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Analytic and algebraic problems in contemporary Yang-Mills theory

Problemas algébricos e analíticos na Teoria de Yang–Mills contemporânea

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Supervisor: Henrique N. Sá Earp

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Resumo

Esta tese aborda dois problemas em Teoria de Calibres, relacionados à existência e comportamento de certas conexões especiais em fibrados vetoriais complexos, e por isso está dividida em duas partes.

A primeira parte foca em problemas variacionais relacionados ao funcional de ε-Yang–Mills–Higgs abeliano (YMH_ε). Analisamos o comportamento de seus pontos críticos estáveis, estendendo os resultados em [Che21c, NO24], que demonstraram que tais estados satisfazem a equação de vórtice em superfícies de Riemann. Em $\mathbb{CP}^n$, obtemos inicialmente critérios de estabilidade para pontos críticos *redutíveis*. Para o caso *irredutível*, apresentamos resultados parciais que mostram que a estabilidade implica solução de vórtice sob suposições técnicas adicionais. O primeiro caso é obtido computando o traço da segunda variação, inspirado em [Che21c], e o segundo através da busca por direções potencialmente desestabilizadoras dadas pela equação de vórtice linearizada. Finalmente, no contexto de uma variedade Riemanniana fechada qualquer, provamos a ausência de pontos críticos não triviais para ε suficientemente grande, levando a um gap de energia que complementa os resultados existentes para ε pequeno, cf. [PS21].

A segunda parte é motivada principalmente por [Cor23], e tem como construir exemplos de fibrados vetoriais indecomponíveis que admitam soluções para as equações de Hermite–Yang–Mills deformadas (dHYM), mas não para as equações clássicas de Hermite–Yang–Mills (HYM). Para isso, estudamos a condição de Z-estabilidade assintótica (abreviada por a.Z-estabilidade) em superfícies projetivas, e apresentamos novos exemplos explícitos de fibrados de posto 3 que são a.Z-estáveis, mas não μ-estáveis. A a.Z-estabilidade é uma condição que de existência de soluções para as equações dHYM no limite de grande volume, cf. [DMS24]. Considerando extensões apropriadas de fibrados de retas por fibrados de posto 2 do tipo Hartshorne–Serre, que são μ-estáveis, obtemos exemplos sobre $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ e $\mathrm{Bl}_q\mathbb{P}^2$.

Keywords: teorias de calibres; Yang–Mills–Higgs; vortices; instantons deformados; fibrados vetoriais estáveis; cálculo variacional.

Abstract

This thesis approaches two problems in Gauge Theory, concerning existence and behavior of some special connections on complex vector bundles, so it is divided into two parts.

The first part focuses on variational problems related to the abelian ε-Yang–Mills–Higgs (YMH_ε) functional. We analyze the behavior of its stable critical points, extending the results on [Che21c, NO24], which established that such states satisfy the vortex equation over Riemann surfaces. On $\mathbb{CP}^n$, we first obtain stability criteria for *reducible* critical points. For the *irreducible* case, we give partial results showing that stability implies vortex solution under additional technical assumptions. The former is obtained by tracing the second variation, inspired in [Che21c], and the latter by seeking potentially destabilizing directions given the linearized vortex equation. Finally, in the setting of a general Riemannian manifold, we prove the absence of nontrivial critical points for sufficiently large ε, leading to an energy gap that complements existing results for small ε, as in [PS21].

The second part is mainly motivated by [Cor23], and our aim is to construct examples of undecomposable vector bundles admitting solutions to the deformed Hermitian–Yang–Mills (dHYM) equations, but not to the standard HYM equations. We do this by studying the asymptotic Z-stability (a.Z-stabilty for short) condition on projective surfaces, and provide some new explicit examples of rank 3 bundles which are a.Z-stable but not μ-stable. The a.Z-stability measures the existence of solutions to the dHYM equations in the large volume limit, c.f. [DMS24]. Taking suitable extensions of line bundles by rank 2, μ-stable Hartshorne-Serre bundles, we obtain examples on $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Bl}_q\mathbb{P}^2$.

Keywords: gauge theory; Yang–Mills–Higgs; vortices; deformed instantons; stable vector bundles; variational calculus.

Contents

Introduction

Let (X, g) be a closed Riemannian manifold, and $E \to X$ be a smooth complex vector bundle. For each connection $\nabla : \Omega^0(E) \to \Omega^1(E)$ we define its Yang–Mills energy as

$$\mathrm{YM}(\nabla) = \frac{1}{2} \int_X |F_\nabla|^2 \mathrm{vol}_g,$$

where F_∇ is the curvature of ∇. The study of Yang–Mills energy leads to many interesting results on the geometry and topology of the underlying manifold X. An important approach is to study its critical points, which are described by the Yang–Mills equation

$$d_\nabla^* F_\nabla = 0.$$

In order for a critical point to be an actual local minimizer, we must ask for the second variation of the Yang–Mills energy to be non-negative, in the sense that

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathrm{YM}(\nabla + tA) \geqslant 0,$$

for every $A \in \Omega^1(X, \mathrm{End}(E))$. A critical point ∇ satisfying this property is called a stable critical point.

If our manifold has additional geometric structure, e.g. a Kähler structure, we may actually be able to find global minimizers to the functional. In these cases, we usually refer to these minimizers as *instanton connections*. More precisely, the existence of a geometric structure on M implies the existence of some function $\mathcal{I}(F_\nabla)$ such that we can rewrite the Yang–Mills energy as

$$\mathrm{YM}(\nabla) = \mathcal{T}(E) + \int_X |\mathcal{I}(F_\nabla)|^2 \mathrm{vol}_g, \tag{0.0.1}$$

where $\mathcal{T}(E)$ is a non-negative quantity depending only on the topology of E. Hence, the functional YM is bounded below by this quantity $\mathcal{T}(E)$, and this bound is attained at solutions to the *instanton equation* $\mathcal{I}(F_\nabla) = 0$. The fundamental example for the theory of instantons is the case where X is a 4-manifold and E is a SU(2)-bundle. Here we have that

$$\mathcal{I}(F_\nabla) = F_\nabla \pm *F_\nabla \in \Omega^2(X, \mathrm{End}(E)), \quad \mathcal{I}(E) = \pm 4\pi c_2(E), \tag{0.0.2}$$

and the solutions to the instanton equation are the *(anti-)self-dual* connections. The study of the space of solutions to the ASD equations leads to several interesting results in the theory of 4-manifolds in the 1980s, see [DK07] for a survey on the subject. This approach can also be generalized for a $\mathrm{U}(n)$-bundle E over a higher dimensional Kähler manifold X, where we adopt

$$\mathcal{I}(F_\nabla) = \begin{pmatrix} 2F_\nabla^{0,2} \\ \Lambda_\omega F_\nabla - \lambda(E)\mathrm{Id}_E \end{pmatrix} \in \Omega^{0,2}(X, \mathrm{End}(E)) \oplus \Omega^0(X, \mathrm{End}(E)) \tag{0.0.3}$$

with the same $\mathcal{T}(E) = 4\pi c_2(E)$. In this case the instanton equation is referred to as the *Hermitian–Yang–Mills* (HYM) equation. The existence of solutions to the instanton equation, in this case, it is related to a topological property – namely the μ-stability, of the underlying vector bundle. More precisely, let $E \to X$ be a complex vector bundle over X, so we can define the degree of E as $\deg_\omega(E) = \dfrac{i}{2\pi}\displaystyle\int_X c_1(E) \wedge \omega^{n-1}$, and its slope as $\mu(E) = \dfrac{\deg_\omega(E)}{\mathrm{rk}(E)}$. A vector bundle E is said to be μ-stable if, for every proper subsheaf $F \subset E$, the inequality $\mu(F) < \mu(E)$ holds. The celebrated Donaldson–Uhlenbeck–Yau Theorem [UY86] establishes that solutions to the Hermitian–Yang–Mills (HYM) equations on E exist if, and only if, E is a direct sum of μ-stable vector bundles with identical slope, i.e., E is a polystable vector bundle. This profound result unveiled the algebraic-geometric essence of the HYM equations, paving the way for the exploration of algebraic conditions on vector bundles that guarantee the existence of solutions to other gauge-theoretic equations. The idea of linking the existence problem for solutions of gauge-theoretic partial differential equations to stability conditions has proven remarkably fruitful, inspiring analogous notions such as K-stability for constant scalar curvature Kähler metrics [Don02, Sto09], J-stability for deformed Hermitian–Yang–Mills equations on line bundles [Che21a], and, more recently, Z-stability for Z-critical equations [DMS24].

Building upon the preceding discussion of the Yang–Mills functional and its minimizers, we can identify two fundamental questions that naturally arise from the analysis. These questions are of the same nature as those that are addressed in this thesis and are stated, in the case where X is endowed with a special geometric structure leading to an instanton equation, as follows:

Question 1. *How can we describe the local minimizers, i.e. stable critical points, of the Yang–Mills functional? In particular, under what conditions can we ensure that a local minimizer is an instanton?*

Question 2. *Aiming to construct solutions to the instanton equations, how can construct examples of vector bundles that are stable in some sense?*

The first question has been extensively studied in the literature, even considering functionals distinct from the Yang–Mills energy. For instance, in the case where X is a 4-manifold, it was shown in [BL81] that stable critical points of the Yang–Mills functional on S^4 are either ASD or SD connections when the structure group of E is $\mathrm{SU}(2)$, $\mathrm{U}(2)$, or

SU(3). Furthermore, the same work establishes that for compact orientable homogeneous 4-manifolds, a stable critical point of the Yang–Mills functional must be either ASD, SD, or reducible to a U(1)-connection. This result was later generalized in [Ste08], where it was shown that the same conclusion holds for any compact structure group G. Additionally, conditions on the derivatives of the curvature of the critical point were provided to extend the result to non-negatively curved manifolds instead of compact ones. In higher-dimensional settings, such as when X is a G_2-manifold or a Calabi-Yau 3-fold, it was shown in [Hua17] that stable critical points of the Yang–Mills functional are instantons, provided certain technical conditions on the curvature are satisfied.

Since the instanton equation is in general a first-order PDE on the connection, and the stability of a critical point is a second-order condition on the connection, the phenomenon above, of stable critical points being actually instantons, is called *first-order reduction*. This kind of results appear also in other contexts, as in the study of minimal submanifolds. To point out another interesting first-order reduction result, we refer to [LS73], where it was proved that every stable minimal submanifold of $\mathbb{CP}^n$ is actually a complex submanifold.

The second question is an area of active research, concerning the study of moduli spaces of stable vector bundles in various senses. In the case of μ-stability, we point out the study of stable rank 2 bundles over surfaces, see [HL10]. It is well known that the moduli space of rank 2 μ-stable vector bundles with fixed Chern classes over a surface X is non-empty if we ask c_2 to be large enough, and such bundles are obtained via the Hartshorne-Serre construction. In this work we consider a different stability condition for vector bundles and obtain examples of stable vector bundles in this sense, based on the construction above.

In this thesis, we address the aforementioned questions by considering generalizations of the original concepts. In the first part, we investigate the following functional, defined on a Hermitian line bundle $L \to X$ in terms of a pair $(u, \nabla) \in \Omega^0(X, L) \times \mathcal{A}(L)$:

$$\mathrm{YMH}_\varepsilon(u, \nabla) = \int_X \varepsilon^2 |F_\nabla|^2 + |\nabla u|^2 + \frac{1 - |u|^2}{2\varepsilon^2} \operatorname{vol}_g, \tag{0.0.4}$$

referred to as the Abelian ε-Yang–Mills–Higgs functional. This functional can still be rewritten in same form of (0.0.1), with[1]

$$\mathcal{I}(u, \nabla) = \begin{pmatrix} F_\nabla^{0,2} \\ \nabla^{0,1} u \\ i\Lambda_\omega F_\nabla - \dfrac{1 - |u|^2}{2\varepsilon^2} \end{pmatrix} \in \Omega^{0,2}(X) \oplus \Omega^{0,1}(X, L) \oplus \Omega^0(X). \tag{0.0.5}$$

[1] In order for the function to have this shape, we need to consider the norm on the space $\Omega^{0,2}(X) \oplus \Omega^{1,0}(X, L) \oplus \Omega^0(X)$ to be given by $|(\alpha, \beta, \gamma)|^2 = 4\varepsilon^2|\alpha|^2 + 2|\beta|^2 + \varepsilon^2|\gamma|^2$

and $\mathcal{T}(L) = 2\pi \deg_\omega(L) - 8\pi\varepsilon^2 \mathrm{Ch}_2(L,\omega)$ when $\deg_\omega(L) \geqslant 0$. In this environment, the "instanton equation" $\mathcal{I}(u,\nabla) = 0$ is referred to as the vortex equation[2]. The first-order reduction in this context reads "every stable critical point of the functional YMH_ε is a vortex solution", and this was proved to be true on compact Riemann surfaces, see [Che21c, NO24]. Our goal is to generalize this result for higher dimensional Kähler manifolds, specially $\mathbb{CP}^n$. Since the techniques we use are deeply inspired in the arguments presented in [Che21c, NO24], we will briefly summarize the main ideas of these works. In [Che21c], this reduction is proved for $X = S^2 = \mathbb{CP}^1$ and $X = T^2$, leveraging the symmetries of the underlying manifold and the fact that a Killing vector field V induces deformations of the configurations (u,∇). By substituting these deformations into the second variation formula and employing stability, a non-negative quadratic form on the space of Killing vector fields is obtained, whose trace involves one of the vortex equations. The non-negativity of this form, along with some a priori estimate implies the vanishing of the trace, hence the stable critical point is actually a vortex solution. A more general approach in [NO24] extends this result to any compact Riemann surface by demonstrating the existence of an almost complex structure $\mathcal{I}$ on the configuration space. It is shown that the average of the second variation of the functional, applied to directions (v,a) and $\mathcal{I}(v,a)$, is non-positive if (v,a) satisfies the linearized vortex equation, using some a priori estimate. For a stable critical point (u,∇) this average must vanish, implying that (u,∇) is indeed a vortex solution.

For the second part of this thesis, we examine a perturbation of the Hermitian–Yang–Mills (HYM) equations on a holomorphic vector bundle $\mathscr{E}$ over a Kähler manifold (X,g,J,ω), known as the deformed Hermitian–Yang–Mills (dHYM) equation, which is expressed as:

$$\mathrm{Im}\left(e^{-i\varphi_k(\mathscr{E})}\left(k\omega\otimes \mathrm{Id}_{\mathcal{E}} - \frac{F_A}{2\pi}\right)^n\right) = 0, \tag{0.0.6}$$

where

$$\varphi_k(\mathscr{E}) = \arg\left(\int_X \mathrm{tr}\left(k\omega\otimes \mathrm{Id}_{\mathscr{E}} - \frac{F_A}{2\pi}\right)^n\right). \tag{0.0.7}$$

Our starting point is a theorem by Dervan, McCarthy, and Sektnan [DMS24], which provides an analogue of the Donaldson–Uhlenbeck–Yau theorem for equation (0.0.6). This result asserts that the existence of solutions to the dHYM equations for sufficiently large k is equivalent to an asymptotic stability condition on the underlying vector bundle. In this thesis, we address the problem of constructing explicit examples of such asymptotically stable vector bundles. The main inspiration is the so-called *Maruyama* example [OSS80b, Pg. 90] of a rank 3 holomorphic vector bundle on $\mathbb{CP}^2$ that is Gieseker stable but not μ-stable, as the extension of the structural sheaf $\mathscr{O}_{\mathbb{CP}^2}$ by some μ-stable, rank 2 vector bundle E with $c_1(E) = 0$. We present an adapted version of this construction in the case

[2] There is also a version of the equations for $\deg_\omega(L) < 0$, we give more details in Chapter 1

where X is a polycyclic polarized surface, and show that the problem boils down to the construction of rank 2, μ-stable vector bundles over such surfaces. The tools for doing this construction are the Hartshorne-Serre correspondence for rank 2 vector bundles and the Hoppe criterion for μ-stability of rank 2 vector bundles over polycyclic varieties, proved in [JMPSE17].

The thesis is organized as follows. We divide the text into two main parts, each one with its own introduction, where we present the highlighted original results and a contextualized discussion of the subject. The first part consists of Chapters 3 and 4 and concerns the dHYM equations and its associated stability condition. The second part consists of Chapter 1 and 2 and concerns the study of the Abelian Yang–Mills–Higgs functional on $\mathbb{CP}^n$, its variational properties and the first-order reduction problem.

Chapter 3 is dedicated to some basic properties concerning a.Z-stability and μ-stability over polarized surfaces. We start by comparing, in Proposition 3.1.6, the a.Z-stability with Gieseker stability and showing that they coincide in the case where X is a Del Pezzo surface with the polarization given by the anti-canonical divisor. We state the usual Hoppe criterion for μ-stability and use it to study μ-stability of Harthsorne-Serre bundles in Lemma 3.2.2. Finally, we prove a version of Hoppe criterion for a.Z-stability in the case where the vector bundle have rank 2 or 3 in Proposition 3.2.5.

In Chapter 4 we present, in the form of Theorem 4.1.1, the construction of rank 3 a.Z-stable vector bundles over polarized surfaces which are not μ-stable, providing, in Theorem 4.1.4, cohomological constraints for the construction to work. We also present explicit examples of such a.Z-stables over: $\mathbb{P}^2$ in Theorem 4.2.1, $\mathbb{P}^1 \times \mathbb{P}^1$ in Theorem 4.2.3 and $\mathrm{Bl}_q\mathbb{P}^2$ in Theorems 4.2.5 and 4.2.6.

In Chapter 1 is dedicated to the first-order reduction problem for the abelian Yang–Mills–Higgs functional. We present a priori estimates for critical points of the YMH functional, followed by a deduction of its second variation and a discussion about special Killing vector fields on $\mathbb{CP}^n$. Next we provide in Theorem 1.2.3 a full description of the stability of reducible critical points, and end the chapter proving Theorem 1.3.8 and Theorem 1.4.8, regarding the first-order reduction problem. Each of these results states that a stable critical point of the YMH functional is actually a vortex solution, provided some extra conditions on its derivatives are satisfied.

In Chapter 2 we study the existence of critical points of the YMH functional, over any Riemannian manifold (X, g), in the large ε regime. We show in Theorem 2.2.2 that there is a threshold in ε beyond which any critical point of the YMH functional is either reducible of gauge-equivalent to $(1, d)$ – in which case the line bundle L is trivial. Finally, the estimates we obtain to prove the above statement allows us to obtain, in the form of Theorem 2.2.3, a gap theorem for the YMH functional in the large ε setup.

In this thesis, all original results are marked by the suffix “L.” following their numbering. This indicates that the result is due to the author. In the case of joint work, the names (or initials) of the collaborators are also included to acknowledge shared authorship.

Part I

The Yang–Mills–Higgs functional

Context and main results

Let M be a closed Riemannian manifold and $L \to M$ a complex line bundle equipped with a Hermitian metric $\langle \cdot, \cdot \rangle$, so that

$$\langle \alpha\xi, \beta\eta \rangle = \alpha\bar{\beta} \langle \xi, \eta \rangle ,$$

for any $x \in M$, $\xi, \eta \in L_x$ and $\alpha, \beta \in \mathbb{C}$. We shall denote by $\operatorname{Re}\langle \cdot, \cdot \rangle$ the real metric on L induced by $\langle \cdot, \cdot \rangle$. Also, because of the presence of various indices here and there in this chapter, we shall adopt the notation $\sqrt{-1}$ instead of i for the imaginary unity.

Given $\varepsilon > 0$, a section $u : M \to L$ and a metric connection ∇ on L, the abelian ε-Yang–Mills–Higgs functional is defined by

$$E_\varepsilon(u, \nabla) = \int_M \varepsilon^2 |F_\nabla|^2 + |\nabla u|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2} \mathrm{vol}_g, \tag{0.0.8}$$

where F_∇ denotes the curvature of ∇ and is a 2-form with values in $i\,\mathbb{R}$. Among the most prominent features of E_ε is the fact that

$$E_\varepsilon(u, \nabla) = E_\varepsilon(s \cdot u, s \cdot \nabla \cdot s^{-1}), \text{ for all } s : M \to U(1). \tag{0.0.9}$$

A direct computation yields the Euler–Lagrange equations of E_ε:

$$\begin{cases} \nabla^* \nabla u = \dfrac{1 - |u|^2}{2\varepsilon^2} u, \\ d^*(\sqrt{-1}\, F_\nabla) = \dfrac{1}{\varepsilon^2} \operatorname{Re}\left\langle \sqrt{-1}\, u, \nabla u \right\rangle . \end{cases} \tag{YMH$_\varepsilon$}$$

If the second variation of E_ε at a solution to (YMH$_\varepsilon$) is non-negative definite, we say that the solution is *stable*. (See §1.1 for a more precise definition of stability.) Also, a solution to (YMH$_\varepsilon$) is called *reducible* if $u \equiv 0$, in which case F_∇ is harmonic.

As demonstrated in the recent work of Pigati–Stern [PS21] as well as follow-up works by Parise–Pigati–Stern [PPS24], a remarkable link exists between critical points of E_ε and codimension-two minimal submanifolds in M. For example, to paraphrase one of the main results of [PS21], suppose one is given a solution $(u_\varepsilon, \nabla_\varepsilon)$ of (YMH$_\varepsilon$) for each $\varepsilon > 0$, and that $E_\varepsilon(u_\varepsilon, \nabla_\varepsilon)$ is bounded uniformly in ε. Then, along a sequence $\varepsilon_j \to 0$, the solutions $(u_{\varepsilon_j}, \nabla_{\varepsilon_j})$ exhibit energy concentration on a (generalized) minimal submanifold of codimension two. More precisely, the corresponding sequence of Radon measures

$$\frac{1}{2\pi} \Big[\varepsilon_j^2 |F_{\nabla_{\varepsilon_j}}|^2 + |\nabla_{\varepsilon_j} u_{\varepsilon_j}|^2 + \frac{(1 - |u_{\varepsilon_j}|^2)^2}{4\varepsilon_j^2} \Big] \mathrm{vol}_g$$

converges weakly to the weight measure of a stationary, rectifiable $(n-2)$-varifold with integer multiplicity. This, of course, raises the question of how far this analogy can be pushed, and here we are particularly interested in the case where M is a Kähler manifold, in which case both E_ε and the volume functional admit absolute minimizers characterized by first-order equations.

Specifically, on the one hand, the classical Wirtinger inequality implies that a $\pm$-complex submanifold (that is, complex up to changing orientation) in a Kähler ambient space minimizes volume among homologous competitors. Moreover, a converse of sorts holds in the case M is $\mathbb{CP}^n$, where a famous theorem due to Lawson and Simons [LS73] states that in fact all submanifolds that locally minimize volume are $\pm$-complex. On the other hand, by rewriting the integrand in (0.0.8) as done in the pioneering works of Bradlow [Bra90] and García-Prada [GP93], one arrives at the following *vortex equations*, which, similar to the condition of being complex, come in a $\pm$-pair:

$$\begin{cases} F_\nabla^{0,2} = 0, \\ \nabla^{0,1} u = 0, \\ \sqrt{-1}\,\Lambda F_\nabla = \dfrac{1-|u|^2}{2\varepsilon^2} \end{cases} \tag{vor$_+$}$$

$$\begin{cases} F_\nabla^{0,2} = 0, \\ \nabla^{1,0} u = 0, \\ \sqrt{-1}\,\Lambda F_\nabla = -\dfrac{1-|u|^2}{2\varepsilon^2} \end{cases} \tag{vor$_-$}$$

In §1.4 below, we give a quick review of the derivation of the (vor$_+$) and (vor$_-$) following [Bra90]. In particular, as one shall see, depending on the first Chern class of L, absolute minimizers of E_ε on a given line bundle are given by solutions to one of the above systems. In view of the line of investigation initiated by the work [PS21] of Pigati and Stern, it is reasonable to expect that the local minimizers of E_ε are related to the vortex equations in the same form as the local minimizers of the volume relates to the $\pm$-complex submanifolds. Since Lawson–Simons theorem says that, in $\mathbb{CP}^n$, the local minimizers are $\pm$-complex, the natural question on the other side is the following:

Question 3. *Are all local minimizers of E_ε, or more precisely stable critical points of E_ε, solutions of the vortex equations when $M = \mathbb{CP}^n$?*

Since (vor$_+$) and (vor$_-$) are first-order while (YMH$_\varepsilon$) is second-order, we henceforth refer to the phenomenon demanded by Question 3 as *first-order reduction*. The work of Cheng [Che21c] present an affirmative answer for $\mathbb{CP}^1$, and also gives the analogous result on the torus T^2. It is shown in the following way: assuming without loss of generality that $\deg L \geqslant 0$, we start with a stable critical point (u, ∇) and let $\{X_1, \cdots, X_d\}$ be an orthonormal basis of the space of Killing vector fields on $M \in \{\mathbb{CP}^1, T^2\}$. Take $(a_j, v_j) \in \Omega^1(M) \oplus \Gamma(L)$ to be the directions given by

$$(a_j, v_j) = \left(X_j \lrcorner \left(\sqrt{-1}\Lambda F_\nabla - \frac{1-|u|^2}{2\varepsilon^2} \right), \nabla^{0,1}_{X_j} u \right).$$

Since (u, ∇) is stable we has

$$0 \leqslant \sum_j \frac{d^2}{dt^2} E_\varepsilon(u + tv_j, \nabla + \sqrt{-1}ta_j) = -\int_M \left(\sqrt{-1}\Lambda F_\nabla - \frac{1 - |u|^2}{2\varepsilon^2} \right) |u|^2 d\mathrm{vol},$$

and then it is proven that the above quantity is actually not positive, hence zero. This implies that (u, ∇) satisfies the third equation in (vor$_+$), which is shown to be equivalent, for a critical point, to the second equation. Finally, since the first equation is always true in complex dimension 1, we conclude that (u, ∇) is a solution to (vor$_+$). We follow a general kind of this approach in §1.3 for the case of $M = \mathbb{CP}^n$ with $n \geqslant 2$.

Another positive result in the same lines of Question 3 is given in [NO21], where the authors present a positive answer for the case where M is a compact Riemann surface, generalizing the results of [Che21c], but using a different technique. Here it is shown that if there exists a non-trivial $(v, a) \in \Omega^1(M) \oplus \Gamma(L)$ satisfying the linearized vortex equations

$$\begin{cases} \sqrt{-1}\varepsilon\Lambda da + h(u, v) & = 0 \\ \nabla^{0,1} v + a^{0,1} u & = 0 \end{cases} \tag{0.0.10}$$

then either (v, a) destabilizes E_ε at (u, ∇) or (u, ∇) is a vortex. Finally, the existence of solutions to these linearized equation is shown by computing its index, as it is an elliptic equation. In §1.3 we follow an analogue of this approach for higher dimensions, which turns out to be a hard task, since the linearized equation fails to be elliptic.

On another note, all the positive results for the question when $n = 1$ hold under the additional assumption that

$$\varepsilon^{-2} \geqslant |d|, \tag{0.0.11}$$

where d is the degree of the line bundle $L \to \mathbb{CP}^1$, but condition (0.0.11) happens to also be necessary and sufficient for the vortex equations to admit solutions. Moreover, the threshold for ε in (0.0.11) is optimal in the sense that when $\varepsilon^{-2} < |d|$, the normal-phase solutions $(0, \nabla)$ are stable but do not satisfy the vortex equations. In this thesis, we address Question 3 in the case $n > 1$ under some additional assumptions, following the two techniques mentioned above.

Going in a different direction, but still inspired by the work of Pigati and Stern, we can question about the behavior of the solutions to equations (YMH$_\varepsilon$) when the parameter ε is increasing. The YMH functional can be seen as a line bundle generalization of the Allen-Cahn functional, so we can base our expectations on what happens in this case. But for the Allen-Cahn functional, the large ε regime gives rise to no non-trivial critical points [GG18], so this is likely to happen with the YMH functional too. In this work, we show that, although harder to prove, this is indeed the case.

Main results and contributions

Our first result is the observation that, as is the case when $n = 1$, on higher-dimensional projective spaces the range of ε for which the reducible solutions $(0, \nabla)$ are stable but not satisfy the vortex equations is precisely the complement of the range where vortex solutions exists. More precisely, given a Hermitian line bundle L on a closed Kähler manifold M, the analogue of the degree is given by the quantity

$$C_1(L, \omega) = \int_M c_1(L) \wedge \frac{\omega^{n-1}}{(n-1)!},$$

where $c_1(L)$ is the first Chern class of L. In this context, it is shown by Bradlow [Bra90] and García-Prada [GP94] that the vortex equations admit solutions if, and only if,

$$\varepsilon^{-2} \geqslant \frac{4\pi |C_1(L, \omega)|}{\mathrm{Vol}(M, \omega)}.$$

Here and throughout this text, the term *vortex equations* is understood to refer to (vor$_+$) if $C_1(L, \omega) \geqslant 0$, and to (vor$_-$) if $C_1(L, \omega) \leqslant 0$. Our first main result relates this threshold to the stability of the normal-phase solutions.

Theorem (1.2.3, L, D. Cheng and D. Fadel)**.** *Suppose $L \to \mathbb{CP}^n$ is a Hermitian line bundle, and ∇ is a metric connection on L with harmonic curvature F_∇, so that $(0, \nabla)$ is a critical point of E_ε. Assuming further that $C_1(L, \omega) > 0$, we have*

(a) $(0, \nabla)$ *is a stable critical point of E_ε if, and only if,*

$$\varepsilon^{-2} \leqslant \frac{4\pi C_1(L, \omega)}{\mathrm{Vol}(\mathbb{CP}^n)}.$$

(b) $(0, \nabla)$ *is a solution to* (vor$_+$) *if, and only if, equality holds in the (a).*

Our next two main results contain positive answers to Question 3 obtained using different methods and under additional assumptions that are admittedly not the most natural. The proof of Theorem 1.3.8 involves the averaging technique employed in the work of Lawson–Simons [LS73] and Bourguignon–Lawson [BL81], while for Theorem 1.4.8 we adapt the approach of Nagy–Oliveira [NO21], using index theory to produce destabilizing variations when (u, ∇) is not a solution to the vortex equations.

Theorem (1.3.8, L and D. Cheng)**.** *Suppose (u, ∇) is a smooth, stable critical point of E_ε on $\mathbb{CP}^n$ and that $\varepsilon^{-2} \geqslant \dfrac{4\pi |C_1(L, \omega)|}{\mathrm{Vol}(\mathbb{CP}^n)}$. Assume in addition that*

(i) F_∇ *is type* $(1, 1)$*;*

(ii) upon splitting $\sqrt{-1}F_\nabla$ *orthogonally into* $\dfrac{\sqrt{-1}\Lambda F_\nabla}{n}\omega + \xi$*, with the second term satisfying* $\Lambda\xi = 0$*, there holds*

$$\int_{\mathbb{CP}^n} \Big(\frac{n-1}{n}\Big)^2 \frac{|\mathrm{Re}\,\langle u, \nabla u\rangle\,|^2}{\varepsilon^2} - 2\xi_{e_i,e_j}\mathrm{Re}\left\langle \sqrt{-1}\nabla_{e_i}u, \nabla_{e_j}u\right\rangle \geqslant 0.$$

Then (u,∇) *is a solution to either* (vor$_+$) *or* (vor$_-$)*.*

Coming to Theorem 1.4.8, the statement involves a subspace $\mathcal{T}_{(u,\nabla)} \subset \Omega^0(L) \oplus \Omega^{0,1} \oplus \Omega^{0,2}(L)$ whose precise definition requires a certain amount of preparation and thus is postponed to §1.4, where we prove Theorem 1.4.8. (See Definition 1.4.6.) Here we only describe $\mathcal{T}_{(u,\nabla)}$ in rough terms. Given a critical point (u,∇) of E_ε such that $F_\nabla \in \Omega^{1,1}$ and hence $\overline{\partial}_\nabla$ is compatible with a holomorphic structure on L, the space $\mathcal{T}_{(u,\nabla)}$ consists of triples (v, b, ξ) where $\xi \in \Omega^{0,2}(L)$ is holomorphic and $(v, b) \in \Omega^0(L) \oplus \Omega^{0,1}$ almost satisfies the linearization of the vortex equations at (u,∇), up to an error term related to $\overline{\partial}^*_\nabla\xi$.

Theorem (1.4.8 L, D.Cheng and D. Fadel)**.** *Suppose* (u,∇) *is a smooth, stable critical point of* E_ε *on* $\mathbb{CP}^n$ *with* F_∇ *being type* $(1,1)$*. Assume further that* $C_1(L,\omega) > 0$ *and that there exists* $(v, b, \xi) \in \mathcal{T}_{(u,\nabla)}$ *such that* $b \wedge \overline{\partial}_\nabla u = 0$*. Then* (u,∇) *satisfies* (vor$_+$)*.*

In the large ε regime, we obtain two interesting results that hold for any Hermitian line bundle over any closed Riemannian manifold. The first result addresses the existence of interesting critical points, while the second – arising as a consequence of the estimates obtained for the first – concerns the behavior of the functional's energy level for ε beyond a certain threshold.

Theorem (2.2.2, L and D. Cheng)**.** *Suppose* $L \to M$ *is a Hermitian line bundle over a closed Riemannian manifold* (M, g)*. There exists* $\varepsilon_0 = \varepsilon_0(M, L) \geqslant 1$ *such that if* $\varepsilon > \varepsilon_0$ *then any critical point* (u,∇) *of* E_ε *is either reducible, or gauge-equivalent to* $(1, d)$*. (The latter can only occur if* L *is trivial.)*

Theorem (2.2.3 L and D. Cheng)**.** *Given any* $\delta > 0$*, there exists a real constant* $C_\delta > 0$ *such that for every* $\varepsilon \geqslant \delta$ *and any critical point* (u,∇) *of* E_ε *we have either* $(u,\nabla) \sim (1, d)$*, in which case* L *is trivial, or*

$$\varepsilon^2 E_\varepsilon(u,\nabla) \geqslant C_\delta. \tag{0.0.12}$$

Theorem 1.2.3 is based on two observations. The first (Proposition 1.2.1) is that, due primarily to the form of the second variation formula of E_ε, determining the stability of $(0,\nabla)$ reduces to comparing ε^{-2} with the first eigenvalue $\mu_1(\nabla)$ of the connection Laplacian $\nabla^*\nabla$ on L. The second observation (Proposition 1.2.2), also mentioned in [NO24], is that if L admits a section v satisfying $\overline{\partial}_\nabla v = 0$, then $\mu_1(\nabla)$ can essentially be expressed in terms of topological data.

The idea behind the proof of Theorem 1.3.8, as already mentioned, is to follow [LS73] and take advantage of the abundance of Killing fields on $\mathbb{CP}^n$ to construct infinitesimal variations along which to evaluate the second derivative of E_ε, which is non-negative definite by the stability assumption. The hope is that by taking an average of these second derivatives over a suitable basis of the space of Killing fields, one obtains a certain vanishing condition that implies the desired first-order reduction. Apart from the work of Lawson–Simons [LS73] and the earlier, foundational work of Simons [Sim68], this approach has been applied with success to establish first-order reduction in many other contexts, including minimal submanifolds in some other ambient spaces [Mic84, Ohn86], harmonic mappings [BBDBR89], and Yang–Mills connections [BL81]. In all the cited instances, the variations chosen share the property that they lie in the kernel of the Hessian of the relevant functional when the desired first-order reduction occurs, while for a general critical point they may not. In [Che21c], the author follows this same strategy to produce a positive answer to Question 3 in the case of $\mathbb{CP}^1$. In the case of $\mathbb{CP}^n$ with $n > 1$, although we have yet to identify variations with the above-mentioned special property, we still managed to apply the averaging method to make some progress.

The proof of Theorem 1.4.8 is also based on averaging, but of a very different kind. Instead of the space of Killing fields, the average is carried out with respect to a certain complex structure acting on the space of infinitesimal variations. This is also a widely applied technique, the most notable examples include [MM88, MW93, Fra03]. See also [Tau83] for an example where the average is done over a quaternionic action. At any rate, in these examples, such an averaging process exposes a first-order elliptic operator which acts on infinitesimal variations and whose kernel elements, in favorable situations, are destabilizing directions. Index theorems can then be brought to bear to produce these directions. In [NO21], Nagy and Oliveira used this approach to generalize the main theorem in [Che21c] to all compact Riemann surfaces. In our present context, the first-order operator that arises fails to be elliptic. By a trick we learned from Ákos Nagy, we are able to partially remedy this by introducing extra line bundles and augmenting the operator to gain ellipticity, leading to the field ξ appearing in Theorem 1.4.8.

The proof of Theorem 2.2.2 follows the same idea of the analogous result for the Allen-Cahn functional. On the one hand, if (u, ∇) is a critical point of E_ε then $\|\nabla u\|_2 \sim \varepsilon^{-2}$ and on the other hand we prove that $\|\nabla u\|_2$ have a positive lower bound, even for varying $\varepsilon \geqslant 1$. The last assertion is obtained after some steps: we first use the estimates on [PS21] and obtain higher order estimates on the critical points, depending on a prescribed upper bound Λ in the curvature of ∇, then we show that this dependence on Λ can be dropped if $\varepsilon \geqslant 1$. With this we get bounds on the $W^{k,2}$ norms of F_∇ and ∇u for every critical point (u, ∇) and $k \geqslant 0$. To show that the positive lower bound exists we suppose that there exists a sequence of critical points (u_i, ∇_i) for E_{ε_i} satisfying $\|\nabla_i u_i\| \to 0$, then taking

$v_i = u_i/\|u_i\|_2$, the $W^{k,2}$ bounds above implies that $(v_i, \nabla_i) \to (v, \nabla)$ smoothly, up to a subsequence, such that $\|v\|_2 = 1$, $\nabla v \equiv 0$ and there is a $x_0 \in M$ such that $v(x_0) = 0$. This is a contradiction since $\nabla v = 0$ implies $|v|$ is constant, and $v(x_0) = 0$ says that $v \equiv 0$, in contrast with $\|v\|_2 = 1$. The final Theorem 2.2.3 is a consequence of an analogous argument, taking advantage of the higher order estimates we got for the first result.

1 Stable critical points of the YMH functional on $\mathbb{CP}^n$

This chapter will be dedicated study the behavior of stable critical points of the Yang–Mill–Higgs functional over $\mathbb{CP}^n$. In §1.1 we derive the second variation of the YMH functional, give a priori estimates for its critical points, useful later in the text for detecting vortex solutions, and finally recall some properties of real holomorphic vector fields on $\mathbb{CP}^n$. In §1.2 we study the stability of reducible critical points, starting with some properties of the first eigenvalue of the rough Laplacian defined by the connection part of such critical points and using this to proof Theorem 1.2.3. The content of §1.3 concerns our approach to the first-order reduction problem following the averaging approach of [Che21c]. We first infer some special variations from real holomorphic vector fields on $\mathbb{CP}^n$, compute all the quantities needed in the second variation formula for these variations and finally use the data to derive Theorem 1.3.8. Finally, in §1.4 address the first-order reduction problem following the *linearization* technique of [NO24]. We begin the section by recalling the derivation of the vortex equations, from a rewriting of the YMH functional analogous to [Bra90], use this to get another expression for the second variation and then identify some possible decreasing directions for the energy coming from solutions to the linearized vortex equations. We arrive at Theorem 1.4.8 by finding conditions for the existence of such solutions.

Throughout this and the next Chapter we use the standard notation $\Omega^p(M)$ to mean the space of p-forms over M. Given a complex line bundle $L \to M$, we use $\Omega^p(M; L)$ to denote the space of p-forms with values in L. When the choice of base manifold is clear from the context, we drop it from the notation and write simply Ω^p and $\Omega^p(L)$. As already mentioned in the beginning, our convention for the linearity properties over $\mathbb{C}$ of a Hermitian metric $\langle \cdot, \cdot \rangle$ on L is that

$$\langle \alpha\xi, \beta\eta \rangle = \alpha\bar{\beta} \langle \xi, \eta \rangle .$$

Also, for convenience, given $x \in M$ and $\xi, \eta \in L_x$, we define

$$\xi \times \eta = \mathrm{Re} \left\langle \sqrt{-1}\xi, \eta \right\rangle .$$

Analogously, given $\alpha, \beta \in \Omega^1(L)$, we define $\alpha \times \beta \in \Omega^2$ by

$$(\alpha \times \beta)_{v,w} = \mathrm{Re} \left\langle \sqrt{-1}\alpha_v, \beta_w \right\rangle - \mathrm{Re} \left\langle \sqrt{-1}\alpha_w, \beta_v \right\rangle .$$

Fixing a Hermitian metric h on L, we let $\mathcal{A} = \mathcal{A}^h$ denote the space of metric compatible connections on L. A connection $\nabla \in \mathcal{A}^h$ induces an exterior derivative $d_\nabla : \Omega^*(L) \to$

$\Omega^{*+1}(L)$, whose adjoint with respect to h and the Riemannian metric on M we denote by d^*_∇. Also, F_∇ denotes the curvature of ∇. Given any $\nabla, \nabla' \in \mathcal{A}^h$ we have

$$\nabla' = \nabla - \sqrt{-1}a, \quad F_{\nabla'} = F_\nabla - \sqrt{-1}da$$

for some real-valued 1-form a on M. In terms of the above notation, the domain of E_ε is then $\Omega^0(L) \times \mathcal{A}^h$, and we shall often drop the h if the choice of metric on L does not need to be explicitly mentioned. As an application of the cross product notation, we have the following useful identity which can easily be verified using the fact that d^2_∇ is precisely F_∇ and that the latter is purely imaginary:

$$d(u \times \nabla u) = \nabla u \times \nabla u - \sqrt{-1}F_\nabla |u|^2. \tag{1.0.1}$$

In the case where the base is a Kähler manifold (M, g, J, ω), for brevity we define

$$\omega^{[k]} = \frac{\omega^k}{k!}, \text{ for } k = 1, \cdots, \dim_{\mathbb{C}} M.$$

Also, we use superscripts to denote the various components of a differential form, either scalar-valued or vector-valued, with respect to the bi-degree decomposition induced by the complex structure. Thus for instance given $u \in \Omega^0(L)$ and $\nabla \in \mathcal{A}^h$ we have

$$\nabla u = \nabla^{1,0}u + \nabla^{0,1}u,$$

where the two terms on the right hand side are defined by

$$\nabla^{1,0}u = \frac{1}{2}\Big(\nabla u - \sqrt{-1}\nabla u \circ J\Big), \quad \nabla^{0,1}u = \frac{1}{2}\Big(\nabla u + \sqrt{-1}\nabla u \circ J\Big). \tag{1.0.2}$$

The operator $\nabla^{0,1} : \Omega^0(L) \to \Omega^{0,1}(L)$ induces an exterior derivative

$$\overline{\partial}_\nabla : \Omega^{p,q}(L) \to \Omega^{p,q+1}(L).$$

When $\overline{\partial}_\nabla$ is acting on $\Omega^0(L)$, we use it interchangeably with $\nabla^{0,1}$. Similar remarks apply to the relationship between ∂_∇ and $\nabla^{1,0}$. Of course, in general we do not have $\overline{\partial}^2_\nabla = 0$. That occurs, however, when the connection $\nabla \in \mathcal{A}^h$ has curvature lying in $\Omega^{1,1}$, in which case the Koszul–Malgrange Theorem yields a holomorphic structure on L compatible with $\nabla^{0,1}$, in the sense that the holomorphic sections are described by the condition $\nabla^{0,1}v = 0$. Also, following standard practice in the literature we write $\Omega^{1,1}_0(M)$ for the subspace of $\Omega^{1,1}(M)$ consisting of forms ξ such that $\Lambda\xi = 0$, where Λ denotes contraction with ω.

1.1 Preliminary computations and estimates

1.1.1 General setting

Second variation formula

In this section we recall the second variation formula for the abelian Yang–Mills–Higgs functional. Suppose $L \to M$ is a Hermitian line bundle over a closed Riemannian manifold, and that (u, ∇) is a smooth solution to (YMH$_\varepsilon$). Given $(v, a) \in \Omega^0(L) \oplus \Omega^1$, a straightforward computation shows that

$$\begin{aligned}(\delta^2 E_\varepsilon)_{(u,\nabla)}(v,a) &:= \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} E_\varepsilon(u + tv, \nabla - t\sqrt{-1}a) \\ &= \int_M \Big[\varepsilon^2|da|^2 + |\nabla v|^2 - 2a \cdot (u \times \nabla v + v \times \nabla u) + |a|^2|u|^2 \\ &\quad + \frac{(\mathrm{Re}\langle u, v\rangle)^2}{\varepsilon^2} + \frac{|u|^2 - 1}{2\varepsilon^2}|v|^2\Big]\mathrm{vol}_g.\end{aligned} \tag{1.1.1}$$

Polarizing and then formally integrating by parts exposes the operators:

$$\begin{aligned}\mathcal{J}^1_{(u,\nabla)}(v,a) &= \nabla^*\nabla v - (d^*a)\sqrt{-1}u + 2a \cdot \sqrt{-1}\nabla u + \frac{\mathrm{Re}\langle u, v\rangle}{\varepsilon^2}u + \frac{|u|^2 - 1}{2\varepsilon^2}v, \\ \mathcal{J}^2_{(u,\nabla)}(v,a) &= \varepsilon^2 d^*da - u \times \nabla v - v \times \nabla u + |u|^2 a,\end{aligned} \tag{1.1.2}$$

which we henceforth refer to as the *Jacobi operators* and which have the property that

$$(\delta^2 E_\varepsilon)_{(u,\nabla)}(v,a) = \int_M \Big[\mathrm{Re}\left\langle \mathcal{J}^1_{(u,\nabla)}(v,a), v\right\rangle + \mathcal{J}^2_{(u,\nabla)}(v,a) \cdot a\Big]\mathrm{vol}_g$$

We now define the notion of stability, mentioned at the start of the paper, in more precise terms.

Definition 1.1.1. *Let (u, ∇) be a smooth critical point of E_ε. We say that (u, ∇) is stable if*

$$(\delta^2 E_\varepsilon)_{(u,\nabla)}(v,a) \geqslant 0, \ \textit{for all } (v,a) \in \Omega^0(L) \oplus \Omega^1.$$

Otherwise, (u, ∇) is said to be unstable.

The failure of $\mathcal{J}^2_{(u,\nabla)}$ to be elliptic can be remedied by adding a term to $(\delta^2 E_\varepsilon)_{(u,\nabla)}(v, a)$ that penalizes variations along the gauge orbit. This is a standard trick in gauge theory. In our context, we observe that

$$(\delta^2 E_\varepsilon)_{(u,\nabla)}(v,a) + \int_M \Big|\varepsilon d^*a + \frac{u \times v}{\varepsilon}\Big|^2 \mathrm{vol}_g = \int_M \mathrm{Re}\left\langle \widetilde{\mathcal{J}}^1_{(u,\nabla)}(v,a), v\right\rangle + \widetilde{\mathcal{J}}^2_{(u,\nabla)}(v,a) \cdot a, \tag{1.1.3}$$

where

$$\begin{aligned}\widetilde{\mathcal{J}}^1_{(u,\nabla)}(v,a) &= \nabla^*\nabla v + 2a \cdot \sqrt{-1}\nabla u + \frac{3|u|^2 - 1}{2\varepsilon^2}v, \\ \widetilde{\mathcal{J}}^2_{(u,\nabla)}(v,a) &= \varepsilon^2 \Delta a - 2v \times \nabla u + |u|^2 a.\end{aligned} \tag{1.1.4}$$

When the solution (u, ∇) under consideration is clear from the context, we drop the subscripts and simply write $\widetilde{\mathcal{J}}^1, \widetilde{\mathcal{J}}^2$. The following observation is a trivial consequence of (1.1.3) and the definition of stability.

Lemma 1.1.2. *Suppose (u, ∇) is a smooth, stable critical point of E_ε. Then for all $(v, a) \in \Omega^0(L) \oplus \Omega^1$ we have*

$$\int_M \mathrm{Re}\left\langle \widetilde{\mathcal{J}}^1_{(u,\nabla)}(v,a), v\right\rangle + \widetilde{\mathcal{J}}^2_{(u,\nabla)}(v,a) \cdot a \geqslant \int_M \Big|\varepsilon d^*a + \frac{u \times v}{\varepsilon}\Big|^2 \mathrm{vol}_g.$$

A priori estimates for critical points

Now we collect various estimates on solutions of (YMH$_\varepsilon$), the most important being Corollary 1.1.6. We first recall the version of the maximum principle that will be needed below.

Proposition 1.1.3. *Let M be a connected Riemannian manifold and let $\Delta = d^*d$ be the (positive-definite) Laplace-Beltrami operator on M. Suppose that $m \in C^\infty(M;\mathbb{R})$ satisfies $(\Delta + c)m \leqslant 0$ everywhere on M, where $c : M \to [0,\infty)$ is a non-negative continuous function. Under these assumptions, if m attains a non-negative interior maximum, then m is constant.*

Lemma 1.1.4. *Suppose that (M, g, J, ω) is a closed Kähler manifold of complex dimension n and $L \to M$ is a Hermitian line bundle. The for all $u \in \Omega^0(L)$ there holds*

$$\Lambda(\nabla \times \nabla) = |\nabla^{1,0}u|^2 - |\nabla^{0,1}|^2. \tag{1.1.5}$$

Proof. First, choose a local orthonormal frame $e_1, \cdots, e_{2n}$ such that $e_{2k} = e_{2k-1}$ for all $k = 1, \cdots, n$. On one hand we have that

$$\Lambda(\nabla u \times \nabla u) = \operatorname{Re}\langle \nabla u \times \nabla u, \omega\rangle = 2\sum_{k=1}^{n} \operatorname{Re}\left\langle \sqrt{-1}\nabla_{e_{2k-1}}u, \nabla_{e_k}u\right\rangle. \tag{1.1.6}$$

On the other hand,

$$\begin{aligned} 4|\nabla^{1,0}u|^2 &= |\nabla u|^2 + |\nabla u \circ J|^2 + 2\operatorname{Re}\left\langle \sqrt{-1}\nabla u, \nabla u \circ J\right\rangle \\ 4|\nabla^{0,1}u|^2 &= |\nabla u|^2 + |\nabla u \circ J|^2 - 2\operatorname{Re}\left\langle \sqrt{-1}\nabla u, \nabla u \circ J\right\rangle. \end{aligned}$$

Taking the difference we get

$$|\nabla^{1,0}u|^2 - |\nabla^{0,1}|^2 = \operatorname{Re}\left\langle \sqrt{-1}\nabla u, \nabla u \circ J\right\rangle = 2\sum_{k=1}^{n} \operatorname{Re}\left\langle \sqrt{-1}\nabla_{e_{2k-1}}u, \nabla_{e_k}u\right\rangle, \tag{1.1.7}$$

and we are done. □

The next proposition is a consequence of the techniques found in the book by Jaffe and Taubes [JT80], and serves to simplify the task of detecting solutions to the vortex equations.

Proposition 1.1.5. *Let M and L be as in Lemma 1.1.4 and let $(u, \nabla) \in \Omega^0(L) \times \mathcal{A}^h$ be a smooth critical point of E_ε such that $u \not\equiv 0$. Define*

$$f := \varepsilon\sqrt{-1}\Lambda F_\nabla \qquad\qquad s := \frac{1 - |u|^2}{2\varepsilon}. \tag{1.1.8}$$

Then we have the following.

(a) *$\pm f \leqslant s$, and if equality holds at a point, it holds at every point.*

(b) *$f = s$ if, and only if, $\nabla^{0,1}u = 0$. Similarly, $f = -s$ if, and only if, $\nabla^{1,0}u = 0$.*

(c) *If $\nabla^{0,1}u = 0$ or $\nabla^{1,0}u = 0$, then $F^{0,2}_\nabla = 0$.*

Proof. We first notice by the second equation in (YMH$_\varepsilon$) and the identity (1.0.1) that

$$\varepsilon^2\Delta(\sqrt{-1}F_\nabla) = \varepsilon^2 dd^*(\sqrt{-1}F_\nabla) = \nabla u \times \nabla u - |u|^2\sqrt{-1}F_\nabla. \tag{1.1.9}$$

(a) We are going to use Proposition 1.1.3 for $c = \dfrac{|u|^2}{\varepsilon}$ and $m = \pm f - s$. Since the Laplacian commutes with Λ, we see from (1.1.9) that

$$\Delta f = \varepsilon\Lambda d(\sqrt{-1}d^*F_\nabla) = \frac{1}{\varepsilon}\Lambda(\nabla u \times \nabla u) - f\frac{|u|^2}{\varepsilon^2}. \tag{1.1.10}$$

On the other hand, the first equation in (YMH$_\varepsilon$) gives

$$\Delta(-s) = \frac{1}{2\varepsilon}\Delta|u|^2 = \frac{1}{\varepsilon}d^*\mathrm{Re}\,\langle u, \nabla u\rangle = \frac{|u|^2}{\varepsilon^2}s - \frac{|\nabla u|^2}{\varepsilon}. \tag{1.1.11}$$

Adding $\pm$ (1.1.10) and (1.1.11), we get

$$\Delta(\pm f - s) = \frac{|u|^2}{\varepsilon^2}(s \mp f) + \frac{1}{\varepsilon}(\pm\Lambda(\nabla u \times \nabla u) - |\nabla u|^2).$$

Applying Lemma 1.1.4 we get

$$\left(\Delta + \frac{|u|^2}{\varepsilon^2}\right)(\pm f - s) \leqslant 0. \tag{1.1.12}$$

Now, since M is compact, $\pm f - s$ attains a maximum somewhere, and we see by Proposition 1.1.3 that either this maximum is negative, in which case $\pm f - s < 0$ on M, or $\pm f - s$ is constant, in which case we still have $\pm f - s \leqslant 0$ thanks to (1.1.12) and the fact that $|u|$ is positive somewhere by assumption. This proves the inequality in (a). If equality is achieved somewhere, then another application Proposition 1.1.3 shows that $\pm f - s$ must be identically zero.

(b) Using the first equation in (YMH$_\varepsilon$) and the Bochner-Kodaira-Nakano identity, namely

$$\nabla^*\nabla u = 2(\nabla^{0,1})^*\nabla^{0,1}u + (\sqrt{-1}\Lambda F_\nabla)u, \tag{1.1.13}$$

we can conclude that

$$(\nabla^{0,1})^*\nabla^{0,1}u = \frac{1}{2\varepsilon}(s - f)u$$

and hence

$$\int_M |\nabla^{0,1}u|^2 = \frac{1}{2\varepsilon}\int_M (s - f)|u|^2$$

If $f = s$, this implies $\nabla^{0,1}u = 0$. On the other hand if $\nabla^{0,1}u = 0$ then

$$\int_M (s-f)|u|^2 = 0.$$

Since u is not identically zero and $s-f \geqslant 0$, this implies that $s-f=0$ at some point, and hence must be identically zero by part (a). There is an analogous argument for the second case, where we can use the identity

$$\nabla^*\nabla u = 2(\nabla^{1,0})^*\nabla^{1,0}u - (\sqrt{-1}\Lambda F_\nabla)u$$

instead of (1.1.13).

(c) Since Δ preserves the space of (p,q)-forms, we get from (1.1.9) that

$$(\varepsilon^2\Delta + |u|^2)(\sqrt{-1}F_\nabla^{0,2}) = (\nabla u \times \nabla u)^{0,2}. \tag{1.1.14}$$

Now, by assumption, we have either $\nabla u \circ J = \pm\sqrt{-1}\nabla u$ identically. Therefore, in both cases, we find that

$$2\mathrm{Re}\left\langle \sqrt{-1}\nabla_{Je_i}u, \nabla_{Je_j}u\right\rangle = 2\mathrm{Re}\left\langle \sqrt{-1}\nabla_{e_i}u, \nabla_{e_j}u\right\rangle,$$

that is, $\nabla u \times \nabla u$ is of type $(1,1)$. Combining this with (1.1.14), we conclude after a simple integration by parts argument that

$$|u|^2|F_\nabla^{0,2}|^2 = 0 \text{ on } M. \tag{1.1.15}$$

Since u is not identically zero, it follows from the first equation in (YMH$_\varepsilon$) and standard unique continuation results (see for instance [Kaz88, Theorem 1.8]) that $\{x \in M \mid u(x) \neq 0\}$ is dense in M, but then $F_\nabla^{0,2} = 0$ by (1.1.15) and continuity.

Alternatively, to prove (c), e.g. assuming that $\overline{\partial}_\nabla u \equiv 0$, note that applying $\overline{\partial}_\nabla$ again to the last equation we obtain $F_\nabla^{0,2}u \equiv 0$. Then, since $u \neq 0$, it follows that $F_\nabla^{0,2}$ vanishes in the non-empty open set $M \setminus u^{-1}(0)$. Also, it follows from the second critical point equation in (YMH$_\varepsilon$), and the fact that $\overline{\partial}F_\nabla^{0,2} = 0$ (by Bianchi), that we have $\Delta F_\nabla^{0,2} \equiv 0$, so by unique continuation we get $F_\nabla^{0,2} \equiv 0$.

□

As a consequence of Proposition 1.1.5, we get the vortex detection result alluded to above.

Corollary 1.1.6. *Under the hypotheses of Proposition 1.1.5, if either the second or the third equation in* (vor$_+$) *holds, then all three equations in the system hold. A similar statement is true for* (vor$_-$).

1.1.2 Specializing to $\mathbb{CP}^n$

We denote by $\mathbb{CP}^n$ the n-dimensional complex projective space and view it as the quotient of S^{2n+1} under the S^1-action

$$(z_0, \cdots, z_n) \mapsto (e^{\sqrt{-1}\theta} z_0, \cdots, e^{\sqrt{-1}\theta} z_n). \tag{1.1.16}$$

We normalize the Fubini–Study metric g_{FS} on $\mathbb{CP}^n$ so that, if S^{2n+1} is equipped with the metric of constant curvature 1, then the projection map $\pi : S^{2n+1} \to (\mathbb{CP}^n, \frac{1}{4} g_{FS})$ is a Riemannian submersion. Under this normalization, we have

$$\mathrm{Ric}_{g_{FS}} = \frac{n+1}{2} g_{FS},$$

and the curvature tensor of g_{FS} is given by

$$R_{X,Y}Z = \frac{1}{4}\Big(\langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle JY, Z\rangle JX - \langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ\Big), \tag{1.1.17}$$

where J denotes the complex structure on $\mathbb{CP}^n$. For later use in computations involving the Weitzenböck formula, we note the following straightforward consequence of (1.1.17). Given any 2-form ψ on $\mathbb{CP}^n$, in terms of a local orthonormal frame $e_1, \cdots, e_{2n}$ we have

$$\psi_{e_k, R_{e_i,e_j} e_k} = -\psi^{1,1}_{e_i,e_j} - (\Lambda\psi)\omega_{e_i,e_j}. \tag{1.1.18}$$

Now we would like to recall some useful properties of Killing vector fields and real holomorphic vector fields on $\mathbb{CP}^n$. These fields play an important role in the proof of the Lawson–Simons theorem [LS73] that reduces stable minimal submanifolds of $\mathbb{CP}^n$ to complex ones, and we also make use of them in the proof of Theorem 1.3.8. Below we very briefly summarize the properties of these vector fields which are most relevant to us. Recall that a vector field X is real holomorphic if, and only if, $\mathcal{L}_X J = 0$, or equivalently

$$\nabla_{Jv} X = J\nabla_v X, \text{ for all } v.$$

Since J is integrable, we check easily that if X is real holomorphic, then so is JX. Moreover, because $\mathbb{CP}^n$ is a compact Kähler manifold, an application of the Hodge decomposition theorem shows that any Killing vector field is real holomorphic (see [Mor07, Proposition 15.5]). Thus, letting $\mathcal{K}$ denote the set of Killing vector fields, it follows that any vector field belonging to

$$J\mathcal{K} = \{JX \mid X \in \mathcal{K}\}$$

is real holomorphic. Given $X \in \mathcal{K}$, for all v, w we have

$$\langle \nabla_v JX, w\rangle = \langle J\nabla_v X, w\rangle = \langle \nabla_{Jv} X, w\rangle = -\langle \nabla_w X, Jv\rangle,$$

where the last equality follows because X is a Killing field. Upon noting that $-\langle \nabla_w X, Jv\rangle = \langle \nabla_w JX, v\rangle$, we get that ∇JX is symmetric in the sense that

$$\langle \nabla_v JX, w\rangle = \langle \nabla_w JX, v\rangle. \tag{1.1.19}$$

Next we recall some further facts about Killing vector fields on $\mathbb{CP}^n$, the main reference being the classical text by Kobayashi–Nomizu [KN96, Chapter XI]. Again viewing $\mathbb{CP}^n$ as the quotient of S^{2n+1} by the action (1.1.16), for each given $A \in \mathfrak{su}(n+1)$, we see that the restrictions to S^{2n+1} of the maps e^{tA} descend to isometries of $\mathbb{CP}^n$ which act by

$$\varphi_t([z]) = [e^{tA}z] \text{ for } z \in \mathbb{C}^{n+1},$$

and these produce a Killing field on $\mathbb{CP}^n$ given by

$$W_A = \frac{d}{dt}\Big|_{t=0} \varphi_t.$$

The map $A \mapsto W_A$ from $\mathfrak{su}(n+1)$ to $\mathcal{K}$ turns out to be a Lie algebra isomorphism, and induces an inner product on $\mathcal{K}$ via

$$(W_A, W_B)_{\mathcal{K}} := 2\mathrm{tr}(AB^*), \;\; A, B \in \mathfrak{su}(n+1). \tag{1.1.20}$$

At any $x \in \mathbb{CP}^n$, with respect to the inner product (1.1.20), we have the orthogonal decomposition

$$\mathcal{K} = \mathfrak{f}_x \oplus \mathfrak{p}_x,$$

where $\mathfrak{f}_x = \{X \in \mathcal{K} \mid X|_x = 0\}$ and $\mathfrak{p}_x = \{X \in \mathcal{K} \mid (\nabla X)|_x = 0\}$. The former generates isometries of $\mathbb{CP}^n$ which fix x, while the latter is isometric to the tangent space $T_x\mathbb{CP}^n$ via the evaluation map $X \mapsto X|_x$. Thus, letting $d = \dim \mathcal{K}$, we see that given $x \in \mathbb{CP}^n$, we can choose an orthonormal basis $X_1, \cdots, X_{2n}, \cdots, X_d$ of $\mathcal{K}$ with respect to the metric (1.1.20) such that

$$X_1|_x, \cdots, X_{2n}|_x \text{ form an orthonormal basis for } T_x\mathbb{CP}^n, \tag{1.1.21}$$

while

$$(\nabla X_1)|_x = \cdots = (\nabla X_{2n})|_x = 0, \quad X_{2n+1}|_x = \cdots = X_d|_x = 0. \tag{1.1.22}$$

To conclude, we recall the following result which may be gathered from the calculations in [LS73].

Lemma 1.1.7. *Let $X_1, \cdots, X_d$ be an orthonormal basis for $\mathcal{K}$. Then for any $x \in \mathbb{CP}^n$ and $X, Y, Z, W \in T_x\mathbb{CP}^n$, there holds*

$$\sum_{a=1}^{d} \langle \nabla_X JX_a, Y \rangle \langle \nabla_Z JX_a, W \rangle = \langle R_{X,JY} JZ, W \rangle. \tag{1.1.23}$$

1.2 Stability threshold for reducible solutions

We prove Theorem 1.2.3. Let (M, g, J, ω) be a closed Kähler manifold with complex dimension n and let (L, h) be a Hermitian line bundle over M such that $C_1(L, \omega) \geqslant 0$. For a configuration of the form $(0, \nabla) \in \Omega^0(L) \times \mathcal{A}^h$, equation (YMH$_\varepsilon$) reduces to $d^*F_\nabla = 0$, which means that $(0, \nabla)$ is a critical point of E_ε if, and only if, F_∇ is harmonic, in which case we refer to $(0, \nabla)$ as a *normal phase solution*. We first observe that there is a stability criterion for normal phase solutions in terms of ε and the first eigenvalue of $\nabla^*\nabla$ acting on sections of L. The complex structure on M is irrelevant for this first proposition; the proof is valid for a general closed Riemannian manifold M.

Proposition 1.2.1. *Suppose ∇ is a metric connection on L with harmonic curvature F_∇. Denote by $0 < \mu_1(\nabla) < \mu_2(\nabla) < \ldots \uparrow \infty$ the unbounded increasing sequence of positive eigenvalues of the self-adjoint elliptic operator $\nabla^*\nabla$. Then $(0, \nabla) \in \Omega^0(L) \times \mathcal{A}^h$ is a stable critical point for E_ε if, and only if, $\varepsilon^{-2} \leqslant 2\mu_1(\nabla)$. Moreover, for each $N \in \mathbb{N}$, there is $\varepsilon_N := \dfrac{1}{\sqrt{2\mu_N(\nabla)}}$ such that if $\varepsilon < \varepsilon_N$ then the Morse-index of $(0, \nabla)$ is at least N.*

Proof. ($\Leftarrow$) Suppose $\varepsilon^{-2} \leqslant 2\mu_1(\nabla)$. We first recall that (u, ∇) is a critical point of E_ε since the curvature F_∇ is harmonic. Now, by the variational characterization of $\mu_1(\nabla)$ of L, we have

$$\int_M |\nabla v|^2 \mathrm{vol}_g \geqslant \mu_1(\nabla) \int_M |v|^2 \mathrm{vol}_g, \text{ for all } v \in \Omega^0(L).$$

Now, by (1.1.1), we have for all $(v, a) \in \Omega^0(L) \oplus \Omega^1$ that

$$\begin{aligned} 2(\delta^2 E_\varepsilon)_{(0,\nabla)}(v, a) &= \int_M 2\varepsilon^2 |da|^2 + 2|\nabla v|^2 - \varepsilon^{-2}|v|^2 \mathrm{vol}_g \\ &\geqslant \int_M 2|\nabla v|^2 - \varepsilon^{-2}|v|^2 \mathrm{vol}_g \\ &\geqslant (2\mu_1(\nabla) - \varepsilon^{-2}) \int_M |v|^2 \mathrm{vol}_g \geqslant 0, \end{aligned}$$

and hence $(0, \nabla)$ is stable.

($\Rightarrow$) Now, suppose that $\varepsilon^{-2} > 2\mu_1(\nabla)$ and let $v \in \Omega^0(L)$ be an L^2-unitary eigensection of $\nabla^*\nabla$ associated to the eigenvalue $\mu_1(\nabla)$. Then

$$(\delta^2 E_\varepsilon)_{(0,\nabla)}(v, 0) = \int_M 2|\nabla v|^2 - \varepsilon^{-2}|v|^2 \mathrm{vol}_g = (2\mu_1(\nabla) - \varepsilon^{-2}) < 0,$$

so that $(0, \nabla)$ is unstable.

Finally, let $v_1, \cdots, v_N \in \Omega^0(L)$ be L^2-unitary eigenvalues of $\nabla^*\nabla$ associated to the eigenvalue $\mu_i(\nabla)$, respectively. Taking $\varepsilon < \varepsilon_N$ implies that $\varepsilon^{-2} > 2\mu_N(\nabla)$ and from the discussion above we get

$$(\delta^2 E_\varepsilon)_{(0,\nabla)}(v_i, 0) = (2\mu_i(\nabla) - \varepsilon^{-2}) < 0$$

and since $\{v_1, \cdots, v_N\}$ is linearly independent, it follows that there is at least N decreasing directions at $(0, \nabla)$, i.e. the Morse-index this point is at least N.

□

In cases where $L \to (M, g, J, \omega)$ admits a section v satisfying $\nabla^{0,1} v \equiv 0$, the threshold in the previous proposition can be expressed in terms of the first Chern class of L and the Kähler form. The next result is not new, see for example [NO24, Theorem 3.3], but we include the proof here for convenience.

Proposition 1.2.2. *Again suppose that ∇ is a metric connection with harmonic curvature F_∇. Then we have*

$$\mu_1(\nabla) \geqslant \frac{2\pi C_1(L,\omega)}{\mathrm{Vol}(M,\omega)}. \tag{1.2.1}$$

Moreover, equality holds in (1.2.1) if, and only if, $F_\nabla^{0,2} = 0$ and $\mu_1(\nabla)$-eigensections are holomorphic sections of $(L, \overline{\partial}_\nabla)$.

Proof. Since $\sqrt{-1}F_\nabla$ is a harmonic 2-form and Λ commutes with the Hodge Laplacian, we see that $\sqrt{-1}\Lambda F_\nabla$ is a harmonic function and thus must be *constant*, as M is closed. Integrating over M shows that

$$2\pi C_1(L,\omega) = \int_M \sqrt{-1}F_\nabla \wedge \omega^{[n-1]} = \int_M (\sqrt{-1}\Lambda F_\nabla)\omega^{[n]} = (\sqrt{-1}\Lambda F_\nabla)\mathrm{Vol}(M),$$

and hence $\sqrt{-1}\Lambda F_\nabla = \dfrac{2\pi C_1(L,\omega)}{\mathrm{Vol}(M)}$. Consequently, if we let $0 \neq v \in \Omega^0(L)$ be a $\mu_1(\nabla)$-eigensection of $\nabla^*\nabla$, using the Weitzenböck–Kähler identity

$$2\overline{\partial}_\nabla^*\overline{\partial}_\nabla = \nabla^*\nabla - i\Lambda F_\nabla,$$

and integrating by parts we get

$$0 \leqslant 2\|\overline{\partial}_\nabla v\|_{L^2}^2 = \left(\mu_1(\nabla) - \frac{2\pi C_1(L,\omega)}{\mathrm{Vol}(M,\omega)}\right)\|v\|_{L^2}^2, \tag{1.2.2}$$

which implies (1.2.1). Moreover, equality happens exactly when the $\mu_1(\nabla)$-eigenvectors v are the elements of $\ker(\overline{\partial}_\nabla)$, which implies $F_\nabla^{0,2} = 0$ in a non-empty open set, and so together with the harmonicity $\Delta F_\nabla^{0,2} = 0$ one gets $F_\nabla^{0,2} \equiv 0$. □

Theorem 1.2.3. *Suppose $L \to \mathbb{CP}^n$ is a Hermitian line bundle, and ∇ is a metric connection on L with harmonic curvature F_∇, so that $(0, \nabla)$ is a critical point of E_ε. Assuming further that $C_1(L,\omega) > 0$, we have*

(a) $(0,\nabla)$ *is stable critical point of E_ε if, and only if,*

$$\varepsilon^{-2} \leqslant \frac{4\pi C_1(L,\omega)}{\mathrm{Vol}(\mathbb{CP}^n)}. \tag{1.2.3}$$

(b) $(0,\nabla)$ *is a solution to (vor$_+$) if, and only if, equality holds in (1.2.3).*

Proof. The theorem is a consequence of Propositions 1.2.1 and 1.2.2. Specifically, since we are on $\mathbb{CP}^n$, the assumption that F_∇ is harmonic implies that

$$\sqrt{-1}F_\nabla = \frac{\lambda}{n}\omega, \text{ for some } \lambda \in \mathbb{R}. \tag{1.2.4}$$

In particular, F_∇ is of type $(1,1)$, so that $\nabla^{0,1}$ is compatible with a holomorphic structure on L. The assumption that $C_1(L,\omega) > 0$ together with standard facts on holomorphic line bundles over $\mathbb{CP}^n$ then guarantees the existence of a section $v \not\equiv 0$ such that $\nabla^{0,1}v \equiv 0$, and it follows from Proposition 1.2.2 that

$$\mu_1(\nabla) = \frac{2\pi C_1(L,\omega)}{\mathrm{Vol}(\mathbb{CP}^n)}.$$

Part (a) now follows immediately from Proposition 1.2.1. For part (b), we note by (1.2.4) that the first two equations in (vor$_+$) are automatically satisfied by $(0,\nabla)$. Moreover, integrating (1.2.4) over $\mathbb{CP}^n$ as in the proof of Proposition 1.2.2 reveals that $\sqrt{-1}\Lambda F_\nabla = \lambda = \dfrac{2\pi C_1(L,\omega)}{\mathrm{Vol}(M,\omega)}$. Consequently, $(0,\nabla)$ solves (vor$_+$) if, and only if, $\varepsilon^{-2} = \dfrac{4\pi C_1(L,\omega)}{\mathrm{Vol}(\mathbb{CP}^n)}$, as asserted. □

Remark 1.2.4. With the result of Theorem 1.2.3, we notice that the range of ε for which one has existence of vortices, namely (1.4.4), almost coincides with the set of ε values for which the normal phase solutions $(0,\nabla)$ are unstable, the only exception being the case $\varepsilon^{-2} = \dfrac{4\pi C_1(L,\omega)}{\mathrm{Vol}(\mathbb{CP}^n)}$, and in this case $(0,\nabla)$ is stable and actually a vortex. This shows that first-order reduction holds for normal phase solutions, provided vortices exist. ○

1.3 Describing stable critical points: the averaging approach

We prove Theorem 1.3.8, following the approach taken in the works of Lawson–Simons [LS73] and Bourguignon–Lawson [BL81]. Throughout this section, unless otherwise stated, we assume that (u,∇) is a smooth, stable critical point of E_ε on $\mathbb{CP}^n$ with $F_\nabla \in \Omega^{1,1}$. We shall make use of Lemma 1.1.2 with (v,a) having the form

$$(v,a) = (\sqrt{-1}\nabla_{JX}u, -\iota_X(h\omega)),$$

where X is a Killing vector field on $\mathbb{CP}^n$ and h is a smooth function to be determined. In §1.3, we compute the result of substituting the above choice of (v,a) into $\widetilde{\mathcal{J}}^1_{(u,\nabla)}$ and $\widetilde{\mathcal{J}}^2_{(u,\nabla)}$. Then, in §1.3 we average these computations over the space of Killing fields on $\mathbb{CP}^n$ to arrive at the proof of Theorem 1.3.8.

Basic computations

We begin with computations related to $\widetilde{\mathcal{J}}^1(v,a)$.

Lemma 1.3.1. *Let X be a Killing field on $\mathbb{CP}^n$. Then*

$$\nabla^*\nabla(\nabla_{JX}u) + \frac{3|u|^2-1}{2\varepsilon^2}\nabla_{JX}u = -2(\sqrt{-1}F_{JX,e_i})\cdot\sqrt{-1}\nabla_{e_i}u - 2\left\langle\nabla_{e_i}JX, e_j\right\rangle\nabla^2_{e_i,e_j}u. \tag{1.3.1}$$

Proof. In a local geodesic frame we have,

$$\begin{aligned}
-\nabla^*\nabla(\nabla_{JX}u) &= \nabla_{e_i}\nabla_{e_i}\nabla_{JX}u = \nabla_{e_i}\left(\nabla^2_{e_i,JX}u + \nabla_{\nabla_{e_i}JX}u\right)\\
&= \nabla_{e_i}\left(\nabla^2_{JX,e_i}u + F_{e_i,JX}u + \nabla_{\nabla_{e_i}JX}u\right)\\
&= \nabla^3_{e_i,JX,e_i}u + \nabla^2_{\nabla_{e_i}JX,e_i}u + (\nabla F)_{e_i,e_i,JX}u + F_{e_i,\nabla_{e_i}JX}u\\
&\quad + F_{e_i,JX}\nabla_{e_i}u + \nabla^2_{e_i,\nabla_{e_i}JX}u + \nabla_{J\nabla^2_{e_i,e_i}X}u.
\end{aligned}$$

We now use the identity

$$\nabla^3_{e_i,JX,e_i}u - \nabla^3_{JX,e_i,e_i}u = F_{e_i,JX}\nabla_{e_i}u - \nabla_{R_{e_i,JX}e_i}u,$$

to get

$$\begin{aligned}
-\nabla^*\nabla(\nabla_{JX}u) = {}&\nabla^3_{JX,e_i,e_i}u + 2F_{e_i,JX}\nabla_{e_i}u - (d^*_\nabla F)_{JX}u\\
&+ \nabla^2_{\nabla_{e_i}JX,e_i}u + \nabla^2_{e_i,\nabla_{e_i}JX}u + F_{e_i,\nabla_{e_i}JX}u\\
&- \nabla_{R_{e_i,JX}e_i}u + \nabla_{JR_{e_i,X}e_i}u.
\end{aligned} \tag{1.3.2}$$

Where in the last term we used the following formula,

$$\nabla^2_{V,W}X = R_{V,X}W, \tag{1.3.3}$$

which is valid on any Riemannian manifold as long as X is a Killing vector field. To simplify the curvature terms on the last line of (1.3.2), note that

$$-\nabla_{R_{e_i,JX}e_i}u + \nabla_{JR_{e_i,X}e_i}u = \nabla_{R_{JX,e_i}e_i}u - \nabla_{JR_{X,e_i}e_i} = \nabla_{\mathrm{Ric}(JX)}u - \nabla_{J\mathrm{Ric}(X)} = 0, \tag{1.3.4}$$

where the last equality holds because the Fubini–Study metric is Kähler. Also, using the second equation in (YMH$_\varepsilon$), we have

$$-(d^*_\nabla F)_{JX} = \frac{\sqrt{-1}\mathrm{Re}\left\langle\sqrt{-1}u, \nabla_{JX}u\right\rangle}{\varepsilon^2}. \tag{1.3.5}$$

Finally, by (1.1.19)

$$\begin{aligned}
&\nabla^2_{\nabla_{e_i}JX,e_i}u + \nabla^2_{e_i,\nabla_{e_i}JX}u = \left\langle\nabla_{e_i}JX, e_j\right\rangle(\nabla^2_{e_j,e_i}u + \nabla^2_{e_i,e_j}u) = 2\left\langle\nabla_{e_i}JX, e_j\right\rangle\nabla^2_{e_i,e_j}u,\\
&F_{e_i,\nabla_{e_i}JX} = \left\langle\nabla_{e_i}JX, e_j\right\rangle F_{e_i,e_j} = 0.
\end{aligned} \tag{1.3.6}$$

Substituting (1.3.4), (1.3.5) and (1.3.6) into (1.3.2), we get

$$\begin{aligned}
-\nabla^*\nabla(\nabla_{JX}u) = {}& -\nabla_{JX}(\nabla^*\nabla u) + 2F_{e_i,JX}\nabla_{e_i}u + \frac{\mathrm{Re}\left\langle\sqrt{-1}u, \nabla_{JX}u\right\rangle}{\varepsilon^2}\sqrt{-1}u\\
&+ 2\left\langle\nabla_{e_i}JX, e_j\right\rangle\nabla^2_{e_i,e_j}u.
\end{aligned} \tag{1.3.7}$$

We now use the first equation in (YMH$_\varepsilon$) to get

$$-\nabla_{JX}\Big(\nabla^*\nabla u\Big) = -\nabla_{JX}\Big(\frac{1-|u|^2}{2\varepsilon^2}u\Big) = \frac{\mathrm{Re}\langle u, \nabla_{JX}u\rangle}{\varepsilon^2}u + \frac{|u|^2-1}{2\varepsilon^2}\nabla_{JX}u.$$

Next we substitute this into (1.3.7) and use the following consequence of the fact that $u(x), \sqrt{-1}u(x)$ forms an orthogonal basis for L_x with respect to $\mathrm{Re}\langle\cdot,\cdot\rangle$ where $u(x)\neq 0$:

$$\mathrm{Re}\,\langle u, \nabla_{JX}u\rangle\, u + \mathrm{Re}\left\langle \sqrt{-1}u, \nabla_{JX}u\right\rangle \sqrt{-1}u = |u|^2\nabla_{JX}u.$$

Doing so yields

$$-\nabla^*\nabla(\nabla_{JX}u) = \frac{3|u|^2-1}{2\varepsilon^2}\nabla_{JX}u + 2F_{e_i,JX}\nabla_{e_i}u + 2\,\langle\nabla_{e_i}JX, e_j\rangle\,\nabla^2_{e_i,e_j}u,$$

which immediately gives (1.3.1). □

The next two lemmas will be used in the computation of $\widetilde{\mathcal{J}}^2(v,a)$.

Lemma 1.3.2. *Let ψ be a 2-form and X a Killing field on $\mathbb{CP}^n$. Then*

$$(\Delta\iota_X\psi)_{e_j} = (\Delta\psi)_{X,e_j} + \psi^{1,1}_{X,e_j} + (\Lambda\psi)\omega_{X,e_j} - \langle\nabla_{e_i}X, e_k\rangle\Big((d\psi)_{e_i,e_k,e_j} + (\nabla\psi)_{e_j,e_k,e_i}\Big). \tag{1.3.8}$$

Moreover, in the case $\psi = h\omega$ where h is a smooth function, we have

$$\Delta(h\omega) = (\Delta h)\omega. \tag{1.3.9}$$

Proof. As in the previous proof, we take a local geodesic frame $\{e_i\}$. We begin by noting that

$$d^*\iota_X\psi = -e_i(\psi_{X,e_i}) = -(\nabla\psi)_{e_i,X,e_i} - \psi_{\nabla_{e_i}X,e_i} = -(d^*\psi)_X - \psi_{\nabla_{e_i}X,e_i}, \tag{1.3.10}$$

and hence

$$\begin{aligned}(dd^*\iota_X\psi)_{e_j} &= e_j(d^*\iota_X\psi)\\ &= -e_j(d^*\psi)_X - (\nabla\psi)_{e_j,\nabla_{e_i}X,e_i} - \psi_{e_i,R_{X,e_j}e_i},\end{aligned} \tag{1.3.11}$$

where in the second line we again used (1.3.3). On the other hand,

$$\begin{aligned}(d\iota_X\psi)_{e_i,e_j} &= e_i(\psi_{X,e_j}) - e_j(\psi_{X,e_i})\\ &= (\nabla\psi)_{e_i,X,e_j} + \psi_{\nabla_{e_i}X,e_j} - (\nabla\psi)_{e_j,X,e_i} - \psi_{\nabla_{e_j}X,e_i}\\ &= (d\psi)_{e_i,X,e_j} + (\nabla\psi)_{X,e_i,e_j} + \psi_{\nabla_{e_i}X,e_j} - \psi_{\nabla_{e_j}X,e_i}.\end{aligned}$$

Thus

$$\begin{aligned}(d^*d\iota_X\psi)_{e_j} &= -e_i\Big((d\iota_X\psi)_{e_i,e_j}\Big)\\ &= -e_i\Big((d\psi)_{e_i,X,e_j} + (\nabla\psi)_{X,e_i,e_j} + \psi_{\nabla_{e_i}X,e_j} - \psi_{\nabla_{e_j}X,e_i}\Big)\\ &= (d^*d\psi)_{X,e_j} - (d\psi)_{e_i,\nabla_{e_i}X,e_j} - (\nabla^2\psi)_{e_i,X,e_i,e_j}\\ &\quad - (\nabla\psi)_{\nabla_{e_i}X,e_i,e_j} - (\nabla\psi)_{e_i,\nabla_{e_i}X,e_j} - \psi_{\nabla^2_{e_i,e_i}X,e_j}\\ &\quad + (\nabla\psi)_{e_i,\nabla_{e_j}X,e_i} + \psi_{\nabla^2_{e_i,e_j}X,e_i}.\end{aligned} \tag{1.3.12}$$

For the first two terms on the second-to-last line, we note that since ∇X is anti-symmetric, we get

$$(\nabla\psi)_{\nabla_{e_i}X,e_i,e_j} + (\nabla\psi)_{e_i,\nabla_{e_i}X,e_j} = \langle \nabla_{e_i}X, e_k\rangle\Big((\nabla\psi)_{e_k,e_i,e_j} + (\nabla\psi)_{e_i,e_k,e_j}\Big) = 0.$$

Next, for the term involving the second derivative of ψ, we change the order of differentiation to get

$$\begin{aligned}(\nabla^2\psi)_{e_i,X,e_i,e_j} &= (\nabla^2\psi)_{X,e_i,e_i,e_j} - \psi_{R_{e_i,X}e_i,e_j} - \psi_{e_i,R_{e_i,X}e_j}\\ &= -(\nabla d^*\psi)_{X,e_j} - \psi_{R_{e_i,X}e_i,e_j} - \psi_{e_i,R_{e_i,X}e_j}.\end{aligned}$$

Putting the two above computations back into (1.3.12) and applying (1.3.3) a couple of times yields

$$\begin{aligned}(d^*d\iota_X\psi)_{e_j} &= (d^*d\psi)_{X,e_j} - (d\psi)_{e_i,\nabla_{e_i}X,e_j} + (\nabla d^*\psi)_{X,e_j} + \psi_{R_{e_i,X}e_i,e_j} + \psi_{e_i,R_{e_i,X}e_j}\\ &\quad + \psi_{\mathrm{Ric}(X),e_j} + (d^*\psi)_{\nabla_{e_j}X} - \psi_{R_{X,e_i}e_j,e_i}\\ &= (d^*d\psi)_{X,e_j} - (d\psi)_{e_i,\nabla_{e_i}X,e_j} + (\nabla d^*\psi)_{X,e_j} + (d^*\psi)_{\nabla_{e_j}X}.\end{aligned} \tag{1.3.13}$$

Summing (1.3.13) and (1.3.11), and also noting that

$$(\nabla d^*\psi)_{X,e_j} - e_j(d^*\psi)_X + (d^*\psi)_{\nabla_{e_j}X} = (\nabla d^*\psi)_{X,e_j} - (\nabla d^*\psi)_{e_j,X} = (dd^*\psi)_{X,e_j}$$

leads to

$$(\Delta\iota_X\psi)_{e_j} = (\Delta\psi)_{X,e_j} - \psi_{e_i,R_{X,e_j}e_i} - (d\psi)_{e_i,\nabla_{e_i}X,e_j} - (\nabla\psi)_{e_j,\nabla_{e_i}X,e_i},$$

which easily implies (1.3.8) upon recalling (1.1.18).

To get the second asserted identity, we apply to $h\omega$ the Weitzenböck formula for 2-forms to get:

$$(\Delta(h\omega))_{kl} = (\nabla^*\nabla(h\omega))_{kl} + h\omega_{\mathrm{Ric}(e_k),e_l} + h\omega_{e_k,\mathrm{Ric}(e_l)} + h\omega_{e_i,R_{e_k,e_l}e_i}. \tag{1.3.14}$$

As ω is parallel, we have

$$\nabla^*\nabla(h\omega) = (\Delta h)\omega. \tag{1.3.15}$$

On the other hand, recalling that $\omega = g(J\cdot,\cdot)$ and using the extra symmetry of R available for Kähler metrics, together with the Bianchi identity, we have

$$\begin{aligned}\omega_{e_i,R_{e_k,e_l}e_i} &= \langle R_{e_i,Je_i}e_k, e_l\rangle = -\langle R_{Je_i,e_k}e_i, e_l\rangle - \langle R_{e_k,e_i}Je_i, e_l\rangle\\ &= \langle R_{e_k,Je_i}Je_i, Je_l\rangle + \langle R_{e_k,e_i}e_i, Je_l\rangle\\ &= 2\mathrm{Ric}(e_k, Je_l) = -2\mathrm{Ric}(Je_k, e_l) = -(\omega_{\mathrm{Ric}(e_k),e_l} + \omega_{e_k,\mathrm{Ric}(e_l)}).\end{aligned}$$

Substituting this and (1.3.15) into (1.3.14) gives (1.3.9). The proof is now complete. □

Lemma 1.3.3. *Given $m \in \mathbb{N}$, define*

$$h = \frac{1 - |u|^2}{2m\varepsilon^2}$$

and write ψ for $h\omega$. Then we have

$$\begin{aligned} \varepsilon^2(\Delta\iota_X\psi)_{e_j} + |u|^2(\iota_X\psi)_{e_j} = & \frac{|\nabla u|^2}{m}\omega_{X,e_j} + \varepsilon^2(n+1)h\omega_{X,e_j} \\ & - \varepsilon^2 \langle \nabla_{e_i} X, e_k \rangle \Big((d\psi)_{e_i,e_k,e_j} + (\nabla\psi)_{e_j,e_k,e_i} \Big). \end{aligned} \tag{1.3.16}$$

Proof. Since $\omega^{1,1} = \omega$ and $\Lambda\omega = n$, combining the two identities in the previous lemma gives

$$(\Delta\iota_X\psi)_{e_j} = (\Delta h)\omega_{X,e_j} + (n+1)h\omega_{X,e_j} - \langle \nabla_{e_i} X, e_k \rangle \Big((d\psi)_{e_i,e_k,e_j} + (\nabla\psi)_{e_j,e_k,e_i} \Big). \tag{1.3.17}$$

On the other hand, using the first equation in (YMH$_\varepsilon$), we have

$$\Delta h = \frac{|\nabla u|^2}{m\varepsilon^2} - \frac{|u|^2}{\varepsilon^2} h.$$

Substituting this back into (1.3.17) and multiplying through by ε^2 yields (1.3.16). □

In view of Lemma 1.1.2, we shall also need the following result.

Lemma 1.3.4. *Let X be a Killing field on $\mathbb{CP}^n$ and define $h = \dfrac{1-|u|^2}{2m\varepsilon^2}$ as in Lemma 1.3.3. Then*

$$d^*\iota_X(-h\omega) + \frac{\operatorname{Re}\left\langle \sqrt{-1}u, \sqrt{-1}\nabla_{JX} u \right\rangle}{\varepsilon^2} = (1-m)\langle \nabla h, JX \rangle + h\omega_{\nabla_{e_i} X, e_i}. \tag{1.3.18}$$

Proof. By direct computation, we find that

$$\begin{aligned} d^*\iota_X(h\omega) = & -e_i(h\omega_{X,e_i}) = -e_i(h)\omega_{X,e_i} - h\omega_{\nabla_{e_i} X, e_i} \\ = & -\langle JX, \nabla h \rangle - h\omega_{\nabla_{e_i} X, e_i}. \end{aligned}$$

Since

$$\frac{\operatorname{Re}\left\langle \sqrt{-1}u, \sqrt{-1}\nabla_{JX} u \right\rangle}{\varepsilon^2} = -m \langle \nabla h, JX \rangle,$$

we deduce (1.3.18). □

We now put together the computations contained in the previous lemmas.

Proposition 1.3.5. *Suppose (u, ∇) is a smooth, stable critical point of E_ε and let X be a Killing vector field on $\mathbb{CP}^n$. Then*

$$
\begin{aligned}
&\varepsilon^2 \int_{\mathbb{CP}^n} (m-1)^2 |\langle \nabla h, JX\rangle|^2 - 2(m-1)h \langle \nabla h, JX\rangle \omega_{\nabla_{e_i}X, e_i} + h^2 \Big| \sum_i \omega_{\nabla_{e_i}X, e_i} \Big|^2 \\
\leqslant &\int_{\mathbb{CP}^n} 2\sqrt{-1} F_{JX, e_j} \mathrm{Re} \left\langle \sqrt{-1}\nabla_{JX} u, \nabla_{e_j} u \right\rangle - 4h |\nabla_{JX} u|^2 \\
&+ \int_{\mathbb{CP}^n} \frac{|\nabla u|^2}{m} h |JX|^2 + \varepsilon^2 (n+1) h^2 |JX|^2 \\
&- \int_{\mathbb{CP}^n} 2 \langle \nabla_{e_i} JX, e_j \rangle \mathrm{Re} \left\langle \nabla^2_{e_i, e_j} u, \nabla_{JX} u \right\rangle \\
&- \int_{\mathbb{CP}^n} \varepsilon^2 h \langle \nabla_{e_i} X, e_k \rangle \Big((d\psi)_{e_i, e_k, JX} + (\nabla\psi)_{JX, e_k, e_i} \Big)
\end{aligned}
\tag{1.3.19}
$$

Proof. For brevity, we write

$$(v, a) = (\sqrt{-1}\nabla_{JX} u, \iota_X(-h\omega)).$$

Thanks to Lemma 1.3.1 and Lemma 1.3.3, we get

$$
\begin{aligned}
\widetilde{\mathscr{J}}^1(v,a) =& \nabla^*\nabla(\sqrt{-1}\nabla_{JX} u) + 2(-\iota_X(h\omega)) \cdot \sqrt{-1}\nabla u + \frac{3|u|^2 - 1}{2\varepsilon^2} \sqrt{-1}\nabla_{JX} u \\
=& -2h\sqrt{-1}\nabla_{JX} u + 2\sqrt{-1} F_{JX, e_j} \nabla_{e_j} u - 2 \langle \nabla_{e_i} JX, e_j \rangle \sqrt{-1} \nabla^2_{e_i, e_j} u,
\end{aligned}
\tag{1.3.20}
$$

and

$$
\begin{aligned}
\widetilde{\mathscr{J}}^2(v,a)_{e_j} =& -\varepsilon^2 (\Delta \iota_X(h\omega))_{e_j} + 2\mathrm{Re} \left\langle \nabla_{JX} u, \nabla_{e_j} u \right\rangle - |u|^2 h \omega_{X, e_j} \\
=& -\frac{|\nabla u|^2}{m} \omega_{X, e_j} - \varepsilon^2 (n+1) h \omega_{X, e_j} + 2\mathrm{Re} \left\langle \nabla_{JX} u, \nabla_{e_j} u \right\rangle \\
&+ \varepsilon^2 \langle \nabla_{e_i} X, e_k \rangle \Big((d\psi)_{e_i, e_k, e_j} + (\nabla\psi)_{e_j, e_k, e_i} \Big).
\end{aligned}
\tag{1.3.21}
$$

Therefore,

$$
\begin{aligned}
\mathrm{Re} \left\langle \widetilde{\mathscr{J}}^1(v,a), v \right\rangle =& -2h |\nabla_{JX} u|^2 + 2\sqrt{-1} F_{JX, e_j} \mathrm{Re} \left\langle \sqrt{-1}\nabla_{JX} u, \nabla_{e_j} u \right\rangle \\
&- 2 \langle \nabla_{e_i} JX, e_j \rangle \mathrm{Re} \left\langle \nabla^2_{e_i, e_j} u, \nabla_{JX} u \right\rangle,
\end{aligned}
$$

while

$$
\begin{aligned}
\left\langle \widetilde{\mathscr{J}}^2(v,a), a \right\rangle =& \frac{|\nabla u|^2}{m} h |JX|^2 + \varepsilon^2 (n+1) h^2 |JX|^2 - 2h |\nabla_{JX} u|^2 \\
&- \varepsilon^2 h \langle \nabla_{e_i} X, e_k \rangle \Big((d\psi)_{e_i, e_k, JX} + (\nabla\psi)_{JX, e_k, e_i} \Big).
\end{aligned}
$$

Combining these with Lemma 1.3.4 and recalling Lemma 1.1.2 gives the desired inequality. □

Averaging over real holomorphic vector fields

To continue, we let $X_1, \cdots, X_d$ be any orthonormal basis of the space $\mathcal{K}$ of Killing vector fields on $\mathbb{CP}^n$ with respect to the inner product (1.1.20) and define, in view of the integrands in Proposition 1.3.5, the following trace-like quantities:

$$
\begin{aligned}
Q_1 = & \sum_{a=1}^{d}\Big[2\sqrt{-1}F_{JX_a,e_j}\cdot \mathrm{Re}\left\langle \sqrt{-1}\nabla_{JX_a}u, \nabla_{e_j}u\right\rangle - 4h|\nabla_{JX_a}u|^2\Big] \\
& + \sum_{a=1}^{d}\Big[\varepsilon^2(n+1)h^2|JX_a|^2 + \frac{|\nabla u|^2}{m}h|JX_a|^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
Q_2 = & \sum_{a=1}^{d} 2\left\langle \nabla_{e_i}JX_a, e_j\right\rangle \mathrm{Re}\left\langle \nabla^2_{e_i,e_j}u, \nabla_{JX_a}u\right\rangle \\
& + \varepsilon^2\sum_{a=1}^{d} h\left\langle \nabla_{e_i}X_a, e_k\right\rangle\Big((d\psi)_{e_i,e_k,JX_a} + (\nabla\psi)_{JX_a,e_k,e_i}\Big)
\end{aligned}
$$

$$
\begin{aligned}
Q_3 = & \sum_{a=1}^{d}\Big[(m-1)^2|\left\langle \nabla h, JX_a\right\rangle|^2 - 2(m-1)h\left\langle \nabla h, JX_a\right\rangle \sum_{i=1}^{2n}\omega_{\nabla_{e_i}X_a,e_i} \\
& + h^2\Big|\sum_{i=1}^{2n}\omega_{\nabla_{e_i}X_a,e_i}\Big|^2\Big].
\end{aligned}
$$

Then Q_1, Q_2, Q_3 are all smooth, real-valued functions on $\mathbb{CP}^n$. More importantly, each one is well-defined, i.e invariant under changes of orthonormal basis of $\mathcal{K}$.

Proposition 1.3.6. *We have the following expressions for Q_1, Q_2, Q_3:*

$$
\begin{aligned}
Q_1 &= 2\sqrt{-1}F_{e_i,e_j}\mathrm{Re}\left\langle \sqrt{-1}\nabla_{e_i}u, \nabla_{e_j}u\right\rangle + 2n\varepsilon^2(n+1)h^2 + \frac{2n-4m}{m}h|\nabla u|^2, \\
Q_2 &= 0, \\
Q_3 &= (m-1)^2|\nabla h|^2 + 2n(n+1)h^2.
\end{aligned}
\tag{1.3.22}
$$

Proof. Fix $x \in \mathbb{CP}^n$ and choose an orthonormal basis $X_1, \cdots, X_d$ of $\mathcal{K}$ that satisfies (1.1.21) and (1.1.22). The asserted identities for Q_1 and Q_2 follows easily. For Q_3, the only point that perhaps requires elaboration is how we treat the term

$$
\sum_{a=1}^{d} h^2\Big|\sum_{i=1}^{2n}\omega_{\nabla_{e_i}X,e_i}\Big|^2.
$$

For this we note, with the help of (1.1.23), that

$$
\begin{aligned}
\sum_{a=1}^{d}\Big|\sum_{i=1}^{2n}\omega_{\nabla_{e_i}X_a,e_i}\Big|^2 &= \sum_{a=1}^{d}\sum_{i,j=1}^{2n}\left\langle \nabla_{e_i}JX_a, e_i\right\rangle\left\langle \nabla_{e_j}JX_a, e_j\right\rangle = \sum_{i,j=1}^{2n}\left\langle R_{e_i,Je_i}Je_je_j\right\rangle \\
&= -\sum_{i,j=1}^{2n}\Big[\left\langle R_{Je_i,Je_j}e_i, e_j\right\rangle + \left\langle R_{Je_j,e_i}Je_i, e_j\right\rangle\Big] \\
&= \sum_{i,j=1}^{2n}\left\langle R_{e_i,e_j}e_je_i\right\rangle + \left\langle R_{e_j,Je_i}Je_i, e_j\right\rangle \\
&= 2n(n+1),
\end{aligned}
$$

where we used the Bianchi identity to get the second line. □

The following is an immediate consequence of Proposition 1.3.5 and Proposition 1.3.6.

Corollary 1.3.7. *Let (u, ∇) be a smooth, stable critical point of E_ε and define $h = \dfrac{1-|u|^2}{2m\varepsilon^2}$. Then*

$$\begin{aligned}&\int_{\mathbb{CP}^n} 2\sqrt{-1}F_{e_i,e_j}\mathrm{Re}\left\langle \sqrt{-1}\nabla_{e_i}u, \nabla_{e_j}u\right\rangle + \frac{2n-4m}{m}h|\nabla u|^2 \\ \geqslant &\int_{\mathbb{CP}^n} \Big(\frac{m-1}{m}\Big)^2 \frac{|\mathrm{Re}\,\langle u, \nabla u\rangle\,|^2}{\varepsilon^2}.\end{aligned} \tag{1.3.23}$$

We are now ready to prove the main theorem of the section.

Theorem 1.3.8. *Suppose (u, ∇) is a smooth, stable critical point of E_ε on $\mathbb{CP}^n$ and that $\varepsilon^{-2} \geqslant \dfrac{4\pi|C_1(L,\omega)|}{\mathrm{Vol}(\mathbb{CP}^n)}$. Assume furthermore that*

(i) F_∇ is type $(1,1)$,

(ii) upon splitting $i\,F_\nabla$ orthogonally into $\dfrac{i\,\Lambda F_\nabla}{n}\omega + \xi$, with the second term satisfying $\Lambda\xi = 0$, there holds

$$\int_{\mathbb{CP}^n} \Big(\frac{n-1}{n}\Big)^2 \frac{|\mathrm{Re}\,\langle u, \nabla u\rangle\,|^2}{\varepsilon^2} - 2\xi_{e_i,e_j}\mathrm{Re}\left\langle i\,\nabla_{e_i}u, \nabla_{e_j}u\right\rangle \geqslant 0. \tag{1.3.24}$$

Then (u, ∇) is a solution to either (vor$_+$) or (vor$_-$).

Proof of Theorem 1.3.8

We take $m = n$ in Corollary 1.3.7 to get

$$\int_{\mathbb{CP}^n} 2\sqrt{-1}F_{e_i,e_j}\mathrm{Re}\left\langle \sqrt{-1}\nabla_{e_i}u, \nabla_{e_j}u\right\rangle - 2h|\nabla u|^2 \geqslant \int_{\mathbb{CP}^n} \Big(\frac{n-1}{n}\Big)^2 \frac{|\mathrm{Re}\,\langle u, \nabla u\rangle\,|^2}{\varepsilon^2}. \tag{1.3.25}$$

To treat the term in (1.3.23) involving $\sqrt{-1}F$, we decompose the latter as in the statement of Theorem 1.3.8 into

$$\sqrt{-1}F = \frac{\sqrt{-1}\Lambda F}{n}\omega + \xi =: f\omega + \xi,$$

for some $\xi \in \Omega^{1,1}$ satisfying $\Lambda\xi = 0$. (Remember that $F \in \Omega^{1,1}$ by assumption.) Recalling that

$$\nabla u = \nabla^{1,0}u + \nabla^{0,1}u,$$

and that

$$\nabla^{1,0}u \circ J = \sqrt{-1}\nabla^{1,0}u, \quad \nabla^{0,1}u \circ J = -\sqrt{-1}\nabla^{0,1}u$$

we compute

$$\begin{aligned}\omega_{e_i,e_j}\mathrm{Re}\left\langle\sqrt{-1}\nabla_{e_i}u,\nabla_{e_j}u\right\rangle &= \mathrm{Re}\left\langle\sqrt{-1}\nabla_{e_i}u,\nabla_{Je_i}u\right\rangle \\ &= \mathrm{Re}\left\langle\sqrt{-1}\nabla^{1,0}_{e_i}u+\sqrt{-1}\nabla^{0,1}_{e_i}u,\sqrt{-1}\nabla^{1,0}_{e_i}u-\sqrt{-1}\nabla^{0,1}_{e_i}u\right\rangle \\ &= |\nabla^{1,0}u|^2-|\nabla^{0,1}u|^2.\end{aligned} \tag{1.3.26}$$

Also noting that $|\nabla u|^2 = |\nabla^{1,0}u|^2 + |\nabla^{0,1}u|^2$, we see that the left-hand side of (1.3.25) becomes

$$\int_{\mathbb{CP}^n} 2\xi_{e_i,e_j}\mathrm{Re}\left\langle\sqrt{-1}\nabla_{e_i}u,\nabla_{e_j}u\right\rangle - 2(h-f)|\nabla^{1,0}u|^2 - 2(h+f)|\nabla^{0,1}u|^2,$$

and hence we deduce that

$$\begin{aligned}&-2\int_{\mathbb{CP}^n}(h-f)|\nabla^{1,0}u|^2+(h+f)|\nabla^{0,1}u|^2 \\ &\geqslant \int_{\mathbb{CP}^n}\Big(\frac{n-1}{n}\Big)^2\frac{|\mathrm{Re}\left\langle u,\nabla u\right\rangle|^2}{\varepsilon^2} - 2\xi_{e_i,e_j}\mathrm{Re}\left\langle\sqrt{-1}\nabla_{e_i}u,\nabla_{e_j}u\right\rangle.\end{aligned}$$

Now, in the case $u \not\equiv 0$, then by Proposition 1.1.5(a) we have $h \pm f \geqslant 0$, so that, under the assumption (1.3.24), we conclude that on $\mathbb{CP}^n$ we have

$$(h-f)|\nabla^{1,0}u|^2 \equiv 0, \quad (h+f)|\nabla^{0,1}u|^2 \equiv 0.$$

If $\nabla^{0,1}u \equiv 0$, we have by Corollary 1.1.6 that (u,∇) solves (vor$_+$). Otherwise, Proposition 1.1.5(a) gives $h+f=0$, in which case Corollary 1.1.6 shows that (u,∇) solves (vor$_-$).

Finally, note that if $\varepsilon^{-2} > \dfrac{4\pi|C_1(L,\omega)|}{\mathrm{Vol}(\mathbb{CP}^n)}$, then the stability assumption implies $u \not\equiv 0$ thanks to Theorem 1.2.3(a), and we are done in this case by the previous paragraph. It remains to consider the case where $\varepsilon^{-2} = \dfrac{4\pi|C_1(L,\omega)|}{\mathrm{Vol}(\mathbb{CP}^n)}$ and $u \equiv 0$. In this case, Theorem 1.2.3 shows that normal phase solutions $(0,\nabla)$ are stable and moreover satisfy either (vor$_+$) or (vor$_-$), and thus do not violate the statement of Theorem 1.3.8. This completes the proof.

1.4 Describing stable critical points: the linearization approach

In this section we prove Theorem 1.4.8. We begin in slightly greater generality by letting (M,g,J,ω) be a closed Kähler manifold with complex dimension n, and $L \to M$ a Hermitian line bundle with $C_1(L,\omega) \geqslant 0$. Later in this section we shall specialize to $M = \mathbb{CP}^n$.

We first recall the derivation of the vortex equations, following mostly [Bra90]. It will be useful later in this section to obtain a suitable way to write the second variation formula for the functional E_ε. Let (M, g, J, ω) be a closed Kähler manifold of complex dimension n and $(L, h) \to M$ a Hermitian line bundle. Given $(u, \nabla) \in \Omega^0(L) \times \mathcal{A}^h$, in the notation of (1.0.2), we have the following well-known identity [Bra90, Equation (2.3)]:

$$\int_M (|\nabla^{1,0}u|^2 - |\nabla^{0,1}u|^2)\omega^{[n]} = \int_M \left(\sqrt{-1}\Lambda F_\nabla\right)|u|^2\omega^{[n]}. \tag{1.4.1}$$

On the other hand, the counterpart of (1.4.1) for 2-forms (see [Bra90, Equation (2.4)]) gives

$$\int_M |F_\nabla|^2\omega^{[n]} = \int_M (|\Lambda F_\nabla|^2 + 4|F_\nabla^{0,2}|^2)\omega^{[n]} + \int_M F_\nabla \wedge F_\nabla \wedge \omega^{[n-2]}. \tag{1.4.2}$$

Combining this with (1.4.1) yields

$$\begin{aligned} E_\varepsilon(u, \nabla) = & \int_M \Big[4\varepsilon^2|F_\nabla^{0,2}|^2 + 2|\nabla^{0,1}u|^2 + |\varepsilon\sqrt{-1}\Lambda F_\nabla - \frac{1-|u|^2}{2\varepsilon}|^2\Big]\omega^{[n]} \\ & + \int_M \sqrt{-1}F_\nabla \wedge \omega^{[n-1]} + \varepsilon^2 \int_M F_\nabla \wedge F_\nabla \wedge \omega^{[n-2]} \\ = & \int_M \Big[4\varepsilon^2|F_\nabla^{0,2}|^2 + 2|\nabla^{1,0}u|^2 + |\varepsilon\sqrt{-1}\Lambda F_\nabla + \frac{1-|u|^2}{2\varepsilon}|^2\Big]\omega^{[n]} \\ & - \int_M \sqrt{-1}F_\nabla \wedge \omega^{[n-1]} + \varepsilon^2 \int_M F_\nabla \wedge F_\nabla \wedge \omega^{[n-2]}, \end{aligned} \tag{1.4.3}$$

which shows that $E_\varepsilon(u, \nabla)$ is bounded below by the topological quantity

$$\begin{aligned} & \pm 2\pi \int_M c_1(L) \wedge \omega^{[n-1]} - 8\pi^2\varepsilon^2 \int_M ch_2(L) \wedge \omega^{[n-2]} \\ =: & \pm 2\pi C_1(L, \omega) - 8\pi^2\varepsilon^2 Ch_2(L, \omega). \end{aligned}$$

where $c_1(L)$ and $ch_2(L)$ denote, respectively, the first Chern class and second Chern character of L. Setting the squared terms in the first and third lines of (1.4.3) to zero leads to (vor$_+$) and (vor$_-$), respectively, and (u, ∇) attains the topological lower bound if it satisfies one of the two systems. Focusing on (vor$_+$) for definiteness, we see that the first equation says that $\nabla^{0,1}$ is compatible with a holomorphic structure on L, with respect to which u is holomorphic by the second equation. The fundamental existence and classification result for solutions to the vortex equations is due, independently, to Bradlow [Bra90] and García-Prada [GP94]. In particular, solutions exist if, and only if,

$$\varepsilon^{-2} \geqslant \frac{4\pi\left|C_1(L, \omega)\right|}{\mathrm{Vol}(M, \omega)}. \tag{1.4.4}$$

Moving towards the stable critical points, the idea is to make use of the rewriting (1.4.3) to get another expression for the second variation formula, and from this to extract information about possible decreasing directions for the energy E_ε. If we take a variation $(v, a) \in \Omega^0(L) \oplus \Omega^1$, there holds

$$\begin{aligned} F^{0,2}_{\nabla - t\sqrt{-1}a} &= F_\nabla^{0,2} - t\sqrt{-1}\,\overline{\partial}a^{0,1}, \\ [(\nabla - t\sqrt{-1}a)(u + tv)]^{0,1} &= (\overline{\partial}_\nabla - t\sqrt{-1}a^{0,1})(u + tv) \\ &= \overline{\partial}_\nabla u + t(\overline{\partial}_\nabla v - \sqrt{-1}a^{0,1}u) - t^2\sqrt{-1}a^{0,1}v. \end{aligned}$$

Then we may compute the second variation using (1.4.3) to get

$$\begin{aligned}\delta^2 E_\varepsilon(u,\nabla)(v,a) &:= \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} E_\varepsilon(u+tv, \nabla - t\sqrt{-1}a)\\ &= \int_M \Big[4\varepsilon^2|\overline{\partial}a^{0,1}|^2 + 2|\overline{\partial}_\nabla v - \sqrt{-1}a^{0,1}u|^2 - 4\mathrm{Re}\langle\overline{\partial}_\nabla u, \sqrt{-1}a^{0,1}v\rangle\\ &\quad + \left|\varepsilon\Lambda da + \frac{\mathrm{Re}\langle u,v\rangle}{\varepsilon}\right|^2 + \left(\varepsilon\sqrt{-1}\Lambda F_\nabla - \frac{1-|u|^2}{2\varepsilon}\right)\frac{|v|^2}{\varepsilon}\Big]\mathrm{vol}_g.\end{aligned}$$

The following proposition is an observation made by Nagy and Oliveira in [NO21], and shows that the term $\mathrm{Re}\langle\overline{\partial}_\nabla u, \sqrt{-1}a^{0,1}v\rangle$ in the above formula can be cancelled via an averaging process with respect to a complex structure of sorts, which acts on $\Omega^0(L)\oplus\Omega^1$.

Definition 1.4.1. *Let $(u,\nabla)\in\Omega^0(L)\times\mathcal{A}^h$. For $(v,a)\in\Omega^0(L)\oplus\Omega^1$ we define*

$$\mathrm{Av}_{(u,\nabla)}(v,a) = \frac{1}{2}\Big(\delta^2E_\varepsilon(u,\nabla)(v,a) + \delta^2E_\varepsilon(u,\nabla)(\sqrt{-1}v, -a\circ J)\Big) \tag{1.4.5}$$

Proposition 1.4.2. *Let (u,∇) be a smooth solution to (YMH$_\varepsilon$). For $(v,a)\in\Omega^0_L\oplus\Omega^1$, we have*

$$\begin{aligned}\mathrm{Av}_{(u,\nabla)}(v,a) = &\int_M 4\varepsilon^2|\overline{\partial}a^{0,1}|^2 + 2|\overline{\partial}_\nabla v - \sqrt{-1}a^{0,1}u|^2 + \frac{1}{2}\left|2\varepsilon\overline{\partial}^*a^{0,1} - \sqrt{-1}\frac{\langle v,u\rangle}{\varepsilon}\right|^2\\ &+\int_M\left(\varepsilon\sqrt{-1}\Lambda F_\nabla - \frac{1-|u|^2}{2\varepsilon}\right)\frac{|v|^2}{\varepsilon}.\end{aligned}$$

Proof. We first consider how the term $\left|\varepsilon\Lambda da + \frac{\mathrm{Re}\langle u,v\rangle}{\varepsilon}\right|^2$ in the integrand changes when (v,a) is replaced with $(\sqrt{-1}v, -a\circ J)$. Given $a\in\Omega^1$, in terms of the complex structure J on M we have

$$a^{0,1} = \frac{1}{2}(a+\sqrt{-1}a\circ J),\ \ a^{1,0} = \frac{1}{2}(a-\sqrt{-1}a\circ J)$$

Consequently, the following relations hold:

$$(-a\circ J)^{0,1} = \sqrt{-1}a^{0,1},\ \ (-a\circ J)^{1,0} = -\sqrt{-1}a^{1,0}. \tag{1.4.6}$$

Also, recall that $[\Lambda, d] = \sqrt{-1}(\overline{\partial}^* - \partial^*)$. Hence

$$\Lambda da = \sqrt{-1}(\overline{\partial}^* - \partial^*)a = \sqrt{-1}(\overline{\partial}^*a^{0,1} - \partial^*a^{1,0}), \tag{1.4.7}$$

and therefore

$$\Lambda d(-a\circ J) = -\partial^*a^{1,0} - \overline{\partial}^*a^{0,1}.$$

On the other hand, recalling our convention that the Hermitian metric is conjugate linear in the second variable, we have

$$\mathrm{Re}\langle u, \sqrt{-1}v\rangle = \mathrm{Im}\langle u,v\rangle = -\mathrm{Im}\langle v,u\rangle. \tag{1.4.8}$$

From the two equations above we deduce that

$$\varepsilon\Lambda d(-a\circ J)+\frac{\mathrm{Re}\langle u,\sqrt{-1}v\rangle}{\varepsilon}=\varepsilon(-\partial^* a^{1,0}-\overline{\partial}^* a^{0,1})-\frac{\mathrm{Im}\langle v,u\rangle}{\varepsilon}. \tag{1.4.9}$$

Now since both $\varepsilon\Lambda da+\dfrac{\mathrm{Re}\langle u,v\rangle}{\varepsilon}$ and $\varepsilon\Lambda d(-a\circ J)+\dfrac{\mathrm{Re}\langle u,\sqrt{-1}v\rangle}{\varepsilon}$ are real, we deduce with the help of (1.4.7), (1.4.8) and (1.4.9) that

$$\begin{aligned}
&\left|\varepsilon\Lambda da+\frac{\mathrm{Re}\langle u,v\rangle}{\varepsilon}\right|^2+\left|\varepsilon\Lambda d(-a\circ J)+\frac{\mathrm{Re}\langle u,\sqrt{-1}v\rangle}{\varepsilon}\right|^2\\
&=\left|\varepsilon\Lambda da+\frac{\mathrm{Re}\langle v,u\rangle}{\varepsilon}-\sqrt{-1}\left(\varepsilon\Lambda d(-a\circ J)+\frac{\mathrm{Re}\langle u,\sqrt{-1}v\rangle}{\varepsilon}\right)\right|^2\\
&=\left|2\varepsilon\sqrt{-1}\overline{\partial}^* a^{0,1}+\frac{\langle v,u\rangle}{\varepsilon}\right|^2\\
&=\left|2\varepsilon\overline{\partial}^* a^{0,1}-\sqrt{-1}\frac{\langle v,u\rangle}{\varepsilon}\right|^2.
\end{aligned}$$

This gives the third term on the right-hand side of the asserted formula. Next we turn to the term $\mathrm{Re}\langle\overline{\partial}_\nabla u,\sqrt{-1}a^{0,1}v\rangle$ and note that

$$\mathrm{Re}\langle\overline{\partial}_\nabla u,\sqrt{-1}(-a\circ J)^{0,1}(\sqrt{-1}v)\rangle=-\mathrm{Re}\langle\overline{\partial}_\nabla u,\sqrt{-1}a^{0,1}v\rangle,$$

which explains the absence of $\mathrm{Re}\langle\overline{\partial}_\nabla u,\sqrt{-1}a^{0,1}v\rangle$ in the conclusion. Finally it is clear that the quantities $|\overline{\partial}a^{0,1}|^2$, $|\overline{\partial}_\nabla v-\sqrt{-1}a^{0,1}u|^2$ and $|v|^2$ are left unchanged upon replacing (v,a) with $(\sqrt{-1}v,-a\circ J)$. It follows that the asserted identity holds. □

From Propositions 1.1.5, 1.4.2 and Corollary 1.1.6, we deduce the following consequence.

Corollary 1.4.3. *Let $(u,\nabla)\in\Omega^0(L)\times\mathcal{A}^h$ be a smooth solution to (YMH_ε) with $u\not\equiv 0$ and $F^{0,2}_\nabla=0$, and assume that (u,∇) is stable. If the following system of equations admits a solution $(v,b)\in\Omega^0(L)\oplus\Omega^{0,1}$ with $v\not\equiv 0$, then (u,∇) is actually a solution to (vor_+).*

$$\begin{cases}\overline{\partial}_\nabla v&=\sqrt{-1}bu,\\ \overline{\partial}b&=0,\\ \overline{\partial}^* b&=\dfrac{\sqrt{-1}\langle v,u\rangle}{2\varepsilon^2}.\end{cases}\tag{1.4.10}$$

Remark 1.4.4. Let $\Phi_+:\Omega^0(L)\times\mathcal{A}^h\to\Omega^{0,1}(L)\oplus\Omega^{0,2}\oplus\Omega^0$ denote the operator given by

$$\Phi_+(u,\nabla)=\begin{pmatrix}\overline{\partial}_\nabla u\\ F^{0,2}_\nabla\\ \sqrt{-1}\Lambda F_\nabla-\dfrac{1-|u|^2}{2\varepsilon^2}\end{pmatrix}.$$

Then the solutions to the holomorphic vortex equations (vor_+) are the configurations satisfying $\Phi_+(u,\nabla)=0$. The system (1.4.10) emerges as the equation governing the kernel of the linearized operator $(d\Phi_+)_{(u,\nabla)}$ restricted to variations orthogonal to the gauge orbit of (u,∇). Thus, we call the system (1.4.10) the linearized vortex equations. When (u,∇) is a vortex, the linearized vortex equations characterize the tangent space at (u,∇) of the moduli space of vortices, modulo the gauge group action. ◯

In order to find solutions to system (1.4.10), we want to follow a similar argument to [NO21] using index theory. Unfortunately, over $\mathbb{CP}^n$ with $n>1$, a dimension count shows that (1.4.10) is not an elliptic system. Our response to this is to introduce "ghost fields", a trick communicated to us by Ákos Nagy. The result is the following Proposition.

Proposition 1.4.5. *Suppose $M=\mathbb{CP}^n$ and that $C_1(L,\omega)>0$. Also, let ∇ be an integrable connection on L. Then there exists a pair $(v,b)\in\Omega^0(L)\oplus\Omega^{0,1}$ and a holomorphic $(0,2)$-form $\xi\in\Omega^{0,2}(L)$ such that*

$$\begin{cases} \overline{\partial}_\nabla v+\overline{\partial}^*_\nabla\xi &= \sqrt{-1}bu \\ \overline{\partial}^* b &= \dfrac{\sqrt{-1}\langle v,u\rangle}{2\varepsilon^2} \\ \overline{\partial} b &= 0 \end{cases} \tag{1.4.11}$$

Proof. As $\overline{\partial}^2_\nabla=0$ by assumption, we can consider the Dolbeault complex $\overline{\partial}_\nabla:\Omega^{0,*}(L)\to\Omega^{0,*+1}(L)$ and its cohomology groups $H^q(M;L)$ for $q=0,\cdots,n$. Letting

$$\Omega^{\text{even}}(L)=\bigoplus_q\Omega^{0,2q}(L),\ \ \Omega^{\text{odd}}(L)=\bigoplus_q\Omega^{0,2q+1}(L),$$

it is well-known that $\overline{\partial}_\nabla+\overline{\partial}^*_\nabla:\Omega^{\text{even}}(L)\to\Omega^{\text{odd}}(L)$ is an elliptic operator with index equal to $\sum_{q=0}^n(-1)^q\dim H^{0,q}(M;L)$. Now we specialize to the case where $M=\mathbb{CP}^n$. Since L is assumed to be positive, it is holomorphically equivalent to $\mathcal{O}(m)$ for some $m>0$, and we have (see [Huy05, Example 5.2.5 and Proposition 2.4.1]):

$$\begin{aligned} H^{0,q}(\mathbb{CP}^n;\mathcal{O}(m))&=\{0\},\ \text{for } 0<q\leqslant n,\\ \dim H^{0,0}(\mathbb{CP}^n;\mathcal{O}(m))&=\frac{(n+m)!}{n!\ m!}. \end{aligned} \tag{1.4.12}$$

Therefore we arrive at

$$\operatorname{Ind}(\overline{\partial}_\nabla+\overline{\partial}^*_\nabla)=\frac{(n+m)!}{n!\ m!}. \tag{1.4.13}$$

Similarly, considering instead the Dolbeault complex of scalar-valued $(0,q)$-forms on M (taking L to be the trivial bundle in the above discussion), we have that the index of the operator

$$\overline{\partial}^*+\overline{\partial}:\Omega^{\text{odd}}\to\Omega^{\text{even}}$$

is given by $\operatorname{Ind}(\overline{\partial}^* + \overline{\partial}) = -1$. Hence, the operator

$$\mathcal{D} := (\overline{\partial}_\nabla + \overline{\partial}_\nabla^*, \overline{\partial}^* + \overline{\partial}) : \Omega^{\text{even}}(L) \oplus \Omega^{\text{odd}} \to \Omega^{\text{odd}}(L) \oplus \Omega^{\text{even}}$$

has Fredholm index equal to $\dfrac{(n+m)!}{n!\, m!} - 1$.

Next we define the following auxiliary operator $\mathcal{P} : \Omega^{\text{even}}(L) \oplus \Omega^{\text{odd}} \to \Omega^{\text{odd}}(L) \oplus \Omega^{\text{even}}$ given by

$$\mathcal{P} \begin{pmatrix} b & v \\ \beta_3 & \xi_2 \\ \vdots & \vdots \\ \beta_{2n-1} & \xi_{2n} \end{pmatrix} = \begin{pmatrix} \sqrt{-1} bu & \dfrac{\sqrt{-1}\langle v, u\rangle}{2\varepsilon^2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

As (u, ∇) is smooth by assumption, this operator $\mathcal{P}$, viewed as an operator

$$\mathcal{P} : W^{1,2}(\Omega^{\text{even}}(L) \oplus \Omega^{\text{odd}}) \to L^2(\Omega^{\text{odd}}(L) \oplus \Omega^{\text{even}})$$

is compact, which implies, since $m \geqslant 1$ by assumption, that

$$\operatorname{Ind}(\mathcal{D} - \mathcal{P}) = \operatorname{Ind}(\mathcal{D}) = \frac{(n+m)!}{n!\, m!} - 1 > 0. \tag{1.4.14}$$

Consequently, there exist non-trivial elements in Ker $(\mathcal{D} - \mathcal{P})$. Now, a non-trivial kernel element in this case is a solution to the following systems of equations

$$\begin{cases} \overline{\partial}_\nabla v + \overline{\partial}_\nabla^* \xi_2 & = \sqrt{-1} bu \\ \overline{\partial}_\nabla \xi_{2j} + \overline{\partial}_\nabla^* \xi_{2j+2} & = 0 \end{cases} \qquad \begin{cases} \overline{\partial}^* b & = \dfrac{\sqrt{-1}\langle v, u\rangle}{2\varepsilon^2} \\ \overline{\partial} b + \overline{\partial}^* \beta_3 & = 0 \\ \overline{\partial}\beta_{2j-1} + \overline{\partial}^* \beta_{2j+1} & = 0 \end{cases}$$

for $1 \leqslant j \leqslant n-1$. Applying the Hodge decomposition we can conclude that

$$\begin{aligned} \overline{\partial}_\nabla \xi_{2j} &= 0 && \text{for} && 1 \leqslant j \leqslant n-1 \\ \overline{\partial}_\nabla^* \xi_{2j} &= 0 && \text{for} && 2 \leqslant j \leqslant n \\ \overline{\partial}\beta_{2j-1} &= 0 && \text{for} && 2 \leqslant j \leqslant n \\ \overline{\partial}^* \beta_{2j+1} &= 0 && \text{for} && 1 \leqslant j \leqslant n-1, \end{aligned}$$

and also $\overline{\partial} b = 0$. To finish, we set $\xi := \xi_2$ and see that the triple (v, b, ξ) is a solution to equation (1.4.11) and that $\xi \in \Omega^{0,2}(L)$ is holomorphic, as we wanted. □

Definition 1.4.6. *Let $M = \mathbb{CP}^n$ and ∇ be an integrable connection on the Hermitian line bundle (L, h), and let $(u, \nabla) \in \Omega^0(L) \times \mathcal{A}^h$ be a smooth solution to* (YMH$_\varepsilon$)*. We define*

$$\mathcal{T}_{(u,\nabla)} = \{(v, b, \xi) \in \Omega^0(L) \oplus \Omega^{0,1} \oplus \Omega^{0,2}(L) \mid \textit{satisfies (1.4.11) and } \xi \textit{ is holomorphic}\}. \tag{1.4.15}$$

Remark 1.4.7. Note that $\mathcal{T}_{(u,\nabla)} \neq 0$ and it is finite-dimensional because it is essentially a subspace of the kernel of an elliptic operator, by Proposition 1.4.5. ○

Theorem 1.4.8. *Suppose $M = \mathbb{CP}^n$ and that $(L, h) \to \mathbb{CP}^n$ is a Hermitian line bundle with $C_1(L, \omega) > 0$. Let $(u, \nabla) \in \Omega^0(L) \times \mathcal{A}^h$ be a smooth, stable critical point of E_ε such that $F_\nabla \in \Omega^{1,1}$. Assume in addition that there exists a non-zero element $(v, b, \xi) \in \mathcal{T}_{(u,\nabla)}$ such that*

$$b \wedge \overline{\partial}_\nabla u = 0. \tag{1.4.16}$$

Then the pair (u, ∇) is a solution to (vor$_+$).

Proof. Since $\overline{\partial} b = 0$, we get

$$\overline{\partial}_\nabla (bu) = -b \wedge \overline{\partial}_\nabla u = 0.$$

Since $H^{0,1}(\mathbb{CP}^n, L) = \{0\}$ by (1.4.12), we get that $\sqrt{-1}bu = \overline{\partial}_\nabla w$ for some $w \in \Omega^0(L)$. But then $\overline{\partial}_\nabla v + \overline{\partial}^*_\nabla \xi = \sqrt{-1}bu$ implies that $\overline{\partial}^*_\nabla \xi = \overline{\partial}_\nabla (w - v)$. By Hodge decomposition, this implies that $\overline{\partial}^*_\nabla \xi = 0$ and then $\overline{\partial}_\nabla v = \sqrt{-1}bu$. With this, we get that (v, b) is a solution to equation (1.4.10). To see that $v \not\equiv 0$, we note that since $\overline{\partial}_\nabla \xi = 0$ and $\overline{\partial}^*_\nabla \xi = 0$, and since $H^{0,2}(\mathbb{CP}^n; L) = \{0\}$, we must have $\xi = 0$. The assumption that (v, b, ξ) be non-zero then implies $(v, b) \neq (0, 0)$. Moreover, if $v \equiv 0$, then the last two equations in (1.4.10) and the fact that $\mathbb{CP}^n$ is simply-connected forces $b \equiv 0$ as well, a contradiction. Thus v is not identically zero, and we are done by Corollary 1.4.3. □

2 Behavior of critical points in the large ε regime

The aim of this chapter is to study the behavior of critical points of the ε-Yang–Mills–Functional in the large ε regime. In contrast to the case $\varepsilon \to 0$, we show that this regime provides no non-trivial irreducible solutions. We begin in § 2.1 recalling some estimates from [PS21] for critical points and obtaining in Proposition 2.1.4 estimates on all derivatives of the curvature of the connection part and of the section part, in terms of a given upper bound for the YMH energy. We show in Corollary 2.1.6 and Corollary 2.1.7, using Hodge decomposition and Chern-Weyl theory, that we can remove the upper energy bound assumption. We end the section by showing in Lemma 2.1.8 that the set of pairs (u, ∇) satisfying $\|u\| = 1$, $\{u = 0\} \neq \emptyset$ and estimates we obtained previously have a positive lower bound on $\|\nabla u\|_2$. After the preliminary computations we start Section 2.2 demonstrating in Theorem 2.2.2 that in the large ε regime there is no irreducible critical points for the YMH functional except the ones gauge equivalent to $(1, d)$ in the case where L is the trivial line bundle, running a contradiction argument using Lemma 2.1.8. We end the section by using estimates from the beginning to obtain a gap theorem for the YMH energy in the large ε regime. This is the content of Theorem 2.2.3.

In this chapter, we let (M, g) be a closed Riemannian manifold with real dimension n. If ∇ is a connection on a complex line bundle $L \to M$ and S is a tensor on M with values on L, i.e. a section of $(TM)^{\otimes p} \otimes (TM^*)^{\otimes q} \otimes L$, we still denote by ∇S the covariant derivative of S induced by the connection ∇ and the Levi-Civita connection D of (M, g).

2.1 Preliminary computations and estimates

The goal here is to obtain good estimates for study of the large ε regime of the YMH funcional. To begin with, we recall the Weitzenbock formulas for 1-forms and 2-forms. Let L be a Hermitian line bundle and ∇ be a unitary connection on L. For $\sigma \in \Omega^1(L)$ we define the 1-form $\mathcal{R}_1^\nabla(\sigma)$ by

$$\mathcal{R}_1^\nabla(\sigma)_X = \sum_{i=1}^n (F_\nabla)_{e_i, X} \sigma_{e_i}, \tag{2.1.1}$$

where (e_i) is an orthonormal frame on (M, g). For a 2-form $\varphi \in \Omega^2(M)$ we define the 2-form $\mathcal{R}_2(\varphi)$ given by

$$\mathcal{R}_2(\varphi)_{X,Y} = \varphi_{\mathrm{Ric}(X),Y} + \varphi_{X,\mathrm{Ric}(Y)} + \sum_{i=1}^n \varphi_{e_i, R_{X,Y} e_i}, \tag{2.1.2}$$

where R denotes the Riemann curvature tensor of (M, g). We say that $\mathcal{R}_2 \geqslant 0$ if pointwise on M we have

$$\langle \mathcal{R}_2(\varphi), \varphi \rangle = \sum_{i<j} \left\langle \varphi_{\mathrm{Ric}(e_i),e_j} + \varphi_{e_i,\mathrm{Ric}(e_j)} + \varphi_{e_k,R_{e_i,e_j}e_k}, \varphi_{e_i,e_j} \right\rangle \geqslant 0, \tag{2.1.3}$$

for every $\varphi \in \Omega^2(M)$.

The following are the Weitzenbock formulas we shall need, for a proof we refer to [BL81, Section 3].

Proposition 2.1.1. *Let L be a Hermitian line bundle over M e ∇ be a unitary connection on L. Then, for every $\sigma \in \Omega^1(L)$ and $\varphi \in \Omega^2(M)$, the following formulas hold:*

1. $\Delta^\nabla \sigma = \nabla^*\nabla\sigma + \sigma \circ \mathrm{Ric} + \mathcal{R}_1^\nabla(\sigma)$;
2. $\Delta^\nabla \varphi = \nabla^*\nabla\varphi + \mathcal{R}_2^\nabla(\varphi)$.

2.1.1 Pigati-Stern estimates

We begin this subsection by presenting some estimates on critical points of the YMH functional contained in [PS21], in order to use them to obtain higher order estimates, which will be useful later for compactness arguments. In the proofs below we often denote just by F the curvature F_∇ of a connection ∇, for the sake of conciseness, although still using the subscript ∇ in highlighted statements.

Lemma 2.1.2 ([PS21], Section 3)**.** *Suppose that (u, ∇) is an irreducible critical point of E_ε with $E_\varepsilon(u, \nabla) \leqslant \Lambda$. Then we have the following.*

(a) There holds the pointwise estimate

$$|F_\nabla| \leqslant C_M(\|F_\nabla\|_2 + \varepsilon^{-2}). \tag{2.1.4}$$

(b) If moreover $\mathcal{R}_2 \geqslant 0$, then

$$|F_\nabla| \leqslant \frac{1 - |u|^2}{2\varepsilon^2}. \tag{2.1.5}$$

Proof. For a complete proof we refer to [PS21, Section 3]. For convenience we reframe part of their proof in our notation. For part (a), recall that

$$\nabla^*\nabla\sqrt{-1}F = -\frac{|u|^2}{\varepsilon^2}\sqrt{-1}F + \frac{\nabla u \times \nabla u}{\varepsilon^2} - \mathcal{R}_2(\sqrt{-1}F), \tag{2.1.6}$$

and that the function $f : \Omega^2(M) \to \mathbb{R}$ given by $f(\omega) = -\frac{\langle \mathcal{R}_2(\omega), \omega\rangle}{|\omega|^2}$ have a positive upper bound A_M. Now, Lemma A.1 gives

$$\Delta|F| \leqslant -\frac{|u|^2}{\varepsilon^2}|F| + \frac{|\nabla u|^2}{\varepsilon^2} + A_M|F|, \tag{2.1.7}$$

in the sense of distributions. Also recalling that

$$\Delta\Big(\frac{1-|u|^2}{2}\Big) = |\nabla u|^2 - \frac{|u|^2}{\varepsilon^2}\frac{1-|u|^2}{2}, \tag{2.1.8}$$

and letting $\xi = |F| - \dfrac{1-|u|^2}{2\varepsilon^2}$, we have

$$\Delta\xi \leqslant -\frac{|u|^2}{\varepsilon^2}\xi + A_M|F|. \tag{2.1.9}$$

Now let $G : M \times M \setminus \{(x,x)\}_{x\in M} \to [0,\infty)$ be a non-negative Green's function on M and define $w = G * |F|$. Then $w \geqslant 0$ and

$$\Delta w = |F| - \fint_M |F| =: |F| - k.$$

Moreover, since $|G(x,y)| \leqslant C_M d(x,y)^{2-n}$, Hölder's inequality gives

$$\|w\|_2 \leqslant C_M\|F\|_2, \quad \|w\|_\infty \leqslant C_M\|F\|_{n-1} \leqslant C_M\|F\|_\infty^{\frac{n-3}{n-1}}\|F\|_2^{\frac{2}{n-1}}. \tag{2.1.10}$$

Next we note that $\xi - A_M w$ satisfies

$$\Delta(\xi - A_M w) \leqslant -\frac{|u|^2}{\varepsilon^2}(\xi - A_M w) + A_M k.$$

Lemma A.2 and Remark A.3 then give

$$\begin{aligned}
\|(\xi - A_M w)_+\|_\infty &\leqslant C_M(\|(\xi - A_M w)_+\|_2 + k)\\
&\leqslant C_M(\|\xi\|_2 + \|w\|_2 + k)\\
&\leqslant C_M(\|F\|_2 + \varepsilon^{-2}).
\end{aligned}$$

Consequently, since

$$\xi \leqslant (\xi - A_M w)_+ + A_M w,$$

we get upon also recalling (2.1.10) and the definition of ξ that

$$|F| \leqslant \frac{1-|u|^2}{2\varepsilon^2} + C_M(\|F\|_2 + \varepsilon^{-2}) + C_M\|F\|_\infty^{\frac{n-3}{n-1}}\|F\|_2^{\frac{2}{n-1}},$$

which implies

$$\begin{aligned}
\|F\|_\infty &\leqslant C_M(\|F\|_2 + \varepsilon^{-2}) + C_M\|F\|_\infty^{\frac{n-3}{n-1}}\|F\|_2^{\frac{2}{n-1}}\\
&\leqslant C_M(\|F\|_2 + \varepsilon^{-2}) + C_M\delta\|F\|_\infty + C_M\delta^{-\frac{n-3}{2}}\|F\|_2,
\end{aligned} \tag{2.1.11}$$

where we used Young's inequality in the last line. Choosing $\delta = \dfrac{1}{2C_M}$ gives

$$\|F\|_\infty \leqslant C_M(\|F\|_2 + \varepsilon^{-2}).$$

This proves (2.1.4). Next, under the assumption $\mathcal{R}_2 \geqslant 0$, we deduce from (2.1.6) the following improved version of (2.1.7):

$$\Delta|F| \leqslant -\frac{|u|^2}{\varepsilon^2}|F| + \frac{|\nabla u|^2}{\varepsilon^2}.$$

The inequality (2.1.9) then becomes simply $\left(\Delta + \frac{|u|^2}{\varepsilon^2}\right)\xi \leqslant 0$, and (2.1.5) follows by the maximum principle and the fact that $u \not\equiv 0$. □

Lemma 2.1.3 ([PS21])**.** *Let (u, ∇) be an irreducible critical point of E_ε with $\|F_\nabla\|_2 \leqslant \Lambda$. Then we have the following.*

(a) There holds the L^2-estimate

$$\|\nabla u\|_2 \leqslant \frac{1}{\sqrt{2}\cdot\varepsilon}\|u\|_2 \leqslant \frac{(\mathrm{Vol}(M))^{\frac{1}{2}}}{\sqrt{2}\cdot\varepsilon} \tag{2.1.12}$$

(b) For the L^∞-norm of ∇u we have

$$\begin{aligned}\|\nabla u\|_\infty &\leqslant C_{M,n}(\varepsilon^{-2}+\Lambda+1)^{\frac{n}{4}}\|\nabla u\|_2 \\ &\leqslant C_{M,n}\varepsilon^{-1}(\varepsilon^{-2}+\Lambda+1)^{\frac{n}{4}}.\end{aligned} \tag{2.1.13}$$

(c) Assuming that $\mathcal{R}_2 \geqslant 0$ and $\mathrm{Ric} \geqslant 0$, we have

$$|\nabla u| \leqslant \frac{3(1-|u|^2)}{2\varepsilon}.$$

Proof. As u is not identically zero, we see from (2.1.8) and the maximum principle that

$$1 - |u|^2 \geqslant 0.$$

Thus, from the equation $\nabla^*\nabla u = \frac{1-|u|^2}{2\varepsilon^2}u$ we have

$$\int_M |\nabla u|^2 = \int_M \frac{1-|u^2|}{2\varepsilon^2}|u|^2 \leqslant \frac{1}{2\varepsilon^2}\int_M |u|^2 \leqslant \frac{\mathrm{Vol}(M)}{2\varepsilon^2}.$$

This gives (2.1.12). Next we note using the Yang–Mills–Higgs equations and the Weitzenböck formula that

$$\begin{aligned}\nabla^*\nabla(\nabla u) &= \Delta(\nabla u) - F_{e_i,\cdot}\nabla_{e_i}u - \nabla_{\mathrm{Ric}(\cdot)}u \\ &= d^*_\nabla(Fu) + d_\nabla\left(\frac{1-|u|^2}{2\varepsilon^2}u\right) - F_{e_i,\cdot}\nabla_{e_i}u - \nabla_{\mathrm{Ric}(\cdot)}u \\ &= -\frac{|u|^2}{\varepsilon^2}\nabla u + \frac{1-|u|^2}{2\varepsilon^2}\nabla u - 2F_{e_i,\cdot}\nabla_{e_i}u - \nabla_{\mathrm{Ric}(\cdot)}u.\end{aligned} \tag{2.1.14}$$

Now we want to estimate the inner product of (2.1.14) with ∇u. To do this, we first compute

$$\begin{aligned}
-2\sum_{i,j}\mathrm{Re}\left\langle F_{e_i,e_j}\nabla_{e_i}u,\nabla_{e_j}u\right\rangle &= 2\sum_{i,j}\sqrt{-1}F_{e_i,e_j}\mathrm{Re}\left\langle\sqrt{-1}\nabla_{e_i}u,\nabla_{e_j}u\right\rangle\\
&=\sum_{i,j}\sqrt{-1}F_{e_i,e_j}(\nabla u\times\nabla u)_{e_i,e_j}\\
&=2\sum_{i<j}\sqrt{-1}F_{e_i,e_j}(\nabla u\times\nabla u)_{e_i,e_j}\\
&\leqslant 2|F||\nabla u\times\nabla u|\leqslant 2|F||\nabla u|^2,
\end{aligned}$$

and therefore

$$\langle\nabla^*(\nabla u),\nabla u\rangle\leqslant-\frac{|u|^2}{\varepsilon^2}|\nabla u|^2+\left(\frac{1-|u|^2}{2\varepsilon^2}+2|F|^2|+|\mathrm{Ric}|\right)|\nabla u|^2. \tag{2.1.15}$$

To proceed, Lemma A.1 for $p_0=2$ and $q=\infty$ gives

$$\begin{aligned}
\Delta|\nabla u|\leqslant&\ -\frac{|u|^2}{\varepsilon^2}|\nabla u|+(\frac{1-|u|^2}{2\varepsilon^2}+2|F|+|\mathrm{Ric}|)|\nabla u|\\
\leqslant&\ C_M(\varepsilon^{-2}+\Lambda+1)|\nabla u|,
\end{aligned}$$

where in getting the last inequality we used Lemma 2.1.2. The first inequality in (2.1.13) now follows from Lemma A.2, and the second inequality follows upon recalling (2.1.12).

For part (c), we use Lemma 2.1.2(b) and the non-negative Ricci assumption to get from (2.1.14) that

$$\Delta|\nabla u|\leqslant-\frac{|u|^2}{\varepsilon^2}|\nabla u|+\frac{3(1-|u|^2)}{2\varepsilon^2}|\nabla u|.$$

Combining this with (2.1.8) gives

$$\begin{aligned}
\big(\Delta+\frac{|u|^2}{\varepsilon^2}\big)(|\nabla u|-\frac{3(1-|u|^2)}{2\varepsilon})&\leqslant\frac{|\nabla u|}{\varepsilon}\big(\frac{3(1-|u|^2)}{2\varepsilon}-\frac{3|\nabla u|}{\varepsilon}\big)\\
&\leqslant\frac{|\nabla u|}{\varepsilon}\big(\frac{3(1-|u|^2)}{2\varepsilon}-|\nabla u|\big).
\end{aligned}$$

Consequently,

$$\big(\Delta+\frac{|u|^2}{\varepsilon^2}+\frac{|\nabla u|}{\varepsilon}\big)(|\nabla u|-\frac{3(1-|u|^2)}{2\varepsilon})\leqslant 0,$$

and the desired estimate follows from the maximum principle, since $u\not\equiv 0$. □

With the pointwise estimates on $|F_\nabla|$ and $|\nabla u|$ provided by Lemma 2.1.2 and Lemma 2.1.3, we can begin an iterative process to establish higher-order estimates.

Proposition 2.1.4. *Let (u,∇) be an irreducible critical point of E_ε satisfying $\|F_\nabla\|_2\leqslant\Lambda$ and $\varepsilon\geqslant 1$.*

(a) For all $k \in \mathbb{N} \cup \{0\}$ we have

$$\sum_{i=0}^{k} |\nabla^i F_\nabla| + |\nabla^{i+1} u| \leqslant C_{k,\Lambda,M,L}, \text{ pointwise on } M. \tag{2.1.16}$$

(b) If moreover $\mathcal{R}_2 \geqslant 0$ and Ric $\geqslant 0$*, then the dependence on Λ can be eliminated from the constants in* (2.1.16).

Proof. To prove (a) we argue by induction. The base case is already taken care of by Lemma 2.1.2 and Lemma 2.1.3. Now suppose (2.1.16) holds for some $k \geqslant 0$. By (2.1.6), (2.1.14) and Lemma A.4 we have

$$\begin{aligned}\left|\nabla^*\nabla(\nabla^k F)\right| \leqslant\ & C_{k,n} \sum_{i=0}^{k} \frac{1}{\varepsilon^2} |\nabla^i(|u|^2)||\nabla^{k-i} F| + C_{k,n} \sum_{i=0}^{k} \frac{1}{\varepsilon^2} |\nabla^{i+1} u||\nabla^{k+1-i} u| \\ & + C_{k,n} \sum_{i=0}^{k} |\nabla^i R||\nabla^{k-i} F|,\end{aligned} \tag{2.1.17}$$

and that

$$\begin{aligned}\left|\nabla^*\nabla(\nabla^k \nabla u)\right| \leqslant\ & C_{k,n} \sum_{i=0}^{k} \left|\nabla^i\left(\frac{1-3|u|^2}{2\varepsilon^2}\right)\right| |\nabla^{k+1-i} u| + C_{k,n} \sum_{i=0}^{k} |\nabla^i F||\nabla^{k+1-i} u| \\ & + C_{k,n} \sum_{i=0}^{k} |\nabla^i R||\nabla^{k+1-i} u|.\end{aligned} \tag{2.1.18}$$

Since F and ∇u are differentiated at most k times and $k+1$ times, respectively, on the right-hand sides, the induction hypothesis gives

$$\begin{aligned}\int_M |\nabla^{k+1} F|^2 + |\nabla^{k+2} u|^2 \mathrm{vol}_g &= \int_M \left\langle \nabla^k F, \nabla^*\nabla(\nabla^k F)\right\rangle + \mathrm{Re} \left\langle \nabla^{k+1} u, \nabla^*\nabla(\nabla^{k+1} u)\right\rangle \mathrm{vol}_g \\ &\leqslant C_{k,\Lambda,M,L}.\end{aligned} \tag{2.1.19}$$

Next we use (2.1.17) and (2.1.18) with $k+1$ in place of k to get that

$$\begin{aligned}\Delta|\nabla^{k+1} F| \leqslant\ & C_{k,n}\varepsilon^{-2} \sum_{i=0}^{k+1} |\nabla^i |u|^2||\nabla^{k+1-i} F| + C_{k,n}\varepsilon^{-2} \sum_{i=0}^{k+1} |\nabla^{k+2-i} u||\nabla^{i+1} u| \\ & + C_{k,n} \sum_{i=0}^{k+1} |\nabla^i R||\nabla^{k+1-i} F|,\end{aligned}$$

and that

$$\begin{aligned}\Delta|\nabla^{k+2} u| \leqslant\ & C_{k,n}\varepsilon^{-2} \sum_{i=0}^{k+1} \left|\nabla^i\left(\frac{1-3|u|^2}{2\varepsilon^2}\right)\right| |\nabla^{k+2-i} u| + C_{k,n} \sum_{i=0}^{k+1} |\nabla^i F||\nabla^{k+2-i} u| \\ & + C_{k,n} \sum_{i=0}^{k+1} |\nabla^i R||\nabla^{k+2-i} u|.\end{aligned}$$

Combining the two inequalities and recalling the induction hypothesis, as well as the assumption that $\varepsilon \geqslant 1$, we get

$$\begin{aligned}\Delta\Big(|\nabla^{k+1}F| + |\nabla^{k+2}u|\Big) &\leqslant C_{k,n}\Big(1 + |\nabla u| + |F| + |R|\Big)\Big(|\nabla^{k+1}F| + |\nabla^{k+2}u|\Big) + C_{k,\Lambda,M,L} \\ &\leqslant C_{k,M,L,\Lambda}\Big(|\nabla^{k+1}F| + |\nabla^{k+2}u|\Big) + C_{k,\Lambda,M,L}.\end{aligned}$$

Lemma A.2 then implies

$$\|\nabla^{k+1}F\|_\infty + \|\nabla^{k+2}u\|_\infty \leqslant C_{k,M,L,\Lambda}(1 + \|\nabla^{k+1}F\|_2 + \|\nabla^{k+2}u\|_2),$$

and we are done with the induction step upon recalling (2.1.19). To verify part (b), we instead invoke Lemma 2.1.2(b) and Lemma 2.1.3(c) to establish the induction base, and notice that it is the only place where Λ enters the estimates. $\square$

2.1.2 Improved estimates and lower energy bound

In the previous estimates, we got a pointwise bound on all the derivatives of F_∇ and ∇u for a critical point (u, ∇), but these bounds depend on the bound Λ for the L_2 norm of F_∇, except in the case where $\mathcal{R}_2 \geqslant 0$ and $\mathrm{Ric} \geqslant 0$. Since these hypotheses on $\mathcal{R}_2$ and Ric could are restrictive, our first objective in this subsection is to show that we can actually drop the Λ dependency and the assumptions on the curvature.

To begin we recall the following basic L^2-estimate for the Hodge Laplacian. Given a closed Riemannian manifold M, there exists $C = C_{M,p}$ such that for any $\xi \in \Omega^p(M)$ we have

$$\|\xi\|_{1,2} \leqslant C(\|d\xi\|_2 + \|d^*\xi\|_2 + \|\xi\|_2), \tag{2.1.20}$$

and that

$$\|\xi - h(\xi)\|_{1,2} \leqslant C(\|d\xi\|_2 + \|d^*\xi\|_2), \tag{2.1.21}$$

where $h(\xi)$ denotes the harmonic part in the Hodge decomposition of ξ.

Proposition 2.1.5. *Let $L \to M$ be a Hermitian line bundle over a closed Riemannian manifold. There exists $\Lambda_0 = \Lambda_0(M, L)$ such that whenever (u, ∇) is an irreducible critical point of E_ε with $\varepsilon \geqslant 1$, there holds*

$$\|F_\nabla\|_2 \leqslant \Lambda_0. \tag{2.1.22}$$

Proof. Since $d\sqrt{-1}F = 0$, the Hodge decomposition of $\sqrt{-1}F$ takes the form

$$\sqrt{-1}F = da + h(\sqrt{-1}F). \tag{2.1.23}$$

On the other hand, by Chern–Weil theory we know that

$$\sqrt{-1}F \in 2\pi \cdot c_1(L). \tag{2.1.24}$$

Consequently, letting h_0 be the unique harmonic 2-form in $2\pi \cdot c_1(L)$, we see from (2.1.24) and (2.1.23) that

$$h(\sqrt{-1}F) = h_0.$$

The L^2-estimate (2.1.21) then gives

$$\|\sqrt{-1}F\|_{1,2} \leqslant C(\|d^*\sqrt{-1}F\|_2 + \|h_0\|_{1,2}) \leqslant C(\varepsilon^{-2}\|u \times \nabla u\|_2 + \|h_0\|_2), \tag{2.1.25}$$

where the second inequality follows from the Yang–Mills–Higgs equations and (2.1.20), the latter applied to h_0. We next note that, since

$$|u \times \nabla u| \leqslant |u||\nabla u| \leqslant |\nabla u| \text{ pointwise on } M,$$

Lemma 2.1.3(a) yields

$$\|u \times \nabla u\|_2 \leqslant \|\nabla u\|_2 \leqslant C_M \varepsilon^{-1}.$$

Combining this with (2.1.25) and recalling that $\varepsilon \geqslant 1$,

$$\|F\|_2 \leqslant C_M(1 + \|h_0\|_2) =: \Lambda_0(M, L),$$

where the constant Λ_0 indeed has only the indicated dependency, since h_0 is determined by M and L. □

With Proposition 2.1.5 in hand, we obtain better estimates, in the form of the following Corollary.

Corollary 2.1.6. *Let (u, ∇) be an irreducible critical point of E_ε with $\varepsilon \geqslant 1$. Then, for all $k \in \mathbb{N} \cup \{0\}$,*

$$\sum_{i=0}^{k} |\nabla^i F| + |\nabla^{i+1} u| \leqslant C_{k,M,L}.$$

Proof. We simply note that, given (u, ∇) as in the statement, Proposition 2.1.5 tells us that Proposition 2.1.4(a) is applicable with $\Lambda = \Lambda_0(M, L)$. The constants $C_{k,\Lambda_0,M,L}$ then depend only on k, M and L. □

Now, with Corollary 2.1.6, we may go back to the equation

$$\nabla^*\nabla u = \frac{1 - |u|^2}{2\varepsilon^2} u$$

and also improve the form of the estimates we have on $\nabla^k u$.

Corollary 2.1.7. *Suppose (u, ∇) is an irreducible critical point of E_ε with $\varepsilon \geqslant 1$. Then, for all $k \in \mathbb{N}$,*

$$\|\nabla^k u\|_2 \leqslant C_{k,M,L} \|u\|_2,$$

with perhaps a different set of constants $C_{k,M,L}$ than those in Corollary 2.1.6.

Proof. For brevity we write $V = \dfrac{1-|u|^2}{2\varepsilon^2}$. Since $\varepsilon \geqslant 1$, we see from Corollary 2.1.6 and the bound $|u| \leqslant 1$ that $|\nabla^k V| \leqslant C_{k,M,L}$, and hence

$$|\nabla^k(Vu)| \leqslant C_{n,k}\sum_{i=0}^{k}|\nabla^{k-i}V||\nabla^i u| \leqslant C_{k,M,L}\sum_{i=0}^{k}|\nabla^i u|.$$

Since $\nabla^*\nabla u = Vu$, combining the above with Lemma A.4 while again using Corollary 2.1.6, we obtain

$$|\nabla^*\nabla(\nabla^k u)| \leqslant |\nabla^k(Vu)| + C_{n,k}\sum_{i=0}^{k}(|\nabla^{k-i}F| + |\nabla^{k-i}R|)|\nabla^i u| \leqslant C_{k,M,L}\sum_{i=0}^{k}|\nabla^i u|.$$

In particular, taking the inner product of $\nabla^*\nabla(\nabla^k u)$ with $\nabla^k u$ and integrating gives

$$\int_M |\nabla^{k+1}u|^2 \mathrm{vol}_g = \int_M \left\langle \nabla^k u, \nabla^*\nabla(\nabla^k u)\right\rangle \mathrm{vol}_g \leqslant C_{k,M,L}\|\nabla^k u\|_2 \sum_{i=0}^{k}\|\nabla^i u\|_2,$$

where in the last inequality we used Hölder inequality. Then, the trivial bound $\|\nabla^k u\|_2 \leqslant \sum_{i=0}^{k}\|\nabla^i u\|_2$ allows us to conclude that

$$\|\nabla^{k+1}u\|_2 \leqslant C_{k,M,L}\sum_{i=0}^{k}\|\nabla^i u\|_2.$$

The asserted estimate now follows by induction on k, the base case being provided by Lemma 2.1.3(a). □

Now that we have all the good estimates, we will use them to get a lower L_2-bound of ∇u for a special family of pairs $(u,\nabla) \in \Omega^0(L)\times\mathcal{A}$. First, we present an argument from [PS21, Section 7] for controlling the norm of the harmonic part of the curvature of a connection ∇.

Letting $\mathcal{H}^1(M)$ denote the space of real harmonic 1-forms on M, which is a real $b_1(M)$-dimensional vector space, and introduce the lattice

$$\Gamma = \left\{ h \in \mathcal{H}^1(M) \mid \int_\gamma h \in 2\pi\mathbb{Z}, \text{ for all loops } \gamma : S^1 \to M \right\}$$

of harmonic 1-forms with integral periods. Letting $[\gamma_1], \cdots, [\gamma_{b_1(M)}]$ be a basis for $H_1(M;\mathbb{R})$ the map

$$h \mapsto \left(\int_{\gamma_i} h\right)_{1\leqslant i\leqslant b_1(M)} \tag{2.1.26}$$

is an isomorphism of real vector spaces by the de Rham isomorphism Theorem [Lee12, Theorem 18.14]. Consequently, setting

$$|h|_b := \max\left\{\left|\int_{\gamma_i} h\right| \;:\; 1 \leqslant i \leqslant b_1(M)\right\}$$

defines a norm on $\mathcal{H}^1(M)$ which, by virtue of the fact that $\mathcal{H}^1(M)$ is finite-dimensional, is equivalent to any other norm on it. With respect to $|\cdot|_b$, we may define the distance from a harmonic 1-form to the lattice Γ. In particular we have the trivial bound

$$\mathrm{dist}_b(h, \Gamma) \leqslant \pi,$$

for all $h \in \mathcal{H}^1(M)$. Since (2.1.26) is an isomorphism, given any $h \in \mathcal{H}^1(M)$ we can always find $\xi \in \Gamma$ such that

$$|h - \xi|_b = \mathrm{dist}_b(h, \Gamma) \leqslant \pi.$$

With this, we are ready to obtain our lower energy bound:

Lemma 2.1.8. *Suppose $L \to M$ is a Hermitian line bundle over a closed Riemannian manifold, and let $\mathcal{A}$ denote the space of all metric compatible connections on L. Given a sequence $B = (B_0, \cdots, B_k, \cdots)$ of positive numbers, define $\mathcal{A}_B$ to be the collection of pairs $(v, \nabla) \in \Omega^0(L) \times \mathcal{A}$ such that*

$$\|v\|_2 = 1, \ \{v = 0\} \neq \emptyset, \ \textit{and that } \|\nabla^k v\|_2 + \|\nabla^k F_\nabla\|_2 \leqslant B_k, \ \textit{for } k = 0, 1, 2, \cdots.$$

Assuming further that $\mathcal{A}_B$ is non-empty, then

$$\alpha_B := \inf \left\{ \int_M |\nabla v|^2 \mathrm{vol}_g \mid (v, \nabla) \in \mathcal{A}_B \right\} > 0.$$

Proof. Suppose towards a contradiction that $\alpha_B = 0$. Then there exists a sequence (v_i, ∇_i) in $\mathcal{A}_B$ such that

$$\alpha_i := \int_M |\nabla_i v_i|^2 \mathrm{vol}_g \to 0 \text{ as } i \to \infty.$$

Fixing a reference connection $\nabla_0 \in \mathcal{A}$ on L, we express each ∇_i as

$$\nabla_i = \nabla_0 - \sqrt{-1} A_i$$

for some real 1-form A_i. Since the integral $\int_M |\nabla v|^2$ is invariant under gauge transformations of (v, ∇), we may assume in addition that

$$d^* A_i = 0, \quad |h(A_i)|_b \leqslant \pi,$$

where $h(A_i)$ denotes the harmonic part in the Hodge decomposition of A_i. In particular, for each k, since $|\cdot|_b$ is equivalent to $\|\cdot\|_{k,2}$ on $\mathcal{H}^1(M)$, we have

$$\|h(A_i)\|_{k,2} \leqslant C_k. \tag{2.1.27}$$

Next, by definition of $\mathcal{A}_B$, we have

$$\|F_0 - \sqrt{-1} dA_i\|_{k,2} \leqslant B_k \text{ for all } k, i,$$

and hence

$$\|dA_i\|_{k,2} \leqslant B_k + \|F_0\|_{k,2}.$$

A priori estimates for the operator $(d + d^*)$ then gives

$$\begin{aligned}\|A_i - h(A_i)\|_{k+1,2} &\leqslant C_k(\|dA_i\|_{k,2} + \|d^* A_i\|_{k,2}) \\ &= C_k\|dA_i\|_{k,2} \leqslant C_k(B_k + \|F_0\|_{k,2}).\end{aligned}$$

Since F_0 is fixed, combining this with (2.1.27) gives, with possibly a different set of C_k's,

$$\|A_i\|_{k+1,2} \leqslant C_k, \text{ for all } i, k. \tag{2.1.28}$$

Thus, up to taking a subsequence, (A_i) converges smoothly on M to a limiting 1-form A. Next, for all $m \in \mathbb{N} \cup \{0\}$, from the definition of $\mathcal{A}_B$ we have

$$\|(\nabla_0 - \sqrt{-1}A_i)^m v_i\|_2 \leqslant B_m,$$

with B_m independent of i. Combining this with (2.1.28), we may argue by induction on m that $\|\nabla_0^m v_i\|_2$ is bounded uniformly in i. (See Lemma A.5.) Consequently, up to taking a further subsequence, (v_i) converges smoothly on M to some $v \in \Omega^0(L)$. The limiting section v must vanish somewhere. Indeed, for all i, by the definition of $\mathcal{A}_B$ there is some $x_i \in M$ such that

$$v_i(x_i) = 0.$$

Since M is compact, we may assume that $x_i \to x \in M$. This together with the uniform convergence of v_i to v gives

$$v(x) = 0.$$

Now, with $\nabla := \nabla_0 - \sqrt{-1}A$, we note that

$$\int_M |\nabla v|^2 \mathrm{vol}_g = \lim_{i\to\infty} \int_M |\nabla_i v_i|^2 \mathrm{vol}_g = 0,$$

so that $\nabla v = 0$ identically, which implies that $|v|$ is constant. As v vanishes somewhere, we must have $v \equiv 0$, but this leads to a contradiction, because we also have

$$\|v\|_2 = \lim_{i\to\infty} \|v_i\|_2 = 1.$$

The proof is complete. □

2.2 Non-existence threshold and energy gap

Combining Lemma 2.1.8 with the pointwise estimates in Corollary 2.1.6 and Corollary 2.1.7, and also recalling the bound (2.1.12), which is to be interpreted as lower bound to the least non-zero eigenvalue of $\nabla^*\nabla$, we obtain a non-existence result when ε is too large.

Before going into the main theorem, we recall the following Lemma, which is proved in [PS21, Proposition 7.10], but we include a proof for completeness.

Lemma 2.2.1. *Suppose (u, ∇) is a critical point of E_ε. If $\{u = 0\} = \emptyset$ then L is trivial and (u, ∇) is gauge equivalent to $(1, d)$.*

Proof. As u is a non-vanishing section, it follows immediately that L is trivial, so that we may think of u as a complex-valued function. Then, up to the gauge-transformation determined by $\dfrac{u}{|u|} : M \to S^1$, we may further assume that u is real-valued and positive. In particular, writing $\nabla = d - \sqrt{-1}A$, we have

$$u \times \nabla u = \mathrm{Re}\left\langle \sqrt{-1}u, du - \sqrt{-1}Au \right\rangle = -|u|^2 A,$$

so that the second Yang–Mills–Higgs equation becomes

$$\varepsilon^2 d^* dA + |u|^2 A = 0.$$

Taking the inner product with A and integrating by parts gives

$$\int_M \varepsilon^2 |dA|^2 + |u|^2 |A|^2 \mathrm{vol}_g = 0.$$

Since u is positive, this forces $A \equiv 0$, so that $\nabla = d$.

It remains to show that $u \equiv 1$. To that end, note that since $\nabla = d$, the first Yang–Mills–Higgs equation becomes

$$\Delta u = \frac{1 - |u|^2}{2\varepsilon^2} u,$$

with $\Delta = -\mathrm{tr}\nabla^2$ being just the Laplace–Beltrami operator. Also, as $u > 0$, it makes sense to talk about $\log u$. By a direct computation, we get

$$\Delta(\log u) = \frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2} = \frac{1 - u^2}{2\varepsilon^2} + \frac{|\nabla u|^2}{u^2}.$$

Since (u, ∇) is irreducible and hence $|u| \leqslant 1$, we find after integrating over M that

$$0 = \int_M \Delta(\log u) \mathrm{vol}_g = \int_M \frac{1 - u^2}{2\varepsilon^2} + \frac{|\nabla u|^2}{u^2} \mathrm{vol}_g \geqslant 0,$$

which together with the fact that u is positive forces u to be identically 1. □

Now we prove the main result of the section, using all the results obtained so far.

Theorem 2.2.2. *Suppose $L \to M$ is a Hermitian line bundle over a closed Riemannian manifold (M, g). There exists $\varepsilon_0 = \varepsilon_0(M, L) \geqslant 1$ such that if $\varepsilon > \varepsilon_0$ then any critical point (u, ∇) of E_ε is either reducible, or gauge-equivalent to $(1, d)$. (The latter can only occur if L is trivial.)*

Proof. Suppose (u, ∇) is an irreducible critical point of E_ε which is not gauge-equivalent to $(1, d)$. Then, by Lemma 2.2.1, we know that $\{u = 0\} \neq \emptyset$. Thus, letting

$$v = \frac{u}{\|u\|_2},$$

which is permitted since $u \not\equiv 0$, we deduce from Corollary 2.1.7 that

$$\|\nabla^k v\|_2 = \Big\|\frac{\nabla^k u}{\|u\|_2}\Big\|_2 \leqslant C_{k,M,L}.$$

Combining this with the bounds on F_∇ and its covariant derivatives given by Corollary 2.1.6, we obtain a sequence $B = (B_{k,M,L})_{k\geqslant 0}$ such that $(v, \nabla) \in \mathcal{A}_B$. In particular, Lemma 2.1.8 gives

$$\int_M |\nabla v|^2 \mathrm{vol}_g \geqslant \alpha_B > 0,$$

where we emphasize that α_B depends only on M and L. On the other hand, we infer from (2.1.12) that

$$\int_M |\nabla v|^2 \mathrm{vol}_g \leqslant \frac{1}{2\varepsilon^2}.$$

Combining this with the lower bound gives

$$\varepsilon \leqslant \frac{1}{\sqrt{2\alpha_B}} =: \varepsilon_0,$$

and we are done. □

To end the chapter, we use estimates of Section 2.1 to obtain an energy gap for the YMH energy.

Theorem 2.2.3. *Given any $\delta > 0$, there exists a real constant $C_\delta > 0$ such that for every $\varepsilon \geqslant \delta$ and any critical point (u, ∇) of E_ε we have either $(u, \nabla) \sim (1, d)$, in which case L is trivial, or*

$$\varepsilon^2 E_\varepsilon(u, \nabla) \geqslant C_\delta. \tag{2.2.1}$$

Proof. Suppose, by contradiction, that there exists $\delta > 0$ and a sequence of critical points $(u_i, \nabla_i) \not\sim (1, d)$ of E_{ε_i}, with $\varepsilon_i \geqslant \delta$, satisfying

$$\varepsilon_i^2 E_{\varepsilon_i}(u_i, \nabla_i) \downarrow 0 \text{ as } i \to \infty. \tag{2.2.2}$$

This implies, in particular, that for $i \gg 1$ we must have (u_i, ∇_i) irreducible, i.e. $u_i \neq 0$; indeed, $u_i \equiv 0$ forces the uniform lower bound

$$\varepsilon_i^2 E_{\varepsilon_i}(u_i, \nabla_i) \geqslant \frac{\mathrm{vol}(M)}{4},$$

so our assumption (2.2.2) implies that there must be at most finitely many indices i for which (u_i, ∇_i) is reducible.

In turn, the fact that (u_i, ∇_i) is irreducible for $i \gg 1$ implies that

$$\limsup_{i\to\infty} \varepsilon_i < \infty,$$

because if not, taking a subsequence if necessary, we have $\varepsilon_i \uparrow \infty$ and then Proposition 2.2.2 says that (u_i, ∇_i) is reducible for $i \gg 1$, a contradiction. So, up to taking a subsequence, we can assume $\varepsilon_i \to \varepsilon \geqslant \delta > 0$ and (u_i, ∇_i) is irreducible for all i.

Then, it follows from the estimates in Corollaries 2.1.6 and 2.1.7 that (u_i, ∇_i) converges smoothly on M to some limiting configuration (u, ∇). But then

$$\varepsilon^2 E_\varepsilon(u, \nabla) = \lim_{i\to\infty} \varepsilon_i^2 E_{\varepsilon_i}(u_i, \nabla_i) = 0,$$

which implies that $E_\varepsilon(u, \nabla) = 0$ and hence $(u, \nabla) \sim (1, d)$. Finally, we conclude that for $i \gg 1$ the section u_i is non-vanishing, and therefore $(u_i, \nabla_i) \sim (1, d)$ by Lemma 2.2.1, contradicting our hypothesis. $\square$

Definition 2.2.4. *For each $\varepsilon > 0$ we define*

$$\Gamma_\varepsilon = \inf\{E_\varepsilon(u, \nabla) \mid (u, \nabla) \in \mathrm{Crit}(E_\varepsilon),\ (u, \nabla) \nsim (1, d)\}. \tag{2.2.3}$$

Now, recall from the work of Pigati-Stern [PS21, Proposition 7.11] that there exist constants $\varepsilon_0^{PS} > 0$ and $C^{PS} > 0$, depending only on (M, g), such that for all $0 < \varepsilon \leqslant \varepsilon_0^{PS}$ we have $\Gamma_\varepsilon \geqslant C^{PS}$. On the other hand, taking $\delta = \varepsilon_0^{PS} > 0$ in our Theorem 2.2.3, we get that there is $C > 0$ depending only on (M, g) such that $\Gamma_\varepsilon \geqslant \varepsilon^{-2} C$ for all $\varepsilon \geqslant \varepsilon_0^{PS}$. Thus, we conclude the following:

Corollary 2.2.5. *For each $\varepsilon > 0$,*

$$\Gamma_\varepsilon \geqslant C_{\varepsilon,M} := \min\left\{C^{PS}, \frac{C}{\varepsilon^2}\right\}. \tag{2.2.4}$$

Part II

The deformed Hermitian–Yang–Mills equations

Contex and main results

The deformed Hermitian–Yang–Mills (dHYM) equations first emerged in the context of String Theory, where they appeared as the so-called *Fourier–Mukai* transform of the Special Lagrangian equations for submanifolds of a Calabi-Yau manifold X. The first version of the equation, and the one which has a larger literature in, was stated in terms of the curvature of a line bundle $\mathscr{L} \to X$, as one can see in [LYZ00, MW24]. A more general version of the dHYM equation, depending on the curvature of a higher rank vector bundle $\mathscr{E}$ over any compact Kähler manifold X were proposed by Collins and Yau in [CY18], and can be written in the following form. Let (X, g, J, ω) be a compact Kähler manifold on X, and A be a connection on $\mathscr{E}$. We say that A is a dHYM connection if the following equation holds

$$\operatorname{Im}\left(e^{-i\varphi_k(\mathscr{E})}\left(k\omega \otimes \mathrm{Id}_{\mathscr{E}} - \frac{F_A}{2\pi}\right)^n\right) = 0, \tag{2.2.5}$$

where

$$\varphi_k(\mathscr{E}) = \arg\left(\int_X \operatorname{tr}\left(k\omega \otimes \mathrm{Id}_{\mathscr{E}} - \frac{F_A}{2\pi}\right)^n\right). \tag{2.2.6}$$

In the rank 1 case, the existence of solutions to the dHYM equations has been investigated in a series of works [CXY17, CY18, CS22] from the analytical point of view by means of techniques such as Geometric Invariant Theory. On the other hand, from the perspective of Algebraic Geometry, important progress was made in [Che21a, Che21b], where solutions to the dHYM equations are identified with solutions of the so-called J-equations, and the existence of such solutions is shown to be related to the J-stability of the underlying vector bundle.

In the higher rank case, the work of Eder Correa [Cor23, Cor24] undertook a systematic study of solutions to the dHYM equations on rank 2, decomposable vector bundles over flag varieties. In particular, examples of *strict* dHYM connections were constructed—that is, connections satisfying the dHYM equations but failing to satisfy the classical Hermitian–Yang–Mills equations—highlighting the richness and subtleties of the dHYM theory of connections.

Further insights into the existence problem in higher rank were obtained in [DMS24], where the authors considered a more general class of equations, known as Z-critical equations, encompassing the dHYM equations as a special case. It was shown that, in the so-called large volume limit, the existence of solutions is equivalent to a new stability condition termed *asymptotic Z-stability*, thus providing a powerful bridge between geometric analysis and algebro-geometric stability conditions.

Finally, in [KS24], the focus shifted to the study of Z-critical equations on rank 2 vector bundles over complex surfaces, beyond the large volume regime. In this setting, it was demonstrated that (non-asymptotic) Z-stability becomes a necessary condition for the existence of solutions, underscoring the delicate interplay between curvature, stability, and complex geometry in the theory of Z-critical connections.

The concept of asymptotic Z-stability will be the main subject in this part of the thesis, hence we shall introduce it properly. Let (X, L) be a polarized variety, where $L \in \mathrm{Div}(X)$ is an ample divisor, and $\mathscr{E}$ be a holomorphic vector bundle on it. For each $k \in \mathbb{N}$ we can define the k-central charge of $\mathscr{E}$ by

$$Z_k(\mathscr{E}) = i^{n+1}\mathrm{ch}(\mathscr{E}) \cdot e^{-ikL} \in \mathbb{C}, \tag{2.2.7}$$

where $\mathrm{ch}(\mathscr{E}) \in H^*(X, \mathbb{R})$ denotes the total Chern character of $\mathscr{E}$. Using the central charge, we define the k-slope of $\mathscr{E}$ as

$$\mu_k(\mathscr{E}) = -\frac{\mathrm{Re}(Z_k(\mathscr{E}))}{\mathrm{Im}(Z_k(\mathscr{E}))}. \tag{2.2.8}$$

We say that $\mathscr{E}$ is asymptotically Z-stable (a.Z-stable for short) if for every proper torsion-free subsheaf $\mathscr{F} \hookrightarrow \mathscr{E}$ there is a $k_0 > 0$ such that

$$\mu_k(\mathscr{F}) \leqslant \mu_k(\mathscr{E}), \quad \forall k \geqslant k_0. \tag{2.2.9}$$

It is proved in [DMS24] that this stability condition guarantees the existence of solutions to the equation (2.2.5) for all sufficiently large k.

On the other hand, the existence of standard Hermitian–Yang–Mills (HYM) connections on a vector bundle is related to its Mumford-Takemoto stability, called here μ-stability for short, which is defined in terms of the slope

$$\mu(\mathscr{E}) = \frac{c_1(\mathscr{E}) \cdot L^{n-1}}{\mathrm{rk}(\mathscr{E})} \tag{2.2.10}$$

where similarly we say that the vector bundle $\mathscr{E}$ is μ-semistable if for every proper torsion-free subsheaf $\mathscr{F} \hookrightarrow \mathscr{E}$ we have

$$\mu(\mathscr{F}) \leqslant \mu(\mathscr{E}), \tag{2.2.11}$$

and when the inequality is always strict, we say that $\mathscr{E}$ is μ-stable.

In [DMS24] it is shown that every a.Z-stable vector bundle is automatically μ-semistable, and it is natural to question whether we can construct a.Z-stable bundles which are not μ-stable. This question is also motivated from the analytical point of view. In fact since both are measuring the existence of solutions to the dHYM equations and HYM equations, respectively, this question can be stated in the following way.

Question 4. *How can we construct vector bundles which admit solutions to the dHYM equations which but do not admit solutions to the HYM equations?*

One of the interesting points concerning the question above is that if we have such a bundle, solutions to the dHYM equations represent a genuinely new feature of our setting (X, L). A positive answer to this question is given in [Cor24], where the authors present examples of such solutions on a rank 2 decomposable vector bundle over the full flag $\mathbb{P}(T\mathbb{P}^2)$. With this in mind, we ask ourselves here about the existence of indecomposable bundles satisfying the conditions of Question 4.

Main results and contributions

We start §3.1 by defining the central charge and k-slope for a coherent sheaf $\mathscr{E}$ over a polarized surface (X, L), where $L \in \mathrm{Pic}(X)$ is an ample divisor. As a consequence we obtain an interesting relationship between a.Z-stability and Gieseker stability, in the case X is a fano surface and the polarization is taken to be the anti-canonical divisor. This is the content of Proposition 3.1.6.

In §3.2 we start by stating the Hoppe criterion for μ-stabiliy on polarized varieties which is proved in [JMPSE17]. We use this criterion to study the stability of Hartshorne-Serre bundles in Proposition 3.2.2. Finally, we present a version of Hoppe's criterion for the a.Z-stability of rank 2 and 3 vector bundles in Proposition 3.2.5, the main inspiration being [OSS80a, Lemma 1.2.5].

§4.1 is dedicated to proving our main theorems. The first one being Theorem 4.1.1, which gives a way to construct strictly a.Z-stable bundles, starting from suitable μ-stable vector bundles of rank 2. The construction of Theorem 4.1.4 is a version of Maruyama's example [OSS80b, Pg. 90], and is also considered in [DMS24, KS24], in the context of Z-stability for other central charges. Then we compile Theorem 4.1.1 and Lemma 3.2.2 together to obtain Theorem 4.1.4, which gives a recipe for the construction of strictly a.Z-stable bundles over a polarized surface (X, L), in terms of a pair (Z, D), where $Z \subset X$ is a local complete intersection subscheme of points and $D \in \mathrm{Pic}(X)$ is a divisor.

Theorem (4.1.4, L.)**.** *Let (X, L) be a polycyclic, polarized surface. Let also $R_0 \subset \mathrm{Pic}(X)$ be such that $H^0(X, \mathscr{O}_X(B)) = 0$ for all $B \notin R_0$. For each pair (Z, D), where $D \in \mathrm{Pic}(X)$ and $Z \subset X$ is a local complete intersection subscheme of points, we let*

$$R := \{B \in R_0 \mid B \cdot L \leqslant D \cdot L\}. \tag{2.2.12}$$

Assume the following conditions are satisfied for a pair (Z, D):

(a) $D \cdot L > 0$;

(b) $D^2 < \ell(Z)$;

(c) $H^0(X, \mathscr{I}_Z(B)) = 0$, *for all* $B \in R$;

(d) $H^1(X, \mathscr{O}_X(-D)) \neq 0$ *or* $H^2(X, \mathscr{O}_X(-D)) = 0$ *with* $\ell(Z) > 2\chi(\mathscr{O}_X)$;

(e) $H^2(X, \mathscr{O}_X(-2D)) = 0$.

Then, the rank 3 *sheaf* $\mathscr{F}_{Z,D}$ *obtained as the double extension*

$$\begin{tikzcd} \mathscr{O}_X \arrow[r, hook] & \mathscr{E}_{Z,D} \arrow[r, two heads] \arrow[d, hook] & \mathscr{I}_Z(2D) \\ & \mathscr{F}_{Z,D} \arrow[d, two heads] & \\ & \mathscr{O}_X(D) & \end{tikzcd} \tag{2.2.13}$$

is an indecomposable, holomorphic, strictly a.Z-stable vector bundle.

Based on this construction, we present in §4.2 some new examples of strictly a.Z-stable bundles over some surfaces. The list of examples is given below. For them, we use notation of Theorem 4.1.4, and refer to§4.2 for details.

1. The first example appears on the projective plane $\mathbb{P}^2$, naturally polarized by the hyperplane class $L = H$, which generates $\mathrm{Pic}(\mathbb{P}^2)$. We can take $R_0 = \{mH \in \mathbb{Z} \mid m \geqslant 0\}$ and then
$$R = \{m \in \mathbb{Z} \mid 0 \leqslant m \leqslant k\}.$$
Choosing $D = kH$, the conditions of Theorem 4.1.4 are satisfied for $k = 1$ and Z being the general intersection of two curves in $\mathbb{P}^2$ with degrees at least 2. We also present an example of a.Z-stable bundle over $\mathbb{P}^2$ given by choosing $D = 2H$ and the same Z as before, but this example is not obtained by Theorem 4.1.4.

2. Here we assume that $X = \mathbb{P}^1 \times \mathbb{P}^1$, in which case $\mathrm{Pic}(X) = \mathbb{Z}A \oplus \mathbb{Z}B \cong \mathbb{Z}^2$, where $A, B \in \mathrm{Pic}(X)$ are the classes of each copy of $\mathbb{P}^1 \subset X$, and we can let R_0 be the first quadrant in $\mathbb{Z}^2$. We prove that for every $k, \ell \in \mathbb{Z}$ such that $\ell > k > 0$, Theorem 4.1.4 implies the existence of a.Z-stable bundles with respect to the polarization $L = A+B$. In this case we take $D = -kA + \ell B$ and Z being the intersection of two curves C_i with bi-degrees (a_i, b_i) such that $a_i, b_i > \ell$.

3. For this example we take $X = \mathrm{Bl}_q\mathbb{P}^2$, in which case we have $\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E$, where H is the hyperplane class and E is the exceptional divisor. We produce two examples for the polarization $L = 3H - E$. We first prove in Lemma 4.2.4 that a

possible region R_0 is given by $\{aH + bE \mid a \geqslant 0 \text{ and } a + b \geqslant 0\}$. So in the first example we take $D = -H + 4E$, in which case we have

$$R = \{0, E\}$$

and then we let Z be the (pullback of) intersection of two lines on $\mathbb{P}^2$ that do not pass through the blown-up point q. The second example is given by $D = -H + 5E$, in which case we have

$$R = \{0, E, H - E\}$$

and then we let Z be the intersection of $C_1 = 2H$ with $C_2 = 2H - 2E$.

Outline of the proofs

The main idea of Theorem 4.1.4 is to start with a rank 2 bundle, obtained from Harthsorne-Serre's construction, for some pair (D, Z) as stated in the hypotheses, assure its μ-stability and use it to extend the line bundle $\mathscr{O}_X(D)$, obtaining a rank 3 strictly a.Z-stable bundle. The choice of twisting the ideal sheaf $\mathscr{I}_Z$ by $2D$ is intentionally made to ensure that $c_1(\mathscr{E}_{Z,D}) = 2D$. We would like to tune Z and D in order for $\mathscr{E}$ to be non-trivial, locally-free and μ-stable. It is known that this construction gives rise to rank 2 stable vector bundles if we take $\ell(Z)$ large enough, see [HL10, Theorem 5.1.3], but since this suitable $\ell(Z)$ depends on the dimension of the Hilbert scheme of divisors on X with degree less than $\mu(\mathscr{E}_{Z,D})$, we prefer a more concrete approach to finding good D and Z. The μ-stability is ensured using the Hoppe criterion, see Lemma 3.2.1, and the exact sequence (4.1.7), if we have $H^0(X, \mathscr{O}_X(B)) = H^0(X, \mathscr{I}_Z(B + D)) = 0$ for all $B \in \operatorname{Pic}(X)$ such that $B \cdot L \leqslant -\mu(\mathscr{E}_{Z,D}) = -D \cdot L$. Since $\operatorname{Pic}(X) = \mathbb{Z}^k$, the inequality $B \cdot L \leqslant a \in \mathbb{R}$ defines a semi-space in $\mathbb{Z}^k$ bounded by $a \in \mathbb{R}$, we call it S_a. Choosing $D \in \operatorname{Pic}(X)$ so that $D \cdot L > 0$ forces $H^0(X, \mathscr{O}(B)) = 0$ for $B \in S_{-\mu(\mathscr{E})}$ due to the ampleness of L. Now we notice that $B \in S_{-D \cdot L}$ if and only if $B + D \in S_{D \cdot L}$, and letting $R_0 \subset \operatorname{Pic}(X)$ be a subset such that $H^0(X, \mathcal{O}_X(A)) = 0$ for all $A \notin R_0$, it suffices to show that

$$H^0(X, \mathscr{I}_Z(A)) = 0, \quad \forall A \in R := R_0 \cap S_{\mu(\mathscr{E})}. \tag{2.2.14}$$

We explore examples where the set R is finite, so we can determine the 0-dimensional subscheme $Z \subset X$ as intersection of curves on X, satisfying a finite number of restrictions.

In summary, our construction is given in terms of a pair (Z, D), where Z is a 0-dimensional local complete intersection subscheme of X and $D \in \operatorname{Pic}(X)$ is a divisor. The cohomological properties of X, the choice of the polarization L and the divisor D give rise to a region $R \subset \operatorname{Pic}(X) = \mathbb{Z}^k$. Requiring Z to satisfy equation (2.2.14) gives us the μ-stability of the sheaf $\mathscr{E}_{Z,D}$. We need also to verify that $\mathscr{E}_{Z,D}$ is non-trivial and locally-free.

Thus, we finally get a rank 3 a.Z-stable holomorphic bundle $\mathscr{F}_{Z,D}$, as illustrated in the diagram (2.2.13).

3 dHYM equations and stability conditions on surfaces

In this chapter, (X, L) is a polarized surface, and by this we mean that $L \in \mathrm{Div}(X)$ is an ample divisor on X, providing an embedding $X \hookrightarrow \mathbb{P}^m$ for some m, that is, X is also a projective variety. We also assume that X is polycyclic, i.e., that $\mathrm{Pic}(X) \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$. Since X is a projective variety, we have an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Div}(X)/\sim$, where $\sim$ is the linear equivalence relation on divisors. We will denote by $\mathscr{O}_X(D)$ the line bundle associated to the divisor D, and also we will write D for the divisor class $[D] \in \mathrm{Div}(X)/\sim$.

3.1 Basic facts and definitions

We begin this section by writing down the explicit formulas for the objects we mentioned above. Suppose that $\mathscr{E}$ is a coherent sheaf over X and denote $\alpha := \dfrac{L^2}{2}$, then we can write the central charge of $\mathscr{E}$, defined by equation 2.2.7 as

$$Z_k(\mathscr{E}) = -kc_1(\mathscr{E}) \cdot L + i\left(\alpha k^2 \mathrm{rk}(\mathscr{E}) - \mathrm{ch}_2(\mathscr{E})\right), \tag{3.1.1}$$

and as a consequence we can write the k-slope of $\mathscr{E}$ as

$$\mu_k(\mathscr{E}) = \frac{kc_1(\mathscr{E}) \cdot L}{\alpha k^2 \mathrm{rk}(\mathscr{E}) - \mathrm{ch}_2(\mathscr{E})} \tag{3.1.2}$$

We also have the following Lemma, which is proved in [DMS24], that relates the asymptotic Z-stability with μ-stability.

Lemma 3.1.1. *If $\mathscr{E}$ is a a.Z-stable sheaf, then $\mathscr{E}$ is μ-semistable.*

The above Lemma allows us to characterize a.Z-stability on surfaces in a way which will be usefull later on, for comparing it with the Gieseker stability condition.

Proposition 3.1.2. *Let $\mathscr{E}$ be a coherent sheaf over X. Then $\mathscr{E}$ is a.Z-stable if and only if for every proper, torsion-free sheaf $\mathscr{F} \hookrightarrow \mathscr{E}$ one of the following conditions holds:*

(i) $\mu(\mathscr{F}) < \mu(\mathscr{E})$;

(ii) $\mu(\mathscr{F}) = \mu(\mathscr{E})$ *and* $\mathrm{ch}_2(\mathscr{F})c_1(\mathscr{E}) \cdot L < \mathrm{ch}_2(\mathscr{E})c_1(\mathscr{F}) \cdot L$.

In particular, if $\mathscr{E}$ is a.Z stable and $c_1(\mathscr{E}) \cdot L = 0$, then $\mathscr{E}$ is μ-stable.

Proof. The first conclusion follows from the fact that, for $k \gg 0$, inequality $\mu_k(\mathscr{F}) < \mu_k(\mathscr{E})$ is equivalent to

$$\alpha k^3(\mathrm{rk}(\mathscr{E})c_1(\mathscr{F}) \cdot L - \mathrm{rk}(\mathscr{F})c_1(\mathscr{E}) \cdot L) + k(\mathrm{ch}_2(\mathscr{E})c_1(\mathscr{F}) - \mathrm{ch}_2(\mathscr{F})c_1(\mathscr{E}) \cdot L) > 0.$$

Now, suppose that $c_1(\mathscr{E}) \cdot L = 0$ and $\mathscr{E}$ is a.Z-stable. The first part of the Proposition shows that the only possible μ-destabilizing subsheaf $\mathscr{F} \subset \mathscr{E}$ must satisfy $\mu(\mathscr{F}) = \mu(\mathscr{E})$, but then we have that $c_1(\mathscr{F}) \cdot L = 0$, which in turn implies that $\mathrm{ch}_2(\mathscr{F})c_1(\mathscr{E}) \cdot L = \mathrm{ch}_2(\mathscr{E})c_1(\mathscr{F}) \cdot L = 0$, a contradiction. Thus, there is no μ-destabilizing subsheaf $\mathscr{F}$ for $\mathscr{E}$, and we conclude that $\mathscr{E}$ is μ-stable. □

Now we would like to compare asymptotic Z_k-stability with Gieseker stability. The following defintions and propositions are taken from [Fri98, Section 2.4].

Definition 3.1.3. *Let $\mathscr{E}$ be a coherent sheaf over X and let $\mathscr{L} = \mathscr{O}_X(L)$ be the line bundle defined by the polarization L. We say that $\mathscr{E}$ is Gieseker stable, with respect to L, if for every proper coherent sub-sheaf $\mathscr{F} \hookrightarrow \mathscr{E}$ we have*

$$\frac{\chi(\mathscr{F} \otimes \mathscr{L}^k)}{\mathrm{rk}(\mathscr{F})} < \frac{\chi(\mathscr{E} \otimes \mathscr{L}^k)}{\mathrm{rk}(\mathscr{E})}, \qquad \text{for} \qquad k \gg 0. \tag{3.1.3}$$

Proposition 3.1.4. *A coherent sheaf $\mathscr{E}$ is Gieseker stable if, and only if, for each coherent sub-sheaf $\mathscr{F} \hookrightarrow \mathscr{E}$ one of the following conditions hold:*

(i) $\mu(\mathscr{F}) < \mu(\mathscr{E})$

(ii) $\mu(\mathscr{F}) = \mu(\mathscr{E})$ *and* $\dfrac{\chi(\mathscr{F})}{\mathrm{rk}(\mathscr{F})} < \dfrac{\chi(\mathscr{E})}{\mathrm{rk}(\mathscr{E})}$.

We also have the following useful formula, coming from the Riemann-Roch formula on surfaces.

Proposition 3.1.5. *Let $\mathscr{E}$ be a coherent sheaf over X. Then the Euler characteristic of $\mathscr{E}$ satisfies*

$$\chi(\mathscr{E}) = \mathrm{ch}_2(\mathscr{E}) - \frac{c_1(\mathscr{E}) \cdot K_X}{2} + \mathrm{rk}(\mathscr{E})\chi(\mathscr{O}_X). \tag{3.1.4}$$

With this information, we have the relation below between Gieseker stability and asymptotic Z-stability. The next proposition is a direct consequence of the previous definitions and results.

Proposition 3.1.6. *Suppose that the $(X, L = -K_X)$ is a Del Pezzo surface. Then a coherent sheaf $\mathscr{E}$ with $c_1(\mathscr{E}) \cdot K_X < 0$ is a.Z-stable if and only if it is Gieseker stable.*

Proof. In this case, it follows from Proposition 3.1.5 that for every coherent sheaf $\mathscr{E}$ we have

$$\frac{\chi(\mathscr{E})}{\mathrm{rk}(\mathscr{E})} = \frac{\mathrm{ch}_2(\mathscr{E})}{\mathrm{rk}(\mathscr{E})} + \frac{1}{2}\mu(\mathscr{E}) + \chi(\mathscr{O}_X),$$

And thus, from Proposition 3.1.4 it follows that $\mathscr{E}$ is Gieseker stable if and only if for every coherent sub-sheaf $\mathscr{F} \hookrightarrow \mathscr{E}$ we have either $\mu(\mathscr{F}) < \mu(\mathscr{E})$ or $\mu(\mathscr{F}) = \mu(\mathscr{E})$ and $\frac{\mathrm{ch}_2(\mathscr{F})}{\mathrm{rk}(\mathscr{F})} < \frac{\mathrm{ch}_2(\mathscr{E})}{\mathrm{rk}(\mathscr{E})}$. Comparing with Proposition 3.1.2 we see that the first condition are the both for Gieseker and a.Z-stability. The second condition for a.Z-stability on Proposition 3.1.2 is that in the case where $\mu(\mathscr{F}) = \mu(\mathscr{E})$ we should have

$$\mathrm{ch}_2(\mathscr{F})c_1(\mathscr{E}) \cdot (-K_X) < \mathrm{ch}_2(\mathscr{E})c_1(\mathscr{F}) \cdot (-K_X).$$

Multiplying both sides by $\mathrm{rk}(\mathscr{E})$ and using $\mu(\mathscr{F}) = \mu(\mathscr{E})$, the last inequality is equivalent to

$$-\mathrm{rk}(\mathscr{E})\mathrm{ch}_2(\mathscr{F})c_1(\mathscr{E}) \cdot K_X < -\mathrm{rk}(\mathscr{F})\mathrm{ch}_2(\mathscr{E})c_1(\mathscr{E}) \cdot K_X,$$

which is exactly the condition for Gieseker stability, since $-c_1(\mathscr{E}) \cdot K_X > 0$. □

Remark 3.1.7. Due to this relation between a.Z-stability and Gieseker stability, we expect to be able to adapt the techinques for constructing strictly Gieseker stable bundles to the case of a.Z-stability. In particular, we will construct a.Z-stable bundles on surfaces by using the same techniques as in [OSS80b]. To give the explicit motivation, it is shown in [OSS80b] that if $\mathscr{E}$ is a μ-stable bundle on $\mathbb{P}^2$ satisfying $c_1(\mathscr{E}) = 0$ and $H^1(\mathbb{P}^2, E) \neq 0$, then every holomorphic vector $\mathscr{F}$ arising as a an extension of the form

$$\mathscr{E} \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{O}_{\mathbb{P}^2}$$

is strictly Gieseker stable. Since this construction gives bundles with $c_1(\mathscr{F}) = 0$, there is no hope of obtaining a a.Z-stable bundle, because of Proposition 3.1.2. So we need to adapt it, in order to obtain bundles with $c_1(\mathscr{F}) \neq 0$. ○

3.2 Adapted Hoppe criteria

The main goal of this section is to provide cohomological criteria to ensure the (semi-)stability of vector bundles over polarized varieties in some special settings, i.e. special choices of variety, vector bundles and stability conditions. We begin by using the version of Hoppe's criterion for μ-stability of rank 2 bundles over polycyclic varieties, proved in [JMPSE17], to study the μ-stability of Hartshorne-Serre bundles. This will be useful in Chapter 4 to construct a.Z-stable bundles. We then adapt the version of Hoppe's criterion for μ-stability of vector bundles of low rank presented in [OSS80b, Lemma 1.2.5] to the case of a.Z-stability.

3.2.1 Stability of Hartshorne-Serre bundles

In this section we will study the μ-stability of Hartshorne-Serre sheaves. In particular, we will be interested in vector bundles appearing as extenstions of the following form

$$\mathscr{O}_X \hookrightarrow \mathscr{E} \twoheadrightarrow \mathscr{I}_Z(2D) \tag{3.2.1}$$

for some divisor $D \in \mathrm{Div}(X)$ and a zero dimensional subscheme $Z \subset X$. This will be useful for our construction on Chapter 4. To do this, one useful tool is the following form of Hoppe's criterion, adapted for rank 2 vector bundles over polycyclic varieties.

Lemma 3.2.1 ([JMPSE17]). *Let (X, L) be a polarized variety with dimension n, and $\mathscr{E}$ be a holomorphic vector bundle with rank 2 over X. Then $\mathscr{E}$ is μ-stable if and only if*

$$H^0(X, \mathscr{E}(B)) = 0 \quad \forall B \in \mathrm{Pic}(X) \ \text{ with } \ B \cdot L^{n-1} \leqslant -\mu(\mathscr{E}). \tag{3.2.2}$$

With this in mind, we can prove the following lemma, which is a direct consequence of Lemma 3.2.1.

Lemma 3.2.2 (L.). *Let (X, L) be a polarized surface, and $\mathscr{E}$ be a holomorphic vector bundle over X given as an extension (3.2.1). Let $R_0 \subset \mathrm{Pic}(X)$ be such that $H^0(X, \mathscr{O}_X(B)) = 0$ for all $B \notin R_0$. If $D \cdot L > 0$ and $H^0(X, \mathscr{I}_Z(B)) = 0$, for all $B \in R_0$ such that $B \cdot L \leqslant D \cdot L$, then $\mathscr{E}$ is μ-stable.*

Proof. Let $B \in \mathrm{Pic}(X)$ be such that $B \cdot L \leqslant -\mu(\mathscr{E}) = -D \cdot L$. Since $D \cdot L > 0$ and L is an ample line bundle, we can conclude that $H^0(X, \mathscr{O}_X(B)) = 0$. Now, twisting the extension (3.2.1) by $\mathscr{O}_X(B)$, we get the following short exact sequence

$$\mathscr{O}_X(B) \hookrightarrow \mathscr{E}(B) \twoheadrightarrow \mathscr{I}_Z(2D + B),$$

and from its associated long exact sequence in cohomology, together with the previous fact, we conclude that there exists an inclusion

$$H^0(X, \mathscr{E}(B)) \hookrightarrow H^0(X, \mathscr{I}_Z(2D + B)).$$

Hence, to show that $\mathscr{E}$ is μ-stable we just need to show that the cohomology on the right-hand side vanishes. To this end, notice that $B \cdot L \leqslant -D \cdot L$ is equivalent to $(2D + B) \cdot L \leqslant D \cdot L$. If $2D + B \in R_0$, then $H^0(X, \mathscr{I}_Z(2D + B)) = 0$ by hypothesis, and if $2D + B \notin R_0$ we have

$$H^0(X, \mathscr{I}_Z(2D + B)) \hookrightarrow H^0(X, \mathscr{O}_X(2D + B)) = 0. \qquad \square$$

Remark 3.2.3. The way we use the previous Lemma to get examples is to choose wisely the polarization so that the set of divisors $B \in R_0$ such that $B \cdot L \leqslant D \cdot L$ is finite, so the vanishing of global sections imposes a finite number of constraints on Z. ◯

3.2.2 Version for a.Z-stability

We are now interested in the case of a.Z-stability. We prove a version of Hoppe's criterion for the stability of low rank vector bundles. The result per se will not be used in the rest of the thesis but it could be an useful tool in the future to study stability of vector bundles over surfaces. The proof is a direct adaptation of the proof of Lemma 1.2.5 in [OSS80b] to the case of a.Z-stability.

Lemma 3.2.4. *Let $\mathscr{E}, \mathscr{F}$ be holomorphic vector bundles over (X, L). Suppose we have a short exact sequence*

$$\mathscr{F} \hookrightarrow \mathscr{E} \twoheadrightarrow \mathscr{O}_X(B),$$

for some $B \in \mathrm{Pic}(X)$. Then, for $k \gg 0$, we have

$$\mu_k(\mathscr{F}) < \mu_k(\mathscr{E}) \iff \mu_k(\mathscr{O}_X(B)) > \mu_k(\mathscr{E}) \tag{3.2.3}$$

Proof. Since for every coherent sheaf $\mathscr{E}$ the leading coefficient of $\mathrm{Im}(Z_k(E))$ is $\mathrm{rk}(\mathscr{E}) > 0$, we know that

$$\mathrm{Im}(Z_k(\mathscr{E})) > 0 \qquad \text{and} \qquad \mathrm{Im}(Z_k(\mathscr{E})) - \mathrm{Im}(Z_k(\mathscr{O}(B))) = \mathrm{Im}(Z_k(\mathscr{F})) > 0$$

for $k \gg 0$. Then

$$\begin{aligned} \mu_k(\mathscr{F}) < \mu_k(\mathscr{E}) &\iff \mathrm{Im}(Z_k(\mathscr{E}))\mathrm{Re}(Z_k(\mathscr{F})) > \mathrm{Im}(Z_k(\mathscr{F}))\mathrm{Re}(Z_k(\mathscr{E}) \\ &\iff -\mathrm{Re}(Z_k(\mathscr{O}(B)))\mathrm{Im}(Z_k(\mathscr{E})) > -\mathrm{Re}(Z_k(\mathscr{E}))\mathrm{Im}(Z_k(\mathscr{O}(B))) \\ &\iff \mu_k(\mathscr{O}(B)) > \mu_k(\mathscr{E}). \end{aligned}$$

□

So now we are able to prove the adapted version of Hoppe's criterion, stated in the following form.

Proposition 3.2.5 (L). *Let $\mathscr{E} \to X$ be asymptotically Z-stable vector bundle. Then the exists a k_0 such that, for all $k \geqslant k_0$,*

$$H^0(X, \mathscr{E}(B)) = 0, \quad \forall B \in \mathrm{Pic}(X) \ \textit{ with } \ \mu_k(\mathscr{O}(B)) \leqslant -\mu_k(\mathscr{E}), \tag{3.2.4}$$

and

$$H^0(X, \mathscr{E}^*(B)) = 0, \quad \forall B \in \mathrm{Pic}(X) \ \textit{ with } \ \mu_k(\mathscr{O}(B)) \leqslant \mu_k(\mathscr{E}). \tag{3.2.5}$$

Conversely, and if moreover $rk(\mathscr{E}) = 2, 3$, then equations (3.2.4) and (3.2.5) imply that $\mathscr{E}$ is asymptotically Z-stable.

Proof. First, suppose that there is some $B \in \mathrm{Pic}(X)$ is such that $\mu_k(\mathscr{O}_X(B)) \leqslant -\mu_k(\mathscr{E})$ and $H^0(X, \mathscr{E}(B)) \neq 0$ for all $k > 0$. Then there exists an inclusion $\mathscr{O}_X(-B) \hookrightarrow E$ and

$$\mu_k(\mathscr{O}_X(-B)) = -\mu_k(\mathscr{O}_X(B)) \geqslant \mu_k(\mathscr{E}), \quad \forall k > 0.$$

Thus $\mathscr{O}_X(-B)$ destabilizes $\mathscr{E}$. On the other hand, assume that there exists a $B \in \mathrm{Pic}(X)$ with $H^0(X, \mathscr{E}^*(B)) \neq 0$ and $\mu_k(\mathscr{O}_X(B)) \leqslant \mu_k(\mathscr{E})$ for all $k > 0$. Then we have an inclusion $\mathscr{O}_X(-B) \hookrightarrow \mathscr{E}^*$, which after dualizing leads to a short exact sequence

$$\mathscr{F} \hookrightarrow \mathscr{E} \twoheadrightarrow \mathscr{O}_X(B)$$

where $\mathscr{F} \subset \mathscr{E}$ is a rank $rk(\mathscr{E}) - 1$ sub-bundle with $\mu_k(\mathscr{O}(B)) \leqslant \mu_k(\mathscr{E})$. But Lemma 3.2.4 says that for $k \gg 0$, this last inequality is equivalent to $\mu_k(\mathscr{F}) \geqslant \mu_k(\mathscr{E})$ and then $\mathscr{F}$ destabilizes $\mathscr{E}$. Therefore stability of $\mathscr{E}$ implies equations (3.2.4) and (3.2.5).

For the converse, assume equations (3.2.4) and (3.2.5), and let $\mathscr{F} \hookrightarrow \mathscr{E}$. If $\mathscr{F}$ is rank 1, then it is of the form $\mathscr{F} = \mathscr{O}_X(B)$ for some $B \in \mathrm{Pic}(X)$, and the inclusion $\mathscr{O}_X(B) \hookrightarrow \mathscr{E}$ implies that $H^0(X, \mathscr{E}(-B)) \neq 0$, and thus by hypothesis we get

$$-\mu_k(\mathscr{E}) < \mu_k(\mathscr{O}_X(-B)) = -\mu_k(\mathscr{O}_X(B)) = -\mu_k(\mathscr{F}),$$

i.e. $\mu_k(\mathscr{F}) < \mu_k(\mathscr{E})$. This covers the converse for $rk(\mathscr{E}) = 2$.

Finally, for $\mathrm{rk}(\mathscr{E}) = 3$, we must also consider the case $\mathrm{rk}(\mathscr{F}) = 2$. In this case the $\mathscr{F} \hookrightarrow \mathscr{E}$ yields a short exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{O}_X(B) \to 0,$$

and dualizing we get a monomorphism $\mathscr{O}(-B) \hookrightarrow \mathscr{E}^*$ and thus $H^0(X, \mathscr{E}^*(B)) \neq 0$. Using our hypothesis, we have $\mu_k(\mathscr{O}(B)) > \mu_k(\mathscr{E})$ which is equivalent to $\mu_k(\mathscr{F}) < \mu_k(\mathscr{E})$ for $k \gg 0$. □

Remark 3.2.6. We notice that conditions (3.2.4) and (3.2.5) are equivalent if $\mathrm{rk}(\mathscr{E}) = 2$.
○

4 Constructing asymptotically Z-stable bundles

4.1 Constructing a.Z-stable bundles over surfaces

We present a method for constructing rank 3, a.Z-stable bundles over surfaces. We start by constructing a rank 2 μ-stable bundle $\mathscr{E}$, and use it to obtain a rank 3 bundle $\mathscr{F}$ from and extension of $\mathscr{O}_X(2D)$ by $\mathscr{E}$. The condition for $\mathscr{F}$ to be a.Z-stable is the content of Theorem 4.1.1. The main inspiration for the construction is the example of a strictly Gieseker stable bundle over $\mathbb{P}^2$ presented in Remark 3.1.7.

Theorem 4.1.1 (L)**.** *Let (X, L) be a polarized and polycyclic complex surface. Let $\mathscr{E} \to X$ be a μ-stable holomorphic bundle with $\operatorname{rk}(\mathscr{E}) = 2$ and $c_1(\mathscr{E}) = 2D \in \operatorname{Pic}(X)$, and let $\mathscr{F}$ the vector bundle obtained by an extension*

$$\mathscr{E} \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{O}_X(D). \tag{4.1.1}$$

Then $\mathscr{E}$ is the only possible a.Z-destabilizer for $\mathscr{F}$.

Furthermore, if $D \cdot L > 0$ then the bundle $\mathscr{F}$ is asymptotically Z-stable if, and only if,

$$2\operatorname{ch}_2(\mathscr{O}_X(D)) - \operatorname{ch}_2(\mathscr{E}) > 0,$$

or, equivalently,

$$c_2(\mathscr{E}) > D^2. \tag{4.1.2}$$

Proof. Let $\mathscr{F}'$ be a torsion-free sheaf such that $\mathscr{F}' \subset \mathscr{F}$ and $\operatorname{rk}(\mathscr{F}') < 3$. Assume first that $\operatorname{rk}(\mathscr{F}') = 1$, so that $\mathscr{F}' = \mathscr{O}_X(B)$ for some $B \in \operatorname{Pic}(X)$. As in the proof of Proposition 3.1.2, we see that $\mu_k(\mathscr{F}) - \mu_k(\mathscr{O}(B))$ is a positive multiple of

$$(D \cdot H - B \cdot H)k^3 + O(k^2), \tag{4.1.3}$$

which means that the sign of this difference for $k \gg 0$ is given by the sign of $D \cdot H - B \cdot H$, provided it is non-zero. If $D \cdot H \leqslant B \cdot H$, we use that $\mathscr{E}$ is μ-stable to obtain that $H^0(X, \mathscr{E}(-B)) = 0$ and hence we have from the long exact sequence in cohomology

$$H^0(X, \mathscr{F}(-B)) \hookrightarrow H^0(X, \mathscr{O}_X(D-B)) \xrightarrow{\delta} H^1(X, \mathscr{E}(-B)).$$

Since $B \cdot H \geqslant D \cdot H$, there is a non-trivial morphism $\mathscr{O}_X(B) \hookrightarrow \mathscr{O}_X(D)$ if, and only if, $B = D$. So either $h^0(\mathscr{O}_X(D-B)) = 0$ or $h^0(X, \mathscr{O}(D-B)) = 1$ with δ being a non-trivial map. In both cases we conclude that $H^0(X, \mathscr{F}(-B)) = \ker \delta = 0$, contradiction with the assumption that $\mathscr{O}_X(B) \hookrightarrow \mathscr{F}$. So $\mu_k(\mathscr{F}') < \mu_k(\mathscr{F})$ for $k \gg 0$ and $\mathscr{F}'$ does not destabilize $\mathscr{F}$.

Now if $\mathrm{rk}(\mathscr{F}') = 2$ so that the quotient $\mathscr{Q}$ given by

$$\mathscr{F}' \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{Q}$$

is torsion-free, and let $c_1(\mathscr{F}') = B \in \mathrm{Pic}(X)$. The dual $\mathscr{Q}^*$ is reflexive so it is locally-free[1], then $\mathscr{Q}^* = \mathscr{O}_X(B - 3D)$ and again, we have that $\mu_k(\mathscr{F}) - \mu_k(\mathscr{F}')$ is a positive multiple of

$$(2D \cdot H - B \cdot H)k^3 + O(k^2),$$

and again its sign for $k \gg 0$ depends on the leading term. If we suppose that $2D \cdot H - B \cdot H \leqslant 0$, we can conclude from $\mu(\mathscr{E}^*) = -D \cdot L$ that $(3D - B) \cdot L \leqslant -\mu(\mathscr{E}^*)$, and since $\mathscr{E}^*$ is μ-stable we get $0 = H^0(X, \mathscr{E}^*(3D - B)) = \mathrm{Hom}(\mathscr{Q}^*, \mathscr{E}^*)$, by Lemma 3.2.1. This implies that the composite map $\mathscr{Q}^* \to \mathscr{F}^* \to \mathscr{E}^*$ vanishes, and thus induces a map $\mathscr{Q}^* \hookrightarrow \mathscr{O}_X(-D)$. On the other hand $\mu(\mathscr{Q}^*) \geqslant \mu(\mathscr{O}_X(-D))$ by assumption, which means that the inclusion should be an isomorphism and $B = 2D$, since $\mathscr{O}_X(-D)$ is μ-stable. Furthermore, we get the exact sequence

$$\mathscr{Q}^* \hookrightarrow \mathscr{F}^* \twoheadrightarrow \mathscr{E}^*,$$

and this implies, after dualizing, that $\mathscr{Q} = \mathscr{Q}^{**}$, so that $\mathscr{Q} = \mathscr{O}_X(3D - B) = \mathscr{O}_X(D)$, and finally $\mathscr{F}' = \mathscr{E}$. In this case, we have that $\mu_k(\mathscr{F}) - \mu_k(\mathscr{E})$ is a positive multiple of

$$(D \cdot H)(2\mathrm{ch}_2(\mathscr{O}_X(D)) - \mathrm{ch}_2(\mathscr{E})) > 0, \tag{4.1.4}$$

and then $\mathscr{E}$ does not destabilize $\mathscr{F}$. □

Remark 4.1.2. If $\mathscr{F}$ is obtained as an extension (4.1.1), and we assume that $D \cdot L$, then

$$(D \cdot H)(2\mathrm{ch}_2(\mathscr{O}_X(D)) - \mathrm{ch}_2(\mathscr{E})) > 0 \iff c_2(\mathscr{E}) < D^2,$$

and following the proof of Theorem 4.1.1 we conclude that $\mathscr{F}$ is a.Z-stable if, and only if, $c_2(\mathscr{E}) < D^2$. On the other hand, we are assuming that $\mathscr{E}$ is a μ-stable vector bundle, so we can apply Bogomolov inequality, c.f. [Fri98, Ch. 9], to obtain

$$c_1(\mathscr{E})^2 \leqslant 4c_2(\mathscr{E}) \iff c_2(\mathscr{E}) \geqslant D^2.$$

This implies that in this case $\mathscr{F}$ can not be a.Z-stable. ○

Another useful Lemma is the following consequence of [HL10, Theorem 5.1.1].

Lemma 4.1.3. *Suppose that $\mathscr{L}$ is a line bundle over a projective surface X. If*

$$H^2(X, \mathscr{L}^*) = 0,$$

[1] It is worth to mention that this is a very special feature in dimension 2 varieties.

then for every 0-dimensional subscheme $Z \subset X$, there exists a non-trivial, locally-free extension of the form

$$\mathscr{O}_X \hookrightarrow \mathscr{E} \twoheadrightarrow \mathscr{I}_Z \otimes \mathscr{L}.$$

With these tools, we are ready to prove our highlighted theorem:

Theorem 4.1.4 (L)**.** *Let (X, L) be a polycyclic, polarized surface. Let also $R_0 \subset \mathrm{Pic}(X)$ be such that $H^0(X, \mathscr{O}_X(B)) = 0$ for all $B \notin R_0$. For each pair (Z, D), where $D \in \mathrm{Pic}(X)$ and $Z \subset X$ is a local complete intersection subscheme of points, we let*

$$R := \{B \in R_0 \mid B \cdot L \leqslant D \cdot L\}. \tag{4.1.5}$$

Assume the following conditions are satisfied for a pair (Z, D):

(a) $D \cdot L > 0$*;*

(b) $D^2 < \ell(Z)$*;*

(c) $H^0(X, \mathscr{I}_Z(B)) = 0$*, for all* $B \in R$*;*

(d) $H^1(X, \mathscr{O}_X(-D)) \neq 0$ *or* $H^2(X, \mathscr{O}_X(-D)) = 0$ *with* $\ell(Z) > 2\chi(\mathscr{O}_X)$*;*

(e) $H^2(X, \mathscr{O}_X(-2D)) = 0$*.*

Then, the rank 3 sheaf $\mathscr{F}_{Z,D}$ obtained as the double extension

$$\begin{array}{ccccc} \mathscr{O}_X & \hookrightarrow & \mathscr{E}_{Z,D} & \twoheadrightarrow & \mathscr{I}_Z(2D) \\ & & \downarrow & & \\ & & \mathscr{F}_{Z,D} & & \\ & & \downarrow & & \\ & & \mathscr{O}_X(D) & & \end{array} \tag{4.1.6}$$

is an indecomposable, holomorphic, strictly a.Z-stable vector bundle.

Proof. We start by constructing the rank 2 μ-stable bundle $\mathscr{E}_{Z,D}$ over X, given by the extension

$$\mathscr{O}_X \hookrightarrow \mathscr{E}_{Z,D} \twoheadrightarrow \mathscr{I}_Z(2D). \tag{4.1.7}$$

Since condition (e) is satisfied, we know by Lemma 4.1.3 that the sheaf $\mathscr{E}_{Z,D}$ is non-trivial and locally-free. To see that $\mathscr{E}_{Z,D}$ is μ-stable, we just use condition (c) together with Lemma 3.2.2, then $\mathscr{E}_{Z,D}$ is a rank 2, indecomposable, μ-stable holomorphic bundle over (X, L). Now we apply Lemma 4.1.1 to construct the rank 3 bundle $\mathscr{F}_{Z,D}$, which is given by the extension

$$\mathscr{E}_{Z,D} \hookrightarrow \mathscr{F}_{Z,D} \twoheadrightarrow \mathscr{O}_X(D).$$

To ensure that it is non-trivial, we must compute $H^1(X, \mathscr{E}_{Z,D}(-D))$. From the exact sequence 4.1.7, twisted by $-D$, we get the exact sequence

$$H^0(X, \mathscr{I}_Z(D)) \to H^1(X, \mathscr{O}_X(-D)) \to H^1(X, \mathscr{E}_{Z,D}(-D)),$$

and applying condition (c) again we get that $H^1(X, \mathscr{I}_Z(D)) = 0$, which means that

$$0 \neq H^1(X, \mathscr{O}_X(-D)) \subset H^1(X, \mathscr{E}_{Z,D}(-D)),$$

by the first part of condition (d), which deals with the non-triviality of the extension. Furthermore, if $H^1(X, \mathscr{O}(-D)) = 0$ but $H^2(X, \mathscr{O}(-D)) = 0$ then the same long exact sequence above will give us

$$H^1(X, \mathscr{E}_{Z,D}(-D)) \cong H^1(X, \mathscr{I}_Z(D)),$$

and to prove that it is non-zero we use the twisted restriction sequence

$$\mathscr{I}_Z(D) \hookrightarrow \mathscr{O}_X(D) \twoheadrightarrow \mathscr{O}_Z$$

to obtain

$$h^1\mathscr{I}_Z(D) = \ell(Z) + h^2\mathscr{O}_X(D) - \chi(\mathscr{O}_X(D)) \geqslant \ell(Z) - \chi(\mathscr{O}_X) > 0,$$

by the second part of condition (d). Finally, since $c_2(\mathscr{E}_{Z,D}) = \ell(Z)$, the condition (b) together with Lemma 4.1.1 gives us that $\mathscr{F}_{Z,D}$ is a.Z-stable. □

4.2 Examples

In the computations of this section, the base variety on which we compute cohomology groups of sheaves will be clear in the context. So, in order to save some notation, we shall use the $h^p\mathscr{E}$ for the dimension of the cohomology group $H^p(X, \mathscr{E})$.

4.2.1 The complex projective plane $\mathbb{P}^2$

In this case we are taking $X = \mathbb{P}^2$ and the polarization L to be given by the hyperplane class $H \in \operatorname{Pic}(X)$. Because of this, we shall use the most common notation for line bundles on $\mathbb{P}^2$, which is $\mathscr{O}_{\mathbb{P}^2}(k) := \mathscr{O}_{\mathbb{P}^2}(kH)$. The following proposition gives us a family of examples of strictly asymptotically Z-stable bundles over $\mathbb{P}^2$.

Proposition 4.2.1 (L). *Let $X = \mathbb{P}^2$ with the polarization given by the hyperplane class. The following holds:*

1. *If Z is the generic intersection of two curves with degrees d_1 and d_2 with $d_i \geqslant 2$, then there exists a non-trivial, locally-free sheaf $\mathscr{E}_{Z,2}$ which is given by an extension*

$$\mathscr{O}_X \hookrightarrow \mathscr{E}_{Z,1} \twoheadrightarrow \mathscr{I}_Z(2). \tag{4.2.1}$$

Furthermore, this sheaf is μ-stable (w.r.t. the polarization given).

2. *There exists a non-trivial locally free sheaf $\mathscr{F}$ given by an extension of the form*

$$\mathscr{E}_{Z,2} \hookrightarrow \mathscr{F}_{Z,1} \twoheadrightarrow \mathscr{O}_X(1). \tag{4.2.2}$$

Furthermore, this sheaf is strictly asymptotically Z-stable (w.r.t. the polarization given).

Proof. To show this, we just have to show that the conditions inTheorem 4.1.4 are satisfied. In the notation of Theorem 4.1.4 we are choosing $D = L$, and hence we see that $D \cdot L = 1$, so condition (a) is satisfied. We are taking Z to be the generic intersection of two curves C_1 and C_2 in $\mathbb{P}^2$ with degrees at least 2. By Bézout theorem we have that $\ell(Z) \geqslant 4 > 1 = D^2$, so condition (b) is also verified. To check condition (c), we recall the following exact sequence for the ideal sheaf $\mathscr{I}_Z$:

$$\mathscr{O}_{\mathbb{P}^2}(-d_1 - d_2) \hookrightarrow \mathscr{O}_{\mathbb{P}^2}(-d_1) \oplus \mathscr{O}_{\mathbb{P}^2}(-d_2) \twoheadrightarrow \mathscr{I}_Z., \tag{4.2.3}$$

where d_i is the degree of C_i. Since in this case we can take $R_0 = \{kL \mid k \geqslant 0\}$ and $D \cdot L = 1$, we have that $R = \{0, L\}$, so we must show that $H^0(\mathbb{P}^2, \mathscr{I}_Z) = H^0(\mathbb{P}^2, \mathscr{I}_Z(1)) = 0$. Taking the global section functor on the exact sequence (4.2.3) gives us

$$H^0(\mathbb{P}^2, \mathscr{I}_Z) \subset H^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-d_1 - d_2)) = 0.$$

On the other hand, twisting the exact sequence (4.2.3) by $\mathscr{O}_{\mathbb{P}^2}(1)$ and taking the global section functor, we get

$$H^0(\mathbb{P}^2, \mathscr{I}_Z(1)) \subset H^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-d_1 - d_2 + 1)) = 0,$$

since $d_i \geqslant 2$ implies that $H^0(X, \mathscr{O}(-d_i + 1)) = 0$. Therefore, our setting satisfies condition (c). Now, we have

$$H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-1)) = H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-2)) = 0,$$

by Serre duality, and since $2\chi(\mathscr{O}_{\mathbb{P}^2}) = 2 < 4 \leqslant \ell(Z)$ we also satisfy condition (d). Finally, we get condition (e) by computing directly

$$H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-2)) = H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-1)) = 0.$$

So using Theorem 4.1.4 we conclude the proof. □

Remark 4.2.2. The choice for $D = L$ is exclusive to match condition (e). Indeed, for $D = kL$ with $k > 0$, we have

$$H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-2k)) = H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2k - 3)),$$

by Serre duality and the right-hand side vanishes if, and only if, $k = 1$. It is possible to get holomorphic bundles arising from an extension of the form

$$\mathscr{O}_{\mathbb{P}^2} \hookrightarrow \mathscr{E} \twoheadrightarrow \mathscr{I}_Z(2k), \tag{4.2.4}$$

by just taking Z such $\ell(Z) > h^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2k-3))$, see [HL10, pg. 148]. We can also analyze the stability of the holomorphic bundle $\mathscr{E}$ defined by the exact sequence (4.2.4): since $D \cdot L = k$ in this case we have that $R = \{aL \mid 0 \leqslant a \leqslant k\}$ so that we must verify $H^0(\mathbb{P}^2, \mathscr{I}_Z(a)) = 0$ for all $0 \leqslant a \leqslant k$. Again, assuming that Z is a general intersection of two curves C_1, C_2, with respective degrees d_1 and d_2, we can twist the exact sequence (4.2.3) by $\mathscr{O}_{\mathbb{P}^2}(a)$ and apply the global section functor to obtain the projection

$$H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a-d_1)) \oplus H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(a-d_2)) \twoheadrightarrow H^0(\mathbb{P}^2, \mathscr{I}_Z(a)),$$

and then the desired cohomology vanishes for all $0 \leqslant a \leqslant k$ if we take $d_i \geqslant k+1$.

On the other hand, in order to follow the construction and have a non-trivial holomorphic bundle arising as the extension

$$\mathscr{E} \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{O}_{\mathbb{P}^2}(k),$$

we would like to ensure that $H^2(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-k)) = H^0(\mathbb{P}^2, \mathscr{O}(k-3))$ vanishes, which happens if, and only if, $k = 1, 2$.

We already considered the case $k = 1$, so we proceed to understand the constraints for the case $k = 2$. In this case $D^2 = 4 < 9 \leqslant \ell(Z)$, so the bundle $\mathscr{F}$ is a.Z-stable. The existence of a locally-free extension is ensured by the fact that

$$h^0\mathscr{O}_{\mathbb{P}^2}(2k-3) = h^0\mathscr{O}_{\mathbb{P}^2}(1) = 3 < 9 \leqslant \ell(Z)$$

We conclude that the diagram

$$\begin{tikzcd}
\mathscr{O}_{\mathbb{P}^2} \arrow[r, hook] & \mathscr{E}_{Z,2} \arrow[r, two heads] \arrow[d, hook] & \mathscr{I}_Z(4) \\
& \mathscr{F}_{Z,2} \arrow[d, two heads] & \\
& \mathscr{O}_{\mathbb{P}^2}(2), &
\end{tikzcd} \tag{4.2.5}$$

also gives an example of a strictly a.Z-stable bundle which is not contemplated in Theorem 4.1.4. ○

4.2.2 The product $\mathbb{P}^1 \times \mathbb{P}^1$

To begin with polycyclic varieties, we consider the product of two projective lines, $X = \mathbb{P}^1 \times \mathbb{P}^1$. The Picard group is given by $\mathrm{Pic}(X) = \mathbb{Z}A \oplus \mathbb{Z}B$, where A and B are the generators corresponding to each copy of $\mathbb{P}^1$. The intersection matrix is given by $A^2 = B^2 = 1$ and $A \cdot B = 0$. As we did before, we shall use the notation $\mathscr{O}_X(m,n) \coloneqq \mathscr{O}_X(mA + nB)$. The following proposition gives us a family of examples of strictly asymptotically Z-stable bundles over $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 4.2.3 (L)**.** *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and write* $\mathrm{Pic}(X) = \mathbb{Z}A \oplus \mathbb{Z}B$ *where A and B are the generators corresponding to each copy of $\mathbb{P}^1$. The following holds:*

1. *For every $\ell > k > 0$, if Z is the generic intersection of two curves C_1 and C_2 with bi-degrees (a_i, b_i) such that $a_i, b_i > \ell$, there exists a non-trivial holomorphic vector bundle $\mathscr{E}_{Z,(-k,\ell)}$ which arises as an extension*

$$\mathscr{O}_X \hookrightarrow \mathscr{E}_{Z,(-k,\ell)} \twoheadrightarrow \mathscr{I}_Z(-2k, 2\ell). \tag{4.2.6}$$

   Furthermore, this sheaf is slope stable w.r.t. the polarization $L = A + B$.

2. *There exists a non-trivial holomorphic vector bundle $\mathscr{F}_{Z,(-k,\ell)}$ which arises as an extension of the form*

$$\mathscr{E}_{-2k,2\ell} \hookrightarrow \mathscr{F}_{Z,(-k,\ell)} \twoheadrightarrow \mathscr{O}_X(-k, \ell). \tag{4.2.7}$$

   Furthermore, this vector bundle is strictly asymptotically Z-stable w.r.t. the polarization $L = A + B$.

Proof. In the notation of Theorem 4.1.4, we are choosing, for each $\ell > k > 0$, the divisor $D = -kA + \ell B$ and the subscheme Z to be a generic intersection of two curves C_1 and C_2, each one with bi-degree (a_i, b_i) satisfying that $a_i, b_i > \ell$. Before we get into the verification of the hypotheses of Theorem 4.1.4, we present a useful table of the cohomologies of line bundles over X. Denoting $\mathscr{O}_X(aA + bB) = \mathscr{O}_X(a,b)$, we have the following table:

	$h^0(X, \mathscr{O}_X(a,b))$	$h^1(X, \mathscr{O}_X(a,b))$	$h^2(X, \mathscr{O}_X(a,b))$
$a \geqslant -1, b \geqslant -1$	$(a+1)(b+1)$	0	0
$a \geqslant -1, b \leqslant -1$	0	$(a+1)(-b-1)$	0
$a \leqslant -1, b \geqslant -1$	0	$(-a-1)(b+1)$	0
$a \leqslant -1, b \leqslant -1$	0	0	$(a+1)(b+1)$

Table 1 – Cohomology dimensions for $\mathscr{O}_X(a,b)$ over $X = \mathbb{P}^1 \times \mathbb{P}^1$

Now, we are going to check the conditions of Theorem 4.1.4. The condition (a) is clearly satisfied since $D \cdot L = \ell - k > 0$. To verify condition (b) we will need to compute $\ell(Z)$. First, considering the restriction exact sequence

$$\mathscr{I}_Z \hookrightarrow \mathscr{O}_X \twoheadrightarrow \mathscr{O}_Z, \tag{4.2.8}$$

we apply the global section functor and use the fact that $H^1(X, \mathscr{O}_X) = 0 = H^0(X, \mathscr{I}_Z)$ to obtain

$$\ell(Z) = h^0\mathscr{O}_Z = h^0\mathscr{O}_X + h^1\mathscr{I}_Z = 1 + h^1\mathscr{I}_Z. \tag{4.2.9}$$

Now, since Z is the intersection of the curves C_i, we have the following exact sequence

$$\mathscr{O}_X(-a_1 - a_2, -b_1 - b_2) \hookrightarrow \mathscr{O}_X(-a_1, -b_1) \oplus \mathscr{O}_X(-a_2, -b_2) \twoheadrightarrow \mathscr{I}_Z, \tag{4.2.10}$$

from which we extract that

$$\begin{aligned} h^1\mathscr{I}_Z &= h^2\mathscr{O}_X(-a_1 - a_2, -b_1 - b_2) - h^2\mathscr{O}_X(-a_1, -b_1) - h^2\mathscr{O}_X(-a_2, -b_2) \\ &= a_1b_2 + a_2b_1 - 1. \end{aligned} \tag{4.2.11}$$

Combining (4.2.9) and (4.2.11) gives us

$$\ell(Z) = a_1b_2 + a_2b_1 > 2\ell^2 > k^2 + \ell^2 = D^2,$$

so condition (b) is satisfied. In order to verify condition (c), we take $R_0 = \{aA + bB \mid a, b \geqslant 0\}$. It is clear from the Kunneth formula that R_0 is such that $H^0(X, \mathscr{O}_X(B)) = 0$, for all $B \notin R_0$. Therefore, since $D \cdot L = \ell - k$ we have

$$R = \{aA + bB \mid a, b \geqslant 0 \text{ and } a + b \leqslant \ell - k\},$$

and hence we must show that $H^0(X, \mathscr{I}_Z(a, b)) = 0$ for each $a, b \geqslant 0$ such that $a + b \leqslant \ell - k$. For this, we twist the exact sequence (4.2.10) by $aA + bB$ and get

$$\mathscr{O}_X(-a_1 - a_2 + a, -b_1 - b_2 + b) \hookrightarrow \mathscr{O}_X(-a_1 + a, -b_1 + b) \oplus \mathscr{O}_X(-a_2 + a, -b_2 + b) \twoheadrightarrow \mathscr{I}_Z(a, b),$$

and since $a, b \leqslant \ell - k < \ell < a_i, b_i$ conclude, after taking cohomologies, that we have the inclusion

$$H^0(X, \mathscr{I}_Z(a, b)) \subset H^1(X, \mathscr{O}_X(-a_1 - a_2 + a, -b_1 - b_2 + b)).$$

But the last cohomology vanishes since $-a_1 - a_2 + a < -2\ell + \ell < -1$ and analogously $-b_1 - b_2 + b < -1$. This is enough for condition (c). Condition (d) is satisfied since

$$h^1\mathscr{O}_X(k, -\ell) = (k + 1)(\ell - 1) > k^2 - 1 > 0.$$

Finally, we get (e) from the fact that $h^2\mathscr{O}_X(2k, -2\ell) = 0$, see table 1. $\square$

4.2.3 Blow up of $\mathbb{P}^2$ at a point

Now we seek for examples when the variety is $X = \mathrm{Bl}_q\mathbb{P}^2$, the blow-up of $\mathbb{P}^2$ at a point q. We write $\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E$, where H is the pullback of the hyperplane class of $\mathbb{P}^2$ and E is the exceptional divisor. The intersection form for these generators is given by

$$H^2 = 1 \quad E^2 = -1 \quad H \cdot E = 0.$$

In terms of the above basis, the canonical divisor is $K_X = -3H + E$.

Again, we use the notation $\mathscr{O}_X(aH + bE) \coloneqq \mathscr{O}_X(a, b)$. Before we go on the construction of a.Z-stable bundles in this case, we present a Lemma that provides a suitable set R_0, in the notation of Theorem 4.1.4, for this setup. It is inspired in the computations of [Zha22].

Lemma 4.2.4. *Let $\mathscr{O}_X(a, b)$ be a line bundle over X defined as above. If either $a < 0$ or $b < -a < 0$, then $H^0(X, \mathscr{O}_X(a, b)) = 0$.*

Proof. Suppose first that $a < 0$. First we take the restriction exact sequence for H, which is

$$\mathscr{O}_X(-1, 0) \hookrightarrow \mathscr{O}_X \twoheadrightarrow \mathscr{O}_H, \tag{4.2.12}$$

and since the induced map $H^0(X, \mathscr{O}_X) \to H^0(H, \mathscr{O}_H)$ is injective, we conclude that its kernel, $H^0(X, \mathscr{O}_X(-1, 0))$ vanishes. Twisting the exact sequence (4.2.12) by aH gives us the exact sequence

$$\mathscr{O}_X(a - 1, 0) \hookrightarrow \mathscr{O}_X(a, 0) \twoheadrightarrow \mathscr{O}_H(a), \tag{4.2.13}$$

and since $H^0(H, \mathscr{O}_H(a)) = 0$ we get that $H^0(X, \mathscr{O}_X(a - 1, 0)) = H^0(X, \mathscr{O}_X(a, 0))$ and all these vanish. On the other hand, if we twist the restriction exact sequence for E by the divisor $aH + bE$, with $b \geqslant 0$, we get the exact sequence

$$\mathscr{O}_X(a, b - 1) \hookrightarrow \mathscr{O}_X(a, b) \twoheadrightarrow \mathscr{O}_E(-b), \tag{4.2.14}$$

and since $H^0(E, \mathscr{O}_E(-b)) = 0$ we can conclude that

$$H^0(X, \mathscr{O}_X(a, b)) = H^0(X, \mathscr{O}_X(a, b - 1)) = \cdots = H^0(X, \mathscr{O}_X(a, 0)) = 0.$$

Now, if $b < 0$, we have the inclusion

$$\mathscr{O}_X(a, b) \hookrightarrow \mathscr{O}_X(a, b + 1)$$

which shows that

$$H^0(X, \mathscr{O}_X(a, b)) = H^0(X, \mathscr{O}_X(a, b + 1)) = \cdots = H^0(X, \mathscr{O}_X(a, 0)) = 0,$$

and this proves the first part. Now suppose that $b < -a < 0$. First, the restriction exact sequence for E is

$$\mathscr{O}_X(0, -1) \hookrightarrow \mathscr{O}_X \twoheadrightarrow \mathscr{O}_E. \tag{4.2.15}$$

Since the map $H^0(X, \mathscr{O}_X) \to H^0(X, \mathscr{O}_E)$ is surjective we have $H^0(X, \mathscr{O}_X(0, -1)) = 0$. Also we can conclude that $H^0(X, \mathscr{O}_X(0, b)) = 0$ for all $b < 0$. Now we twist the restriction exact sequence for $A = H - E$ by $aH + bE$, to get the exact sequence

$$\mathscr{O}_X(a - 1, b + 1) \hookrightarrow \mathscr{O}_X(a, b) \twoheadrightarrow \mathscr{O}_A(a + b),$$

and since $a+b<0$ we see that $H^0(A,\mathscr{O}_A(a+b))=0$ and hence $H^0(X,\mathscr{O}_X(a,b))=H^0(X,\mathscr{O}_X(a-1,b+1))$. We do this step a times and obtain that $H^0(X,\mathscr{O}_X(a,b))=H^0(X,\mathscr{O}_X(0,b+a))=0$. □

Now we can present our first example in the form of the following proposition.

Proposition 4.2.5 (L)**.** *Let $X=\mathrm{Bl}_q\mathbb{P}^2$ and write $\mathrm{Pic}(X)=\mathbb{Z}H\oplus\mathbb{Z}E$, where H is corresponds to the hyperplane class, and E is exceptional divisor. The following holds:*

1. *If Z is the generic intersection of two curves C_i, linearly equivalent to H, then there exists a holomorphic vector bundle $\mathscr{E}_{Z,(-1,4)}$ which is given by a non-trivial extension*

$$\mathscr{O}_X\hookrightarrow\mathscr{E}_{Z,(-1,4)}\twoheadrightarrow\mathscr{I}_Z(-2H+8E). \tag{4.2.16}$$

   Furthermore, this bundle is μ-stable w.r.t. the polarization $L=3H-E$.

2. *There exists a holomorphic vector bundle $\mathscr{F}$ given by a non-trivial extension of the form*

$$\mathscr{E}_{Z,(-1,4)}\hookrightarrow\mathscr{F}_{Z,(-1.4)}\twoheadrightarrow\mathscr{O}_X(-H+4E). \tag{4.2.17}$$

   Furthermore, this bundles is strictly asymptotically Z-stable w.r.t. the polarization $L=3H-E$.

Proof. We show that the choice of (Z,D) satisfies the conditions in Theorem 4.1.4. Condition (a) is clearly satisfied since $D\cdot L=-3+4=1$. Condition (b) is also quickly verified since

$$D^2=1-16=-15<0\leqslant\ell(Z),$$

independently of Z. Skipping condition (c) for a moment, for condition (d) we need to compute $H^1(X,\mathscr{O}_X(1,-4))$, which we do using the Riemann-Roch formula:

$$\begin{aligned}h^1(\mathscr{O}_X(1,-4))&=h^0(\mathscr{O}_X(1,-4))+h^0(-4,3)-\frac{1}{2}(H-4E)(4H-3E)-1\\&=3.\end{aligned}$$

Condition (e) is also direct since

$$H^2(X,\mathscr{O}_X(-2,8))=H^0(X,\mathscr{O}_X(-1,-7))=0.$$

Finally, we address condition (c). By Lemma 4.2.4 we can take

$$R_0=\{aH+bE\mid a\geqslant 0\text{ and }a+b\geqslant 0\}, \tag{4.2.18}$$

and since $D\cdot L=1>0$, we have

$$R=\{aH+bE\mid a\geqslant 0,\ a+b\geqslant 0\text{ and }3a+b\leqslant 1\}=\{0,E\}. \tag{4.2.19}$$

Since we already know that $H^0(X, \mathscr{I}_Z) = 0$, we just need to show that $H^0(X, \mathscr{I}_Z(0,1)) = 0$. Recall that Z is the intersection of the pullbacks of two lines in $\mathbb{P}^2$ that do not pass through q, and since each line is represented by H in the Picard group, we have the following exact sequence defining $\mathscr{I}_Z$:

$$\mathscr{O}_X(-2,0) \hookrightarrow \mathscr{O}_X(-1,0)^{\oplus 2} \twoheadrightarrow \mathscr{I}_Z \tag{4.2.20}$$

and after twisting sequence (4.2.20) by E we get

$$\mathscr{O}_X(-2,1) \hookrightarrow \mathscr{O}_X(-1,1)^{\oplus 2} \twoheadrightarrow \mathscr{I}_Z(0,1).$$

Looking at the associated long exact sequence, we obtain

$$H^0(X, \mathscr{I}_Z(0,1)) \hookrightarrow H^1(X, \mathscr{O}_X(-2,1)),$$

since $H^0(X, \mathscr{O}_X(-1,1)) = 0$, and finally as, by the Riemann-Roch formula,

$$h^1(\mathscr{O}_X(-2,1)) = -\frac{1}{2}(-2H+E)\cdot H - 1 = 0,$$

we conclude that $H^0(X, \mathscr{I}_Z(0,1)) = 0$. □

We use the same approach to present the second example, which is given by the following proposition.

Proposition 4.2.6. *Let $X = \mathrm{Bl}_q\mathbb{P}^2$ and write $\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}E$, where H is corresponds to the hyperplane class, and E is the exceptional divisor.*

1. *If Z is the generic intersection of two curves C_i, linearly equivalent to $2H$ and $2H - 2C$ respectively, then there exists a holomorphic vector bundle $\mathscr{E}_{Z,(-1,5)}$ which is given by a non-trivial extension*

$$\mathscr{O}_X \hookrightarrow \mathscr{E}_{Z,(-1,5)} \twoheadrightarrow \mathscr{I}_Z(-2H+10E). \tag{4.2.21}$$

   Furthermore, this bundle is μ-stable w.r.t. the polarization $L = 3H - E$.

2. *There exists a holomorphic vector bundle $\mathscr{F}$ given by a non-trivial extension of the form*

$$\mathscr{E}_{Z,(-1,4)} \hookrightarrow \mathscr{F}_{Z,(-1,5)} \twoheadrightarrow \mathscr{O}_X(-H+5E). \tag{4.2.22}$$

   Furthermore, this bundle is strictly asymptotically Z-stable w.r.t. the polarization $L = 3H - E$.

Proof. We show that the choice of (Z, D) satisfies the conditions in Theorem 4.1.4. Condition (a) is clearly satisfied since $D \cdot L = -3 + 4 = 2$. Condition (b) is also quickly verified since

$$D^2 = 1 - 25 = -24 < 0 \leqslant \ell(Z),$$

independently of Z. Skipping condition (c) for a moment, for condition (d) we need to compute $H^1(X, \mathscr{O}_X(1,-5))$, which we do using the Riemann-Roch formula:

$$\begin{aligned} h^1(\mathscr{O}_X(1,-5)) &= h^0(\mathscr{O}_X(1,-5)) + h^0(-4,6) - \frac{1}{2}(H-5E)(4H-6E) - 1 \\ &= 12. \end{aligned}$$

Condition (e) is also direct since

$$H^2(X, \mathscr{O}_X(-2,10)) = H^0(X, \mathscr{O}_X(-1,-9)) = 0.$$

Finally, for condition (c). By Lemma 4.2.4 we can take

$$R_0 = \{aH + bE \mid a \geqslant 0 \text{ and } a+b \geqslant 0\}, \tag{4.2.23}$$

and since $D \cdot L = 2$, we have

$$R = \{aH + bE \mid a \geqslant 0,\ a+b \geqslant 0 \text{ and } 3a+b \leqslant 2\} = \{0, E, H-E\}. \tag{4.2.24}$$

Since we already know that $H^0(X, \mathscr{I}_Z) = 0$, we just need to show that

$$H^0(X, \mathscr{I}_Z(0,1)) = H^0(X, \mathscr{I}_Z(1,-1)) = 0.$$

Recall that Z is the intersection of two curves $C_1 \sim 2H$ and $C_2 \sim 2H - 2E$ and hence we have the following exact sequence defining $\mathscr{I}_Z$:

$$\mathscr{O}_X(-4,2) \hookrightarrow \mathscr{O}_X(-2,0) \oplus \mathscr{O}_X(-2,2) \twoheadrightarrow \mathscr{I}_Z. \tag{4.2.25}$$

After twisting the sequence (4.2.25) by E we get

$$\mathscr{O}_X(-4,3) \hookrightarrow \mathscr{O}_X(-2,1) \oplus \mathscr{O}_X(-2,4) \twoheadrightarrow \mathscr{I}_Z(0,1).$$

Looking at the associated long exact sequence, we obtain

$$H^0(X, \mathscr{I}_Z(0,1)) \hookrightarrow H^1(X, \mathscr{O}_X(-4,3)),$$

since $H^0(X, \mathscr{O}_X(-2,1)) = 0 = H^0(X, \mathscr{O}_X(-2,4))$. On the other hand, the Riemann-Roch formula gives

$$h^1(\mathscr{O}_X(-4,3)) = -\frac{1}{2}(-4H+3E) \cdot (-H+2E) - 1 = 0,$$

and therefore $H^0(X, \mathscr{I}_Z(0,1)) = 0$. Now we twist the sequence (4.2.25) by $H - E$ and get

$$\mathscr{O}_X(-3,1) \hookrightarrow \mathscr{O}_X(-1,-1) \oplus \mathscr{O}_X(-1,1) \twoheadrightarrow \mathscr{I}_Z(1,-1).$$

The associated long exact sequence gives us that

$$H^0(X, \mathscr{I}_Z(1,-1)) \hookrightarrow H^1(X, \mathscr{O}_X(-3,1)),$$

since $H^0(X, \mathscr{O}_X(-1,-1)) = 0 = H^0(X, \mathscr{O}_X(-1,1))$. On the other hand, the Riemann-Roch formula again gives

$$h^1(\mathscr{O}_X(-3,1)) = h^2\mathscr{O}_X(-3,1)) - \frac{1}{2}(-3H+E) \cdot 0 - 1 = h^0(X, \mathscr{O}_X) - 1 = 0, \qquad \square$$

Remark 4.2.7. The first motivation for considering $X = \mathrm{Bl}_q\mathbb{P}^2$ was to use the construction of Theorem 4.1.1 taking the bundle $\mathscr{E}$ to be a Hartshorne-Serre bundle (3.2.1) without the ideal sheaf $\mathscr{I}_Z$ part. That is, we would like to take the extension of the form

$$\mathscr{O}_X \hookrightarrow \mathscr{E} \twoheadrightarrow \mathscr{O}_X(2D), \tag{4.2.26}$$

which is equivalent to asking $c_2(\mathscr{E}) = 0$. By Theorem 4.1.1 it follows that the divisor D must satisfy $D^2 < 0$. The simplest example of a surface admitting negative self-intersection is $\mathrm{Bl}_q\mathbb{P}^2$. The existence of a non-trivial extension as above is ensured by $h^1\mathscr{O}_X(-2D) \neq 0$, so let us analyze its μ-stability. Take the polarization to be $L = 3H - E$ as we are doing in the examples above. By the same argument as in the proof of Lemma 3.2.2 we can conclude that a bundle $\mathscr{E}$ obtained from the extension (4.2.26) is μ-stable if, and only if

$$\ker(H^0(X, \mathscr{E}(B)) \to H^1(X, \mathscr{O}_X(B - 2D))) = 0, \quad \forall B \in R_0 \text{ s.t. } B \cdot L \leqslant D \cdot L. \tag{4.2.27}$$

For example, if we take $D \cdot L = 1$ we see that the only possible divisors $B \in R_0$ such that $B \cdot L \leqslant 1$ are $B = 0$ and $B = E$. The condition (4.2.27) for $B = 0$ is just that $\mathscr{E}$ be a non-trivial extension, which is satisfied by the construction. The condition for $B = D$ is that $H^0(X, \mathscr{E}(D)) \to H^1(X, \mathscr{O}_X(E - 2D))$ is injective, but these conditions lacks a geometric interpretation. ◯

APPENDIX A – Analytical Tools

Here we list some technical results that were used to get estimates on Chapter 2. All the results are very standard but we include the proofs to make it compatible with our notation. All the results of this sections are taken from the appendix on [CFL25].

Lemma A.1 (Kato's inequality). *Let M be a Riemannian manifold and $E \to M$ a vector bundle equipped with a bundle metric $\langle \cdot, \cdot \rangle$ and a metric-compatible connection ∇. Suppose that $f : M \to E$ is a smooth section. Then, in the sense of distributions, we have $\Delta |f| \leqslant F$, where*

$$F = \begin{cases} \dfrac{\langle f, \nabla^* \nabla f \rangle}{|f|} & \text{on } \{f \neq 0\}, \\ 0 & \text{on } \{f = 0\}. \end{cases}$$

Proof. For $\delta \in (0,1)$ we define $h_\delta = \sqrt{|f|^2 + \delta^2}$. Then by direct computation we have

$$\nabla h_\delta = \frac{\langle f, \nabla f \rangle}{\sqrt{|f|^2 + \delta^2}},$$

and thus

$$\begin{aligned} \Delta h_\delta =& - e_i \Big(\frac{\langle f, \nabla_{e_i} f \rangle}{\sqrt{|f|^2 + \delta^2}} \Big) \\ =& - \frac{|\nabla f|^2}{\sqrt{|f|^2 + \delta^2}} + \frac{|\langle f, \nabla f \rangle|^2}{(|f|^2 + \delta^2)^{\frac{3}{2}}} + \frac{\langle f, \nabla^* \nabla f \rangle}{\sqrt{|f|^2 + \delta^2}} \\ =& - \frac{(|f|^2 + \delta^2)|\nabla f|^2 - |\langle f, \nabla f \rangle|^2}{(|f|^2 + \delta^2)^{\frac{3}{2}}} + \frac{\langle f, \nabla^* \nabla f \rangle}{\sqrt{|f|^2 + \delta^2}} \\ \leqslant& \frac{\langle f, \nabla^* \nabla f \rangle}{\sqrt{|f|^2 + \delta^2}}. \end{aligned} \tag{A.1}$$

Multiplying by a non-negative test function $\varphi \in C_c^\infty(M)$ and integrating by parts, we get

$$\int_M h_\delta \cdot \Delta \varphi \mathrm{vol}_g \leqslant \int_M \varphi \cdot \frac{\langle f, \nabla^* \nabla f \rangle}{\sqrt{|f|^2 + \delta^2}} \mathrm{vol}_g. \tag{A.2}$$

Now since h_δ and $\dfrac{\langle f, \nabla^* \nabla f \rangle}{\sqrt{|f|^2 + \delta^2}}$ converge pointwise to $|f|$ and F, respectively, as $\delta \to 0$, and since

$$0 \leqslant h_\delta \leqslant |f| + 1, \quad \frac{\big| \langle f, \nabla^* \nabla f \rangle \big|}{\sqrt{|f|^2 + \delta^2}} \leqslant |\nabla^* \nabla f|,$$

we see from (A.2) and the dominated convergence theorem that

$$\int_M |f| \cdot \Delta \varphi \mathrm{vol}_g \leqslant \int_M \varphi \cdot F \mathrm{vol}_g,$$

for all non-negative smooth function φ with compact support. $\square$

Now we record a version of Moser's iteration that is used repeatedly in Section 2.1. Let (M, g) be an n-dimensional Riemannian manifold with $n \geqslant 3$, and $B_{2r}(x_0)$ a geodesic normal ball on which

$$\Lambda^{-1} g_{\mathbb{R}^n} \leqslant \exp_{x_0}^* g \leqslant \Lambda g_{\mathbb{R}^n}, \tag{A.3}$$

for some $\Lambda > 0$. Here $g_{\mathbb{R}^n}$ denotes the standard flat metric. Suppose further that $u : B_{2r}(x_0) \to [0, \infty)$ is a non-negative, bounded, Lipschitz function satisfying, in the distributional sense, that

$$\Delta u \leqslant bu + c, \tag{A.4}$$

where $b, c \in L^q(B_{2r}(x_0))$ for some $q \in (\frac{n}{2}, \infty]$. For all $p_0 \in [1, \infty)$, we define

$$\gamma_{n,q,p_0} = \begin{cases} \dfrac{nq}{p_0(2q-n)}, & \text{if } q < \infty, \\ \dfrac{n}{2p_0}, & \text{if } q = \infty. \end{cases}$$

Lemma A.2. *In the above setting, suppose $p_0 > 1$ and let $\tau \in (0, 1)$ be a scaling factor. Then, we have*

$$\begin{aligned} \|u\|_{\infty; B_{\tau r}(x_0)} \leqslant{}& C_{\Lambda,n,q,p_0} (1-\tau)^{-\frac{n}{p_0}} \Big(1 + r^{2-\frac{n}{q}} \|b\|_{q;B_r(x_0)}\Big)^{\gamma_{n,q,p_0}} \\ &\times \Big(r^{-\frac{n}{p_0}} \|u\|_{p_0;B_r(x_0)} + r^{2-\frac{n}{q}} \|c\|_{q;B_r(x_0)}\Big). \end{aligned} \tag{A.5}$$

Proof. Every step of the proof is standard, and we include the details only to keep track of how exactly the constants are affected by $\|b\|_{q;B_r(x_0)}$. Below, when the center of a geodesic ball is not specified, it is understood to be centered at x_0. Also, all the integrals are taken with respect to the volume form of g, which is comparable on $B_{2r}(x_0)$ to the Euclidean volume form due to the assumption (A.3). To begin, let $\varphi : \mathbb{R} \to [0, \infty)$ be a non-negative smooth function such that

$$\varphi(t) = \begin{cases} 0 & , \text{ if } t \geqslant 1, \\ 1 & , \text{ if } t \leqslant 0. \end{cases}$$

Given $k \in (0, \infty)$, $\beta \in [p_0, \infty)$, as well as $0 < \sigma < \rho \leqslant r$, we set $p = \beta - 1$ and define

$$\widetilde{b} = |b| + \frac{|c|}{k}, \quad \zeta = \varphi\Big(\frac{d(\cdot, x_0) - \sigma}{\rho - \sigma}\Big), \quad v = \zeta^2 \cdot [(u+k)^p - k^p],$$

where d denotes the geodesic distance. Testing (A.4) against v gives

$$\begin{aligned} \int_M |b|uv + |c|v \geqslant{}& \int_M p\zeta^2 (u+k)^{p-1} |\nabla u|^2 - 2\zeta[(u+k)^p - k^p] |\nabla \zeta| |\nabla u| \\ \geqslant{}& p \int_M \zeta^2 (u+k)^{p-1} |\nabla u|^2 - 2 \int_M \zeta (u+k)^p |\nabla \zeta| |\nabla u| \\ \geqslant{}& \frac{p}{2} \int_M \zeta^2 (u+k)^{p-1} |\nabla u|^2 - \frac{2}{p} \int_M (u+k)^{p+1} |\nabla \zeta|^2, \end{aligned} \tag{A.6}$$

where the last line follows from Young's inequality. Noting that

$$(u+k)^{p-1}|\nabla u|^2 = \frac{4}{(p+1)^2}\Big|\nabla\Big[(u+k)^{\frac{p+1}{2}}\Big]\Big|^2,$$

and that

$$|b|uv + |c|v \leqslant \widetilde{b}(u+k)v \leqslant \widetilde{b}(u+k)^{p+1}\zeta^2,$$

we deduce from (A.6) that

$$\frac{2p}{(p+1)^2}\int_M \zeta^2\Big|\nabla\Big[(u+k)^{\frac{p+1}{2}}\Big]\Big|^2 \leqslant \frac{2}{p}\int_M (u+k)^{p+1}|\nabla\zeta|^2 + \int_M \widetilde{b}(u+k)^{p+1}\zeta^2,$$

and hence

$$\begin{aligned}\int_M |\nabla[\zeta(u+k)^{\frac{p+1}{2}}]|^2 &\leqslant 2\int_M \zeta^2\Big|\nabla[(u+k)^{\frac{p+1}{2}}]\Big|^2 + 2\int_M |\nabla\zeta|^2(u+k)^{p+1} \\ &\leqslant 2\Big[\Big(\frac{1+p}{p}\Big)^2+1\Big]\int_M |\nabla\zeta|^2(u+k)^{p+1} + \frac{(1+p)^2}{p}\int_M \widetilde{b}(u+k)^{p+1}\zeta^2 \\ &\leqslant C_{p_0}\int_M |\nabla\zeta|^2(u+k)^{p+1} + C_{p_0}\cdot p\int_M \widetilde{b}(u+k)^{p+1}\zeta^2,\end{aligned} \tag{A.7}$$

where in passing to the third line we used the fact that $\dfrac{1+p}{p} \leqslant \dfrac{p_0}{p_0-1}$ whenever $p \geqslant p_0 - 1$.

Next, since the function $\zeta(u+k)^{\frac{p+1}{2}}$ is supported in the geodesic ball $B_{2r}(x_0)$, the assumption (A.3) allows us to invoke the Euclidean Sobolev inequality accompanying the embedding $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$ to deduce that

$$\Big(\int_M [\zeta^2(u+k)^{p+1}]^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \leqslant C_{n,\Lambda}\int_M |\nabla[\zeta(u+k)^{\frac{p+1}{2}}]|^2. \tag{A.8}$$

Hence, upon letting

$$w = (u+k)^{p+1}\zeta^2,$$

we get from (A.8) and (A.7) that

$$\|w\|_{\frac{n}{n-2}} \leqslant C_{n,\Lambda,p_0}\int_M |\nabla\zeta|^2(u+k)^{p+1} + C_{n,\Lambda,p_0}\cdot p\int_M \widetilde{b}w. \tag{A.9}$$

To continue, we set

$$\theta = \begin{cases} \dfrac{n}{2q}, & \text{if } q<\infty, \\ 0, & \text{if } q=\infty. \end{cases}$$

In the case $q<\infty$, by Hölder's inequality, the standard interpolation inequality between L^p-norms, and Young's inequality, we have that

$$\begin{aligned}\int_M \widetilde{b}w &\leqslant \|\widetilde{b}\|_{q;B_r}\cdot\|w\|_{\frac{q}{q-1}} \leqslant \|\widetilde{b}\|_{q;B_r}\cdot\|w\|^{\theta}_{\frac{n}{n-2}}\|w\|_1^{1-\theta} \\ &\leqslant \|\widetilde{b}\|_{q;B_r}\cdot\Big(\theta\delta\|w\|_{\frac{n}{n-2}} + (1-\theta)\delta^{-\frac{\theta}{1-\theta}}\|w\|_1\Big),\end{aligned} \tag{A.10}$$

with $\delta > 0$ to be determined momentarily. Substituting (A.10) back into (A.9), we obtain upon rearranging that

$$(1 - C_{n,\Lambda,p_0} \cdot p\theta \|\widetilde{b}\|_{q;B_r} \cdot \delta)\|w\|_{\frac{n}{n-2}} \leqslant C_{n,\Lambda,p_0} \int_M |\nabla\zeta|^2 (u+k)^{p+1} \\ + C_{n,\Lambda,p_0} \cdot p\|\widetilde{b}\|_{q;B_r}(1-\theta)\delta^{-\frac{\theta}{1-\theta}}\|w\|_1.$$

Making the choice

$$\delta = \frac{1}{2C_{n,\Lambda,p_0} \cdot p\theta \cdot (\|\widetilde{b}\|_{q;B_r} + t)},$$

and then letting $t \to 0^+$, we obtain

$$\|w\|_{\frac{n}{n-2}} \leqslant 2C_{n,\Lambda,p_0} \int_M |\nabla\zeta|^2 (u+k)^{p+1} + \left(2C_{n,\Lambda,p_0} \cdot p\|\widetilde{b}\|_{q;B_r}\right)^{\frac{1}{1-\theta}} \theta^{\frac{\theta}{1-\theta}} \|w\|_1.$$

Recalling the definition of w and our choice of ζ, and using the fact that $p^{\frac{1}{1-\theta}} \geqslant (p_0 - 1)^{\frac{1}{1-\theta}}$, we deduce that

$$\|(u+k)^{p+1}\|_{\frac{n}{n-2};B_\sigma} \leqslant C_{\Lambda,n,q,p_0} \cdot p^{\frac{1}{1-\theta}}(\rho-\sigma)^{-2} \\ \times \left[1 + (\rho-\sigma)^2 \|\widetilde{b}\|_{q;B_r}^{\frac{1}{1-\theta}}\right] \|(u+k)^{p+1}\|_{1;B_\rho}. \tag{A.11}$$

On the other hand, when $q = \infty$, we replace (A.10) by

$$\int_M \widetilde{b} \cdot (u+k)^{p+1}\zeta^2 \leqslant \|\widetilde{b}\|_{\infty;B_r} \cdot \|(u+k)^{p+1}\zeta^2\|_1 \tag{A.12}$$

to deduce from (A.9) that the inequality (A.11) still holds. At any rate, recalling that $\beta = p+1$ and letting $\chi = \dfrac{n}{n-2}$, we get upon taking the β-th root of both sides of (A.11) that

$$\|u+k\|_{\beta\chi;B_\sigma} \leqslant C_{\Lambda,n,q,p_0}^{\frac{1}{\beta}} \cdot \beta^{\frac{1}{(1-\theta)\beta}} (\rho-\sigma)^{-\frac{2}{\beta}} \left[1 + (\rho-\sigma)^2 \|\widetilde{b}\|_{q;B_r}^{\frac{1}{1-\theta}}\right]^{\frac{1}{\beta}} \|u+k\|_{\beta;B_\rho}. \tag{A.13}$$

For $m \in \mathbb{N} \cup \{0\}$, we now define

$$r_m = (\tau + \frac{1-\tau}{2^m})r,$$

and apply (A.13) with

$$\beta = p_0\chi^m, \quad \sigma = r_{m+1}, \quad \rho = r_m,$$

to obtain, with perhaps a different C_{Λ,n,q,p_0},

$$\|u+k\|_{p_0\chi^{m+1};B_{r_{m+1}}} \leqslant \left((C_{\Lambda,n,q,p_0})^{\chi^{-m}} (2^{\frac{2}{p_0}} \chi^{\frac{1}{p_0(1-\theta)}})^{m\chi^{-m}} [(1-\tau)r]^{-\frac{2}{p_0}\chi^{-m}}\right) \\ \times \left(\left[1 + r^2 \|\widetilde{b}\|_{q;B_r}^{\frac{1}{1-\theta}}\right]^{\frac{1}{p_0}\chi^{-m}} \|u+k\|_{p_0\chi^m;B_{r_m}}\right). \tag{A.14}$$

Recalling the elementary inequality

$$1 + t^\alpha \leqslant (1+t)^\alpha, \text{ whenever } t \geqslant 0, \ \alpha \geqslant 1,$$

we have
$$1 + r^2\|\widetilde{b}\|_{q;B_r}^{\frac{1}{1-\theta}} \leqslant \Big(1 + r^{2-\frac{n}{q}}\|\widetilde{b}\|_{q;B_r}\Big)^{\frac{1}{1-\theta}}.$$
Substituting this back into (A.14) and iterating, we get for all $m \geqslant 1$ that
$$\begin{aligned}\|u+k\|_{p_0\chi^m;B_{r_m}} \leqslant{}& \Big((C_{\Lambda,n,q,p_0})^{\sum_{i=0}^{m-1}\chi^{-i}} \cdot (C_{n,q,p_0})^{\sum_{i=0}^{m-1} i\chi^{-i}} \cdot [(1-\tau)r]^{-\frac{2}{p_0}\sum_{i=0}^{m-1}\chi^{-i}}\Big)\\ &\times \Big(\Big[1 + r^{2-\frac{n}{q}}\|\widetilde{b}\|_{q;B_r}\Big]^{\frac{1}{p_0(1-\theta)}\sum_{i=0}^{m-1}\chi^{-i}} \|u+k\|_{p_0;B_r}\Big).\end{aligned}$$
Letting $m \to \infty$ gives
$$\|u+k\|_{\infty;B_{\tau r}} \leqslant C_{\Lambda,n,q,p_0}[(1-\tau)r]^{-\frac{n}{p_0}}\Big(1 + r^{2-\frac{n}{q}}\|\widetilde{b}\|_{q;B_r}\Big)^{\gamma_{n,q,p_0}}(\|u\|_{p_0;B_r} + kr^{\frac{n}{p_0}}),$$
where we used (A.3) to estimate $\|k\|_{p_0;B_r}$. Taking $k = r^{2-\frac{n}{q}}\|c\|_{q;B_r} + \delta$, with $\delta > 0$ to be sent to 0 in a moment, we find that
$$r^{2-\frac{n}{q}}\|\widetilde{b}\|_{q;B_r} \leqslant r^{2-\frac{n}{q}}\|b\|_{q;B_r} + 1.$$
Consequently,
$$\begin{aligned}\|u\|_{\infty;B_{\tau r}} \leqslant{}& C_{\Lambda,n,q,p_0}(1-\tau)^{-\frac{n}{p_0}}\Big(1 + r^{2-\frac{n}{q}}\|b\|_{q;B_r}\Big)^{\gamma_{n,q,p_0}}\\ &\times \Big(r^{-\frac{n}{p_0}}\|u\|_{p_0;B_r} + r^{2-\frac{n}{q}}\|c\|_{q;B_r} + \delta\Big).\end{aligned}$$
Letting $\delta \to 0$ gives the estimate (A.5) we want. □

Corollary A.3. *In the notation of Lemma A.2, if $b \leqslant 0$ then we have*
$$\|u_+\|_\infty \leqslant C_{M,q}(\|u_+\|_2 + \|c\|_q).$$

Proof. Notice that if $b \leqslant 0$, then on the right-hand side of the first line of (A.6) we have
$$buv = bu_+v \leqslant 0,$$
so that in this case
$$\int_M buv + cv \leqslant \int_M \frac{|c|}{k}kv.$$
Following the remainder of the proof gives the desired result. □

Next we recall the following result about commuting covariant derivatives with the connection Laplacian.

Lemma A.4 (Commuting derivatives)**.** *Let $E \to M$ be a real vector bundle of rank r, equipped with a bundle metric $\langle\cdot,\cdot\rangle$ and a metric connection ∇, and let S be a section of $(\otimes^p T^*M) \otimes E$, that is, a smooth p-tensor with values in E. Then we have the following inequality, where $|\cdot|$ denote pointwise tensor norms:*
$$|\nabla^*\nabla(\nabla S) - \nabla(\nabla^*\nabla S)| \leqslant C_{n,p,r}\Big(|\nabla F||S| + |\nabla R||S| + |F||\nabla S| + |R||\nabla S|\Big). \tag{A.15}$$
More generally, for all $m \geqslant 1$ we have
$$|\nabla^*\nabla(\nabla^m S) - \nabla^m(\nabla^*\nabla S)| \leqslant C_{m,n,p,r}\Big(\sum_{i=0}^m |\nabla^i F||\nabla^{m-i}S| + \sum_{i=0}^m |\nabla^i R||\nabla^{m-i}S|\Big). \tag{A.16}$$

Proof. In terms of a geodesic frame $e_1, \cdots, e_n$, we compute, at the center point

$$\begin{aligned}
-[\nabla^*\nabla(\nabla S)]_{e_j,e_{i_1},\cdots,e_{i_p}} &= [\nabla^3_{e_i,e_i,e_j} S]_{e_{i_1},\cdots,e_{i_p}} \\
&= \nabla_{e_i}\big((\nabla^2_{e_i,e_j} S)_{e_{i_1},\cdots,e_{i_p}}\big) \\
&= \nabla_{e_i}\Big((\nabla^2_{e_j,e_i} S)_{e_{i_1},\cdots,e_{i_p}} + F_{e_i,e_j} S_{e_{i_1},\cdots,e_{i_p}} - \sum_{\lambda=1}^{p} S_{\cdots,R_{e_i,e_j}e_{i_\lambda},\cdots}\Big) \\
&= (\nabla^3_{e_i,e_j,e_i} S)_{e_{i_1},\cdots,e_{i_p}} \\
&\quad + (\nabla_{e_i} F)_{e_i,e_j} S_{e_{i_1},\cdots,e_{i_p}} + F_{e_i,e_j}(\nabla_{e_i} S)_{e_{i_1},\cdots,e_{i_p}} \\
&\quad - \sum_{\lambda=1}^{p} (\nabla_{e_i} S)_{\cdots,R_{e_i,e_j}e_{i_\lambda},\cdots} - \sum_{\lambda=1}^{p} S_{\cdots,(\nabla_{e_i}R)_{e_i,e_j}e_{i_\lambda},\cdots}.
\end{aligned}$$

Next we note that

$$\begin{aligned}
(\nabla^3_{e_i,e_j,e_i} S)_{e_{i_1},\cdots,e_{i_p}} &= (\nabla^3_{e_j,e_i,e_i} S)_{e_{i_1},\cdots,e_{i_p}} + F_{e_i,e_j}(\nabla_{e_i} S)_{e_{i_1},\cdots,e_{i_p}} \\
&\quad - (\nabla_{R_{e_i,e_j}e_i} S)_{e_{i_1},\cdots,e_{i_p}} - \sum_{\lambda=1}^{p} (\nabla_{e_i} S)_{\cdots,R_{e_i,e_j}e_{i_\lambda},\cdots}.
\end{aligned}$$

Since

$$(\nabla^3_{e_j,e_i,e_i} S)_{e_{i_1},\cdots,e_{i_p}} = -(\nabla\nabla^*\nabla S)_{e_j,e_{i_1},\cdots,e_{i_p}},$$

we obtain

$$\begin{aligned}
(\nabla^*\nabla(\nabla S))_{e_j,e_{i_1},\cdots,e_{i_p}} &= (\nabla(\nabla^*\nabla S))_{e_j,e_{i_1},\cdots,e_{i_p}} - (\nabla_{e_i} F)_{e_i,e_j} S_{e_{i_1},\cdots,e_{i_p}} + \sum_{\lambda=1}^{p} S_{\cdots,(\nabla_{e_i}R)_{e_i,e_j}e_{i_\lambda},\cdots} \\
&\quad - 2F_{e_i,e_j}(\nabla_{e_i} S)_{e_{i_1},\cdots,e_{i_p}} + 2\sum_{\lambda=1}^{p} (\nabla_{e_i} S)_{\cdots,R_{e_i,e_j}e_{i_\lambda},\cdots} \\
&\quad + (\nabla_{R_{e_i,e_j}e_i} S)_{e_{i_1},\cdots,e_{i_p}},
\end{aligned} \tag{A.17}$$

and hence the first asserted inequality, which also establishes the base step for an induction proof of the second conclusion. For the inductive step, we assume that (A.16) holds for some $m \geqslant 1$, and apply it to ∇S to get

$$|\nabla^*\nabla(\nabla^{m+1} S) - \nabla^m \nabla^*\nabla\nabla S| \leqslant C_{m,n,p,r} \cdot \sum_{i=0}^{m} \Big(|\nabla^i F||\nabla^{m+1-i} S| + |\nabla^i R||\nabla^{m+1-i} S|\Big).$$

Taking (A.17) and covariantly differentiating it m times gives

$$\begin{aligned}
|\nabla^m(\nabla^*\nabla\nabla S - \nabla\nabla^*\nabla S)| &\leqslant C_{m,n,p,r} \cdot \sum_{i=0}^{m} \Big(|\nabla^{i+1} F||\nabla^{m-i} S| + |\nabla^{i+1} R||\nabla^{m-i} S|\Big) \\
&\quad + C_{m,n,p,r} \cdot \sum_{i=0}^{m} \Big(|\nabla^i F||\nabla^{m+1-i} S| + |\nabla^i R||\nabla^{m+1-i} S|\Big).
\end{aligned}$$

Adding this to the previous inequality completes the inductive step. $\square$

Lemma A.5 (Comparing connections)**.** *Let $L \to M$ be a Hermitian line bundle and suppose ∇_0 and $\nabla := \nabla_0 - \sqrt{-1}A$ are two metric-compatible connections on L. Then for all $k \in \mathbb{N}$, we have*

$$|\nabla^k S - \nabla_0^k S| \leqslant C_{n,p,k}\Big(1 + \sum_{j=0}^{k-1} |\nabla^j A|\Big)^k \cdot \sum_{j=0}^{k-1} |\nabla_0^j S|, \tag{A.18}$$

for all $p \in \mathbb{N} \cup \{0\}$ and $S \in \Gamma((\otimes^p T^*M) \otimes L)$.

Proof. We argue by induction on k. The base case $k = 1$ is straightforward since

$$|\nabla S - \nabla_0 S| = |\sqrt{-1}A \otimes S| \leqslant (1 + |A|)|S|.$$

For the induction step, we suppose that for some for $k = m \geqslant 1$, the estimate (A.18) holds for all $p \in \mathbb{N} \cup \{0\}$ and $S \in \Gamma((\otimes^p T^*M) \otimes L)$, and note that

$$\begin{aligned} |\nabla^{m+1} S - \nabla_0^{m+1} S| &\leqslant |(\nabla^m - \nabla_0^m)(\nabla_0 S)| + |(\nabla^m - \nabla_0^m)(\nabla S - \nabla_0 S)| + |\nabla_0^m(\nabla S - \nabla_0 S)| \\ &= |(\nabla^m - \nabla_0^m)(\nabla_0 S)| + |(\nabla^m - \nabla_0^m)(A \otimes S)| + |\nabla_0^m (A \otimes S)| \\ &=: (I) + (II) + (III). \end{aligned} \tag{A.19}$$

For (I), we apply the induction hypothesis to $\nabla_0 S \in \Gamma((\otimes^{p+1} T^*M) \otimes L)$ to get

$$\begin{aligned} |(I)| &\leqslant C_{n,p,m}\Big(1 + \sum_{j=0}^{m-1} |\nabla^j A|\Big)^m \cdot \sum_{j=0}^{m-1} |\nabla_0^{j+1} S| \\ &\leqslant C_{n,p,m}\Big(1 + \sum_{j=0}^{m-1} |\nabla^j A|\Big)^m \cdot \sum_{j=0}^{m} |\nabla_0^j S| \end{aligned}$$

For (II), we again apply the induction hypothesis, this time to $A \otimes S \in \Gamma((\otimes^{p+1} T^*M) \otimes L)$, to obtain

$$\begin{aligned} (II) &\leqslant C_{n,p,m}\Big(1 + \sum_{j=0}^{m-1} |\nabla^j A|\Big)^m \cdot \sum_{j=0}^{m-1} |\nabla_0^j (A \otimes S)| \\ &\leqslant C_{n,p,m}\Big(1 + \sum_{j=0}^{m-1} |\nabla^j A|\Big)^m \Big(\sum_{j=0}^{m-1} |\nabla^j A|\Big)\Big(\sum_{j=0}^{m-1} |\nabla_0^j S|\Big) \\ &\leqslant C_{n,p,m}\Big(1 + \sum_{j=0}^{m-1} |\nabla^j A|\Big)^{m+1} \Big(\sum_{j=0}^{m-1} |\nabla_0^j S|\Big), \end{aligned}$$

where in the second line we used the fact that

$$|\nabla_0^j (A \otimes S)| \leqslant C_{n,j,p} \sum_{i=0}^{j} |\nabla^{j-i} A||\nabla_0^i S| \leqslant C_{n,j,p}\Big(\sum_{i=0}^{j} |\nabla^i A|\Big)\Big(\sum_{i=0}^{j} |\nabla_0^i S|\Big),$$

which also happens to help us bound (III):

$$|(III)| \leqslant C_{n,m,p}\Big(\sum_{i=0}^{m} |\nabla^j A|\Big)\Big(\sum_{i=0}^{m} |\nabla_0^j S|\Big) \leqslant C_{n,m,p}\Big(1 + \sum_{i=0}^{m} |\nabla^j A|\Big)^{m+1} \Big(\sum_{i=0}^{m} |\nabla_0^j S|\Big).$$

Putting the estimates for (I), (II) and (III) back into (A.19) gives

$$|\nabla^{m+1} S - \nabla_0^{m+1} S| \leqslant C_{n,p,m}\Big(1 + \sum_{j=0}^{m} |\nabla^j A|\Big)^{m+1}\Big(\sum_{j=0}^{m} |\nabla_0^j S|\Big).$$

This completes the inductive proof. □

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