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3. Hacer los pasos intermedios para encontrar la regla de Simpson simple 1/3.

$$\int_a^b f(x)\,dx \cong \int_a^b p_2(x)\,dx = \frac{h}{3}\left(f(a) + 4f(x_m) + f(b)\right)$$

$$x_m = \frac{a+b}{2} \qquad b-a = 2h \qquad a-b = -2h$$

$$p_2(x) = \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a) + \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) + \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b), \quad \forall x \in [a,b]$$

$$I = \int_a^b f(x)\,dx = \underbrace{\int_a^b \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a)}_{1} + \underbrace{\frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m)}_{2} + \underbrace{\frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)}_{3}$$

$$\int_a^b \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a)\,dx = \int_a^b \frac{(x-b)\left(x-\left(\frac{a+b}{2}\right)\right)}{(a-b)\left(a-\left(\frac{a+b}{2}\right)\right)} f(a)\,dx = \int_a^b \frac{(x-b)(2x-a-b)}{(a-b)(2a-a-b)} f(a)\,dx$$

$$= \int_a^b \frac{2x^2 - xa - 3xb + ab + b^2}{(a-b)(a-b)} f(a)\,dx = \frac{f(a)}{(a-b)^2}\int_a^b 2x^2 - xa - 3xb + ab + b^2\,dx$$

$$= \frac{f(a)}{(a-b)^2}\left[\frac{2x^3}{3} - \frac{ax^2}{2} - \frac{3bx^2}{2} + abx + b^2x\right]_a^b$$

$$= \frac{f(a)}{(a-b)^2}\left[\left(\frac{2b^3}{3} - \frac{ab^2}{2} - \frac{3b^3}{2} + ab^2 + b^3\right) - \left(\frac{2a^3}{3} - \frac{a^3}{2} - \frac{3ba^2}{2} + a^2b + ab^2\right)\right]$$

$$= \frac{f(a)}{(a-b)^2}\left[\frac{2b^3}{3} - \frac{ab^2}{2} - \frac{3b^3}{2} + ab^2 + b^3 - \frac{2a^3}{3} + \frac{a^3}{2} + \frac{3ba^2}{2} - a^2b - ab^2\right]$$

$$= \frac{f(a)}{(a-b)^2}\left[\frac{b^3}{6} - \frac{ab^2}{2} - \frac{a^3}{6} + \frac{a^2b}{2}\right] = \frac{f(a)}{(a-b)^2}\left[\frac{b^3 - 3ab^2 + 3ab^2 - a^3}{6}\right]$$

$$= \frac{f(a)}{(a-b)^2}\frac{(b-a)^3}{6} = \frac{-f(a)(a-b)^3}{6(a-b)^2} = \frac{-f(a)(a-b)}{6} = \frac{f(a)(b-a)}{6}$$

$$= \frac{f(a)(2h)}{6} = \frac{f(a)h}{3}$$

$$\int_a^b \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) = \int_a^b \frac{(x-a)\ (x-b)}{\left(\left(\frac{a+b}{2}\right)-a\right)\left(\left(\frac{a+b}{2}\right)-b\right)} f(x_m)\,dx = \int_a^b \frac{(x-a)(x-b)}{(a+b-2a)(a+b-2b)} f(x_m)\,dx$$

$$= \frac{f(x_m)}{(b-a)(a-b)} \int_a^b (x-a)(x-b)\,dx = \frac{f(x_m)}{(b-a)(a-b)} \int_a^b x^2 - xb - xa + ab\ \ dx = \frac{f(x_m)}{(b-a)(a-b)} \left[\frac{x^3}{3} - \frac{bx^2}{2} - \frac{ax^2}{2} + abx \right]_a^b$$

$$= \frac{f(x_m)}{(b-a)(a-b)} \left[\left(\frac{b^3}{3} - \frac{b^3}{2} - \frac{ab^2}{2} + ab^2 \right) - \left(\frac{a^3}{3} - \frac{a^2b}{2} - \frac{a^3}{2} + a^2b \right) \right]$$

$$= \frac{f(x_m)}{(b-a)(a-b)} \left[\frac{b^3}{3} - \frac{b^3}{2} - \frac{ab^2}{2} + ab^2 - \frac{a^3}{3} + \frac{a^2b}{2} + \frac{a^3}{2} - a^2b \right]$$

$$= \frac{f(x_m)}{(b-a)(a-b)} \left[-\frac{b^3}{6} + \frac{ab^2}{2} - \frac{a^2b}{2} + \frac{a^3}{6} \right] = \frac{f(x_m)}{(b-a)(a-b)} \left[\frac{-b^3 + 3ab^2 - 3a^2b + a^3}{6} \right]$$

$$= \frac{f(x_m)}{(b-a)(a-b)} \frac{(a-b)^3}{6} = \frac{f(x_m)}{(b-a)(a-b)} \frac{(-2h)^3}{6} = \frac{f(x_m)}{-h^2} \cdot \frac{-8h^3}{6} = \boxed{\frac{4}{3} f(x_m)\,h}$$

$$\int_a^b \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)\,dx = \int_a^b \frac{(x-a)\left(x-\left(\frac{a+b}{2}\right)\right)}{(b-a)\left(b-\left(\frac{a+b}{2}\right)\right)} f(b)\,dx = \int_a^b \frac{(x-a)(2x-a-b)}{(b-a)(2b-a-b)} f(b)\,dx$$

$$= \frac{f(b)}{(b-a)^2} \int_a^b 2x^2 \underbrace{-ax - bx - 2ax}_{-3ax} + a^2 + ab\ \ dx = \frac{f(b)}{(b-a)^2} \left[\frac{2x^3}{3} - \frac{3ax^2}{2} - \frac{bx^2}{2} + a^2x + abx \right]_a^b$$

$$= \frac{f(b)}{(b-a)^2} \left[\frac{2b^3}{3} - \frac{3ab^2}{2} - \frac{b^3}{2} + a^2b + ab^2 - \frac{2a^3}{3} + \frac{3a^3}{2} + \frac{a^2b}{2} - a^3 - a^2b \right]$$

$$= \frac{f(b)}{(b-a)^2} \left[\frac{b^3}{6} - \frac{ab^2}{2} + \frac{a^2b}{2} - \frac{a^3}{6} \right] = \frac{f(b)}{(b-a)^2} \left[\frac{b^3 - 3ab^2 + 3a^2b - a^3}{6} \right] = \frac{f(b)}{(b-a)^2} \frac{(b-a)^3}{6}$$

$$= \frac{f(b)\,(b-a)}{6} = \boxed{\frac{f(b)\,h}{3}}$$

$$\int_a^b P_2(x)\,dx = \frac{h\,f(a)}{3} + \frac{4}{3} h\, f(x_m) + \frac{h}{3} f(b)$$

$$\boxed{= \frac{h}{3} \left[f(a) + 4 f(x_m) + f(b) \right]}$$

4. Verificar el resultado presentado en la ecuación

$$E = \int_a^b E(x)\,dx = \int_a^b \frac{f'''(\xi)}{4!}(x-a)(x-b)(x-(a+b)/2)\,dx = 0 \qquad a \le \xi \le b$$

Suponiendo que $f(x)$ es continua y derivable de clase C^3 en el intervalo $[a,b]$: $f(x) = p_2(x) + E(x)$

$$\int_a^b \frac{f'''(\xi)}{4!}(x-a)(x-b)(x-(a+b)/2)\,dx = \frac{f'''(\xi)}{4!}\int_a^b (x-a)(x-b)(2x-a-b)\,dx$$

$$= \frac{f'''(\xi)}{4}\int_a^b (x^2-bx-ax+ab)(2x-a-b)\,dx$$

$$= \frac{f'''(\xi)}{4!}\int_a^b 2x^3-2bx^2-2ax^2+2abx-ax^2+abx+a^2x-a^2b-bx^2+b^2x+abx-ab^2\,dx$$

$$= \frac{f'''(\xi)}{4!}\int_a^b 2x^3-3bx^2-3ax^2+4abx+a^2x-ab^2+b^2x-ab^2\,dx$$

$$= \frac{f^{(3)}(\xi)}{4!}\int_a^b x^3-\frac{3bx^2}{2}-\frac{3ax^2}{2}+2abx+\frac{a^2x}{2}-\frac{ab^2}{2}+\frac{b^2x}{2}-\frac{ab^2}{2}\,dx$$

$$= \frac{f^{(3)}(\xi)}{4!}\left[\frac{x^4+a^2x^2+bx^2-2bx^3-2ax^3+4abx^2-4ab^2x}{4}\right]_a^b$$

$$= \frac{f^{(3)}(\xi)}{4!}\left[\frac{b^4+a^2b^2+b^3-2b^4-2ab^3+4ab^3-4ab^3-a^4-a^4-a^2b+2a^3b+2a^4-4a^3b-4a^2b^2}{4}\right]$$

$$= \frac{f^{(3)}(\xi)}{4!}[0] \qquad = 0$$