Mathematics 1 - Linear Agebra

Lecture 05 – §3.4 Solution of linear systems of equations

Sabine Le Borne

Systems of linear equations

Example (2×2 LES)

We consider the 2×2 LES

$$\begin{array}{r|rcrcr|}
\cline{2-6}
G_1\colon & x_1 & + & 3x_2 & = & 7 \\
G_2\colon & 2x_1 & - & x_2 & = & 0 \\
\cline{2-6}
\end{array}$$

Elimination of x_1: Subtract twice of G_1 from G_2.

$$\begin{array}{r|rcrcr|l}
\cline{2-6}
G_1\colon & x_1 & + & 3x_2 & = & 7 & \\
G_2\colon & 2x_1 & - & x_2 & = & 0 & -2\cdot G_1 \\
\cline{2-6}
\end{array}
\quad
\begin{array}{c} \rightsquigarrow \\ \rightsquigarrow \end{array}
\quad
\begin{array}{r|rcrcr|}
\cline{2-6}
G_1' := G_1\colon & x_1 & + & 3x_2 & = & 7 \\
G_2' := G_2 - 2\cdot G_1\colon & \boxed{} & - & 7x_2 & = & -14 \\
\cline{2-6}
\end{array}$$

Backward substitution: Solve the second equation for x_2, then the first for x_1.

$$\begin{array}{r|rcrcr|}
\cline{2-6}
G_1'\colon & x_1 & + & 3x_2 & = & 7 \\
G_2'\colon & \boxed{} & - & 7x_2 & = & -14 \\
\cline{2-6}
\end{array}
\quad
\begin{array}{c} \rightsquigarrow \\ \rightsquigarrow \end{array}
\quad
\begin{array}{r|lccl|}
\cline{2-5}
G_1'' := G_1' - 3x_2\colon & x_1 & & = & 7 - 3x_2 = 1 \\
G_2'' := -\frac{1}{7}G_2'\colon & & x_2 & = & 2 \\
\cline{2-5}
\end{array}$$

Solution of systems of linear equations

Example (3×3 LES)

We consider the 3×3 LES

$$
\begin{array}{r|rcrcrcr|l}
G_1: & 2x_1 & + & 3x_2 & - & x_3 & = & 4 & \\
G_2: & 2x_1 & - & x_2 & + & 7x_3 & = & 0 & -1 \cdot G_1 \\
G_3: & 6x_1 & + & 13x_2 & - & 4x_3 & = & 9 & -3 \cdot G_1
\end{array}
$$

Elimination of x_1: Subtract a multiple of G_1 from G_2 and G_3, resp..

$$
\begin{array}{r|rcrcrcr|l}
G_1': & 2x_1 & + & 3x_2 & - & x_3 & = & 4 & \\
G_2': & & - & 4x_2 & + & 8x_3 & = & -4 & \\
G_3': & & + & 4x_2 & - & x_3 & = & -3 & +1 \cdot G_2'
\end{array}
$$

Elimination of x_2: Subtract a multiple of G_2' from G_3'.

$$
\begin{array}{r|rcrcrcr|}
G_1'': & 2x_1 & + & 3x_2 & - & x_3 & = & 4 \\
G_2'': & & - & 4x_2 & + & 8x_3 & = & -4 \\
G_3'': & & & & & 7x_3 & = & -7
\end{array}
$$

Backward substitution: Solve the third equation for x_3, the second for x_2 and then the first for x_1.

Solution of systems of linear equations

Generalization of the solution method to an $m \times n$ LES: Gauß algorithm

- Elimination of variables $\rightsquigarrow$ system in triangular form
  - Subtraction of the multiple of a row from another row
  - Swapping of rows

- Backward substitution
  - Solve the n-th equation for x_n.
  - When $x_{i+1}, \ldots, x_n$ are already known, then substitute them into the i-th equation and solve it for x_i (for $i = n-1, n-2, \ldots, 1$).

Next: Gauß algorithm in matrix form.

Solution of systems of linear equations

Definition 3.15 (extended system matrix $(\mathbf{A}|\mathbf{b})$)

We write the system of linear equations

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}$$

in short as

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right)$$

and call $(\mathbf{A}|\mathbf{b})$ the <u>*extended system matrix*</u>.

Solution of systems of linear equations

Review of the previous two examples: elimination to triangular form

$$\begin{array}{r} G_1: \\ G_2: \end{array} \left(\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & -1 & 0 \end{array}\right) \begin{array}{l} \\ -2\cdot G_1 \end{array} \rightsquigarrow \begin{array}{r} G_1': \\ G_2': \end{array} \left(\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -7 & -14 \end{array}\right)$$

$$\begin{array}{r} G_1: \\ G_2: \\ G_3: \end{array} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 2 & -1 & 7 & 0 \\ 6 & 13 & -4 & 9 \end{array}\right) \begin{array}{l} \\ -1\cdot G_1 \\ -3\cdot G_1 \end{array} \rightsquigarrow \begin{array}{r} G_1': \\ G_2': \\ G_3': \end{array} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 4 & -1 & -3 \end{array}\right) \begin{array}{l} \\ \\ +1\cdot G_2' \end{array} \rightsquigarrow \begin{array}{r} G_1'': \\ G_2'': \\ G_3'': \end{array} \left(\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & 7 & -7 \end{array}\right)$$

Solution of systems of linear equations

Example (4×5 LES)

$$
\begin{array}{r}
G_1\colon \\ G_2\colon \\ G_3\colon \\ G_4\colon
\end{array}
\left(\begin{array}{rrrrr|r}
0 & 0 & 2 & 3 & 12 & 10 \\
4 & 8 & 2 & 3 & 4 & 14 \\
1 & 2 & 0 & 1 & 0 & 3 \\
-3 & -6 & -6 & 8 & 4 & 4
\end{array}\right)
\quad \rightsquigarrow \quad
\begin{array}{r}
G_1'\colon \\ G_2'\colon \\ G_3'\colon \\ G_4'\colon
\end{array}
\left(\begin{array}{rrrrr|r}
1 & 2 & 0 & 1 & 0 & 3 \\
4 & 8 & 2 & 3 & 4 & 14 \\
0 & 0 & 2 & 3 & 12 & 10 \\
-3 & -6 & -6 & 8 & 4 & 4
\end{array}\right)
$$

$$
\begin{array}{r}
\\ G_1'\colon \\ G_2'\colon \\ G_3'\colon \\ G_4'\colon
\end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\
\left(\begin{array}{rrrrr|r}
1 & 2 & 0 & 1 & 0 & 3 \\
4 & 8 & 2 & 3 & 4 & 14 \\
0 & 0 & 2 & 3 & 12 & 10 \\
-3 & -6 & -6 & 8 & 4 & 4
\end{array}\right)
\end{array}
\begin{array}{l}
\\ \\ -4 \cdot G_1' \\ \\ +3 \cdot G_1'
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{r}
\\ G_1''\colon \\ G_2''\colon \\ G_3''\colon \\ G_4''\colon
\end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\
\left(\begin{array}{rrrrr|r}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 2 & 3 & 12 & 10 \\
0 & 0 & -6 & 11 & 4 & 13
\end{array}\right)
\end{array}
$$

Solution of systems of linear equations

Example (4×5 LES), continuation

$$
\begin{array}{r} \\ G_1'': \\ G_2'': \\ G_3'': \\ G_4'': \end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\
\left(\begin{array}{ccccc|c}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 2 & 3 & 12 & 10 \\
0 & 0 & -6 & 11 & 4 & 13
\end{array}\right)
\end{array}
\begin{array}{l} \\ \\ \\ -1\cdot G_2'' \\ +3\cdot G_2'' \end{array}
\rightsquigarrow
\begin{array}{r} \\ G_1''': \\ G_2''': \\ G_3''': \\ G_4''': \end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\
\left(\begin{array}{ccccc|c}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 0 & 4 & 8 & 8 \\
0 & 0 & 0 & 8 & 16 & 19
\end{array}\right)
\end{array}
$$

$$
\rightsquigarrow
\begin{array}{r} \\ G_1'''': \\ G_2'''': \\ G_3'''': \\ G_4'''': \end{array}
\begin{array}{c}
\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\
\left(\begin{array}{ccccc|c}
1 & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & 2 & -1 & 4 & 2 \\
0 & 0 & 0 & 4 & 8 & 8 \\
0 & 0 & 0 & 0 & 0 & 3
\end{array}\right)
\end{array}
$$

The resulting extended matrix is not quite triangular, but in so-called (row) echelon form.

Solution of systems of linear equations

Definition 3.17 (row echelon form, pivot element)

An (extended) matrix is given in *row echelon form* if we can draw a line from top left to right with steps going down such that

- each step is exactly one row high and at least one column wide,
- below the line there are only zeros,
- at the left end of each step there is a (nonzero) *pivot element* (above the line).

$$\left(\begin{array}{ccccc|c} \circledast & * & * & * & * & * \\ 0 & 0 & \circledast & * & * & * \\ 0 & 0 & 0 & \circledast & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{array}\right) \qquad \left(\begin{array}{cc|c} \circledast & * & * \\ 0 & \circledast & * \\ 0 & 0 & * \\ 0 & 0 & * \end{array}\right)$$

$\circledast$: nonzero pivot entries

$*$: arbitrary entries, zero or nonzero

The pivot elements are the entries that are used in the elimination step to zero out all entries below them. They must be nonzero.
The pivot elements of the 4×5 example were 1, 2 and 4.

Solution of systems of linear equations

Theorem 3.18 (admissible operations for row echelon form)

Every LES – whether square or rectangular, solvable or not solvable – can be transformed into row echelon form using the following admissible operations:

(i) subtract a multiple of one row from *another* row,
(ii) swap two rows.

The solution set does not change under these transformations. They are so-called equivalence transformations.

What is the benefit of the row echelon form?

- It tells us whether a solution exists.
- If yes:
  - It tells us whether the solution is unique.
  - It is a good starting point to determine the solution(s).

Solution of systems of linear equations

Our 4×5 example in row echelon form

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & -1 & 4 & 2 \\ 0 & 0 & 0 & \boxed{4} & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right) \end{array} \rightsquigarrow$$

equation of the last line

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 3$$

impossible/always wrong!

LES has no solution if a row $(0\ 0\ \ldots\ 0 \mid c)$ with $c \neq 0$ exists.

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & -1 & 4 & 2 \\ 0 & 0 & 0 & \boxed{4} & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array} \rightsquigarrow$$

Choose $x_5 = 0$. Then it follows that

$x_4 = 2,$

$x_3 = 2.$

Choose $x_2 = 0$. Then it follows that

$x_1 = 1.$

LES has infinitely many solutions (x_2, x_5 free to choose, determine x_1, x_3, x_4).

Solution of systems of linear equations

Theorem 3.19 (Solvability of an LES)

An LES is **unsolvable** if its row echelon form has a row $(0\ 0 \cdots 0 \mid c)$ with $c \neq 0$.
If this is not the case, then the system is **solvable**.

Definition 3.20 (free and dependent variables)

Variables that have a pivot element in their column are called *dependent variables*.
All other variables (columns without pivot elements) are called *free variables*.

In the example, x_1, x_3, x_4 are dependent, x_2, x_5 are free:

$$\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left(\begin{array}{ccccc|c}
\boxed{1} & 2 & 0 & 1 & 0 & 3 \\
0 & 0 & \boxed{2} & -1 & 4 & 2 \\
0 & 0 & 0 & \boxed{4} & 8 & 8
\end{array}\right)
\end{array}$$

Theorem (solutions of an LES)

If all variables of a solvable LES are dependent, then the solution is unique.
If a solvable LES has at least one free variable, then there are infinitely many solutions.

True or false?

1. Every linear system of equations can be transformed into a uniquely determined row echelon form using admissible transformations.
2. The following LES are in row echelon form: $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 4 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$, $\left(\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right)$
3. The LES $\left(\begin{array}{cccc|c} 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 2 \end{array}\right)$ has three dependent and two free variables.
4. The LES $\left(\begin{array}{cc|c} 2 & 2 & 1 \\ 0 & 2 & 2 \end{array}\right)$ has a unique solution.
5. If an LES $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution, then the matrix $\mathbf{A}$ must have been a square matrix.

Solution of systems of linear equations

How can we express an infinite solution set elegantly?

Consider the LES.

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & \boxed{2} & -1 & 4 & 2 \\ 0 & 0 & 0 & \boxed{4} & 8 & 8 \end{array}\right) \iff$$

Write all equations explicitly.

$$\begin{array}{cccccc} \boxed{1}\, x_1 & +2\, x_2 & +0\, x_3 & +1\, x_4 & +0\, x_5 & = 3 \\ 0\, x_1 & +0\, x_2 & +\boxed{2}\, x_3 & -1\, x_4 & +4\, x_5 & = 2 \\ 0\, x_1 & +0\, x_2 & +0\, x_3 & +\boxed{4}\, x_4 & +8\, x_5 & = 8 \end{array}$$

Move all free variables to the right hand side and write it in matrix form:

$$\begin{array}{cccccc} \boxed{1}\, x_1 & +0\, x_3 & +1\, x_4 & = 3 & -2\, x_2 & -0\, x_5 \\ 0\, x_1 & +\boxed{2}\, x_3 & -1\, x_4 & = 2 & -0\, x_2 & -4\, x_5 \\ 0\, x_1 & +0\, x_3 & +\boxed{4}\, x_4 & = 8 & -0\, x_2 & -8\, x_5 \end{array} \iff \begin{array}{c} \begin{array}{ccc|c} x_1 & x_3 & x_4 & \end{array} \\ \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 3-2\, x_2 \\ 0 & \boxed{2} & -1 & 2-4\, x_5 \\ 0 & 0 & \boxed{4} & 8-8\, x_5 \end{array}\right) \end{array}$$

Perform backward substitution (next slide).

Solution of systems of linear equations

How can we express an infinite solution set elegantly?

$$\begin{array}{c}
\begin{array}{ccc} \quad x_1 & \quad x_3 & \quad x_4 \end{array} \\
\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3-2\,x_2 \\ 0 & 2 & -1 & 2-4\,x_5 \\ 0 & 0 & 4 & 8-8\,x_5 \end{array}\right)
\end{array}$$

Perform backward substitution to determine the dependent variables.

third equation	$4x_4 = 8 - 8x_5$	$\Longrightarrow$	$x_4 = 2 - 2x_5$
second equation:	$2x_3 - x_4 = 2 - 4x_5$	$\Longrightarrow$	$x_3 = \frac{1}{2}(2 - 4x_5 + x_4) = \frac{1}{2}(4 - 6x_5) = 2 - 3x_5$
first equation	$x_1 + x_4 = 3 - 2x_2$	$\Longrightarrow$	$x_1 = 3 - 2x_2 = 1 - 2x_2 + 2x_5$

Solution set:

$$\mathcal{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 - 2x_2 + 2x_5 \\ x_2 \\ 2 - 3x_5 \\ 2 - 2x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix} : x_2, x_5 \in \mathbb{R} \right\}$$

Solution of systems of linear equations

1. **Solution of several systems** with the same matrix but different right hand sides:

$$\mathbf{Ax} = \mathbf{b},\ \mathbf{Ax} = \mathbf{c}: \qquad (\mathbf{A}|\mathbf{b}|\mathbf{c}) \rightsquigarrow (\mathbf{A}'|\mathbf{b}'|\mathbf{c}')$$

   Elimination only once, then backward substitution for the respective right hand sides.

2. **Complexity estimate:** How many operations $(+, -, \cdot, \div)$ does Gauß elimination require for an $n \times n$ system?
   - Subtract multiples of the first row from $n-1$ further rows: $2(n-1)^2$
   - Subtract multiples of the second row from $n-2$ further rows: $2(n-2)^2$
   - $\vdots$
   - Subtract a multiple of the last but one row from the last row: 2.

$$\textbf{sum:}\quad 2(n-1)^2 + 2(n-2)^2 + \ldots + 2\cdot 1^2$$
$$= 2\sum_{i=1}^{n-1} i^2 = \frac{2}{6}(n-1)\ n\ (2n-1) = \frac{2}{3}n^3 + \ldots$$

Hence: Doubling the system's size n leads to eight times the computation time!

Theory of systems of linear equations

Definition 3.22 (rank of $\mathbf{A}$)

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and let $\mathbf{A} \rightsquigarrow \mathbf{A}'$ be a transition of $\mathbf{A}$ to row echelon form $\mathbf{A}'$ by elimination.

One can show: The number of pivot elements in $\mathbf{A}'$ is always the same – regardless of the particular choice of elimination taken among many different possibilities.

The number of pivot elements in $\mathbf{A}'$ is called *rank of* $\mathbf{A}$ and denoted by $\operatorname{rank}(\mathbf{A})$.

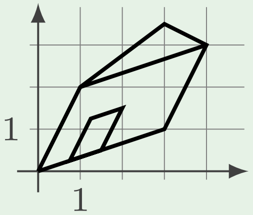
Theorem 3.23 (rank and matrix size)

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then there hold $\operatorname{rank}(\mathbf{A}) \leqslant m$ and $\operatorname{rank}(\mathbf{A}) \leqslant n$.

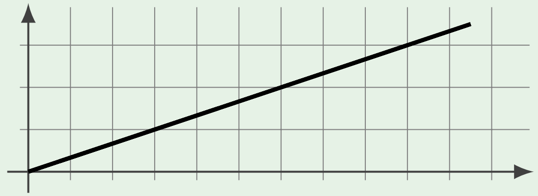
Proof. Let $\mathbf{A} \rightsquigarrow \mathbf{A}'$ be any elimination of $\mathbf{A}$ to row echelon form $\mathbf{A}'$.
Then there still holds $\mathbf{A}' \in \mathbb{R}^{m \times n}$.
Each row of $\mathbf{A}'$ contains at most one pivot element. Hence there holds $\operatorname{rank}(\mathbf{A}) \leqslant m$.
Each column of $\mathbf{A}'$ contains at most one pivot element. Hence there holds $\operatorname{rank}(\mathbf{A}) \leqslant n$. □

Good houses, bad houses

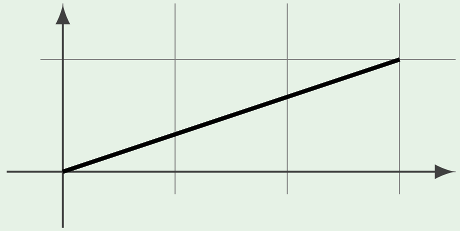
$\mathbf{K} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ $\text{rank}(\mathbf{K}) = 2$



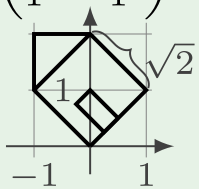
$\mathbf{L} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ $\text{rank}(\mathbf{L}) = 1$



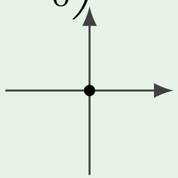
$\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{M}) = 1$



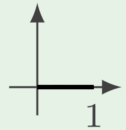
$\mathbf{N} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\text{rank}(\mathbf{N}) = 2$



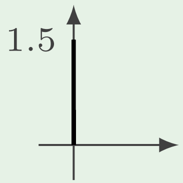
$\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{O}) = 0$



$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{P}) = 1$

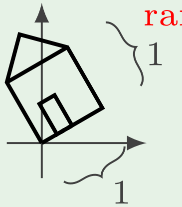


$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\text{rank}(\mathbf{Q}) = 1$

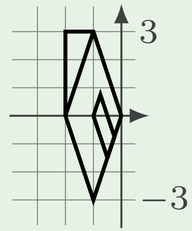


$\mathbf{R} = \begin{pmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{pmatrix}$

$\text{rank}(\mathbf{U}) = 2$



$\mathbf{S} = \begin{pmatrix} -1 & -1 \\ -3 & 3 \end{pmatrix}$ $\text{rank}(\mathbf{S}) = 2$



Theorem 3.24 (rank and (unique) solvability of $\mathbf{Ax}=\mathbf{b}$)

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then there holds

a) The system $\mathbf{Ax} = \mathbf{b}$ is **solvable** if and only if there holds $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b})$.
b) There are exactly $n - \operatorname{rank}(\mathbf{A})$ free variables and $\operatorname{rank}(\mathbf{A})$ dependent variables.
c) $\mathbf{Ax} = \mathbf{b}$ is **uniquely solvable** if and only if there holds $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b}) = n$.

Proof **a)** $\operatorname{rank}(\mathbf{A})$ and $\operatorname{rank}(\mathbf{A}|\mathbf{b})$ are the number of nonzero rows in an arbitrary row echelon form of $\mathbf{A}$ and $(\mathbf{A}|\mathbf{b})$, resp.. Hence it follows that $\operatorname{rank}(\mathbf{A}) \leqslant \operatorname{rank}(\mathbf{A}|\mathbf{b})$.

Theorem 3.19: $\mathbf{Ax} = \mathbf{b}$ is unsolvable. $\iff$ Its extended row echelon form has a row $(0\ 0 \cdots 0 \mid c)$ with $c \neq 0$, i.e., $\operatorname{rank}(\mathbf{A}) < \operatorname{rank}(\mathbf{A}|\mathbf{b})$.

$\mathbf{Ax} = \mathbf{b}$ is solvable. $\iff$ No such row exists, i.e., $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b})$.

b) The matrix $\mathbf{A}$ has a total of n columns. Of these, $\operatorname{rank}(\mathbf{A})$ have a pivot element and the remaining $n - \operatorname{rank}(\mathbf{A})$ do not have a pivot. Each column with a pivot corresponds to a dependent variable, each column without a pivot to a free variable.

c) This follows from **a)** and **b)**: The system is uniquely solvable if and only if

- it is solvable, i.e., $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b})$,
- and the number of free variables is zero, i.e., $n = \operatorname{rank}(\mathbf{A})$. $\square$

Theory of systems of linear equations

Definition 3.25 (homogeneous and inhomogeneous system, kernel of $\mathbf{A}$)

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

The system $\mathbf{Ax} = \mathbf{b}$ is called *homogeneous* if the right hand side is the zero vector, $\mathbf{b} = \mathbf{o}$. Otherwise it is called *inhomogeneous*.

The solution set of the homogeneous LES is called the *kernel* (or *nullspace*) of $\mathbf{A}$:

$$\mathrm{Ker}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{Ax} = \mathbf{o}\}.$$

Recall the example (4×5 LES)

$$\underbrace{\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 3 & 12 & 10 \\ 4 & 8 & 2 & 3 & 4 & 14 \\ 1 & 2 & 0 & 1 & 0 & 3 \\ -3 & -6 & -6 & 8 & 4 & 1 \end{array}\right)}_{(\mathbf{A}|\mathbf{b})} \rightsquigarrow \quad \mathcal{L} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix}}_{\mathbf{v}_0} + x_2 \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{v}_1} + x_5 \underbrace{\begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix}}_{\mathbf{v}_2} : x_2, x_5 \in \mathbb{R} \right\}$$

There holds $\quad \mathbf{Av}_0 = \mathbf{b}, \quad \mathbf{Av}_1 = \mathbf{o}, \quad \mathbf{Av}_2 = \mathbf{o}.$

Theory of systems of linear equations

Theorem 3.26 (solution set of homogeneous and inhomogeneous system)

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ be such that $\mathbf{Ax} = \mathbf{b}$ is solvable.
The Gauß algorithm yields the solution set $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b}\}$ in the form

$$\mathcal{L} \;=\; \{\mathbf{v}_0 + \lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k \;:\; \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}.$$

Here, $\lambda_1, \ldots, \lambda_k$ with $k = n - \operatorname{rank}(\mathbf{A})$ correspond to the free variables, and $\mathbf{v}_0 \in \mathbb{R}^n$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n \setminus \{\mathbf{o}\}$ are certain vectors which satisfy the following:

a) $\mathbf{A}\mathbf{v}_0 = \mathbf{b}$, hence $\mathbf{v}_0 \in \mathcal{L}$;

b) $\mathbf{A}\mathbf{v}_1 = \ldots = \mathbf{A}\mathbf{v}_k = \mathbf{o}$, hence $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \operatorname{Ker}(\mathbf{A})$;

c) $\operatorname{Ker}(\mathbf{A}) = \{\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k : \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$;

d) $\mathcal{L} = \{\mathbf{v}_0 + \mathbf{x}_0 : \mathbf{x}_0 \in \operatorname{Ker}(\mathbf{A})\} =: \mathbf{v}_0 + \operatorname{Ker}(\mathbf{A})$.

In short: Each solution of the inhomogeneous LES is composed of a particular solution $\mathbf{v}_0$ of the inhomogeneous LES and the general solution of the homogeneous LES.

Theory of systems of linear equations

Proof. Let

$$\mathcal{L} = \{\mathbf{v}_0 + \lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k \colon \lambda_1, \ldots, \lambda_k \in \mathbb{R}\} \qquad (*)$$

be the solution set obtained with the Gauß algorithm, and let $\mathbf{x} = \mathbf{v}_0 + \lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k$ with arbitrary $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Then there holds $\mathbf{x} \in \mathcal{L}$ and

$$\mathbf{b} = \mathbf{A}\mathbf{x} = \mathbf{A}(\mathbf{v}_0 + \lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k) = \mathbf{A}\mathbf{v}_0 + \lambda_1 \mathbf{A}\mathbf{v}_1 + \ldots + \lambda_k \mathbf{A}\mathbf{v}_k.$$

a) Show $\mathbf{A}\mathbf{v}_0 = \mathbf{b}$, i.e., $\mathbf{v}_0 \in \mathcal{L}$: Set $\lambda_1 = ... = \lambda_k = 0$, then there follows $\mathbf{b} = \mathbf{A}\mathbf{v}_0$.

b) Show $\mathbf{A}\mathbf{v}_1 = \ldots = \mathbf{A}\mathbf{v}_k = \mathbf{o}$: Let $i \in \{1, ..., k\}$ be arbitrary. Set $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \in \{1, ..., k\} \setminus \{i\}$. Then there follows

$$\mathbf{b} = \mathbf{A}\mathbf{v}_0 + \mathbf{A}\mathbf{v}_i \overset{a)}{=} \mathbf{b} + \mathbf{A}\mathbf{v}_i,$$

hence $\mathbf{A}\mathbf{v}_i = \mathbf{o}$.

Theory of systems of linear equations

Proof (continuation). Let

$$\mathcal{L} = \{\mathbf{v}_0 + \lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k \colon \lambda_1, \ldots, \lambda_k \in \mathbb{R}\} \qquad (\,*\,)$$

be the solution set obtained with the Gauß algorithm.

c) Show $\operatorname{Ker}(\mathbf{A}) = \{\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k \colon \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$:

"$\subset$": Let $\mathbf{x} \in \operatorname{Ker}(\mathbf{A})$. Then there holds $\mathbf{A}(\mathbf{v}_0 + \mathbf{x}) = \mathbf{A}\mathbf{v}_0 + \mathbf{A}\mathbf{x} = \mathbf{b} + \mathbf{o} = \mathbf{b}$. It follows that $\mathbf{v}_0 + \mathbf{x} \in \mathcal{L}$. In view of the representation $(\,*\,)$ of $\mathcal{L}$, $\mathbf{x}$ must be of the form $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k$ with certain $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, i.e. $\mathbf{x} \in \{\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k \colon \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$.

"$\supset$": Now let $\mathbf{x} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k$ with certain $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. With **b)** $(\mathbf{A}\mathbf{v}_j = \mathbf{o})$ it follows that

$$\mathbf{A}\mathbf{x} = \mathbf{A}(\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k) = \lambda_1 \underbrace{\mathbf{A}\mathbf{v}_1}_{\mathbf{o}} + \ldots + \lambda_k \underbrace{\mathbf{A}\mathbf{v}_k}_{\mathbf{o}} = \mathbf{o}, \quad \text{hence } \mathbf{x} \in \operatorname{Ker}(\mathbf{A}).$$

d) Show $\mathcal{L} = \{\mathbf{v}_0 + \mathbf{x}_0 : \mathbf{x}_0 \in \operatorname{Ker}(\mathbf{A})\}$: This follows directly from $(\,*\,)$ and c). $\square$

Recall the example (4×5 LES): **Alternative computation** of $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -1 & 4 & 2 \\ 0 & 0 & 0 & 4 & 8 & 8 \end{array}\right) \end{array} \rightsquigarrow \quad \mathcal{L} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix}}_{\mathbf{v}_0 \in \mathcal{L}} + x_2 \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{v}_1 \in \mathrm{Ker}(\mathbf{A})} + x_5 \underbrace{\begin{pmatrix} 2 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix}}_{\mathbf{v}_2 \in \mathrm{Ker}(\mathbf{A})} : x_2, x_5 \in \mathbb{R} \right\}$$

$\mathbf{v}_0$: Set all free variables to zero and solve inhomogebeous system

$$\begin{array}{c} \begin{array}{ccc|c} x_1 & x_3 & x_4 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 4 & 8 \end{array}\right) \end{array}.$$

$\mathbf{v}_1$: Set $\mathbf{x}_2 = 1$ and all other free variables to zero and solve homogebeous system.

$\mathbf{v}_2$: Set $\mathbf{x}_2 = 1$, all other free variables to zero, solve homogebeous system.

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & 1 & x_3 & x_4 & 0 & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 4 & 0 \\ 0 & 0 & 0 & 4 & 8 & 0 \end{array}\right) \end{array} \iff \begin{array}{c} \begin{array}{ccc|c} x_1 & x_3 & x_4 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & -2 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right) \end{array}$$

Determine the solution sets of the following three LES.

$$\left(\begin{array}{rr|r} 1 & 2 & 0 \\ -2 & -3 & -1 \\ 2 & 4 & 1 \end{array}\right), \quad \left(\begin{array}{rr|r} 1 & 2 & 0 \\ -2 & -3 & -1 \\ 2 & 4 & 0 \end{array}\right), \quad \left(\begin{array}{rr|r} 1 & 2 & 0 \\ -2 & -4 & 0 \\ 2 & 4 & 0 \end{array}\right)$$