2º teste de Cálculo Vetorial

25/05/2021 Correção

1 a) $\frac{\partial x}{\partial s} = 2st^2$, $\frac{\partial x}{\partial t} = 2s^2t$, $\frac{\partial y}{\partial s} = 2s$, $\frac{\partial y}{\partial t} = 2t$

$$J_{(s,t)}f = \begin{pmatrix} 2st^2 & 2s^2t \\ 2s & 2t \end{pmatrix}$$

b) $\frac{\partial u}{\partial x} = \cos y$, $\frac{\partial u}{\partial y} = -x\,\text{sen}\,y$

$$\frac{\partial u}{\partial t}(s,t) = \frac{\partial u}{\partial x}\Big|_{(x(s,t),\,y(s,t))} \frac{\partial x}{\partial t}\Big|_{(s,t)} + \frac{\partial u}{\partial y}\Big|_{(x(s,t),\,y(s,t))} \frac{\partial y}{\partial t}\Big|_{(s,t)}$$

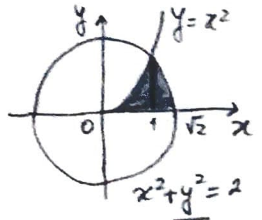
$$= \cos y\big|_{(s^2t^2,\,s^2+t^2)} \cdot 2st^2 - x\,\text{sen}\,y\big|_{(s^2t^2,\,s^2+t^2)} \cdot 2t$$

$$= 2st^2\cos(s^2+t^2) - s^2t^2 \cdot 2t\,\text{sen}(s^2+t^2)$$

2 $R = \{(x,y)\in\mathbb{R}^2 : 0 \le y \le 1,\ \sqrt{y} \le x \le \sqrt{2-y^2}\}$

$\begin{cases} x \ge 0 \\ x = \sqrt{y} \end{cases} \Leftrightarrow \begin{cases} x \ge 0 \\ x^2 = y \end{cases}$ $\begin{cases} x \ge 0 \\ x = \sqrt{2-y^2} \end{cases} \Leftrightarrow \begin{cases} x \ge 0 \\ x^2+y^2 = 2 \end{cases}$

a)



b) $I = \int_0^1 \int_0^{x^2} 2x\,dy\,dx + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} 2x\,dy\,dx$

uma vez que, se $y \ge 0$ e $x^2+y^2=2$ então $y = \sqrt{2-x^2}$

c) $I = \int_0^1 \int_{\sqrt{y}}^{\sqrt{2-y^2}} 2x\,dx\,dy = \int_0^1 \left[x^2\right]_{\sqrt{y}}^{\sqrt{2-y^2}} dy = \int_0^1 (2-y^2-y)\,dy = \left[2y - \frac{y^3}{3} - \frac{y^2}{2}\right]_0^1 = \frac{7}{6}$

d) $\begin{cases} x \ge 0 \\ y \ge 0 \\ y \le x^2 \\ x^2+y^2 \le 2 \end{cases} \Leftrightarrow \begin{cases} r\cos\theta \ge 0 \\ r\,\text{sen}\,\theta \ge 0 \\ r\,\text{sen}\,\theta \le r^2\cos^2\theta \Leftrightarrow r \ge \frac{\text{sen}\,\theta}{\cos^2\theta} \\ r^2 \le 2 \Leftrightarrow r \le \sqrt{2} \end{cases}$ (first two) $\Rightarrow 0 \le \theta \le \pi/2$

Pela figura, vê-se que $0 \le \theta \le \pi/4$. Se quisermos concluir o mesmo analiticamente, devemos notar que temos a restrição $\frac{\text{sen}\,\theta}{\cos^2\theta} \le \sqrt{2}$, o que é equivalente a $\text{tg}\,\theta \sec\theta \le \sqrt{2}$ e, resolvendo, obtemos $\theta \le \pi/4$

$$I = \int_0^{\pi/4} \int_{\frac{\text{sen}\,\theta}{\cos^2\theta}}^{\sqrt{2}} 2r^2\cos\theta\,dr\,d\theta$$

3 a) $\begin{cases} x \ge 0 \\ y \ge 0 \\ z \ge 0 \\ z \le 4-(x^2+y^2) \end{cases}$ Como $x \ge 0$ e $y \ge 0$, o maior valor que z pode tomar é 4. Então

Variação de z: $0 \le z \le 4$

Corte por z = constante ; Variação de x: $0 \le x \le \sqrt{4-z}$; variação de y: $0 \le y \le \sqrt{4-z-x^2}$

Então $I = \int_0^4 \int_0^{\sqrt{4-z}} \int_0^{\sqrt{4-z-x^2}} (x^2+y^2)\,dy\,dx\,dz$

b) $\begin{cases} r\cos\theta \ge 0 \\ r\,\text{sen}\,\theta \ge 0 \\ z \ge 0 \\ z \le 4-r^2 \end{cases}$ (first two) $\Leftrightarrow 0 \le \theta \le \pi/2$ Fixado θ = constante

Variação de r: $0 \le r \le 2$
Variação de z: $0 \le z \le 4-r^2$

Então $I = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} r \cdot r^2\,dz\,dr\,d\theta$

c) $I = \int_0^{\pi/2} \int_0^2 \left[r^3 z\right]_0^{4-r^2} dr\,d\theta = \int_0^{\pi/2} \int_0^2 (4r^3 - r^5)\,dr = \int_0^{\pi/2} \left[r^4 - \frac{r^6}{6}\right]_0^2 d\theta = \int_0^{\pi/2} \left(16 - \frac{64}{6}\right) d\theta = \frac{16}{3}\left[\theta\right]_0^{\pi/2} = \frac{8\pi}{3}$

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④ a) Uma parametrização do segmento que une $(1,1,0)$ a $(0,1,1)$ é

$c(t)=(1,1,0)+t((0,1,1)-(1,1,0))=(1,1,0)+t(-1,0,1)$, $t\in[0,1]$, ou ~~seja~~

$$c:[0,1]\longrightarrow\mathbb{R}^3 \qquad t\longmapsto(1-t,1,t)$$

Então $c'(t)=(-1,0,1)$

$$\int_c F\,ds=\int_0^1 F(c(t))\cdot c'(t)\,dt=\int_0^1 (t+1,\ (1-t)t+1,\ 1-t+1+2t)\cdot(-1,0,1)\,dt$$

$$=\int_0^1(-t-1+2+t)\,dt=\left[t\right]_0^1=1$$

b) <u>Resolução 1</u>: $F_1(x,y,z)=yz+1$, $F_2(x,y,z)=xz+1$, $F_3(x,y,z)=xy+1+2z$

$$\frac{\partial F_1}{\partial y}=z=\frac{\partial F_2}{\partial x}=z\ ;\quad \frac{\partial F_1}{\partial z}=y=\frac{\partial F_3}{\partial x}=y\ ;\quad \frac{\partial F_2}{\partial z}=x=\frac{\partial F_3}{\partial y}=x$$

Então, por um Teorema, o campo de vetores F é conservativo

<u>Resolução 2</u>: vejamos que existe $f:\mathbb{R}^3\to\mathbb{R}$ tal que $\nabla f=F$

$$\begin{cases}\dfrac{\partial f}{\partial x}=yz+1\\[2mm] \dfrac{\partial f}{\partial y}=xz+1\\[2mm] \dfrac{\partial f}{\partial z}=xy+1+2z\end{cases}$$

$\dfrac{\partial f}{\partial x}=yz+1\Rightarrow f(x,y,z)=xyz+x+g(y,z)$. Então devemos ter $xz+\dfrac{\partial g}{\partial y}=xz+1$, logo $\dfrac{\partial g}{\partial y}=1$, ou seja $g(y,z)=y+h(z)$. Assim, $f(x,y,z)=xyz+x+y+h(z)$ e, consequentemente, devemos ter $xy+h'(z)=xy+1+2z$, isto é, $h(z)=z+z^2+c$, $c\in\mathbb{R}$

Tomando $f:\mathbb{R}^3\longrightarrow\mathbb{R}$, $(x,y,z)\longmapsto xyz+x+y+z+z^2$, verifica-se que $\nabla f=F$. Então, sendo F é um campo de gradientes, F é conservativo.

⑤ $\begin{cases}x\geq 0\\ y\geq 0\\ 0\leq z\leq 1\\ x^4+y^4\leq 1\end{cases}$ Fazendo $x=u^2$ e $y=v^2$, ou seja, considerando a mudança de variável

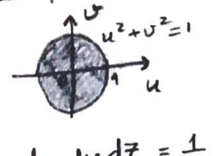
$$\phi:\mathbb{R}_0^+\times\mathbb{R}_0^+\times\mathbb{R}_0^+\longrightarrow\mathbb{R}_0^+\times\mathbb{R}_0^+\times\mathbb{R}_0^+,\qquad (u,v,z)\longmapsto(u^{1/2},v^{1/2},z)$$ temos

$$\begin{cases}u\geq 0\\ v\geq 0\\ 0\leq z\leq 1\\ u^2+v^2\leq 1\end{cases}\qquad J_{(u,v,z)}\phi=\begin{pmatrix}\frac12 u^{-1/2} & 0 & 0\\ 0 & \frac12 v^{-1/2} & 0\\ 0 & 0 & 1\end{pmatrix}\ \text{e}\ \left|\det J_{(u,v,z)}\phi\right|=\frac{1}{4\sqrt{u}\sqrt{v}}$$

<u>Variação de z</u>: $0\leq z\leq 1$ $\rightsquigarrow$ <u>corte por z=constante</u>

<u>Variação de u</u>: $0\leq u\leq 1$

<u>Variação de v</u>: $0\leq v\leq\sqrt{1-u^2}$



$$\iiint_V x^3y\,d(x,y,z)=\int_0^1\int_0^1\int_0^{\sqrt{1-u^2}}u^{3/2}v^{1/2}\frac{1}{4u^{1/2}v^{1/2}}\,dv\,du\,dz=\frac14\int_0^1\int_0^1\int_0^{\sqrt{1-u^2}}u\,dv\,du\,dz$$

$$=\frac14\int_0^1\int_0^1\left[uv\right]_0^{\sqrt{1-u^2}}du\,dz=\frac14\int_0^1\int_0^1-\frac12(-2u)\sqrt{1-u^2}\,du\,dz$$

$$=-\frac18\int_0^1\left[\frac23(1-u^2)^{3/2}\right]_0^1dz=-\frac1{12}\int_0^1(0-1)\,dz=\frac1{12}\left[z\right]_0^1=\frac1{12}$$