Soluções da Folha 7

1 a) $c'(t) = (\cos t, -\operatorname{sen} t, 1)$ $\|c'(t)\| = \sqrt{\cos^2 t + \operatorname{sen}^2 t + 1} = \sqrt{2}$

$$\int_c f\,ds = \int_0^{2\pi} f(\operatorname{sen} t, \cos t, t)\,\|c'(t)\|\,dt = \sqrt{2}\int_0^{2\pi} (\operatorname{sen} t + \cos t + t)\,dt$$

$$= \sqrt{2}\left[-\cos t + \operatorname{sen} t + \frac{t^2}{2}\right]_0^{2\pi} = \sqrt{2}\left[\frac{4\pi^2}{2}\right] = 2\sqrt{2}\pi^2$$

b) $$\int_c f\,ds = \int_0^{2\pi} f(\cos t, \operatorname{sen} t, t)\sqrt{(-\operatorname{sen} t)^2 + \cos^2 t + 1}\,dt$$

$$= \int_0^{2\pi} \cos t \cdot \sqrt{2}\,dt = \sqrt{2}\left[\operatorname{sen} t\right]_0^{2\pi} = 0$$

c) $c'(t) = (0,0,2t)$, $\|c'(t)\| = \sqrt{4t^2} = 2t$ (Recordem que $t \in [0,1]$)

$$\int_c f\,ds = \int_0^1 e^{\sqrt{t^2}} \cdot 2t\,dt = 2\int_0^1 te^t\,dt = [te^t]_0^1 - \int_0^1 e^t\,dt = e - [e^t]_0^1$$

$u = t$ $u' = 1$
$v' = e^t$ $v = e^t$

$= e - e + 1 = 1$

d) $c'(t) = (1,3,2)$ $\|c'(t)\| = \sqrt{1+9+4} = \sqrt{14}$

$$\int_c f\,ds = \int_1^3 3t.\,2t.\,\sqrt{14} = 6\sqrt{14}\int_1^3 t\,dt = 6\sqrt{14}\left[\frac{t^2}{2}\right]_1^3 = 3\sqrt{14}(9-1) = 24\sqrt{14}$$

2 $\gamma : [1,2] \longrightarrow \mathbb{R}^2$ $\gamma'(t) = (1, \frac{1}{t})$ $\|\gamma'(t)\| = \sqrt{1 + \frac{1}{t^2}}$

$t \longmapsto (t, \ln t)$

$$L(\gamma) = \int_1^2 \|\gamma'(t)\|\,dt = \int_1^2 \left(1 + \frac{1}{t^2}\right)^{1/2} dt = \int_{1^2}^2 \tfrac{1}{2}\cdot 2t\,(1+t^2)^{1/2}\,dt$$

$$= \frac{1}{2}\left[(1+t^2)^{3/2}\right]_1^2 = \frac{1}{2}\left(5^{3/2} - 2^{3/2}\right)$$

3 $c'(t) = (2t, 1, 0)$ $\|c'(t)\| = \sqrt{4t^2+1}$

$$L(c) = \int_0^1 \|c'(t)\|\,dt = \int_0^1 \sqrt{1+4t^2}\,dt = \int_0^{\operatorname{arctg}(2)} \sqrt{1+\operatorname{tg}^2\theta}\,\frac{\sec^2\theta\,d\theta}{2}$$

$t=0 \Rightarrow \theta = 0$ | $2t = \operatorname{tg}\theta$
$t=1 \Rightarrow \theta = \operatorname{arctg}(2)$ | $2dt = \sec^2\theta$

$$= \frac{1}{2}\int_0^{\operatorname{arctg}(2)} \sec^3\theta\,d\theta = \frac{1}{2}\int_0^{\operatorname{arctg}(2)} \frac{1}{\cos^3\theta}\,d\theta$$

E não vale a pena perder tempo com o cálculo deste integral.

(4) $c(t) = (\cos t, \operatorname{sen} t)$

Resolução 1

$F(x,y) = (-y, x)$

$c'(t) = (-\operatorname{sen} t, \cos t)$

$$\int_C x\,dy - y\,dx = \int_0^{2\pi} F(c(t)) \cdot c'(t)\,dt$$

$$= \int_0^{2\pi} (-\operatorname{sen} t, \cos t) \cdot (-\operatorname{sen} t, \cos t)\,dt = \int_0^{2\pi} (\operatorname{sen}^2 t + \cos^2 t)\,dt$$

$$= [t]_0^{2\pi} = 2\pi$$

Resolução 2

$$\int_C x\,dy - y\,dx =$$

$$= \int_0^{2\pi} \cos t \cdot \cos t\,dt - \operatorname{sen} t(-\operatorname{sen} t\,dt)$$

$$= \int_0^{2\pi} 1\,dt = [t]_0^{2\pi} = 2\pi$$

$c(t) = (\cos t, \operatorname{sen} t)$, $x(t) = \cos t$, $y(t) = \operatorname{sen} t$

$x'(t)\,dt = -\operatorname{sen} t\,dt$ $\quad$ $y'(t)\,dt = \cos t\,dt$

ou

$dx = -\operatorname{sen} t\,dt$ $\quad$ $dy = \cos t\,dt$

b) $x(t) = \cos(\pi t)$ $\quad$ $y(t) = \operatorname{sen}(\pi t)$

$dx(t) = -\pi \operatorname{sen}(\pi t)\,dt$ $\quad$ $dy(t) = \pi \cos(\pi t)\,dt$

$$\int_C x\,dx + y\,dy = \int_0^2 -\pi \cos(\pi t)\operatorname{sen}(\pi t)\,dt + \pi \operatorname{sen}(\pi t)\cos(\pi t)\,dt = 0$$

c) $\int_C yz\,dx + xz\,dy + xy\,dz$

$F(x,y,z) = (yz, xz, xy)$

$$\int_C F \cdot ds = \int_0^1 F(C_1(t)) \cdot C_1'(t) + \int_0^1 F(C_2(t)) \cdot C_2'(t)$$

$C = C_1 \cup C_2$

$C_1(t) = (1,0,0) + t\,((0,1,0) - (1,0,0))$, $t \in [0,1]$

$= (1-t, t, 0)$

$C_2(t) = (0,1,0) + t\,((0,0,1) - (0,1,0))$, $t \in [0,1]$

$= (0, 1-t, t)$

$$= \int_0^1 F(1-t, t, 0) \cdot (-1,1,0)\,dt + \int_0^1 F(0, 1-t, t) \cdot (0,-1,1)\,dt$$

$$= \int_0^1 (0, 0, (1-t)t) \cdot (-1,1,0)\,dt + \int_0^1 ((1-t)t, 0, 0) \cdot (0,-1,1)\,dt = 0$$

ou

O campo de vectores F é conservativo porque

$$\frac{\partial F_1}{\partial y} = z = \frac{\partial F_2}{\partial x} \qquad \frac{\partial F_1}{\partial z} = y = \frac{\partial F_3}{\partial x} = y \qquad \frac{\partial F_2}{\partial z} = x = \frac{\partial F_3}{\partial y},$$

sendo $F = (F_1, F_2, F_3)$

Então existe $f: \mathbb{R}^3 \to \mathbb{R}$ tal que $\nabla f = F$. É fácil verificar que $f(x,y,z) = xyz$. Assim,

$$\int_C F \cdot ds = \int_C \nabla f \cdot ds = f(0,0,1) - f(1,0,0) = 0$$

d) $F(x,y,z) = (x^2, -xy, 1)$ Este campo de vectores não é conservativo. ③

$C: [-1,1] \longrightarrow \mathbb{R}^3$

$t \longmapsto (t, 0, t^2)$

$x(t) = t$ $\quad y(t)=0$ $\quad z(t) = t^2$

$dx(t) = dt$ $\quad dy(t) = 0dt$ $\quad dz(t) = 2tdt$

$$\int_C F.ds = \int_{-1}^{1} x^2dx - xydy + dz = \int_{-1}^{1} t^2dt - 0 + 2tdt = \left[\frac{t^3}{3} + t^2\right]_{-1}^{1}$$

$$= \frac{1}{3} + 1 - \frac{-1}{3} - 1 = \frac{2}{3}$$

⑤ a) $\nabla f(x,y,z) = (e^y\cos(\pi z), xe^y\cos(\pi z), -\pi xe^y sen(\pi z))$

b) $\int_C F.ds = f(c(\pi)) - f(c(0)) = f(3,0,0) - f(3,0,0) = 0$

Se calculássemos directamente daria muitos cálculos:

$$\int_C F.ds = \int e^y\cos(\pi z)\,dx + xe^y\cos(\pi z)\,dy - \pi xe^y sen(\pi z)\,dz$$

$x(t) = 3\cos^4 t$ $\quad y(t) = 5\,sen^7 t$ $\quad z(t) = 0$

$dx(t) = 12\cos^3 t\,(-sen t)dt$ $\quad dy(t) = 35\,sen^6 t\cos t\,dt$ $\quad dz(t) = 0dt$

$$= \int_0^{\pi} -e^{5sen^7 t}\cos(0).\,12\cos^3 t\,sen t\,dt + \int_0^{\pi} 3\cos^4 t\, e^{5sen^7 t}\cos(0)\,35 sen^6 t\cos t\,dt + 0$$

$= \dots$ etc.

⑥ $F(x,y,z) = (y^2, 2xy + e^{3z}, 3ye^{3z})$

a) Para verificar que F é campo de gradientes podemos proceder de 2 maneiras:

(i) Verificar que $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$ e $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

$\frac{\partial F_1}{\partial y} = 2y = \frac{\partial F_2}{\partial x}$; $\quad \frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}$

$\frac{\partial F_2}{\partial z} = 3e^{3z} = \frac{\partial F_3}{\partial y}$

(ii) Encontrar $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ verificando $\nabla f = F$

$\frac{\partial f}{\partial x} = y^2 \Rightarrow f(x,y,z) = xy^2 + g(y,z) \Rightarrow \frac{\partial f}{\partial y} = 2xy + \frac{\partial g}{\partial y} = 2xy + e^{3z}$

$\Rightarrow \frac{\partial g}{\partial y} = e^{3z} \Rightarrow g(y,z) = ye^{3z} + h(z) \Rightarrow f(x,y,z) = xy^2 + ye^{3z} + h(z)$

$\Rightarrow \frac{\partial f}{\partial z} = 3ye^{3z} + h'(z) = 3ye^{3z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C,\ C \in \mathbb{R}$

Então $f(x,y,z) = xy^2 + ye^{3z} + C$, $C \in \mathbb{R}$ é tal que $\nabla f = F$

(Tomamos $C = 0$)

b) Resolução 1: Sabendo que o campo F é conservativo, podemos escolher uma parametrização de uma qualquer curva que una (1,0,1) a (0,1,0), uma vez que o integral não depende do caminho.

Seja $c(t) = (1,0,1) + t\,((0,1,0) - (1,0,1))$, $t \in [0,1]$

$$= (1,0,1) + t(-1,1,-1)$$

$$= (1-t,\, t,\, 1-t) \quad \text{(parametrização do segmento)}$$

Então

$$\int_C F \cdot ds = \int_C y^2 dx + (2xy + e^{3z})dy + 3ye^{3z}dz$$

$$= \int_0^1 t^2(-dt) + (2(1-t)t + e^{3(1-t)})dt + 3te^{3(1-t)}(-dt)$$

$$= \int_0^1 (-t^2 + 2t - 2t^2 + e^{3(1-t)} - 3t\, e^{3(1-t)})\, dt$$

$$= \int_0^1 (-3t^2 + 2t + (1-3t)e^{3(1-t)})dt = ⊛$$

e este integral dá algum trabalho a calcular, por causa da última parcela...

Resolução 2

$$\int_C F \cdot ds = f(0,1,0) - f(1,0,1) = (0 + e^0) - (0+0) = 1$$

Decidi resolver o integral... (mais umas continhas)

$$⊛ = \left[-t^3 + t^2\right]_0^1 + \int_0^1 \underbrace{(1-3t)}_{u}\, \underbrace{e^{3(1-t)}}_{v'} =$$

$$u = 1-3t \qquad u' = -3$$

$$v' = e^{3(1-t)} \qquad v = -\frac{1}{3}e^{3(1-t)}$$

$$= (-1+1) - 0 + \left[-\frac{1}{3}(1-3t)e^{3(1-t)}\right]_0^1 + \int_0^1 e^{3(1-t)}dt$$

$$= -\frac{1}{3}\cdot(-2)e^0 + \frac{1}{3}e^3 - \left[\frac{1}{3}e^{3(1-t)}\right]_0^1 = -\frac{2}{3} + \frac{1}{3}e^3 - \frac{1}{3}e^0 + \frac{1}{3}e^3$$

Então, se $f(x,y) = x^2y + x\,\text{sen}\,y + C$, $C \in \mathbb{R}$, temos que (5)
$\nabla f = F$, pelo que F é um campo de vectores conservativo.

b) $G(x,y) = \left(\dfrac{x-2y}{(x^2+y^2+1)^{1/2}} , \dfrac{x-2}{(x^2+y^2+1)^{1/2}} \right)$

$$\frac{\partial G_1}{\partial y} = \frac{-2(x^2+y^2+1)^{1/2} - (x-2y)\frac{1}{2}(x^2+y^2+1)^{-1/2}2y}{x^2+y^2+1}$$

$$= \frac{-2(x^2+y^2+1)^{1/2} - (x-2y)y(x^2+y^2+1)^{-1/2}}{x^2+y^2+1}$$

$$\frac{\partial G_2}{\partial x} = \frac{(x^2+y^2+1)^{1/2} - (x-2)\frac{1}{2}(x^2+y^2+1)^{-1/2}2x}{x^2+y^2+1}$$

$$\frac{\partial G_1}{\partial y} = \frac{\partial G_2}{\partial x} \Leftrightarrow -2(x^2+y^2+1)^{1/2} - (x-2y)y(x^2+y^2+1)^{-1/2}$$
$$= (x^2+y^2+1)^{1/2} - (x-2)x(x^2+y^2+1)^{-1/2},$$

uma vez que os denominadores são iguais. Substituindo no ponto $(2,1)$, obtemos $-2\sqrt{6} = \sqrt{6}$, o que é falso. Então G não é conservativo

Alternativamente, queremos verificar se existe g tal que $\nabla g = G$, ou seja

$$\frac{\partial g}{\partial x} = G_1 \quad e \quad \frac{\partial g}{\partial y} = G_2$$

Como podem verificar, os cálculos que vou apresentar são bastante complexos

$$\frac{\partial g}{\partial y} = \frac{x-2}{\sqrt{1+x^2+y^2}} = \frac{x-2}{\sqrt{(1+x^2)\left(1+\frac{y^2}{1+x^2}\right)}} = \frac{x-2}{\sqrt{1+x^2}\sqrt{1+\left(\frac{y}{\sqrt{1+x^2}}\right)^2}}$$

$$= (x-2)\frac{\frac{1}{\sqrt{1+x^2}}}{\sqrt{1+\left(\frac{y}{\sqrt{1+x^2}}\right)^2}} \Rightarrow g(x,y) = (x-2)\,\text{arg sh}\left(\frac{y}{\sqrt{1+x^2}}\right) + h(x)$$

$$\Rightarrow \frac{\partial g}{\partial x} = \text{arg sh}\left(\frac{y}{(1+x^2)^{1/2}}\right) + (x-2)\frac{-\frac{1}{2}y(1+x^2)^{-3/2}2x}{\sqrt{1+\left(\frac{y}{\sqrt{1+x^2}}\right)^2}} + h'(x)$$

$$= \frac{x-2y}{\sqrt{1+x^2+y^2}}$$ e é extremamente complicado verificar que esta igualdade é falsa!
Resumindo: devemos usar a 1ª resolução

(8) $P(x,y) = x$, $Q(x,y) = xy$ $\quad D = D((0,0),1)$

$$\int_{C^+} P\,dx + Q\,dy \qquad C^+(t) = (\cos t, \operatorname{sen} t),\ t\in[0,2\pi]$$

$$= \int_{C^+} x\,dx + xy\,dy = \int_0^{2\pi} \cos t(-\operatorname{sen} t\,dt) + \cos t \operatorname{sen} t \cos t\,dt$$

$$= \int_0^{2\pi} -\operatorname{sen} t \cos t\,dt + \int_0^{2\pi} \operatorname{sen} t \cos^2 t\,dt$$

$$= \left[\frac{\cos^2 t}{2}\right]_0^{2\pi} - \left[\frac{\cos^3 t}{3}\right]_0^{2\pi} = 0$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d(x,y) = \iint_D y\,d(x,y) = \int_0^{2\pi}\int_0^1 r^2 \operatorname{sen}\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^1 \operatorname{sen}\theta\,d\theta = \frac{1}{3}\left[-\cos\theta\right]_0^{2\pi} = 0$$

(9) $P(x,y) = \dfrac{-y}{x^2+y^2}$, $Q(x,y) = \dfrac{x}{x^2+y^2}$

a) $$\frac{\partial P}{\partial y} = \frac{-(x^2+y^2) + y\cdot 2y}{x^2+y^2} = \frac{-x^2+y^2}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{x^2+y^2 - x\cdot 2x}{x^2+y^2} = \frac{-x^2+y^2}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \equiv 0$$

b) $$\int_C F\cdot ds = \int_C P\,dx + Q\,dy \qquad C(t) = (\cos t, \operatorname{sen} t),\ t\in[0,2\pi]$$

$$= \int_0^{2\pi} \frac{-\operatorname{sen} t}{\cos^2 t + \operatorname{sen}^2 t}(-\operatorname{sen} t\,dt) + \frac{\cos t}{\operatorname{sen}^2 t + \cos^2 t}\cos t\,dt$$

$$= \int_0^{2\pi} (\operatorname{sen}^2 t + \cos^2 t)\,dt = \int_0^{2\pi} 1\,dt = [t]_0^{2\pi} = 2\pi$$

c) Notem que $\int_C P\,dx + Q\,dy = 2\pi \neq 0 \text{ "=" } \iint_D F\cdot ds$, o que não contraria o Teorema de Green, uma vez que a função F não está definida em $(0,0)$.