Mathematics 1 - Linear Algebra

Lecture 08 – §9.1 LU decomposition

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The transposed matrix

Definition 3.68 (transposed matrix)

Let $\mathbf{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{m\times n}$. Then the matrix $\mathbf{A}^\mathsf{T} := \begin{pmatrix} a_{1,1} & \dots & a_{m,1} \\ \vdots & & \vdots \\ a_{1,n} & \dots & a_{m,n} \end{pmatrix} \in \mathbb{R}^{n\times m}$ is the so-called *transposed matrix* (or just the transpose) of $\mathbf{A}$.
Hence, we obtain the transpose by writing the columns as rows (and vice versa).

Example 3.69 (transposed matrix)

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix} \in \mathbb{R}^{2\times 3} \quad \Rightarrow \quad \mathbf{A}^\mathsf{T} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 2},$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \quad \Rightarrow \quad \mathbf{u}^\mathsf{T} \cdot \mathbf{v} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 4 = \langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle,$$

$$\Rightarrow \quad \mathbf{u} \cdot \mathbf{v}^\mathsf{T} = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \\ 6 & 3 & 0 \end{pmatrix} \quad (\neq \mathbf{v} \cdot \mathbf{u}^\mathsf{T}).$$

The transposed matrix

Theorem 3.72 (rules to compute with transposed matrices)

a) For all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$ there holds: $(\mathbf{A} + \mathbf{B})^\mathsf{T} = \mathbf{A}^\mathsf{T} + \mathbf{B}^\mathsf{T}$.
b) For all $\mathbf{A} \in \mathbb{R}^{m \times n}$ and for all $\alpha \in \mathbb{R}$ there holds: $(\alpha \cdot \mathbf{A})^\mathsf{T} = \alpha \cdot \mathbf{A}^\mathsf{T}$.
c) For all $\mathbf{A} \in \mathbb{R}^{m \times n}$ there holds: $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$.
d)
e)
f) $\mathrm{rank}(\mathbf{A}^\mathsf{T}) = \mathrm{rank}(\mathbf{A})$.
g) For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ there holds: $\mathbf{u}^\mathsf{T} \cdot \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{v}^\mathsf{T} \cdot \mathbf{u}$.

Proof. a), b), c) and g) can easily be checked/computed.

f) This follows from "row rank = column rank" (Theorem 3.54).

The transposed matrix

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b) For all $\mathbf{A} \in \mathbb{R}^{m \times n}$ and for all $\alpha \in \mathbb{R}$ there holds: $(\alpha \cdot \mathbf{A})^\mathsf{T} = \alpha \cdot \mathbf{A}^\mathsf{T}$.
c) For all $\mathbf{A} \in \mathbb{R}^{m \times n}$ there holds: $(\mathbf{A}^\mathsf{T})^\mathsf{T} = \mathbf{A}$.
d) For all $\mathbf{A} \in \mathbb{R}^{m \times n}$ and for all $\mathbf{B} \in \mathbb{R}^{n \times r}$ there holds: $(\mathbf{A} \cdot \mathbf{B})^\mathsf{T} = \mathbf{B}^\mathsf{T} \cdot \mathbf{A}^\mathsf{T}$.
e) If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible, then $\mathbf{A}^\mathsf{T}$ is also invertible with $(\mathbf{A}^\mathsf{T})^{-1} = (\mathbf{A}^{-1})^\mathsf{T}$.
f) $\operatorname{rank}(\mathbf{A}^\mathsf{T}) = \operatorname{rank}(\mathbf{A})$.
g) For all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ there holds: $\mathbf{u}^\mathsf{T} \cdot \mathbf{v} = \mathbf{v}^\mathsf{T} \cdot \mathbf{u} = \langle \mathbf{u}, \mathbf{v} \rangle$.

Proof. d) There holds $\mathbf{AB} \in \mathbb{R}^{m \times r}$, hence $(\mathbf{AB})^\mathsf{T} \in \mathbb{R}^{r \times m}$. There also holds $\mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T} \in \mathbb{R}^{r \times m}$. For both $r \times m$ matrices we compute the entry in the i-th row and j-th column:

$$\left((\mathbf{A} \cdot \mathbf{B})^\mathsf{T}\right)_{ij} = (\mathbf{A} \cdot \mathbf{B})_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki},$$

$$\left(\mathbf{B}^\mathsf{T} \cdot \mathbf{A}^\mathsf{T}\right)_{ij} = \sum_{k=1}^{n} b^T_{ik} a^T_{kj} = \sum_{k=1}^{n} b_{ki} a_{jk} = (\mathbf{A} \cdot \mathbf{B})_{ji}.$$

e) There holds $\mathbf{I} = \mathbf{I}^\mathsf{T} = \left(\mathbf{A}\mathbf{A}^{-1}\right)^\mathsf{T} \overset{\text{d)}}{=} \left(\mathbf{A}^{-1}\right)^\mathsf{T} \mathbf{A}^\mathsf{T}$, hence $(\mathbf{A}^\mathsf{T})^{-1} = (\mathbf{A}^{-1})^\mathsf{T}$. $\square$

The transposed matrix

Theorem 3.73 (the role of $\mathbf{A}^\mathsf{T}$ in the scalar product)

For all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$ there holds

$$\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{A}^\mathsf{T}\mathbf{y} \rangle.$$

Proof. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$ there holds $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\mathsf{T}\mathbf{v}$.
Hence for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$ there holds

$$\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = (\mathbf{A}\mathbf{x})^\mathsf{T}\mathbf{y} = (\mathbf{x}^\mathsf{T}\mathbf{A}^\mathsf{T})\mathbf{y} = \mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T}\mathbf{y}) = \langle \mathbf{x}, \mathbf{A}^\mathsf{T}\mathbf{y} \rangle.$$

$\square$

Definition 3.74 (symmetric and skewsymmetric matrices)

A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called <u>*symmetric*</u> if there holds $\mathbf{A}^\mathsf{T} = \mathbf{A}$.
It is called <u>*skewsymmetric*</u> if there holds $\mathbf{A}^\mathsf{T} = -\mathbf{A}$.

LU decomposition

Example for the solution of an LES with LU decomposition

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ -4 \\ 7 \end{pmatrix}.$$

First solution approach: Gauß-Elimination

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 4 & 6 & -2 & -4 \\ -1 & -4 & 2 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -2 & 2 & 8 \\ 0 & -2 & 1 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -2 & 2 & 8 \\ 0 & 0 & -1 & -4 \end{array}\right)$$

$$\rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

LU decomposition

Example for the solution of an LES with LU decomposition

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 \\ -4 \\ 7 \end{pmatrix}.$$

Second solution approach:

Write $\mathbf{A}$ as the product of two triangular matrices $\mathbf{L}, \mathbf{U}$:

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix}}_{\mathbf{A}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}}_{=:\,\mathbf{L}} \underbrace{\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{pmatrix}}_{=:\,\mathbf{U}}$$

Forward/backward substitution with $\mathbf{L}, \mathbf{U}$

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \Longrightarrow \quad \mathbf{L}(\underbrace{\mathbf{U}\mathbf{x}}_{\mathbf{y}:=}) = \mathbf{b} \quad \Longrightarrow$$

1. Solve $\mathbf{L}\mathbf{y} = \mathbf{b}$ with forward substitution.
2. Solve $\mathbf{U}\mathbf{x} = \mathbf{y}$ with backward substitution.

LU decomposition

The most important admissible operation in Gauss elimination is:

Subtract α_{ij} times the j-th row from the i-th row.

This operation can be expressed as a matrix-matrix multiplication:

$$\underbrace{\underbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & -\alpha_{ij} & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}}_{\substack{\text{identity matrix with } -\alpha_{ij} \\ \text{in row } i \text{ and column } j}}}_{\mathbf{L}_{i,j,\alpha_{ij}} :=} \begin{pmatrix} \text{---}\mathbf{a}_1\text{---} \\ \vdots \\ \text{---}\mathbf{a}_{i-1}\text{---} \\ \text{---}\mathbf{a}_i\text{---} \\ \text{---}\mathbf{a}_{i+1}\text{---} \\ \vdots \\ \text{---}\mathbf{a}_n\text{---} \end{pmatrix} = \begin{pmatrix} \text{---}\mathbf{a}_1\text{---} \\ \vdots \\ \text{---}\mathbf{a}_{i-1}\text{---} \\ \text{---}\ \mathbf{a}_i - \alpha_{ij}\mathbf{a}_j\ \text{---} \\ \text{---}\mathbf{a}_{i+1}\text{---} \\ \vdots \\ \text{---}\mathbf{a}_n\text{---} \end{pmatrix}$$

LU decomposition

Elimination of entries in the first column

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ -1 & -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \boxed{\text{-4}} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ -1 & -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \boxed{\text{-4}} & 1 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

LU decomposition

Elimination of the entries in the first column:

$$\underbrace{\underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ -\alpha_{n,1} & \cdots & 0 & 1 \end{pmatrix}}_{\mathbf{L}_{n,1,\alpha_{n,1}}} \cdots \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\alpha_{2,1} & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}}_{\mathbf{L}_{2,1,\alpha_{2,1}}}}_{= \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\alpha_{2,1} & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ -\alpha_{n,1} & 0 & \cdots & 0 & 1 \end{pmatrix} =: \mathbf{L}_1} \begin{pmatrix} \text{—}\mathbf{a}_1\text{—} \\ \vdots \\ \text{—}\mathbf{a}_i\text{—} \\ \vdots \\ \text{—}\mathbf{a}_n\text{—} \end{pmatrix} = \begin{pmatrix} \text{—}\mathbf{a}_1\text{—} \\ \text{—}\mathbf{a}_2 - \alpha_{2,1}\mathbf{a}_1\text{—} \\ \text{—}\mathbf{a}_3 - \alpha_{3,1}\mathbf{a}_1\text{—} \\ \vdots \\ \text{—}\mathbf{a}_n - \alpha_{n,1}\mathbf{a}_1\text{—} \end{pmatrix}$$

with $\qquad \alpha_{j,1} = \dfrac{a_{j,1}}{a_{1,1}}$ (subtract $\alpha_{j,1}$ times row 1 from row j).

LU decomposition

Analogously, all entries below the diagonal are eliminated successively:

$$\mathbf{L}_{n-1}\mathbf{L}_{n-2}\cdots\mathbf{L}_j \underbrace{\mathbf{L}_{j-1}\ldots\mathbf{L}_1\mathbf{A}}_{\mathbf{A}^{(j-1)}:=} = \mathbf{U},$$

$$\text{with } \mathbf{L}_j := \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ & \ddots & \ddots & & & \vdots \\ & & 1 & \ddots & & \vdots \\ & & -\alpha_{j+1,j} & 1 & \ddots & \vdots \\ & & \vdots & & \ddots & 0 \\ & & -\alpha_{n,j} & & & 1 \end{pmatrix}, \quad \alpha_{k,j} = \frac{a_{k,j}^{(j-1)}}{a_{j,j}^{(j-1)}}.$$

Finally we compute

$$\mathbf{A} = \underbrace{\mathbf{L}_1^{-1}\cdots\mathbf{L}_{n-1}^{-1}}_{?}\mathbf{U}$$

LU decomposition

Example

$$\begin{pmatrix} 1 & 0 & 0 \\ \boxed{\text{-4}} & 1 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{\text{-1}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \boxed{\text{-4}} & 1 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{\text{-4}} & 1 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{\text{-1}} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

LU decomposition

$$\mathbf{A} = \underbrace{\mathbf{L}_1^{-1} \cdots \mathbf{L}_{n-1}^{-1}}_{?} \mathbf{U} = \mathbf{LU}$$

There holds

$$\mathbf{L}_j^{-1} = \begin{pmatrix} 1 & 0 & \cdots & & \cdots & \cdots & 0 \\ & \ddots & \ddots & & & & \vdots \\ & & 1 & & \ddots & & \vdots \\ & & +\alpha_{j+1,j} & & 1 & \ddots & \vdots \\ & & \vdots & & & \ddots & 0 \\ & & +\alpha_{n,j} & & & & 1 \end{pmatrix} \qquad \text{(check that } \mathbf{L}_j^{-1}\mathbf{L}_j = \mathbf{I}\text{),}$$

$$\mathbf{L} := \mathbf{L}_1^{-1} \cdots \mathbf{L}_{n-1}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha_{2,1} & 1 & \ddots & & \vdots \\ \alpha_{3,1} & \alpha_{3,2} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \alpha_{n,1} & \cdots & \cdots & \alpha_{n,n-1} & 1 \end{pmatrix}.$$

LU decomposition

Example

$$\begin{pmatrix} 1 & 2 & -1 \\ 4 & 6 & -2 \\ -1 & -4 & 2 \end{pmatrix} = \underbrace{\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \boxed{\text{-4}} & 1 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix}^{-1}}_{\begin{pmatrix} 1 & 0 & 0 \\ \boxed{4} & 1 & 0 \\ \boxed{\text{-1}} & 0 & 1 \end{pmatrix}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{\text{-1}} & 1 \end{pmatrix}^{-1}}_{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{1} & 1 \end{pmatrix}}}_{\begin{pmatrix} 1 & 0 & 0 \\ \boxed{4} & 1 & 0 \\ \boxed{\text{-1}} & \boxed{1} & 1 \end{pmatrix}} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{A} = \mathbf{LU}$$

LU decomposition

Definition 9.1 (LU factorization/decomposition)

We have an *LU factorization* of $\mathbf{A} \in \mathbb{R}^{m\times n}$ if the following holds:

- $\mathbf{A}$ can be written as a matrix product $\mathbf{L}\cdot\mathbf{U}$.
- Here, $\mathbf{L} \in \mathbb{R}^{m\times m}$ is square and $\mathbf{U} \in \mathbb{R}^{m\times n}$ has the same numbers of rows and columns as $\mathbf{A}$.
- $\mathbf{L}$ is a normed **l**ower triangular matrix: it has ones along the diagonal, zeros above the diagonal, and below the diagonal in position (i, j) it shows how often row i is subtracted from row j in Gauss elimination.
- $\mathbf{U}$ is the result of these elimination steps in row echelon form. If $\mathbf{A}$ is square, then $\mathbf{U}$ is an **u**pper triangular matrix (i.e., all entries below the diagonal are zero).

Example:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 4 & 8 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 12 \\ -3 & -6 & -6 & 8 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & -3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

LU decomposition

Example - computation of $\mathbf{L}, \mathbf{U}$

$$
\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ \boxed{4} & 8 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 12 \\ -3 & -6 & -6 & 8 & 4 \end{pmatrix}
\rightsquigarrow
\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 4 \\ \boxed{0} & 0 & 2 & 3 & 12 \\ \boxed{-3} & -6 & -6 & 8 & 4 \end{pmatrix}
\qquad
\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 4 \\ 0 & 0 & \boxed{2} & 3 & 12 \\ 0 & 0 & -6 & 11 & 4 \end{pmatrix}
\rightsquigarrow
\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & \boxed{-6} & 11 & 4 \end{pmatrix}
\qquad
\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & & 1 & 0 \\ -3 & & & 1 \end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & \boxed{8} & 16 \end{pmatrix}
\qquad
\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & -3 & & 1 \end{pmatrix}
$$

$$
\rightsquigarrow
\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{U},
\qquad
\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & -3 & 2 & 1 \end{pmatrix}
$$

LU decomposition

True or false?

1. $\mathbf{A} + \mathbf{A}^T$ is symmetric.
2. If $\mathbf{C} \in \mathbb{R}^{n \times n}$ is a lower triangular matrix, then $\mathbf{C}^T$ is an upper triangular matrix.
3. The following matrix products are LU decompositions:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

4. Let $\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{A} \in \mathbb{R}^{4 \times 4}$. $\mathbf{A}$ and $\mathbf{AL}$ differ only in the third column.

LU decomposition

The second admissible operation in Gauss elimination is

> Exchange the i-th and j-th row.

This operation can also be expressed as a matrix-matrix multiplication:

Let $\mathbf{P}_{i,j}$ be the matrix that results from exchanging the i-th and j-th row of the $n \times n$ identity matrix $\mathbf{I}$. Then there holds

$$\mathbf{P}_{i,j} \begin{pmatrix} \text{—}\mathbf{a}_1\text{—} \\ \vdots \\ \text{—}\mathbf{a}_i\text{—} \\ \vdots \\ \text{—}\mathbf{a}_j\text{—} \\ \vdots \\ \text{—}\mathbf{a}_n\text{—} \end{pmatrix} = \begin{pmatrix} \text{—}\mathbf{a}_1\text{—} \\ \vdots \\ \text{—}\mathbf{a}_j\text{—} \\ \vdots \\ \text{—}\mathbf{a}_i\text{—} \\ \vdots \\ \text{—}\mathbf{a}_n\text{—} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & | & & | & \\ \dots & \mathbf{b}_i & \dots & \mathbf{b}_j & \dots \\ & | & & | & \end{pmatrix} \mathbf{P}_{i,j} = \begin{pmatrix} & | & & | & \\ \dots & \mathbf{b}_j & \dots & \mathbf{b}_i & \dots \\ & | & & | & \end{pmatrix}.$$

The matrix $\mathbf{P}_{i,j}$ is invertible:

$$\mathbf{P}_{i,j}^{-1} = \mathbf{P}_{i,j}.$$

LU decomposition

Example for swapping two rows/columns

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{\mathbf{P}_{1,3}} \begin{pmatrix} 0 & 2 & -1 \\ 4 & 6 & -2 \\ 1 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 \\ 4 & 6 & -2 \\ 0 & 2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & -1 \\ 4 & 6 & -2 \\ 1 & -4 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{\mathbf{P}_{1,3}} = \begin{pmatrix} -1 & 2 & 0 \\ -2 & 6 & 4 \\ 2 & -4 & 1 \end{pmatrix}$$

LU decomposition

So far: Gauss elimination with only the first admissible operation (Subtraction of the multiple of one row from another row, $\mathbf{L}_j$):

$$\mathbf{L}_{n-1} \cdots \mathbf{L}_1 \mathbf{A} = \mathbf{U} \qquad \Longrightarrow \qquad \mathbf{A} = \mathbf{L}_1^{-1} \cdots \mathbf{L}_{n-1}^{-1} \mathbf{U} = \mathbf{L}\mathbf{U}$$

Now: Gauss elimination with both admissible operations (i.e., also row exchanges, $\mathbf{P}_{ij}$):

$$\mathbf{L}_{n-1} \mathbf{P}_{n-1,j_{n-1}} \mathbf{L}_{n-2} \cdots \mathbf{L}_2 \mathbf{P}_{2,j_2} \mathbf{L}_1 \mathbf{P}_{1,j_1} \mathbf{A} = \mathbf{U} \qquad \Longrightarrow \qquad \mathbf{A} = \mathbf{?}$$

We will show:

$$\mathbf{L}_{n-1} \mathbf{P}_{n-1,j_{n-1}} \mathbf{L}_{n-2} \cdots \mathbf{L}_1 \mathbf{P}_{1,j_1} \mathbf{A} = \mathbf{L}_{n-1} \mathbf{L}'_{n-2} \cdots \mathbf{L}'_1 \underbrace{\mathbf{P}_{n-1,j_{n-1}} \cdots \mathbf{P}_{1,j_1}}_{\mathbf{P}:=} \mathbf{A} = \mathbf{U}$$

$$\Longrightarrow \qquad \mathbf{P}\mathbf{A} = \underbrace{(\mathbf{L}'_1)^{-1} \cdots \mathbf{L}_{n-1}^{-1}}_{\mathbf{L}:=} \mathbf{U} = \mathbf{L}\mathbf{U}$$

LU decomposition

We will use the following property: For $k, \ell > j$ there holds

$$\mathbf{P}_{k,\ell}\mathbf{L}_j = \mathbf{P}_{k,\ell}\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & & 0 \\ & \ddots & \ddots & & & & \vdots \\ & -\alpha_{j+1,j} & 1 & \ddots & & & \vdots \\ & \vdots & & \ddots & \ddots & & \vdots \\ & -\alpha_{k,j} & & & 1 & 0 & \vdots \\ & -\alpha_{\ell,j} & & & & 1 \quad 0 & \vdots \\ & \vdots & & & & \ddots & 0 \\ & -\alpha_{n,j} & & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & & 0 \\ & \ddots & \ddots & & & & \vdots \\ & -\alpha_{j+1,j} & 1 & \ddots & & & \vdots \\ & \vdots & & \ddots & \ddots & & \vdots \\ & -\alpha_{\ell,j} & & & 0 & 1 & \vdots \\ & -\alpha_{k,j} & & & 1 & 0 & \vdots \\ & \vdots & & & & \ddots & 0 \\ & -\alpha_{n,j} & & & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & & 0 \\ & \ddots & \ddots & & & & \vdots \\ & -\alpha_{j+1,j} & 1 & \ddots & & & \vdots \\ & \vdots & & \ddots & \ddots & & \vdots \\ & -\alpha_{\ell,j} & & & 1 & 0 & \vdots \\ & -\alpha_{k,j} & & & 0 & 1 & \vdots \\ & \vdots & & & & \ddots & 0 \\ & -\alpha_{n,j} & & & & & 1 \end{pmatrix}\mathbf{P}_{k,\ell}$$

LU decomposition

Hence it follows that

$$\begin{aligned}
&\mathbf{L}_{n-1}\left(\mathbf{P}_{n-1,j_{n-1}}\mathbf{L}_{n-2}\right)\mathbf{P}_{n-2,j_{n-2}}\mathbf{L}_{n-3}\cdots\mathbf{L}_1\mathbf{P}_{1,j_1}\mathbf{A}\\
&=\mathbf{L}_{n-1}\mathbf{L}'_{n-2}\left(\mathbf{P}_{n-1,j_{n-1}}\mathbf{P}_{n-2,j_{n-2}}\mathbf{L}_{n-3}\right)\cdots\mathbf{L}_1\mathbf{P}_{1,j_1}\mathbf{A}\\
&=\mathbf{L}_{n-1}\mathbf{L}'_{n-2}\mathbf{L}'_{n-3}\mathbf{P}_{n-1,j_{n-1}}\mathbf{P}_{n-2,j_{n-2}}\cdots\mathbf{L}_1\mathbf{P}_{1,j_1}\mathbf{A}\\
&=\cdots\\
&=\mathbf{L}_{n-1}\mathbf{L}'_{n-2}\cdots\mathbf{L}'_1\underbrace{\mathbf{P}_{n-1,j_{n-1}}\cdots\mathbf{P}_{1,j_1}}_{\mathbf{P}:=}\mathbf{A}=\mathbf{U}
\end{aligned}$$

and hence

$$\mathbf{P}\mathbf{A}=(\mathbf{L}'_1)^{-1}\cdots(\mathbf{L}'_{n-2})^{-1}\mathbf{L}_{n-1}^{-1}\mathbf{U}=\mathbf{L}\mathbf{U}.$$

$\mathbf{P}$ is a (row) permutation matrix, $\mathbf{L}$ a normed lower and $\mathbf{U}$ an upper triangular matrix.

LU decomposition

Practical computation of $\mathbf{PA} = \mathbf{LU}$

Write the multipliers α_{ij} (entries of the lower triangular matrix $\mathbf{L}$) directly into the place of the just eliminated zero entries, well separated from the actual matrix entries. This way the multipliers α_{ij} are automatically exchanged as well when two rows are exchanged.

In addition, one keeps track of the numbering of the original rows in order to obtain the permutation matrix $\mathbf{P}$ at the end.

The multipliers α_{ij} participate in the row exchanges but **not** in the subtraction of a multiple of one row from another.

$$
\begin{array}{r}1:\\2:\\3:\\4:\end{array}
\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 4 & 8 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 12 \\ -3 & -6 & -6 & 8 & 4 \end{pmatrix}
\rightsquigarrow \quad \cdots \quad \rightsquigarrow
\begin{array}{r}1:\\2:\\3:\\4:\end{array}
\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ \boxed{4} & 0 & 2 & -1 & 4 \\ \boxed{0} & 0 & \boxed{1} & 4 & 8 \\ \boxed{-3} & 0 & \boxed{-3} & \boxed{2} & 0 \end{pmatrix}
$$

LU decomposition

Example 9.2 (LU factorization with row exchange)

$$\begin{array}{r} 1: \\ 2: \\ 3: \\ 4: \end{array}\underbrace{\begin{pmatrix} 0 & 0 & 0 & 3 & 2 \\ 2 & -1 & 1 & -1 & -2 \\ 4 & -1 & 3 & 0 & -3 \\ 6 & -7 & 4 & 0 & -2 \end{pmatrix}}_{\mathbf{A}:=} \overset{\mathbf{P}_{1,2}\cdot}{\rightsquigarrow} \begin{array}{r} 2: \\ 1: \\ 3: \\ 4: \end{array}\begin{pmatrix} 2 & -1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 3 & 2 \\ 4 & -1 & 3 & 0 & -3 \\ 6 & -7 & 4 & 0 & -2 \end{pmatrix} \overset{\mathbf{L}_1\cdot}{\rightsquigarrow} \begin{array}{r} 2: \\ 1: \\ 3: \\ 4: \end{array}\begin{pmatrix} 2 & -1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 3 & 2 \\ 2 & 1 & 1 & 2 & 1 \\ 3 & -4 & 1 & 3 & 4 \end{pmatrix}$$

$$\overset{\mathbf{P}_{2,3}\cdot}{\rightsquigarrow} \begin{array}{r} 2: \\ 3: \\ 1: \\ 4: \end{array}\begin{pmatrix} 2 & -1 & 1 & -1 & -2 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 3 & -4 & 1 & 3 & 4 \end{pmatrix} \overset{\mathbf{L}_2\cdot}{\rightsquigarrow} \begin{array}{r} 2: \\ 3: \\ 1: \\ 4: \end{array}\begin{pmatrix} 2 & -1 & 1 & -1 & -2 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 3 & 2 \\ 3 & -4 & 5 & 11 & 8 \end{pmatrix} \overset{\mathbf{P}_{3,4}\cdot}{\rightsquigarrow} \begin{array}{r} 2: \\ 3: \\ 4: \\ 1: \end{array}\begin{pmatrix} 2 & -1 & 1 & -1 & -2 \\ 2 & 1 & 1 & 2 & 1 \\ 3 & -4 & 5 & 11 & 8 \\ 0 & 0 & 0 & 3 & 2 \end{pmatrix}$$

$$\Longrightarrow \underbrace{\begin{pmatrix} \text{---}\mathbf{e}_2\text{---} \\ \text{---}\mathbf{e}_3\text{---} \\ \text{---}\mathbf{e}_4\text{---} \\ \text{---}\mathbf{e}_1\text{---} \end{pmatrix}}_{\mathbf{P}} \underbrace{\begin{pmatrix} 0 & 0 & 0 & 3 & 2 \\ 2 & -1 & 1 & -1 & -2 \\ 4 & -1 & 3 & 0 & -3 \\ 6 & -7 & 4 & 0 & -2 \end{pmatrix}}_{\mathbf{A}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} 2 & -1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 5 & 11 & 8 \\ 0 & 0 & 0 & 3 & 2 \end{pmatrix}}_{\mathbf{U}}$$

Example 9.2 (continued)

Use the LU factorization for the solution of $\mathbf{Ax} = \begin{pmatrix} 5 \\ -1 \\ 3 \\ 1 \end{pmatrix}$.

$$\mathbf{Ax} = \mathbf{b} \iff \mathbf{PAx} = \mathbf{Pb} \iff \mathbf{L}\underbrace{(\mathbf{Ux})}_{\mathbf{y}:=} = \mathbf{Pb}$$

1. Solve $\mathbf{Ly} = \mathbf{Pb}$ with forward substitution:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 5 \end{pmatrix} \implies \begin{array}{l} y_1 = -1 \\ y_2 = 3 - 2y_1 = 5 \\ y_3 = 1 - 3y_1 + 4y_2 = 24 \\ y_4 = 5 \end{array}$$

2. Solve $\mathbf{Ux} = \mathbf{y}$ with backward substitution (and free variable $x_5 =: \lambda$)

$$\begin{pmatrix} 2 & -1 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 5 & 11 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 24 \\ 5 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ 1 \\ 8 \\ 2 \end{pmatrix} \implies \mathcal{L} = \left\{ \frac{1}{30} \begin{pmatrix} 1 \\ 16 \\ 34 \\ 50 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 29/30 \\ 7/15 \\ -2/15 \\ -2/3 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

LU decomposition

True or false?

1. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and let $\mathbf{P}_{i,j}$ be the matrix that results from exchanging the i-th and j-th row of the identity matrix $\mathbf{I}_n$.
   Then multiplication from the left, $\mathbf{P}_{i,j}\mathbf{A}$, exchanges the i-th and j-th row whereas multiplication from the right, $\mathbf{A}\mathbf{P}_{i,j}$, exchanges the i-th and j-th column of $\mathbf{A}$.
2. Every matrix $\mathbf{A}$ has an LU factorization in the form $\mathbf{A} = \mathbf{L}\mathbf{U}$.
3. Every matrix $\mathbf{A}$ has an LU factorization in the form $\mathbf{P}\mathbf{A} = \mathbf{L}\mathbf{U}$ where $\mathbf{P}$ denotes a (row) permutation matrix.