Formulário Matemático

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Conteúdo

Formulário matemático

Trigonometria

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)-\sin(\alpha+\beta)\right]$$

$$\cos\alpha+\cos\beta = 2\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)$$
$$\cos\alpha-\cos\beta = -2\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\alpha+\beta)$$
$$\sin\alpha\pm\sin\beta = 2\sin\frac{1}{2}(\alpha\pm\beta)\cos\frac{1}{2}(\alpha\mp\beta)$$

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$
$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

$$\cos 2\alpha = \cos^2\alpha-\sin^2\alpha$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$$
$$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

$$\cos(nx) = \sum_{k\,par}(-1)^{k/2}\binom{n}{k}\cos^{n-k}x\sin^k x$$
$$\sin(nx) = \sum_{k\,impar}(-1)^{(k-1)/2}\binom{n}{k}\cos^{n-k}x\sin^k x$$

Regra dos senos num triângulo

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Regra dos cossenos num triângulo

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Integrais

Indefinidos

$$\int x\sin(\alpha x)\,dx = \frac{1}{\alpha^2}\sin(\alpha x) - \frac{1}{\alpha}x\cos(\alpha x)$$

$$\int x\cos(\alpha x)\,dx = \frac{1}{\alpha^2}\cos(\alpha x) + \frac{1}{\alpha}x\sin(\alpha x)$$

$$\int x^n\sin(\alpha x)\,dx = -\frac{1}{\alpha}x^n\cos(\alpha x) + \frac{n}{\alpha}\int x^{n-1}\cos(\alpha x)$$

$$\int x^n\cos(\alpha x)\,dx = \frac{1}{\alpha}x^n\sin(\alpha x) - \frac{n}{\alpha}\int x^{n-1}\sin(\alpha x)$$

$$\int \sin^2(\alpha x)\,dx = \frac{x}{2} - \frac{\sin(2\alpha x)}{4\alpha}$$

$$\int \cos^2(\alpha x)\,dx = \frac{x}{2} + \frac{\sin(2\alpha x)}{4\alpha}$$

$$\int xe^{\alpha x}dx = \frac{\alpha x - 1}{\alpha^2}e^{\alpha x}$$

$$\int x^2e^{\alpha x}dx = \frac{x^2}{\alpha}e^{\alpha x} - \frac{2}{\alpha^2}xe^{\alpha x} + \frac{2}{\alpha^3}e^{\alpha x}$$

$$\int x^ne^{\alpha x}dx = \frac{1}{\alpha}x^ne^{\alpha x} - \frac{n}{\alpha}\int x^{n-1}e^{\alpha x}dx$$

$$\int \frac{e^{\alpha x}}{x}dx = \ln x + \alpha x + \frac{(\alpha x)^2}{2.2!} + \frac{(\alpha x)^3}{3.3!} + \cdots = \ln x + \sum_{n=1}^{\infty}\frac{(\alpha x)^n}{n.n!}$$

$$\int \frac{e^{\alpha x}}{x^n}dx = \frac{1}{n-1}\left[-\frac{e^{\alpha x}}{x^{n-1}} + \alpha\int\frac{e^{\alpha x}}{x^{n-1}}dx\right] \qquad n > 1$$

Definidos

$$\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

$$\int_{-\infty}^\infty e^{i\omega x - \alpha x^2} dx = e^{\frac{-\omega^2}{4\alpha}}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\pi \sin^2 x\ dx = \int_0^\pi \cos^2 x\ dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \frac{\sin x^2}{x^2} dx = \frac{\pi}{2}$$

$$\int_0^\infty e^{-\alpha x^2} \cos(\beta x) dx = \frac{\sqrt{\pi}}{2\alpha} e^{-\frac{\beta^2}{4\alpha^2}}$$

Integração por partes

$$d(uv) = udv + vdu$$

$$\int d(uv) = uv = \int udv + \int vdu$$

$$\int udv = uv - \int vdu$$

$$\int uv' dx = uv - \int vu' dx$$

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$

Derivadas

$$\nabla^2 e^{-\beta r} = \left(\beta^2 - \frac{2\beta}{r}\right) e^{-\beta r}$$

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta(\boldsymbol{r})$$

Expansões

Teorema binomial

$$(1+x)^n = 1 + \frac{n\,x}{1!} + \frac{n\,(n-1)\,x^2}{2!} + \cdots x^n$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

em que

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Expansão da exponencial

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Expansão do logaritmo

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

Expansões trigonométricas

$$\begin{aligned}
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \\
\tan x &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots
\end{aligned}$$

Função delta de Dirac

Pode ser vista como a derivada da função degrau de Heaviside:

$$\delta(x) = \frac{d}{dx}\Theta(x)$$

ou

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-a)} dp$$

Propriedade fundamental:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

Outras propriedades:

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$
$$\delta(x^2 - a^2) = \frac{1}{2|a|}\left(\delta(x+a) + \delta(x-a)\right)$$
$$\delta(x-a) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos[n(x-a)]$$

Se $g(x)$ é uma função com raízes x_i

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|}\delta(x - x_i)$$

Derivadas da função delta definem-se por:

$$\int f(x)\delta^{(n)}(x)dx = -\int \frac{df(x)}{dx}\delta^{(n-1)}(x)dx$$

e

$$\int xg(x)\delta'(x)dx = -\int g(x)\delta(x)dx$$

Em geral:

$$x^n\delta^{(n)}(x) = (-1)^n n!\delta(x)$$

$$x\frac{d\delta(x)}{dx} = -\delta(x)$$

Outras igualdades:

$$\int_{-1}^{1}\delta\left(\frac{1}{x}\right)dx = 0$$
$$\nabla^2\frac{1}{r} = -4\pi\delta^3(\boldsymbol{r})$$

Identidade de Dirac

$$\frac{1}{x+i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) - i\pi\delta(x)$$

Definições como limite:

$$\delta(x) = $$
$$\lim_{\epsilon\to 0} \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}$$
$$\lim_{\epsilon\to 0} \frac{1}{2}\epsilon|x|^{\epsilon-1}$$
$$\lim_{\epsilon\to 0^+} \frac{1}{2\sqrt{\pi\epsilon}}e^{-x^2/(4\epsilon)}$$
$$\lim_{\epsilon\to 0} \frac{1}{\pi x}\sin\left(\frac{x}{\epsilon}\right)$$
$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\mathrm{Ai}\left(\frac{x}{\epsilon}\right)$$
$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}J_{1/\epsilon}\left(\frac{x+1}{\epsilon}\right)$$
$$\lim_{\epsilon\to 0} |\frac{1}{\epsilon}e^{-x^2/(\epsilon)}L_n\left(\frac{2x}{\epsilon}\right)$$

em que $\mathrm{Ai}(x)$ é a função de Airy, $J_n(x)$ é a função de Bessel de primeira ordem, e $L_n(x)$ é o polinómio de Laguerre de ordem positiva inteira arbitrária.

$$\delta(x) = \lim_{n\to\infty}\frac{1}{2\pi}\frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{1}{2}x\right)}$$

Em duas dimensões:

$$\delta^2(x,y) = \delta(x)\delta(y)$$

Em coordenadas polares:

$$\delta^2(x,y) = \frac{\delta(r)}{\pi|r|}$$

Em três dimensões:

$$\delta^3(x,y,z) = \delta(x)\delta(y)\delta(z)$$

Em coordenadas cilíndricas:

$$\delta^3(r,\theta,z) = \frac{\delta(r)\delta(z)}{\pi r}$$

Em coordenadas esféricas:

$$\delta^3(r,\theta,\phi) = \frac{\delta(r)}{2\pi r^2}$$

Coordenadas esféricas

$$\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{e}_r &= \sin\theta\cos\phi\boldsymbol{e}_x + \sin\theta\sin\phi\boldsymbol{e}_y + \cos\theta\boldsymbol{e}_z \\
\boldsymbol{e}_\theta &= \cos\theta\cos\phi\boldsymbol{e}_x + \cos\theta\sin\phi\boldsymbol{e}_y - \sin\theta\boldsymbol{e}_z \\
\boldsymbol{e}_\phi &= -\sin\phi\boldsymbol{e}_x + \cos\phi\boldsymbol{e}_y \\
d\boldsymbol{r} &= dr\boldsymbol{e}_r + rd\theta\boldsymbol{e}_\theta + r\sin\theta d\phi\boldsymbol{e}_\phi
\end{aligned}$$

$$\boldsymbol{\nabla} = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\boldsymbol{e}_\phi$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{a} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta a_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}$$

$$\boldsymbol{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$=\boldsymbol{\nabla}\times\boldsymbol{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta\boldsymbol{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ a_r & ra_\theta & r\sin\theta a_\phi \end{vmatrix}$$

Outra forma:

$$\boldsymbol{\nabla}^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Coordenadas cilíndricas

$$\begin{aligned}
x &= \rho\cos\phi \\
y &= \rho\sin\phi \\
z &= z
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{e}_\rho &= \cos\phi \boldsymbol{e}_x + \sin\phi \boldsymbol{e}_y \\
\boldsymbol{e}_\theta &= -\rho \sin\phi \boldsymbol{e}_x + \rho\cos\phi \boldsymbol{e}_y \\
\boldsymbol{e}_z &= \boldsymbol{e}_z \\
d\boldsymbol{r} &= d\rho \boldsymbol{e}_\rho + \rho d\phi \boldsymbol{e}_\phi + dz \boldsymbol{e}_z
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \rho}{\partial x} &= \cos\phi \\
\frac{\partial \rho}{\partial y} &= \sin\phi \\
\frac{\partial \phi}{\partial x} &= -\frac{\sin\phi}{\rho} \\
\frac{\partial \phi}{\partial y} &= \frac{\cos\phi}{\rho}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \cos\phi \frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial y} &= \sin\phi \frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}
\end{aligned}
$$

$$
\boldsymbol{\nabla} = \frac{\partial}{\partial \rho}\boldsymbol{e}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\boldsymbol{e}_\phi + \frac{\partial}{\partial z}\boldsymbol{e}_z
$$

$$
\boldsymbol{\nabla} \cdot \boldsymbol{a} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho a_\rho\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi} + \frac{\partial}{\partial z}\boldsymbol{e}_z
$$

$$
\boldsymbol{\nabla}^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}
$$

$$
\boldsymbol{\nabla} \times \boldsymbol{a} = \frac{1}{\rho}\begin{vmatrix} \boldsymbol{e}_r & \rho\boldsymbol{e}_\phi & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix}
$$

Funções esféricas de Bessel e de Newmann

São as soluções da equação diferencial:

$$x^2 \frac{d^2}{dx^2} f(x) + 2x \frac{d}{dx} f(x) + \left[x^2 - l(l+1)\right] f(x) = 0$$

Tem duas soluções:

Funções esféricas de Bessel

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$$

Funções esféricas de Newmann

$$n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$$

$$\begin{aligned}
j_0(x) &= \frac{\sin x}{x} \\
n_0(x) &= -\frac{\cos x}{x} \\
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\
n_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x}
\end{aligned}$$

$$\begin{aligned}
j_l(x) &\to \frac{1}{x}\sin\left(x - \frac{1}{2}l\pi\right) \qquad x \to \infty \\
n_l(x) &\to -\frac{1}{x}\cos\left(x - \frac{1}{2}l\pi\right) \qquad x \to \infty \\
j_l(x) &\to \frac{x^l}{(2l+1)!!} \qquad x \to 0 \\
n_l(x) &\to \frac{(2l+1)!!}{x^{l+1}} \qquad x \to 0
\end{aligned}$$

Polinómios de Legendre

Os polinómios de Legendre têm a propriedade:

$$\int_{-1}^{+1} P_l(x) P_m(x) dx = \frac{2}{2l+1}\delta_{lm}$$

$$
\begin{aligned}
P_l(1) &= 1 \\
P_l(-1) &= (-1)^l
\end{aligned}
$$

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\
P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right)
\end{aligned}
$$

Polinómios de Hermite

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Satisfazem a equação diferencial:

$$\frac{d^2 H_n(x)}{dx} - 2x \frac{dH_n(x)}{dx} + 2nH_n(x) = 0$$

E as seguintes relações recursivas:

$$
\begin{aligned}
&H_{n+1} - 2xH_n + 2nH_{n-1} = 0 \\
&H_{n+1} + \frac{dH_n}{dx} - 2xH_n = 0
\end{aligned}
$$

$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x
\end{aligned}
$$

Matrizes

Traço de matrizes

$$\text{Tr}\,(A \pm B) = \text{Tr}\,(A) \pm \text{Tr}\,(B)$$

$$\text{Tr}\,(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j a_{ij} b_{ji} = \sum_j (BA)_{jj} = \text{Tr}\,(BA)$$

O traço de um comutador é sempre zero:

$$\text{Tr}\,([A, B]) = 0$$

$$\text{Tr}\,(ABC) = \text{Tr}\,(BCA) = \text{Tr}\,(CAB)$$

$$\text{Tr}\left(A^T\right) = \text{Tr}\,(A)$$

$$\text{Tr}\left(A^\dagger\right) = (\text{Tr}\ A)^*$$

Determinantes

O determinante é o 'volume'; a três dimensões:

$$|A| = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$$

$$|A^T| = |A|$$

$$|A^\dagger| = |(A^*)^T| = |A^*| = |A|^*$$

Matrix $N \times N$:

$$|\lambda A| = \lambda^N |A|$$

Troca de duas linhas ou duas colunas faz trocar o sinal do determinante.

Duas linhas ou coulnas idênticas: $|A| = 0$

$$|AB| = |A||B| = |BA|$$

Inversa

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

$$\left(A^\dagger\right)^{-1} = \left(A^{-1}\right)^\dagger$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

Cálculo vectorial

Relações entre produto externo e interno

$$
\begin{aligned}
\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) &= \boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A}) = \boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}) \\
\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) &= (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \\
(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} &= (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{A} \\
\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) &= (\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} + \boldsymbol{B}\times(\boldsymbol{A}\times\boldsymbol{C}) \\
\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) + \boldsymbol{C}\times(\boldsymbol{A}\times\boldsymbol{B}) &+ \boldsymbol{B}\times(\boldsymbol{C}\times\boldsymbol{A}) = 0 \\
(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D}) &= (\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D}) - (\boldsymbol{B}\cdot\boldsymbol{C})(\boldsymbol{A}\cdot\boldsymbol{D}) \\
(\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}))\boldsymbol{D} &= (\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\times\boldsymbol{C}) + (\boldsymbol{B}\cdot\boldsymbol{D})(\boldsymbol{C}\times\boldsymbol{A}) + (\boldsymbol{C}\cdot\boldsymbol{D})(\boldsymbol{A}\times\boldsymbol{B}) \\
(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D}) &= (\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{D}))\boldsymbol{C} - (\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}))\boldsymbol{D}
\end{aligned}
$$

Gradiente

$$
\begin{aligned}
\boldsymbol{\nabla}(\psi\phi) &= \psi\boldsymbol{\nabla}\phi + \phi\boldsymbol{\nabla}\psi \\
\boldsymbol{\nabla}(\boldsymbol{A}\cdot\boldsymbol{B}) &= (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) \\
\boldsymbol{\nabla}\left(\frac{\psi}{\phi}\right) &= \frac{\phi\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\phi}{\phi^2}
\end{aligned}
$$

Divergência

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot(\psi\boldsymbol{A}) &= \psi\boldsymbol{\nabla}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{\nabla}\psi \\
\boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) \\
\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{A}}{\phi}\right) &= \frac{\phi(\boldsymbol{\nabla}\cdot\boldsymbol{A}) - \boldsymbol{A}\cdot\boldsymbol{\nabla}\phi}{\phi^2}
\end{aligned}
$$

Rotacional

$$
\begin{aligned}
\boldsymbol{\nabla}\times(\psi\boldsymbol{A}) &= \psi(\boldsymbol{\nabla}\times\boldsymbol{A}) + \boldsymbol{\nabla}\psi\times\boldsymbol{A} \\
\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{A}(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \\
\boldsymbol{\nabla}\times(\psi\boldsymbol{\nabla}\phi) &= \boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\phi \\
\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{A}}{\phi}\right) &= \frac{\phi(\boldsymbol{\nabla}\times\boldsymbol{A}) + \boldsymbol{A}\times\boldsymbol{\nabla}\phi}{\phi^2}
\end{aligned}
$$

Segundas derivadas

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) &= 0 \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \psi) &= 0 \\
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \psi) &= \boldsymbol{\nabla}^2 \psi \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} \\
\boldsymbol{\nabla} \cdot (\phi \boldsymbol{\nabla} \psi) &= \phi \boldsymbol{\nabla}^2 \psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi \\
\psi \boldsymbol{\nabla}^2 \phi - \phi \boldsymbol{\nabla}^2 \psi &= \boldsymbol{\nabla} \cdot (\psi \boldsymbol{\nabla}\phi - \phi \boldsymbol{\nabla}\psi) \\
\boldsymbol{\nabla}^2(\phi\psi) &= \phi \boldsymbol{\nabla}^2 \psi + 2\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi + \psi \boldsymbol{\nabla}^2 \phi \\
\boldsymbol{\nabla}^2(\psi \boldsymbol{A}) &= \boldsymbol{A} \boldsymbol{\nabla}^2 \psi + 2(\boldsymbol{\nabla}\psi \cdot \nabla)\boldsymbol{A} + \psi \boldsymbol{\nabla}^2 \boldsymbol{A} \\
\boldsymbol{\nabla}^2(\boldsymbol{A} \cdot \boldsymbol{B}) &= \boldsymbol{A} \cdot \boldsymbol{\nabla}^2 \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{\nabla}^2 \boldsymbol{A} + 2\boldsymbol{\nabla} \cdot ((\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + \boldsymbol{B} \times \nabla \times \boldsymbol{A})
\end{aligned}
$$

Terceiras derivadas

$$
\begin{aligned}
\boldsymbol{\nabla}^2(\boldsymbol{\nabla}\psi) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\psi)) = \boldsymbol{\nabla}(\boldsymbol{\nabla}^2 \psi) \\
\boldsymbol{\nabla}^2(\boldsymbol{\nabla} \cdot \boldsymbol{A}) &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A})) = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}^2 \boldsymbol{A}) \\
\boldsymbol{\nabla}^2(\boldsymbol{\nabla} \times \boldsymbol{A}) &= -\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})) = \boldsymbol{\nabla} \times (\boldsymbol{\nabla}^2 \boldsymbol{A})
\end{aligned}
$$

Operador biharmónico

$$
\nabla^4 = \Delta^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + 2\frac{\partial^4}{\partial y^2 \partial z^2} + 2\frac{\partial^4}{\partial z^2 \partial x^2}
$$

$$
\nabla^4 \left(\frac{1}{r}\right) = \frac{3\left(15 - 8n + n^2\right)}{r^5}
$$

em que n é a dimensão do espaço.

Teoremas

$d\boldsymbol{v} = dxdydz = d^3\boldsymbol{r}$ é o elemento do volume V.

$d\boldsymbol{a} = \boldsymbol{n}da$ é o elemento de superfície de S que é a fronteira de V.

$d\boldsymbol{l}$ é o elemento de linha que percorre C, que é a fronteira de S.

Teorema do gradiente

$$
\int_{\boldsymbol{a}}^{\boldsymbol{b}} (\boldsymbol{\nabla} f) \cdot d\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a})
$$

O integral de linha de um gradiente não depende do percurso.

$$
\int_V (\boldsymbol{\nabla} f) \cdot d\boldsymbol{v} = \int_S f\boldsymbol{n}da
$$

Teorema da divergência ou de Gauss ou de Green

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{A}) d\boldsymbol{v} = \oint_S \boldsymbol{A} \cdot \boldsymbol{n} da$$

O integral num volume é igual ao valor da função na fronteira (que é uma sperfície fechada).

O lado direito da equação representa o fluxo da função $\boldsymbol{A}$, enquanto a divergência do lado esquerdo representa o espalhamento dos vectores, uma fonte ou survedouro

Teorema do rotacional ou de Stokes

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{n} da = \oint_C \boldsymbol{A} \cdot d\boldsymbol{l}$$

O integral do rotacional numa superfície é igual ao integral da função na fronteira da superfície. Não depende da superfície, apenas da fronteira. Se a superfície for fechada, o integral da direita é zero.

Primeira identidade de Green

$$\int_V (f \nabla^2 g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g) d\boldsymbol{v} = \int_S f \boldsymbol{n} \cdot \boldsymbol{\nabla} g da$$

Teorema de Green

$$\int_V (f \nabla^2 g - g \nabla^2 f) d\boldsymbol{v} = \int_S (f \boldsymbol{\nabla} g - g \boldsymbol{\nabla} f) \cdot \boldsymbol{n} da$$

Outros

$$\int_S \boldsymbol{n} \times \boldsymbol{\nabla} f da = \oint_C f d\boldsymbol{l}$$

$$\oint_S (\hat{\boldsymbol{n}} \times \boldsymbol{A}) da = \int_V (\boldsymbol{\nabla} \times \boldsymbol{A}) d\boldsymbol{v}$$

Tranformadas de Fourier e expansões de ondas planas

Expansão de uma onda plana em harmónicos esféricos:

$$e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} (i)^l j_l(kr) Y_{lm}^*(\boldsymbol{\kappa}) Y_{lm}(\boldsymbol{\rho}) \tag{1}$$

em que

$$\boldsymbol{\kappa} = \frac{\boldsymbol{k}}{k} \qquad\qquad \boldsymbol{\rho} = \frac{\boldsymbol{r}}{r} \tag{2}$$

e $j_l(kr)$ é a função de Bessel esférica de ordem l.

Uma onda plana a propagar-se na direcção z pode escrever-se em coordenadas esféricas:

$$e^{ikz} = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr) P_l(\cos\theta) \tag{3}$$