Homework 5 for
Mathematics I (winter term 25/26)

Submit your solutions to **Problems 2 and 4** until Sunday, **November 30**, 11:59 pm at the latest, using **abGabi**. Only one of these problems will be chosen by us to be corrected and graded. We strongly suggest that you **submit in pairs**. Please state the names and matriculation numbers of both persons on your sumbissions and only submit once per group (the other person will still receive credit). Submission in larger groups is not permitted.

Problem 1 *Sequences*

For each of the following sequences $(x_n)_{n\in\mathbb{N}}$, determine all accumulation points, limit inferior and limit superior. Also, indicate whether it converges (and if so, to which limit) or diverges.

(a) $x_n = \sin\left(\frac{n\pi}{2}\right)$,

(b) $x_n = \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$,

(c) $x_n = \frac{(-1)^n 2n^3+2n-1}{4n^3-n+2}$.

Problem 2 *Series*

Determine whether the following series converge or diverge.

(a) $\sum_{k=1}^{\infty} \frac{1}{3^k}$,

(b) $\sum_{k=1}^{\infty} \frac{k+1}{k+2}$,

(c) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{4k-1}}$.

Problem 3 *Linear maps*

Which of the following functions are linear? Justify your decision.

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \cdot x_2 \\ x_1 + x_2 \end{pmatrix}$
- $g : \mathbb{R} \to \mathbb{R}^2$ with $x \mapsto \begin{pmatrix} x \\ -x \end{pmatrix}$
- $h : \mathbb{R} \to \mathbb{R}$ with $x \mapsto 2x + 1$

Problem 4 *System of linear equations*

Let the following two systems of linear equations be given:

$$\begin{array}{rcrcrcr} 5x_1 & + & 3x_2 & + & x_3 & = & 2, \\ 20x_1 & + & 15x_2 & + & 6x_3 & = & 12, \\ 15x_1 & + & 3x_2 & + & x_3 & = & -1, \end{array} \tag{LES 1}$$

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & - & 3x_3 & + & x_4 & = & 1, \\ 8x_1 & + & 12x_2 & - & 12x_3 & + & 4x_4 & = & 4, \\ -4x_1 & - & 6x_2 & + & 15x_3 & + & 4x_4 & = & 1, \\ 6x_1 & + & 10x_2 & - & 12x_3 & + & x_4 & = & 3. \end{array} \tag{LES 2}$$

(a) Write these systems as an augmented matrix.

(b) Bring these systems to a row echelon form.

(c) Mark all pivot elements.

(d) Determine the solution sets of the given systems.

2.

a)

$\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

first term $a = \frac{1}{3}$

ratio $r = \frac{1}{3}$

$|r| = \left|\frac{1}{3}\right| = \frac{1}{3}$

The absolute value of the common ratio is less than 1

That means it converges.

b) $\sum_{k=1}^{\infty} \frac{k+1}{k+2} = \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

$$\lim_{k\to\infty} \frac{k+1}{k+2} = \lim_{k\to\infty} \frac{\frac{k}{k}+\frac{1}{k}}{\frac{k}{k}+\frac{2}{k}} = \lim_{k\to\infty} \frac{1+\frac{1}{k}}{1+\frac{2}{k}}$$

Divergence: $\lim_{k\to\infty} a_k \neq 0$

$$\lim_{k\to\infty} a_k = \frac{1+0}{1+0} = 1$$

Due to $\lim_{k\to\infty} a_k$ being 1 and not 0, means it diverges.

c)

$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{4k-1}}$

↳ $b_k = \frac{1}{\sqrt{4k-1}}$

↳ $\lim_{k\to\infty} = \frac{1}{\sqrt{4k-1}}$

limit of b_k is 0?

$$\lim_{k\to\infty} \frac{1}{\sqrt{4k-1}} = 0$$

b_k is decreasing?

$\sqrt{4k-1}$ increases?

$4(k+1)-1 = 4k+3$

↳ since $4k+3 > 4k-1$

$\sqrt{4(k+1)-1} > \sqrt{4k-1}$

$= b_{k+1} < b_k$

due to both conditions being satisfied, the series converges.

Problem 4 *System of linear equations*

Let the following two systems of linear equations be given:

$$\begin{array}{rcrcrcr} 5x_1 &+& 3x_2 &+& x_3 &=& 2, \\ 20x_1 &+& 15x_2 &+& 6x_3 &=& 12, \\ 15x_1 &+& 3x_2 &+& x_3 &=& -1, \end{array} \qquad \text{(LES 1)}$$

$$\begin{array}{rcrcrcrcr} 2x_1 &+& 3x_2 &-& 3x_3 &+& x_4 &=& 1, \\ 8x_1 &+& 12x_2 &-& 12x_3 &+& 4x_4 &=& 4, \\ -4x_1 &-& 6x_2 &+& 15x_3 &+& 4x_4 &=& 1, \\ 6x_1 &+& 10x_2 &-& 12x_3 &+& x_4 &=& 3. \end{array} \qquad \text{(LES 2)}$$

(a) Write these systems as an augmented matrix.

(b) Bring these systems to a row echelon form.

(c) Mark all pivot elements.

(d) Determine the solution sets of the given systems.

LES 1:

(a)

$$\begin{array}{c} \\ (I) \\ (II) \\ (III) \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array} \\ \left(\begin{array}{ccc|c} 5 & 3 & 1 & 2 \\ 20 & 15 & 6 & 12 \\ 15 & 3 & 1 & -1 \end{array}\right) \end{array}$$

(b)

$(I)\ (5\ \ 3\ \ 1\ |\ 2)\ |\cdot(-4) = (-20\ \ -12\ \ -4\ |\ -8) \to (II)+(I) =$

$(0\ \ 3\ \ 2\ |\ 4)$

$(I)\ (5\ \ 3\ \ 1\ |\ 2)\ |\cdot(-3) = (-15\ \ -9\ \ -3\ |\ -6) \to (III)+(I) =$

$(0\ \ -6\ \ -2\ |\ -7)$

$(II)\ (0\ \ 3\ \ 2\ |\ 4)\ |\cdot 2 = (0\ \ 6\ \ 4\ |\ 8) \to (III)+(II) =$

$(0\ \ 0\ \ 2\ |\ 1)$

(c) $\left(\begin{array}{ccc|c} 5 & 3 & 1 & 2 \\ 0 & 3 & 2 & 4 \\ 0 & 0 & 2 & 1 \end{array}\right)$

(d)

(III): $2x_3 = 1 \quad | :2$

$x_3 = 0{,}5$

(II): $3x_2 + 1 = 4 \quad | -1$

$3x_2 = 3 \quad | :3$

$x_2 = 1$

(I): $5x_1 + 3 + 0{,}5 = 2 \quad | -3{,}5$

$5x_1 = 5{,}5 \quad | :5$

$x_1 = 1{,}1$

$v = \begin{pmatrix} 1{,}1 \\ 1 \\ 0{,}5 \end{pmatrix}$

LES 2:

(a)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left(\begin{array}{cccc|c} 2 & 3 & -3 & 1 & 1 \\ 8 & 12 & -12 & 4 & 4 \\ -4 & -6 & 15 & 4 & 1 \\ 6 & 10 & -12 & 1 & 3 \end{array}\right)$$

(b)

$(I) \cdot (-4) = (-8 \quad -12 \quad 12 \quad -4 \quad (-4)) \to (II) + (I) =$

$(0 \quad 0 \quad 0 \quad 0 \mid 0) \to$ infinite solutions, treat x_4 as arbitrary $z \in \mathbb{R}$

$(I) \cdot 2 = (4 \quad 6 \quad -6 \quad 2 \mid 2) \to (III) + (I) =$

$(0 \quad 0 \quad 9 \quad 6 \mid 3)$

$(I) \cdot (-3) = (-6 \quad -9 \quad 9 \quad -3 \mid -3) \to (IV) + (I) =$

$(0 \quad 1 \quad -3 \quad -2 \mid 0)$

rearrange LES:

(c) $\left(\begin{array}{cccc|c} 2 & 3 & -3 & 1 & 1 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 9 & 6 & 3 \end{array}\right)$

$(III): \; 9x_3 + 6x_4 = 3 \quad | -6x_4$

$9x_3 = 3 - 6x_4 \quad | :9$

$x_3 = \frac{1}{3} - \frac{2}{3}x_4$

$(II): x_2 - 3\left(\frac{1}{3} - \frac{2}{3}x_4\right) = 0$

$x_2 - 1 + 2x_4 = 0 \quad | +1 \quad | -2x_4$

$x_2 = 1 - 2x_4$

$(I): 2x_1 + 3(1 - 2x_4) + \left(\frac{1}{3} - \frac{2}{3}x_4\right) = 1$

$2x_1 + 3 - 6x_4 + \frac{1}{3} - \frac{2}{3}x_4 = 1$

$\frac{10}{3} + 2x_1 - 6\frac{2}{3}x_4 = 1 \quad | -\frac{10}{3}$

$2x_1 - 6\frac{2}{3}x_4 = -\frac{7}{3} \quad | +6\frac{2}{3}x_4$

$2x_1 = -\frac{7}{3} + 6\frac{2}{3}x_4 \quad | :2$

(d)

$x_1 = -\frac{7}{6} + 3\frac{1}{3}x_4$

$$v = \begin{pmatrix} -\frac{7}{6} + 3\frac{1}{3}x_4 \\ 1 - 2x_4 \\ \frac{1}{3} - \frac{2}{3}x_4 \\ x_4 \end{pmatrix} \text{ with } x_4 \in \mathbb{R}$$