1. Calcule os valores próprios e os subespaços próprios das seguintes matrizes:

a) $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

$det(A - \lambda I) \;=\; \det \begin{bmatrix} 1-\lambda & 1 & -1 \\ 2 & 2-\lambda & -2 \\ -1 & -1 & 1-\lambda \end{bmatrix} \overset{C_2+C_3}{=} \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 2 & 2-\lambda & -\lambda \\ -1 & -1 & -\lambda \end{bmatrix} \overset{L_3-L_2}{=}$
$\det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 2 & 2-\lambda & -\lambda \\ -3 & \lambda-3 & 0 \end{bmatrix} = -(-\lambda)\det \begin{bmatrix} 1-\lambda & 1 \\ -3 & \lambda-3 \end{bmatrix} = \lambda((1-\lambda)(\lambda-3)+3)=$
$\lambda^2\,(\lambda-4) = 0$

$\lambda = 0, \lambda = 0, \lambda = 4.$

$V_{\lambda=0} = \{x \in \mathbb{R}^3 : Ax = 0x\} = \{x \in \mathbb{R}^3 : \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & 1 \end{bmatrix} x = 0\}$

$= < (-1,1,0),(1,0,1) > .$

$V_{\lambda=4} = \{x \in \mathbb{R}^3 : Ax = 4x\} = \{x \in \mathbb{R}^3 : \begin{bmatrix} 1-4 & 1 & -1 \\ 2 & 2-4 & -2 \\ -1 & -1 & 1-4 \end{bmatrix} x = 0\}$

$= < (-1,-2,1) >$

b) $B = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

$det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 & -1 & 0 \\ 0 & 2-\lambda & -2 & 1 \\ 0 & 0 & 1-\lambda & -1 \\ 0 & 0 & 0 & -2-\lambda \end{bmatrix} = (\lambda-2)\,(\lambda+2)\,(\lambda-1)^2 = 0.$

$\lambda = 2, \lambda = -2, \lambda = 1, \lambda = 1.$

$V_{\lambda=2} = \{x \in \mathbb{R}^4 : Ax = 2x\} = \{x \in \mathbb{R}^4 : \begin{bmatrix} 1-2 & 1 & -1 & 0 \\ 0 & 2-2 & -2 & 1 \\ 0 & 0 & 1-2 & -1 \\ 0 & 0 & 0 & -2-2 \end{bmatrix} x = 0\}$

$= < (1,1,0,0) >$

$V_{\lambda=-2} = \{x \in \mathbb{R}^4 : Ax = -2x\} = \{x \in \mathbb{R}^4 : \begin{bmatrix} 1+2 & 1 & -1 & 0 \\ 0 & 2+2 & -2 & 1 \\ 0 & 0 & 1+2 & -1 \\ 0 & 0 & 0 & -2+2 \end{bmatrix} x = 0\}$

$= < (5,-3,12,36) >$

$V_{\lambda=1} = \{x \in \mathbb{R}^4 : Ax = 1x\} = \{x \in \mathbb{R}^4 : \begin{bmatrix} 1-1 & 1 & -1 & 0 \\ 0 & 2-1 & -2 & 1 \\ 0 & 0 & 1-1 & -1 \\ 0 & 0 & 0 & -2-1 \end{bmatrix} x = 0\}$

$= < (1,0,0,0) >$

c) $C = \begin{bmatrix} 5 & -2 & 1 \\ 2 & 1 & 1 \\ -4 & 2 & 0 \end{bmatrix}$

$$det(A-\lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 1 \\ 2 & 1-\lambda & 1 \\ -4 & 2 & 0-\lambda \end{bmatrix} \overset{L_2-L_1}{=} \det \begin{bmatrix} 5-\lambda & -2 & 1 \\ \lambda-3 & 3-\lambda & 0 \\ -4 & 2 & -\lambda \end{bmatrix}$$

$$= (\lambda-3)\det \begin{bmatrix} 5-\lambda & -2 & 1 \\ 1 & -1 & 0 \\ -4 & 2 & -\lambda \end{bmatrix} \overset{C_1+C_2}{=} (\lambda-3)\det \begin{bmatrix} 5-\lambda & 3-\lambda & 1 \\ 1 & 0 & 0 \\ -4 & -2 & -\lambda \end{bmatrix}$$

$$= -(\lambda-3)\det \begin{bmatrix} 3-\lambda & 1 \\ -2 & -\lambda \end{bmatrix} = -(\lambda-3)(\lambda-1)(\lambda-2) = 0.$$

$\lambda = 1, \lambda = 2, \lambda = 3.$

$$V_{\lambda=1} = \{x \in \mathbb{R}^3 : Ax = 1x\} = \{x \in \mathbb{R}^3 : \begin{bmatrix} 5-1 & -2 & 1 \\ 2 & 1-1 & 1 \\ -4 & 2 & 0-1 \end{bmatrix} x = 0\}$$

$= < (-1,-1,2) > .$

$$V_{\lambda=2} = \{x \in \mathbb{R}^3 : Ax = 2x\} = \{x \in \mathbb{R}^3 : \begin{bmatrix} 5-2 & -2 & 1 \\ 2 & 1-2 & 1 \\ -4 & 2 & 0-2 \end{bmatrix} x = 0\}$$

$= < (-1,-1,1) > .$

$$V_{\lambda=3} = \{x \in \mathbb{R}^3 : Ax = 3x\} = \{x \in \mathbb{R}^3 : \begin{bmatrix} 5-3 & -2 & 1 \\ 2 & 1-3 & 1 \\ -4 & 2 & 0-3 \end{bmatrix} x = 0\}$$

$= < (-2,-1,2) > .$

2. Considere a matriz: $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

a) Calcule os valores próprios da matriz A.

$$det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{bmatrix} = (2-\lambda)\det \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}$$

$= (2-\lambda)(\lambda-2)(\lambda-3) = 0$

$\lambda = 2, \lambda = 2, \lambda = 3.$

b) Seja B a matriz obtida de A pela operação elementar: $l_3 \to l_3 - 2l_2$. Calcule os valores próprios de B e compare-os com os valores próprios de A.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 6 \end{bmatrix}$$

$\lambda = 2, \lambda = 1, \lambda = 6.$

3. Seja $\alpha \in \mathbb{R}$ e $A_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \alpha \\ 0 & 1 & 1 \end{bmatrix}.$

a) Determine α de modo que A_α admita o valor próprio 2.

$$det(A_\alpha - 2I) = \det \begin{bmatrix} 1-2 & 0 & 0 \\ 0 & \alpha-2 & \alpha \\ 0 & 1 & 1-2 \end{bmatrix} = 0$$

$\iff \; 2\alpha - 2 = 0.$

$\iff \alpha = 1.$

b) Mostre que o vector $(1,0,0)$ é vector próprio de A_α independentemente do valor de α.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \alpha \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

4. Seja A uma matriz quadrada de ordem n e x um vector próprio de A associado ao valor próprio λ. Mostre que:

   a) Dado $\alpha \in \mathbb{R}$, x é um vector próprio de αA associado ao valor próprio $\alpha\lambda$.

   $\alpha A x = (\alpha\lambda)x.$

   b) O vector x é um vector próprio de A^2 associado ao valor próprio λ^2.

   $A^2x = A(Ax) = A(\lambda x) = \lambda A x = \lambda\lambda x = \lambda^2 x.$

5. Dados $a, b \in \mathbb{R}$, considere a matriz real:

$$A_{a,b} = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 1 \\ 1 & 0 & a \end{bmatrix}$$

   a) Mostre que, para todos $a, b \in \mathbb{R}$, 1 é valor próprio de $A_{a,b}$.

$$\det(A_{a,b} - 1I) = \det \begin{bmatrix} 1-1 & 0 & b \\ 0 & 1-1 & 1 \\ 1 & 0 & a-1 \end{bmatrix} = 0.$$

   b) Determine os valores próprios de $A_{1,b}$.

$$\det(A_{1,b} - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & b \\ 0 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)\det \begin{bmatrix} 1-\lambda & b \\ 1 & 1-\lambda \end{bmatrix}$$

   $= (1-\lambda)(\lambda^2 - 2\lambda - b + 1) = 0.$

   $\lambda = 1, \lambda = 1 + \sqrt{b}, \lambda = 1 - \sqrt{b}.$

   c) Determine os valores próprios da matriz $A_{a,0}$.

$$\det(A_{a,0} - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & 0 & a-\lambda \end{bmatrix} = (1-\lambda)\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & a-\lambda \end{bmatrix}$$

   $= (1-\lambda)(1-\lambda)(a-\lambda) = 0$

   $\lambda = 1, \lambda = 1, \lambda = a.$

   d) Determine os vectores próprios da matriz $A_{1,0}$.

   $(A_{1,0} - 1I)x = 0 \iff \; x = \alpha(0,1,0), \alpha \in \mathbb{R} \setminus \{0\}$