The Dynamics of Solitons

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The aim of this project was to study the dynamics of solitons by finding numerical solutions to the Korteweg-de Vries (KdeV) equation. A normal mode solution to the equation was initialised and propagated in time using a fourth order Runge-Kutta scheme and the inter-dependence of wave properties such as height and speed was studied. The relationship between numerical stability and the time and space step sizes as well as wave velocity was also studied. In addition, different initial solutions were propagated according to the same scheme to study different aspects of soliton behaviour, such as collisions between two solitons, wave breaking and, when the dispersion is ignored, shock waves.

1 Introduction

1.1 Background

Solitary waves or "solitons" are pulses that propagate for large distances without changing their shape. The phenomenon was first observed by John Scott Russell in 1832, as he followed a solitary wave along the Edinburgh-Glasgow canal [1]. In the 1870's, Lord Rayleigh and Boussinesq both independently deduced the functional form of the solitary wave [2] and, in 1895, Korteweg and de Vries derived the equation that describes the evolution and propagation of these solitons and that is satisfied by the form found by Rayleigh and Boussinesq [3]. This equation is known as the Korteweg de Vries (KdeV) equation, and is as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0\ , \tag{1}$$

with x and t both measured in arbitrary, dimensionless units [4]. The equation includes both a dispersive term and a non-linear term, which happen to cancel out identically, leading to soliton behaviour i.e. the preservation of the pulse's shape over long distances [4].

1.2 Normal Mode Solution

The normal mode solution of the KdeV equation, as initially modelled by Rayleigh and Boussinesq [2] is:

$$u = 12\alpha^2\,\mathrm{sech}^2[\alpha(\mathrm{x} - 4\alpha\mathrm{t})]\ . \tag{2}$$

One can show that this is indeed a solution to the KdeV equation:

First, to simplify notation, write:

$$v = \alpha(x - 4\alpha^2 t). \tag{3}$$

Then, after differentiating (2), one can write:

$$\frac{\partial u}{\partial t} = 96\alpha^5 \tanh v\,\mathrm{sech}^2\, v \tag{4a}$$

$$\frac{\partial u}{\partial x} = -24\alpha^3 \tanh v \operatorname{sech}^2 v$$
$$\Rightarrow u\frac{\partial u}{\partial x} = -288\alpha^5 \tanh v \operatorname{sech}^4 v \tag{4b}$$

$$\frac{\partial^3 u}{\partial x^3} = -48\alpha^5(\cosh(2v) - 5) \tanh v \operatorname{sech}^4 v \tag{4c}$$

Summing equations (4a-c), one obtains:

$$\begin{aligned}
96\alpha^5 \tanh v \operatorname{sech}^2 v - 288\alpha^5 \tanh v \operatorname{sech}^4 v - 48\alpha^5(\cosh(2v) - 5) \tanh v \operatorname{sech}^4 v &= \\
= 48\alpha^5 \tanh v \operatorname{sech}^2 v\,(2 - 6\operatorname{sech}^2 v - (\cosh(2v) - 5)\operatorname{sech}^2 v) \\
= 48\alpha^5 \tanh v \operatorname{sech}^2 v(2 - 6\operatorname{sech}^2 v - (2\cosh^2 v - 6)\operatorname{sech}^2 v) \\
= 48\alpha^5 \tanh v \operatorname{sech}^2 v(2 - 6\operatorname{sech}^2 v - 2\cosh^2 v \operatorname{sech}^2 v + 6\operatorname{sech}^2 v) \\
= 48\alpha^5 \tanh v \operatorname{sech}^2 v(2 - 6\operatorname{sech}^2 v - 2 + 6\operatorname{sech}^2 v) \\
= 0
\end{aligned}$$

Therefore, equation (2) is indeed a solution to the KdeV equation.

2 Discretisation

2.1 Forward Time, Centred Space

One way to discretise the KdeV equation is to use centre difference approximations for the spatial derivatives and a forward difference approximation for the time derivative [4]. This is shown below:

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}^j - u_{i-1}^j}{2h} \tag{5a}$$

$$u \approx \frac{u_{i+1}^j + u_{i-1}^j}{2} \tag{5b}$$

By multiplying together (5a) and (5b), one obtains:

$$u\frac{\partial u}{\partial x} \approx \frac{1}{4h}\left[\left(u_{i+1}^j\right)^2 - \left(u_{i-1}^j\right)^2\right] \tag{5c}$$

Finally, the third order spatial derivative can be approximated as:

$$\frac{\partial^3 u}{\partial x^3} \approx \frac{u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j}{2h^3} \tag{5d}$$

Then, using a forward difference approximation for the first order time derivative, the discretised form of the KdeV equation becomes:

$$u_i^{j+1} = u_i^j - \frac{1}{4}\frac{\Delta t}{h}\left[\left(u_{i+1}^j\right)^2 - \left(u_{i-1}^j\right)^2\right] - \frac{1}{2}\frac{\Delta t}{h^3}\left[u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j\right], \tag{6}$$

where h is a small step in x and Δt is a small step in t.

2.1.1 Consistency

It can be shown that equation (6) is consistent with the KdeV equation. This is done by Taylor expanding the u terms in (6) around u_j^i as follows:

$$u_i^{j+1} = u_j^i + \Delta t\frac{\partial u}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2} + O(\Delta t^3) \tag{7a}$$

$$u_{i\pm1}^j = u_j^i \pm h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} \pm \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + O(h^4) \tag{7b}$$

$$u_{i\pm2}^j = u_j^i \pm 2h\frac{\partial u}{\partial x} + \frac{(2h)^2}{2}\frac{\partial^2 u}{\partial x^2} \pm \frac{(2h)^3}{6}\frac{\partial^3 u}{\partial x^3} + O(h^4) \tag{7c}$$

All the derivatives are evaluated at i and j, so that $\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_i^j$ etc.

Using equations (7b) and (7c), one can write:

$$\left(u_{i+1}^j\right)^2 - \left(u_{i-1}^j\right)^2 = \left(u_{i+1}^j - u_{i-1}^j\right)\left(u_{i+1}^j + u_{i-1}^j\right) = 4hu_j^i\frac{\partial u}{\partial x} + O(h^3) \tag{8a}$$

and

$$u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j = 2h^3\frac{\partial^3 u}{\partial x^3} + O(h^3) \tag{8b}$$

Then, substituing equations (7a), (8a) and (8b) into (6), one can retrieve the modified differential equation:

$$\frac{\partial u}{\partial t} + O(\Delta t) = u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} + O(h^2), \tag{9}$$

which clearly reduces to the KdeV equation when $h, \Delta t \to 0$. Therefore the proposed scheme is consistent with the equation it is trying to solve.

2.1.2 Stability - von Neumann Analysis

The stability of the proposed discretisation scheme in the limit of small u can be tested using the so-called von Neumann (Fourier) method, which checks whether the scheme causes a Fourier mode to grow exponentially in amplitude [5]. A Fourier mode is substituted for $u(x,t)$ as follows:

$$u_i^j = e^{i(kx_i - \omega t^j)} = g^j e^{ikx_i}, \tag{10}$$

where g is the amplification factor and g^j signifies g raised to the power j [5]. The relations between u_i^j and adjacent points on the grid are:

$$u_i^{j+1} = g u_i^j, \qquad u_{i\pm1}^j = u_i^j e^{\pm ikh} \qquad \text{and} \qquad u_{i\pm2}^j = u_i^j e^{\pm 2ikh}.$$

Substituting into (6) yields (the limit of small u is considered here, so the terms of order u^2 can be ignored):

$$\begin{aligned} g u_i^j &= u_i^j - \frac{\Delta t}{2h^3}\left[u_i^j e^{2ikh} - 2u_i^j e^{ikh} + 2u_i^j e^{-ikh} - u_i^j e^{-2ikh}\right] \\ g &= 1 - \frac{\Delta t}{2h^3}\left[e^{2ikh} - e^{-2ikh} - 2\left(e^{ikh} + e^{-ikh}\right)\right] \\ g &= 1 - i\frac{\Delta t}{h^3}\left[\sin(2kh) - 2\sin(kh)\right] \end{aligned}$$

Therefore, taking the square of the modulus of g:

$$|g|^2 = 1 + \frac{\Delta t^2}{h^6}\left[\sin(2kh) - 2\sin(kh)\right]^2 \tag{11}$$

The condition for stability is $|g| \leq 1$, which implies $|g|^2 \leq 1$. However, $[\sin(2kh) - 2\sin(kh)]^2 \geq 0$ and $\Delta t, h > 0$. Therefore, from equation 11, one can deduce that $|g|^2 \geq 1$ for all values of $\Delta t, h$ and therefore the proposed scheme is unconditionally unstable.

2.2 Alternative Discretisation - Runge Kutta

Since the method proposed in §2.1 is unconditionally unstable, an alternative scheme for the time discretisation must be utilised. A fourth order Runge Kutta method is chosen for this purpose. A centred difference scheme is still used for the discretisation of the spatial derivatives. Therefore, the KdeV equation can be written as:

$$\left(\frac{\partial u}{\partial t}\right)_i^j = -\frac{1}{4h}\left[\left(u_{i+1}^j\right)^2 - \left(u_{i-1}^j\right)^2\right] - \frac{1}{2h^3}\left[u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j\right], \tag{12}$$

or, more generally,

$$\frac{\partial \boldsymbol{u}^j}{\partial t} = \boldsymbol{F}\left(\boldsymbol{u}^j\right), \tag{13}$$

where: $\boldsymbol{u}^j = \left(u_0^j, u_1^j, \ldots, u_N^j\right)$

and $\boldsymbol{F}\left(\boldsymbol{u}^j\right) = -\frac{1}{4h}\left[\left(u_{i+1}^j\right)^2 - \left(u_{i-1}^j\right)^2\right] - \frac{1}{2h^3}\left[u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j\right].$

Therefore, the fourth order Runge Kutta scheme for the KdeV equation can be written as:

$$\boldsymbol{u}^{j+1} = \boldsymbol{u}^j + \frac{1}{6}\left(\boldsymbol{k}_1 + \boldsymbol{2k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4\right), \tag{14}$$

where:

$$\begin{aligned} \boldsymbol{k}_1 &= \Delta t \boldsymbol{F}\left(\boldsymbol{u}^j\right), \\ \boldsymbol{k}_2 &= \Delta t \boldsymbol{F}\left(\boldsymbol{u}^j + \frac{1}{2}\boldsymbol{k}_1\right), \end{aligned}$$

$$\boldsymbol{k}_3 = \Delta t\boldsymbol{F}\left(\boldsymbol{u}^j + \frac{1}{2}\boldsymbol{k}_2\right) \text{ and}$$
$$\boldsymbol{k}_4 = \Delta t\boldsymbol{F}\left(\boldsymbol{u}^j + \boldsymbol{k}_3\right).$$

This is the scheme used to discretise the KdeV equation and propagate the initial waveforms in time for the whole project.

2.3 Stability Arguments

2.3.1 h and α

The pulse is characterised physically by its "soliton parameter", α, as seen in equation (2). By looking at equation (2), one can immediately deduce that this parameter controls the amplitude as well as the speed of the travelling pulse. As a result, the choice of value of parameter α has implications on the choice of step sizes h and Δt.

First, the effect of amplitude across the spatial domain is considered. Since the pulse is divided into a spatial grid, one condition that should be fulfilled is that the discretised pulse is sufficiently "smooth". In other words, the spatial step size h should be sufficiently small so that the amplitude of the wave varies slowly as it moves to adjacent points on the spatial grid. An approximate relationship between h and α can therefore be found by dictating an arbitrary tolerance for how much the amplitude is allowed to vary from one grid point to the next. First, from equation (2), one deduces the following relation between the amplitude, A, and x at a set time t:

$$A \propto \operatorname{sech}^2(\alpha x) = \left(\frac{2}{e^{\alpha x} + e^{-\alpha x}}\right)^2 = 4\left(\frac{2}{e^{2\alpha x} + 2 + e^{-2\alpha x}}\right)$$

If large values of x are considered, this can be approximated as:

$$A \propto \frac{1}{e^{2\alpha x}} \tag{15}$$

Then, it is decided that the difference in amplitude at one grid point, at x, and its adjacent grid point, at $x + h$, should not exceed an arbitrary tolerance of 10%. This can be expressed as:

$$\left|\frac{A(x+h) - A(x)}{A(x)}\right| < 0.1$$

Using the expression in (15) yields:

$$\left|e^{-2\alpha h} - 1\right| < 1,$$

which leads to the following approximate expression relating h and α:

$$h < \frac{\ln\left(\frac{10}{9}\right)}{2\alpha}. \tag{16}$$

This approximate relation can therefore be used to find an appropriate step size h, given a value of the soliton paramater α.

2.3.2 h and t

As concluded in the previous section, specifying a wave amplitude or velocity, or in other words a value of α, dictates the permissible values of h. Similarly, one would expect that a choice of α and/or h would in turn dictate what values of Δt are allowed. An approximate expression for Δt is found by imposing a simple condition that the amplitude of a wave should not increase exponentially as it is propagated through time. By examining the relation between amplitudes of the pulse at adjacent points in the temporal grid discretisation scheme in equations (12) and (14), it is noted that there are two terms that contribute to the change in amplitude.

The first term is , the discretisation of the non-linear term in the KdeV equation and is of the form $-\frac{\Delta t}{4h}\left[\left(u_{i+1}^{j}\right)^2 - \left(u_{i-1}^{j}\right)^2\right]$. The second term is the discretisation of the dispersion term in the KdeV equation and is of the form $-\frac{\Delta t}{2h^3}\left[u_{i+2}^{j} - 2u_{i+1}^{j} + 2u_{i-1}^{j} - u_{i-2}^{j}\right]$. The first term is proportional to $\frac{1}{h}$, while the second term is proportional to $\frac{1}{h^3}$. Since h will always be small (as dictated by the value of α), the magnitude of the first term is negligible compared to the magnitude of the second term, so only the second term is considered.

The simple condition is that the change in amplitude from one point on the temporal grid to the next should be very small. Therefore, a simple relation that can be used as a guideline for choosing an appropriate value of Δt is:

$$\frac{\Delta t}{2h^3} \ll 1 \, . \tag{17}$$

If this condition is not satisfied, then the amplitude change due to the non-dispersive term would increase multiplicatively with each time step, purely due to act of discretising the time domain. This would make the amplitude rise exponentially, thereby making the method unstable. If the condition is satisfied, the change in amplitude caused by the discretisation of time should stay sufficiently small.

Naturally, neither of the equations (16) and (17) is an exact or strict condition for stability, they merely provide guidelines for the choice of values of h and Δt as well as providing some insight and intuition into the factors affecting the stability of the numerical methods for this physical problem.

3 Dynamics

3.1 Single Soliton

The normal mode solution given by equation (2) outlined in §1.2 is used to initialise a single pulse and the scheme described in §2.2 is used to propagate it in time, for different values of α, h and Δt and with periodic boundary conditions. This section displays and describes the results of this investigation through Figures 1-6. Note that the KdeV equation solved in this project is already expressed in reduced, dimensionless units. Therefore, the units of the u, x and t axes in all the figures in this report are arbitrary.

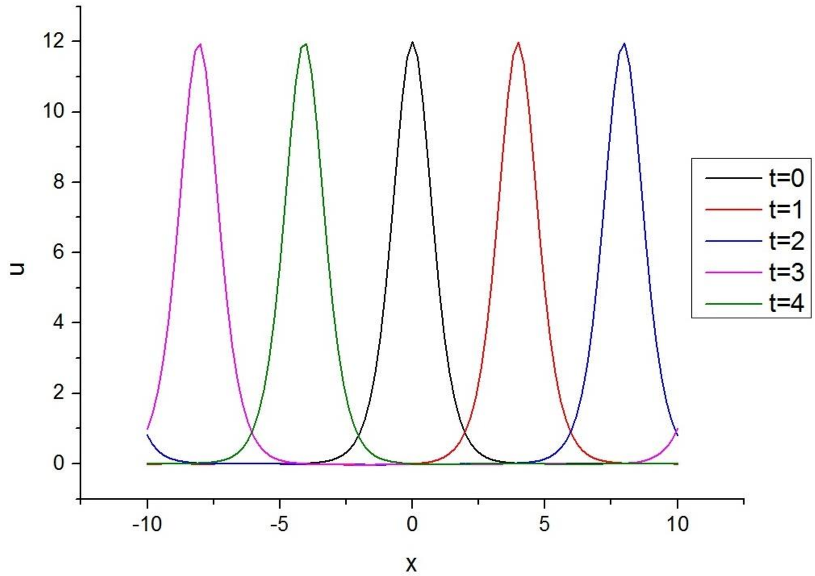

Figure 1: A plot of $u(x)$ on the interval $x \in [-10,10]$ for five different values of t. $\alpha = 1$, $h = 0.2$, $\Delta t = 0.001$. It can be seen that the pulse retains its shape for one whole transit across the spatial domain. The effect of the periodic boundary conditions can also be observed in this figure.

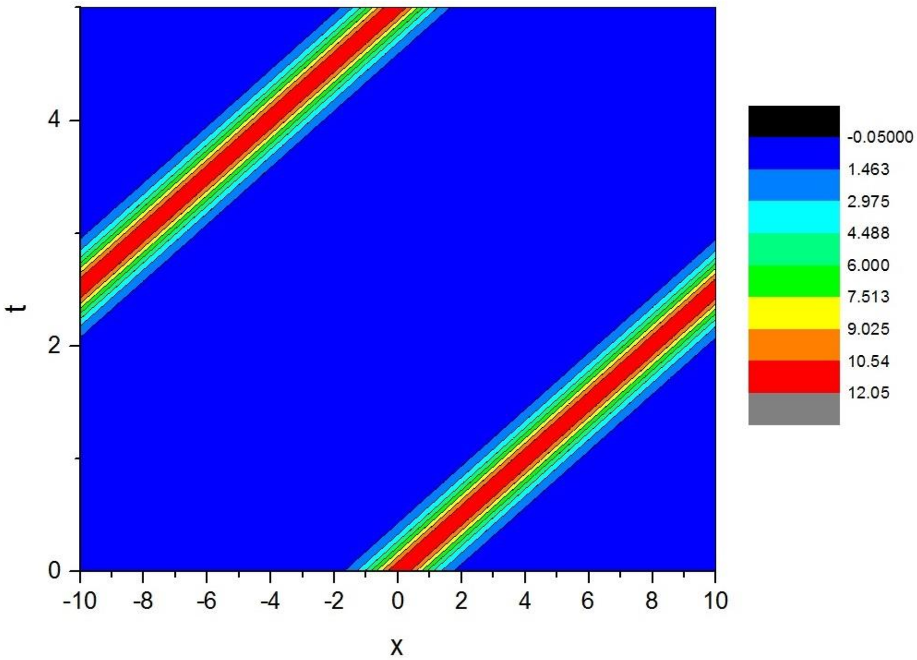

Figure 2: The same pulse as in Figure 1 plotted on a spacetime diagram on the intervals $x \in [-10,10]$ and $t \in [0,5]$. The preservation of the height and width of the pulse, as well as the periodic boundary conditions, can be observed in this plot. The height, $u(x)$, is encoded in the colour, as described in the key.

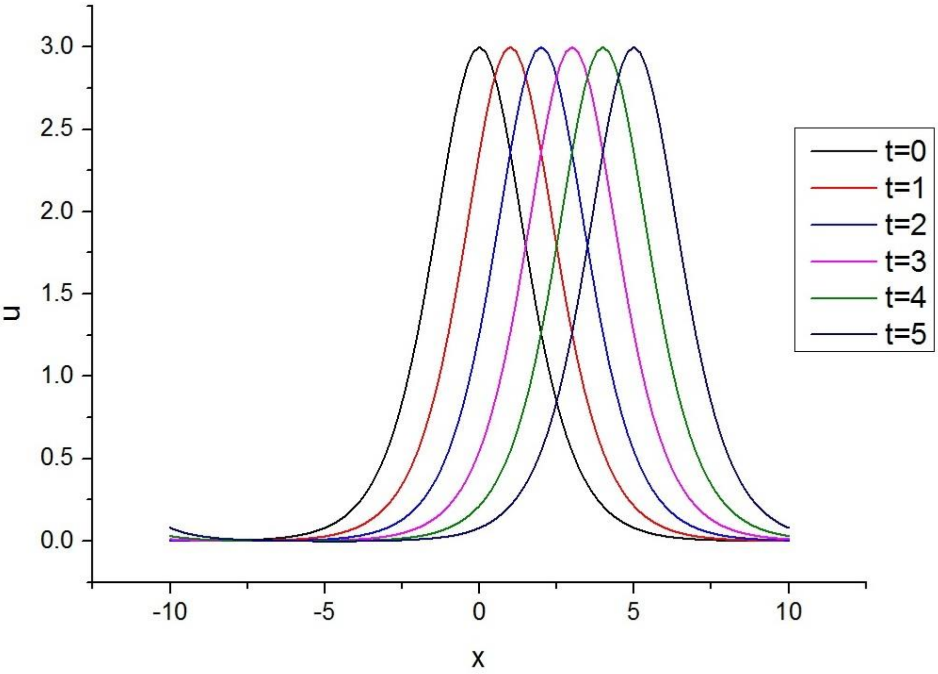

Figure 3: A plot of $u(x)$ *on the interval* $x \in [-10,10]$ *for six different values of* t. $\alpha = 0.5$, $h = 0.2$, $\Delta t = 0.001$.

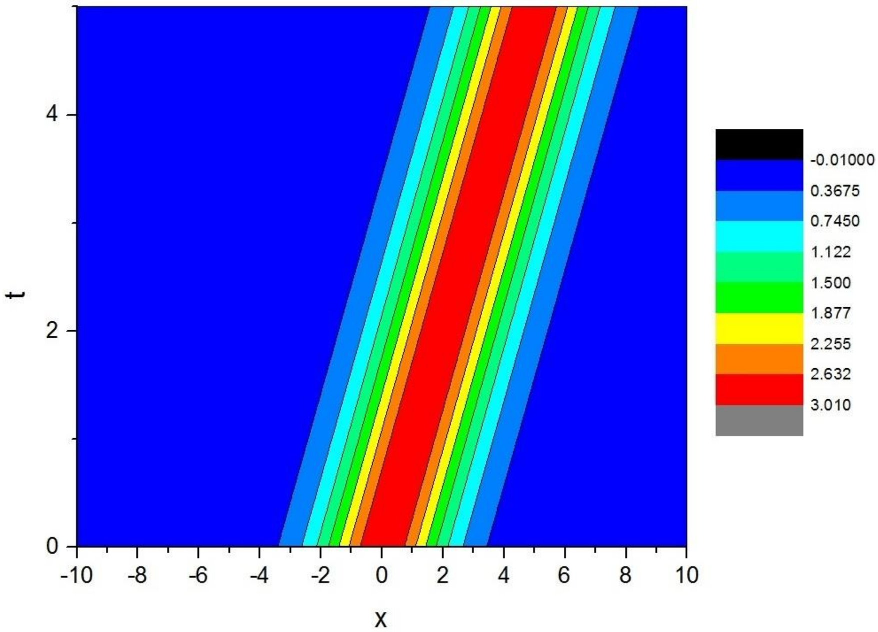

Figure 4: The same pulse as in Figure 3 plotted on a spacetime diagram on the intervals $x \in [-10,10]$ *and* $t \in [0,5]$. *The height,* $u(x)$, *is encoded in the colour, as described in the key.*

Comparing Figures 1-2 with Figures 3-4, which all have the same values of h and Δt, but show pulses with two different values of α, one can conclude that a taller pulse (one with a larger α) travels faster than a smaller one. Therefore, the speed and amplitue of a soliton are inter-dependent and are both described by the soliton parameter α.

One can verify numerically the validity of the statements made in §2.3 by comparison to the solution in Figures 1 and 2, which is evidently stable in the given space and time domains. First, by keeping α and Δt the same as in Figures 1 and 2, but increasing h, one can verify the validity of the statement expressed in equation (16) in §2.3.1. However, the value of h used in plotting Figures 1 and 2 is already above the critical value of h predicted by (16). Therefore, a larger value is chosen until the pulse is observed to no longer be "smooth". Then, by keeping α and h the same as in Figures 1 and 2, but increasing Δt, one can verify the validity of the statement expressed in equation (17) in §2.3.2. This is shown in Figures 5 and 6 below:

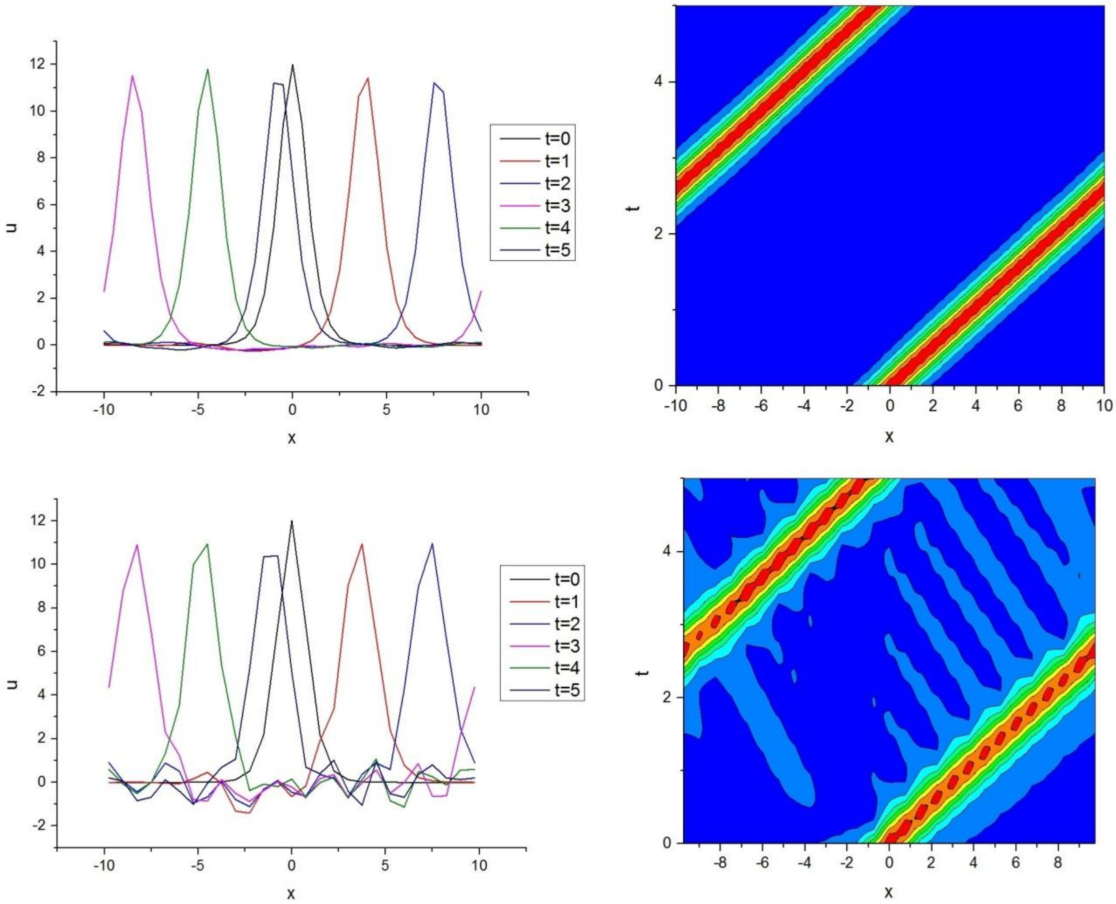

Figure 5: Top: $\alpha = 1,\ \Delta t = 0.001,\ h = 0.5$; bottom: $\alpha = 1,\ \Delta t = 0.001,\ h = 0.75$. Plots of $u(x)$ for different values of t (left) as well as spacetime diagrams (right) for two different values of h. For $h = 0.5$, one can already begin to see that the pulse does not maintain its shape completely, while for $h = 0.75$, the shape is definitely not maintained as the pulse evolves in time. Furthermore, the pulse is visibly too discrete, as is evident from the sharpness of the peaks.

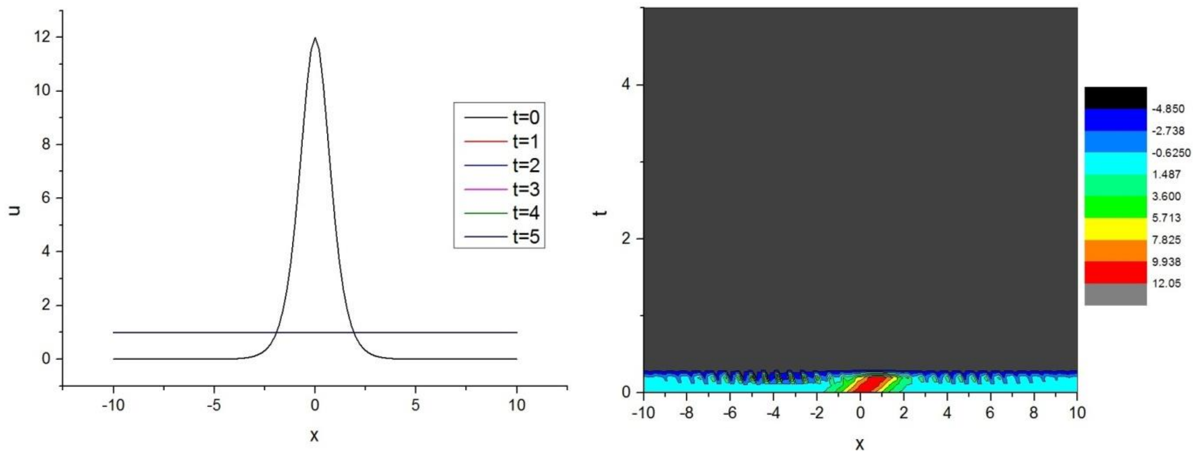

Figure 6: Plot of $u(x)$ for different values of t (left) as well as a spacetime diagram (right) for $\alpha = 1$, $h = 0.2$, $\Delta t = 0.01$. The solution becomes unstable almost immediately, as the amplitude of the pulse already exceeds the maximum value of the variable type double at $t = 1$ (hence the plot on the left shows $u = 0$ for all t) .

Figure 5 does not confirm equation (16) exactly, as there is a numerical discrepancy between the values determined numerically and the values set by the arbitrary tolerance. However, it does confirm the general idea, which is that choosing a value of α sets an upper limit on the value of h. Figure 6 confirms equation (17) as, even though $\Delta t = 0.01$ is less than $2h^3 = 0.016$, the solution becomes unstable almost immediately.

3.2 Collisions of Solitons

As the KdeV equation is non-linear, one does not expect simple behaviour from a collision of two solitons, as there will be no linear superposition. This section investigates the behaviour of two solitons undergoing a collision by initialising two solitons of different heights (and, equivalently, speeds) and propagating them using the same discretisation scheme as before. This is done for two pairs of solitons - ones with very different speeds and ones with slightly different speeds.

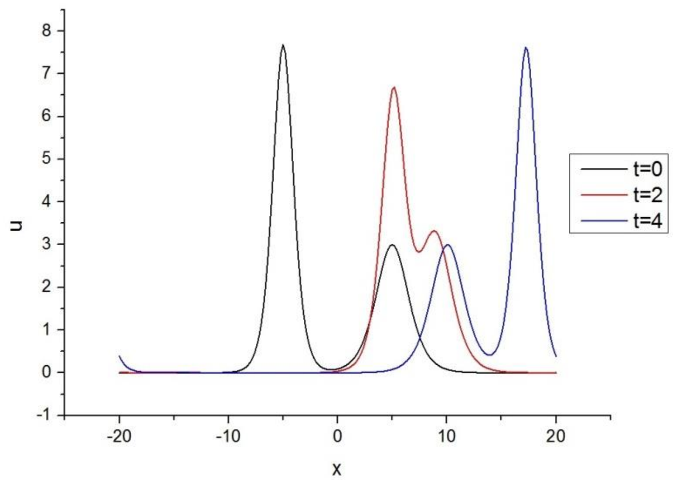

Figure 7: Plot of $u(x)$ against x for 3 different values of t, showing a collision between two solitons in the interval $x \in [-10,10]$. The faster soliton starts at $x = -5$ and has $\alpha = 0.8$. The slower soliton starts at $x = 5$ and has $\alpha = 0.5$. $h = 0.2$, $\Delta t = 0.001$.

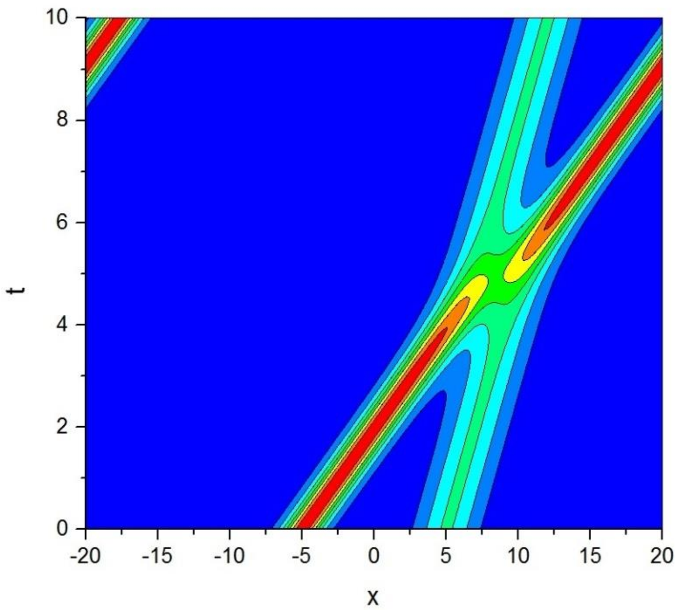

Figure 8: A spacetime diagram of the collision in Figure 7. The faster pulse catches up with the slower pulse and overtakes it. In this diagram, it can be seen that both solitons undergo a phase shift at the point of collision. The fast wave is moved further along, while the slower wave is set back.

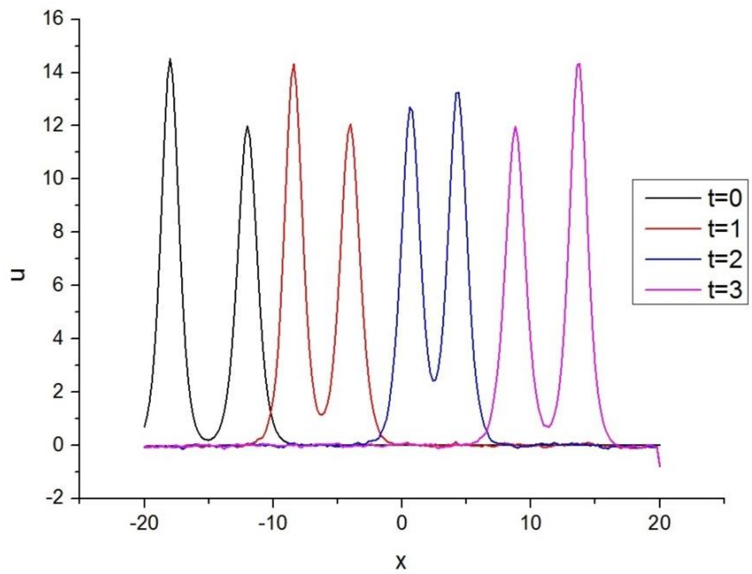

Figure 9: Plot of $u(x)$ against x for 4 different values of t, showing a collision between two solitons in the intervals $x \in [-10,10]$. The faster soliton starts at $x = -18$ and has $\alpha = 1.1$. The slower soliton starts at $x = -12$ and has $\alpha = 1$. $h = 0.2$, $\Delta t = 0.001$.

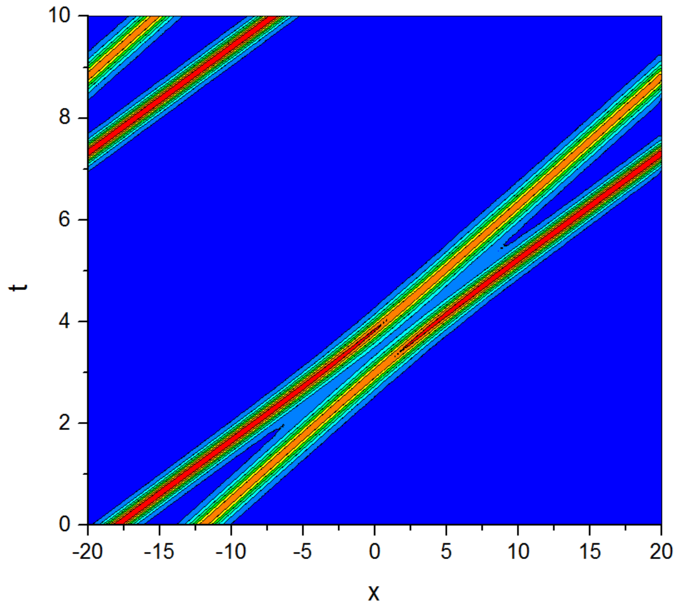

Figure 10: A spacetime diagram of the collision in Figure 9. The same phenomenon that was observed in Figures 7 and 8 is observed here, although it is less obvious that the faster wave overtakes the slower wave, as there is no direct intersection between the crests on the diagram.

As expected, the amplitudes of colliding solitons do not add linearly and height is not conserved. A faster soliton will catch up and overtake a slower soliton and they will both experience a phase change during the collision, with the faster wave pushed even further forwards and the slower wave set further back. At the moment of collision, waves of very different speeds will form a singular crest. This is not observed for waves of only slightly different speeds.

3.3 Wave Breaking

In the analysis so far, only normal mode solutions to the KdeV equation have been considered. In this section, a different initial waveform is studied. In particular, an initial solution of the form:

$$u(x) = 12\alpha^2 \cos(Kx), \tag{18}$$

where the factor of $12\alpha^2$ ensures the same amplitude as the normal mode solution, and K is the wavenumber of the solution, such that the period in the spatial domain (the wavelength) is $\frac{2\pi}{K}$.

Initial solutions with different wavelengths relative to their amplitudes are investigated and the phenomenon of so-called "wave breaking", wherein an initial oscillatory solution breaks up into multiple pulses as it propagates in time [6], is studied. $\alpha = 1$, $h = 0.2$ and $\Delta t = 0.001$ for all plots in this section and the domains considered are $x \in [-10,10]$ and $t \in [0,20]$. Since α is kept constant, the amplitude is also constant and equal to 12. Therefore, the only variable in this section is the wavenumber K.

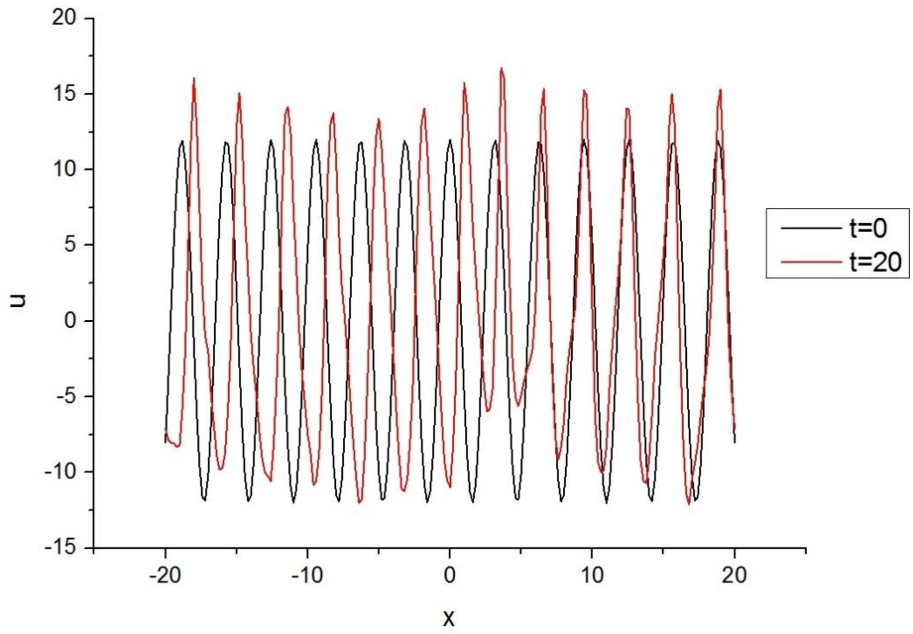

Figure 11: Plot of $u(x)$ against x for 2 different values of t. $K = 2$ and therefore the wavelength is very small compared to the amplitude. No wave breaking is observed.

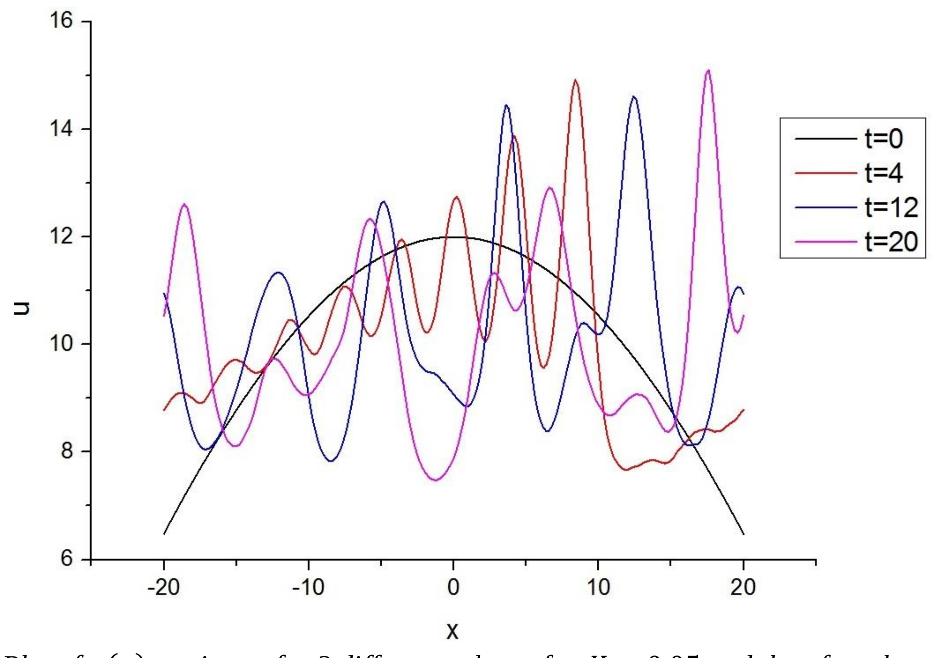

Figure 12: Plot of $u(x)$ against x for 2 different values of t. $K = 0.05$ and therefore the wavelength is very large compared to the amplitude. Wave breaking is observed within the given time period - it can be seen how the oscillatory waveform decomposes into multiple pulses as it is propagated in time.

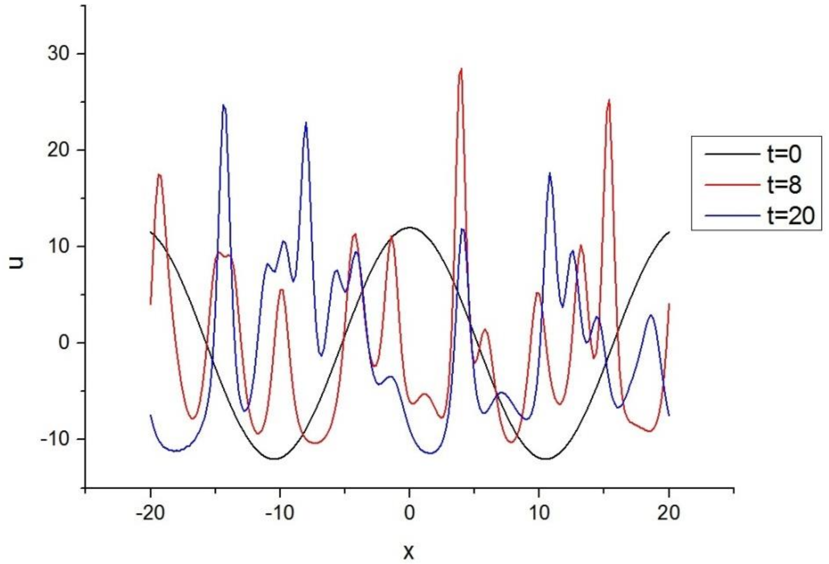

Figure 12: Plot of $u(x)$ against x for 3 different values of t. $K = 0.3$ and therefore the wavelength is approximately equal to the amplitude. Wave breaking is not yet observed within the given time period, as there is still a very dominant oscillatory portion of the final waveforms.

From this investigation, it is clear that the KdeV equation evolves oscillatory initial conditions such that they decompose into multiple pulses that resemble normal mode solitons as they evolve in time. How soon they decompose depends on the ratio of the wavelength and amplitude of the initial wavefor, with wave breaking occuring sooner the larger the wavelength in comparison to the amplitude.

3.4 Shock Waves

The KdeV equation can propagate stable solutions over extended periods of time because its non-linearity is exactly cancelled out by its dispersion. If the dispersive term in the KdeV equation is removed, one obtains what is known as a "shock wave" [5]. In this section, a normal mode solution to the KdeV equation is initialised and propagated as a shock wave in order to study the behaviour of these waves.

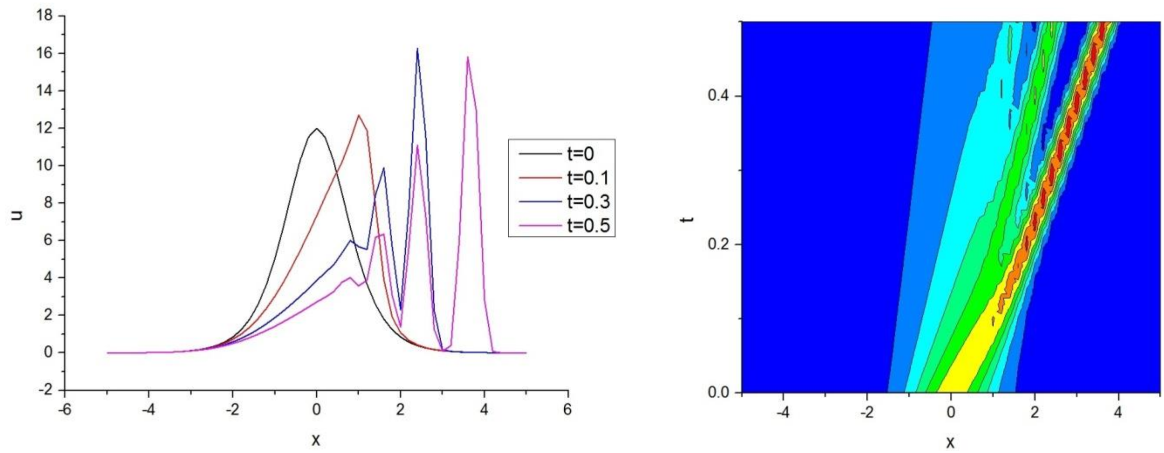

Figure 13: Plot of $u(x)$ against x for 4 different values of t (left) and spacetime diagram (right) of a shockwave. $x \in [-5,5]$ and $t \in [0,1]$. $\alpha = 1$, $h = 0.2$ and $\Delta t = 0.001$.

As can be seen in Figure 13, the shockwave propagates a very short distance from its starting point, as the solution becomes unstable rapidly. A potential way of alleviating this is by adding a diffusive term into the equation i.e. a term that is $\propto \frac{\partial^2 u}{\partial x^2}$.

4 Conclusion

The Korteweg de Vries equation, which describes the behavious of solitary waves, or solitons, was successfully discretised in space and propagated in time numerically using a fourth order Runge-Kutta scheme. By changing the initial conditions, a number of properties of solitons and their interactions, as well as properties of the numerical method, were observed and studied.

The inter-dependence of a solitons height and velocity was observed when a single normal mode was propagated and it was found that taller solitons always travel faster. In addition, using the single normal mode, the stability conditions for the discretisation method were verified numerically and were found to agree with the theoretical predictions. It was found that solitons do not add linearly when they collide and undergo a phase shift upon collision, causing the faster one to overtake the slower one. It was found that an oscillatory solution decomposes into multiple solitons of varying amplitudes, depending on the ratio between the initial solutions wavelength and amplitude. Finally, it was found that, by removing the dispersive term in the KdeV equation, one can obtain a simple model for shock waves.

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