MATH1231 Algebra Summary

Subspaces

A subspace S is only a subspace of V over a field $\mathbb{F}$ if:

- The zero vector $\vec{0}$ belongs to S
- S is closed under vector addition
- S is closed under scalar multiplication

Linear Combinations and Spans

Given a set of vectors $S = \{v_1, v_1 \ldots v_n\}$:

- A **linear combination** of S is a sum of scalar multiples of the form $\lambda_1 v_1 + \ldots \lambda_n v_n$
- A **span** of S is the set of all linear combinations of S
- A **spanning set** is the finite set S such that every vector in a vector space V can be expressed as a linear combination of vectors in S

Linear Independence

Linearly Independent

A set of vectors S is said to be linearly independent if for each vector in S, the only values of the scalars $\lambda_1, \lambda_2 \ldots \lambda_n$ for which $\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0$, are 0.

Linearly Dependent

A set of vectors S is said to be linearly dependent if it is not a linearly independent set. In other words, there exists a λ_n that is not 0.

Basis and Dimension

A set of vectors S is a basis for a vector space V if:

- S is a linearly independent set
- S is a spanning set for V

Linear Maps

A function $T : V \to W$ is called a **linear transformation** if:

- Addition is preserved: $T(\mathbf{v} + \mathbf{u}) = T(\mathbf{v}) + T(\mathbf{u})$
- Scalar multiplication is preserved: $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$

Linear Map Subspaces

Kernel

Definition: $ker(A) = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = 0\}$
Calculate: row reduce A, solve for λ's and form a span
Basis: form a set using the smallest number of linearly independent vectors from $ker(A)$

Nullity

Definition: $nullity(A) = dim(ker(A))$
Calculate: find $ker(A)$ and take its dimension OR take $dim(A) - rank(A)$

Image

Definition: $im(A) = \{\mathbf{b} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$
Calculate: row reduce $(A|b)$
Basis: row reduce, find the linearly independent columns and take those columns from the original matrix as a span

Rank

Definition: $rank(A) = dim(im(A))$
Calculate: find $im(A)$ and take its dimension OR row reduce A and take the number of leading columns

Eigenvalues and Eiegnvectors

Eigenvalue

Definition: λ's such that $A\mathbf{v} = \lambda\mathbf{v}$
Calculate: solve $det(A - \lambda I) = 0$ where I is the identity matrix for A

Eigenvector

Definition: $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue λ
Calculate: solve $ker(A - \lambda I)$ for each eigenvalue λ

Diagonalisation

A square matrix A is said to be diagonalisable if there exists an invertible matrix M and a diagonal matrix D such that $M^{-1}AM = D$.

Given n linearly independent eigenvectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$, we let

$$M = (\mathbf{v_1}|\ldots|\mathbf{v_n}), \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

such that $M^{-1}AM = D$ holds.

Systems of Differential Equations

The system $\begin{cases} \frac{dx}{dt} = a_1 y_1 + b_1 y_2 \\ \frac{dy}{dt} = a_2 y_2 + b_2 y_2 \end{cases}$ can be written as $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ where $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \mathbf{v_1} e^{\lambda_1 t} + c_2 \mathbf{v_2} e^{\lambda_2 t}$ given that $\lambda_k, \mathbf{v_k}$ are eigenvalue-eigenvector pairs.

Probability

Rules and Conditions

1. $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
2. $P(A^c) = 1 - P(A)$
3. $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$
4. Mutual exclusion if $A \cap B = \emptyset$
5. Statistically independent if $P(A \cap B) = P(A) \times P(B)$

Random Variables

Probability Distribution

To show a sequence p_k is a probability distribution the following properties must be proven

1. $p_k \geq 0$
2. $\sum_k^\infty p_k = 1$

Expected Value

$E(X) = \sum_{\text{all k}} kp_k$
$E(X^2) = \sum_{\text{all k}} k^2 p_k$

Variance

$Var(X) = E(X^2) - E(X)^2$

Standard Deviation

$SD(X) = \sqrt{Var(X)}$

Special Distributions

Binomial Distribution

$B(n, p, k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}$ where $k = 0, 1, \ldots n$

Geometric Distribution

$G(p, k) = (1-p)^{k-1} p$

Continuous Random Variables

A random variable X is continuous if $F_X(x)$ is continuous.

Probability Density Function

The probability density function of a continuous random variable X is defined by

$$f(x) = f_X(x) = \frac{d}{dx} F(x) \ , \ x \in \mathbb{R}$$

if $F(x)$ is differentiable, and $\lim_{x \to a^-} \dfrac{d}{dx} F(x)$ if $F(x)$ is not differentiable at $x = a$.

Expected Value

$E(X) = \displaystyle\int_{-\infty}^{\infty} x \ f(x) \ dx$

Variance

$Var(X) = E(X^2) - (E(X))^2$

Special Continuous Distributions

Normal Distribution

A continuous random variable X has a normal distribution $N(\mu, \sigma^2)$ if it has a probability density $\phi(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$ where $-\infty < x < \infty$.

Exponential Distribution

A continuous random variable T has an exponential distribution $Exp(\lambda)$ if it has a probability distribution density

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if t} \geq 0 \\ 0 & \text{if t} < 0 \end{cases}$$