Prep Course Mathematics

Differentiation

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   - Rules for derivatives
   - Higher derivatives
2. Application of differentiation
   - Monotonicity
   - Curvature behaviour
   - Extrema
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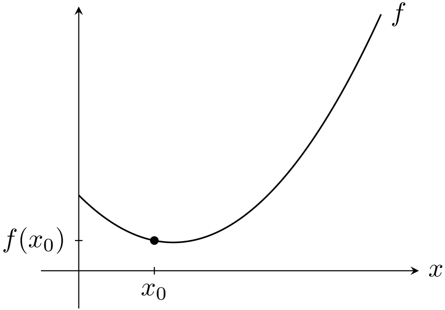
Differentiation

Slopes of secants and tangents, differentiability

For interval D and function $f\colon D \to \mathbb{R}$:



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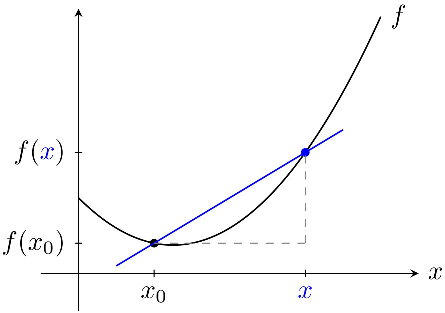

slope of secant between x_0 and x:

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

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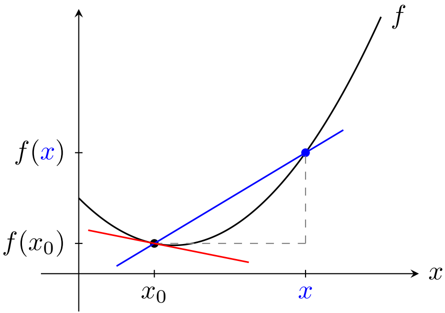

slope of secant between x_0 and x:

$$\frac{f(x) - f(x_0)}{x - x_0}.$$

slope of tangent at x_0:
approximation by secant slopes for x tending to x_0.
notation: $f'(x_0)$ or $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$ for slope of tangent;
first derivative of f at x_0.
Then f differentiable at x_0.

Example

For function f given by $f(x) := x^2$ for $x \in \mathbb{R}$:

At first $x_0 = 3$:

- slope of secant between x and 3:

$$\frac{f(x) - f(3)}{x - 3} = \frac{x^2 - 3^2}{x - 3} = \frac{(x+3)(x-3)}{x - 3} = x + 3.$$

- slope of tangent at 3:
  slope of secant $x + 3$ tends to 6 for x tending to 3, hence $f'(3) = 6$.

Example

For function f given by $f(x) := x^2$ for $x \in \mathbb{R}$:

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- slope of secant between x and 3:

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Now general $x_0 \in \mathbb{R}$:

- slope of secant between x and x_0:

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{(x+x_0)(x-x_0)}{x-x_0} = x + x_0.$$

- slope of tangent at x_0:
  slope of secant $x + x_0$ tends to $2x_0$ for x tending to x_0, hence $f'(x_0) = 2x_0$.

Exercise

Show that the function f given by $f(x) := \sqrt{x}$ for $x > 0$ is differentiable at $x_0 = 1$.

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Slope of secant between x and 1:

$$\frac{f(x) - f(1)}{x - 1} = \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \frac{\sqrt{x} - \sqrt{1}}{(\sqrt{x} - \sqrt{1})(\sqrt{x} + \sqrt{1})} = \frac{1}{\sqrt{x} + \sqrt{1}}.$$

Therefore, slope of tangent at 1:
slope of secant $\frac{1}{\sqrt{x}+\sqrt{1}}$ tends to $\frac{1}{2}$ for x tending to 1, hence $f'(1) = \frac{1}{2}$.

Derivatives of elementary functions and rules for derivatives

$f(x)$	$f'(x)$
$c \quad (c \in \mathbb{R})$	0
$x^\alpha \quad (\alpha \neq 0)$	$\alpha \cdot x^{\alpha-1}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
$a^x \quad (a > 0)$	$\log(a) \cdot a^x$
$\log(x)$	$\frac{1}{x}$
$\log_a(x) \quad (a > 0)$	$\frac{1}{\log(a) \cdot x}$

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For differentiable f, g:

Constant factor rule: for $c \in \mathbb{R}$

$$(c \cdot f)'(x) = c \cdot f'(x)$$

Sum rule:

$$(f \pm g)'(x) = f'(x) \pm g'(x)$$

Product rule:

$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient rule:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Chain rule:

$$(f(g))'(x) = f'(g(x)) \cdot g'(x)$$

Example

First derivative of $f\colon \mathbb{R} \to \mathbb{R}$ given by:

- $f(x) := 3x^4 + 5x^3 - 2x^2 + x + 8$:

$$f'(x) = 12x^3 + 15x^2 - 4x + 1.$$

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- $f(x) := 3x^4 + 5x^3 - 2x^2 + x + 8$:

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- $f(x) := x^2 \sin(x)$:
  We set $u(x) := x^2$, $v(x) := \sin(x)$. Then $u'(x) = 2x$, $v'(x) = \cos(x)$, and

$$f'(x) = u'(x)v(x) + u(x)v'(x) = 2x\sin(x) + x^2\cos(x).$$

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$$f'(x) = u'(x)v(x) + u(x)v'(x) = 2x\sin(x) + x^2\cos(x).$$

- $f(x) := \frac{x^2+7}{\cos(x)+2}$: We set $u(x) := x^2 + 7$, $v(x) := \cos(x) + 2$. Then $u'(x) = 2x$, $v'(x) = -\sin(x)$, and

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{2x(\cos(x) + 2) + (x^2 + 7)\sin(x)}{(\cos(x) + 2)^2}.$$

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First derivative of $f\colon \mathbb{R} \to \mathbb{R}$ given by:

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$$f'(x) = u'(x)v(x) + u(x)v'(x) = 2x\sin(x) + x^2\cos(x).$$
- $f(x) := \frac{x^2+7}{\cos(x)+2}$: We set $u(x) := x^2 + 7$, $v(x) := \cos(x) + 2$. Then $u'(x) = 2x$, $v'(x) = -\sin(x)$, and
$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{2x(\cos(x) + 2) + (x^2 + 7)\sin(x)}{(\cos(x) + 2)^2}.$$
- $f(x) := \mathrm{e}^{3x+\sin(x)}$: We set $u(x) := \mathrm{e}^x$, $v(x) := 3x + \sin(x)$. Then $u'(x) = \mathrm{e}^x$, $v'(x) = 3 + \cos(x)$, and
$$f'(x) = u'(v(x))v'(x) = \mathrm{e}^{3x+\sin(x)}(3 + \cos(x)).$$

Exercise

Compute the first derivative of the function f:

- $f(x) := x^2 \mathrm{e}^x$ for $x \in \mathbb{R}$.
- $f(x) := \frac{\cos(x)}{\mathrm{e}^x}$ for $x \in \mathbb{R}$.
- $f(x) := \sin(\mathrm{e}^x)$ for $x \in \mathbb{R}$.

Exercise

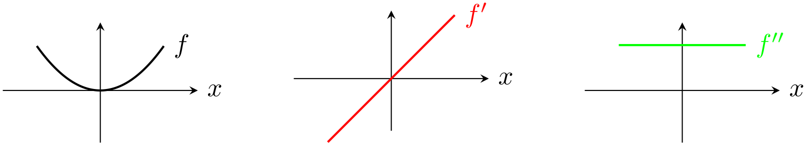
Compute the first derivative of the function f:

- $f(x) := x^2 \mathrm{e}^x$ for $x \in \mathbb{R}$.
  We set $u(x) := x^2$, $v(x) := \mathrm{e}^x$. Then $u'(x) = 2x$, $v'(x) = \mathrm{e}^x$, and
  $$f'(x) = u'(x)v(x) + u(x)v'(x) = 2x\mathrm{e}^x + x^2\mathrm{e}^x = (x^2 + 2x)\mathrm{e}^x.$$
- $f(x) := \frac{\cos(x)}{\mathrm{e}^x}$ for $x \in \mathbb{R}$.
  We set $u(x) := \cos(x)$, $v(x) := \mathrm{e}^x$. Then $u'(x) = -\sin(x)$, $v'(x) = \mathrm{e}^x$, and
  $$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = \frac{-\sin(x)\mathrm{e}^x - \cos(x)\mathrm{e}^x}{\mathrm{e}^{2x}} = \frac{-\sin(x) - \cos(x)}{\mathrm{e}^x}.$$
- $f(x) := \sin(\mathrm{e}^x)$ for $x \in \mathbb{R}$.
  We set $u(x) := \sin(x)$, $v(x) := \mathrm{e}^x$. Then $u'(x) = \cos(x)$, $v'(x) = \mathrm{e}^x$, and
  $$f'(x) = u'(v(x))v'(x) = \cos(\mathrm{e}^x)\mathrm{e}^x.$$

Higher derivatives

For interval D and function $f\colon D \to \mathbb{R}$:
If f differentiable at all $x \in D$, then derivative $f'\colon D \to \mathbb{R}$.

For $x_0 \in D$:
If f' differentiable at x_0, then $f''(x_0) := (f')'(x_0)$ second derivative of f at x_0.
Analogously: f''', $f^{(4)}$, $f^{(5)}$ etc.



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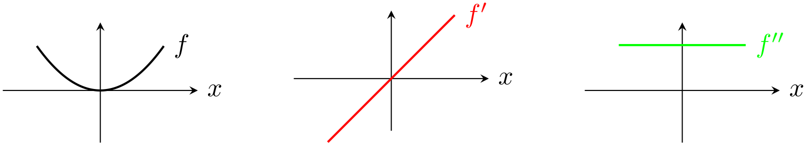

Exercise: Function f given by $f(x) := x^2 \sin(x)$ for $x \in \mathbb{R}$. Then, for $x \in \mathbb{R}$:

$$f'(x) =$$

$$f''(x) =$$

$$f'''(x) =$$

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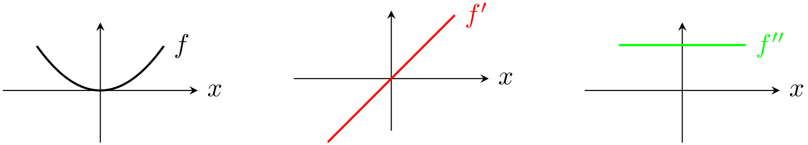


Exercise: Function f given by $f(x) := x^2 \sin(x)$ for $x \in \mathbb{R}$. Then, for $x \in \mathbb{R}$:

$$f'(x) = 2x\sin(x) + x^2\cos(x),$$

$$f''(x) = (2 - x^2)\sin(x) + 4x\cos(x),$$

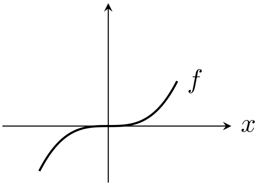
$$f'''(x) = -6x\sin(x) + (6 - x^2)\cos(x).$$

Application of differentiation

Monotonicity and first derivative

For interval D and function $f\colon D \to \mathbb{R}$:
Then f

- monotonically increasing, if $f(x_1) \leqslant f(x_2)$ for all $x_1, x_2 \in D$ such that $x_1 \leqslant x_2$,
- monotonically decreasing, if $f(x_1) \geqslant f(x_2)$ for all $x_1, x_2 \in D$ such that $x_1 \leqslant x_2$.



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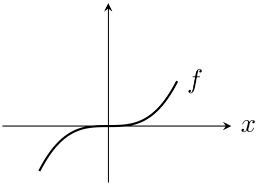
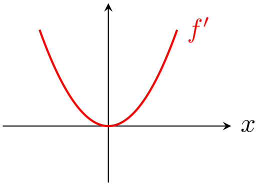
For f differentiable:

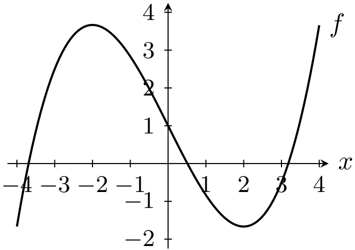
- f monotonically increasing if and only if $f'(x) \geqslant 0$ for all $x \in D$,
- f monotonically decreasing if and only if $f'(x) \leqslant 0$ for all $x \in D$.





Example

Determination of the regions of monotonicity of function f given by $f(x) := \frac{1}{6}x^3 - 2x + 1$ for $x \in \mathbb{R}$.



We calculate:

$$f'(x) = \frac{1}{2}x^2 - 2,$$

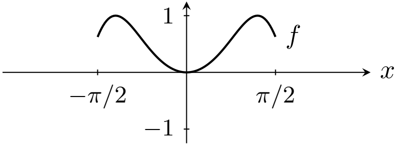
and solve $f'(x) \stackrel{!}{=} 0$. The solutions are $x = \pm 2$. Since f' parabola opening to the top:

- $f'(x) \geqslant 0$ for $x \leqslant -2$ and for $x \geqslant 2$,
- $f'(x) \leqslant 0$ for $-2 \leqslant x \leqslant 2$.

Thus, f monotonically increasing for $x \leqslant -2$ and for $x \geqslant 2$, and monotonically decreasing for $-2 \leqslant x \leqslant 2$.

Exercise

Determine the regions of monotonicity of the function f given by $f(x) := \sin(x^2)$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.



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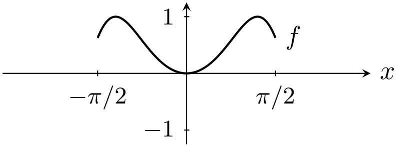


We calculate:

$$f'(x) = \cos(x^2) \cdot 2x.$$

Hence:

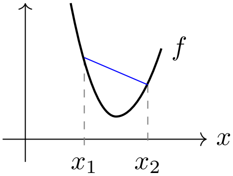
- f monotonically increasing for $-\frac{\pi}{2} \leqslant x \leqslant -\sqrt{\frac{\pi}{2}}$ and for $0 \leqslant x \leqslant \sqrt{\frac{\pi}{2}}$,
- f monotonically decreasing for $-\sqrt{\frac{\pi}{2}} \leqslant x \leqslant 0$ and for $\sqrt{\frac{\pi}{2}} \leqslant x \leqslant \frac{\pi}{2}$.

Curvature behaviour and second derivative

For interval D and function $f\colon D \to \mathbb{R}$:

Then f

- (strictly) left curved or (strictly) convex, if f goes through a left turn, i.e. for all $x, x_1, x_2 \in D$ such that $x_1 \leqslant x \leqslant x_2$ the point $(x, f(x))$ is (strictly) below the secant to $(x_1, f(x_1))$ and $(x_2, f(x_2))$.



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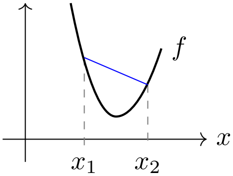
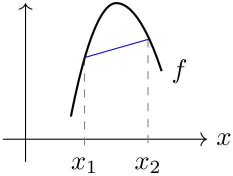
- (strictly) right curved or (strictly) concave, if f goes through a right turn, i.e. for all $x, x_1, x_2 \in D$ such that $x_1 \leqslant x \leqslant x_2$ the point $(x, f(x))$ is (strictly) above the secant to $(x_1, f(x_1))$ and $(x_2, f(x_2))$.





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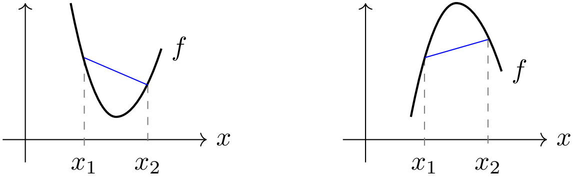
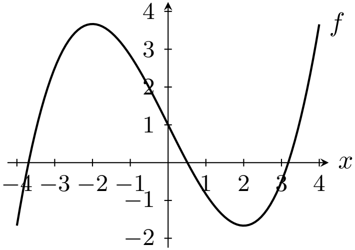


For f two times differentiable:

- f left curved if and only if $f''(x) \geqslant 0$ for all $x \in D$.
- f right curved if and only if $f''(x) \leqslant 0$ for all $x \in D$.

Example

Determination of the curvature behaviour of function f given by $f(x) := \frac{1}{6}x^3 - 2x + 1$ for $x \in \mathbb{R}$.



We calculate:

$$f'(x) = \frac{1}{2}x^2 - 2, \qquad f''(x) = x,$$

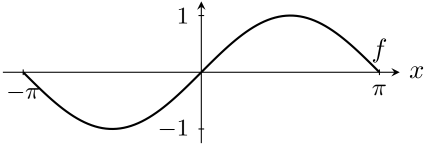
and solve $f''(x) \stackrel{!}{=} 0$. The solution is $x = 0$. Since f'' increasing affine function:

- $f''(x) \geqslant 0$ for $x \geqslant 0$,
- $f''(x) \leqslant 0$ for $x \leqslant 0$.

Hence, f right curved for $x \leqslant 0$, and left curved for $x \geqslant 0$.

Exercise

Determine the curvature behaviour of the function f given by $f(x) := \sin(x)$ for $x \in [-\pi, \pi]$.



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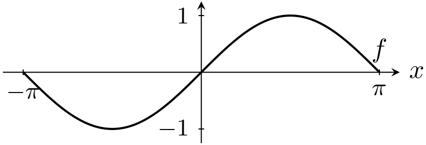

We calculate:

$$f'(x) = \cos(x),$$
$$f''(x) = -\sin(x).$$

Hence:

- f left curved for $x \in [-\pi, 0]$,
- f right curved for $x \in [0, \pi]$.

Local extrema

For interval D and function $f\colon D \to \mathbb{R}$:
Then $x_0 \in D$

- local maximum, if $f(x) \leqslant f(x_0)$ for all x close to x_0.
- local minimum, if $f(x) \geqslant f(x_0)$ for all x close to x_0.
- local extremum, if local maximum or minimum.

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- global maximum, if $f(x) \leqslant f(x_0)$ for all $x \in D$.
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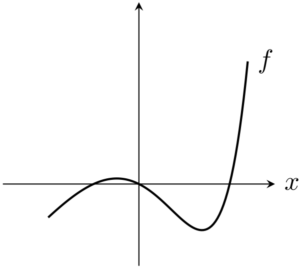

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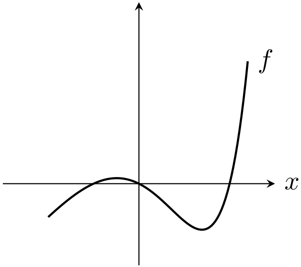
- global minimum, if $f(x) \geqslant f(x_0)$ for all $x \in D$.



For x_0 interior point, f differentiable in x_0:
If f has a local extremum at x_0, then $f'(x_0) = 0$.

- $x_0 \in D$ stationary point, if $f'(x_0) = 0$.

Local extrema

For interval D and function $f\colon D \to \mathbb{R}$:
Then $x_0 \in D$

- local maximum, if $f(x) \leqslant f(x_0)$ for all x close to x_0.
- local minimum, if $f(x) \geqslant f(x_0)$ for all x close to x_0.
- local extremum, if local maximum or minimum.
- global maximum, if $f(x) \leqslant f(x_0)$ for all $x \in D$.
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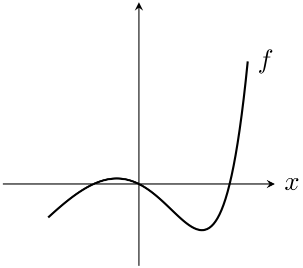


For x_0 interior point, f differentiable in x_0:
If f has a local extremum at x_0, then $f'(x_0) = 0$.

- $x_0 \in D$ stationary point, if $f'(x_0) = 0$.

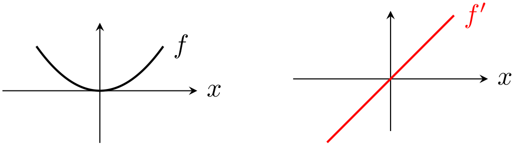
⚠ For local extrema at boundary points x_0 we can have $f'(x_0) \neq 0$.

Criteria for extrema

For open interval D, function $f\colon D \to \mathbb{R}$, and stationary point $x_0 \in D$:

...using the first derivative: for f differentiable:

- If $f'(x) \geqslant 0$ for $x < x_0$ and $f'(x) \leqslant 0$ for $x > x_0$, then f has a local maximum in x_0.
- If $f'(x) \leqslant 0$ for $x < x_0$ and $f'(x) \geqslant 0$ for $x > x_0$, then f has a local minimum in x_0.



Criteria for extrema

For open interval D, function $f\colon D \to \mathbb{R}$, and stationary point $x_0 \in D$:

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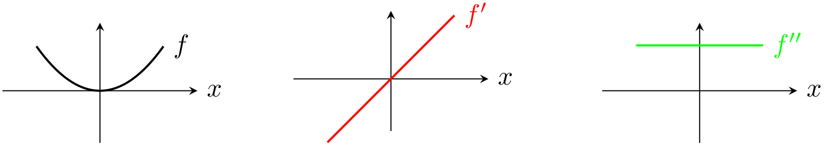
- If $f'(x) \geqslant 0$ for $x < x_0$ and $f'(x) \leqslant 0$ for $x > x_0$, then f has a local maximum in x_0.
- If $f'(x) \leqslant 0$ for $x < x_0$ and $f'(x) \geqslant 0$ for $x > x_0$, then f has a local minimum in x_0.

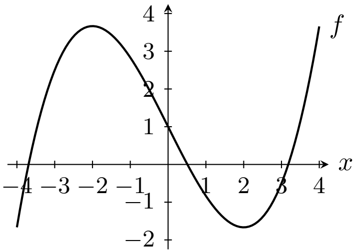
...using the second derivative: For f two times differentiable:

- If $f''(x_0) < 0$, then f has a local maximum in x_0.
- If $f''(x_0) > 0$, then f has a local minimum in x_0.



Example

Determination of extrema of function f given by $f(x) := \frac{1}{6}x^3 - 2x + 1$ for $x \in \mathbb{R}$.



We calculate:

$$f'(x) = \frac{1}{2}x^2 - 2,$$

and solve $f'(x) \stackrel{!}{=} 0$. The solutions are $x = \pm 2$.
We compute

$$f''(x) = x.$$

Since $f''(-2) = -2 < 0$: $x = -2$ local maximum.
Since $f''(2) = 2 > 0$: $x = 2$ local minimum.

Exercise

Determine the local extrema of the function f given by $f(x) := x^2 \mathrm{e}^{-x^2}$ for $x \in \mathbb{R}$.

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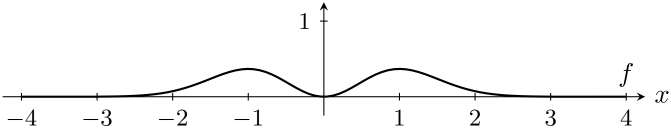


We calculate:

$$f'(x) = 2xe^{-x^2} + x^2 e^{-x^2} \cdot (-2x) = 2x(1 - x^2)e^{-x^2},$$

and solve $f'(x) \overset{!}{=} 0$. The solutions are $x = 0$ und $x = \pm 1$.
We compute

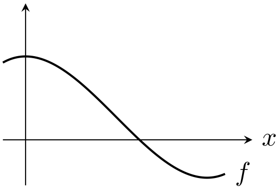
$$f''(x) = (4x^4 - 10x^2 + 2)e^{-x^2}.$$

- Since $f''(0) = 2 > 0$: $x = 0$ local minimum.
- Since $f''(\pm 1) = -4e^{-1} < 0$: $x = \pm 1$ local maximum.

Inflection points

For open interval D, function $f\colon D \to \mathbb{R}$, and $x_0 \in D$:

- x_0 inflection point of f, if f changes curvature behaviour strictly at x_0.



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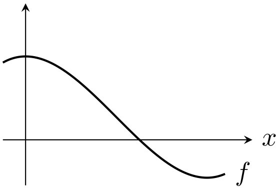
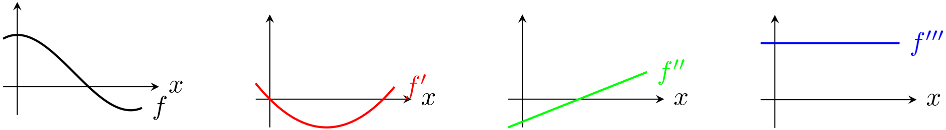


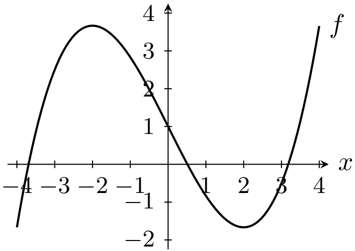
Criteria for inflection points:

- If f is two times differentiable, x_0 inflection point, then $f''(x_0) = 0$.
- If f is three times differentiable, $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 inflection point.



Example

Determination of inflection points for function f given by $f(x) := \frac{1}{6}x^3 - 2x + 1$ for $x \in \mathbb{R}$.



We calculate:

$$f'(x) = \frac{1}{2}x^2 - 2, \qquad f''(x) = x,$$

and solve $f''(x) \stackrel{!}{=} 0$. The solution is $x = 0$.
We compute

$$f'''(x) = 1.$$

Since $f'''(0) = 1 \neq 0$: $x = 0$ inflection point.

Exercise

Determine the inflection points of the function f given by $f(x) := \mathrm{e}^{-x^2}$ for $x \in \mathbb{R}$.

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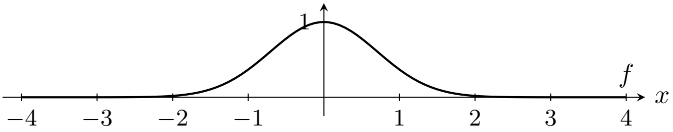
Determine the inflection points of the function f given by $f(x) := \mathrm{e}^{-x^2}$ for $x \in \mathbb{R}$.



We calculate:

$$f'(x) = -2x\mathrm{e}^{-x^2},$$
$$f''(x) = (4x^2 - 2)\mathrm{e}^{-x^2},$$

and solve $f''(x) \stackrel{!}{=} 0$. The solutions are $x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$.
We compute

$$f'''(x) = (-8x^3 + 12x)\mathrm{e}^{-x^2}.$$

Since

- $f'''\left(-\frac{\sqrt{2}}{2}\right) = -4\sqrt{2}\mathrm{e}^{-\frac{1}{2}} \neq 0$: $x = -\frac{\sqrt{2}}{2}$ inflection point.
- $f'''\left(+\frac{\sqrt{2}}{2}\right) = +4\sqrt{2}\mathrm{e}^{-\frac{1}{2}} \neq 0$: $x = +\frac{\sqrt{2}}{2}$ inflection point.