Exercise 6

Exercise 1 (Sums):

Let I and J be sets such that a_i is a number for every $i \in I \cup J$.

(a) Assume that I and J are disjoint. Then show $\sum_{i\in I} a_i + \sum_{i\in J} a_i = \sum_{i\in I\cup J} a_i$.

(b) Now show that $\sum_{i\in I} a_i + \sum_{i\in J} a_i = \sum_{i\in I\cup J} a_i + \sum_{i\in I\cap J} a_i$ in general. If $a_i = 1$, what do we get as a corollary?

Exercise 2 (Counting):

For the following cases pick the correct formula for counting and compute the solution. Hints are in the footnotes if you need them.

(a) In a volleyball club, a team of size 6 has to be selected from a list of 11 players. How many possible teams can there be? Remember than in volley ball team, all player positions are indistinguishable. [1]

(b) One person goes to the bar and orders drinks for them and their 3 friends. There are 7 possible options for the drinks. How many different possible orders are there? [2] $\binom{7}{3}$

(c) You want to go on a trip and visit 5 EU countries. The order in which you visit those countries *does* matter to you. How many possible trips are there, knowing that the EU has 27 member countries? [3] 27^5

(d) You told your friend that your TUHH password consists of uppercase letters from the English alphabet and has length 7. In the worst possible case, how many possible passwords would your friend have to try to hack into your TUHH account? [4] 26^7

(e) For numbers below 10^6, are there more numbers with or without a 9 in their decimal representation? [5]

(f) How many distinct words can be written by rearranging the letters of the word "function"? [6] $8!$

Exercise 3 (Counting with unions of two sets):

Remember that $|A \cup B| = |A| + |B| - |A \cap B|$.

(a) How many numbers divisible by 2 or 3 in $\{1, \ldots, 99\}$ are there?

(b) How many words of length 10 (using the English alphabet) are there that contain both 'a' and 'e'?

Exercise 4 (Combinatorial Proofs):

(a) Show that $\binom{2n}{2} = 2 \cdot \binom{n}{2} + n^2$ using a combinatorial proof.

[1] Here we pick a set of players without replacement.
[2] Here we pick a multiset of drinks with replacement.
[3] Here we select a tuple of countries, implicitly without replacement.
[4] A password is a tuple of letters, taken with replacement.
[5] Is is simpler to count the number of numbers without 9 than with.
[6] When placing the occurrences of each distinct letter, we're picking a set of positions without replacement.

(b) Show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ holds for all $1 \leq k \leq n$ using a combinatorial proof.

(c) Show that $\binom{n+m}{k} = \sum_{l=0}^{k} \binom{n}{l}\binom{m}{k-l}$ using a combinatorial proof. [7]

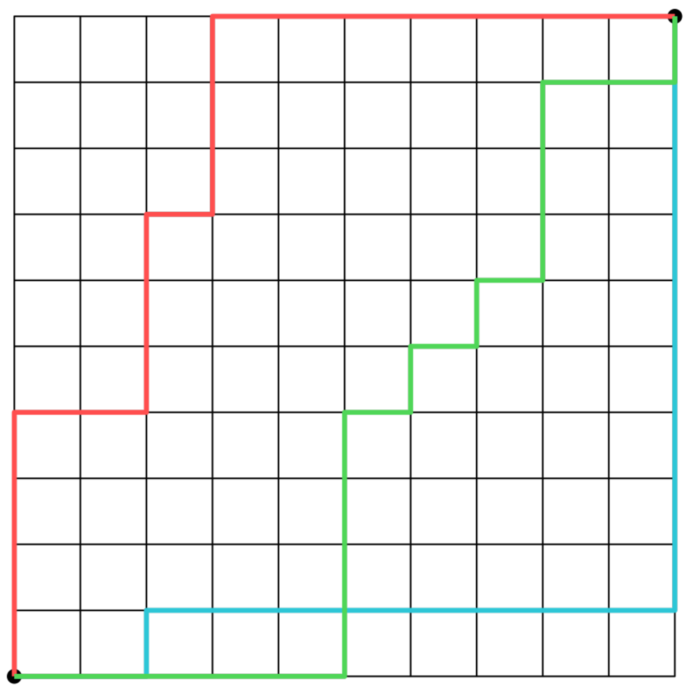
Exercise 5 (More counting):

To do at home.

In all the questions below, you are asked not only to give the right result but also to describe how you arrived at the result using the concepts of drawing with/without replacement, and drawing a tuple, set, or multiset.

(a) How many binary symmetric relations $R \subseteq A \times A$ are there if $|A| = 10$. Give a combinatorial proof for your result, and not only the number.

(b) In the 10×10 grid below, one considers walks from the lower left corner to the upper right corner where the walks only go "up" or "right" at every junction. 3 possible such walks are draw in the picture. How many possible walks are there in total? Give a proof for your answer.



(c) Show $\sum_{i=0}^{m} \binom{n+i}{i} = \binom{m+n+1}{m}$ using a combinatorial proof. [8]

(d) Let $n \geq 6$ be an arbitrary **even** number. How many subsets $S \subseteq \{1, \ldots, n\}$ are there of size 5 and containing exactly 2 even numbers? Give a proof for your answer.

[7]Think about this equality as selecting k balls from an urn of $n + m$ (numbered) balls where n are blue and the remaining m are red.

[8]Hint: generalize your results from 5b

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(b) Now show that $\sum_{i \in I} a_i + \sum_{i \in J} a_i = \sum_{i \in I \cup J} a_i + \sum_{i \in I \cap J} a_i$ in general. If $a_i = 1$, what do we get as a corollary?

(a) Thesis: $\sum_{i \in I} a_i + \sum_{i \in J} a_i = \sum_{i \in I \cup J} a_i$ (I, J are disjoint)

1. By definition (left side):
   iterates over and sums a_i for $\{i \mid i \in I \oplus i \in J\}$ ($\oplus$ follows from disjoint I, J)
   1.1. by def. of $\oplus$ => sums $\{a_i \mid i \in I \cup J\}$
2. -> equivalent to right side expression
   Q.E.D.

(b) Thesis: $\sum_{i \in I} a_i + \sum_{i \in J} a_i = \sum_{i \in I \cup J} a_i + \sum_{i \in I \cap J} a_i$

1. (a) fails because I, J may not be disjoint -> possibly missing sums for double inclusions
   1.1. => $\sum_{i \in I} a_i + \sum_{i \in J} a_i \neq \sum_{i \in I \cup J}$
2. $\sum_{i \in I \cap J} a_i$ describes sum of $\forall a_i$ for $\{i \mid i \in I \cap J\}$ (meaning: double included)
3. => adding back this sum to RHS restores LHS = RHS
   (i.e. [see theorem])
   Q.E.D.

corollary $a_i = 1$: $|I| + |J| = |I \cup J| + |I \cap J|$

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(a) G: $n=11$; $k=6$ | no replacement | no order   S: $\circ$

$$\circ = \binom{n}{k} = \frac{11^{\underline{6}}}{6!} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{11\cdot 7}{6}\cdot\frac{10}{5}\cdot\frac{9}{3}\cdot\frac{8}{4}\cdot\frac{6}{2}$$

$$= \frac{77}{6}\cdot 2\cdot 3\cdot 2\cdot 3$$

$$= \frac{77}{6}\cdot 36$$

$$= \frac{77}{\frac{1}{6}} = 77\cdot 6 = 385+77$$

$$= 462$$

(e) assuming all $(1,10^n) \in (1,10^n) \cap \mathbb{N}$

for $i \in (1,10^n)$, i has n digits

so appearance of a specific number down all digits of i (let j):

$$j_n = 10^n \cdot \frac{1}{10} = 10^{n-1}$$

$$10^5 \cdot 9 > 10^5 \cdot 1$$