\})) \leq C(\Lambda, n, \mathcal{N}).$$

where $\text{Min}_\rho^k(\cdot)$ is the Minkowski content.

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The Minkowski dimension estimate: $\dim_{\text{Min}}(\text{sing}(u) \cap \{t\}) \leq n-2$

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implying that the spatial $(n - 1)$ -symmetry of tangent measures indicate the smoothness.

- We do not require the uniqueness of the tangent flow or measure.
- Assuming $\partial_t u \equiv 0$, the properties degenerate to those for stationary harmonic maps in [Naber-Valtorta, 2017, *Ann. Math.*].

Extension to Ginzburg-Landau Model

One can extend our results to the Ginzburg-Landau model

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \varepsilon^{-2} f(u_\varepsilon), \quad f(p) = -D_p[\chi(\text{dist}^2(p, \mathcal{N}))],$$

where χ is an appropriate cut-off function.

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- **Concentration set:**

$$\Sigma := \bigcap_{r>0} \left\{ X \in P_1 : \liminf_{i \rightarrow +\infty} \left(r^{-n} \int_{P_r(X)} |\nabla u_i|^2 \right) \geq \frac{1}{2} \varepsilon_0(n, \mathcal{N}) \right\}.$$

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$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \varepsilon^{-2} f(u_\varepsilon), \quad f(p) = -D_p[\chi(\text{dist}^2(p, \mathcal{N}))],$$

where χ is an appropriate cut-off function.

- **Convergence:** If $\sup_{\varepsilon > 0} \|\nabla u_\varepsilon\|_{L^1(P_1)} \leq \Lambda$, then $\exists \varepsilon_i \rightarrow 0^+$ s.t.

- $u_i := u_{\varepsilon_i} \rightharpoonup u$ weakly in $H^1(P_1)$;
- $\frac{1}{2} |\nabla u_i|^2 dx dt \rightharpoonup^* \mu$ in $\mathcal{M}(P_1)$.

- **Concentration set:**

$$\Sigma := \bigcap_{r>0} \left\{ X \in P_1 : \liminf_{i \rightarrow +\infty} \left(r^{-n} \int_{P_r(X)} |\nabla u_i|^2 \right) \geq \frac{1}{2} \varepsilon_0(n, \mathcal{N}) \right\}.$$

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- [Lin-Wang, 2002, *AHIP*]: Rectifiability for a.e. t .

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- **Note:** If ψ is a quasi-harmonic k -sphere, then

$$u(x, t) = \psi\left(\frac{x}{\sqrt{-t}}\right)$$

solves the harmonic map flow for $t < 0$.

Main Theorem: Regularity Improvements

Theorem (Fu-W.-Wu-Zhang, *arXiv:2504.14880*)

Assume \mathcal{N} admits neither harmonic nor quasi-harmonic 2-spheres and $u : P_2 \rightarrow \mathcal{N}$ is a suitable solution. Then

$$\sup_{t \in (-1, 1)} \|\nabla u(\cdot, t)\|_{L^{3, \infty}(B_1)} \leq C(\Lambda, n, \mathcal{N}).$$

The Lorentz space $L^{p, \infty}(E)$ for $p \in [1, \infty)$ and measurable $E \subset \mathbb{R}^n$ is

$$\|f\|_{L^{p, \infty}(E)} := \sup_{t > 0} t [\mathcal{L}^n(\{x \in E : |f(x)| > t\})]^{\frac{1}{p}} < +\infty.$$

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Remark

Since $L^{p,\infty} \subset L^q$ for each $q \in [1, 3)$, we have

$$\sup_{t \in (-1,1)} \|\nabla u(\cdot, t)\|_{L^q(B_1)} \leq C(\Lambda, n, \mathcal{N}, q) \quad \forall q \in [1, 3).$$

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- **Regularity improvements:**

- *Static case*: [Naber-Valtorta, 2017, *Ann. Math.*] proved regularity for Dirichlet energy minimizers.
- *Dynamic case*: We use properties of \mathcal{N} for sharp regularity bounds.

Strategies

Key Idea: Quantitative stratification

- [Cheeger-Naber, 2013, *Invent. Math.*]: Gromov-Hausdorff limit.
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Assume $0 < r < R \leq 1$, $k \in \mathbb{Z} \cap [0, n-1]$, and $u : P_4 \rightarrow \mathcal{N}$ is a suitable solution. Let

$$\Sigma_{\varepsilon; r, R}^k(u, t) := \{x \in B_2 : u \text{ is not spatially} \\ (k, \varepsilon)\text{-sym in } B_s(x) \text{ at } t, \forall s \in [r, R]\}.$$

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We have $\Sigma^k(u, t) = \bigcup_{\varepsilon > 0} \Sigma_{\varepsilon, R}^k(u, t)$.

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- Intuitively, compared to $\Sigma^k(u, t)$, $\Sigma_{\varepsilon; r, R}^k(u, t)$ has more information, so it's easier to get volume estimates: for a suitable solution in P_4 ,

$$\mathcal{L}^n(B_r(\Sigma_{\varepsilon; r, R}^k(u, t) \cap B_1)) \leq Cr^{n-k},$$

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- Rectifiability is preserved under the countable union. Then the rectifiability of $\Sigma^k(u, t)$ reduces to $\Sigma_{\varepsilon; R}^k(u, t)$.
- For the top stratum, through compactness arguments, we always have an ε -room to improve, i.e., for some $\varepsilon > 0$,

$$\Sigma_{\varepsilon; 0, \varepsilon}^{n-2}(u, t) = \Sigma^{n-2}(u, t).$$

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 - When applied to flows, one may need the uniqueness of the blow-ups ([Huang-Jiang, *arXiv:2406.05877*] and [Fang-Li, *arXiv:2504.09811*]) or more information about the tangent flows [Huang-Jiang, *arXiv:2510.17060*].

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- [Edelen-Engelstein, 2019, *TAMS*].

The Key Covering Result

Theorem (Covering of quantitative strata)

Let $k \in \mathbb{Z} \cap [1, n]$, $R \in (0, \frac{1}{10})$, and $(x_0, t_0) \in P_2$. Assume $u : P_4 \rightarrow \mathcal{N}$ is a suitable solution with $\|\nabla u\|_{L^2(P_4)} \leq \Lambda$. Then $\exists \eta = \eta(\varepsilon, \Lambda, n, \mathcal{N}) > 0$ s.t. when $r \in (0, \eta^2)$, the following properties hold.

① $\exists \{B_{Rr}(y)\}_{y \in \mathcal{C}}$ s.t.

$$\Sigma_{\varepsilon; \eta Rr, 1}^k(u, t_0) \cap B_r(x_0) \subset \bigcup_{y \in \mathcal{C}} B_r(y).$$

② We have $(\#\mathcal{C})R^k \leq C(\varepsilon, \Lambda, n, \mathcal{N})$.

The Reifenberg Theorem

Definition (k -dimensional displacement)

Let $k \in \mathbb{Z} \cap [0, n]$, $0 < r \leq 1$, and $U \subset \mathbb{R}^n$ be a bounded open set. Assume that μ is a finite Radon measure on U , namely, $\mu(U) < +\infty$. For $x_0 \in U$ and $0 < r < \text{dist}(x_0, U^c)$, we define the k -dimensional displacement as

$$D_\mu^k(x_0, r) := \min_{L \in \mathbb{A}(n, k)} \left(r^{-k-2} \int_{B_r(x_0)} \text{dist}^2(y, L) d\mu(y) \right).$$

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Remark

$D_\mu^k(x_0, r) = 0$ if and only if $\text{supp } \mu \cap B_r(x_0) \subset L$ for some $L \in \mathbb{A}(n, k)$.

The Reifenberg theorem

Theorem (Naber-Valtorta, 2017, *Ann. Math.*)

Let $k \in \mathbb{Z} \cap [0, n]$ and $x_0 \in \mathbb{R}^n$. Assume $\{B_{r_y}(y)\}_{y \in \mathcal{D}} \subset B_{2r}(x_0)$ is a collection of pairwise disjoint balls, $\mathcal{D} \subset B_r(x_0)$, and

$$\mu := \sum_{y \in \mathcal{D}} \omega_k r_y^k \delta_y,$$

where ω_k denotes the volume of a k -dimensional unit ball. There exist $\delta_R(n), C_R(n) > 0$ s.t. if

$$\int_{B_t(x)} \left(\int_0^t D_\mu^k(y, s) \frac{ds}{s} \right) d\mu(y) < \delta_R t^k, \quad \forall B_t(x) \subset B_{2r}(x_0),$$

then $\mu(B_r(x_0)) \leq C_R r^k$.

Different forms of similar results: [Azzam-Tolsa, 2015, *GAFA*].

Parabolic L^2 -best estimates

Proposition

Assume $u : P_4 \rightarrow \mathcal{N}$ is a suitable solution with $\|\nabla u\|_{L^2(P_4)} \leq \Lambda$. $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, \Lambda, n, \mathcal{N}) > 0$ s.t. if u is spatially $(0, \delta)$ -sym but not spatially $(k+1, \varepsilon)$ -sym in $B_{2r}(x_0)$ at t_0 , then

$$D_\mu^k(x_0, r) \leq \frac{C}{r^k} \int_{B_r(x_0)} \left[\mathcal{W}\left(u, (y, t_0), 2r, \frac{r}{2}\right) + r \right] d\mu(y),$$

where

$$\mathcal{W}\left(u, (y, t_0), 2r, \frac{r}{2}\right) := \Psi(u, (y, t_0), 2r) - \Psi\left(u, (y, t_0), \frac{r}{2}\right)$$

represents the difference of densities in scales $2r$ and $\frac{r}{2}$.

This proposition establish the connection between the monotonicity formula and the displacements. It is new and key observation.

Main covering

Lemma (Main covering)

Let $k \in \mathbb{Z} \cap [1, n]$, $R \in (0, \frac{1}{10})$, and $(x_0, t_0) \in P_2$. Let $u : P_4 \rightarrow \mathcal{N}$ be a suitable solution with $\|\nabla u\|_{L^2(P_4)} \leq \Lambda$. $\exists \eta = \eta(\varepsilon, \Lambda, n, \mathcal{N}) < \frac{1}{10}$ s.t. if $r \in (0, \eta^2)$, then $\exists \{B_{r_y}(y)\}_{y \in \mathcal{C}}$ with $r_y \in [Rr, \frac{r}{10}] \forall y \in \mathcal{C}$, and

$$\Sigma_{\varepsilon; \eta Rr, 1}^k(u, t_0) \cap B_r(x_0) \subset \bigcup_{y \in \mathcal{C}} B_{r_y}(y), \quad \mathcal{C} \subset \Sigma_{\varepsilon; \eta Rr}^k(t_0) \cap B_r(x_0).$$

① $\exists C_M = C_M(\varepsilon, \Lambda, n, \mathcal{N}) > 0$ s.t. $\sum_{y \in \mathcal{C}} r_y^k \leq C_M r^k$.

② For any $y \in \mathcal{C}$, either $r_y = Rr$ or

$$\sup_{z \in B_{2r_y}(y)} \Psi(u, (z, t_0), 2r_y) \leq \sup_{z \in B_{2r}(x_0)} \Psi(u, (z, t_0), 2r) - \frac{\eta}{3}.$$

Thank you for listening!