2.16. Seja $A \in M_{m,n}(\mathbb{C})$, sua DFT $\hat{A} \in M_{m,n}(\mathbb{C})$ é dada por

$$\hat{A}_{k,l} = \sum_{r=0}^{m-1} \sum_{s=0}^{n-1} a_{r,s}\, e^{-2\pi i (kr/m + ls/n)} \quad , \text{com } 0 \leq k \leq m-1 \text{ e } 0 \leq l \leq n-1$$

Daí se segue

$$\hat{A}_{0,0} = \sum_{r=0}^{1} \sum_{s=0}^{1} a_{r,s}\, e^{-2\pi i (0.r/m + 0.s/n)} = 1 \cdot e^{0} + (-1) \cdot e^{0} + 2 \cdot e^{0} + 0 \cdot e^{0} = 2$$

$$\hat{A}_{0,1} = \sum_{r=0}^{1} \sum_{s=0}^{1} a_{r,s}\, e^{-2\pi i (0.r/m + 1.s/n)} = 1 \cdot e^{0} + (-1)\, e^{-2\pi i \cdot \frac{1}{2}} + 2 \cdot e^{0} + 0 \cdot e^{-2\pi i \cdot \frac{1}{2}} = 1 + (-1)(-1) + 2 =$$

$$= 4$$

$$\hat{A}_{1,0} = \sum_{r=0}^{1} \sum_{s=0}^{1} a_{r,s} \cdot e^{-2\pi i (1.r/m + 0.s/n)} = 1 \cdot e^{0} + (-1) \cdot e^{0} + 2 \cdot e^{-2\pi i \cdot \frac{1}{2}} + 0 \cdot e^{-2\pi i \cdot \frac{1}{2}} = 1 + (-1) + 2 \cdot (-1) =$$

$$= (-2)$$

$$\hat{A}_{1,1} = \sum_{r=0}^{1} \sum_{s=0}^{1} a_{r,s}\, e^{-2\pi i (1.r/m + 1.s/n)} = 1 \cdot e^{0} + (-1)\, e^{-2\pi i \frac{1}{2}} + 2 \cdot e^{-2\pi i \frac{1}{2}} + 0 \cdot e^{-2\pi i} = 1 + (-1)(-1) + 2 \cdot (-1) =$$

$$= 0$$

Portanto

$$\hat{A} = \begin{bmatrix} 2 & 4 \\ -2 & 0 \end{bmatrix}$$

Agora para computar a transformada inversa, fazemos

$$a_{r,s} = \frac{1}{mm} \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} \hat{A}_{k,l}\, e^{2\pi i (kr/m + ls/m)} \quad , \text{ com } 0 \leq r \leq m-1 \text{ e } 0 \leq s \leq m-1$$

$$a_{0,0} = \frac{1}{4} \sum_{k=0}^{1} \sum_{l=0}^{1} \hat{A}_{k,l}\, e^{2\pi i (k\cdot 0/m + l\cdot 0/m)} = \frac{1}{4}\left(2\cdot e^{0} + 4 e^{0} + (-2) e^{0} + 0\, e^{0}\right) = 1$$

$$a_{0,1} = \frac{1}{4} \sum_{k=0}^{1} \sum_{l=0}^{1} \hat{A}_{k,l}\, e^{2\pi i (k\cdot 0/m + l\cdot 1/m)} = \frac{1}{4}\left(2\cdot e^{0} + 4\cdot e^{2\pi i \frac{1}{2}} + (-2)\, e^{0} + 0\, e^{2\pi i \frac{1}{2}}\right) = \frac{1}{4}\cdot 4(-1) = -1$$

$$a_{1,0} = \frac{1}{4}\sum_{k=0}^{1}\sum_{l=0}^{1} \hat{a}_{k,l}\, e^{2\pi i\left(k\cdot 1/m + l\cdot 0/n\right)} = \frac{1}{4}\left(2e^{0} + 4e^{0} + (-2)e^{2\pi i \frac{1}{2}} + 0\, e^{2\pi i \frac{1}{2}}\right) = \frac{1}{4}\cdot 8 = 2$$

$$a_{1,1} = \frac{1}{4}\sum_{k=0}^{1}\sum_{l=0}^{1} \hat{a}_{k,l}\, e^{2\pi i\left(k\cdot 1/m + l\cdot 1/n\right)} = \frac{1}{4}\left(2e^{0} + 4e^{2\pi i \frac{1}{2}} + (-2)e^{2\pi i \frac{1}{2}} + 0\, e^{2\pi i}\right) = \frac{1}{4}\cdot\left(2-4+2\right) = 0$$

Portanto

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

2.17. a.

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{bmatrix} \quad e \quad y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$x y^T = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{bmatrix} \begin{bmatrix} y_0 & y_1 & \cdots & y_{n-1} \end{bmatrix} = \begin{bmatrix} x_0 y_0 & x_0 y_1 & \cdots & x_0 y_{n-1} \\ x_1 y_0 & x_1 y_1 & \cdots & x_1 y_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m-1} y_0 & x_{m-1} y_1 & \cdots & x_{m-1} y_{n-1} \end{bmatrix} = Z$$

Portanto $x y^T = Z$ vem da própria definição da multiplicação de matrizes b.

$$\hat{z}_{k,l} = \sum_{n=0}^{m-1}\sum_{s=0}^{m-1} z_{n,s}\, e^{-2\pi i(kn/m + ls/m)} = \sum_{n=0}^{m-1}\sum_{s=0}^{m-1} x_n y_s\, e^{-2\pi i kn/m} \cdot e^{-2\pi i ls/m} =$$

$$= \sum_{n=0}^{m-1} x_n e^{-2\pi i kn/m} \sum_{s=0}^{m-1} y_s e^{-2\pi i ls/m} = \hat{x}_k \hat{y}_l$$

2.19.
$$\hat{y}_n = \sum_{k=0}^{N-1} y_k e^{-2\pi i nk/N} = y_0 e^{-2\pi i n0/N} + y_1 e^{-2\pi i n1/N} + \ldots + y_{N-1} e^{-2\pi i n(N-1)/N} =$$

$$= x_{(0+m)\bmod N} e^{-2\pi i n0/N} + x_{(1+m)\bmod N} e^{-2\pi i n1/N} + \ldots + x_{(N-1+m)\bmod N} e^{-2\pi i n(N-1)/N}$$

Como estamos trabalhando com índices mod N, podemos tomar $0 \leq m \leq N-1$ sem perda de generalidade.

Daí se segue:

Para $m = 0$

$$\hat{y}_n = x_0\, e^{-2\pi i n 0/N} + x_1\, e^{-2\pi i n 1/N} + \dots + x_{N-1}\, e^{-2\pi i n (N-1)/N} = \hat{x}_n = e^{0}\hat{x}_n = e^{2\pi i n 0/N}\hat{x}_n$$

Para $m > 0$, tome qualquer índice j da somatória $\hat{y}_n$:

Se $m + j < N$, então

$$y_j e^{-2\pi i n j/N} = x_{j+m} e^{-2\pi i n j/N}$$

Se $m + j \geq N$, e sabendo que $m \leq N-1$ e $j \leq N-1$, temos que $m + j \leq 2N-2$ e, portanto,

$m + j \bmod n = m + j - N$. Daí, temos

$$y_j e^{-2\pi i n j/N} = x_{j+m-N} e^{-2\pi i n j/N}$$

Em ambos os casos o fator j da somatória de y_n está a $e^{2\pi i n m/N}$ do fator da somatória $\hat{x}_n$.

Para o primeiro caso:

$$\frac{y_j e^{-2\pi i n j/N}}{e^{2\pi i n m/N}} = x_{j+m} e^{-2\pi i n (j+m)/N}$$

Para o segundo caso:

$$\frac{y_j e^{-2\pi i n j/N}}{e^{2\pi i n m/N}} = x_{j+m-N} e^{-2\pi i n (j+m)/N} = x_{j+m-N} e^{-2\pi i n (j+m-N)/N}$$

Dessa forma, vimos que $\hat{X}_n$ é dado por $\dfrac{\sum_{k=0}^{N-1} y_k e^{-2\pi i n k/N}}{e^{2\pi i n m/N}} = \dfrac{\hat{y}_n}{e^{2\pi i n m/N}}$ e, portanto, $\hat{y}_n = e^{2\pi i n m/N} \hat{X}_n$

Agora sabendo da igualdade anterior e utilizando da propriedade $|z_1 z_2| = |z_1||z_2|$ para $z_1, z_2 \in \mathbb{C}$, temos que

$|\hat{y}_n| = |e^{2\pi i n m/N}||\hat{X}_n|$, mas $e^{2\pi i n m/N} = \cos(2\pi n m/N) + i \operatorname{sen}(2\pi n m/N)$, então $|e^{2\pi i n m/N}| = \sqrt{\cos^2(2\pi n m/N) + \operatorname{sen}^2(2\pi n m/N)} = 1$

Daí segue $|\hat{y}_n| = |\hat{X}_n|$.