Derivada da função composta

- Sejam $\mathcal{U}$ um aberto do $\mathbb{R}^n$, $f: \mathcal{U} \longrightarrow \mathbb{R}^m$ função, sendo $x \longmapsto (f_1(x), \dots, f_m(x))$

$x = (x_1, \dots, x_n)$.

- Sejam $\mathcal{V}$ um aberto de $\mathbb{R}^m$, $g: \mathcal{V} \longrightarrow \mathbb{R}^k$ função, sendo $Y \longmapsto (g_1(Y), \dots, g_k(Y))$

$Y = (y_1, \dots, y_m)$.

- Suponhamos que $f(\mathcal{U}) \subseteq \mathcal{V}$. Então podemos considerar a função composta $g \circ f: \mathcal{U} \longrightarrow \mathbb{R}^k$
$x \longmapsto (g_1(f(x)), \dots g_k(f(x)))$

- Suponhamos que f e g admitem derivadas parciais de 1ª ordem todas contínuas.

Então, à semelhança do que acontece em $\mathbb{R}$, existe uma fórmula para calcular $\frac{\partial}{\partial x_s}(g_i \circ f)(x)$, sendo

$i \in \{1, \dots, k\}$, $j \in \{1, \dots, m\}$, $s \in \{1, \dots, n\}$. Notem que, quando calculamos em $x_0 = (x_1^0, \dots, x_n^0)$ a derivada em ordem a x_s da função $g_i \circ f_j$, estamos a derivar a função de uma variável (a variável x_s, as outras estão fixas)

$$\begin{aligned}]x_s^0 - \varepsilon, x_s^0 + \varepsilon[&\longrightarrow \mathbb{R} \\ x_s &\longmapsto g_i(f_1(x_0^1, \dots, x_s, \dots, x_0^n), \dots, g_i(f_m(x_0^1, \dots, x_s, \dots x_0^n \end{aligned}$$

Recordem que, se têm duas funções φ e ψ de uma variável e escalares (conjunto de chegada $\mathbb{R}$), se ambas forem deriváveis e se as pudermos compor, então

$$(\varphi \circ \psi)'(x) = \varphi'(\psi(x))\, \psi'(x).$$

Aqui a situação é um pouco mais complexa. A função g_i tem conjunto de chegada $\mathbb{R}$, mas depende de m variáveis, $y_1, \dots, y_m$. E quando compomos com f, g_i vai depender de x_s na 1ª variável, que passa a ser $f_1(x_0^1, \dots, x_s, \dots, x_0^n)$, ..., na m-ésima variável, que passa a ser $f_m(x_0^1, \dots, x_s, \dots, x_0^n)$.
Então teremos

$$\boxed{\frac{\partial (g_i \circ f)}{\partial x_s}(x_0) = \frac{\partial g_i}{\partial y_1}(f(x_0)) \frac{\partial f_1}{\partial x_s}(x_0) + \dots + \frac{\partial g_i}{\partial y_m}(f(x_0)) \frac{\partial f_m}{\partial x_s}(x_0)}$$

$$= \sum_{j=1}^{m} \frac{\partial g_i}{\partial y_j}(f(x_0)) \frac{\partial f_j}{\partial x_s}(x_0)$$

Podes escrever que

$$J_{x_0}(g \circ f) = \underset{\substack{\uparrow \\ \text{produto de matrizes}}}{J_{f(x_0)} g \cdot J_{x_0} f}$$

notem que $J_{x_0}(g\circ f) \in \mathcal{M}_{k,n}$, $J_{f(x_0)} g \in \mathcal{M}_{k,m}$ e $J_{x_0} f \in \mathcal{M}_{m,n}$ e, se multiplicarmos uma matriz $k\times m$ por uma $m\times n$, obtemos uma matriz $k\times n$, o que "bate certo". Por outro lado,

$$\underbrace{\begin{pmatrix} \frac{\partial(g_1\circ f)}{\partial x_1}(x_0) & \cdots & \frac{\partial(g_1\circ f)}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial(g_k\circ f)}{\partial x_1}(x_0) & \cdots & \frac{\partial(g_k\circ f)}{\partial x_n}(x_0) \end{pmatrix}}_{J_{f(x_0)}(g\circ f)} \overset{}{=} \underbrace{\begin{pmatrix} \frac{\partial g_1}{\partial y_1}(f(x_0)) & \cdots & \frac{\partial g_1}{\partial y_m}(f(x_0)) \\ \vdots & & \\ \frac{\partial g_k}{\partial y_1}(f(x_0)) & \cdots & \frac{\partial g_k}{\partial y_m}(f(x_0)) \end{pmatrix}}_{J_{f(x_0)} g} \cdot \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}}_{J_{x_0} f}$$

Atendendo à definição do produto de matrizes, concluímos que

$$\underbrace{\frac{\partial(g_i\circ f)}{\partial x_s}(x_0)}_{\text{elemento na linha } i \text{ e coluna } s} = \frac{\partial g_i}{\partial y_1}(f(x_0))\frac{\partial f_1}{\partial x_s}(x_0) + \cdots + \frac{\partial g_i}{\partial y_m}(f(x_0))\frac{\partial f_m}{\partial x_s}(x_0),$$

que é precisamente a fórmula referida na folha anterior.

<u>Exemplo</u>:

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$(x,y) \longmapsto (e^{xy}, x\,\text{sen}\,y, x^2y)$$

$$g: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto (xyz, x-y+z)$$

Comecemos por calcular $g\circ f: \mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{aligned} g\circ f(x,y) &= g(e^{xy}, x\,\text{sen}\,y, x^2y) \\ &= (e^{xy}\cdot x\,\text{sen}\,y\cdot x^2y,\ e^{xy} - x\,\text{sen}\,y + x^2y) \\ &= (x^3y\,\text{sen}\,y\,e^{xy},\ e^{xy} - x\,\text{sen}\,y + x^2y) \end{aligned}$$

Pretende-se, neste exemplo, calcular $\frac{\partial(g_2\circ f)}{\partial y}(x,y)$, isto é, a derivada em ordem a y da segunda função componente de $g\circ f$

Calculemos

$$\frac{\partial(g_2 \circ f)}{\partial y}(x,y) = xe^{xy} - x\cos y + x^2$$

Por outro lado

$$J_{(x,y)}f = \begin{pmatrix} ye^{xy} & xe^{xy} \\ \operatorname{sen} y & x\cos y \\ 2xy & x^2 \end{pmatrix} \qquad J_{(x,y,z)}g = \begin{pmatrix} yz & xz & xy \\ 1 & -1 & 1 \end{pmatrix}$$

$$J_{f(x,y)}g = J_{(e^{xy},\, x\operatorname{sen} y,\, x^2y)}g = \begin{pmatrix} x^3y\operatorname{sen} y & x^2ye^{xy} & x\operatorname{sen} y\, e^{xy} \\ 1 & -1 & 1 \end{pmatrix}$$

$$J_{f(x,y)}g \cdot J_{(x,y)}f = \begin{pmatrix} x^3y\operatorname{sen} y & x^2ye^{xy} & x\operatorname{sen} y\, e^{xy} \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} ye^{xy} & xe^{xy} \\ \operatorname{sen} y & x\cos y \\ 2xy & x^2 \end{pmatrix}$$

$$= \begin{pmatrix} x^3y^2\operatorname{sen} y e^{xy} + x^2y\operatorname{sen} y e^{xy} + 2x^2y\operatorname{sen} y e^{xy} & x^3\operatorname{sen} y e^{xy} + x^3y\cos y\, e^{xy} + x^4y\operatorname{sen} y\, e^{xy} \\ ye^{xy} - \operatorname{sen} y + 2xy & \boxed{xe^{xy} - x\cos y + x^2} \end{pmatrix} = (A)$$

$$\shortparallel$$

$$\frac{\partial(g_2 \circ f)}{\partial y}(x,y)$$

notem que bastaria termos calculado o elemento da matriz dentro do rectângulo, que resulta do produto seguinte

$$\begin{pmatrix} \times & \times & \times \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \times & xe^{xy} \\ \times & x\cos y \\ \times & x^2 \end{pmatrix} = 1.\, xe^{xy} + (-1).\, x\cos y + 1.\, x^2$$

Verificámos, então, neste caso particular, que se verifica a fórmula

$$J_{(x,y)}g\circ f = J_{f(x,y)}g \cdot J_{(x,y)}f,$$

Comparando (A) com

$$(B) = J_{(x,y)}(g\circ f) = \begin{pmatrix} 3x^2y\operatorname{sen} y e^{xy} + x^3y\operatorname{sen} y e^{xy} y & x^3\operatorname{sen} y e^{xy} + x^3y\cos y e^{xy} + x^3y\operatorname{sen} y e^{xy} x \\ ye^{xy} - \operatorname{sen} y + 2xy & ye^{xy} - x\cos y + x^2 \end{pmatrix}$$

Utilização de notação que simplifica as contas

$$f(x,y) = (e^{xy}, x\,\text{sen}\,y, x^2y) = (u(x,y), v(x,y), w(x,y))$$
$$g(u,v,w) = (uvw, u - v + w)$$

$$\frac{\partial(g_2 \circ f)}{\partial y}(x,y) = \frac{\partial g_2}{\partial u}(u(x,y), v(x,y), w(x,y))\frac{\partial u}{\partial y}(x,y)$$
$$+ \frac{\partial g_2}{\partial v}(u(x,y), v(x,y), w(x,y))\frac{\partial v}{\partial y}(x,y)$$
$$+ \frac{\partial g_2}{\partial w}(u(x,y), v(x,y), w(x,y))\frac{\partial w}{\partial y}(x,y) = ⊛$$

$$\frac{\partial g_2}{\partial u} = 1, \quad \frac{\partial g_2}{\partial v} = -1, \quad \frac{\partial g_2}{\partial w} = -1$$

$$\frac{\partial u}{\partial y} = ye^{xy}, \quad \frac{\partial v}{\partial y} = x\cos y, \quad \frac{\partial w}{\partial y} = x^2$$

$$⊛ = 1 \cdot ye^{xy} - 1 \cdot x\cos y + 1 \cdot x^2$$

Exemplo 2

Exercício 4 da Folha 4

b) $w = r^2 + s^2$, $r = pq^2$, $s = p^2\,\text{sen}\,q$

Calcular $\dfrac{\partial w}{\partial p}$ e $\dfrac{\partial w}{\partial q}$

nota: temos as funções $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$
$$(p,q) \longmapsto (pq^2, p^2\,\text{sen}\,q)$$

e $g: \mathbb{R}^2 \longrightarrow \mathbb{R}$
$$(r,s) \longmapsto r^2 + s^2$$

$$g \circ f(p,q) = g(pq^2, p^2\,\text{sen}\,q) = (pq^2)^2 + (p^2\,\text{sen}\,q)^2$$

$g \circ f: \mathbb{R}^2 \longrightarrow \mathbb{R}$
$$(p,q) \longmapsto (pq^2)^2 + (p^2\,\text{sen}\,q)^2$$

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial r}\Big|_{(pq^2, p^2\text{sen}\,q)} \frac{\partial r}{\partial p}\Big|_{(p,q)} + \frac{\partial w}{\partial s}\Big|_{(pq^2, p^2\text{sen}\,q)} \frac{\partial s}{\partial p}\Big|_{(p,q)}$$

Notem que $\dfrac{\partial w}{\partial r}\Big|_{(pq^2, p^2\text{sen}\,q)}$ significa $\dfrac{\partial w}{\partial r}$ calculado no ponto $(pq^2, p^2\,\text{sen}\,q)$ que é, precisamente $f(p,q)$

Façamos, então, as contas

$\frac{\partial r}{\partial q} = 2pq \quad \frac{\partial s}{\partial q} = p^2 \cos q$

$\frac{\partial w}{\partial r} = 2r \quad \frac{\partial w}{\partial s} = 2s \quad \frac{\partial r}{\partial p} = q^2 \quad \frac{\partial s}{\partial p} = 2p\,\mathrm{sen}\,q$

Então

$$\frac{\partial w}{\partial p} = 2r\Big|_{(pq^2,\, p^2 \mathrm{sen}\, q)} q^2 + 2s\Big|_{(pq^2,\, p^2 \mathrm{sen}\, q)} 2p\,\mathrm{sen}\,q$$
$$= 2pq^2 \cdot q^2 + 2p^2 \mathrm{sen}\,q \cdot 2p\,\mathrm{sen}\,q$$
$$= 2pq^4 + 4p^3 \mathrm{sen}^2 q$$

$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial r}\Big|_{(pq^2,\, p^2 \mathrm{sen}\, q)} \frac{\partial r}{\partial q}\Big|_{(p,q)} + \frac{\partial w}{\partial s}\Big|_{(pq^2,\, p^2 \mathrm{sen}\, q)} \frac{\partial s}{\partial q}(p,q)$$
$$= 2r\Big|_{(pq^2,\, p^2 \mathrm{sen}\, q)} 2pq + 2s\Big|_{(pq^2,\, p^2 \mathrm{sen}\, q)} p^2 \cos q$$
$$= 2pq^2 \cdot 2pq + 2p^2 \mathrm{sen}\,q \cdot p^2 \cos q$$
$$= 4p^2q^3 + 2p^4 \mathrm{sen}\,q \cos q$$

Agora repito as contas compondo primeiro e derivando depois:

$$w(r(p,q), s(p,q)) = p^2q^4 + p^4 \mathrm{sen}^2 q$$

$$\frac{\partial w}{\partial p} = 2pq^4 + 4p^3 \mathrm{sen}^2 q$$

$$\frac{\partial w}{\partial q} = 4p^2q^3 + 2p^4 \mathrm{sen}\,q \cos q$$

a) $u(x,y) = \ln\left(\mathrm{sen}\,\frac{x}{y}\right)$, $x(t) = 3t^2$, $y(t) = (1+t^2)^{1/2}$

$u(x(t), y(t))$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\Big|_{(3t^2, \sqrt{1+t^2})} \frac{dx}{dt}(t) + \frac{\partial u}{\partial y}\Big|_{(3t^2, \sqrt{1+t^2})} \frac{dy}{dt}(t) = (*)$$

$$\frac{\partial u}{\partial x} = \frac{\cos\frac{x}{y} \cdot \frac{1}{y}}{\mathrm{sen}\,\frac{x}{y}} \qquad \frac{\partial u}{\partial y} = \frac{\cos\frac{x}{y}\left(-\frac{x}{y^2}\right)}{\mathrm{sen}\,\frac{x}{y}} \qquad \frac{dx}{dt} = 6t, \quad \frac{dy}{dt} = \frac{1}{\cancel{2}}(1+t^2$$

$$(*) = \frac{\cos\frac{x}{y} \cdot \frac{1}{y}}{\mathrm{sen}\,\frac{x}{y}}\Big|_{(3t^2, \sqrt{1+t^2})} \cdot 6t - \frac{\cos\frac{x}{y} \cdot \frac{x}{y^2}}{\mathrm{sen}\,\frac{x}{y}}\Big|_{(3t^2, \sqrt{1+t^2})}$$

$$= \frac{\cos\left(\frac{3t^2}{\sqrt{1+t^2}}\right) \frac{6t}{\sqrt{1+t^2}}}{\mathrm{sen}\left(\frac{3t^2}{\sqrt{1+t^2}}\right)} - \frac{\cos\left(\frac{3t^2}{\sqrt{1+t^2}}\right) \frac{3t^2}{\sqrt{1+t^2}}}{\mathrm{sen}\left(\frac{3t^2}{\sqrt{1+t^2}}\right)} \cdot \frac{t}{\sqrt{1+t^2}}$$