Resolução Folha 6

1 a) $\iiint_D (x+y+z)\, d(x,y,z) = \int_0^2\int_0^2\int_0^2 (x+y+z)\,dx\,dy\,dz$

$= \int_0^2\int_0^2 \left[\frac{x^2}{2} + yx + zx\right]_0^2 dy\,dz = \int_0^2\int_0^2 (2+2y+2z-0)\,dy\,dz$

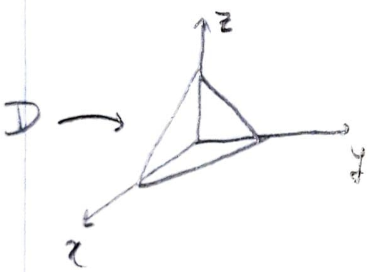
$= \int_0^2 \left[2y + y^2 + 2zy\right]_0^2 dz = \int_0^2 (4+4+4z-0)\,dz = \left[8z+2z^2\right]_0^2 = 16+8=24$

b) $\iiint_D z e^{x+y}\, d(x,y,z) = \int_0^1\int_0^1\int_0^1 z e^{x+y}\,dz\,dy\,dx = \int_0^1\int_0^1 \left[\frac{z^2}{2} e^{x+y}\right]_0^1 dy\,dx$

$= \int_0^1\int_0^1 \left(\frac{1}{2} e^x . e^y - 0\right) dy\,dx = \int_0^1 \left[\frac{1}{2} e^x e^y\right]_0^1 dx = \int_0^1 \left(\frac{1}{2} e^x . e - \frac{1}{2} e^x . e^0\right) dx$

$= \left[\left(\frac{e}{2} - \frac{1}{2}\right) e^x\right]_0^1 = \left(\frac{e}{2} - \frac{1}{2}\right) e - \left(\frac{e}{2} - \frac{1}{2}\right) e^0 = \frac{e-1}{2}\,(e-1)$

c) $D = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$



Vamos resolver o exercício imaginando que desconhecemos a região D.

Passo 1: variação total de uma das variáveis, escolhemos x, por exemplo:

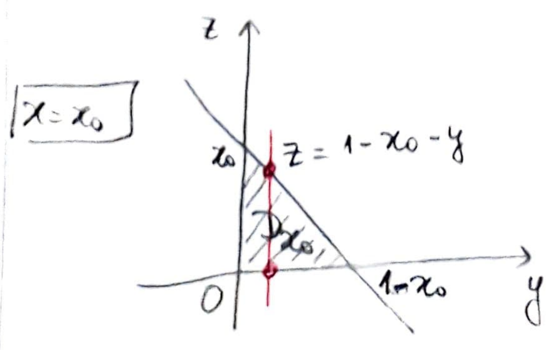
- menor valor possível de x: 0
- maior valor possível de x: quando $y = z = 0$ e, então $x \leq 1$

Logo $\boxed{0 \leq x \leq 1}$

Passo 2: Fixemos $x = x_0$ e façamos um corte da região D pelo plano $x = x_0$:

$D_{x_0} = D \cap \{(x_0, y, z) : y, z \in \mathbb{R}\} = \{(x_0,y,z) : y \geq 0, z \geq 0, y+z \leq 1 - x_0\}$ (paralelo ao plano OYZ)

Desenhemos D_{x_0} no plano $x = x_0$



Variação do y: $0 \leq y \leq 1 - x_0$

Variação do z: $0 \leq z \leq 1 - x_0 - y$

Pensamos agora em x, em vez de x_0, pois fixar $x=x_0$ foi meramente auxiliar. Então

$$\iiint_D xy\,d(x,y) = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[xyz\right]_0^{1-x-y} dy\,dx$$

$$=\int_0^1\int_0^{1-x}\left(xy(1-x-y)-0\right)dy\,dx = \int_0^1\int_0^{1-x}\left(xy-x^2y-xy^2\right)dy\,dx$$

$$=\int_0^1\left[x\frac{y^2}{2}-x^2\frac{y^2}{2}-x\frac{y^3}{3}\right]_0^{1-x}dx = \int_0^1\left(\frac{x(1-x)^2}{2}-\frac{x^2(1-x)^2}{2}-\frac{x(1-x)^3}{3}-0\right)dx$$

$= \dots$

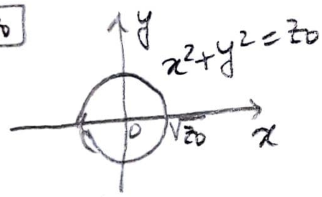
d) $D=\{(x,y,z)\in\mathbb{R}^3: 0\leq z\leq 3,\ x^2+y^2\leq z\}$

Variação de z: $0\leq z\leq 3$

Corte por $z=z_0$:

$D_{z_0}=\{(x,y,z_0)\in\mathbb{R}^3: x^2+y^2\leq z_0\}$

$\boxed{z=z_0}$



Então temos

Variação de x: $-\sqrt{z_0}\leq x\leq\sqrt{z_0}$

Variação de y: $-\sqrt{z_0-x^2}\leq y\leq\sqrt{z_0-y^2}$

$$\iiint_D x\,d(x,y,z) = \int_0^3\int_{-\sqrt{z}}^{\sqrt{z}}\int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} x\,dy\,dx\,dz$$

$$=\int_0^3\int_{-\sqrt{z}}^{\sqrt{z}}\left[xy\right]_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}}dx\,dz = \int_0^3\int_{-\sqrt{z}}^{\sqrt{z}}\left(x\sqrt{z-x^2}+x\sqrt{z-x^2}\right)dx\,dz$$

$$=\int_0^3\int_{-\sqrt{z}}^{\sqrt{z}} 2x(z-x^2)^{-1/2}\,dx\,dz = -\int_0^3\int_{-\sqrt{z}}^{\sqrt{z}} -2x(z-x^2)^{-1/2}\,dx\,dz$$

$$=-\int_0^3\left[\frac{(z-x^2)^{3/2}}{3/2}\right]_{-\sqrt{z}}^{\sqrt{z}}dz = -\int_0^3(0-0)\,dz = 0$$

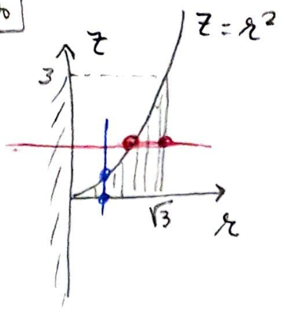
<u>Vejamos este integral em coordenadas cilíndricas</u>

$$\begin{cases}0\leq z\leq 3\\ 0\leq x^2+y^2\leq z\end{cases}\qquad\begin{cases}0\leq z\leq 3\\ 0\leq r^2\leq z\end{cases}$$

Então θ varia entre 0 e 2π (nenhuma desigualdade depende de θ)

E $0\leq z\leq 3$, $0\leq r\leq\sqrt{z}$. Neste caso nem vale a pena fazer o esboço do corte pelo semi-plano $\theta=\theta_0$. Mas vou fazê-lo

$\theta = \theta_0$



Este desenho elucida-nos sobre a forma do domínio D, que se obtém rodando a região a tracejado em torno do eixo Oz ③

$$\iiint_D x\, d(x,y,z) = \int_0^{2\pi}\int_0^3\int_{\sqrt{z}}^{\sqrt{3}} r \cdot r\cos\theta \, dr\, dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 \left[\frac{r^3}{3}\cos\theta\right]_{\sqrt{z}}^{\sqrt{3}} dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 \frac{1}{3}\left(3\sqrt{3}\cos\theta - \frac{z^{3/2}}{3}\cos\theta\right) dz\, d\theta$$

$$= \int_0^{2\pi}\left[\sqrt{3}\cos\theta\, z - \frac{1}{9}\frac{z^{5/2}}{5/2}\cos\theta\right]_0^3 d\theta$$

$$= \int_0^{2\pi}\left(3\sqrt{3}\cos\theta - \frac{2\cdot\cancel{3^2}\sqrt{3}}{5\cdot\cancel{9}}\cos\theta\right) d\theta$$

$$= \left[\left(3 - \frac{2}{5}\right)\sqrt{3}\,\mathrm{sen}\,\theta\right]_0^{2\pi} = 0$$

$$\| $$

$$\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{r^2} r\cdot r\cos\theta\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\left[r^2\cos\theta\, z\right]_0^{r^2} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} r^4\cos\theta\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^5}{5}\cos\theta\right]_0^{\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} \frac{9\sqrt{3}}{5}\cos\theta\, d\theta$$

$$= \left[\frac{9\sqrt{3}}{5}\,\mathrm{sen}\,\theta\right]_0^{2\pi} = 0$$

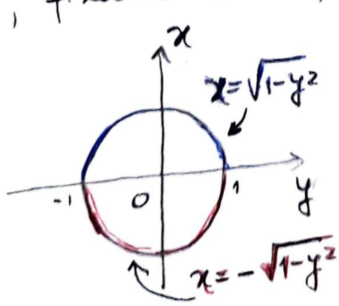
---

② $$\int_0^1\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} z(x^2+y^2)\, dx\, dy\, dz = (*)$$

O domínio de integração é definido pelas desigualdades seguintes:

$0 \leq z \leq 1$, $-1 \leq y \leq 1$, $-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}$

Então, fixado $z = z_0$, temos nesse plano o conjunto



(atenção à posição dos eixos OX e OY)

Então temos

$0 \leq z \leq 1$, $0 \leq x^2 + y^2 \leq 1$

e, em coordenadas polares temos

$0 \le z \le 1, \quad 0 \le r^2 \le 1$

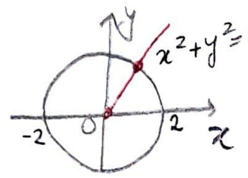
donde se conclui que $0 \le \theta \le 2\pi$, uma vez que não há restrições a θ.

$$(*) = \int_0^{2\pi}\int_0^1\int_0^1 r.z.r^2\,dr\,dz\,d\theta = \int_0^{2\pi}\int_0^1 \left[z\frac{r^4}{4}\right]_0^1 dz\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 \frac{z}{4}\,dz\,d\theta = \int_0^{2\pi}\left[\frac{z^2}{2}\right]_0^1 d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \frac{1}{2}\left[\theta\right]_0^{2\pi} = \pi$$

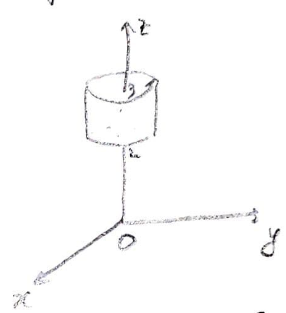
③ $D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 4,\ 2 \le z \le 3\}$

Fixado $z_0 \in [2,3]$, desenhe-se $D_{z_0} = D \cap \{(x,y,z_0) : x,y \in \mathbb{R}\}$



$\boxed{\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{matrix}}$

O conjunto D é o cilindro



$$\iiint_D z e^{x^2+y^2}\,d(x,y,z)$$

$$= \int_2^3\int_0^{2\pi}\int_0^2 r z e^{r^2}\,dr\,d\theta\,dz$$

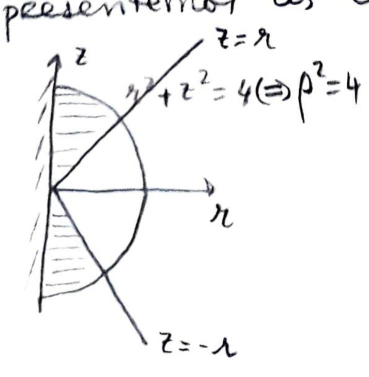
$$= \int_2^3\int_0^{2\pi}\left[z\frac{e^{r^2}}{2}\right]_0^2 d\theta\,dz = \int_2^3\int_0^{2\pi} z\left(\frac{e^4}{2}-\frac{e^0}{2}\right)d\theta\,dz = \frac{e^4-1}{2}\int_0^3 \left[z\theta\right]_0^{2\pi} dz$$

$$= \frac{e^4-1}{2}\int_0^3 2\pi z\,dz = \frac{e^4-1}{2}\,\pi\left[z^2\right]_0^3 = \frac{9}{2}\pi(e^4-1)$$

④ Para resolver um integral em coordenadas esféricas prefiro, se possível, visualizar o conjunto usando coordenadas cilíndricas.

$$\begin{cases} z^2 = x^2+y^2 \\ x^2+y^2+z^2 = 4 \end{cases} \quad \begin{cases} r^2 = z^2 \\ r^2+z^2 = 4 \end{cases} \quad \begin{cases} z = r \lor z = -r \\ r^2+z^2 = 4 \end{cases}$$

Representemos as linhas no semi-plano $\theta = \theta_0$



O nosso sólido obtém-se por rotação da região a tracejado em torno do eixo Oz

Notem que:

- $0 \le \theta \le 2\pi$ (é um sólido de revolução)
- olhando só para a parte acima do plano Oxy, temos
  $0 \le \varphi \le \pi/4$ (a outra parte do domínio corresponde a $\frac{3\pi}{4} \le \varphi \le \frac{\pi}{2}$)
- $0 \le \rho \le 2$

Volume $(D) = 2\iiint_{D^+} d(x,y,z)$, sendo $D^+ = D \cap \{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0\}$ (5)

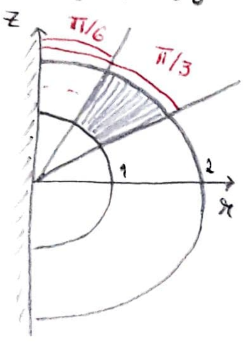
Volume $(D) = 2\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} \rho^2 \operatorname{sen}\varphi \, d\rho\, d\varphi\, d\theta = 2\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^2 \operatorname{sen}\varphi\, d\varphi\, d\theta$

$= 2\cdot\frac{8}{3}\int_0^{2\pi}\left[-\cos\varphi\right]_0^{\pi/4} d\theta = \frac{16}{3}\int_0^{2\pi}\left[-\frac{\sqrt{2}}{2}+1\right] d\theta = \frac{16}{3}\frac{(2-\sqrt{2})}{2}\left[\theta\right]_0^{2\pi}$

$= \frac{16(2-\sqrt{2})\pi}{3}$

5-a) Em coordenadas esféricas temos $\begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ \pi/6 \leq \varphi \leq \pi/3 \end{cases}$

Façamos a representação deste domínio quando $\theta = \theta_0$



O sólido em questão consiste no sólido de revolução obtido por rotação em torno do eixo OZ da região a tracejado.

b) $\iiint_D e^{(x^2+y^2+z^2)^{3/2}} d(x,y,z) = \int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_1^2 e^{(\rho^2)^{3/2}} \rho^2 \operatorname{sen}\varphi\, d\rho\, d\varphi\, d\theta$

$= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \frac{1}{3}\operatorname{sen}\varphi \cdot 3\rho^2 e^{\rho^3} d\rho\, d\varphi\, d\theta = \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \frac{1}{3}\operatorname{sen}\varphi \left[e^{\rho^3}\right]_1^2 d\varphi\, d\theta$

$= \frac{e^8-e}{3}\int_0^{2\pi}\left[-\cos\varphi\right]_{\pi/6}^{\pi/3} d\theta = \frac{e^8-e}{3}\int_0^{2\pi}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right) d\theta = \frac{(e^8-e)(\sqrt{3}-1)}{3\cdot 2}\left[\theta\right]_0^{2\pi}$

$= \frac{(e^8-e)(\sqrt{3}-1)}{3}$

(6) $D = \{(x,y,z) \in \mathbb{R}^3 : z \leq 4-y^2,\ x \geq 0, y \geq 0, z \geq 0, x \leq 6\}$

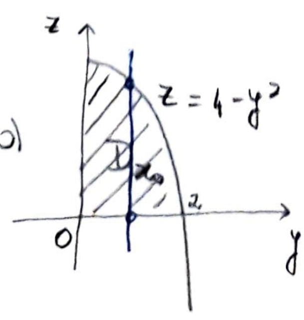
Coordenadas cartesianas

Este integral é demasiado difícil em coordenadas cilíndricas e em coordenadas esféricas, mas apresento aqui a resolução, para vosso conhecimento

- Variação de x: $0 \leq x \leq 6$
- Corte por $x = x_0$ $\quad D_{x_0} = \{(x_0, y, z) : z \leq 4-y^2,\ y \geq 0, z \geq 0\}$

$\boxed{x = x_0}$

$O = (x_0, 0, 0)$



- Variação de y: $0 \leq y \leq 2$
- Variação de z: $0 \leq z \leq 4-y^2$

Volume $(D) = \int_0^6\int_0^2\int_0^{4-y^2} dz\, dy\, dx$

$= \int_0^6\int_0^2 \left[z\right]_0^{4-y^2} dy\, dx = \int_0^6\int_0^2 (4-y^2)\, dy\, dx$

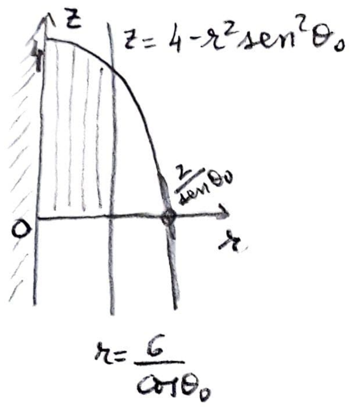
$= \int_0^6 \left[4y - \frac{y^3}{3}\right]_0^2 dx = \int_0^6 \left(8-\frac{8}{3}\right) dx = \left[\frac{16}{3}x\right]_0^6 = 32$

Coordenadas cilíndricas (VER SLIDES - penúltima página)

$$\begin{cases} x \geq 0 \\ x \leq 6 \\ y \geq 0 \\ z \geq 0 \\ z \leq 4 - y^2 \end{cases} \qquad \begin{cases} r\cos\theta \geq 0 \\ r\cos\theta \leq 6 \\ r\,\text{sen}\,\theta \geq 0 \\ z \geq 0 \\ z \leq 4 - r^2\,\text{sen}^2\theta \end{cases} \qquad \begin{matrix} \cancel{r}\cos\theta \geq 0 \\ \cancel{r}\,\text{sen}\,\theta \geq 0 \end{matrix} \Bigg| \Rightarrow \theta \in [0, \pi/2]$$

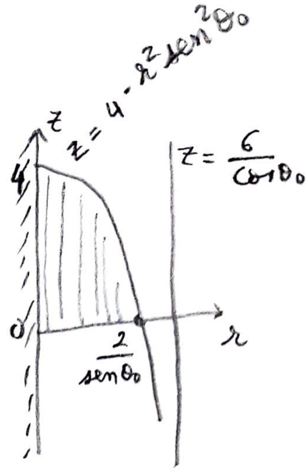
Corte por $\theta = \theta_0$

(I)



Quando $\dfrac{6}{\cos\theta_0} \leq \dfrac{2}{\text{sen}\,\theta_0}$, ou seja, $\text{tg}\,\theta_0 \leq \frac{1}{3}$

(II)



Quando $\dfrac{6}{\cos\theta_0} > \dfrac{2}{\text{sen}\,\theta_0}$, ou seja $\text{tg}\,\theta_0 > 1/3$

$$\text{Volume}(D) = \int_0^{\text{arctg}\frac{1}{3}} \int_0^{\frac{6}{\cos\theta}} \int_0^{4 - r^2\text{sen}^2\theta} r\,dz\,dr\,d\theta + \int_{\text{arctg}\frac{1}{3}}^{\pi/2} \int_0^{\frac{2}{\text{sen}\,\theta}} \int_0^{4 - r^2\text{sen}^2\theta} r\,dz\,dr\,d\theta$$

$$= \int_0^{\text{arctg}\frac{1}{3}} \int_0^{\frac{6}{\cos\theta}} \Big[rz\Big]_0^{4 - r^2\text{sen}^2\theta} dr\,d\theta + \int_{\text{arctg}\frac{1}{3}}^{\pi/2} \int_0^{\frac{2}{\text{sen}\,\theta}} \Big[rz\Big]_0^{4 - r^2\text{sen}^2\theta} dr\,d\theta$$

$$= \int_0^{\text{arctg}\frac{1}{3}} \int_0^{\frac{6}{\cos\theta}} (4r - r^3\text{sen}^2\theta)\,dr\,d\theta + \int_{\text{arctg}\frac{1}{3}}^{\pi/2} \int_0^{\frac{2}{\text{sen}\,\theta}} (4r - r^3\text{sen}^2\theta)\,dr\,d\theta$$

$$= \int_0^{\text{arctg}\frac{1}{3}} \left[2r^2 - \frac{r^4}{4}\text{sen}^2\theta\right]_0^{\frac{6}{\cos\theta}} d\theta + \int_{\text{arctg}1/3}^{\pi/2} \left[2r^2 - \frac{r^4}{4}\text{sen}^2\theta\right]_0^{\frac{2}{\text{sen}\,\theta}} d\theta$$

$$= \int_0^{\text{arctg}1/3} \left(\frac{72}{\cos^2\theta} - 324\frac{\text{sen}^2\theta}{\cos^4\theta}\right)d\theta + \int_{\text{arctg}1/3}^{\pi/2} \left(\frac{8}{\text{sen}^2\theta} - \frac{4}{\text{sen}^2\theta}\right)d\theta = (*)$$

Recordem que:

$\sec\theta = \dfrac{1}{\cos\theta}$, $(\text{tg}\,\theta)' = \sec^2\theta$, $\sec^2\theta = 1 + \text{tg}^2\theta$, $\text{cotg}\,\theta = \dfrac{1}{\text{tg}\,\theta} = \dfrac{\cos\theta}{\text{sen}\,\theta}$

$\text{cosec}\,\theta = \dfrac{1}{\text{sen}\,\theta}$, $(\text{cotg}\,\theta)' = -\text{cosec}^2\theta$

$$(*) = \int_0^{\text{arctg}1/3} 72\sec^2\theta\,d\theta - \int_0^{\text{arctg}1/3} 324\,\text{tg}^2\theta\sec^2\theta\,d\theta + \int_{\text{arctg}1/3}^{\pi/2} 4\,\text{cosec}^2\theta\,d\theta$$

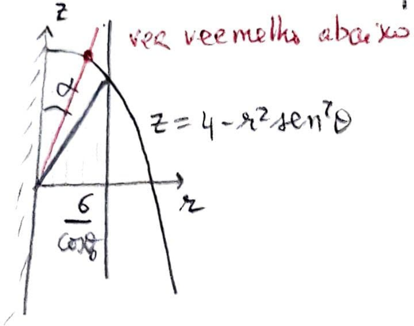
$$= \left[72\,tg\,\theta\right]_0^{arctg\,1/3} - \left[324\,\frac{tg^3\theta}{3}\right]_0^{arctg\,1/3} - \left[cotg\,\theta\right]_0^{arctg\,1/3}$$

$$= 72\,tg(arctg\,1/3) - 108\left(tg(arctg\,1/3)\right)^3 + \frac{1}{tg(arctg\,1/3)} = \frac{72}{3} - \frac{108}{3^3} +$$

$= 32$

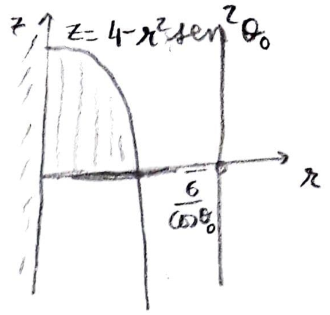
Coordenadas esféricas: vou repetir o desenho (em cilíndricas) da folha anterior

(I)



$tg\,\theta_0 \leq 1/3$

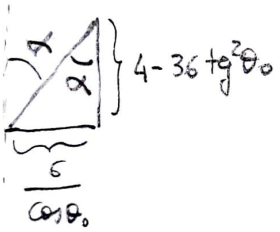
(II)



$tg\,\theta_0 > 1/3$

(VER SLIDES – última página!)

$\boxed{\text{Em (I)}}$



$$\begin{cases} z = 4 - r^2 sen^2\theta_0 \\ r = \frac{6}{\cos\theta_0} \end{cases} \quad \begin{cases} z = 4 - 36\,\frac{sen^2\theta_0}{\cos^2\theta_0} \\ \ldots \end{cases} \quad \begin{cases} z = 4 - 36\,tg^2 \\ \end{cases}$$

$tg\,\alpha = \frac{\text{medida do cateto oposto}}{\text{medida do cateto adjacente}}$

$$tg\,\alpha = \frac{\frac{6}{\cos\theta_0}}{4 - 36\,tg^2\theta_0} = \frac{6}{\cos\theta_0\,(4 - 36\,tg^2\theta_0)} = \frac{3\cos\theta_0}{2\cos^2\theta_0 - 18\,sen^2\theta_0}$$

$$= \frac{3\cos\theta_0}{2\cos^2\theta_0 - 18(1 - \cos^2\theta_0)} = \frac{3\cos\theta_0}{20\cos^2\theta_0 - 18}$$

Então $\alpha = arctg\,\frac{3\cos\theta_0}{20\cos^2\theta_0 - 18}$

Recordem que
$\begin{cases} z = \rho\cos\varphi_0 \\ r = \rho\,sen\,\varphi_0 \end{cases}$

- $0 \leq \varphi_0 \leq \alpha$

$0 \leq \rho \leq$ equação "$z = 4 - r^2 sen^2\theta_0$" em esféricas

$$z = 4 - r^2 sen^2\theta_0 \iff \rho\cos\varphi_0 = 4 - (\rho\,sen\,\varphi_0)^2 sen^2\theta_0$$

$$\iff \rho^2 sen^2\varphi_0\, sen^2\theta_0 + \rho\cos\varphi_0 - 4 = 0$$

$$\iff \rho = \frac{-\cos\varphi_0 \pm \sqrt{\cos^2\varphi_0 + 16\,sen^2\varphi_0\,sen^2\theta_0}}{2\,sen^2\varphi_0\,sen^2\theta_0}$$

Devemos escolher o sinal + antes da raiz pois $\rho \geq 0$

- $\alpha \leq \varphi_0 \leq \pi/2$

$0 \leq \rho \leq$ equação " $r = \frac{6}{\cos\theta_0}$ " em esféricas

$$r = \frac{6}{\cos\theta_0} \iff \rho \operatorname{sen}\varphi_0 = \frac{6}{\cos\theta_0} \iff \rho = \frac{6}{\cos\theta_0 \operatorname{sen}\varphi_0}$$

$\boxed{\text{Em (II)}}$

- $0 \leq \varphi_0 \leq \pi/2$
- $0 \leq \rho \leq \dfrac{-\cos\varphi_0 + \sqrt{\cos^2\varphi_0 + 16\operatorname{sen}^2\varphi_0 \operatorname{sen}^2\theta_0}}{2\operatorname{sen}^2\varphi_0 \operatorname{sen}^2\theta_0}$

Então

$$\text{Volume}(D) = \iiint_D 1 \, d(x,y,z) = \iiint_{\phi^{-1}(D)} \rho^2 \operatorname{sen}\varphi \, d(\rho,\theta,\varphi)$$

$\boxed{\phi \text{ mudança de variável para esféricas}}$

$$= \int_0^{\operatorname{arctg} 1/3} \int_0^{\operatorname{arctg} \frac{3\cos\theta}{20\cos^2\theta - 18}} \int_0^{\frac{-\cos\varphi + \sqrt{\cos^2\varphi + 16\operatorname{sen}^2\varphi \operatorname{sen}^2\theta}}{2\operatorname{sen}^2\varphi \operatorname{sen}^2\theta}} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta$$

$$+ \int_0^{\operatorname{arctg} 1/3} \int_{\operatorname{arctg} \frac{3\cos\theta}{20\cos^2\theta - 18}}^{\pi/2} \int_0^{\frac{6}{\cos\theta \operatorname{sen}\varphi}} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta$$

$$+ \int_{\operatorname{arctg} 1/3}^{\pi/2} \int_0^{\pi/2} \int_0^{\frac{-\cos\varphi + \sqrt{\cos^2\varphi + 16\operatorname{sen}^2\varphi \operatorname{sen}^2\theta}}{2\operatorname{sen}^2\varphi \operatorname{sen}^2\theta}} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta$$

UFA! Como é evidente, não vou calcular o integral!

7 a) $x^2+y^2+z^2=9$ e $x^2+y^2+z^2=3$

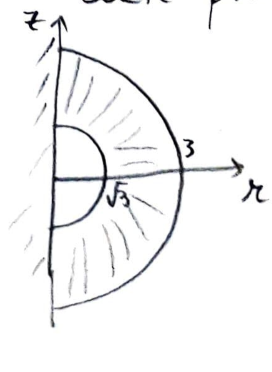
É "evidente" que $D = \{(x,y,z) \in \mathbb{R}^3 : 3 \leq x^2+y^2+z^2 \leq 9\}$

Mas, se não temos a certeza, passemos para coordenadas cilíndricas

$$x^2+y^2+z^2=9 \iff r^2+z^2=9$$
$$x^2+y^2+z^2=3 \iff r^2+z^2=3$$

Como θ não intervém, então $0 \leq \theta \leq 2\pi$

Corte por $\theta = \theta_0$



Então está confirmado

$$\begin{cases} x^2+y^2+z^2 \leq 9 \\ x^2+y^2+z^2 \geq 3 \end{cases} \quad \begin{cases} \rho^2 \leq 9 \\ \rho^2 \geq 3 \end{cases} \quad \begin{cases} \rho \leq 3 \\ \rho \geq \sqrt{3} \end{cases} \quad \text{(esféricas)}$$

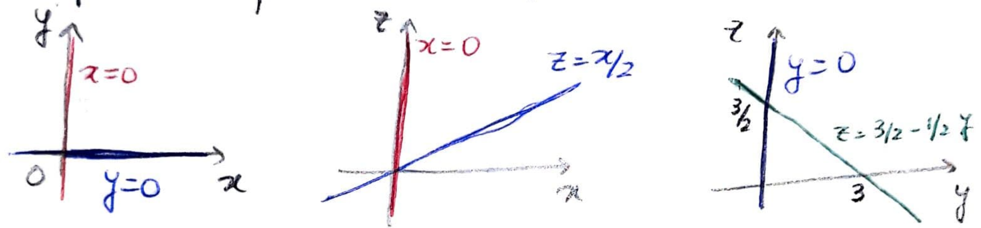
$$\text{Volume}(D) = \iiint_D 1\, d(x,y,z) = \int_0^{2\pi}\int_0^{\pi}\int_{\sqrt{3}}^{3} \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^3}{3} \operatorname{sen}\varphi\right]_{\sqrt{3}}^{3} d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi} (9-\sqrt{3}) \operatorname{sen}\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \left[-(9-\sqrt{3})\cos\varphi\right]_0^{\pi} d\theta = 2(9-\sqrt{3}) \cdot 2\pi = 4\pi(9-\sqrt{3})$$

b) $x=0,\ y=0,\ x=2z,\ y+2z=3$

O que se passa nos 3 planos coordenados?



Parece que devemos considerar $x \geq 0, y \geq 0, z \geq \frac{x}{2}, z \leq \frac{3}{2} - \frac{1}{2}y$

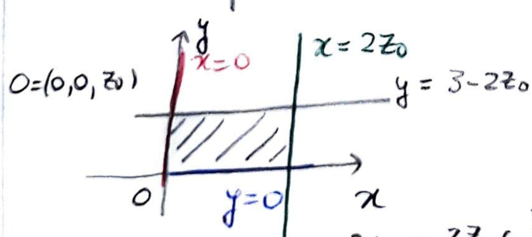
Vejamos:

- Variação total de z:

  $z \geq 0$ (porque $z \geq \frac{x}{2}$ e $x \geq 0$)

  $z \leq \frac{3}{2}$ (porque $z \leq \frac{3}{2} - \frac{1}{2}y$ e $y \geq 0$)

- Corte por $z = z_0$



$$\text{Volume}(D) = \int_0^{3/2}\int_0^{2z}\int_0^{3-2z} dy\,dx\,dz = \int_0^{3/2}\int_0^{2z} [y]_0^{3-2z} dx\,dz$$

$$= \int_0^{3/2}\int_0^{2z} (3-2z)\,dx\,dz = \int_0^{3/2} [(3-2z)x]_0^{2z} dz = \int_0^{3/2} (6z-4z^2)\,dz = \left[3z^2 - \frac{4z^3}{3}\right]_0^{3/2} \ldots$$

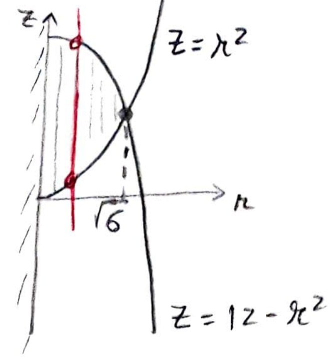
c) $z = x^2+y^2$ e $z = 12-x^2-y^2$

$z = x^2+y^2 \Leftrightarrow z = r^2$

$z = 12-x^2-y^2 \Leftrightarrow z = 12-r^2$

Como θ não aparece nas equações, temos $0 \leq \theta \leq 2\pi$

Corte por $\theta = \theta_0$



$D = \{(x,y,z) \in \mathbb{R}^3 : z \geq x^2+y^2, z \leq 12 - x^2 - y^2\}$

$\begin{cases} z = r^2 \\ z = 12 - r^2 \end{cases} \begin{cases} - \\ r^2 = 12 - r^2 \end{cases} \begin{cases} - \\ r^2 = 6 \end{cases} \begin{cases} - \\ r = \sqrt{6} \end{cases}$

$$\text{Volume}(D) = \iiint_D 1 \, d(x,y,z) = \int_0^{2\pi} \int_0^{\sqrt{6}} \int_{r^2}^{12-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{6}} [rz]_{r^2}^{12-r^2} dr d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{6}} (r(12 - r^2) - r^3) \, dr \, d\theta = \int_0^{2\pi} \int_0^{\sqrt{6}} (12r - 2r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[6r^2 - \frac{r^4}{2}\right]_0^{\sqrt{6}} d\theta = [18\theta]_0^{2\pi} = 36\pi$$

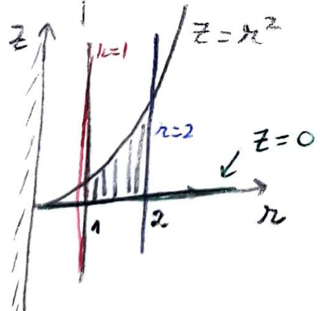
d) $z = 0$, $z = x^2+y^2$, $x^2+y^2 = 1$, $x^2+y^2 = 4$

Em coordenadas cilíndricas

$z = 0 \iff z = 0$ $\quad x^2+y^2 = 1 \iff r^2 = 1 \iff r = 1$

$z = x^2+y^2 \iff z = r^2$ $\quad x^2+y^2 = 4 \iff r^2 = 4 \iff r = 2$

- Variação de θ: $0 \leq \theta \leq 2\pi$
- Corte por $\theta = \theta_0$



$D = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0, x^2+y^2 \leq z, 1 \leq x^2+y^2 \leq 4\}$

$$\text{Volume}(D) = \iiint_D 1 \, d(x,y,z)$$

$$= \int_0^{2\pi} \int_1^2 \int_0^{r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_1^2 [rz]_0^{r^2} dr d\theta$$

$$= \int_0^{2\pi} \int_1^2 r^3 \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^4}{4}\right]_1^2 d\theta = \int_0^{2\pi} \left(4 - \frac{1}{4}\right) d\theta = \left[\frac{15}{4}\theta\right]_0^{2\pi} = \frac{15}{2}\pi$$

(8) $D = \{(x,y,z) \in \mathbb{R}^3 : x, y \geq 0, \sqrt{x+y} + 1 \leq z \leq 2\}$

$$\phi : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$(u, v, w) \longmapsto (u^2, v^2, w)$$

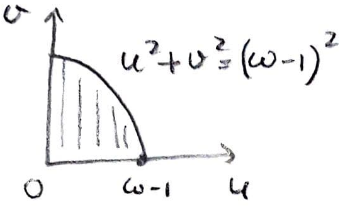
$$J_{(u,v,w)}\phi = \begin{pmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$|\det J_{(u,v,w)} \phi| = 4uv$

$$\begin{cases} x = u^2 \\ y = v^2 \\ w = z \end{cases} \qquad \begin{cases} x \geq 0 \\ y \geq 0 \\ \sqrt{x+y} + 1 \leq z \\ z \leq 2 \end{cases} \iff \begin{cases} u^2 \geq 0 \\ v^2 \geq 0 \\ \sqrt{u^2+v^2} + 1 \leq w \\ w \leq 2 \end{cases} \quad \begin{cases} u \geq 0 \\ v \geq 0 \quad (\text{ver } D\phi) \\ u^2+v^2 \leq (w-1)^2 \\ w \leq 2 \end{cases}$$

- Variação de w: $1 \leq w \leq 2$ (pois $w \geq 1 + \sqrt{u^2+v^2} \geq 1$ pois $u=0$ e $v=0$ ~~são~~ permitidos)

- Corte por $w = w_0$



Recordem que $u \geq 0$ e $v \geq 0$

$$\iiint_D \frac{1}{\sqrt{xy}}\, d(x,y,z) = \int_1^2 \int_0^{w-1} \int_0^{\sqrt{(w-1)^2-u^2}} \frac{4uv}{\sqrt{u^2v^2}}\, dv\, du\, dw$$

$$= \int_1^2 \int_0^{w-1} \int_0^{\sqrt{(w-1)^2-u^2}} \frac{4\cancel{uv}}{\cancel{uv}}\, dv\, du\, dw = (*)$$

Notem que estou a integrar num domínio em que o corte por $w = w_0$ é um quarto de círculo, com a função integrando 4. É mais fácil passar para cilíndricas. Já sabemos que $1 \leq w \leq 2$. Olhando para a figura acima, temos $0 \leq \theta \leq \pi/2$ e $0 \leq r \leq \sqrt{(w-1)^2} = w-1$

$$(*) = \int_1^2 \int_0^{\pi/2} \int_0^{w-1} r\, dr\, d\theta\, dw = \int_1^2 \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{w-1} d\theta\, dw$$

$$= \int_1^2 \int_0^{\pi/2} \frac{(w-1)^2}{2}\, d\theta\, dw = \int_1^2 \left[\frac{(w-1)^2}{2}\theta\right]_0^{\pi/2} = \frac{\pi}{4}\left[\frac{(w-1)^3}{3}\right]_1^2$$

$$= \frac{\pi}{12}$$