Mathematics 1 - Linear Algebra

Lecture 12

§5.5 Orthogonal systems

§5.6 Orthogonal matrices: Rotations and reflections

Sabine Le Borne

TUHH

INSTITUTE OF MATHEMATICS

Orthogonal systems and bases

Things get easier when the basis vectors of a subspace are orthogonal to each other.

Definition 5.19 (OGS, ONS, OGB, ONB)

Let U be a subspace of $\mathbb{R}^n$. A family $\mathcal{F} = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ of vectors in U is called

- *orthogonal system* (OGS), if all vectors in $\mathcal{F}$ are pairwise orthogonal to each other, i.e., $\mathbf{u}_i \perp \mathbf{u}_j$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$;
- *orthonormal system* (ONS), if in addition there holds $\|\mathbf{u}_i\| = 1$ for all $i \in \{1, \ldots, k\}$, i.e.,

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \qquad \text{for all } \ i, j \in \{1, \ldots, k\};$$

- *orthogonal basis* (OGB), if it is an OGS and also a basis of U;
- *orthonormal basis* (ONB), if it is an ONS and also a basis of U.

Orthogonal systems and bases

Examples for OGS and ONB

a) The standard basis

$$\mathbf{e}_1 = (1, 0, \ldots, 0)^\mathsf{T}, \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0)^\mathsf{T}, \quad \ldots, \quad \mathbf{e}_n = (0, \ldots, 0, 1)^\mathsf{T}$$

of $\mathbb{R}^n$ is an orthonormal basis (ONB) of $U = \mathbb{R}^n$.

b) The family $\mathcal{F} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ with

$$\mathbf{u}_1 = (1, 0, 1)^\mathsf{T}, \quad \mathbf{u}_2 = (1, 0, -1)^\mathsf{T}, \quad \mathbf{u}_3 = (0, 1, 0)^\mathsf{T}$$

is an orthogonal basis (OGB) of $\mathbb{R}^3$ since

- $\mathcal{F}$ is an OGS (check $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ for $i \neq j$),
- $\mathcal{F}$ is a basis since $\mathcal{F}$ contains three linearly independent vectors of $\mathbb{R}^3$ (see Theorem 5.21) and hence is also a generating system of $\mathbb{R}^3$ (see Theorem 3.56).

Orthogonal systems and bases

Theorem 5.21 (Every OGS is linearly independent.)

Let $\mathcal{F} = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ be an OGS in $\mathbb{R}^n$ with $\mathbf{u}_i \neq \mathbf{o}$ for $i = 1, \ldots, k$.
Then $\mathcal{F}$ is linearly independent.

Proof. Let $\mathcal{F}$ be an OGS. We prove the linear independence of $\mathcal{F}$ by showing that

$$\alpha_1 \mathbf{u}_1 + \ldots + \alpha_k \mathbf{u}_k = \mathbf{o} \quad \Longrightarrow \quad \alpha_1 = \ldots = \alpha_k = 0.$$

We form the scalar product (of the left equation) with all $\mathbf{u}_i,\ i = 1, \ldots, k$, and obtain

$$\begin{aligned} 0 &= \langle \mathbf{o}, \mathbf{u}_i \rangle = \langle \alpha_1 \mathbf{u}_1 + \ldots + \alpha_k \mathbf{u}_k, \mathbf{u}_i \rangle \\ &= \alpha_1 \underbrace{\langle \mathbf{u}_1, \mathbf{u}_i \rangle}_{0} + \ldots + \alpha_{i-1} \underbrace{\langle \mathbf{u}_{i-1}, \mathbf{u}_i \rangle}_{0} + \alpha_i \underbrace{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}_{\|\mathbf{u}_i\|^2} + \alpha_{i+1} \underbrace{\langle \mathbf{u}_{i+1}, \mathbf{u}_i \rangle}_{0} + \ldots + \alpha_k \underbrace{\langle \mathbf{u}_k, \mathbf{u}_i \rangle}_{0} \\ &= \alpha_i \|\mathbf{u}_i\|^2. \end{aligned}$$

A final division by $\|\mathbf{u}_i\|^2$ ($\neq 0$ since $\mathbf{u}_i \neq \mathbf{o}$) shows $\alpha_i = 0$. $\square$

Theorem 5.22 (Gram matrix in the case of an OGB/ONB)

The Gram matrix $G(\mathcal{B})$ of an OGB $\mathcal{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ is a diagonal matrix:

$$G(\mathcal{B}) = \begin{pmatrix} \langle \mathbf{u}_1, \mathbf{u}_1 \rangle & \ldots & \langle \mathbf{u}_k, \mathbf{u}_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{u}_1, \mathbf{u}_k \rangle & \ldots & \langle \mathbf{u}_k, \mathbf{u}_k \rangle \end{pmatrix} = \begin{pmatrix} \|\mathbf{u}_1\|^2 & & 0 \\ & \ddots & \\ 0 & & \|\mathbf{u}_k\|^2 \end{pmatrix}.$$

If $\mathcal{B}$ is even an ONB, then $G(\mathcal{B}) = \mathbf{I}$. For $\mathbf{x} \in \mathbb{R}^n$, the LES

$$G(\mathcal{B}) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}, \mathbf{u}_1 \rangle \\ \vdots \\ \langle \mathbf{x}, \mathbf{u}_k \rangle \end{pmatrix} \qquad \text{has the solution } \alpha_1 = \frac{\langle \mathbf{x}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2},\ \alpha_2 = \frac{\langle \mathbf{x}, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}, \ldots, \alpha_k = \frac{\langle \mathbf{x}, \mathbf{u}_k \rangle}{\|\mathbf{u}_k\|^2}.$$

Hence

$$\mathbf{x}_{\downarrow U} = \frac{\langle \mathbf{x}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \ldots + \frac{\langle \mathbf{x}, \mathbf{u}_k \rangle}{\|\mathbf{u}_k\|^2}\mathbf{u}_k \qquad \text{and} \qquad \mathbf{x}_{\downarrow U^\perp} = \mathbf{x} - \mathbf{x}_{\downarrow U}$$

are the orthogonal projections $\mathbf{x}_{\downarrow U}, \mathbf{x}_{\downarrow U^\perp}$ of $\mathbf{x}$ onto $U = \mathrm{span}(\mathcal{B}) \subset \mathbb{R}^{\mathrm{n}}$ and $U^\perp$, resp..

If $\mathcal{B}$ is even an ONB, then the denominators $\|\mathbf{u}_i\|^2$ are all equal to 1.

Orthogonal systems and bases

Corollary 5.23 (Fourier expansion wrt. an OGB or ONB)

Let U be a subspace of $\mathbb{R}^n$ and $\mathcal{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ an OGB of U. Then the (unique) representation of the vector $\mathbf{x} \in U$ with respect to the basis $\mathcal{B}$ is given by

$$\mathbf{x} = \alpha_1 \mathbf{u}_1 + \ldots + \alpha_k \mathbf{u}_k \qquad \text{with} \qquad \alpha_i = \frac{\langle \mathbf{x}, \mathbf{u}_i \rangle}{\|\mathbf{u}_i\|^2} \text{ for all } i \in \{1, \ldots, k\}.$$

This representation of $\mathbf{x}$ is called a *Fourier expansion* of $\mathbf{x}$ wrt. the basis $\mathcal{B}$, and the α_i are the corresponding *Fourier coefficients*.

If $\mathcal{B}$ is even an ONB, then there holds $\alpha_i = \langle \mathbf{x}, \mathbf{u}_i \rangle$ for all $i = 1, \ldots, k$.

In the case $U = \mathbb{R}^n$ the above holds with $k = n$.

Proof. The statement is a special case of Theorem 5.22 in which $\mathbf{x} \in U$ holds so that $\mathbf{x}_{\downarrow U} = \mathbf{x}$. The coordinates of the projection $\mathbf{x}_{\downarrow U}$ are the coordinates of $\mathbf{x}$ wrt. the OGB $\mathcal{B}$. $\square$

Orthogonal systems and bases

Question: Let $\mathcal{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ be linearly independent.
How can one generate an ONS $\mathcal{C} = (\mathbf{w}_1, \ldots, \mathbf{w}_k)$ with $\operatorname{span}(\mathcal{B}) = \operatorname{span}(\mathcal{C})$?

Example

$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} \right)$$ is a basis of $\operatorname{span}(\mathcal{B})$, but not an ONS.

Is there an ONS $\mathcal{C}$ with $\operatorname{span}(\mathcal{B}) = \operatorname{span}(\mathcal{C})$?

Yes: $$\mathcal{C} = \left(\frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right)$$

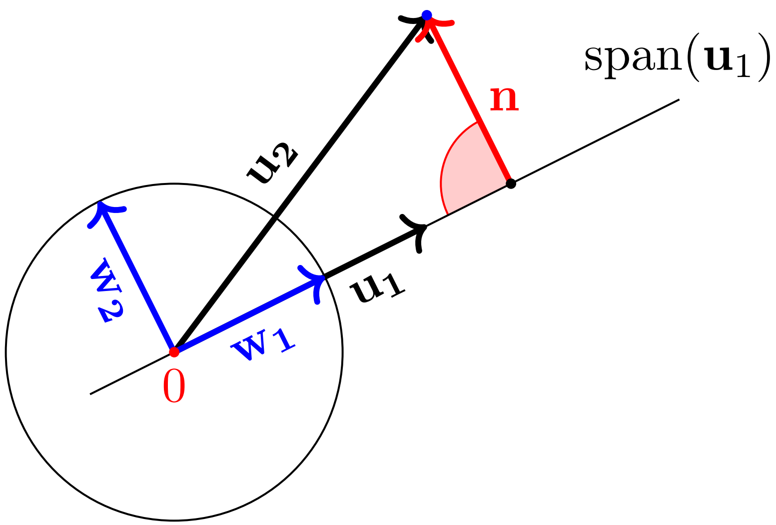
Orthogonal systems and bases

Question: Let $\mathcal{B} = (\mathbf{u}_1, \dots, \mathbf{u}_k)$ be linearly independent.
How can one generate an ONS $\mathcal{C} = (\mathbf{w}_1, \dots, \mathbf{w}_k)$ with $\operatorname{span}(\mathcal{B}) = \operatorname{span}(\mathcal{C})$?

Start with $k = 1$: Set $\mathbf{w}_1 = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1$. Then $\mathcal{C} = (\mathbf{w}_1)$ is an ONS with

$$\operatorname{span}(\mathcal{B}) = \operatorname{span}(\mathbf{u}_1) = \operatorname{span}(\mathbf{w}_1) = \operatorname{span}(\mathcal{C}).$$

Now let $k = 2$: Set $\mathbf{w}_1$ as for $k = 1$. We are looking for $\mathbf{w}_2$ with
(a) $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2) = \operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)$,
(b) $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2)$ is an ONS, i.e., $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0$ and $\|\mathbf{w}_2\| = 1$.



$\mathbf{w}_2$ can be computed as follows:

1. Project $\mathbf{u}_2$ onto $U := \operatorname{span}(\mathbf{w}_1)$.
2. Compute the normal component

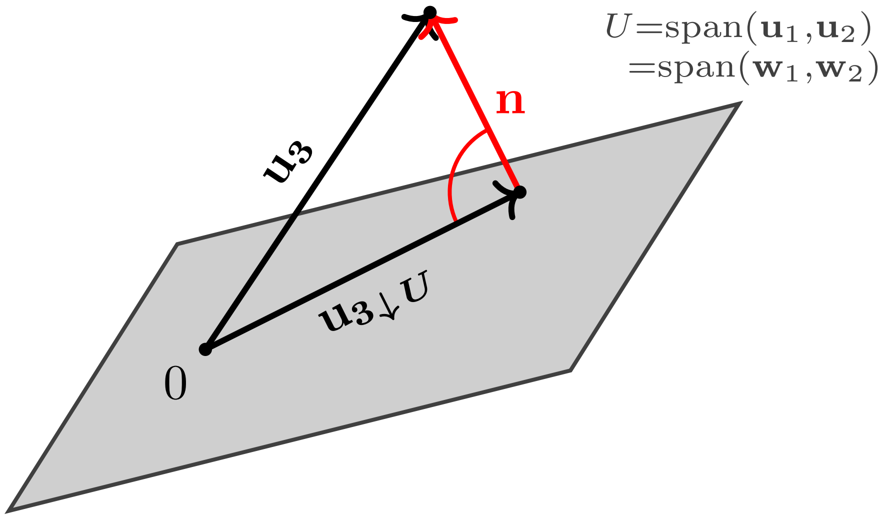
$$\mathbf{n} = \mathbf{u}_2 - \mathbf{u}_{2\downarrow U} = \mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{w}_1 \rangle \mathbf{w}_1.$$

3. Set $\mathbf{w}_2 = \frac{1}{\|\mathbf{n}\|_2}\mathbf{n}$.

Orthogonal systems and bases

Question: Let $\mathcal{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_k)$ be linearly independent.
How can one generate an ONS $\mathcal{C} = (\mathbf{w}_1, \ldots, \mathbf{w}_k)$ with $\mathrm{span}(\mathcal{B}) = \mathrm{span}(\mathcal{C})$?

Now let $k = 3$: Set $\mathbf{w}_1, \mathbf{w}_2$ as for $k = 2$. We are looking for $\mathbf{w}_3$ with
(a) $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \mathrm{span}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$,
(b) $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ is ONS.



$\mathbf{w}_3$ can be computed as follows:

1. Project $\mathbf{u}_3$ onto $U := \mathrm{span}(\mathbf{w}_1, \mathbf{w}_2)$.
2. Compute the normal component

$$\begin{aligned} \mathbf{n} &= \mathbf{u}_3 - \mathbf{u}_{3\downarrow U} \\ &= \mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{u}_3, \mathbf{w}_2 \rangle \mathbf{w}_2. \end{aligned}$$

3. Set $\mathbf{w}_3 = \frac{1}{\|\mathbf{n}\|_2} \mathbf{n}$.

Gram-Schmidt orthonormalization

Let U be a subspace of $\mathbb{R}^n$ with a basis $(\mathbf{u}_1, \ldots, \mathbf{u}_k)$.
The following method produces an orthonormal basis $(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ of U.

1.) The first vector just has to be scaled to length one: $\mathbf{w}_1 := \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1$.

2.) Now take the normal component of $\mathbf{u}_2$ wrt. $\operatorname{span}(\mathbf{w}_1)$

$$\mathbf{n}_2 := \mathbf{u}_2 - \underbrace{\langle \mathbf{u}_2, \mathbf{w}_1\rangle \mathbf{w}_1}_{\mathbf{u}_{2\downarrow \operatorname{span}(\mathbf{w}_1)}} \quad \text{and scale it to length one:} \quad \mathbf{w}_2 := \frac{1}{\|\mathbf{n}_2\|}\mathbf{n}_2.$$

3.) Now take the normal component of $\mathbf{u}_3$ wrt. $\operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)$

$$\mathbf{n}_3 := \mathbf{u}_3 - \underbrace{\big(\langle \mathbf{u}_3, \mathbf{w}_1\rangle \mathbf{w}_1 + \langle \mathbf{u}_3, \mathbf{w}_2\rangle \mathbf{w}_2\big)}_{\mathbf{u}_{3\downarrow \operatorname{span}(\mathbf{w}_1, \mathbf{w}_2)}} \quad \text{and scale it to length one:} \quad \mathbf{w}_3 := \frac{1}{\|\mathbf{n}_3\|}\mathbf{n}_3.$$

$\vdots$

k.) At last, take the normal component of $\mathbf{u}_k$ wrt. $\operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_{\mathrm{k}-1})$

$$\mathbf{n}_k := \mathbf{u}_k - \underbrace{\sum_{i=1}^{k-1} \langle \mathbf{u}_k, \mathbf{w}_i\rangle \mathbf{w}_i}_{\mathbf{u}_{k\downarrow \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_{\mathrm{k}-1})}} \quad \text{and once more scale it to length one:} \quad \mathbf{w}_k := \frac{1}{\|\mathbf{n}_k\|}\mathbf{n}_k.$$

Orthogonal systems and bases

Example 5.25 (Gram-Schmidt orthonormalization)

Determine an ONB of $U = \operatorname{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \operatorname{span}\left(\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}3\\1\\-1\\1\end{pmatrix}, \begin{pmatrix}3\\-1\\1\\3\end{pmatrix}\right)$.

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix},$$

$$\mathbf{n}_2 = \mathbf{u}_2 - \underbrace{\langle \mathbf{u}_2, \mathbf{w}_1\rangle}_{=2}\mathbf{w}_1 = \begin{pmatrix}3\\1\\-1\\1\end{pmatrix} - 2\cdot\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\\-2\\0\end{pmatrix}, \quad \mathbf{w}_2 = \frac{1}{\|\mathbf{n}_2\|}\mathbf{n}_2 = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},$$

$$\mathbf{n}_3 = \mathbf{u}_3 - \underbrace{\langle \mathbf{u}_3, \mathbf{w}_1\rangle}_{=3}\mathbf{w}_1 - \underbrace{\langle \mathbf{u}_3, \mathbf{w}_2\rangle}_{=\sqrt{2}}\mathbf{w}_2 = \frac{1}{2}\begin{pmatrix}1\\-5\\1\\3\end{pmatrix}, \quad \mathbf{w}_3 = \frac{1}{\|\mathbf{n}_3\|}\mathbf{n}_3 = \frac{1}{3}\mathbf{n}_3 = \frac{1}{6}\begin{pmatrix}1\\-5\\1\\3\end{pmatrix}.$$

$(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ is an ONB of U.

True or false? Let

$$\mathbf{v} = \frac{1}{5}\begin{pmatrix}0\\3\\4\end{pmatrix}, \ \mathbf{w} = \begin{pmatrix}1\\4\\-3\end{pmatrix}, \ \mathbf{x} = \begin{pmatrix}3\\0\\1\end{pmatrix}, \text{ and two additional vectors } \mathbf{y}, \mathbf{z} \in \mathbb{R}^3.$$

1. $(\mathbf{v})$ is an ONS.
2. $(\mathbf{v}, \mathbf{w})$ is an ONS.
3. $(\mathbf{v}, \mathbf{w}, \mathbf{x})$ is an OGB.
4. $\left(\frac{1}{\|\mathbf{w}\|}\mathbf{w}, \frac{1}{\|\mathbf{x}\|}\mathbf{x}\right)$ is an ONS.
5. There exists an ONS $\mathcal{B}$ with $\operatorname{span} \mathcal{B} = \operatorname{span}(\mathbf{v}, \mathbf{x})$.
6. If $\mathbf{y}, \mathbf{z}$ are linearly independent, then $(\mathbf{y}, \mathbf{y} \times \mathbf{z})$ is an OGS.
7. If $\mathbf{y}, \mathbf{z}$ are linearly independent, then $(\mathbf{y}, \mathbf{y} \times \mathbf{z}, \mathbf{z})$ is an OGS.
8. If $\mathbf{y}, \mathbf{z}$ are linearly independent, then $(\mathbf{y}, \mathbf{y} \times \mathbf{z}, (\mathbf{y} \times \mathbf{z}) \times \mathbf{z})$ is an OGS.

QR decomposition

Let

$$\mathbf{A} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_k \\ | & & | \end{pmatrix} \in \mathbb{R}^{m \times k}$$

be a matrix with linearly independent columns $\mathbf{a}_1, \dots, \mathbf{a}_k$ in $\mathbb{R}^m$. Then there holds $k \leqslant m$.

Recall the Gram-Schmidt orthonormalization of $(\mathbf{a}_1, \dots, \mathbf{a}_k)$

1.) The first vector just has to be scaled to length one: $\mathbf{q}_1 := \frac{1}{\|\mathbf{a}_1\|}\mathbf{a}_1$.

j.) For $j = 2, \dots, k$, take the normal component of $\mathbf{a}_j$ wrt. $\operatorname{span}(\mathbf{q}_1, \dots, \mathbf{q}_{\mathrm{j}-1})$,

$$\mathbf{n}_j := \mathbf{a}_j - \mathbf{a}_{j\downarrow \operatorname{span}(\mathbf{q}_1,\dots,\mathbf{q}_{\mathrm{j}-1})} = \mathbf{a}_j - \sum_{i=1}^{j-1} \langle \mathbf{a}_j, \mathbf{q}_i \rangle \mathbf{q}_i, \quad \text{and scale it:} \quad \mathbf{q}_j := \frac{1}{\|\mathbf{n}_j\|}\mathbf{n}_j.$$

Solving for $\mathbf{a}_j$ yields:

$$\mathbf{a}_1 = \|\mathbf{a}_1\|\mathbf{q}_1,$$

$$\mathbf{a}_j = \|\mathbf{n}_j\|\mathbf{q}_j + \sum_{i=1}^{j-1} \langle \mathbf{a}_j, \mathbf{q}_i \rangle \mathbf{q}_i, \quad j = 2, \dots, k.$$

QR decomposition

Solving for $\mathbf{a}_j$ yields:

$$
\begin{aligned}
\mathbf{a}_1 &= \|\mathbf{a}_1\|\mathbf{q}_1 \\
\mathbf{a}_2 &= \|\mathbf{n}_2\|\mathbf{q}_2 + \langle \mathbf{a}_2, \mathbf{q}_1\rangle \mathbf{q}_1 \\
&\vdots \\
\mathbf{a}_k &= \|\mathbf{n}_k\|\mathbf{q}_k + \langle \mathbf{a}_k, \mathbf{q}_1\rangle \mathbf{q}_1 + \ldots + \langle \mathbf{a}_k, \mathbf{q}_{k-1}\rangle \mathbf{q}_{k-1}
\end{aligned}
$$

Matrix notation:

$$
\underbrace{\begin{pmatrix} | & & | \\ \mathbf{a}_1 & \ldots & \mathbf{a}_k \\ | & & | \end{pmatrix}}_{\mathbf{A}} = \underbrace{\begin{pmatrix} | & & | \\ \mathbf{q}_1 & \ldots & \mathbf{q}_k \\ | & & | \end{pmatrix}}_{\mathbf{Q}} \underbrace{\begin{pmatrix} \|\mathbf{a}_1\| & \langle \mathbf{a}_2, \mathbf{q}_1\rangle & \cdots & \langle \mathbf{a}_k, \mathbf{q}_1\rangle \\ 0 & \|\mathbf{n}_2\| & & \vdots \\ \vdots & & \ddots & \langle \mathbf{a}_k, \mathbf{q}_{k-1}\rangle \\ 0 & \cdots & 0 & \|\mathbf{n}_k\| \end{pmatrix}}_{\mathbf{R}}
$$

QR decomposition

QR decomposition (compare to Example 5.25)

Let $\mathbf{A} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3) = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$. Determine a QR decomposition of $\mathbf{A}$.

$$\mathbf{q}_1 = \frac{1}{\|\mathbf{a}_1\|}\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{n}_2 = \mathbf{a}_2 - \underbrace{\langle \mathbf{a}_2, \mathbf{q}_1 \rangle}_{=2} \mathbf{q}_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix} - 2 \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \qquad \mathbf{q}_2 = \frac{1}{\|\mathbf{n}_2\|}\mathbf{n}_2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix},$$

$$\mathbf{n}_3 = \mathbf{a}_3 - \underbrace{\langle \mathbf{a}_3, \mathbf{q}_1 \rangle}_{=3} \mathbf{q}_1 - \underbrace{\langle \mathbf{a}_3, \mathbf{q}_2 \rangle}_{=\sqrt{2}} \mathbf{q}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ -5 \\ 1 \\ 3 \end{pmatrix}, \qquad \mathbf{q}_3 = \frac{1}{\|\mathbf{n}_3\|}\mathbf{n}_3 = \frac{1}{3}\mathbf{n}_3 = \frac{1}{6}\begin{pmatrix} 1 \\ -5 \\ 1 \\ 3 \end{pmatrix},$$

$$\Longrightarrow \quad \begin{pmatrix} 1 & 3 & 3 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{5}{6} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{6} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \\ 0 & 2\sqrt{2} & \sqrt{2} \\ 0 & 0 & 3 \end{pmatrix} =: \mathbf{QR}$$

QR decomposition

QR decomposition: important facts at one glance

- In a QR decomposition, $\mathbf{A}$ is represented as a product $\mathbf{Q} \cdot \mathbf{R}$ of a matrix $\mathbf{Q}$ with orthonormal columns and a **r**ight upper triangular matrix $\mathbf{R}$.
- The matrix $\mathbf{Q}$ can be computed column by column by applying the Gram-Schmidt method to the columns of $\mathbf{A}$.
- The matrix $\mathbf{R}$ can be constructed simultaneously during the Gram-Schmidt method. Alternatively, it is computed afterwards via $\mathbf{R} = \mathbf{Q}^\mathsf{T}\mathbf{Q}\mathbf{R} = \mathbf{Q}^\mathsf{T}\mathbf{A}$.
  The diagonal entries r_{jj} are all positive ($r_{jj} = \|\mathbf{n}_j\| > 0$ is the length of the normal vector by which one divides to obtain $\|\mathbf{q}_j\| = 1$).
  Hence $\mathbf{R}$ is invertible.
- Form the point of view of numerical stability, there exist better methods than Gram-Schmidt to determine $\mathbf{Q}$ and $\mathbf{R}$ (Householder reflections, Givens rotations; $\nearrow$ Numerical Mathematics lectures)

Orthogonal matrices

Definition 5.26 (orthogonal matrix)

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with the property $\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{I}$ is called *orthogonal*.

$$\mathbf{A}^\mathsf{T}\mathbf{A} = \begin{pmatrix} \text{---} & {\mathbf{a}_1}^\mathsf{T} & \text{---} \\ & \vdots & \\ \text{---} & {\mathbf{a}_n}^\mathsf{T} & \text{---} \end{pmatrix} \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} {\mathbf{a}_1}^\mathsf{T}\mathbf{a}_1 & \cdots & {\mathbf{a}_1}^\mathsf{T}\mathbf{a}_n \\ \vdots & & \vdots \\ {\mathbf{a}_n}^\mathsf{T}\mathbf{a}_1 & \cdots & {\mathbf{a}_n}^\mathsf{T}\mathbf{a}_n \end{pmatrix}$$

$$= \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{a}_1 \rangle \\ \vdots & & \vdots \\ \langle \mathbf{a}_1, \mathbf{a}_n \rangle & \cdots & \langle \mathbf{a}_n, \mathbf{a}_n \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

The columns $(\mathbf{a}_1, \cdots, \mathbf{a}_n)$ of an orthogonal matrix form an ONS.
$\mathbf{A}^\mathsf{T}$ is the inverse of the orthogonal matrix $\mathbf{A}$.

Orthogonal matrices

Theorem 5.27 (properties of orthogonal matrices)

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix. Then there hold:

a) $\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{I}$.

b) $\mathbf{A}\mathbf{A}^\mathsf{T} = \mathbf{I}$.

c) $\mathbf{A}^{-1} = \mathbf{A}^\mathsf{T}$.

d) $\mathbf{A}^\mathsf{T}$ is an orthogonal matrix.

e) The columns of $\mathbf{A}$ form an ONB of $\mathbb{R}^n$.

f) The rows of $\mathbf{A}$ form an ONB of $\mathbb{R}^n$.

Proof.

a), b) c) and e) are straightforward.
d) follows from b) together with $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$.
f) follows from d).

Orthogonal matrices

Theorem 5.27 (properties of orthogonal matrices)

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix. Then there hold:

g) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ there holds $\langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.

h) For all $\mathbf{x} \in \mathbb{R}^n$ there holds $\|\mathbf{A}\mathbf{x}\| = \|\mathbf{x}\|$. h') $\mathbf{A}$ preserves the length.

i) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ there holds $\angle(\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y}) = \angle(\mathbf{x}, \mathbf{y})$. i') $\mathbf{A}$ preserves the angle.

j) $\det(\mathbf{A}) = \pm 1$.

Proof.

g) follows from calculation: $\langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y} \rangle = (\mathbf{A}\mathbf{x})^\mathsf{T} \mathbf{A}\mathbf{y} = \mathbf{x}^\mathsf{T} \mathbf{A}^\mathsf{T} \mathbf{A}\mathbf{y} = \mathbf{x}^\mathsf{T} \mathbf{I}\mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle$.

h) follows with g): $\|\mathbf{A}\mathbf{x}\|^2 = \langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle = \|\mathbf{x}\|^2$.

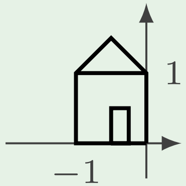
i) follows from $\angle(\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y}) = \arccos\left(\frac{\langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y} \rangle}{\|\mathbf{A}\mathbf{x}\| \|\mathbf{A}\mathbf{y}\|}\right) \overset{\text{g),h)}}{=} \arccos\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}\right) = \angle(\mathbf{x}, \mathbf{y})$.

j) follows in view of $1 = \det(\mathbf{I}) \overset{\text{a)}}{=} \det(\mathbf{A}^\mathsf{T}\mathbf{A}) \overset{\text{Thm. 4.15}}{=} \det(\mathbf{A}^\mathsf{T}) \det(\mathbf{A}) \overset{\text{Thm. 4.9}}{=} \det(\mathbf{A})^2$. □

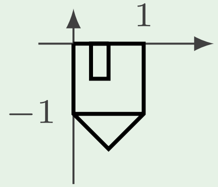
Orthogonal matrices

Houses and orthogonal matrices $\mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{I}, \mathbf{J}, \mathbf{R}$

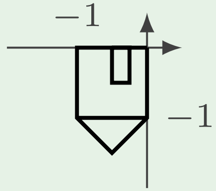
$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



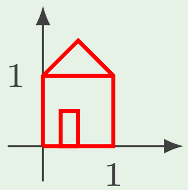
$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



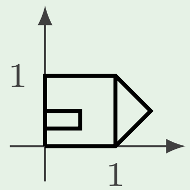
$$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$



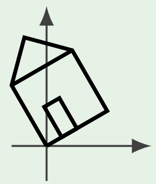
$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad H' = H \qquad f_I = \mathsf{id}$$



$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$\mathbf{R} = \begin{pmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{pmatrix}$$



Lengths, angles and areas do not change under matrix-vector multiplication with orthogonal matrices.

$\mathbf{F}, \mathbf{I}, \mathbf{R}$: Houses are rotated about the origin. Matrices have determinant equal to 1.

$\mathbf{D}, \mathbf{E}, \mathbf{J}$: Houses have been reflected (wrt. y-axis, x-axis or the diagonal $y = x$).
Matrices have determinant equal to -1.

Orthogonal matrices

Definition 5.28 (rotations and reflections)

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be an orthogonal matrix. By Theorem 5.27 j), there holds $\det(\mathbf{A}) = \pm 1$.
In the case $\det(\mathbf{A}) = 1$, $\mathbf{A}$ is called a *rotation*, in the case $\det(\mathbf{A}) = -1$ a *reflection*.

The matrices $\mathbf{F}, \mathbf{I}, \mathbf{R}$ are rotations whereas $\mathbf{D}, \mathbf{E}, \mathbf{J}$ are reflections.

Terminology: Rotation and reflection

a) Not every matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ with $\det(\mathbf{A}) = 1$ (or $\det(\mathbf{A}) = -1$) is a rotation or a reflection. $\mathbf{A}$ also has to be orthogonal.

b) For $n \geqslant 3$, the term "reflection" in Definition 5.28 differs from the intuitive understanding of a reflection with respect to a (hyper-) plane. The 3×3 matrix $-\mathbf{I}$ with the mapping $f_{-\mathbf{I}} : \mathbf{x} \mapsto -\mathbf{x}$ in $\mathbb{R}^3$ is an example of a reflection in the sense of Definition 5.28 but cannot be realized as the reflection wrt. a plane in $\mathbb{R}^3$ (only as a rotary reflection or a reflection wrt. to a point).

Orthogonal matrices

We now take a closer look at orthogonal matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ (i.e., rotations and reflections) for $n = 1, 2, 3$.

We start with $n = 1$.

Rotations and reflections in $\mathbb{R}^{1\times 1}$

The only orthogonal matrices in $\mathbb{R}^{1\times 1}$ are

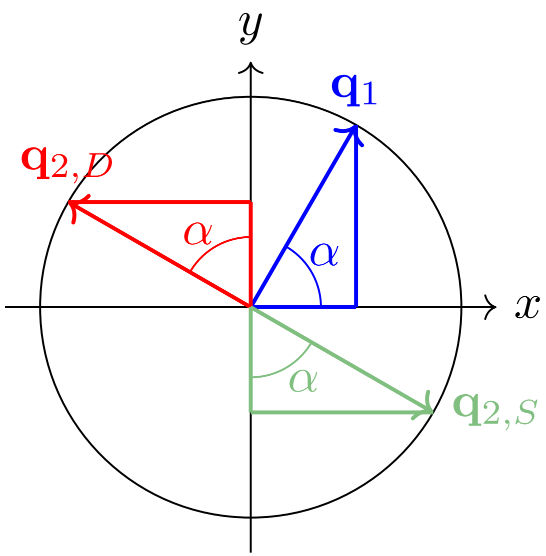
$$\begin{aligned} \mathbf{Q} &= \begin{pmatrix} 1 \end{pmatrix}, && \text{(rotation since } \det Q = 1\text{)}, \\ \mathbf{Q} &= \begin{pmatrix} -1 \end{pmatrix}, && \text{(reflection since } \det Q = -1\text{)}. \end{aligned}$$

Orthogonal matrices

We now consider $n = 2$, i.e., orthogonal matrices in $\mathbb{R}^{2\times 2}$.

The columns of $\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{pmatrix}$ must satisfy:

- $\|\mathbf{q}_1\| = 1 = \|\mathbf{q}_2\| \quad \Longrightarrow \quad \mathbf{q}_1 = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ for an $\alpha \in (-\pi, \pi]$.
- $\langle \mathbf{q}_1, \mathbf{q}_2 \rangle = 0 \quad \Longrightarrow \quad \mathbf{q}_2 = \mathbf{q}_{2,D} := \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix}$ or $\mathbf{q}_2 = \mathbf{q}_{2,S} := \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}$.



Theorem 5.29 (Rotations and reflections in $\mathbb{R}^{2\times 2}$)

Orthogonal matrices in $\mathbb{R}^{2\times 2}$ have the form

$$\mathbf{Q}_D = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad \text{(rotation since } \det Q_D = 1\text{), or}$$

$$\mathbf{Q}_S = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, \quad \text{(reflection since } \det Q_S = -1\text{)}$$

for an $\alpha \in [-\pi, \pi)$.

Orthogonal matrices

Rotation (by the angle α)

$$\mathbf{Q}_D = \begin{pmatrix}\mathbf{q}_1 & \mathbf{q}_2\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix},$$

$$\mathbf{Q}_D\mathbf{e}_1 = \mathbf{q}_1, \quad \mathbf{Q}_D\mathbf{e}_2 = \mathbf{q}_2,$$

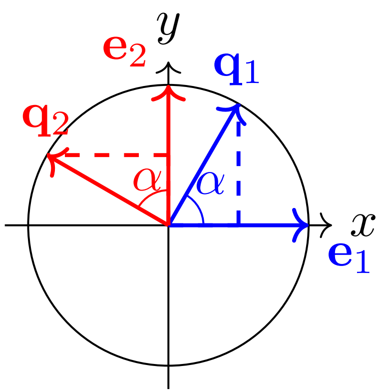
$$\begin{aligned}\mathbf{Q}_D\mathbf{x} = \mathbf{Q}_D\begin{pmatrix}r\cos\varphi \\ r\sin\varphi\end{pmatrix} &= \begin{pmatrix}r(\cos\alpha\cos\varphi - \sin\alpha\sin\varphi) \\ r(\sin\alpha\cos\varphi + \cos\alpha\sin\varphi)\end{pmatrix} \\ &= \begin{pmatrix}r\cos(\varphi+\alpha) \\ r\sin(\varphi+\alpha)\end{pmatrix}\end{aligned}$$

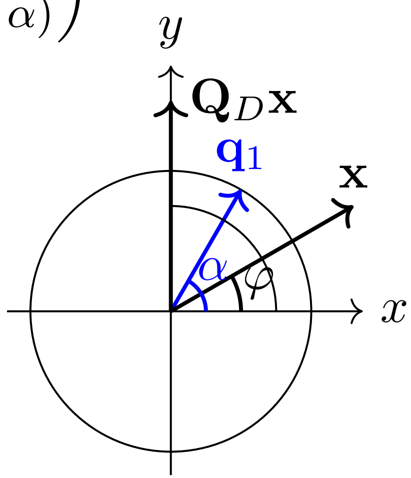
Reflection (wrt. the line with angle $\frac{\alpha}{2}$)

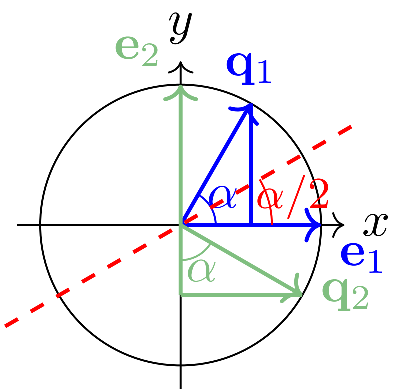
$$\mathbf{Q}_S = \begin{pmatrix}\mathbf{q}_1 & \mathbf{q}_2\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha\end{pmatrix},$$

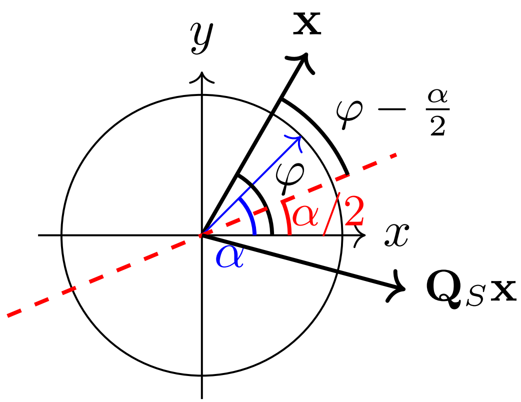
$$\mathbf{Q}_S\mathbf{e}_1 = \mathbf{q}_1, \quad \mathbf{Q}_S\mathbf{e}_2 = \mathbf{q}_2,$$

$$\begin{aligned}\mathbf{Q}_S\mathbf{x} = \mathbf{Q}_S\begin{pmatrix}r\cos\varphi \\ r\sin\varphi\end{pmatrix} &= \begin{pmatrix}r(\cos\alpha\cos\varphi + \sin\alpha\sin\varphi) \\ r(\sin\alpha\cos\varphi - \cos\alpha\sin\varphi)\end{pmatrix} \\ &= \begin{pmatrix}r\cos(\alpha-\varphi) \\ r\sin(\alpha-\varphi)\end{pmatrix} = \begin{pmatrix}r\cos\left(\varphi - 2(\varphi - \frac{\alpha}{2})\right) \\ r\sin\left(\varphi - 2(\varphi - \frac{\alpha}{2})\right)\end{pmatrix}\end{aligned}$$









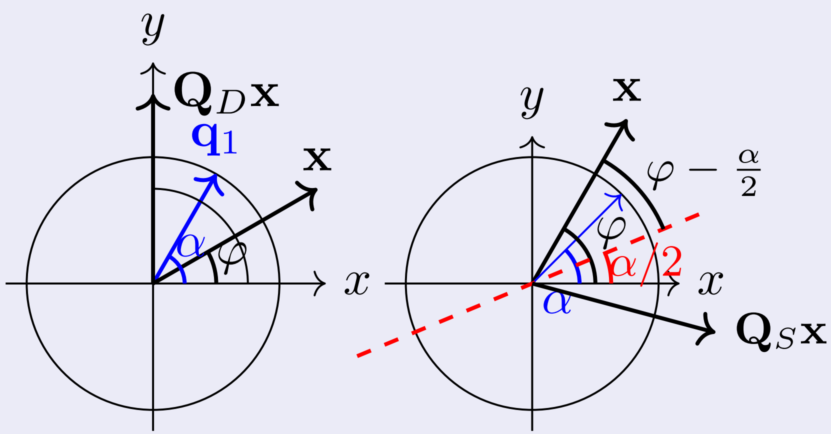
Orthogonal matrices

Extension for Theorem 5.29 (Rotations and reflections in $\mathbb{R}^{2\times 2}$)

Orthogonal matrices in $\mathbb{R}^{2\times 2}$ have the form

$$\mathbf{Q}_D = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad \text{(rotation), or}$$

$$\mathbf{Q}_S = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}, \qquad \text{(reflection)}$$



for an $\alpha \in [-\pi, \pi)$.

Multiplication with $\mathbf{Q}_D$ rotates a vector $\mathbf{x}$ by the angle α.
For $\alpha \in (-\pi, 0) \cup (0, \pi)$ there holds $\mathbf{Q}_D\mathbf{x} \neq \pm\mathbf{x}$ for all $\mathbf{x} \neq \mathbf{o}$.

Multiplication with $\mathbf{Q}_S$ reflects a vector $\mathbf{x}$ wrt. the line that forms the angle $\frac{\alpha}{2}$ with the x-axis.
Each vector along this line, $\mathbf{x} = r\begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \end{pmatrix}$, satisfies $\mathbf{Q}_S\mathbf{x} = \mathbf{x}$.
Each vector orthogonal to this line, $\mathbf{y} = r\begin{pmatrix} \sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2} \end{pmatrix}$, satisfies $\mathbf{Q}_S\mathbf{y} = -\mathbf{y}$.

Orthogonal matrices

At last, we consider $n = 3$, i.e., orthogonal matrices in $\mathbb{R}^{3\times 3}$.

Theorem 5.30a) (properties of orthogonal 3×3 matrices)

Let $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix.

a) If $\mathbf{Q}$ is a rotation, then there exists a vector $\mathbf{v} \in \mathbb{R}^3 \setminus \{\mathbf{o}\}$ with $\mathbf{Q}\mathbf{v} = \mathbf{v}$.
b) If $\mathbf{Q}$ is a reflection, then there exists a vector $\mathbf{v} \in \mathbb{R}^3 \setminus \{\mathbf{o}\}$ with $\mathbf{Q}\mathbf{v} = -\mathbf{v}$.

Proof. There holds

$$\begin{aligned}\det(\mathbf{Q}-\mathbf{I}) &= \det(\mathbf{Q}-\mathbf{Q}\mathbf{Q}^{\mathsf{T}}) = \det(\mathbf{Q}(\mathbf{I}-\mathbf{Q}^{\mathsf{T}}))\\ &= \det\mathbf{Q}\cdot\det(\mathbf{I}-\mathbf{Q}) = \det\mathbf{Q}\cdot(-1)^3\det(\mathbf{Q}-\mathbf{I})\\ &= -\det\mathbf{Q}\cdot\det(\mathbf{Q}-\mathbf{I}).\end{aligned}$$

A rotation satisfies $\det\mathbf{Q} = 1$, and $\det(\mathbf{Q}-\mathbf{I}) = -\det(\mathbf{Q}-\mathbf{I})$ then implies $\det(\mathbf{Q}-\mathbf{I}) = 0$. Hence $\mathbf{Q}-\mathbf{I}$ is singular, i.e., there exists a $\mathbf{v} \neq \mathbf{o}$ with $(\mathbf{Q}-\mathbf{I})\mathbf{v} = \mathbf{o}$ or $\mathbf{Q}\mathbf{v} = \mathbf{I}\mathbf{v} = \mathbf{v}$, resp..
The proof for a reflection follows analogously with
$\det(\mathbf{Q}+\mathbf{I}) = \det\mathbf{Q}\cdot\det(\mathbf{I}+\mathbf{Q}) = -\det(\mathbf{I}+\mathbf{Q})$, hence $\det(\mathbf{I}+\mathbf{Q}) = 0$. $\square$

Orthogonal matrices

Theorem 5.30b) (further properties of orthogonal 3×3 matrices)

Let $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix.

a) If $\mathbf{Q}$ is a rotation, then there exists an ONS $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with
   - $\mathbf{Q}\mathbf{v}_1 = \mathbf{v}_1$,
   - $\mathbf{Q}\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$ for all $\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$.

b) If $\mathbf{Q}$ is a reflection, then there exists an ONS $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with
   - $\mathbf{Q}\mathbf{v}_1 = -\mathbf{v}_1$,
   - $\mathbf{Q}\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$ for all $\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$.

Proof. a) $\mathbf{v}_1$ can be taken as the vector in Theorem 5.30a) normed to length one.
Then $\mathbf{Q}\mathbf{v}_1 = \mathbf{v}_1$ implies $\mathbf{v}_1 = \mathbf{Q}^\mathsf{T}\mathbf{Q}\mathbf{v}_1 = \mathbf{Q}^\mathsf{T}\mathbf{v}_1$.
Let $\mathbf{v}_2, \mathbf{v}_3$ be two (arbitrary) vectors that complement $\mathbf{v}_1$ to an ONS. In particular, then there holds $\mathbf{v}_1^\perp = \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$ (plane through the origin with normal vector $\mathbf{v}_1$). Let $\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$. Then there holds

$$\langle \mathbf{v}_1, \mathbf{Q}\mathbf{w} \rangle = \langle \mathbf{Q}^\mathsf{T}\mathbf{v}_1, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle = 0,$$

and hence $\mathbf{Q}\mathbf{w} \in \mathbf{v}_1^\perp = \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$. The proof of b) follows analogously. $\square$

Orthogonal matrices

Theorem 5.30b) (further properties of orthogonal 3×3 matrices)

Let $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix.

a) If $\mathbf{Q}$ is a rotation, then there exists an ONS $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with
- $\mathbf{Q}\mathbf{v}_1 = \mathbf{v}_1$, ($\mathbf{v}_1$ is the so-called rotation axis),
- $\mathbf{Q}\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$ for all $\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$.

b) If $\mathbf{Q}$ is a reflection, then there exists an ONS $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with
- $\mathbf{Q}\mathbf{v}_1 = -\mathbf{v}_1$,
- $\mathbf{Q}\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$ for all $\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$.

regarding a) It can (later) be shown that the rotation $\mathbf{Q}$ rotates a point $\mathbf{x}$ about the axis $\mathbf{v}_1$ by an angle α. Let $\mathbf{x}$ be represented as a linear combination of the ONS $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$:

$$\mathbf{x} = \underbrace{\langle \mathbf{x}, \mathbf{v}_1 \rangle}_{\gamma_1 :=} \mathbf{v}_1 + \underbrace{\langle \mathbf{x}, \mathbf{v}_2 \rangle}_{\gamma_2 :=} \mathbf{v}_2 + \underbrace{\langle \mathbf{x}, \mathbf{v}_3 \rangle}_{\gamma_3 :=} \mathbf{v}_3 .$$

Then there holds

$$\mathbf{Q}\mathbf{x} = \mathbf{Q}(\gamma_1 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2 + \gamma_3 \mathbf{v}_3) = \gamma_1 \mathbf{v}_1 + \begin{pmatrix} | & | \\ \mathbf{v}_2 & \mathbf{v}_3 \\ | & | \end{pmatrix} \overbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}^{\mathbf{Q}_D \in \mathbb{R}^{2\times 2}} \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix} .$$

Orthogonal matrices

Theorem 5.30b) (further properties of orthogonal 3×3 matrices)

Let $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix.

a) If $\mathbf{Q}$ is a rotation, then there exists an ONS $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with
   - $\mathbf{Q}\mathbf{v}_1 = \mathbf{v}_1$,
   - $\mathbf{Q}\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$ for all $\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$.

b) If $\mathbf{Q}$ is a reflection, then there exists an ONS $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with
   - $\mathbf{Q}\mathbf{v}_1 = -\mathbf{v}_1$, ($\mathbf{v}_1$ is the normal vector of the reflector/mirror),
   - $\mathbf{Q}\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$ for all $\mathbf{w} \in \operatorname{span}(\mathbf{v}_2, \mathbf{v}_3)$.

regarding b) A reflection $\mathbf{Q}$ for $n = 3$ can be visualized as a rotary reflection: A point $\mathbf{x}$ is rotated about the axis $\mathbf{v}_1$ by an angle α and reflected wrt. the plane through the origin with normal vector $\mathbf{v}_1$. Let $\mathbf{x}$ be represented as a linear combination

$$\mathbf{x} = \underbrace{\langle \mathbf{x}, \mathbf{v}_1 \rangle}_{\gamma_1 :=} \mathbf{v}_1 + \underbrace{\langle \mathbf{x}, \mathbf{v}_2 \rangle}_{\gamma_2 :=} \mathbf{v}_2 + \underbrace{\langle \mathbf{x}, \mathbf{v}_3 \rangle}_{\gamma_3 :=} \mathbf{v}_3$$

of the ONS $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Then there holds:

$$\mathbf{Q}\mathbf{x} = \mathbf{Q}(\gamma_1 \mathbf{v}_1 + \gamma_2 \mathbf{v}_2 + \gamma_3 \mathbf{v}_3) = -\gamma_1 \mathbf{v}_1 + \begin{pmatrix} | & | \\ \mathbf{v}_2 & \mathbf{v}_3 \\ | & | \end{pmatrix} \overbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}^{\mathbf{Q}_D \in \mathbb{R}^{2\times 2}} \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix}.$$

True or false?

1. Let $\mathbf{A} \in \mathbb{R}^{5\times 3}$ have a QR decomposition $\mathbf{A} = \mathbf{QR}$. Then there holds $\mathbf{R} = \mathbf{Q}^\mathsf{T}\mathbf{A}$.

2. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be regular with orthogonal columns. Then the matrix $\mathbf{R}$ of the QR factorization of $\mathbf{A}$ is a diagonal matrix.
3. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be regular with QR factorization $\mathbf{A} = \mathbf{QR}$. Then $\mathbf{A}$ has an LU factorization $\mathbf{A} = \mathbf{LU}$ with an upper triangular matrix $\mathbf{U}$ that is equal to $\mathbf{R}$.
4. The following matrices are orthogonal:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

5. Let $\mathbf{Q}_{S,1}, \mathbf{Q}_{S,2} \in \mathbb{R}^{2\times 2}$ be two reflections. Then the product $\mathbf{Q}_{S,2} \cdot \mathbf{Q}_{S,1}$ (i.e., the composition of two reflections) is again a reflection.