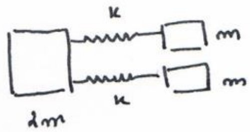
Aula TP 12/5/2020

1.



Movimento restringido a 1 dimensão

Sem atrito

Molas ideais

Obtenha: qualitativamente:

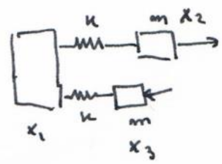
a) As frequências dos modos normais de vibração

b) As equações de movimento das massas, dadas as condições iniciais.

(a) Uma versão qualitativa: todas as massas oscilam c/ mesmo ω e φ num modo normal.

Há 3 modos normais de vibração (há 3 graus de liberdade)

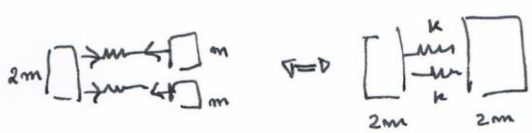
A.



$x_1 = 0$; $x_2 = -x_3$ anti simétrico

$\omega_A^2 = \frac{k}{m}$ (óbvio)

B.



$F = -2k\ 2\Delta x$ (simétrico)

$\omega_B^2 = \frac{2k}{m}$

C.

translacional

$\omega_C = 0$

(transl. c. massa)

temos ∴ 6 equações para 6 incógnitas:

$$0 = C + B\cos\phi_B$$

$$0 = C - B\cos\phi_B + A\cos\phi_A \quad \Rightarrow \quad C - B\cos\phi_B = -A\cos\phi_A$$

$$a = C - B\cos\phi_B - A\cos\phi_A \quad \Rightarrow \quad a = -A\cos\phi_A - A\cos\phi_A$$

$$a = -2A\cos\phi_A \ ; \ \phi_A = \pi$$

$$\left.\begin{aligned} 0 &= V_0 - B\omega_B\sin\phi_B \\ 0 &= V_0 + B\omega_B\sin\phi_B \end{aligned}\right\} \Rightarrow V_0 = 0 \quad \phi_B = 0 \text{ ou } \pi$$

$$0 = V_0 + B\omega_B\sin\phi_B + \omega_A A\sin\phi_A \quad \Rightarrow \quad \phi_A = 0, \pi$$

$$\left\{\begin{aligned} C &= -B\cos\phi_B \\ 0 &= C - B\cos\phi_B - \frac{a}{2} \end{aligned}\right. \qquad \left\{ -2B\cos\phi_B - \frac{a}{2} = 0 \Rightarrow \right.$$

$$\Rightarrow$$

$$C = B\cos\phi_B - A\cos\phi_A = -\frac{a}{4} + \frac{a}{2} = +\frac{a}{4}$$

Logo:

$$\left\{\begin{aligned} x_1(t) &= \frac{a}{4} - \frac{a}{4}\cos(\omega_B t) \\ x_2(t) &= \frac{a}{4} + \frac{a}{4}\cos(\omega_B t) - \frac{a}{2}\cos(\omega_A t + \pi) \\ x_3(t) &= \frac{a}{4} + \frac{a}{4}\cos(\omega_B t) + \frac{a}{2}\cos(\omega_A t + \pi) \end{aligned}\right.$$

O movimento geral das massas será uma sobreposição dos 3 modos:

$$\begin{array}{lllll} & & \text{C} & \text{B} & \overset{\text{Modo}}{\text{A}} \\ x_1 & = & C + v_0 t & + B \cos(\omega_B t + \phi_B) & + 0 \\ x_2 & = & C + v_0 t & - B \cos(\omega_B t + \phi_B) & + A \cos(\omega_A t + \phi_A) \\ x_3 & = & C + v_0 t & - B \cos(\omega_B t + \phi_B) & - A \cos(\omega_A t + \phi_A) \end{array}$$

b) $x_1 = x_3 = 0$; $\dot{x}_1 = \dot{x}_3 = \dot{x}_2 = 0$; $x_2 = a$ (Exemplo)

$$0 = C + B \cos\phi_B$$

$$0 = C - B \cos\phi_B + A \cos\phi_A$$

$$a = C - B \cos\phi_B - A \cos\phi_A$$

$$\dot{x}_1(t) = V_0 - B\omega_B \sin(\omega_B t + \phi_B)$$

$$\dot{x}_2(t) = V_0 + B\omega_B \sin(\omega_B t + \phi_B) - A\omega_A \sin(\omega_A t + \phi_A)$$

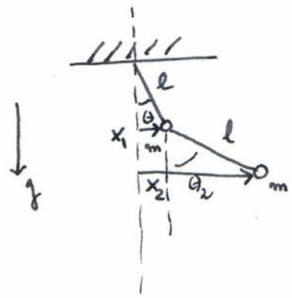
$$\dot{x}_3(t) = V_0 + B\omega_B \sin(\omega_B t + \phi_B) + A\omega_A \sin(\omega_A t + \phi_A)$$

$$\dot{x}_1(0) = V_0 - B\omega_B \sin\phi_B = 0$$

$$\dot{x}_2(0) = V_0 + B\omega_B \sin\phi_B = 0$$

$$\dot{x}_3(0) = V_0 + B\omega_B \sin\phi_B + \omega_A A \sin\phi_A = 0$$

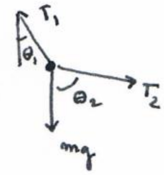
P2 - Determine os modos normais de vibração do pendulo duplo (para $\theta <<< 1$)



$\sin\theta \sim \theta \qquad \cos\theta \sim 1$ (Não há movimento vertical.)

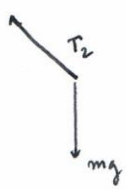
$\underline{d=1}$

massa 1:



$$\text{horiz.}\begin{cases} m\ddot{x}_1 = T_2 \sin\theta_2 - T_1 \sin\theta_1 \\ m\ddot{x}_2 = -T_2 \sin\theta_2 \end{cases}$$

massa 2



$$\text{Vertic}\begin{cases} 0 = T_1 \cos\theta_1 - T_2 \cos\theta_2 - mg \\ 0 = T_2 \cos\theta_2 - mg \end{cases}$$

$\downarrow$

Então, para $\theta << 1$

$$T_2 \simeq mg \quad ; \quad T_1 = 2mg$$

$$\cancel{x_2} \quad \sin\theta_1 = \frac{x_1}{\ell}$$

$$\sin\theta_2 = \frac{x_2 - x_1}{\ell}$$

Logo:

$$\begin{cases} m\ddot{x}_1 = mg\,\dfrac{x_2 - x_1}{\ell} - 2mg\,\dfrac{x_1}{\ell} \\ m\ddot{x}_2 = -mg\,\dfrac{(x_2 - x_1)}{\ell} \end{cases}$$

$$\ddot{x}_1 = \frac{g}{l} x_2 - 3\frac{g}{l} x_1 \qquad \left\{ \begin{array}{l} \ddot{x}_1 = -3\omega_0^2 x_1 + \omega_0^2 x_2 \\ \\ \ddot{x}_2 = \omega_0^2 x_1 - \omega_0^2 x_2 \end{array} \right.$$

$$\ddot{x}_2 = -\frac{g}{l} x_2 + \frac{g}{l} x_1$$

Método-1

$$\frac{d^2}{dt^2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} e^{i\alpha t}$$

$$-\alpha^2 \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -3\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow$$

$$\Leftrightarrow \begin{bmatrix} \alpha^2 - 3\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \alpha^2 - \omega_0^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0$$

$$(\alpha^2 - 3\omega_0^2)(\alpha^2 - \omega_0^2) - \omega_0^4 = 0 \Leftrightarrow$$

$$\Leftrightarrow \alpha^4 - \alpha^2\omega_0^2 - 3\omega_0^2\alpha^2 + 3\omega_0^4 - \omega_0^4 = 0$$

$$\alpha^4 - 4\omega_0^2\alpha^2 + 2\omega_0^4 = 0 \rightarrow \alpha^2 = \frac{4\omega_0^2 \pm \sqrt{16\omega_0^4 - 8\omega_0^4}}{2} =$$

$$\alpha_{\pm}^2 = 2\omega_0^2 \pm \sqrt{2}\,\omega_0^2 = (2 \pm \sqrt{2})\,\omega_0^2$$

Cada uma destas frequências determina o quociente entre A e B. Por exemplo

$$\alpha_+^2 = (2+\sqrt{2})\,\omega_0^2 \Rightarrow \begin{bmatrix} \alpha_+^2 - 3\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \alpha_+^2 - \omega_0^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (\sqrt{2}-1)\,\omega_0^2 & \omega_0^2 \\ \omega_0^2 & (\sqrt{2}+1)\,\omega_0^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \begin{cases} (\sqrt{2}-1)\,A = -B \\ A = -(\sqrt{2}+1)\,B \end{cases}$$

$$\frac{A}{B} = \frac{-1}{\sqrt{2}-1} \stackrel{!}{\equiv} -(\sqrt{2}+1)$$

$$\hookrightarrow \frac{1}{1-\sqrt{2}} = \frac{(1+\sqrt{2})}{-1}$$

$$\omega_+^2 = (2+\sqrt{2})\,\omega_0^2 \rightarrow \boxed{A = -(1+\sqrt{2})\,B}$$

De forma semelhante:

$$\omega_-^2 = (2-\sqrt{2})\,\omega_0^2 \rightarrow \boxed{A = -(1-\sqrt{2})\,B}$$

Os modos normais correspondem pois à projecção no eixo real das funções:

$$\vec{X}_+ = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix}_+ = B_+ \begin{bmatrix} -(1+\sqrt{2}) \\ 1 \end{bmatrix} e^{i(2+\sqrt{2})^{1/2}\omega_0 t}$$

$$\vec{X}_- = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_- = B_- \begin{bmatrix} -(1-\sqrt{2}) \\ 1 \end{bmatrix} e^{i(2-\sqrt{2})^{1/2}\omega_0 t}$$

O movimento geral das massas será:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = B_+ \begin{bmatrix} -(1+\sqrt{2}) \\ 1 \end{bmatrix} \cos\left[(2+\sqrt{2})^{1/2}\omega_0 t + \varphi_+\right] +$$

$$+ B_- \begin{bmatrix} -(1-\sqrt{2}) \\ 1 \end{bmatrix} \cos\left[(2-\sqrt{2})^{1/2}\omega_0 t + \varphi_-\right]$$

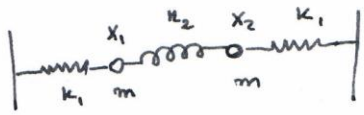
As 4 constantes (B_+, B_-, φ_+ e φ_-) devem ser determinadas dadas as condições iniciais.

Deixo como exercício determiná-las supondo que

$\dot{x}_1(0) = \dot{x}_2(0)$; $x_1(0) = 0$; $x_2(0) = a$

□

P3



Obtenha as equações de movimento admitindo que $K_2 << K_1$

e supondo as condições iniciais $x_1(0) = a$ $x_2(0) = 0$

$\dot{x}_1(0) = \dot{x}_2(0) = 0$

Solução:

$$m\,\ddot{x}_1 = -K_1 x_1 + K_2(x_2 - x_1)$$

$$m\,\ddot{x}_2 = -K_1 x_2 + K_2(x_1 - x_2)$$

Somando e subtraindo estas equações:

$$m(\ddot{x}_1 + \ddot{x}_2) = -K_1(x_1 + x_2)$$

$$m(\ddot{x}_1 - \ddot{x}_2) = -K_1(x_1 - x_2) - 2K_2(x_1 - x_2)$$

As frequências destes modos de vibração são:

$$\omega_+^2 = \frac{K}{m} \quad e \quad \omega_-^2 = \frac{K_1 + 2K_2}{m}$$

Logo:

$$X_+(t) = A\cos\left(\sqrt{\frac{K}{m}}\,t + \varphi_+\right) = x_1 + x_2$$

$$X_-(t) = B\cos\left(\sqrt{\frac{K_1 + 2K_2}{m}}\,t + \varphi_-\right) = x_1 - x_2$$

Correspondentemente:

$$x_1(t) = \frac{A}{2}\cos\left(\frac{\sqrt{k}}{m}t + \varphi_+\right) + \frac{B}{2}\cos\left(\sqrt{\frac{k_1+2k_2}{m}}\,t + \varphi_-\right)$$

$$x_2(t) = \frac{A}{2}\cos\left(\sqrt{\frac{k}{m}}\,t + \varphi_+\right) - \frac{B}{2}\cos\left(\sqrt{\frac{k_1+2k_2}{m}}\,t + \varphi_-\right)$$

As condições iniciais dão origem a:

$$\begin{cases} x_1(0) = \frac{A}{2}(\cos\varphi_+) + \frac{B}{2}\cos\varphi_- = a \\ x_2(0) = \frac{A}{2}\cos\varphi_+ - \frac{B}{2}\cos\varphi_- = 0 \\ \dot{x}_1(0) = -\sqrt{\frac{k}{m}}\,\frac{A}{2}\sin\varphi_+ - \sqrt{\frac{k_1+2k_2}{m}}\,\frac{B}{2}\sin\varphi_- = 0 \\ \dot{x}_2(0) = \sqrt{\frac{k}{m}}\,\frac{A}{2}\sin\varphi_+ + \sqrt{\frac{k_1+2k_2}{m}}\,\frac{B}{2}\sin\varphi_- = 0 \end{cases}$$

$$\Rightarrow \quad \varphi_+ = \varphi_- = 0 \quad \text{e} \quad A = B = a$$

Logo:

$$x_1(t) = \frac{a}{2}\left[\cos\left(\overbrace{\sqrt{\frac{k}{m}}}^{\omega_1}t\right) + \cos\left(\overbrace{\sqrt{\frac{k_1+2k_2}{m}}}^{\omega_2}t\right)\right]$$

$$x_2(t) = \frac{a}{2}\left[\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \cos\left(\sqrt{\frac{k_1+2k_2}{m}}\,t\right)\right]$$

$$\left.\begin{aligned} \omega_1 &= \frac{\omega_1+\omega_2}{2} - \frac{\omega_2-\omega_1}{2} \\ \omega_2 &= \frac{\omega_2-\omega_1}{2} + \frac{\omega_2+\omega_1}{2} \end{aligned}\right\} \Rightarrow$$

$$X_1 = \frac{a}{2}\left[\cos\left[\frac{\omega_2+\omega_1}{2}t - \frac{\omega_2-\omega_1}{2}t\right] + \cos\left[\frac{\omega_2-\omega_1}{2}t + \frac{\omega_2+\omega_1}{2}t\right]\right]$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$x_1(t) = a\left[\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cdot\cos\left(\frac{\omega_2-\omega_1}{2}t\right)\right]$$

De forma semelhante:

$$X_2(t) = a\left[\sin\left(\frac{\omega_2+\omega_1}{2}t\right)\cdot\sin\left(\frac{\omega_2-\omega_1}{2}t\right)\right]$$

$$\omega_2 = \left(\frac{K_1+2K_2}{m}\right)^{1/2} = \sqrt{\frac{K_1}{m}}\left[1+2\frac{K_2}{K_1}\right]^{1/2} \simeq \omega_1\left[1+\frac{1}{2}\cdot 2\cdot\frac{K_2}{K_1}+\cdots\right]$$

$$\simeq \omega_1\left[1+\varepsilon\right] \qquad (\varepsilon \ll 1)$$

Logo $\omega_1+\omega_2 \simeq 2\omega_1$

$\omega_2-\omega_1 \simeq \omega_1\varepsilon$

$$X_1(t) \simeq a\left[\cos(\omega_1 t)\cdot\cos(\omega_1\varepsilon t)\right]$$

$$X_2(t) \simeq a\left[\sin(\omega_1 t)\cdot\sin(\omega_1\varepsilon t)\right]$$

(Batimentos) $\rightarrow$ Amplitude modulada!