Universidade do Minho
Departamento de Matemática

LFIS e MIEFIS

Cálculo Vetorial

Folha 5 — abril de 2020

Exercício 1. Seja $\mathcal{R}$ o rectângulo $[0,1]\times[0,1]$. Calcule os integrais:

a) $\displaystyle\iint_{\mathcal{R}} (x^3+y^2)\ d(x,y)$;

b) $\displaystyle\iint_{\mathcal{R}} ye^{xy}\ d(x,y)$;

c) $\displaystyle\iint_{\mathcal{R}} (xy)^2\cos x^3\ d(x,y)$;

d) $\displaystyle\iint_{\mathcal{R}} \ln\left[(x+10)(y+1)\right]\ d(x,y)$.

Exercício 2. Calcule os seguintes integrais:

a) $\displaystyle\int_0^1\int_0^{x^2} dy\,dx$;

b) $\displaystyle\int_1^2\int_{2x}^{3x+1} dy\,dx$;

c) $\displaystyle\int_0^1\int_1^{e^y} (x+y)\ dx\,dy$;

d) $\displaystyle\int_0^1\int_{x^3}^{x^2} y\ dy\,dx$.

Exercício 3. Calcule $\displaystyle\iint_{\mathcal{D}} f\ d(x,y)$, usando as duas possíveis ordens de integração, quando f e $\mathcal{D}$ são:

a) $f(x,y)=xy,\ \mathcal{D}=\{(x,y)\in\mathbb{R}^2: 0\le x\le 2,\ 0\le y\le x^2\}$;

b) $f(x,y)=x\,\mathrm{sen}\,(x+y),\ \mathcal{D}=\{(x,y)\in\mathbb{R}^2: 0\le y\le \pi,\ 0\le x\le 1\}$;

c) $f(x,y)=e^{x+y},\ \mathcal{D}=\{(x,y)\in\mathbb{R}^2: |x|+|y|\le 1\}$;

d) $f(x,y)=x^2+y^2,\ \mathcal{D}=\{(x,y)\in\mathbb{R}^2: 0\le y\le \mathrm{sen}\,x,\ 0\le x\le \frac{\pi}{2}\}$.

Exercício 4. Inverta a ordem de integração nos seguintes integrais:

a) $\displaystyle\int_{-2}^2\int_0^{4-x^2} f(x,y)\ dy\,dx$;

b) $\displaystyle\int_{-1}^1\int_0^{4-x^2} f(x,y)\ dy\,dx$;

c) $\displaystyle\int_0^2\int_{-\sqrt{4-y^2}}^{2-y} f(x,y)\ dx\,dy$;

d) $\displaystyle\int_{-3}^2\int_{-4+y^2}^{2-y} f(x,y)\ dx\,dy$;

e) $\displaystyle\int_{-2}^2\int_{-4+y^2}^{2-y} f(x,y)\ dx\,dy$;

f) $\displaystyle\int_1^{e^2}\int_{\ln x}^{x} f(x,y)\ dy\,dx$;

g) $\displaystyle\int_{-2}^2\int_0^{-|y|+2} f(x,y)\ dx\,dy$;

h) $\displaystyle\int_0^1\int_0^{\sqrt{x}} f(x,y)\ dy\,dx+\int_1^2\int_0^{-x+2} f(x,y)\ dy\,dx$.

Exercício 5. Represente graficamente o conjunto $\mathcal{D}$ e calcule, recorrendo a integrais duplos, a sua área:

a) $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2,\, x \leq y \leq x^2\}$;

b) $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : -y^2 \leq x \leq y^2,\, 0 \leq y \leq 1\}$.

Exercício 6. Seja $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x,\ 0 \leq x \leq 1\}$.

a) Calcule $\displaystyle\iint_{\mathcal{D}} (x + y)\ d(x, y)$.

b) Calcule o integral da alínea anterior, fazendo a mudança de variáveis $x = u + v$, $y = u - v$.

Exercício 7. Calcule os seguintes integrais, usando uma mudança de variáveis adequada:

a) $\displaystyle\iint_{\mathcal{D}} (x^2 + y^2)\ d(x, y)$, onde $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : -1 \leq x - y \leq 0,\, -1 \leq x + y \leq 0\}$;

b) $\displaystyle\iint_{\mathcal{D}} (x - y)\ d(x, y)$, onde $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : -x \leq y \leq 3 - x,\, 2x - 2 \leq y \leq 2x\}$.

Exercício 8. Determine a área limitada pelas curvas $x^2 + y^2 = 2x$, $x^2 + y^2 = 4x$, $y = x$ e $y = 0$.

Exercício 9. Calcule os seguintes integrais, usando coordenadas polares.

a) $\displaystyle\int_0^{2R} \int_0^{\sqrt{2Rx - x^2}} (x^2 + y^2)\ dy\, dx$;

b) $\displaystyle\int_0^1 \int_{x^2}^{x} dy\, dx$.

Exercício 10. Calcule a área dos seguintes subconjuntos de $\mathbb{R}^2$:

a) $\{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 \leq 4\}$;

b) $\{(x, y) \in \mathbb{R}^2 : x^2 + (y - 3)^2 \leq 9\}$;

c) $\{(x, y) \in \mathbb{R}^2 : x^2 + (y - 3)^2 \leq 4\}$;

d) $\{(x, y) \in \mathbb{R}^2 : x^2 + (y - 2)^2 \leq 9\}$;

e) $\{(x, y) \in \mathbb{R}^2 : y \leq 3,\ x^2 - 1 \leq y \leq 2x + 2)\}$;

f) $\{(x, y) \in \mathbb{R}^2 : x^2 - 4x + y^2 \leq 0,\ x^2 - 2x + y^2 \geq 0\}$;

g) $\{(x, y) \in \mathbb{R}^2 : x^2 - 2x + y^2 \leq 0,\ (x - 1)^2 + y^2 \geq 1\}$;

h) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9,\ y^2 - x^2 \leq 1\}$;

i) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9,\ y^2 - x^2 \leq 1\}$;

j) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq xy\}$;

k) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1,\ x^2 + y^2 \leq 9,\ y \leq x + 3,\ y \geq x - 3\}$;

l) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq by\}$ (com $b > 0$);

m) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq by,\ x^2 + y^2 \leq ax\}$ (com $a, b > 0$);

n) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2,\ (x - b)^2 + y^2 \leq b^2\}$ (com $a, b > 0$);

o) $\{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 \leq 3x^2 - y^2\}$;

p) $\{(x, y) \in \mathbb{R}^2 : (x^2 + y^2)^2 \geq 8(x^2 - y^2),\ x^2 + y^2 - 4y \leq 0\}$.