MATH1231 Calculus Summary

Partial Differentiation

To find the partial derivative of a function with two variables x and y, we can treat one of the variables as a constant and differentiate with respect to the other.

Tangent Plane To Surfaces

Suppose F is a function of two variables, and P is a point (x_0, y_0, z_0) that lies on the surface $z = F(x, y)$.

Tangent Plane Of Surface

$z = z_0 + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$

Normal Vector To Surface

$\begin{pmatrix} F_x(x_0, y_0) \\ F_y(x_0, y_0) \\ -1 \end{pmatrix}$

Total Differential Approximation

$\triangle F \approx F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$

Chain Rule

For a function F with two variables x and y, the chain rule can be defined as

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

Don't forget to substitute in x and y after finding the derivatives.

Functions Of Two Or More Variables

Partial Derivatives

For a function F of three variables x, y and z, the partial derivatives of F can be defined as

$$F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}, \quad F_z = \frac{\partial F}{\partial z}$$

Chain Rule

For a function F of three variables x, y and z, where x and y are each functions of both u and v, the chain rule for F can be defined as

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial x}{\partial v}$$

Integration Techniques

Trigonometric Integrals

Considers integrals of the form

$$\int \cos^m x \ \sin^n x \ dx$$

Cases:

1. m or n or both are odd: $u = \sin x$, $du = \cos x \ dx$
2. m and n are even: $\cos^2 x = \dfrac{1 + \cos 2x}{2}$, $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

Ordinary Differential Equations

Seperable ODEs

Seperable ODEs are differential equations where two variables are involved (usually x and y) that can be seperated so that all the y's are on one side, and all the x's are on the other. The tend to be in the form

$$f(y)\frac{dy}{dx} = g(x)$$

To solve:

1. Move all the y's to one side, and the x's to the other

   So $f(y)\dfrac{dx}{dy} = g(x)$ becomes $f(y)dy = g(x)dx$

2. Integrate both sides with respect to the respective variable

   $\int f(y)dy = \int g(x)dx$

3. Solve for y

First Order Linear ODEs

First Order Linear ODEs are differential equations that involve functions of a single variable. They can be written in the form

$$\frac{dy}{dx} + f(x)y = g(x)$$

To solve:

1. Write the differential equation as above
2. Find the integrating factor $e^{\int f(x)dx}$, this is denoted by $h(x)$
3. Multiply both sides by the integrating factor $h(x)$ to obtain $\dfrac{d}{dx}(h(x)y) = g(x)h(x)$
4. Integrate both sides with respect to x, and solve for y

Exact ODEs

Exact ODEs are differentiable equations involving functions with two or more variables. They are typically of the form

$$F(x,y) + G(x,y)\frac{dy}{dx} = 0$$

and are said to be *exact* if

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$$

To solve:

1. Show that a differential equation is exact by proving the above property
2. Look for a function $H(x,y)$ such that

$$\frac{\partial H}{\partial x} = F(x,y) \qquad (1)$$

$$\frac{\partial H}{\partial y} = G(x,y) \qquad (2)$$

3. Integrate (1) with respect to x to find $H(x,y) = f(x,y) + C(y)$
4. To find $C(y)$, partially differentiate $H(x,y)$ with respect to y (leaving all of the x components constant) and compare that with the partial derivative of H with respect to y. This gives $C'(y)$ and thus allows to find $C(y)$

Second Order Linear ODEs

A second order linear ODE with constant coefficients is said to be homogeneous if it is of the form

$$y'' + ay' + by = 0$$

where a and b are real numbers.

Characteristic Equation

The characteristic equation of a second order linear ODE is given by

$$\lambda^2 + a\lambda + b = 0$$

Taylor Polynomial

For a differentiable function f, the Taylor polynomial p_n of order n at $x = a$ is

$$p_n(x) = f(a) = f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor's Theorem

$$f(x) = p_n(x) + R_{n+1}(x)$$

where

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Sequences

When evaluating limits, functions and sequences are identical. This is shown below

$$\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L$$

A sequence diverges when $\lim_{n\to\infty} a_n \pm \infty$ or $\lim_{n\to\infty} a_n$ does not exist. Otherwise, the sequence converges.

Properties Of Sequences

Given a sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the following properties hold

- *increasing* if $a_n < a_{n+1}$ for each $n \in \mathbb{N}$
- *non-decreasing* if $a_n \leq a_{n+1}$ for each $n \in \mathbb{N}$

- *decreasing* if $a_n > a_{n+1}$ for each $n \in \mathbb{N}$
- *non-increasing* if $a_n \geq a_{n+1}$ for each $n \in \mathbb{N}$
- M is a *upper bound* if $a_n \leq M$ for each $n \in \mathbb{N}$
- M is a *lower bound* if $a_n \geq M$ for each $n \in \mathbb{N}$

Infinite Series

The kth Term Divergence Test

$\sum_{k=1}^{\infty} a_k$ diverges if $\lim_{n\to\infty} a_k$ fails to exist, or is non-zero.

Integral Test

Comparison Test

Suppose that $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ are two positive sequences such that $ak \leq bk$ for every natural number k.

- If $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges
- If $\sum_{k=0}^{\infty} b_k$ diverges, then $\sum_{k=0}^{\infty} a_k$ diverges

Usually used for series of the form $\sum_{k=1}^{\infty} \dfrac{1}{k^p}$, such that this series converges if $p > 1$ and diverges if $p \leq 1$.

Ratio Test

Suppose that $\sum a_k$ is an infinite series with positive terms and that $\lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} = r$.

- If $r < 1$ then $\sum a_k$ converges
- If $r > 1$ then $\sum a_k$ diverges
- If $r = 1$ this test is inconclusive

Alternating Series Test

Taylor Series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Power Series

Given a sequence $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers and that $a \in \mathbb{R}$, then

$$\sum_{k=0}^{\infty} a_k x^k$$

is the power series of x, and

$$\sum_{k=0}^{\infty} a_k (x-a)^k$$

is a power series of $x - a$.