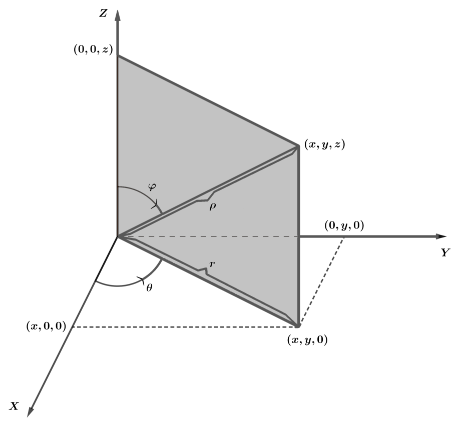
Coordenadas esféricas

Dado um ponto $(x, y, z) \in \mathbb{R}^3 \setminus (\{0\} \times \{0\} \times \mathbb{R})$, consideremos:

- $\rho = \sqrt{x^2 + y^2 + z^2}$, ou seja, ρ é a distância de (x, y, z) à origem;
- θ com o mesmo significado que nas coordenadas cilíndricas;
- φ é o ângulo que o semi-eixo positivo do eixo OZ faz com a semi-recta que une $(0, 0, 0)$ a (x, y, z). Note-se que $\varphi \in [0, \pi]$.



Temos assim a função

$$\begin{array}{rccc} \Phi: & \mathbb{R}_0^+ \times [0, 2\pi[\times [0, \pi] & \longrightarrow & \mathbb{R}^3. \\ & (r, \theta, \varphi) & \mapsto & (x, y, z) \end{array}$$

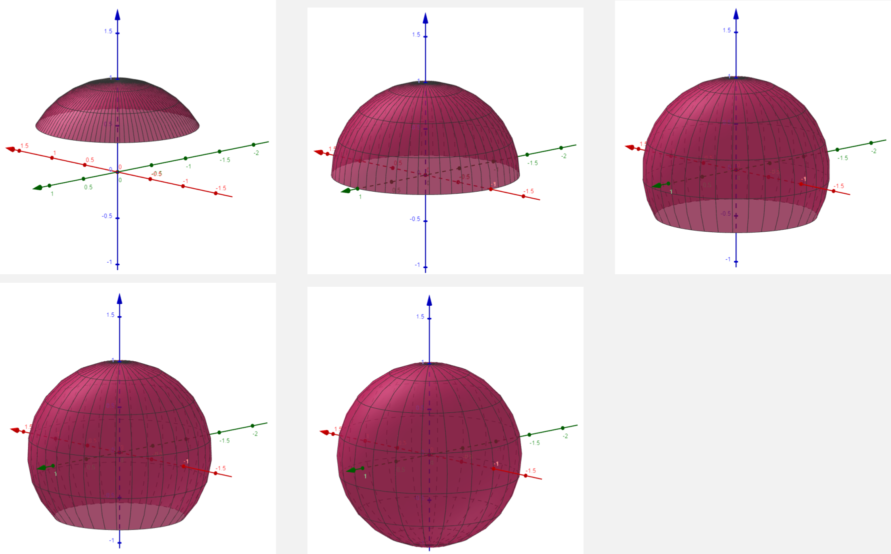
$$\text{Com} \begin{cases} x = \underbrace{\rho \operatorname{sen} \varphi}_{=r} \cos \theta \\ y = \underbrace{\rho \operatorname{sen} \varphi}_{=r} \operatorname{sen} \theta \\ z = \rho \cos \varphi \end{cases}$$

Coordenadas esféricas

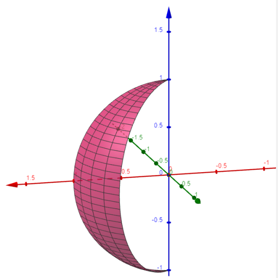
Vejamos algumas observações:

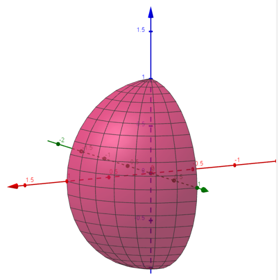
- como nas coordenadas polares e nas coordenadas cilíndricas, poderíamos ter considerado a variação de θ em qualquer intervalo de amplitude 2π;
- a restrição de Φ a $\mathbb{R}^+ \times]0, 2\pi[\times]0, \pi[$ é uma bijecção de classe C^1 sobre $\mathbb{R}^3 \setminus (\mathbb{R}_0^+ \times \{0\} \times \mathbb{R})$ (ou seja, o complementar do semiplano de equação $y = 0$, $x \geq 0$);
- para efeitos de cálculo de integrais, podemos "pensar" em Φ como uma mudança de variável em $\mathbb{R}^3$, uma vez que o conjunto $\mathbb{R}_0^+ \times \{0\} \times \mathbb{R}$ tem "volume zero".

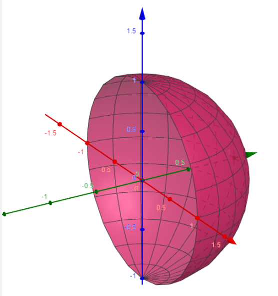
$\rho = \rho_0, \quad \varphi \in [0, \varphi_0]$ e $\quad \theta$ qualquer

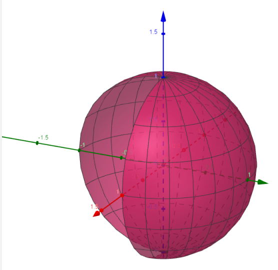


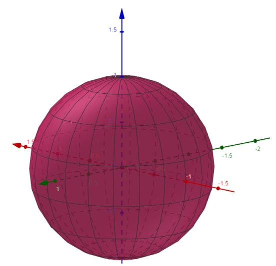
$\rho = \rho_0, \quad \theta \in [0, \theta_0]$ e $\quad \varphi$ qualquer





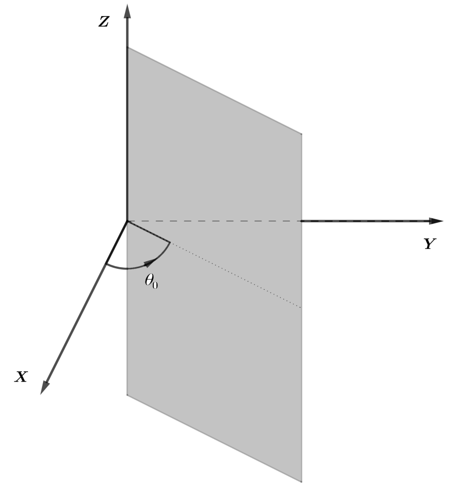






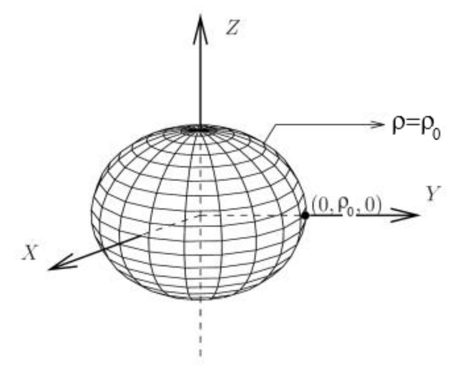
$$\boxed{\theta = \theta_0}$$

O conjunto dos pontos (x, y, z) "cujo" θ é igual a θ_0 (ou seja, o conjunto $\Phi(\mathbb{R}^+ \times \{\theta_0\} \times \mathbb{R})$) é o semi-plano



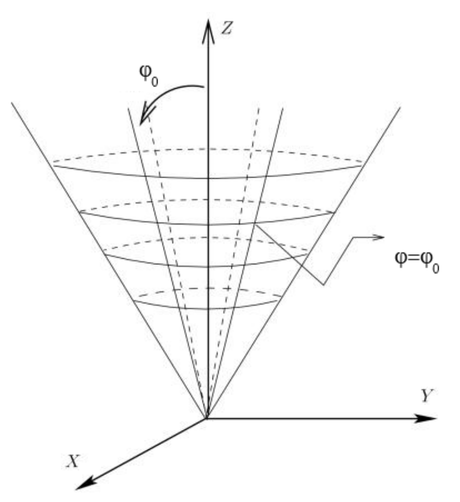
$\boxed{\rho = \rho_0}$

O conjunto dos pontos (x, y, z) "cujo" ρ é igual a ρ_0 (ou seja, o conjunto $\Phi([0, 2\pi[\times\{\rho_0\} \times \mathbb{R}))$ é a superfície esférica centrada na origem e de raio ρ_0

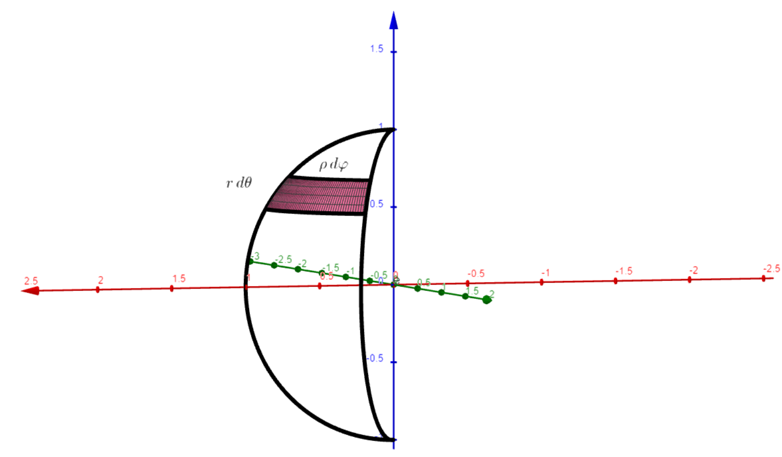


$$\boxed{\varphi = \varphi_0}$$

Se fixarmos $\varphi = \varphi_0$, obtemos o cone vertical infinito cujo vértice é a origem e em que o ângulo que a "altura" faz com a "geratriz" é φ_0

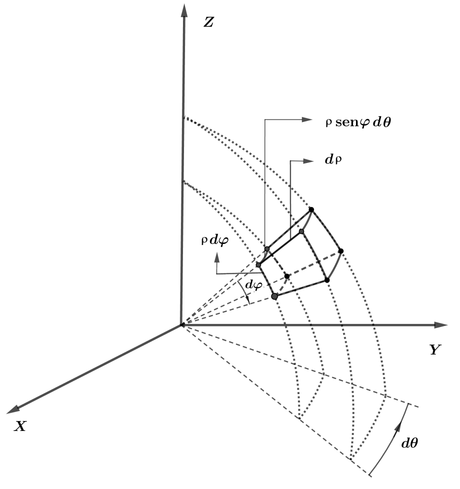


Unidade de volume (1)



Área da região sombreada $\approx \rho\, d\varphi\, r\, d\theta$.

Unidade de volume (2)



Unidade de volume $= \rho^2 \operatorname{sen} \varphi \, d\theta \, d\varphi \, d\rho$.

Unidade de volume (3)

$$J_{(\rho,\theta,\varphi)}\Phi = \begin{pmatrix} \operatorname{sen}\varphi\cos\theta & -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\cos\varphi\cos\theta \\ \operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta \\ \cos\varphi & 0 & -\rho\operatorname{sen}\varphi \end{pmatrix}$$

e $|\det J\,\Phi| = \rho^2\operatorname{sen}\varphi$.

Como nos casos anteriores, esta última observação permite-nos concluir que, se $\Phi(B) = A$ (sendo A e B conjuntos com volume) e $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ uma função integrável em B, então

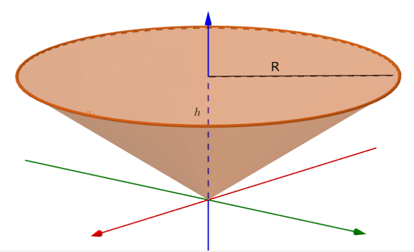
$$\iiint_A f(x,y,z)\,dx\,dy\,dz = \iiint_B \rho^2\operatorname{sen}\varphi\, f(\rho\operatorname{sen}\varphi\cos\theta, \rho\operatorname{sen}\varphi\operatorname{sen}\theta, \rho\cos\varphi)d\rho\,d\theta\,d\varphi.$$

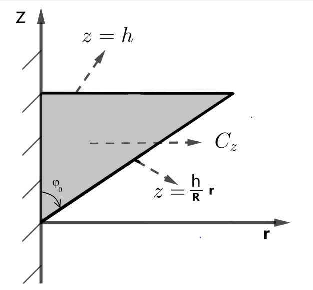
Volume da esfera

A esfera centrada na origem e raio R é definida em coordenadas esféricas por $\rho \leq R$.

$$\operatorname{vol}(S_R) = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \operatorname{sen} \varphi \, d\rho \, d\varphi \, d\theta = \frac{4}{3}\pi R^3.$$

Volume do cone





Em coordenadas esféricas temos $\rho \leq \frac{h}{\cos\varphi}$ e $0 \leq \varphi \leq \varphi_0$ (atenção!).

$$\begin{aligned}\mathrm{vol}(C_{h,\rho}) &= \int_0^{2\pi}\int_0^{\varphi_0}\int_0^{\frac{h}{\cos\varphi}} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta = 2\pi \int_0^{\varphi_0} \frac{h^3}{3\cos^3\varphi} \operatorname{sen}\varphi \, d\varphi \\ &= \frac{\pi h^3}{3}\left[\frac{1}{\cos^2(\varphi_0)} - 1\right] = \frac{\pi h^3}{3}\operatorname{tg}^2(\varphi_0) = \frac{1}{3}\pi R^2 h,\end{aligned}$$

uma vez que $\operatorname{tg}(\varphi_0) = \frac{R}{h}$.

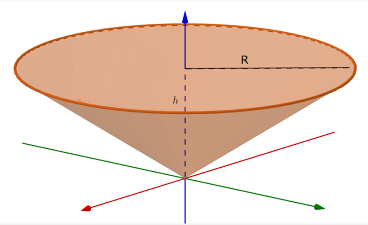
Método - coordenadas esféricas *versus* coordenadas cilíndricas

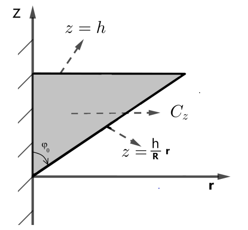
Calculamos a variação máxima de θ (se for possível).

- fazemos o desenho de S_θ no semi-plano OZr;
- trabalhamos neste semi-plano como se fossem coordenadas cartesianas em $\mathbb{R}^2$ e obtemos os limites de integração originais em coordenadas cilíndricas, **ou**;
- trabalhamos neste semi-plano como se fossem coordenadas polares com a excepção de que o ângulo considerado ser medido a começar no semi-eixo positivo OZ em vez de no semi-eixo Or e obtemos os limites de integração originais em coordenadas esféricas.

Note-se que conhecer φ é o mesmo que conhecer $\cos\varphi$ pois $\varphi \in [0, \pi]$.

Volume do cone $C_{h,R} = \{(x,y,z) \in \mathbb{R}^3 : \frac{h}{R}\sqrt{x^2+y^2} \leq z \leq h\}$





Em coordenadas:

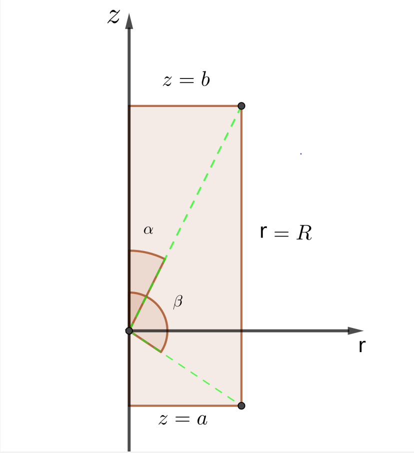
- cilíndricas temos $\frac{h}{R}r \leq z \leq h$;
- esféricas temos $0 \leq \theta \leq \theta_0$ e $\rho \leq \frac{h}{\cos\varphi}$, em que $\cos(\varphi_0) = \frac{h}{\sqrt{h^2+R^2}}$.

$$\mathrm{vol}(C_{h,R}) = \int_0^{2\pi}\int_0^{\varphi_0}\int_0^{\frac{h}{\cos\varphi}} r^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta = \tfrac{1}{3}\pi R^2 h.$$

$$\mathrm{vol}(C_{h,R}) = \int_0^{2\pi}\int_0^{R}\int_{\frac{h}{R}r}^{h} r \, dz \, dr \, d\theta = \tfrac{1}{3}\pi R^2 h.$$

Coordenadas esféricas: $x^2+y^2 \leq R^2$, $a \leq z \leq b$, com $a<0<b$ e $0<R$

Em coordenadas esféricas temos $\rho \operatorname{sen}\varphi \leq R$, $a \leq \rho\cos\varphi \leq b$. Deste modo $\varphi \in [0, 2\pi[$.

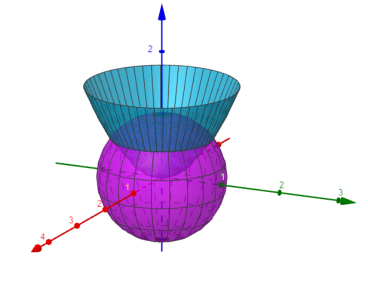


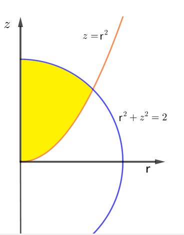
Note-se que $\operatorname{tg}(\alpha) = \frac{R}{b}$, $\operatorname{tg}\beta = \frac{R}{a}$, etc..
Olhando apenas para o desenho obtemos

$$\int_0^{2\pi}\int_0^{\alpha}\int_0^{\frac{b}{\cos\varphi}} \rho^2 \operatorname{sen}\varphi f(\cdots)\, d\rho\, d\varphi\, d\theta$$
$$+\int_0^{2\pi}\int_{\alpha}^{\beta}\int_0^{\frac{R}{\operatorname{sen}\varphi}} \rho^2 \operatorname{sen}\varphi f(\cdots)\, d\rho\, d\varphi\, d\theta$$
$$+\int_0^{2\pi}\int_{\beta}^{\pi}\int_0^{\frac{a}{\cos\varphi}} \rho^2 \operatorname{sen}\varphi f(\cdots)\, d\rho\, d\varphi\, d\theta$$

Experimente não usar a figura!

$S = \{(x,y,z) : x^2+y^2+z^2 \le 2,\ z \le x^2+y^2\}$



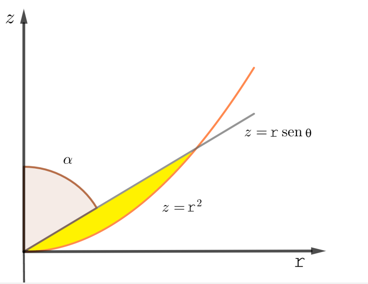


O ponto de intersecção das duas linhas do desenho da direita corresponde a $\rho = 1$ e $z = 1$ e daqui obtemos que.

$$\begin{aligned}
\mathrm{vol}(S) &= \int_0^{2\pi} \left(\int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\cos\varphi}{\operatorname{sen}^2\varphi}} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \right) d\theta \\
&= \cdots = 2\pi \left(\tfrac{2}{3}\sqrt{2} - \tfrac{7}{12} \right) \\
\mathrm{vol}(S) &= \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r \, dz \, dr \, d\theta.
\end{aligned}$$

$S = \{(x,y,z) : x^2 + y^2 \leq z \leq y\}$

S é definido por $r^2 \leq z \leq r \operatorname{sen}\theta$ e, portanto, o "desenho" vai depender de θ. É também claro que $\theta \in [0, \pi]$.

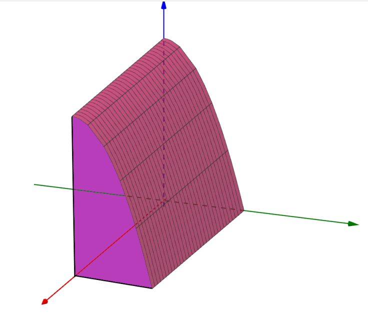


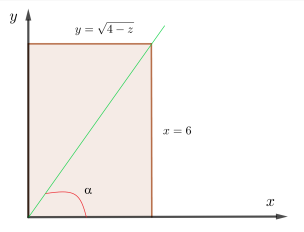
Note-se que $\cos\alpha = \frac{\operatorname{sen}\theta}{\sqrt{1+\operatorname{sen}^2\theta}}$ e $\operatorname{sen}\alpha = \frac{1}{\sqrt{1+\operatorname{sen}^2\theta}}$

$$\begin{aligned} \operatorname{vol}(S) &= \int_0^\pi \int_\alpha^{\frac{\pi}{2}} \int_0^{\frac{\cos\varphi}{\operatorname{sen}^2\varphi}} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta = \cdots = \tfrac{1}{32}\pi \\ \operatorname{vol}(S) &= \int_0^\pi \int_0^{\operatorname{sen}\theta} \int_{r^2}^{r\operatorname{sen}\theta} r \, dz \, dr \, d\theta. \end{aligned}$$

$B = \{(x,y,z) : x,y,z \geq 0,\ z \leq 4 - y^2,\ x \leq 6\}$ - cartesianas (z)

É claro que $0 \leq z \leq 4$.

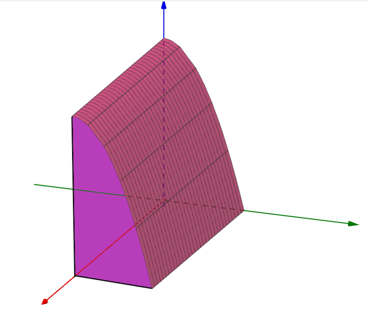


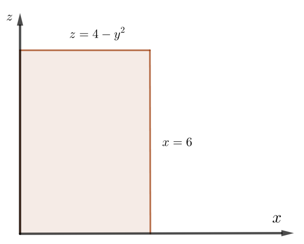


$$\mathrm{vol}(B) = \int_0^4 \int_0^6 \int_0^{\sqrt{4-z}} dy\, dx\, dz = \cdots = 32.$$

$B = \{(x,y,z) : x,y,z \geq 0,\ z \leq 4 - y^2,\ x \leq 6\}$ - cartesianas (y)

É claro que $0 \leq y \leq 2$.

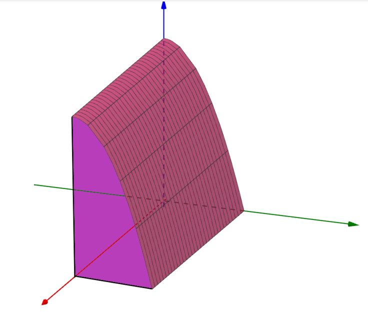


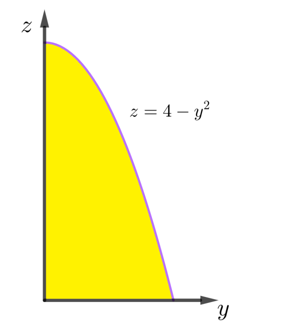


$$\mathrm{vol}(B) = \int_0^2 \int_0^6 \int_0^{4-y^2} dz\, dx\, dy = \cdots = 32.$$

$B = \{(x,y,z) : x,y,z \geq 0,\ z \leq 4 - y^2,\ x \leq 6\}$ - cartesianas (x)

É claro que $0 \leq x \leq 6$.

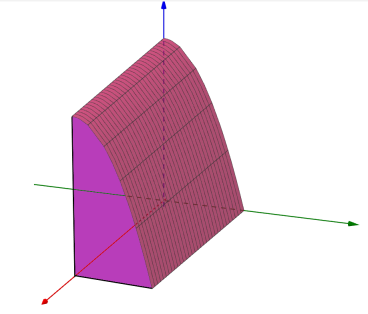


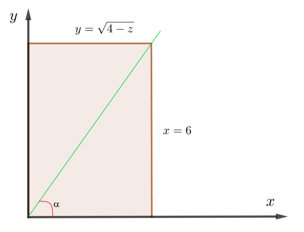


$$\mathrm{vol}(B) = \int_0^6 \int_0^2 \int_0^{4-y^2} dz\, dy\, dx = \cdots = 32.$$

$B = \{(x,y,z) : x,y,z \geq 0,\ z \leq 4-y^2,\ x \leq 6\}$ - cilíndricas (z)

É claro que $0 \leq z \leq 2$.

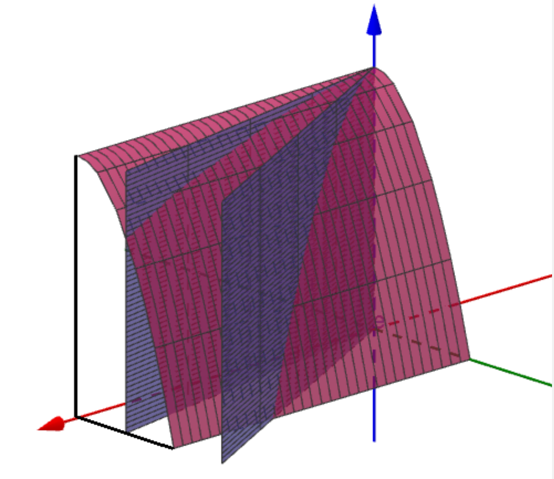




$$\operatorname{vol}(B) = \int_0^2 \left(\int_0^{\operatorname{arctg} \frac{\sqrt{4-z}}{6}} \int_0^{\frac{6}{\cos\theta}} r\, dr\, d\theta + \int_{\operatorname{arctg} \frac{\sqrt{4-z}}{6}}^{\frac{\pi}{2}} \int_0^{\frac{\sqrt{4-z}}{\operatorname{sen}\theta}} r\, dr\, d\theta \right) dz = \cdots = 32.$$

$B = \{(x, y, z) : x, y, z \geq 0,\ z \leq 4 - y^2,\ x \leq 6\}$ - cilíndricas (θ)

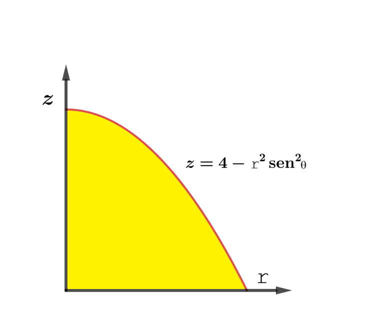
Temos $0 \leq z \leq 4 - r^2 \operatorname{sen}^2 \theta$, $r \cos\theta \leq 6$ e $0 \leq \theta \leq \frac{\pi}{2}$.

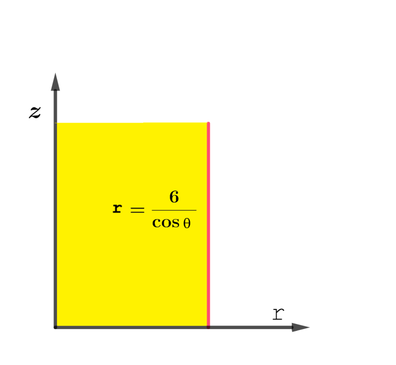


Separação:$\operatorname{arctg} \frac{1}{3}$.

$B = \{(x,y,z) : x,y,z \geq 0,\ z \leq 4 - y^2,\ x \leq 6\}$ - (θ) - continuação

Temos $0 \leq z \leq 4 - r^2 \operatorname{sen}^2 \theta$, $r \cos\theta \leq 6$ e $0 \leq \theta \leq \frac{\pi}{2}$.

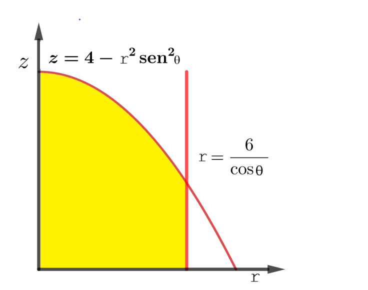


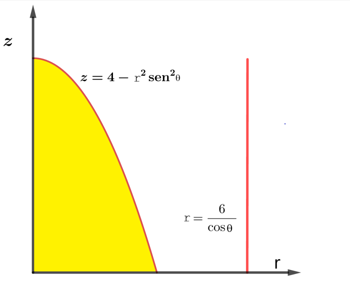


Separação: $\operatorname{arctg} \frac{1}{3}$.

$B=\{(x,y,z): x,y,z\geq 0,\ z\leq 4-y^2,\ x\leq 6\}$ - (θ) - continuação

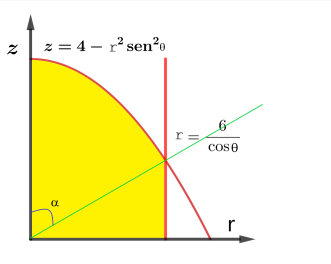
Temos $0 \leq z \leq 4 - r^2 \operatorname{sen}^2\theta$, $r\cos\theta \leq 6$ e $0 \leq \theta \leq \frac{\pi}{2}$.

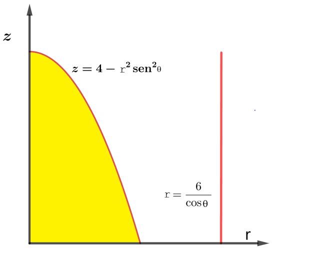




$$\operatorname{vol}(B) = \int_0^{\operatorname{arctg}\frac{1}{3}} \int_0^{\frac{6}{\cos\theta}} \int_0^{4-r^2\operatorname{sen}^2\theta} r\,dr\,dz\,d\theta + \int_{\operatorname{arctg}\frac{1}{3}}^{\frac{\pi}{2}} \int_0^{\frac{2}{\operatorname{sen}\theta}} \int_0^{4-r^2\operatorname{sen}^2\theta} r\,dr\,dz\,d\theta.$$

$B = \{(x,y,z) : x,y,z \geq 0,\ z \leq 4 - y^2,\ x \leq 6\}$ - esféricas





$$
\begin{aligned}
\mathrm{vol}(B) \quad = \quad & \int_0^{\operatorname{arctg}\frac{1}{3}} \int_0^{\operatorname{arctg}\frac{3\cos\theta}{20\cos^2\theta - 18}} \int_0^{\frac{-\cos\varphi + \sqrt{\cos^2\varphi + 16\,\mathrm{sen}^2\varphi\,\mathrm{sen}^2\theta}}{2\,\mathrm{sen}^2\varphi\,\mathrm{sen}^2\theta}} \rho^2\,\mathrm{sen}\,\varphi\, d\rho\, d\varphi\, d\theta \\
& + \int_0^{\operatorname{arctg}\frac{1}{3}} \int_{\operatorname{arctg}\frac{3\cos\theta}{20\cos^2\theta - 18}}^{\frac{\pi}{2}} \int_0^{\frac{6}{\cos\theta\,\mathrm{sen}\,\varphi}} \rho^2\,\mathrm{sen}\,\varphi\, d\rho\, d\varphi\, d\theta \\
& + \int_{\operatorname{arctg}\frac{1}{3}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{-\cos\varphi + \sqrt{\cos^2\varphi + 16\,\mathrm{sen}^2\varphi\,\mathrm{sen}^2\theta}}{2\,\mathrm{sen}^2\varphi\,\mathrm{sen}^2\theta}} \rho^2\,\mathrm{sen}\,\varphi\, d\rho\, d\varphi\, d\theta.
\end{aligned}
$$