Lexical Analysis

Jim Royer

CIS 352

February 10, 2016

http://xkcd/208

➤ The Syntactic Side of Languages

Natural Languages

stream of phonemes $\xrightarrow[\text{analysis}]{\text{via lexical}}$ stream of words $\xrightarrow{\text{via parsing}}$ sentences

Artificial Languages

stream of characters $\xrightarrow[\text{analysis}]{\text{via lexical}}$ stream of tokens $\xrightarrow{\text{via parsing}}$ abstract syntax

What is a token?

Variable names, numerals, operators (e.g., $+$, $/$, etc.), key-words, ...

Lexical structure is typically specified via *regular expressions*.

➤ Regular Expressions *(S. Kleene, mid-1950s)*

Definition

A regular expression has one of six forms:

$\emptyset$	—	matches no string§
ϵ	—	matches the empty string
x	—	matches the character 'x'
$(r_1 \mid r_2)$	—	matches the strings matched by r_1 or r_2
$(r_1 r_2)$	—	matches the strings $w_1 w_2$ where w_1 matches r_1 and w_2 matches r_2
$(r)^*$	—	matches ϵ and the strings $w_1 \dots w_k$ where $k > 0$ and each w_i matches r

We omit the parens in $(r_1|r_2)$, $(r_1 r_2)$, and $(r)^*$ when we can.

§Both Thompson and Mogensen omit this form, and henceforth, so shall we. ($\emptyset$ is very handy in algebraic treatments of regular languages.)

Regular Expressions: Examples

1. Sheep Language = { *ba*!, *baa*!, *baaa*!, *baaaa*!, ... }.
   $baa^*! = (b((a(a^*))!))$ matches exactly the Sheep Language strings.
2. $(0|1)^*$ matches exactly the strings over 0 and 1, including ϵ.
3. $(\epsilon|(1(0|1)^*))1$ matches exactly the binary representation of odd integers.
   —*more examples shortly*—

Notation

$r \Downarrow s \;\equiv_{\text{def}}\;$ regular expression r matches string s.

Big-Step Rules for RegEx Matching

$$\epsilon\text{-match: } \dfrac{}{\epsilon \Downarrow \epsilon} \qquad \text{Literal-match: } \dfrac{}{x \Downarrow x}$$

$$|\text{-match}_1\text{: } \dfrac{r_1 \Downarrow s}{(r_1|r_2) \Downarrow s} \qquad |\text{-match}_2\text{: } \dfrac{r_2 \Downarrow s}{(r_1|r_2) \Downarrow s}$$

$${++}\text{-match: } \dfrac{r_1 \Downarrow s_1 \quad r_2 \Downarrow s_2}{(r_1 r_2) \Downarrow s} \; (s = s_1{++}s_2)$$

$$*\text{-match}_1\text{: } \dfrac{}{r^* \Downarrow \epsilon} \qquad *\text{-match}_2\text{: } \dfrac{r \Downarrow s_1 \quad r^* \Downarrow s_2}{r^* \Downarrow s} \; (s = s_1{++}s_2)$$

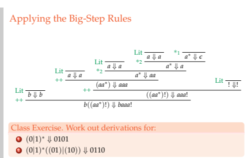
[**Stage direction:** Copy these onto the board, but leave some room.]

Applying the Big-Step Rules

$$\dfrac{\text{Lit}\,\dfrac{}{b \Downarrow b} \quad \dfrac{\dfrac{\text{Lit}\,\dfrac{}{a \Downarrow a} \quad \dfrac{\text{Lit}\,\dfrac{}{a \Downarrow a} \quad \dfrac{\text{Lit}\,\dfrac{}{a \Downarrow a} \quad *_1\,\dfrac{}{a^* \Downarrow \epsilon}}{a^* \Downarrow a}\,{*_2}}{a^* \Downarrow aa}\,{*_2}}{(aa^*) \Downarrow aaa}\,{++} \quad \text{Lit}\,\dfrac{}{! \Downarrow !}}{((aa^*)!) \Downarrow aaa!}\,{++}}{b((aa^*)!) \Downarrow baaa!}\,{++}$$

Class Exercise. Work out derivations for:

1. $(0|1)^* \Downarrow 0101$
2. $(0|1)^*((01)|(10)) \Downarrow 0110$



Class Exercise 1

$$\dfrac{\dfrac{\text{Lit}\,\dfrac{}{0 \Downarrow 0}}{(0|1) \Downarrow 0}\,|_1 \quad \dfrac{\dfrac{\text{Lit}\,\dfrac{}{1 \Downarrow 1}}{(0|1) \Downarrow 1}\,|_2 \quad \dfrac{\dfrac{\text{Lit}\,\dfrac{}{0 \Downarrow 0}}{(0|1) \Downarrow 0}\,|_1 \quad \dfrac{\dfrac{\text{Lit}\,\dfrac{}{1 \Downarrow 1}}{(0|1) \Downarrow 1}\,|_2 \quad *_1\,\dfrac{}{(0|1)^* \Downarrow \epsilon}}{(0|1)^* \Downarrow 1}\,{*_2}\ \text{"1"="1"++}\epsilon}{(0|1)^* \Downarrow 01}\,{*_2}\ \text{"01"="0"++"1"}}{(0|1)^* \Downarrow 101}\,{*_2}\ \text{"101"="1"++"01"}}{(0|1)^* \Downarrow 0101}\,{*_2}\ \text{"0101"="0"++"101"}$$

Class Exercise 2

$$\dfrac{\dfrac{\dfrac{\text{Lit}\,\dfrac{}{0 \Downarrow 0}}{(0|1) \Downarrow 0}\,|_1 \quad \dfrac{\dfrac{\text{Lit}\,\dfrac{}{1 \Downarrow 1}}{(0|1) \Downarrow 1}\,|_2 \quad *_1\,\dfrac{}{(0|1)^* \Downarrow \epsilon}}{(0|1)^* \Downarrow 1}\,{*_2}}{(0|1)^* \Downarrow 01}\,{*_2} \quad \dfrac{\dfrac{\text{Lit}\,\dfrac{}{1 \Downarrow 1} \quad \text{Lit}\,\dfrac{}{0 \Downarrow 0}}{10 \Downarrow 10}\,{++}}{((01)|(10)) \Downarrow 10}\,|_2}{(0|1)^*((01)|(10)) \Downarrow 0110}\,{++}$$

Matching Regular Expressions in Haskell, I

```
data Reg = Epsilon
         | Literal Char
         | Or Reg Reg
         | Then Reg Reg
         | Star Reg
           deriving (Eq)
```

```
matches ::  Reg -> String -> Bool

matches Epsilon st       = (st == "")
matches (Literal ch) st = (st == [ch])
matches (Or r1 r2) st   = matches r1 st || matches r2 st
                        ⋮
```

Credits/Pointers

- The code here is based on work by Simon Thompson.
  See: http://www.haskellcraft.com/craft3e/Reg_exps.html

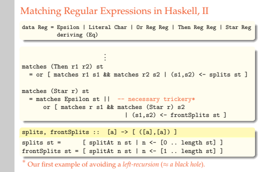
Matching Regular Expressions in Haskell, II

```
data Reg = Epsilon | Literal Char | Or Reg Reg | Then Reg Reg | Star Reg
           deriving (Eq)
```

```
                          ⋮
matches (Then r1 r2) st
  = or [ matches r1 s1 && matches r2 s2 | (s1,s2) <- splits st ]

matches (Star r) st
  = matches Epsilon st ||  -- necessary trickery*
      or [ matches r s1 && matches (Star r) s2
                                | (s1,s2) <- frontSplits st ]
```

```
splits, frontSplits ::  [a] -> [ ([a],[a]) ]
splits st =       [ splitAt n st | n <- [0 .. length st] ]
frontSplits st = [ splitAt n st | n <- [1 .. length st] ]
```

* Our first example of avoiding a *left-recursion* (≈ *a black hole*).



Replace the case for `Star` with

```
matches (Star r) st
   = matches Epsilon st ||
      or [ matches r s1 && matches (Star r) s2
                          | (s1,s2) <- splits st ]
```

and try

```
(Star (Epsilon ‘Or‘ (Literal ’a’))) "aa"
```

Regular Expressions and the Languages They Name

Definition

Suppose r is a regular expression and A and B are sets of strings.

(a) $L(r)$ = the set of strings matched by r.

(b) $A \cdot B = \{ w_a w_b \mid w_a \in A,\ w_b \in B \}$.

(c) $A^0 = \{ \epsilon \},\ \ A^1 = A,\ \ A^2 = A \cdot A,\ \ A^3 = A \cdot A \cdot A,\ \ldots$

Thus:

$$\begin{aligned}
L(\epsilon) &= \{ \epsilon \} \\
L(\mathtt{x}) &= \{ \mathtt{x} \} \\
L(r_1|r_2) &= L(r_1) \cup L(r_2) \\
L(r_1 r_2) &= L(r_1) \cdot L(r_2) \\
L(r^*) &= \{ \epsilon \} \cup L(r) \cdot L(r^*) \quad = \quad \bigcup_{i \geq 0} L(r)^i
\end{aligned}$$

Short Cuts (Mogensen, §2.1.1)

- We can write $(0|1|2|3|4|5|6|7|8|9)$ as $[0123456789]$ or $[0-9]$.
- $r^+ = r\,r^*$, i.e.,

$$\begin{aligned}
r^* &\equiv \text{0 more more matches of } r \\
r^+ &\equiv \text{1 more more matches of } r
\end{aligned}$$

- $r? = r|\epsilon \quad \equiv \quad$ 0 or 1 matches of r.

Examples

- $[a-zA-Z]$ = all alphabetic characters
- $(0|([1-9][0-9]^*))$ = all natural number constants
- $[a-zA-Z\_][a-zA-Z0-9]^* \equiv$ C variable names
- $"\big([a-zA-Z0-9]|\backslash[a-zA-Z0-9]\big)^*" \equiv$ C string constants

Regular Expressions with Their Work Boots On

- grep, egrep, fgrep — print lines matching a pattern
  See `http://en.wikipedia.org/wiki/Grep`
- Also see `tr`, `sed`, ...
  (The original Unix developers knew their automata theory cold.)
- See `http://perldoc.perl.org/perlre.html`.
  (Folks in bioinformatics know their pattern matching cold.)
- See `http://en.wikipedia.org/wiki/List_of_regular_expression_software`.

➤ Non-deterministic Finite Automata

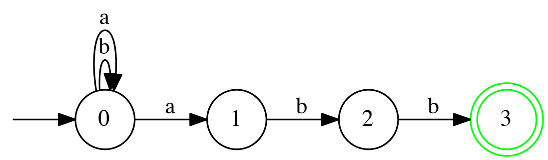
A *Non-deterministic Finite Automaton* (abbreviated *NFA*) consists of:

- A finite set of states, S.
- A finite set of moves *(labeled edges between states)*
  (Moves are labeled by either ϵ or a $c \in \Sigma =$ the input alphabet)
- A start state (in S).
- A set of terminal or final states (a subset of S).

Example

$S = \{0,1,2,3\}$, start state = 0, final sets = $\{3\}$

moves = $\{0 \xrightarrow{a} 0,\ 0 \xrightarrow{b} 0,\ 0 \xrightarrow{a} 1,\ 1 \xrightarrow{b} 2,\ 2 \xrightarrow{b} 3\}$



The Data.Set Module

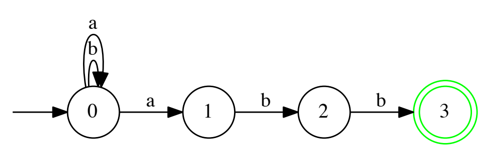
To implement NFA's we need a module for representing sets.
We use:
`http://hackage.haskell.org/packages/archive/containers/latest/doc/html/Data-Set.html`

```
Prelude> :browse Data.Set

         empty :: Set a
      fromList :: Ord a => [a] -> Set a
  intersection :: Ord a => Set a -> Set a -> Set a
  Data.Set.map :: (Ord a, Ord b) => (a -> b) -> Set a -> Set b
     singleton :: a -> Set a
          size :: Set a -> Int
        toList :: Set a -> [a]
         union :: Ord a => Set a -> Set a -> Set a
              etc.
```


NFAs represented in Haskell

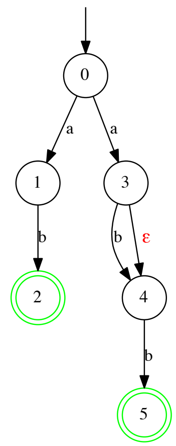
```
data Move a = Move a Char a | Emove a a
              deriving (Eq,Ord,Show)

data Nfa a  = NFA (Set a) (Set (Move a)) a (Set a)
              deriving (Eq,Show)
```

```
machM ::  Nfa Int
machM = NFA
         (S.fromList [0..3])
         (S.fromList [Move 0 'a' 0,  Move 0 'a' 1,  Move 0 'b' 0,
                      Move 1 'b' 2,  Move 2 'b' 3] )
         0
         (S.singleton 3)
```



Another Example NFA

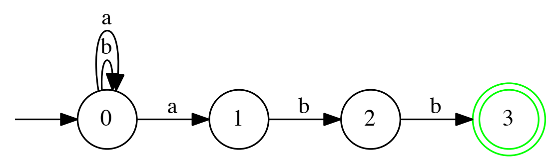
```
machN = NFA
        (S.fromList [0..5])
        (S.fromList [Move 0 'a' 1,
                     Move 1 'b' 2,
                     Move 0 'a' 3,
                     Move 3 'b' 4,
                     Emove 3 4,
                     Move 4 'b' 5])
        0
        (S.fromList [2,5])
```



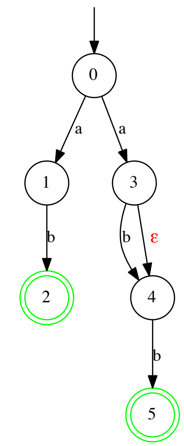
Note the two sorts of nondeterminism this machine exhibits.

Accepting and rejecting strings

- What is the accepting path of *abb* through M?
- What other paths are possible?
- What are the accepting paths of *ab* through N?
- What happens with N and *aa*?



Machine M



Machine N

A small-step semantics for an NFA

Notation

For $M = (States, Moves, start, Final)$:

- $M \vdash s \overset{a}{\Longrightarrow} s' \;\equiv_{\text{def}}\; (s,a,s') \in Moves.$
- $M \vdash s \overset{\epsilon}{\Longrightarrow} s' \;\equiv_{\text{def}}\; (s,\epsilon,s') \in Moves.$

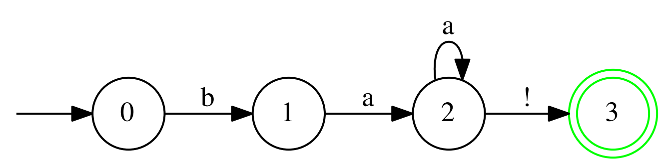
$$\frac{}{M \vdash s \overset{a}{\Longrightarrow} s'} \; ((s,a,s') \in Moves)$$

$$\frac{}{M \vdash s \overset{\epsilon}{\Longrightarrow} s'} \; ((s,\epsilon,s') \in Moves)$$

[**Stage direction:** Copy these onto the board.]

Applying the Small-Step Rules, 1

$$M = (\{0,1,2,3\}, \{0 \overset{b}{\to} 1,\ 1 \overset{a}{\to} 2,\ 2 \overset{a}{\to} 2,\ 2 \overset{!}{\to} 3\}, 0, \{3\})$$

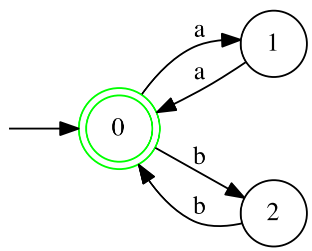


An accepting path for *baaa!*:

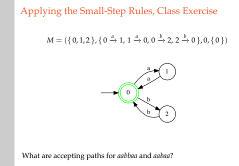
$$0 \overset{b}{\Longrightarrow} 1 \overset{a}{\Longrightarrow} 2 \overset{a}{\Longrightarrow} 2 \overset{a}{\Longrightarrow} 2 \overset{!}{\Longrightarrow} 3$$

Applying the Small-Step Rules, Class Exercise

$$M = (\{0,1,2\}, \{0 \xrightarrow{a} 1,\ 1 \xrightarrow{a} 0,\ 0 \xrightarrow{b} 2,\ 2 \xrightarrow{b} 0\}, 0, \{0\})$$



What are accepting paths for *aabbaa* and *aabaa*?



- *aabbaa* $0 \xRightarrow{a} 1 \xRightarrow{a} 0 \xRightarrow{b} 2 \xRightarrow{b} 0 \xRightarrow{a} 1 \xRightarrow{a} 0$
- *aabaa* $0 \xRightarrow{a} 1 \xRightarrow{a} 0 \xRightarrow{b} 2$ Stuck!

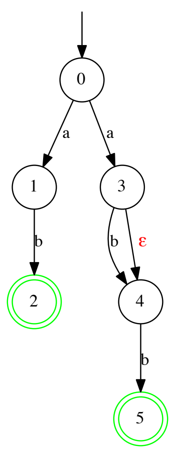
➤ NFAs implemented in Haskell

```
-- (trans nfa str)
--      = the set of states reachable in nfa by following str
trans :: Ord a => Nfa a -> String -> Set a
```

See http://www.cis.syr.edu/courses/cis352/code/RegExp/ImplementNfa.hs

`trans machN "a" =` $\{1,3,4\}$

- ϵ-moves are a problem
- The ϵ-closure of a set of states S = the set of states accessible from S via ϵ-moves



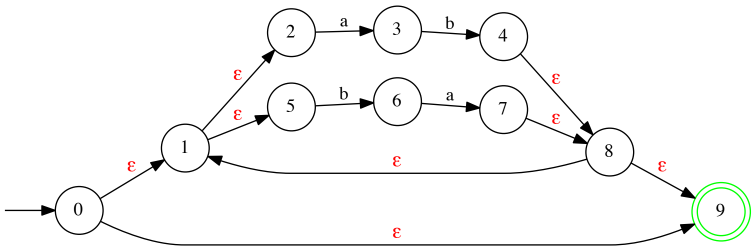
Machine N

Handling ϵ-Closures

```
setlimit ::  Eq a => (Set a -> Set a) -> Set a -> Set a
setlimit f s = let next = f s
                in if s==next
                      then s
                      else setlimit f next
```

```
closure ::  Ord a => Nfa a -> Set a -> Set a
closure (NFA states moves start term) s = setlimit add s
    where
      add stateset = S.union stateset (S.fromList accessible)
          where
            accessible = [ s | x <- S.toList stateset ,
                                 Emove y s <- S.toList moves ,
                                 y==x ]
```

Example: An NFA for $(ab|ba)^*$




```
*Top> closure m (singleton 2)
fromList [2]

*Top> closure m (singleton 1)
fromList [1,2,5]

*Top> closure m (singleton 0)
fromList [0,1,2,5,9]
```


Taking one step

```
onemove :: Ord a => Nfa a -> Char -> Set a -> Set a
onemove (NFA states moves start term) c x
    = S.fromList [ s | t <- S.toList x ,
                       Move z d s <- S.toList moves ,
                       z==t , c==d ]
```

```
onetrans :: Ord a => Nfa a -> Char -> Set a -> Set a
onetrans mach c x = closure mach (onemove mach c x)
```


Taking many steps

```
trans ::  Ord a => Nfa a -> String -> Set a
trans mach str = foldl step startset str
    where
      step set ch = onetrans mach ch set
      startset    = closure mach S.singleton (startstate mach))
```

```
foldl ::  (a -> b -> a) -> a -> [b] -> a
foldl step s (c1:c2:...:ck:[])
   = (... ((s `step` c1) `step` c2) `step` ...  `step` ck)
```


➤ RegExps → NFAs

$M(r)$ = an NFA for accepting $L(r)$.

$\rightarrow ① \xrightarrow{\epsilon} ②$

Figure: $M(\epsilon)$

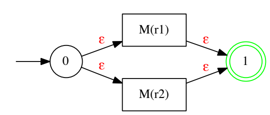


Figure: $M(r_1|r_2)$

$\rightarrow ① \xrightarrow{x} ②$

Figure: $M(x)$

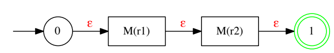


Figure: $M(r_1 r_2)$

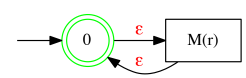
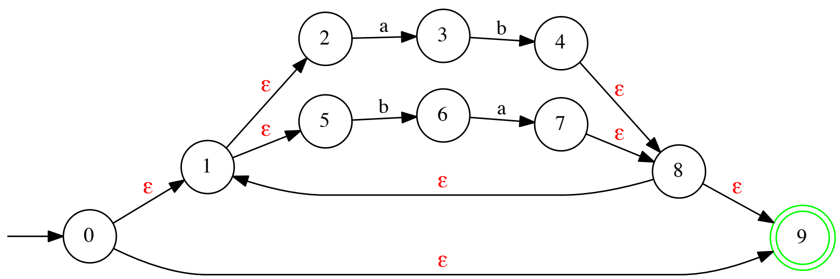


Figure: $M(r^*)$

Example: The NFA for $(ab|ba)^*$



The translation in Haskell

See `BuildNfa.hs`.

```
build :: Reg -> Nfa Int

build Epsilon      = NFA (S.fromList [0..1])
                          S.singleton(Emove 0 1)
                          0
                          S.singleton 1

build (Literal c)  = NFA (S.fromList [0..1])
                          S.singleton(Move 0 c 1)
                          0
                          S.singleton 1

build (Or r1 r2)   = m_or (build r1) (build r2)

build (Then r1 r2) = m_then (build r1) (build r2)

build (Star r)     = m_star (build r)
```

`m_or`, `m_then`, and `m_star` are on the ugly side.

Theory Break: Regular Languages

Definition

The *regular languages* are the languages described by regular expressions ($= \{ L(r) : r$ is a reg. exp. $\}$).

Theorem

The regular languages $\subseteq$ the languages accepted by NFAs.

Proof: We need to show the reg.-exp.→NFA translation is correct — which is a not-too-hard structural induction.

Theorem

The regular languages $\supseteq$ the languages accepted by NFAs.

Proof: There turns out to be an NFA→reg.-exp. translation (which we'll skip here).

➤ Deterministic Finite Automata

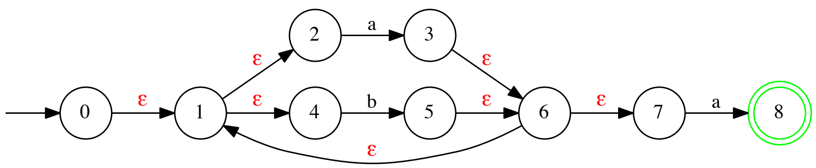
Definition

A *deterministic finite automata* (abbreviated DFA) is a NFA that

- contains no ϵ-moves, and
- has at most one arrow labelled with a particular symbol leaving any given state.

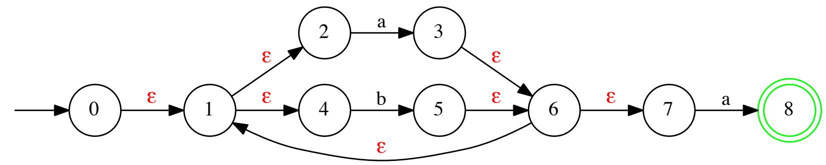
- So in a DFA there is *at most one possible move in any situation*.
- The DFAs also characterize the regular languages.

Example NFA → DFA Translation, I

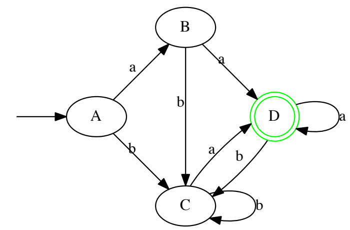


- $A = \epsilon\text{-closure}(\{\,0\,\}) = \{\,0,1,2,4\,\}$.
- $B = \epsilon\text{-closure}(\{\,s \;:\; s' \xrightarrow{a} s,\ s' \in A\,\}) = \{\,1,2,3,4,6,7\,\}$. $(A \xrightarrow{a} B)$
- $C = \epsilon\text{-closure}(\{\,s \;:\; s' \xrightarrow{b} s,\ s' \in A\,\}) = \{\,1,2,4,5,6,7\,\}$. $(A \xrightarrow{b} C)$
- $D = \epsilon\text{-closure}(\{\,s \;:\; s' \xrightarrow{a} s,\ s' \in B\,\}) = \{\,1,2,4,5,6,7,8\,\}$. $(B \xrightarrow{b} D)$
- $C = \epsilon\text{-closure}(\{\,s \;:\; s' \xrightarrow{b} s,\ s' \in B\,\}) = \{\,1,2,4,5,6,7\,\}$. $(B \xrightarrow{b} C)$
- Similarly, $C \xrightarrow{a} D$, $C \xrightarrow{b} C$, $D \xrightarrow{a} D$, $D \xrightarrow{b} C$.

Example NFA → DFA Translation, II



$\Downarrow$



The NFA to DFA algorithm in Haskell

```
make_deterministic :: Nfa Int -> Nfa Int
make_deterministic = number . make_deter
```

```
number :: Nfa (Set Int) -> Nfa Int
number (NFA states moves start finish)
     = NFA states' moves' start' finish'
       where ...
```

```
make_deter :: Nfa Int -> Nfa (Set Int)
make_deter mach = deterministic mach (alphabet mach)
```

Switch to `NfaToDfa.hs`.

Extra Topics

Minimizing DFAs, 1

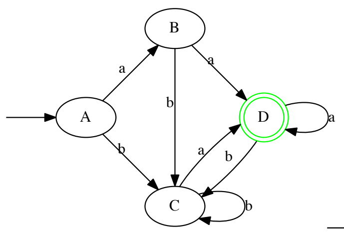
Definition

Suppose s and s' are states in a DFA M.

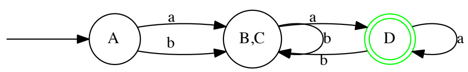
- s and s' are *distinguished* by x when

  M started in s run on x accepts $\iff$ M started in s run on x rejects
- s and s' are *indistinguishable* when no string x distinguishes them.

  So, we can treat merge s and s' safely into a single state.



- ϵ distinguishes D and each of A, B, C
- a distinguishes A and each of B and C.
- B and C turn out to be indistinguishable.
- The result of merging B and C is:



Minimizing DFAs, 2

See Tom Henzinger's notes on the Myhill-Nerode Theorem
`http://engineering.dartmouth.edu/~d25559k/ENGS122_files/Lectures_Notes/Henzinger-Nerode-7.pdf.`
(Much handier than the Pumping Lemma for regular languages)

Regular Definitions, 1

- In building a compiler or interpreter, you want to specify the lexical part of the language (e.g., token) by *regular definitions* (hopped-up regular expressions). E.g.:

$$\begin{aligned} IF &= \texttt{if} \\ ID &= [\texttt{a-zA-Z}][\texttt{a-zA-Z0-9}]^* \\ NUM &= [-+][\texttt{0-9}]^* \\ FLOAT &= \textit{a nasty mess} \end{aligned}$$

- Then you translate the entire collection of these to an NFA. E.g.:

Regular Definitions, 2

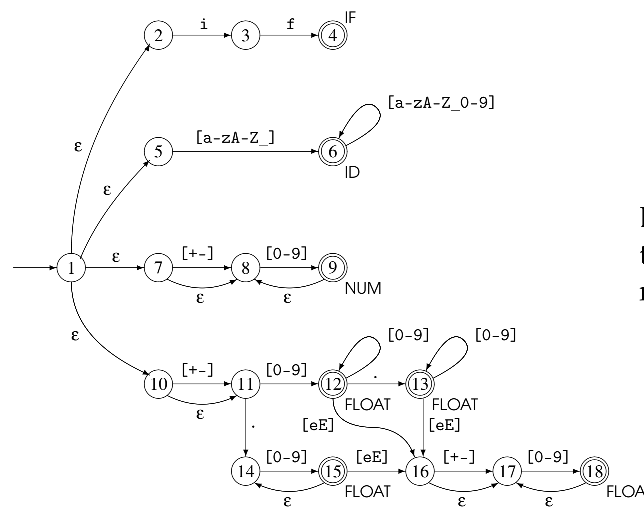


Figure 2.12: Combined NFA for several tokens

By convention, you take the longest match of a string.

Regular Definitions, 3

Then you translate the NFA to a DFA with which you scan through the input and spit out tokens with lightening speed.

See §2.9 of Mogensen for details.

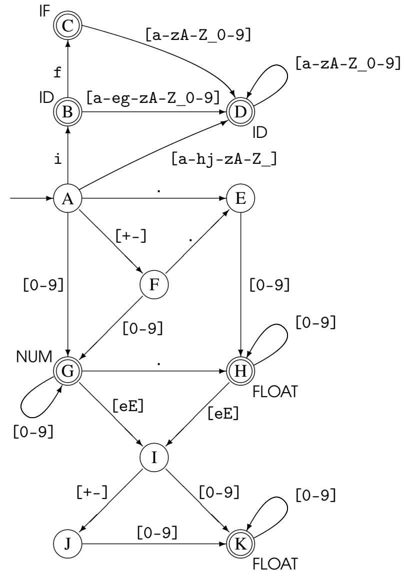


Figure 2.13: Combined DFA for several tokens

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