2.b

We start by computing the mean of the distribution. So by definition, we have:

$$E[X] = \sum_{k=0}^{\infty} kp(X = k) = \sum_{k=0}^{\infty} k\frac{\lambda^k e^{-\lambda}}{k!} =$$

Evaluating the summation on $k = 0$ and summing the rest, we have:

$$= 0 + \sum_{k=1}^{\infty} k\frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-1)!} =$$

Using again the fact that $e^{-\lambda}$ is a constant to the summation over k and splitting λ^k in a convenient way, we have:

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda\lambda^{k-1}}{(k-1)!} = e^{-\lambda}\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} =$$

Now changing variables $j = k - 1$ and using the given Taylor expansion of the exponential function again, we get:

$$= e^{-\lambda}\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

So the mean of the Poisson distribution is $E[X] = \lambda$.

Now for the variance, we start again by the definition:

$$\sigma^2(X) = E[X^2] - E[X]^2$$

Since we have $E[x]$, we know that $E[x]^2 = \lambda^2$, so we just need to compute $E[X^2]$:

$$E[X^2] = \sum_{k=0}^{\infty} k^2 p(X = k) = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} =$$

Separating $e^{-\lambda}$ from the summation again, splitting the first term, $k = 0$, of the summation and also splitting λ^k, we have:

$$= e^{-\lambda}(0 + \sum_{k=1}^{\infty} k \frac{\lambda \lambda^{k-1}}{(k-1)!}) = e^{-\lambda}\lambda(\sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}) =$$

Manipulating the expression to get a $(k-1)$ term, we have:

$$= e^{-\lambda}\lambda \sum_{k=1}^{\infty}((k-1)\frac{\lambda^{k-1}}{(k-1)!} + \frac{\lambda^{k-1}}{(k-1)!}) = e^{-\lambda}\lambda(\sum_{k=1}^{\infty}(k-1)\frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}) =$$

Now computing the next term, $k = 1$, of the first summation, $\sum_{k=1}^{\infty}(k-1)\frac{\lambda^{k-1}}{(k-1)!}$, we get:

$$= e^{-\lambda}\lambda(0 + \sum_{k=2}^{\infty}(k-1)\frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}) = e^{-\lambda}\lambda(\lambda \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!})) =$$

Changing variables $j = k - 2$ and $z = k - 1$, and finally using the given Taylor expansion of the exponential function again, we have:

$$= e^{-\lambda}\lambda(\lambda \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + \sum_{z=0}^{\infty} \frac{\lambda^z}{z!})) = e^{-\lambda}\lambda(\lambda e^{\lambda} + e^{\lambda}) = \lambda^2 + \lambda$$

Going back to the definition of variance, we get:

$$\sigma^2(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

So the variance of the Poisson distribution is $\sigma^2(X) = \lambda$.