Waves and Pulses in Cables

Shyam Krishnan (20596670) and Luka Filipovic (20494344)

***Abstract*— In this experiment, we examine the normal modes and beat frequencies of a pair of coupled pendula.**

I. Introduction

For small angles of displacement from its mean position, a simple pendulum behaves like a harmonic oscillator. When two such pendula are coupled (by means of a spring between them), it allows the transfer of energy between the two pendula and the force due to the spring changes the motion of the system. Each individual pendulum is now driven by the spring. As a result, the initial conditions i.e. the initial angular displacements of the pendula from their mean positions has an effect on their frequency $\omega_i(t)$ which depends now depends on the time. This is in contrast to the case of two uncoupled simple pendula, whose frequencies are time-independent. Two particular sets of initial conditions (see Section II) correspond to the 'normal modes' of the system. For these initial conditions, both pendula move sinusoidally with the same frequency and have a definite phase difference (Marion, J. & Thornton, S. 2004).

Coupled oscillators are used to model atoms in theories of radiation absorption/emission, which makes the study of coupled oscillators important. The normal modes of a coupled oscillator system are important because an arbitrary set of initial conditions of the system can be thought of as a superposition of the normal mode oscillations (Gardiner, J. 2013) (Marion, J. & Thornton, S. 2004).

II. Theory

It's a well known fact that the motion of a simple pendulum is simple harmonic (assuming the angular displacement θ is small). The equation of motion for a simple pendulum is therefore:

$$\tau = -I\ddot{\theta} \tag{II.1}$$

This has the solution:

$$\theta = \theta_0 cos(\omega t) \tag{II.2}$$

$$\omega_0 = \sqrt{g/L} \tag{II.3}$$

Where θ_0 is the initial angular displacement and ω_0 is the angular frequency of the oscillation.

For the case of two simple pendula coupled by a spring of spring constant k, the equations of motion for small θ_i are (as given in the lab handout):

$$\ddot{\theta}_1 = -\frac{g}{L}\theta_1 + \frac{kl^2}{mL^2}(\theta_2 - \theta_1) \tag{II.4}$$

$$\ddot{\theta}_2 = -\frac{g}{L}\theta_2 + \frac{kl^2}{mL^2}(\theta_1 - \theta_2) \tag{II.5}$$

Where l is the 'coupling length' of the pendula. This coupling length refers to the point along the length of each pendulum at which the spring connects the two pendula.

As mentioned in Section I, there are two sets of initial conditions of interest when analyzing coupled oscillators. The first is when $\theta_1 = \theta_2$ and $\dot{\theta}_1 = \dot{\theta}_2$ i.e. when both pendula are started with the same angular displacement and angular velocity. This inital condition corresponds to the 'even' normal mode. In this case, the spring has no effect on the pendula because it does not undergo extension or compression. The pendula oscillate in phase with the same angular frequency ω_0 as in the single pendulum case.

The second set of initial conditions is when $\theta_1 = -\theta_2$ and $\dot{\theta}_1 = -\dot{\theta}_2$. In this case, the pendula are started with the same amplitude, but in different directions. The pendula oscillate with a phase difference of $180\deg$ i.e. exactly out of phase. The angular frequency of oscillation of each pendulum is then given by:

$$\omega^2 = \omega_0^2 + \frac{2kl^2}{mL^2} \tag{II.6}$$

$$\omega = \omega_0 + \frac{kl^2}{\omega_0 mL^2} \tag{II.7}$$

Where the approximations made include: the spring constant k is small, the angular displacements are small, binomial approximation.

The above sets of initial conditions correspond to the normal modes of the coupled oscillator system. These normal modes are important because any arbitrary set of initial conditions can be represented as a superposition of the normal modes. Consider the example given in the lab handout: When one pendulum is kept at rest with the other displaced by some angle θ_2, the difference in the angular frequencies between the two pendula are given by:

$$\Delta\omega = \frac{kl^2}{\omega_o mL^2} \tag{II.8}$$

Here, $\Delta\omega$ is the beat frequency of the system. The energy oscillates back and forth between the two pendula such that after some time $T/2$, the pendulum that was initially oscillating stops whereas the pendulum that was initially at rest oscillates with amplitude θ_2. The beat period T and the beat frequency ω are related by the usual relation:

$$\omega = \frac{2\pi}{T} \tag{II.9}$$

Therefore if the constants $m, \omega_0, L, kandl$ are known, the beat frequency can be determined.

III. Experimental Design and Procedure

The experimental apparatus involved two pendula (1 and 2) of masses $m_1 = 2931g \pm 1g$ and $m_2 = 2946g \pm 1g$ and lengths $L_1 = 84cm \pm 1cm$ and $L_2 = 83.5cm \pm 1cm$ respectively. Four different springs of different spring constant k_i were used to couple the pendula (see Section IV for calculation of spring constants). We determined the spring constant for each spring by hanging different weights on each spring and measuring the subsequent extension. Plots of the hung mass vs the extension were used to calculate the spring constant.

The pendula have 4 mounting points along their length for the spring (as described in the lab handout). The lengths of each of these mounting points were measured from the hinge of each pendulum.

For analyzing normal modes and beats, we plot an amplitude vs time signal for each pendulum. This is done by means of a system consisting of a DC voltage generator, electrical contacts at the hinges of each pendulum and an analog to digital converter that is connected to a computer running a LabView tool that visualizes the signals.

For normal modes, we plotted amplitude vs time signals for each pendulum for each initial condition (in-phase or out-of-phase) and every possible spring constant k_i and mounting location l_i. We followed a similar arrangement for beat frequency, except that the initial conditions were changed such that pendulum 1 was held fixed at rest while pendulum 2 was displaced by some small angle. In both cases, we used a sampling rate of $5000Hz$

For odd normal modes, we made plots of the squared angular frequency ω^2 vs k_i and of ω^2 vs l_i (the coupling lengths) and used (6) to calculate the theoretical ω^2 values.

The theoretical beat frequency was calculated using (8).

IV. Analysis

A. Determining Spring Constants

By measuring spring displacements for various masses and for each of the 4 springs, we get the following:

Spring 1		Spring 2	
Mass/g	Position/cm	Mass/g	Position/cm
0	34.55	0	31.65
20	35.23	20	32.60
40	35.85	40	33.60
60	36.55	50	34.05
80	37.15	60	34.60
100	37.80	70	35.00
120	38.35	80	35.50
Spring 3		Spring 4	
0	31.50	0	33.30
20	31.60	20	36.75
100	32.05	40	40.15
120	32.15	50	41.70
150	32.35	60	43.45
200	32.75	70	45.00
250	33.00	80	46.85

Fig. 1: Bottom position of hanging springs extended under masss

We also introduce an error in position measurement of $\pm 0.2cm$. Error in weights is ignored, due to it being insignificant when compared to position measurements.

To calculate the spring constant for each spring - k_1, k_2, k_3 and k_4 for springs 1,2,3,4, respectively, we use the least square method to find the slope of mass vs position. By Hooke's Law, $k = -\frac{gm}{x}$, so the spring constant is simply slope times the gravitational constant.

The average masses in all four cases, respectively, are:

$$M_1 = 60g$$

$$M_2 = 45.71g$$

$$M_3 = 120g$$

$$M_4 = 45.71g$$

And the average position is:

$$X_1 = 36.5 \pm 0.08cm$$

$$X_2 = 33.86 \pm 0.08cm$$

$$X_3 = 32.20 \pm 0.08cm$$

$$X_4 = 41.03 \pm 0.08cm$$

Since the error is $0.2cm$ for every measurement, the error on the average value of 7 measurements is simply $\frac{\sqrt{7}\cdot 0.2}{7}cm = 0.08cm$.

Now, we just find the slope as in formula (reference). This gives us $s_1 = 31.43g/cm, s_2 = 20.72g/cm, s_3 = 162.78g/cm$ and $s_4 = 5.95g/cm$.

As far as error goes, in the formula $s = \frac{\sum(m_i - M)(x_i - X)}{\sum(x_i - X)^2}$ masses have no significant error, and the error on each $x_i - X$ is $\sqrt{0.2^2 + 0.08^2} = 0.22$, so the total error on the numerator is

$$e_n = 0.22\sqrt{\sum(m_i - M)^2}$$

The error on $(x_i - X)^2$ is $\sqrt{2} \cdot 0.22 \cdot (x_i - X) = 0.31(x_i - X)$, so the error on denominator equals

$$e_d = 0.31\sqrt{\sum (x_i - X)^2}$$

This gives us total error on the slope $\%e = \sqrt{(\%e_n)^2 + (\%e_d)^2)}$.

Notice that, if by d we denote the value of the denominator, then $e_d = 0.31\sqrt{d}$, so

$$\%e_d = \frac{0.31\sqrt{d}}{d} = \frac{0.31}{\sqrt{d}}$$

Let us go through the whole calculation for spring 1. $\sum (m_i - M_1)^2 = 11200$, so

$$e_n = 0.22 \cdot \sqrt{11200} = 23.28$$

The actual value of the numberator is $\sum (m_i - M)(x_i - X) = 356$, so

$$\%e_n = \frac{23.28}{365} = 0.06378$$

Similarly, $\sum (x_i - X)^2 = 11.325$, and hence

$$\%e_d = \frac{0.31}{\sqrt{11.325}} = 0.02737$$

This gives $\%e = \sqrt{0.06378^2 + 0.02737^2} = 0.06940$, and since the value of the slope is $s_1 = 31.43 g/cm$, we get:

$$s_1 = (31.43 \pm 0.0694 \cdot 31.43) g/cm = (31.43 \pm 2.18) g/cm$$

Proceeding with identical calculations, we obtain the following:

$$s_2 = (20.72 \pm 2.36) g/cm$$

$$s_3 = (162.78 \pm 44.93) g/cm$$

$$s_4 = (5.95 \pm 0.19) g/cm$$

As discussed at the beginning of this section, the spring constant is simply slope times the gravitational constant, g. Since we want the slope in kg/m, we simply scale results in g/cm by $\frac{1}{10}$. The spring constants are:

$$k_1 = (30.83 \pm 2.14) N/m$$

$$k_2 = (20.33 \pm 2.32) N/m$$

$$k_3 = (159.69 \pm 44.08) N/m$$

$$k_4 = (5.84 \pm 0.19) N/m$$

B. Normal Modes and Coupling Frequency

Our apparatus has 4 mounting locations for springs, each at a different distance from the hinge. We'll call this distance $l_1 - l_4$, starting from the location closest to the hinge. Finally, total lenght of the pendulum is denoted by L.

	Pendulum 1	Pendulum 2	Average
l_1	13	13.5	13.25
l_2	22.5	22.5	22.5
l_3	32	31	31.5
l_4	42.5	42	42.25
L	56.5	56	56.25

The mass of each pendulum is $m = 2.938 kg$.

The angular frequency is hence, by (II.3),

$$\omega_0 = \sqrt{g/L} = (4.18 \pm 0.04) s^{-1}$$

Uncertainty was easily calculated, since relative error in ω_0 is half of relative error in L.

Now we can use (II.6) to find the angular frequency, ω, in the case of the odd mode, for each combination of the spring constant and mounting position:

	Position 1	Position 2	Position 3	Position 4
Spring 1	4.32	4.56	4.90	5.41
Spring 2	4.27	4.44	4.67	5.03
Spring 3	4.85	5.90	7.18	8.88
Spring 4	4.21	4.26	4.33	4.44

Fig. 2: ω in s^{-1}

C. Beat Frequency

Due to small coupling approximation assumptions, we've used spring 2 and 4 for this part, because they have the smallest spring constant k.

Using (II.7), we get the following theoretical values for beat frequencies:

	Beat frequency/s^{-1}
Spring 2 Position 1	0.092 ± 0.018
Spring 2 Position 2	0.265 ± 0.040
Spring 4 Position 1	0.026 ± 0.008
Spring 4 Position 2	0.076 ± 0.004

Fig. 3: Theoretically predicted beat frequencies

Error is calculated by

$$\%\Delta\omega = \sqrt{(\%k)^2 + (2\%l)^2 + (\%\omega_0)^2 + (2\%L)^2}$$

For example, in the first case, Spring 2 Position 1, we'd have

$$\%\Delta\omega = \sqrt{(\frac{2.32}{20.33})^2 + (\frac{2}{13.25})^2 + (\frac{0.04}{4.18})^2 + (\frac{2}{56.25})^2} = 0.1928$$

and hence

$$\Delta\omega = (0.092 \pm 0.1928 \cdot 0.092)s^{-1} = (0.092 \pm 0.018)s^{-1}$$

V. Conclusions

Hello.