Prep Course Mathematics

Elementary functions

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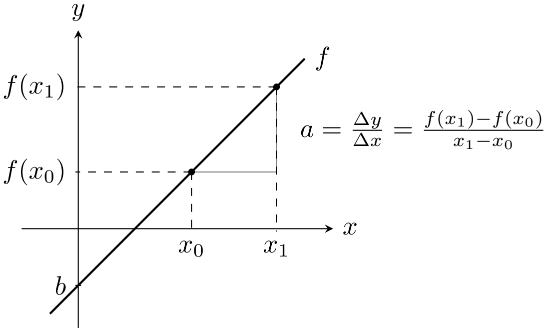
Classes of elementary functions

Affine functions

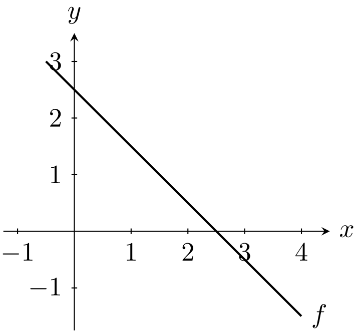
... functions f acting as $f(x) := ax + b$ for $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$

- a slope, b intercept



Example

We determine the term describing the affine function f:



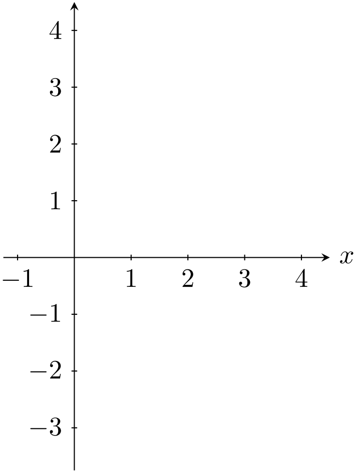
We read off:

- $b = \frac{5}{2}$
- $a = \frac{0-\frac{5}{2}}{\frac{5}{2}-0} = -1$.

Hence $f(x) = -x + \frac{5}{2}$.

Exercise

Draw a sketch of f and g given by $f(x) := \frac{1}{2}x + 1$ and $g(x) := -2x + 4$ for $x \in \mathbb{R}$.



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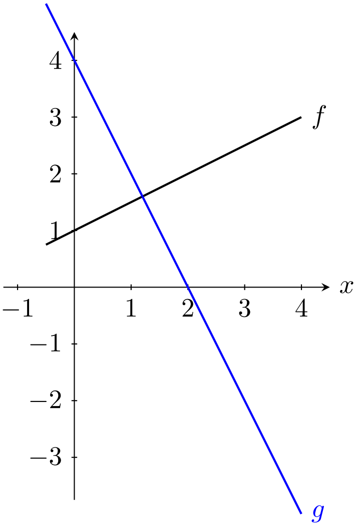


Quadratic functions

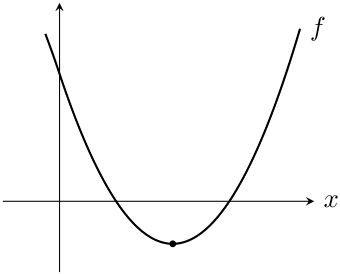
... functions f acting as $f(x) := ax^2 + bx + c$ for $x \in \mathbb{R}$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$

- $a > 0$: opening to the top
- $a < 0$: opening to the bottom

Vertex form via completing the square:

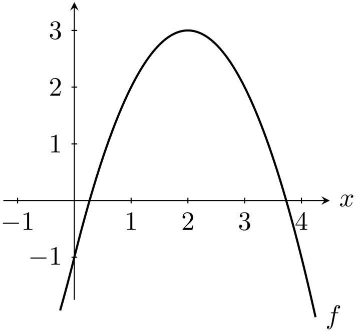
$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - a\frac{b^2}{4a^2}.$$

Vertex $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.



Example

We calculate the vertex form of f given by $f(x) := -x^2 + 4x - 1$ for $x \in \mathbb{R}$.

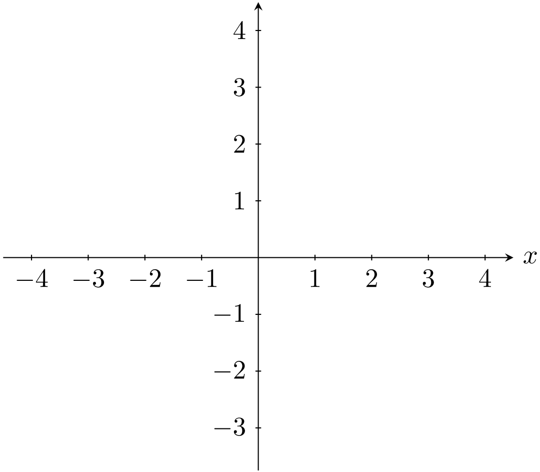


2 possible methods:

- read of the vertex: $(2, 3)$.
- completing the square: $f(x) = -(x-2)^2 + 3$, hence vertex $(2, 3)$.

Exercise

Sketch the function f given by $f(x) := \frac{1}{2}x^2 + x - 1$ for $x \in \mathbb{R}$.



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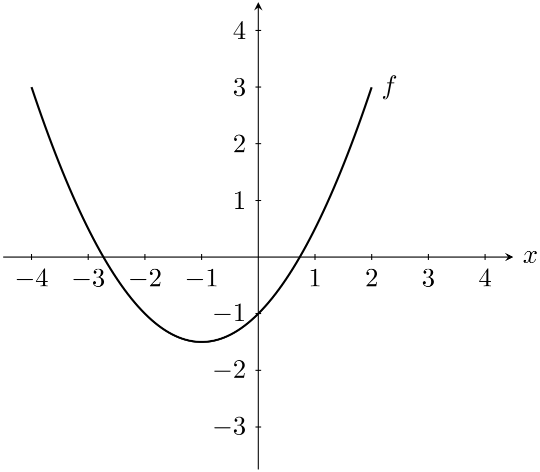
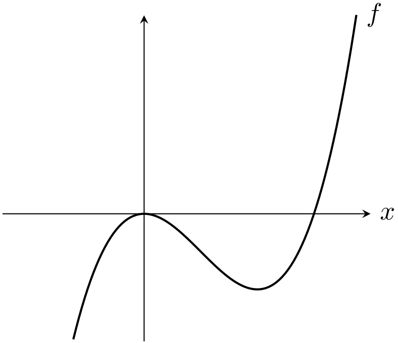


Polynomial functions

... functions f acting as $f(x) := a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ for $x \in \mathbb{R}$, where $a_0, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$

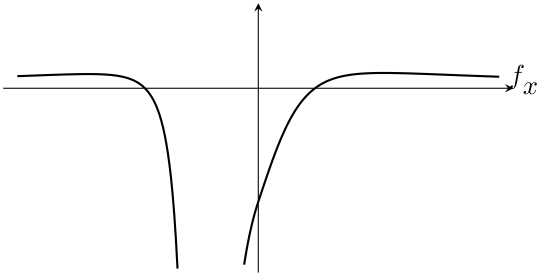


Example:

- f given by $f(x) := x^3 - 3x^2$ for $x \in \mathbb{R}$.
- f given by $f(x) := x^7 - 5x^6 + 3x^3 - 2x^2 - x + 3$ for $x \in \mathbb{R}$.

Rational functions

... functions f acting as $f(x) := \frac{p(x)}{q(x)}$ for $x \in \mathbb{R}$ such that $q(x) \neq 0$, where p, q are polynomial functions.



Example:

- f given by $f(x) := \frac{x^3 - 3x^2}{(x^2+1)(x-1)}$ for $x \in \mathbb{R}$, $x \neq 1$.
- f given by $f(x) := \frac{x^2+1}{(x+2)(x-1)(x-4)}$ for $x \in \mathbb{R}$, $x \neq -2, 1, 4$.

Power functions, root functions

... functions f acting as $f(x) := x^{m/n} = \sqrt[n]{x^m} = \sqrt[n]{x}^m$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$

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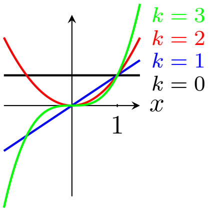
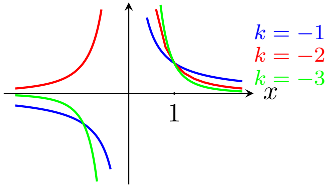
For $k := \frac{m}{n}$ integer:

- $k \geqslant 0$: f defined on $\mathbb{R}$



- $k < 0$: f defined for $x \in \mathbb{R}$, $x \neq 0$



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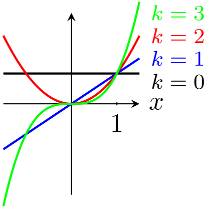

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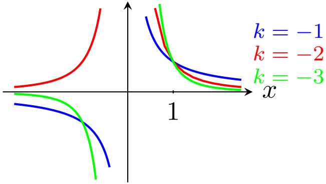
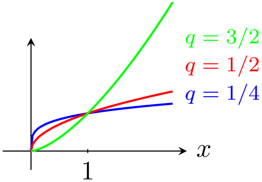
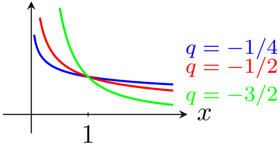


For $q := \frac{m}{n}$ non-integer: f root function

- $q > 0$: f defined for $x \in \mathbb{R}$, $x \geqslant 0$



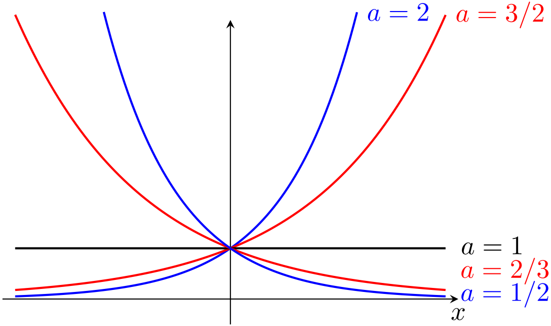
- $q < 0$: f defined for $x \in \mathbb{R}$, $x > 0$



Exponential functions

... functions f acting as $f(x) := a^x$ for $x \in \mathbb{R}$, where $a > 0$.

- $0 < a < 1$: f decreasing
- $a = 1$: f constant
- $a > 1$: f increasing



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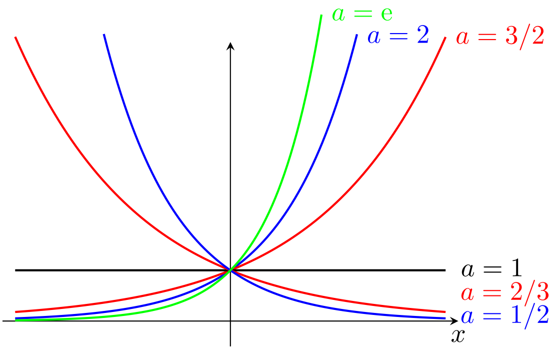

Important special case: $a = \mathrm{e} \approx 2.7182818285$, Euler number. Then $f(x) = \mathrm{e}^x = \exp(x)$.
Every exponential function of this form: $a^x = \mathrm{e}^{x \ln a}$.

Exercise

Growth of bacteria: doubling every 20 minutes, i.e., eightfold increase per hour. With N_0 bacteria at time $t = 0$: amount of bacteria at time t in hours:

$$N(t) = N_0 \cdot 8^t.$$

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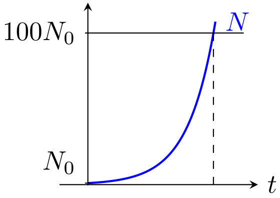
How long does it take to have 100 times as many bacteria?



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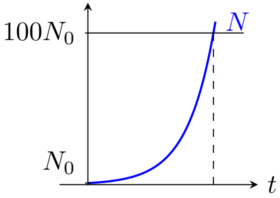


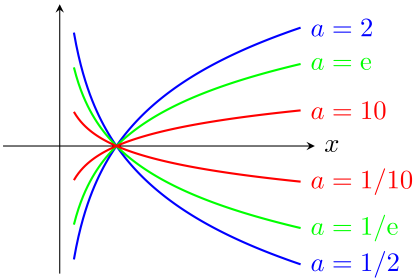
$$N(t) = N_0 \cdot 8^t \stackrel{!}{=} 100N_0,$$

i.e., $8^t = 100$. Hence

$$t = \frac{\ln 100}{\ln 8} \approx 2{,}2146 \text{ hours.}$$

Logarithm functions

... functions f acting as $f(x) := \log_a(x)$ for $x > 0$, where $a > 0$.



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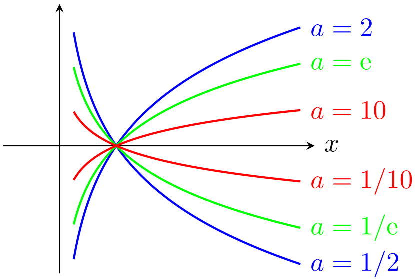
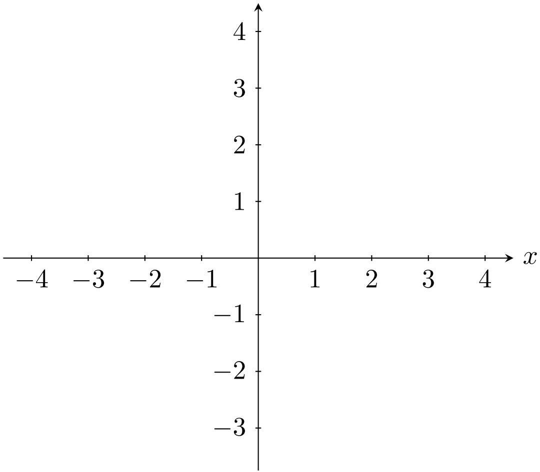


For $a = \mathrm{e}$: $f(x) = \log_{\mathrm{e}}(x) = \ln(x)$, natural logarithm.

Every logarithm function of this form: $\log_a(x) = \frac{\ln x}{\ln a}$.

Exercise

Sketch the functions f, g, h given by $f(x) := \mathrm{e}^x$ and $g(x) := x$ for $x \in \mathbb{R}$, and $h(x) := \ln(x)$ for $x > 0$.



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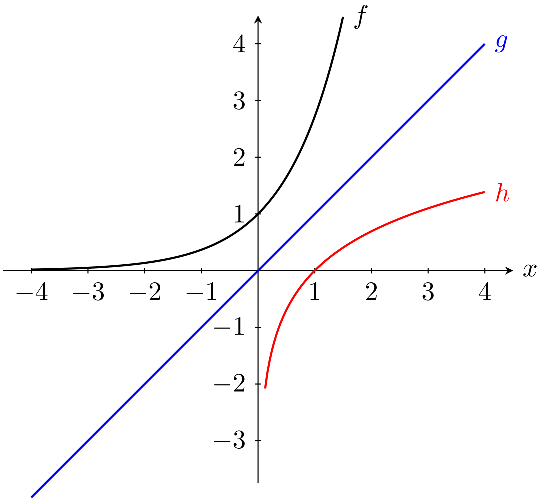
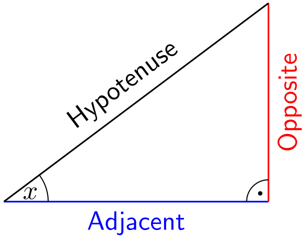


Trigonometric functions

... functions f based on the sine function $\sin$ and the cosine function $\cos$.



$$\cos(x) = \frac{\text{Adjacent}}{\text{Hypotenuse}},$$

$$\sin(x) = \frac{\text{Opposite}}{\text{Hypotenuse}},$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{\text{Opposite}}{\text{Adjacent}}.$$

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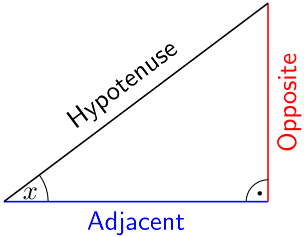

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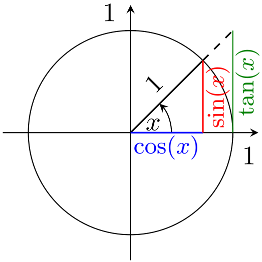
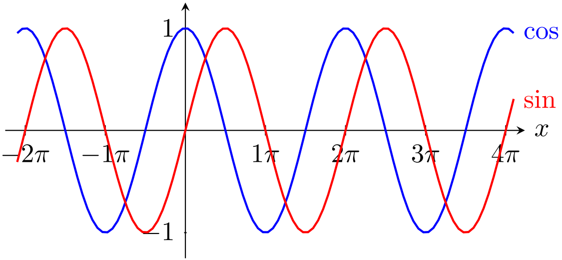


$$\cos(x)^2 + \sin(x)^2 = 1$$

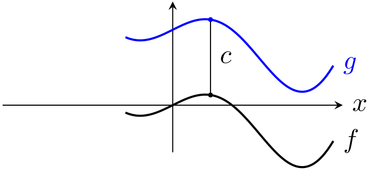


Transformation of functions

Translations

... vertical shift:

For function $f: D \to \mathbb{R}$ and $c \in \mathbb{R}$: $g: D \to \mathbb{R}$ with $g(x) := f(x) + c$ for $x \in D$.



- $c > 0$: shift upwards
- $c = 0$: no shift
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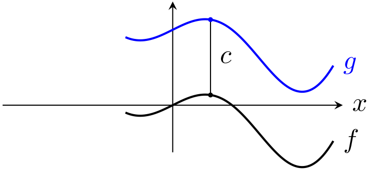

- $c > 0$: shift upwards
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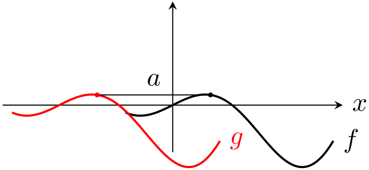
... horizontal shift:

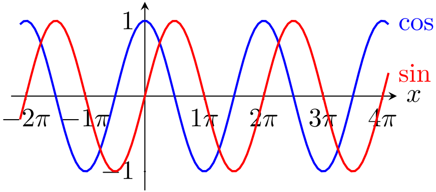
For function $f\colon D \to \mathbb{R}$ and $a \in \mathbb{R}$: $g\colon \{x \in \mathbb{R};\ x + a \in D\} \to \mathbb{R}$ with $g(x) := f(x + a)$ for $x \in \mathbb{R}$ such that $x + a \in D$.



- $a > 0$: shift to the left
- $a = 0$: no shift
- $a < 0$: shift to the right

Exercise

Starting from a sketch, determine $a \in \mathbb{R}$ such that $\cos(x) = \sin(x + a)$ for all $x \in \mathbb{R}$.



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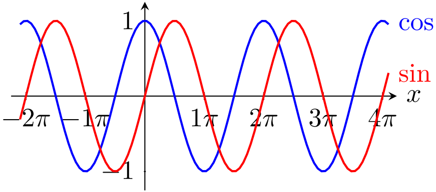
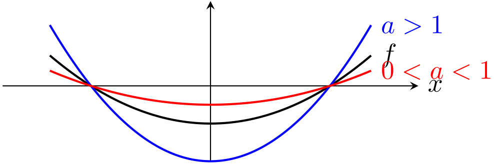


We read off that $\cos$ is a left shift of $\sin$ by $\frac{\pi}{2}$, so $a = \frac{\pi}{2}$.

Scaling

... in vertical direction:

For function $f\colon D \to \mathbb{R}$ and $a > 0$: $g\colon D \to \mathbb{R}$ with $g(x) := af(x)$ for $x \in D$.



- $0 < a < 1$: shrinkage
- $a = 1$: no stretching/shrinkage
- $a > 1$: stretching

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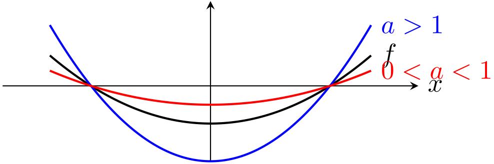

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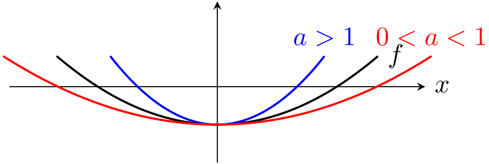
... in horizontal direction:

For function $f\colon D \to \mathbb{R}$ and $a > 0$: $g\colon \{x \in \mathbb{R};\ ax \in D\} \to \mathbb{R}$ with $g(x) := f(ax)$ for $x \in \mathbb{R}$ such that $ax \in D$.



- $0 < a < 1$: stretching
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Exercise

For the given function f with $f(x) := x^2$ for $x \in [-2, 2]$ determine the action of the function g given by $g(x) := f(2x)$ and make a sketch.

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$$g(x) = f(2x) = (2x)^2 = 4x^2$$

for $x \in [-1, 1]$.

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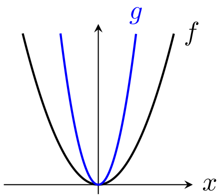
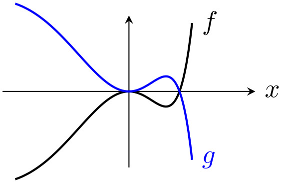


Reflections

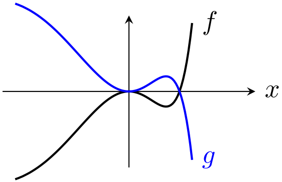
... over the horizontal axis

For function $f\colon D \to \mathbb{R}$:

$g\colon D \to \mathbb{R}$ with $g(x) := -f(x)$ for $x \in D$.



Reflections



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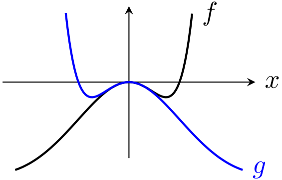

... over the vertical axis

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Reflections

... over the horizontal axis

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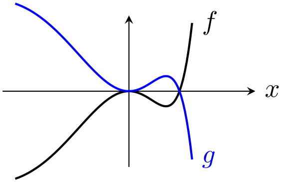
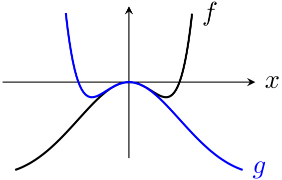


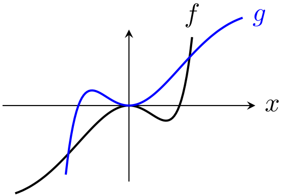
... over the vertical axis

For function $f: D \to \mathbb{R}$:
$g: \{x \in \mathbb{R};\ -x \in D\} \to \mathbb{R}$ with $g(x) := f(-x)$ for $x \in \mathbb{R}$ such that $-x \in D$.



... over the origin

For function $f: D \to \mathbb{R}$:
$g: \{x \in \mathbb{R};\ -x \in D\} \to \mathbb{R}$ with $g(x) := -f(-x)$ for $x \in \mathbb{R}$ such that $-x \in D$.



Compositions

For functions $f\colon \widetilde{D} \to \mathbb{R}$ and $g\colon D \to \widetilde{D}$:
$f(g) = f \circ g\colon D \to \mathbb{R}$, $(f \circ g)(x) := f(g(x))$ for $x \in D$, composition of f and g.

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Example:

- vertical shift of g: $f(x) := x + c$, then $f(g(x)) = g(x) + c$
- horizontal shift of f: $g(x) := x + a$, then $f(g(x)) = f(x + a)$

Compositions

For functions $f\colon \widetilde{D} \to \mathbb{R}$ and $g\colon D \to \widetilde{D}$:
$f(g) = f \circ g\colon D \to \mathbb{R}$, $(f \circ g)(x) := f(g(x))$ for $x \in D$, composition of f and g.

Example:

- vertical shift of g: $f(x) := x + c$, then $f(g(x)) = g(x) + c$
- horizontal shift of f: $g(x) := x + a$, then $f(g(x)) = f(x + a)$
- stretching/shrinkage in vertical direction of g: $f(x) = ax$, then $f(g(x)) = ag(x)$
- stretching/shrinkage in horizontal direction of f: $g(x) = ax$, then $f(g(x)) = f(ax)$

Compositions

For functions $f\colon \widetilde{D} \to \mathbb{R}$ and $g\colon D \to \widetilde{D}$:
$f(g) = f \circ g\colon D \to \mathbb{R}$, $(f \circ g)(x) := f(g(x))$ for $x \in D$, composition of f and g.

Example:

- vertical shift of g: $f(x) := x + c$, then $f(g(x)) = g(x) + c$
- horizontal shift of f: $g(x) := x + a$, then $f(g(x)) = f(x + a)$
- stretching/shrinkage in vertical direction of g: $f(x) = ax$, then $f(g(x)) = ag(x)$
- stretching/shrinkage in horizontal direction of f: $g(x) = ax$, then $f(g(x)) = f(ax)$
- reflection over the horizontal axis of g: $f(x) = -x$, then $f(g(x)) = -f(x)$
- reflection over the vertical axis of f: $g(x) = -x$, then $f(g(x)) = f(-x)$
- reflection over the origin of f: $g(x) = -x$, then $g(f(g(x))) = -f(-x)$

Exercise

For the function f given by $f(x) := \sin(x)$ for $x \in \mathbb{R}$ draw a sketch of the function g given by $g(x) := 2f\left(\frac{1}{2}x + \frac{\pi}{2}\right) + 1$ for $x \in \mathbb{R}$.

Exercise

For the function f given by $f(x) := \sin(x)$ for $x \in \mathbb{R}$ draw a sketch of the function g given by $g(x) := 2f\left(\frac{1}{2}x + \frac{\pi}{2}\right) + 1$ for $x \in \mathbb{R}$.