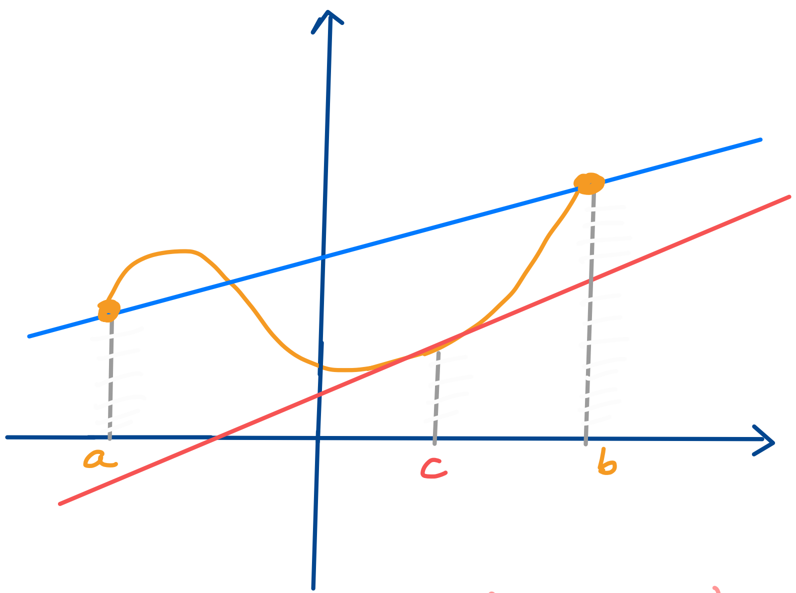
Cálculo EC - aula 5

25. Aplicando o teorema de Lagrange à função $f : [0; 0,1] \to \mathbb{R}$ dada por $f(t) = \ln(1+t)$, mostre que $0 < \ln(1,1) < 0,1$.

Teorema de Lagrange

Seja $f : [a,b] \longrightarrow \mathbb{R}$ uma função contínua em $[a,b]$ e derivável em $]a,b[$

então existe $c \in]a,b[$: $\dfrac{f(b)-f(a)}{b-a} = f'(c)$



$\dfrac{d}{dx} \ln(u(x)) = \dfrac{u'(x)}{u(x)}$

Tomamos $b = 0.1$ e $a = 0$

$\dfrac{f(b)-f(a)}{b-a} = \dfrac{\ln(1.1)-\ln(1)}{0.1} = 10 \ln(1.1)$

$\exists\, c \in]0, 0.1[: \; f'(c) = 10 \cdot \ln(1.1)$

$f'(c) = \dfrac{1}{1+c}$

Temos $0 < c < 0.1 \Rightarrow 1 < 1+c < 1.1$

$$\Rightarrow \frac{1}{1.1} < \frac{1}{1+c} < 1 \Rightarrow \frac{10}{11} < \underbrace{\frac{1}{1+c}}_{= f'(c)} < 1$$

Substituindo obtemos:

$$\frac{10}{11} < 10 \ln(1.1) < 1 \Rightarrow \frac{1}{11} < \ln(1.1) < \frac{1}{10}.$$

26. Calcule, se existirem, os seguintes limites:

$$(a)\ \lim_{x\to 0}\frac{1-\cos 2x}{\operatorname{tg} x}\quad (b)\ \lim_{x\to+\infty}\frac{2^x}{x}\quad (c)\ \lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}$$
$$(d)\ \lim_{x\to+\infty} xe^{-x^2+1}\quad (e)\ \lim_{x\to 0} x\ln x\quad (f)\ \lim_{x\to 0} x^x$$

a

$$\lim_{x\to 0}\frac{1-\cos(2x)}{\operatorname{tg}(x)} = \frac{0}{0} = \lim_{x\to 0}\frac{(1-\cos(2x))'}{(\operatorname{tg} x)'} =$$

Regra de L'Hôpital

$$= \lim_{x\to 0}\frac{2\operatorname{sen}(2x)}{\frac{1}{\cos^2(x)}} = \frac{0}{1} = 0$$

b

$$\lim_{x\to+\infty}\frac{2^x}{x} = \frac{\infty}{\infty} = \lim_{x\to+\infty}\frac{(2^x)'}{x'} = \lim_{x\to+\infty}\frac{\left(e^{(\ln 2)x}\right)'}{x'} =$$

regra de L' Hôpital

$$= \lim_{x\to+\infty}\frac{\ln(2)\,2^x}{1} = +\infty$$

c) $\lim_{x \to 0} \frac{e^{x} + e^{-x} - 2}{1 - \cos x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^{x} - e^{-x}}{\operatorname{sen}(x)} = \frac{0}{0} = \lim_{x \to 0} \frac{e^{x} + e^{-x}}{\cos(x)} = \frac{2}{1} = 2$

regra de L'Hôpital

d) $\lim_{x \to +\infty} x\, e^{-x^2+1} = \infty \cdot 0 = \lim_{x \to +\infty} \frac{x}{e^{x^2-1}} = \lim_{x \to +\infty} \frac{1}{2x\, e^{x^2-1}} = 0$

e) $\lim_{x \to 0} x \ln x = 0(-\infty) = \lim_{x \to 0} \frac{\ln(x)}{1/x} = \frac{\infty}{\infty} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} \frac{-x^2}{x} = \lim_{x \to 0} -x = 0$

f) $\lim_{x \to 0} x^{x} = 0^{0} = \lim_{x \to 0} e^{x \ln x} = e^{\lim_{x \to 0} x \ln x} = e^{0} = 1$

$f(x) = e^{x}$ é contínua

pela alínea e)

27. Considere a função $f :]-\pi,\pi[\to \mathbb{R}$ dada por

$$f(x) = \frac{1-\cos x}{\operatorname{sen} x} \quad \text{se} \quad x \neq 0 \quad \text{e} \quad f(0) = 0.$$

Mostre que f é derivável em 0 e que $f'(0) = \frac{1}{2}$.

Queremos calcular: $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ e verificar que é igual a $\frac{1}{2}$.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1 - \cos x}{x(\operatorname{sen} x)} = \frac{0}{0} =$$

$$= \lim_{x \to 0} \frac{\operatorname{sen} x}{\operatorname{sen} x + x \cos x} = \frac{0}{0} = \lim_{x \to 0} \frac{\cos x}{\cos x + \cos x - x \operatorname{sen} x} = \frac{1}{2}$$

Regra de L'Hôpital

Primitivas

Definição:

Seja $f: I \longrightarrow \mathbb{R}$ uma função, I intervalo de $\mathbb{R}$

$F: I \longrightarrow \mathbb{R}$ diz-se uma primitiva de f se $F' = f$.

Teorema

Se $f: I \longrightarrow \mathbb{R}$ é contínua então f tem primitiva.

Teorema

Seja $f: I \longrightarrow \mathbb{R}$ uma função primitivável e sejam F e $G: I \longrightarrow \mathbb{R}$ duas primitivas de f então

$$F(x) = G(x) + C \quad , \text{ para algum } C \in \mathbb{R}$$

Dem: $(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0$

$\Rightarrow F(x) - G(x) = C$, para algum $C \in \mathbb{R}$. (I intervalo)

Observação:

A função f' do exercício 20 não é contínua (como vimos) mas tem primitiva (obviamente a função f do enunciado).

29. Sabendo que $\operatorname{argsh}'(x) = \dfrac{1}{\sqrt{x^2+1}}$ e que $\operatorname{argsh}(0) = 0$, mostre que, para todo $x \in \mathbb{R}$, $\operatorname{argsh}(x) = \ln(x + \sqrt{x^2+1})$.

$$sh: \mathbb{R} \longrightarrow \mathbb{R} \quad \text{é bijetiva} \qquad sh(0) = 0$$
$$x \longmapsto \frac{e^{x} - e^{-x}}{2}$$

e a função inversa denota-se $\operatorname{argsh}: \mathbb{R} \longrightarrow \mathbb{R}$

$$\ln(x+\sqrt{x^2+1})' = \frac{(x+\sqrt{x^2+1})'}{(x+\sqrt{x^2+1})} = \frac{1+\frac{1}{2}2x(x^2+1)^{-1/2}}{x+\sqrt{x^2+1}} =$$
$$= \frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}} = \frac{\cancel{\sqrt{x^2+1}+x}}{\cancel{(x+\sqrt{x^2+1})}\,(\sqrt{x^2+1})} = \frac{1}{\sqrt{x^2+1}}$$

Fazendo $f(x) = \ln(x+\sqrt{x^2+1})$ então

$$f'(x) = \operatorname{argsh}'(x)$$

Logo $f(x) = \operatorname{argsh}(x) + \mathcal{C}$, $\quad \mathcal{C} \in \mathbb{R}$

Como $\operatorname{argsh}(0) = 0$ vem

$$f(0) = \operatorname{argsh}(0) + \mathcal{C} \iff \ln(1) = \mathcal{C} \iff \mathcal{C} = 0$$

Conclusão: $\operatorname{argsh}(x) = \ln(x+\sqrt{x^2+1})$

30. Determine as seguintes primitivas:

$$1)\int\left(x^2-4x+\frac{5}{x}\right)dx \quad 2)\int\frac{2x+1}{x^2+x+3}\,dx \quad 3)\int\frac{3}{2x-1}\,dx$$

$$4)\int\frac{1}{x}\cos(\ln x)\,dx \quad 5)\int\frac{\sqrt{1+2\ln x}}{x}\,dx \quad 6)\int \operatorname{sen} x\cos^4 x\,dx$$

1) $\displaystyle\int\left(x^2-4x+\frac{5}{x}\right)dx =$

$\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1}+\mathcal{C}$, $n\neq -1$

$\displaystyle= \int x^2\,dx - 4\int x\,dx + 5\int\frac{1}{x}\,dx =$

$\displaystyle\int\frac{1}{x}\,dx = \ln|x| + \mathcal{C}$

$\displaystyle= \frac{x^3}{3} - 4\,\frac{x^2}{2} + 5\ln|x| + \mathcal{C}, \quad \mathcal{C}\in\mathbb{R}$

$\displaystyle= \frac{x^3}{3} - 2x^2 + 5\ln|x| + \mathcal{C}, \ \mathcal{C}\in\mathbb{R}$

Verificação: $\displaystyle\left(\frac{x^3}{3}-2x^2+5\ln|x|+\mathcal{C}\right)' = \ \cdots \ = x^2-4x+\frac{5}{x}$

2) $\displaystyle\int \frac{2x+1}{x^2+x+3}\,dx = (*)$

$$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C \qquad C \in \mathbb{R}$$

$u(x) = x^2+x+3 \Rightarrow u'(x) = 2x+1$

$(*) = \ln|x^2+x+3| + C, \quad C \in \mathbb{R}$

3) $\displaystyle\int \frac{3}{2x-1}\,dx =$

$u(x) = 2x-1$
$u'(x) = 2$

$$= \frac{3}{2}\int \frac{2}{2x-1}\,dx = \frac{3}{2}\ln|2x-1| + C, \quad C \in \mathbb{R}$$

4) $\displaystyle\int \frac{1}{x}\cos(\ln x)\,dx$

$u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x}$

$$\int u'(x)\cos(u(x))\,dx = \text{sen}(u(x)) + C \qquad C \in \mathbb{R}$$

$= \text{sen}(\ln x) + C, \quad C \in \mathbb{R}$

$$\int u'(x)\,\text{sen}(u(x))\,dx = -\cos(u(x)) + C, \quad C \in \mathbb{R}.$$

5) $\displaystyle\int \frac{\sqrt{1+2\ln x}}{x}\,dx$

$$\int u'(x)\,u^n(x)\,dx = \frac{u^{n+1}(x)}{n+1} + \mathcal{C} \qquad \mathcal{C}\in\mathbb{R},\ n\neq -1$$

$$= \int \frac{1}{x}\,(1+2\ln(x))^{1/2}\,dx =$$

$u(x) = 1+2\ln(x) \Rightarrow u'(x) = \frac{2}{x}$

$$= \frac{1}{2}\int \underbrace{\frac{2}{x}}_{u'}\,\underbrace{(1+2\ln(x))^{1/2}}_{u^{1/2}}\,dx = \frac{1}{2}\,\frac{u(x)^{1/2+1}}{\frac{1}{2}+1} + \mathcal{C} = \frac{1}{2}\,\frac{(1+2\ln(x))^{3/2}}{3/2} + \mathcal{C}$$

$$= \frac{1}{3}\,(1+2\ln(x))^{3/2} + \mathcal{C}, \qquad \mathcal{C}\in\mathbb{R}.$$

6) $$\int \operatorname{sen}x\,\cos^4(x)\,dx = -\int \underbrace{-\operatorname{sen}(x)}_{u'}\,\underbrace{\cos^4(x)}_{u^4}\,dx = -\frac{u^5(x)}{5} + \mathcal{C} =$$

$u(x) = \cos(x) \Rightarrow u'(x) = -\operatorname{sen}(x)$

$$= -\frac{\cos^5(x)}{5} + \mathcal{C} \qquad \mathcal{C}\in\mathbb{R}$$