Mathematics 1 - Linear Algebra

Lecture 03 – §2 Vectors in $\mathbb{R}^n$

Sabine Le Borne

TUHH INSTITUTE OF MATHEMATICS

Vectors - for what?

Vectors are used in physics and engineering

- to describe quantities that cannot be grasped by a single number
- but are characterized by a direction and size,
- e.g. displacements, velocities or forces.

Vectors in two dimensions (2D)

Formally: A 2D vector is an object with two components.

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

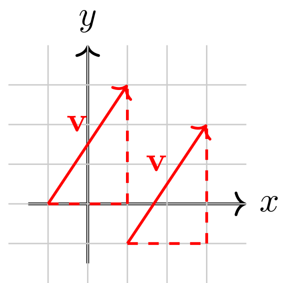
Example: The vector

$$\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

has the first component 2 and the second component 3.

Visually: A 2D vector is an arrow (direction and length).

Example: The vector



goes 2 steps to the right and 3 steps up. The starting point is arbitrary.

In 2D, we call the first component x- and the second y-direction/coordinate.

Vectors in three dimensions (3D)

Formally: A 3D vector is an object with three components.

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

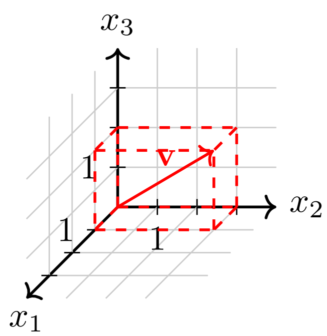
Example: The vector

$$\mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

has the first component 1, the second component 3 and the third component 2.

Visually: A 3D vector is an arrow (direction and length).

Example: The vector



goes 1 step to the front, 3 steps to the right and 2 steps up. The starting point is arbitrary.

In 3D, we call the first component x-, the second y- and the third z-direction/coordinate.

Vectors in n dimensions (nD)

Definition 2.1 (Vectors in n dimensions)

Let $n \in \mathbb{N}$. Every object $\mathbf{v}$ of the form

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{with} \quad v_1, v_2, \ldots, v_n \in \mathbb{R}$$

is called a real vector.
The set of all such vectors is denoted by $\mathbb{R}^n$.
The entries $v_1, v_2, \ldots, v_n$ are called components, v_i is the ith component.

Notes:

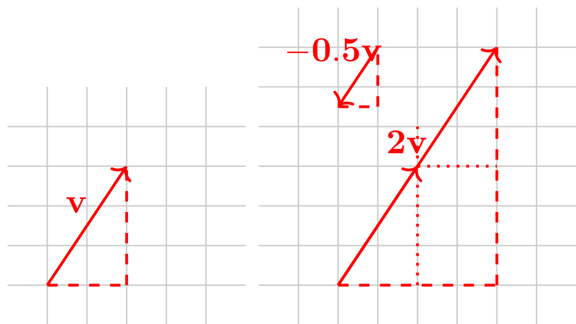
- Vectors in two or three dimensions are only special cases that we obtain for $n \in \{2, 3\}$.
- In different lectures you may see different notations for vectors: $\mathbf{v}$, $\vec{v}$, $\underline{v}$ etc.

Calculating with vectors: scalar multiplication

Example: Let $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Then we have

$$2\mathbf{v} = \begin{pmatrix} 4 \\ 6 \end{pmatrix},$$
$$-0.5\mathbf{v} = \begin{pmatrix} -1 \\ -1.5 \end{pmatrix}.$$

Stretching: same (or opposite) direction, new length

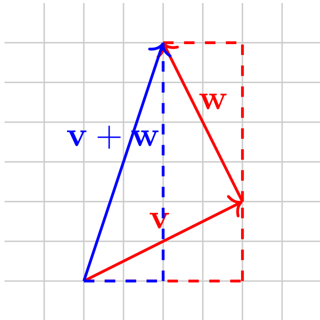


Definition 2.2 (Scalar multiplication; scalar times vector = vector)

Let $\alpha \in \mathbb{R}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. Then we define $\alpha\mathbf{v} = \alpha \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{pmatrix}$.

Calculating with vectors: vector addition

Addition: concatenation of vectors



Example: Let $\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$.

Then we have

$$\mathbf{v} + \mathbf{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4-2 \\ 2+4 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

Definition 2.3 (vector addition; vector + vector = vector)

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$. Then we define $\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} := \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix}$.

Computation rules for scalar multiplication and vector addition

Let $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then there hold

$$(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u},$$
$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},$$
$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}.$$

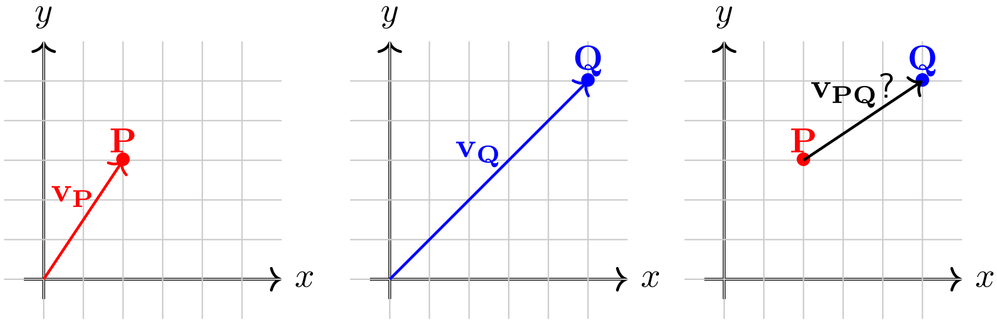
Proof. The equalities result from componentwise application of the distributive and commutative laws for real numbers. $\square$

Points and position vectors

A point P is given as a *tuple* $(v_1, v_2, \dots, v_n)$, e.g., $P = (2, 3)$.

Definition (position vector)

Let O be the origin (of the coordinate system). If P is a point, then the vector going from O to P is called the position vector of P. We often identify points with their position vectors.



$$\mathbf{P} = (2, 3)$$

$$\mathbf{v}_P = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \mathbf{v}_{PQ} &= -\mathbf{v}_P + \mathbf{v}_Q \\ &= \mathbf{v}_Q - \mathbf{v}_P \end{aligned}$$

Vectors between two points

Let P and Q be two points. The vector $\mathbf{v}_{PQ}$ going from P to Q can be obtained by subtracting the position vector of P from the position vector of Q.

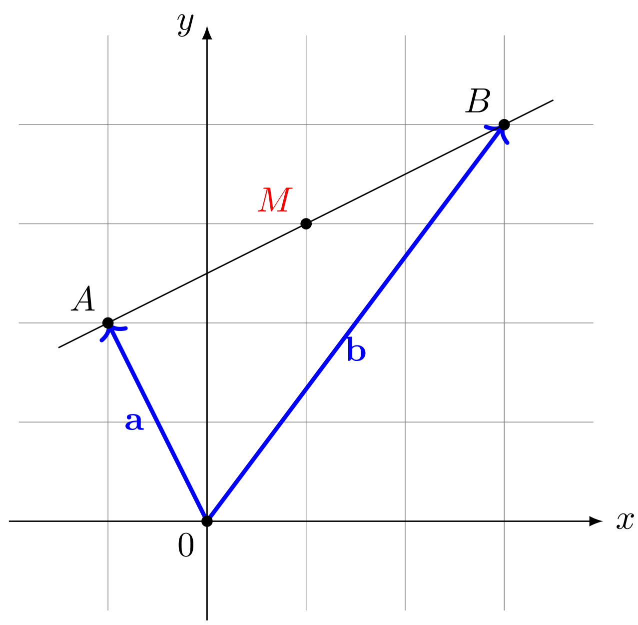
Some special vectors

Definition 2.4 (zero vector and canonical unit vectors)

For $i = 1, \dots, n$, let $\mathbf{e}_i \in \mathbb{R}^n$ be the ith *canonical unit vector* which is the vector whose ith component is 1 and all other components are 0.
Furthermore, let $\mathbf{o} \in \mathbb{R}^n$ be the *zero vector* which is the vector with all components equal to 0:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \dots, \ \mathbf{e}_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{o} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Affine and convex combinations



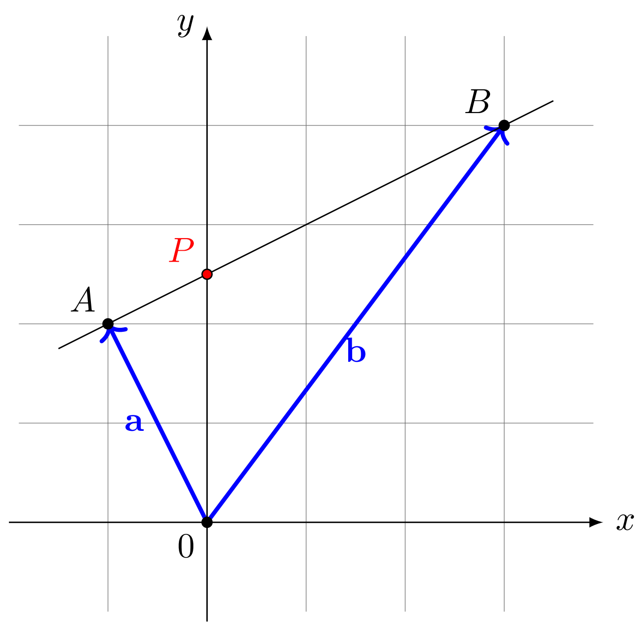
Task: Find the midpoint M between two given points A and B.

Solution: Determine the vector $\mathbf{v}_{AB}$ from A to B and add half of it to the position vector $\mathbf{a}$ of A.

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

$$\begin{aligned} \mathbf{m} = \mathbf{v}_M &= \mathbf{a} + 0.5\mathbf{v}_{AB} \\ &= \mathbf{a} + 0.5\,(\mathbf{b} - \mathbf{a}) \\ &= 0.5\,(\mathbf{a} + \mathbf{b}) \\ &= 0.5\left(\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}. \end{aligned}$$

Affine and convex combinations



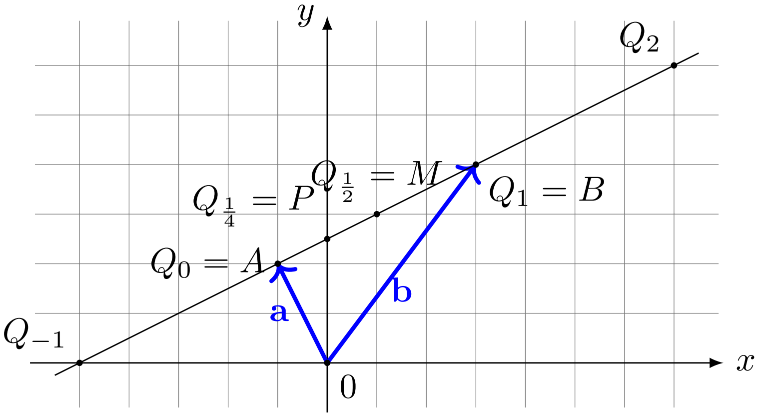
Task: Find the point P that is reached after a quarter of the path from A to B.

Solution: Determine the vector $\mathbf{v}_{AB}$ from A to B and add a quarter of it to the position vector $\mathbf{a}$ of A.

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

$$\begin{aligned} \mathbf{p} = \mathbf{v}_P &= \mathbf{a} + 0.25\mathbf{v}_{AB} \\ &= \mathbf{a} + 0.25\,(\mathbf{b} - \mathbf{a}) \\ &= 0.75\mathbf{a} + 0.25\mathbf{b} \\ &= \begin{pmatrix} 0 \\ 2.5 \end{pmatrix}. \end{aligned}$$

Affine and convex combinations



In general:

$$\begin{aligned} \mathbf{q} = \mathbf{v}_Q &= \mathbf{a} + \lambda \mathbf{v}_{AB} \\ &= \mathbf{a} + \lambda\,(\mathbf{b} - \mathbf{a}) \\ &= (1-\lambda)\mathbf{a} + \lambda \mathbf{b} \end{aligned}$$

Special cases:

$$\begin{aligned} \lambda = 0 &\Rightarrow \mathbf{q} = \mathbf{a} \\ \lambda = 1/4 &\Rightarrow \mathbf{q} = \mathbf{p} \\ \lambda = 1/2 &\Rightarrow \mathbf{q} = \mathbf{m} \\ \lambda = 1 &\Rightarrow \mathbf{q} = \mathbf{b} \end{aligned}$$

$\lambda \in [0,1] \;\Rightarrow\;$ $\mathbf{Q}$ lies on the line segment from $\mathbf{A}$ to $\mathbf{B}$

$\lambda > 1 \;\Rightarrow\;$ $\mathbf{Q}$ lies on the line through $\mathbf{A}$ and $\mathbf{B}$ (behind $\mathbf{B}$)

$\lambda < 0 \;\Rightarrow\;$ $\mathbf{Q}$ lies on the line through $\mathbf{A}$ and $\mathbf{B}$ (before $\mathbf{A}$)

Affine and convex combinations

Definition 2.5 (convex and affine combination of two vectors)

For points A and B in $\mathbb{R}^n$ with respective position vectors $\mathbf{a}$ and $\mathbf{b}$, we call

$$(1-\lambda)\mathbf{a} + \lambda \mathbf{b}$$

a *convex combination* of $\mathbf{a}$ and $\mathbf{b}$ (or of A and B) if $\lambda \in [0,1]$ and
an *affine combination* of $\mathbf{a}$ and $\mathbf{b}$ (or of A and B) if $\lambda \in \mathbb{R}$.

Convex and affine combinations lie on the line through points A and B.

Linear combination

True or false?

1. $\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$
2. $\begin{pmatrix}1\\2\end{pmatrix} \cdot \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$
3. Let points $A = (0,3),\ B = (2,6),\ C = (6,9)$ be given. Then B lies on the line segment between A and C.
4. The vector v_{PQ} from point P to point Q is given by $v_P + v_Q$.
5. If the origin $(0,0)$ is an affine combination of two nonzero vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2 \setminus \{\mathbf{o}\}$, then there exists an $\alpha \in \mathbb{R}$ such that $\mathbf{b} = \alpha\mathbf{a}$.

Linear combination

We next consider more general combinations of the type $\alpha\mathbf{a} + \beta\mathbf{b}$ for $\alpha, \beta \in \mathbb{R}$.

Comparison of apples and pears:

An **apple** contains 12mg vitamin C and 5mg magnesium.
A **pear** contains 5mg vitamin C and 8mg magnesium.

$$\text{apple} \quad \mathbf{a} = \begin{pmatrix}12\\5\end{pmatrix}\begin{matrix}\text{vitC}\\\text{Mg}\end{matrix}, \qquad \text{pear} \quad \mathbf{b} = \begin{pmatrix}5\\8\end{pmatrix}\begin{matrix}\text{vitC}\\\text{Mg}\end{matrix}$$

Fruit salad: How much vitamin C and magnesium do I take if I eat 2 apples and 3 pears?

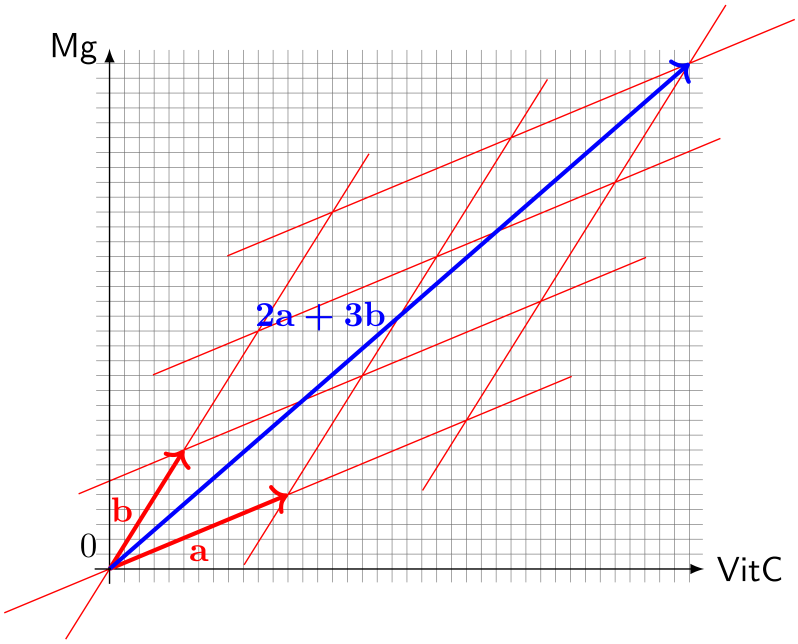
Answer:

$$2\mathbf{a} + 3\mathbf{b} = 2\begin{pmatrix}12\\5\end{pmatrix} + 3\begin{pmatrix}5\\8\end{pmatrix} = \begin{pmatrix}2\cdot 12 + 3\cdot 5\\2\cdot 5 + 3\cdot 8\end{pmatrix} = \begin{pmatrix}39\\34\end{pmatrix}\begin{matrix}\text{VitC}\\\text{Mg}\end{matrix}$$

Linear combination

apple $\mathbf{a} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} \begin{matrix} \text{VitC} \\ \text{Mg} \end{matrix}$,

pear $\mathbf{b} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} \begin{matrix} \text{VitC} \\ \text{Mg} \end{matrix}$



Linear combination

Definition 2.6 (linear combination)

Let $\mathbf{v}_1, \dots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$ with $m, n \in \mathbb{N}$. Then we call each sum

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m \qquad \text{with} \qquad \alpha_1, \dots, \alpha_m \in \mathbb{R}$$

a *linear combination* of $\mathbf{v}_1, \dots, \mathbf{v}_m$ with *coefficients* $\alpha_1, \dots, \alpha_m$.

Theorem 2.7

Each vector is a linear combination of the canonical unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + v_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n$$

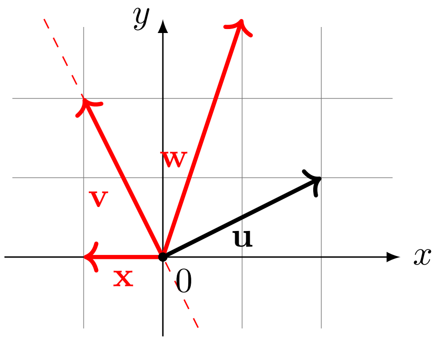
Scalar product

Definition 2.8 (scalar product: $\langle \text{vector}, \text{vector} \rangle$ = scalar)

For two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{we call} \qquad \langle \mathbf{u}, \mathbf{v} \rangle := u_1 v_1 + \ldots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

(standard-) *scalar product* of $\mathbf{u}$ and $\mathbf{v}$.



Examples:

$$\langle \mathbf{x}, \mathbf{u} \rangle = \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = -1 \cdot 2 + 0 \cdot 1 = -2$$

$$\langle \mathbf{v}, \mathbf{u} \rangle = \left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = -1 \cdot 2 + 2 \cdot 1 = 0$$

$$\langle \mathbf{w}, \mathbf{u} \rangle = \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = 1 \cdot 2 + 3 \cdot 1 = 5$$

Orthogonality and size/norm/length of a vector

Definition 2.9 (orthogonality of two vectors in $\mathbb{R}^n$)

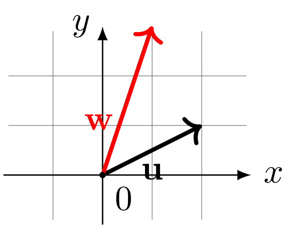
Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are called *orthogonal* to each other if there holds $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. We also use the short notation $\mathbf{u} \perp \mathbf{v}$.

Definition 2.10 (size/norm/length of a vector in $\mathbb{R}^n$)

For a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{we call} \qquad \|\mathbf{v}\| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + \ldots + v_n^2}$$

the *size*, the *norm* or simply the *length* of $\mathbf{v}$.



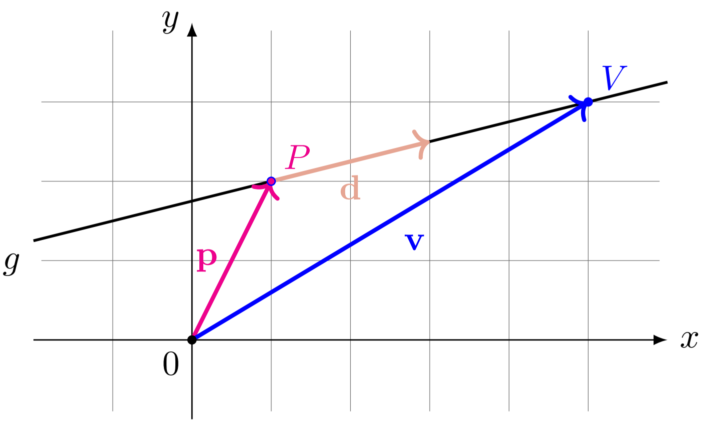
Examples:

$$\|\mathbf{w}\| = \sqrt{w_1^2 + w_2^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$$

Lines in $\mathbb{R}^2$

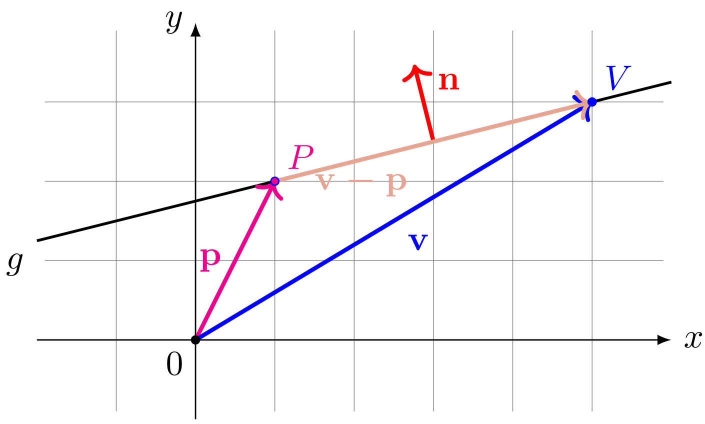
Description with point and direction



Given: position vector $\mathbf{p} \in \mathbb{R}^2$,
direction vector $\mathbf{d} \in \mathbb{R}^2 \setminus \{\mathbf{o}\}$

Line $g = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = \mathbf{p} + \lambda\mathbf{d},\ \lambda \in \mathbb{R}\}$

Description with point and normal vector



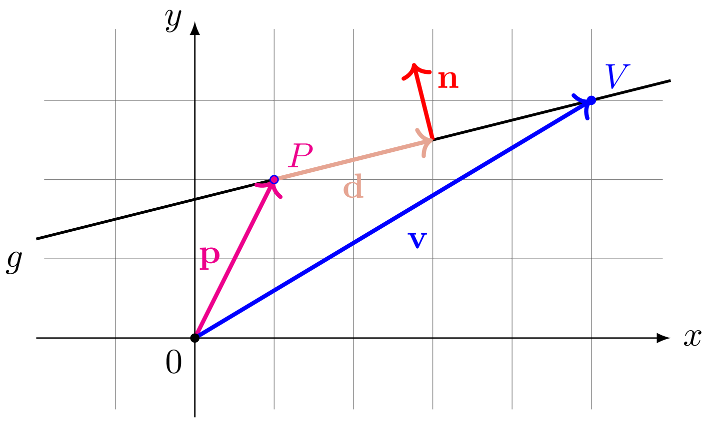
Given: position vector $\mathbf{p} \in \mathbb{R}^2$,
normal vector $\mathbf{n} \in \mathbb{R}^2 \setminus \{\mathbf{o}\}$

Line $g = \{\mathbf{v} \in \mathbb{R}^2 : \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}$

Lines in $\mathbb{R}^2$

Example:

$$p = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ d = \begin{pmatrix} 2 \\ 0.5 \end{pmatrix},\ n = \begin{pmatrix} -0.25 \\ 1 \end{pmatrix}$$



Line $g = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = \mathbf{p} + \lambda\mathbf{d},\ \lambda \in \mathbb{R}\}$
$= \{\mathbf{v} \in \mathbb{R}^2 : \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}$

Question: Does $V = (5, 3)$ lie on the line g?

Solution:

1. Point-direction form

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0.5 \end{pmatrix}$$

$\lambda = 2$ satisfies the equation, hence $\mathbf{v} \in g$.

2. Point-normal form

$$\left\langle \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -0.25 \\ 1 \end{pmatrix} \right\rangle \stackrel{?}{=} 0$$

$$\iff (5-1) \cdot (-0.25) + (3-2) \cdot 1 \stackrel{?}{=} 0$$

is satisfied, hence $\mathbf{v} \in g$.

Lines in $\mathbb{R}^2$

Line in $\mathbb{R}^2$ in normal form

Let $\mathbf{n} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq \mathbf{o}$, $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$. A line is defined as the set

$$g = \{\mathbf{v} \in \mathbb{R}^2 : \langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = 0\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \alpha x + \beta y = \delta \right\}$$

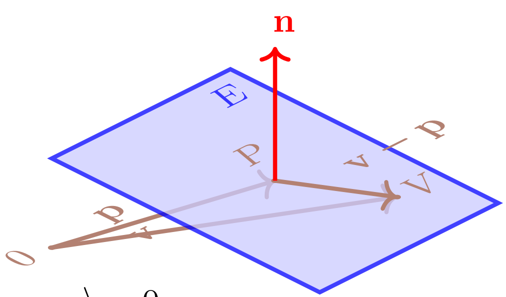
with $\delta := \alpha p_1 + \beta p_2 = \left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\rangle = \langle \mathbf{n}, \mathbf{p} \rangle$. If $\mathbf{0} \in g$, then $\delta = 0$ (choose $\mathbf{p} = \mathbf{o}$).

Transition to 3D: Geometric interpretation of

$$\{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = 0\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \alpha x + \beta y + \gamma z = \delta \right\}$$

with $\delta := \alpha p_1 + \beta p_2 + \gamma p_3 = \left\langle \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \right\rangle = \langle \mathbf{n}, \mathbf{p} \rangle$?

Planes in $\mathbb{R}^3$



$\langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = 0$

Plane in $\mathbb{R}^3$ in normal form

A plane is defined as the set

$$\begin{aligned} E &= \{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{n}, \mathbf{v} - \mathbf{p} \rangle = 0\} \\ &= \{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{n}, \mathbf{v} \rangle = \langle \mathbf{n}, \mathbf{p} \rangle\} \\ &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \alpha x + \beta y + \gamma z = \delta \right\}, \end{aligned}$$

where

- $\mathbf{n} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \neq \mathbf{o}$ is a normal vector of the plane E,
- $\mathbf{p}$ is the position vector of a (arbitrary) point of E

and $\delta := \langle \mathbf{n}, \mathbf{p} \rangle$.
In the case $\mathbf{0} \in E$ one obtains $\delta = 0$ (choose $\mathbf{p} = \mathbf{o}$).

The cross or vector product (only!) in $\mathbb{R}^3$

Definition 2.11 (Cross or vector product: vector $\times$ vector $=$ vector)

The *cross* or *vector product* of two vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \quad \text{is defined as} \quad \mathbf{u} \times \mathbf{v} := \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \in \mathbb{R}^3.$$

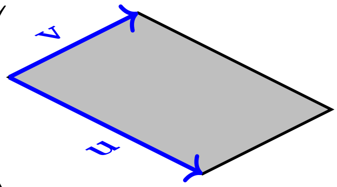
Mnemonic

$$\begin{array}{cc} \cancel{u_1} & \cancel{v_1} \\ u_2 & v_2 \\ u_3 & v_3 \\ u_1 & v_1 \\ u_2 & v_2 \\ \cancel{u_3} & \cancel{v_3} \end{array} \quad \begin{array}{c} \Longrightarrow \\ \Longrightarrow \\ \Longrightarrow \end{array} \quad \begin{pmatrix} +u_2 v_3 - u_3 v_2 \\ +u_3 v_1 - u_1 v_3 \\ +u_1 v_2 - u_2 v_1 \end{pmatrix}$$

The cross or vector product (only!) in $\mathbb{R}^3$

The cross product $\mathbf{u} \times \mathbf{v}$ is the uniquely determined vector in $\mathbb{R}^3$ with the following three properties:

1.) $\mathbf{u} \times \mathbf{v} \ \perp \ \mathbf{u}, \mathbf{v}$

2.) $\|\mathbf{u} \times \mathbf{v}\| =$ area $\Bigg($  $\Bigg)$

3.) Orientation: "right hand rule"