Prep Course Mathematics

Lecture 6 – Vectors and systems of linear equations

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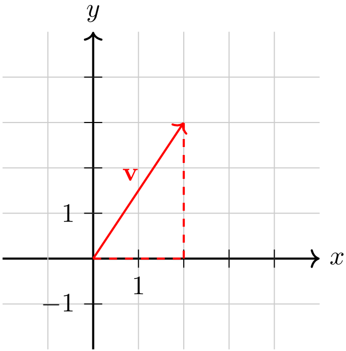
INSTITUTE OF MATHEMATICS

Content

Vectors

Vectors in two dimensions (2D)

Plane: two coordinate axes (e.g. x- and y-axis) intersecting in the zero-point (origin)



Example: the vector

$$\mathbf{v} := \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

starts at an arbitrary point and goes 2 steps in the direction of the x-axis and 3 steps in the direction of the y-axis.

A vector in this plane is an arrow of which only the elongation in the direction of the x- and y-axes is known.

Vectors in two dimensions (2D)

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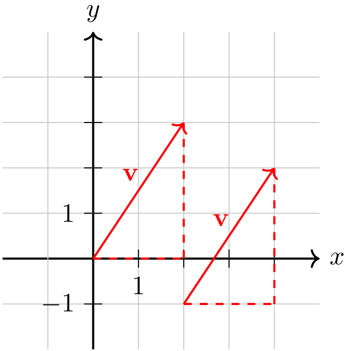

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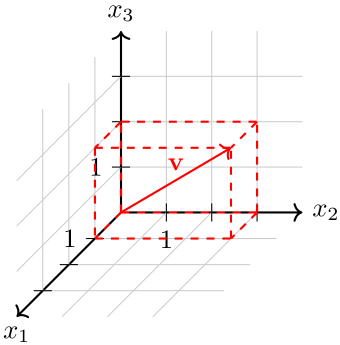
Vectors in three dimensions (3D)

Space: three coordinate axes (e.g., x_1-, x_2- and x_3-axis) intersecting in the zero-point (origin)



Example: the vector

$$\mathbf{v} := \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

goes 1 step in the direction of the x_1-axis,
3 steps in the direction of the x_2-axis and
2 steps in the direction of the x_3-axis.

A vector in this space is an arrow of which only the elongation in the direction of the x_1-, x_2- and x_3-axes is known.

Vectors in n dimensions (nD)

Vectors in n dimensions

Let $n \in \mathbb{N}$. Every object $\mathbf{v}$ of the form

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{with} \quad v_1, v_2, \ldots, v_n \in \mathbb{R}$$

is called a real vector.

Vectors in n dimensions (nD)

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Let $n \in \mathbb{N}$. Every object $\mathbf{v}$ of the form

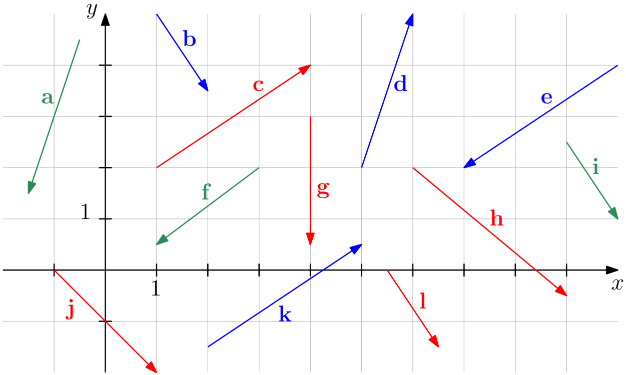
$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{with} \quad v_1, v_2, \ldots, v_n \in \mathbb{R}$$

is called a real vector. The set of all such vectors is denoted by $\mathbb{R}^n$. The entries $v_1, v_2, \ldots, v_n$ are called components.

Notes:

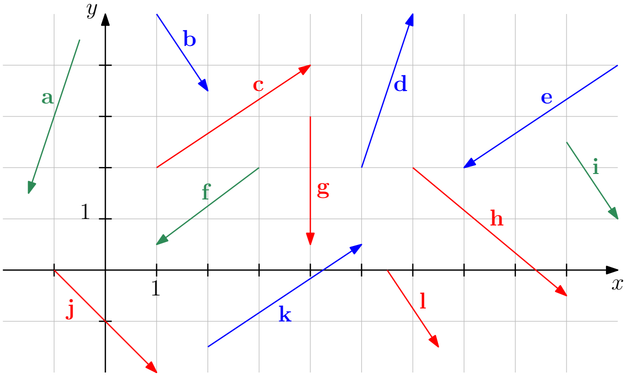
- *Today we mainly consider the case $n \in \{2, 3\}$. However, in Linear Algebra we will often work with larger $n \in \mathbb{N}$.*
- *In different lectures you may see different notation for vectors: $\mathbf{v}$, $\vec{v}$, $\underline{v}$ etc.*

Exercise



Exercise: Which of the given vectors are equal? Give their representations with coordinates.

Exercise



Exercise: Which of the given vectors are equal? Give their representations with coordinates.

Solution:

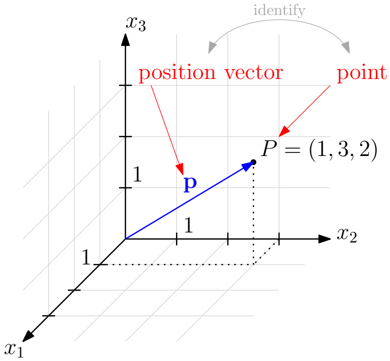
$$\mathbf{c} = \mathbf{k} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \mathbf{i} = \mathbf{l} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}.$$

Points and position vectors

A point P is given as a *tuple* $(v_1, v_2, \ldots, v_n)$, e.g., $P = (1, 3, 2)$.

Definition (position vector)

Let O be the origin (of the coordinate system). If P is a point, then the vector going from O to P is called the position vector of P. We often identify points with their position vectors.



Length of a vector

Length of a vector

Consider any vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n.$$

The length of this $\mathbf{v}$ is given by

$$\|\mathbf{v}\| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\,.$$

Length of a vector

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The length of this $\mathbf{v}$ is given by

$$\|\mathbf{v}\| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\,.$$

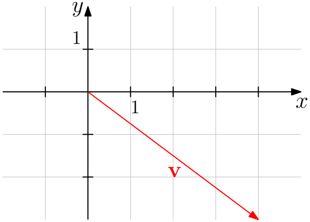
Example: the length of the vector

$$\mathbf{v} := \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

is

$$\|\mathbf{v}\| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5.$$

See the following picture:



Addition/subtraction of vectors

Addition/subtraction of vectors in $\mathbb{R}^n$

The addition/subtraction of two vectors in $\mathbb{R}^n$ is done componentwise.
Given two vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{R}^n,$$

we define

$$\mathbf{v} + \mathbf{w} := \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{pmatrix}, \quad \mathbf{v} - \mathbf{w} := \begin{pmatrix} v_1 - w_1 \\ v_2 - w_2 \\ \vdots \\ v_n - w_n \end{pmatrix}.$$

Addition/subtraction of vectors

Examples:

$$(2) + (5) = (7),$$

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -5 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix} =$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

Addition/subtraction of vectors

Examples:

$$(2) + (5) = (7), \qquad \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -5 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix},$$

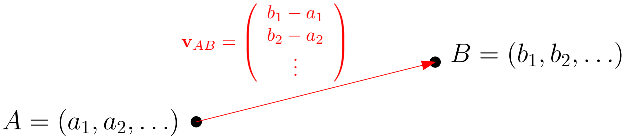
$$\begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} -v_1 \\ \vdots \\ -v_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

Vectors between two points

Vectors between two points

Let A and B be two points. The vector $\mathbf{v}_{AB}$ going from A to B can be obtained by subtracting the position vector of A from the position vector of B.



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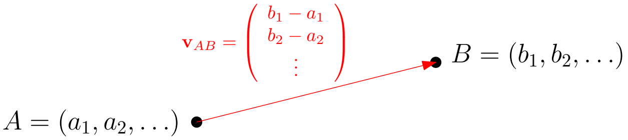
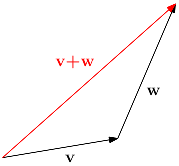


Example: The vector going from $A := (2, 4, -6)$ to $B := (3, -1, 9)$ is

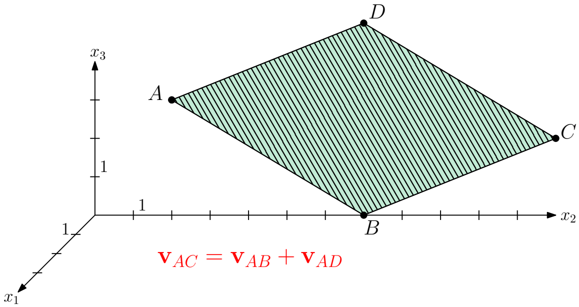
$$\mathbf{v}_{AB} = \begin{pmatrix} 3 \\ -1 \\ 9 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} = \begin{pmatrix} 3-2 \\ -1-4 \\ 9+6 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 15 \end{pmatrix}.$$

Addition/subtraction of vectors

Addition of vectors:
The addition of vectors corresponds to a composition of movements that are described by the vectors.



Example:

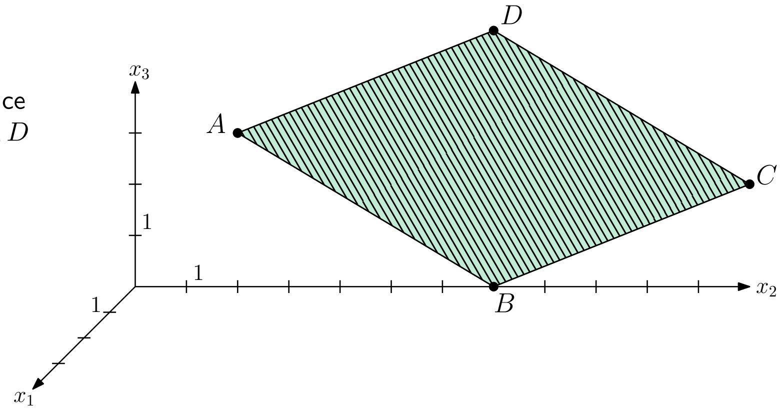


$$\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{AD}$$

Exercise

Of a parallelogramm in space with corner points A, B, C, D three are given as follows:

$$A := (2,3,4),$$
$$B := (0,7,0),$$
$$D := (-2,6,4).$$



(i) Determine the vectors $\mathbf{v}_{AB}$ and $\mathbf{v}_{AD}$ which lead from A to B and from A to D.

(ii) Which coordinates does the point C have?

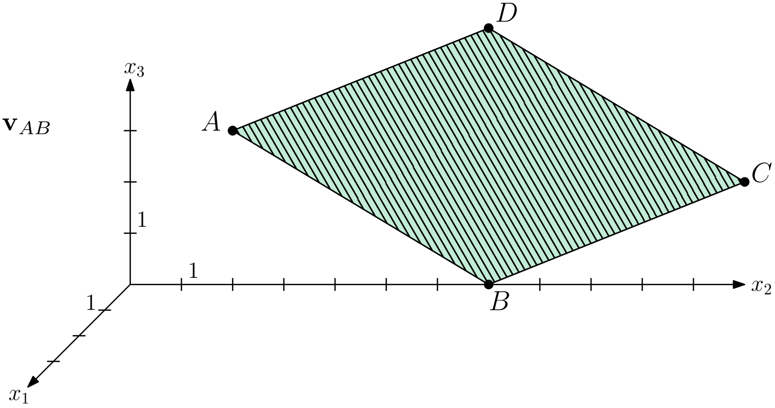
(iii) Determine the circumference U of the parallelogramm.

(iv) Determine the length d_{AC} of the diagonal AC.

Solution

(i) Determine the vectors $\mathbf{v}_{AB}$ and $\mathbf{v}_{AD}$.

$$A := (2, 3, 4),$$
$$B := (0, 7, 0),$$
$$D := (-2, 6, 4).$$

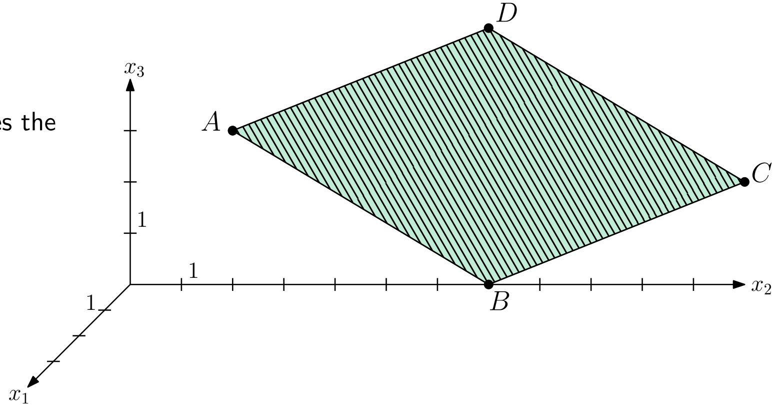


$$\mathbf{v}_{AB} = \begin{pmatrix} 0 \\ 7 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix} = \mathbf{v}_{DC}, \quad \mathbf{v}_{AD} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} = \mathbf{v}_{BC}.$$

Solution

(ii) Which coordinates does the point C have?

$$A := (2,3,4),$$
$$B := (0,7,0),$$
$$D := (-2,6,4).$$

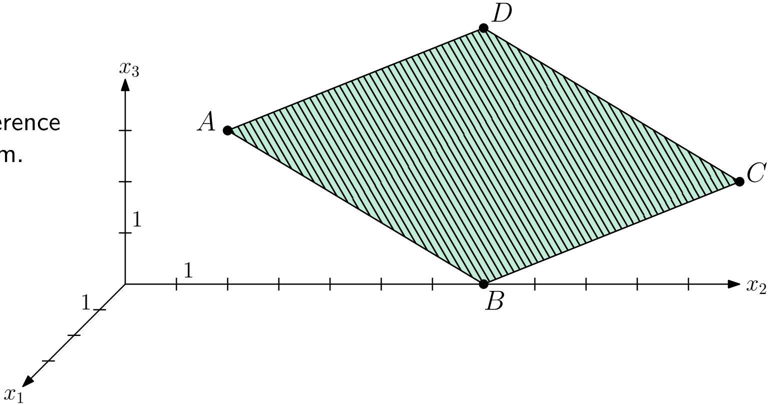


$$\mathbf{v}_{OC} = \mathbf{v}_{OD} + \mathbf{v}_{DC} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ 10 \\ 0 \end{pmatrix}, \quad C = (-4, 10, 0).$$

Solution

(iii) Determine the circumference U of the parallelogramm.

$$A := (2, 3, 4),$$
$$B := (0, 7, 0),$$
$$D := (-2, 6, 4).$$

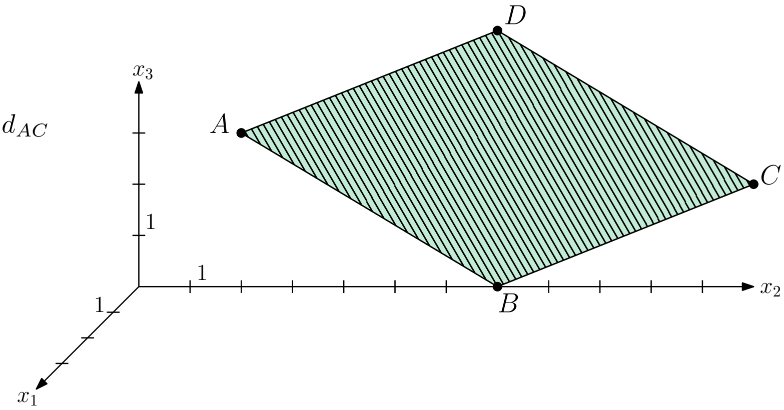


$$\|\mathbf{v}_{AB}\| = \left\| \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix} \right\| = 6, \quad \|\mathbf{v}_{AD}\| = \left\| \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} \right\| = 5, \quad U = 2(\|\mathbf{v}_{AB}\| + \|\mathbf{v}_{AD}\|) = 22.$$

Solution

(iv) Determine the length d_{AC} of the diagonal AC.

$$A := (2, 3, 4),$$
$$B := (0, 7, 0),$$
$$D := (-2, 6, 4).$$



$$\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{BC} = \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ -4 \end{pmatrix}, \quad d_{AC} = \|\mathbf{v}_{AC}\| = \sqrt{36 + 49 + 16} = \sqrt{101}.$$

Scalar multiplication of vectors

Scalar multiplication of vectors in $\mathbb{R}^n$

The scalar multiplication of vectors in $\mathbb{R}^n$ is done componentwise.
Given a real number (scalar) $\lambda \in \mathbb{R}$ and a vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n,$$

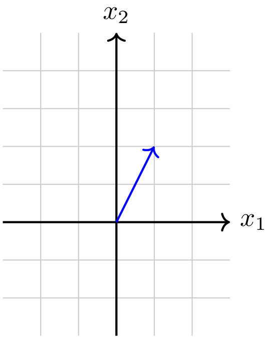
we define

$$\lambda \cdot \mathbf{v} := \begin{pmatrix} \lambda \cdot v_1 \\ \lambda \cdot v_2 \\ \vdots \\ \lambda \cdot v_n \end{pmatrix} \in \mathbb{R}^n.$$

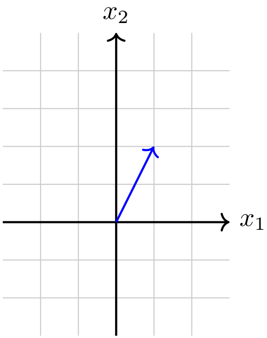
Scalar multiplication of vectors

Examples: Consider the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We determine and draw the following three vectors:

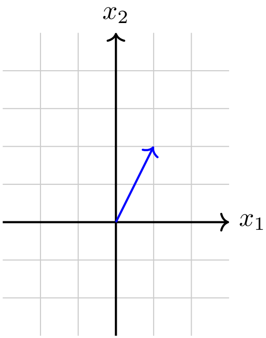
(i) $-1 \cdot \mathbf{v}$



(ii) $2 \cdot \mathbf{v}$



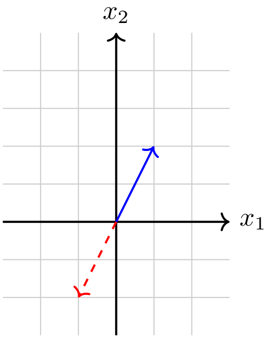
(iii) $\frac{1}{2} \cdot \mathbf{v}$



Scalar multiplication of vectors

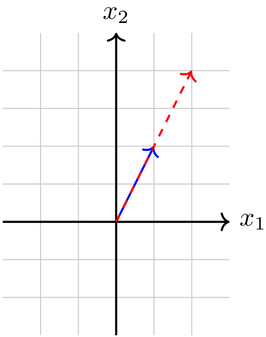
Examples: Consider the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We determine and draw the following three vectors:

(i) $-1 \cdot \mathbf{v} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$



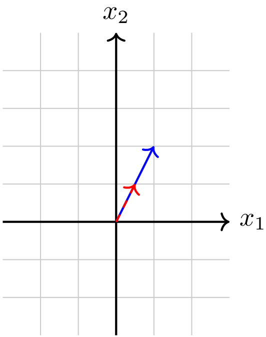
change of direction

(ii) $2 \cdot \mathbf{v} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$



elongation

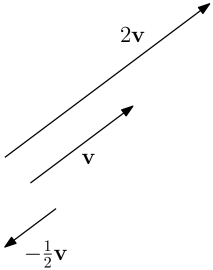
(iii) $\frac{1}{2} \cdot \mathbf{v} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$



shortening

Scalar multiplication of vectors

scalar multiplication of vectors: the scalar multiplication of a vector with a constant α corresponds to a stretching by the factor $|\alpha|$, while the direction is switched if α is negative.



Examples

$$3 \cdot \begin{pmatrix} 13 \\ -12 \end{pmatrix} - 2 \cdot \begin{pmatrix} 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 39 \\ -36 \end{pmatrix} - \begin{pmatrix} 10 \\ -8 \end{pmatrix} = \begin{pmatrix} 29 \\ -28 \end{pmatrix},$$

$$2 \cdot \left[\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix} \right] - \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3 \\ 7 \\ -6 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ -12 \end{pmatrix} - \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 16 \\ -14 \end{pmatrix},$$

$$-1 \cdot \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 2 \cdot \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} - 1 \cdot \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ -7 \end{pmatrix} + \begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix} + \begin{pmatrix} -3 \\ -6 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Rules of calculation of vector addition and scalar multiplication

Rules of calculation in $\mathbb{R}^n$

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be vectors and $\alpha, \beta \in \mathbb{R}$ be scalars. Then the following hold:

(i) Commutativity: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

(ii) Associativity: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) Distributivity I: $\alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v}$

(iv) Distributivity II: $(\alpha + \beta) \cdot \mathbf{v} = \alpha \cdot \mathbf{v} + \beta \cdot \mathbf{v}$

Exercises

(a) Calculate one solution of

$$\text{(i)} \quad 4 \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} - 5 \cdot \begin{pmatrix} -3 \\ 8 \\ -2 \end{pmatrix}, \qquad \text{(ii)} \quad \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} - \left[\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right].$$

(b) Find a vector $\mathbf{x}$ which satisfies one of the given equations:

$$\text{(i)} \quad 2\mathbf{x} - 3 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 4\mathbf{x} + \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}, \qquad \text{(ii)} \quad 2 \cdot \left[\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \mathbf{x} \right] = \mathbf{x} + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}.$$

(c) Find scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that the following equation is true:

$$\alpha_1 \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}.$$

Solutions

(a) Calculate

$$\text{(i)}\quad 4\cdot\begin{pmatrix}3\\0\\-2\end{pmatrix}-5\cdot\begin{pmatrix}-3\\8\\-2\end{pmatrix},\qquad \text{(ii)}\quad \begin{pmatrix}0\\3\\-5\end{pmatrix}-\left[\begin{pmatrix}2\\1\\-3\end{pmatrix}-\begin{pmatrix}-1\\1\\-2\end{pmatrix}\right].$$

Solution:

$$4\cdot\begin{pmatrix}3\\0\\-2\end{pmatrix}-5\cdot\begin{pmatrix}-3\\8\\-2\end{pmatrix} = \begin{pmatrix}12\\0\\-8\end{pmatrix}+\begin{pmatrix}15\\-40\\10\end{pmatrix}=\begin{pmatrix}27\\-40\\2\end{pmatrix}.$$

$$\begin{pmatrix}0\\3\\-5\end{pmatrix}-\left[\begin{pmatrix}2\\1\\-3\end{pmatrix}-\begin{pmatrix}-1\\1\\-2\end{pmatrix}\right] = \begin{pmatrix}0\\3\\-5\end{pmatrix}-\begin{pmatrix}3\\0\\-1\end{pmatrix}=\begin{pmatrix}-3\\3\\-4\end{pmatrix}.$$

Solutions

(b) Find a vector $\mathbf{x}$ which satisfies the given equation:

$$\text{(i)}\quad 2\mathbf{x} - 3\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} = 4\mathbf{x} + \begin{pmatrix}1\\7\\2\end{pmatrix}, \qquad \text{(ii)}\quad 2\cdot\left[\begin{pmatrix}3\\1\\1\end{pmatrix} - \mathbf{x}\right] = \mathbf{x} + \begin{pmatrix}0\\3\\1\end{pmatrix}.$$

Solution:

$$2\mathbf{x} + \begin{pmatrix}-3\\3\\0\end{pmatrix} = 4\mathbf{x} + \begin{pmatrix}1\\7\\2\end{pmatrix}$$

$$\begin{pmatrix}-3\\3\\0\end{pmatrix} - \begin{pmatrix}1\\7\\2\end{pmatrix} = \begin{pmatrix}-4\\-4\\-2\end{pmatrix} = 2\mathbf{x}$$

$$\begin{pmatrix}-2\\-2\\-1\end{pmatrix} = \mathbf{x}.$$

$$\begin{pmatrix}6\\2\\2\end{pmatrix} - 2\mathbf{x} = \mathbf{x} + \begin{pmatrix}0\\3\\1\end{pmatrix}$$

$$\begin{pmatrix}6\\2\\2\end{pmatrix} - \begin{pmatrix}0\\3\\1\end{pmatrix} = \begin{pmatrix}6\\-1\\1\end{pmatrix} = 3\mathbf{x}$$

$$\begin{pmatrix}2\\-1/3\\1/3\end{pmatrix} = \mathbf{x}.$$

Solutions

(c) Find scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that the following equation is true:

$$\alpha_1 \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \alpha_3 \cdot \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}.$$

Solution:

$$\begin{array}{r} \text{I:} \\ \text{II:} \\ \text{III:} \end{array} \quad \begin{pmatrix} 2\alpha_1 + 1\alpha_2 + 4\alpha_3 \\ 3\alpha_2 - 2\alpha_3 \\ -2\alpha_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ -1 \end{pmatrix}$$

$$\text{III:} \quad -2\alpha_3 = -1 \rightsquigarrow \boxed{\alpha_3 = 1/2}$$

$$\text{II:} \quad 3\alpha_2 = -7 + 2\alpha_3 = -7 + 1 = -6 \rightsquigarrow \boxed{\alpha_2 = -2}$$

$$\text{I:} \quad 2\alpha_1 = 3 - 4\alpha_3 - 1\alpha_2 = 3 - 2 + 2 = 3 \rightsquigarrow \boxed{\alpha_1 = 3/2}$$

Linear combination

Linear combination

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ be given. Then we call the vector

$$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k$$

a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. We also say that $\mathbf{v}$ can be generated by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

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Examples:

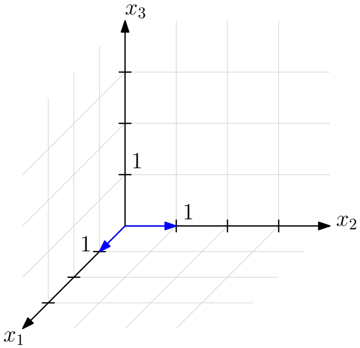
- $\mathbf{a} + \mathbf{b} = 1 \cdot \mathbf{a} + 1 \cdot \mathbf{b}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a} = 1 \cdot \mathbf{a} + 0 \cdot \mathbf{b}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{b} = 0 \cdot \mathbf{a} + 1 \cdot \mathbf{b}$ is a linear combination of $\mathbf{a}$ and $\mathbf{b}$.
- $\begin{pmatrix} 16 \\ 2 \\ 3 \end{pmatrix} = 2 \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + 5 \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.
- The zero vector in $\mathbb{R}^n$ is a linear combination of every tuple of vectors in $\mathbb{R}^n$.

Linear combination

Example:

Every point (position vector) $\mathbf{x} \in \mathbb{R}^3$ lying in the (x_1, x_2)-plane can be generated by the vectors

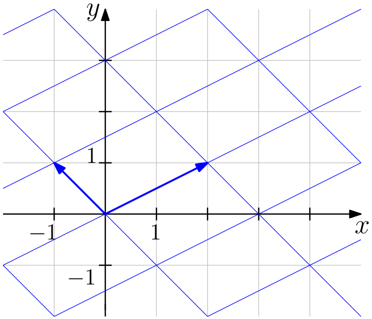
$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$



Example

Every point (position vector) $\mathbf{x} \in \mathbb{R}^2$ can be generated by the vectors

$$\mathbf{v}_1 := \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 := \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$



Span

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Consider vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. The set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is called span (or linear hull) of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Shortly:

$$\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) := \{\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k : \ \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}.$$

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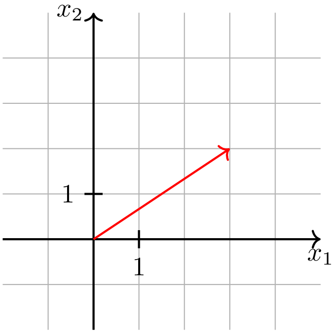
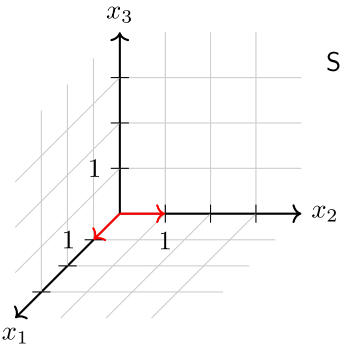
$$\mathrm{Span}\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right)$$





$$\mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

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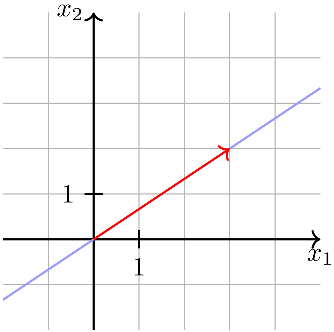
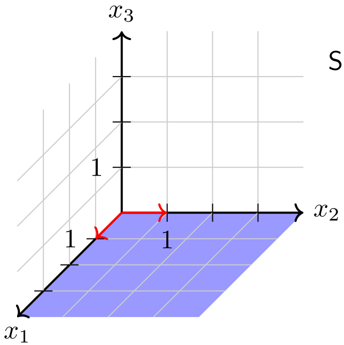


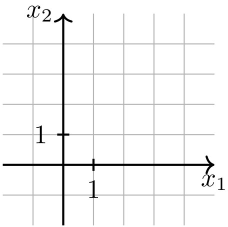


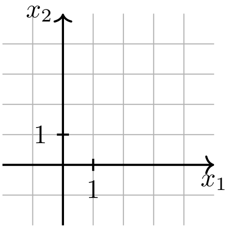
$$\mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

Exercise

Sketch the following spans:

$$\mathrm{Span}\left(\begin{pmatrix}-1\\-2\end{pmatrix}\right)$$





$$\mathrm{Span}\left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right)$$

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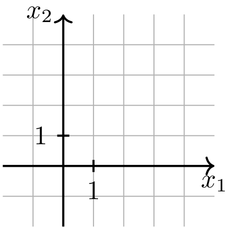

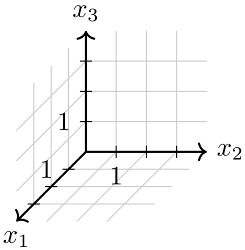

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Sketch the following spans:

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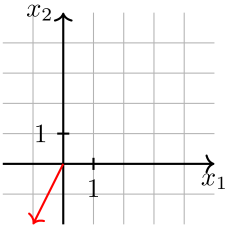

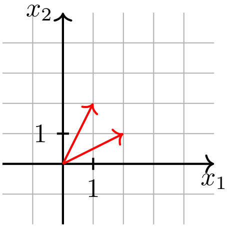

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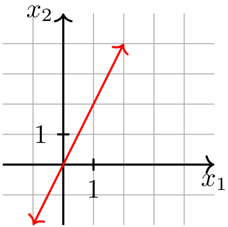

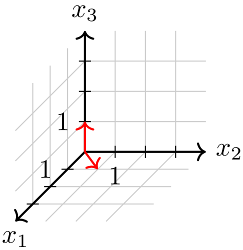

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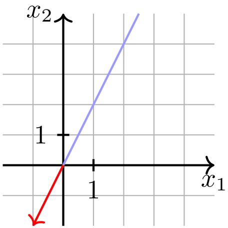

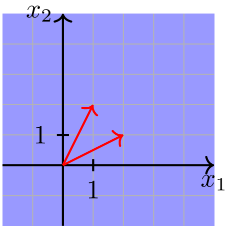

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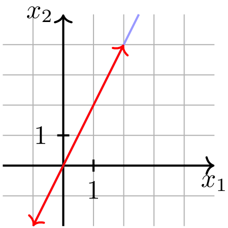

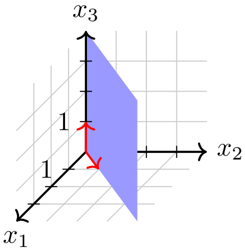

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Linear combination $\to$ system of linear equations

The following question appears frequently in Linear Algebra: "Given a fixed vector $\mathbf{v}$, can it be written as linear combination of some other given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots$?"

$$\begin{pmatrix} 8 \\ 17 \\ 0 \end{pmatrix} = \boxed{} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \boxed{} \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix} + \boxed{} \begin{pmatrix} 5 \\ 9 \\ -4 \end{pmatrix}$$

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This corresponds to asking whether a certain system of linear equations has a solution:

$$\begin{array}{rcrcrcr} 8 & = & 1x_1 & - & 3x_2 & + & 5x_3 \\ 17 & = & 2x_1 & - & 5x_2 & + & 9x_3 \\ 0 & = & -1x_1 & + & 5x_2 & - & 4x_3 \end{array}$$

Systems of linear equations

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Let $m, n \in \mathbb{N}$. A system of linear equations (LES) in the variables $x_1, x_2, \ldots, x_n$ is of the form

$$\begin{aligned} a_{1,1}\,x_1 + a_{1,2}\,x_2 + \cdots + a_{1,n}\,x_n &= b_1, \\ a_{2,1}\,x_1 + a_{2,2}\,x_2 + \cdots + a_{2,n}\,x_n &= b_2, \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m. \end{aligned}$$

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with $a_{i,j}$ and b_i being (usually real) numbers. An assignment of values for $x_1, \ldots, x_n$ such that all equations are satisfied is called a solution of this system of equations. Such a solution is written as a vector.

System of linear equations

Example: the system

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Check:

$$7 \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 3 \cdot \begin{pmatrix} -3 \\ -5 \\ 5 \end{pmatrix} + 2 \cdot \begin{pmatrix} 5 \\ 9 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ -7 \end{pmatrix} + \begin{pmatrix} -9 \\ -15 \\ 15 \end{pmatrix} + \begin{pmatrix} 10 \\ 18 \\ -8 \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \\ 0 \end{pmatrix}.$$

Method of equalization

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In the method of equalization we isolate the same variable in 2 equations and use the obtained expressions as left- and right-hand side. This way we obtain an equation which does not use one of the given variables.

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Computation:

$$\begin{array}{rcl} 2x_1 = 15 + 4x_2 & \Rightarrow & x_1 = 15/2 + 2 \cdot x_2 \\ -4x_1 = -9 - 2x_2 & \Rightarrow & x_1 = 9/4 + 1/2 \cdot x_2 \end{array} \quad \Rightarrow \quad \frac{15}{2} + 2x_2 = \frac{9}{4} + \frac{1}{2}x_2,$$

thus,

$$\left(2 - \frac{1}{2}\right)x_2 = \frac{3}{2}x_2 = \frac{9}{4} - \frac{15}{2} = -\frac{21}{4}, \quad x_2 = -\frac{7}{2}, \quad x_1 = \frac{15}{2} + 2x_2 = \frac{15}{2} - \frac{14}{2} = \frac{1}{2}.$$

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Computation like before:

$$2x_1 = 15 + 4x_2 \quad \Rightarrow \quad x_1 = \frac{15}{2} + 2x_2.$$

Substitution of first variable in second equation:

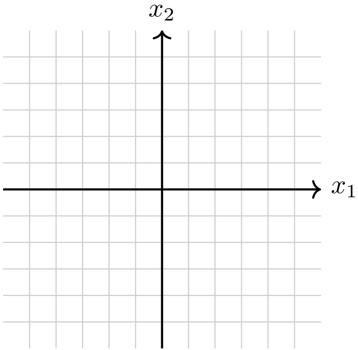
$$-4\left(\frac{15}{2} + 2x_2\right) + 2x_2 = -30 - 8x_2 + 2x_2 = -9 \quad \Rightarrow \quad -6x_2 = 21 \quad \Rightarrow \quad x_2 = -\frac{7}{2}.$$

First variable like before: $x_1 = 1/2$.

Solving by graphing

Solving by graphing (2 variables)

Consider a system of linear equations with 2 variables. Then each equation can be represented by a line in a 2-dimensional coordinate system. The set of solutions then is represented by the intersection of all lines.



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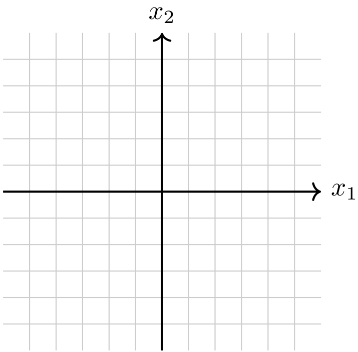

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$$2x_2 = -6 + 4x_1,$$
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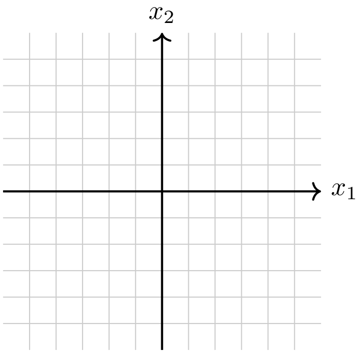

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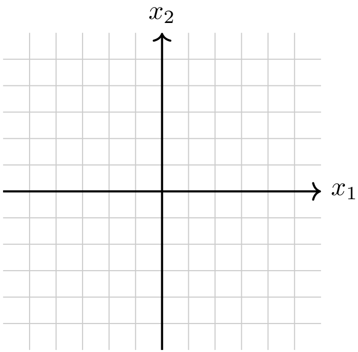
$$x_2 = -3 + 2x_1,$$
$$x_2 = 5 - 2x_1.$$



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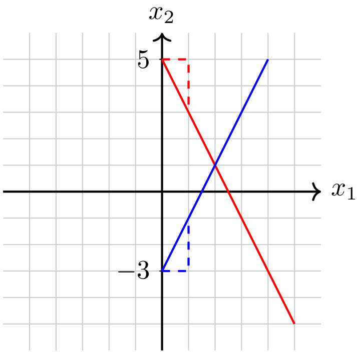

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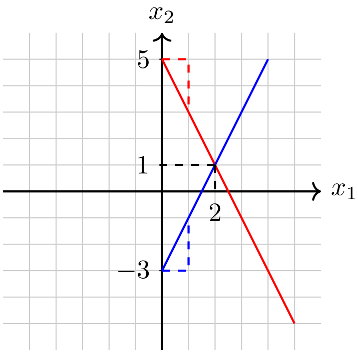
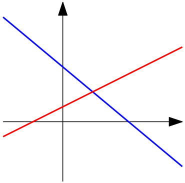
$$\begin{aligned} {\color{blue} x_2 = -3 + 2x_1}, \\ {\color{red} x_2 = 5 - 2x_1}, \end{aligned} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

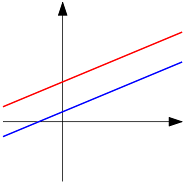


Solvability: number of solutions

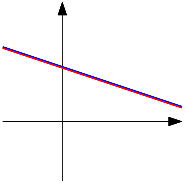
For any lines there exist three kinds of intersections:



intersection point
(one solution)



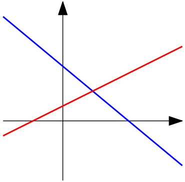
parallel
(no solution)



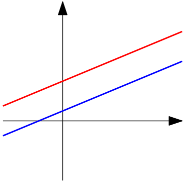
identical
(infinitely many solutions)

Solvability: number of solutions

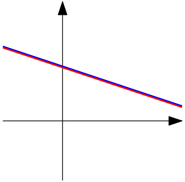
For any lines there exist three kinds of intersections:



intersection point
(one solution)



parallel
(no solution)



identical
(infinitely many solutions)

In general the following is true:

Solvability of systems of linear equations

Every system of linear equations has either (i) exactly one solution or (ii) no solution or (iii) infinitely many solutions.

Method of elimination

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In the method of elimination two equations (or multiples of them) are added/subtracted such that at least one variable is eliminated.

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Example 1: we solve the following system with the method of elimination:

$$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & 15 \\ -4x_1 & + & 2x_2 & = & -9 \end{array}$$

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$$\begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & 15 \\ -4x_1 & + & 2x_2 & = & -9 \end{array} \quad \Rightarrow \quad \begin{array}{rcrcr} 2x_1 & - & 4x_2 & = & 15 \\ & - & 6x_2 & = & 21 \end{array}$$

Solution:

- We subtract the first equation $-2 = \frac{-4}{2}$ times from the second, changing the second equation.
- We solve the new second equation:

$$x_2 = -\frac{21}{6} = -\frac{7}{2}.$$

- We utilize the first equation:

$$2x_1 = 15 + 4x_2 = 15 - 14 = 1 \quad \Rightarrow \quad x_1 = \frac{1}{2}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -7/2 \end{pmatrix}.$$

Method of elimination

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In the method of elimination two equations (or multiples of them) are added/subtracted such that at least one variable is eliminated.

Example 2: we solve the following system with the method of elimination:

$$\begin{array}{rcrcr} 4x_1 & - & 5x_2 & = & 2 \\ -8x_1 & + & 10x_2 & = & -4 \end{array}$$

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Example 2: we solve the following system with the method of elimination:

$$\begin{array}{rcrcr} 4x_1 & - & 5x_2 & = & 2 \\ -8x_1 & + & 10x_2 & = & -4 \end{array} \quad \Rightarrow \quad \begin{array}{rcrcr} 4x_1 & - & 5x_2 & = & 2 \\ & & 0x_2 & = & 0 \end{array}$$

Solution:

- We subtract the first equation $-2 = \frac{-8}{4}$ times from the second, changing the second equation.
- The new second equation is always satisfied:

$$0 \cdot x_2 = 0 \quad \Rightarrow \quad x_2 \in \mathbb{R}.$$

- The first equation yields as solution a line:

$$x_1 = \frac{1}{2} + \frac{5}{4}x_2 \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{5}{4}x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + x_2 \cdot \begin{pmatrix} \frac{5}{4} \\ 1 \end{pmatrix}, \quad x_2 \in \mathbb{R}.$$

Exercises

Solve the following systems with the method of elimination:

(a) $$\begin{array}{rcrcr} 1x_1 & - & 4x_2 & = & 1 \\ 2x_1 & - & 2x_2 & = & 3 \end{array}$$

(b) $$\begin{array}{rcrcr} -3x_1 & + & 2x_2 & = & 5 \\ 9x_1 & - & 6x_2 & = & 13 \end{array}$$

Solution of part (a):

Exercises

Solve the following systems with the method of elimination:

(a) $\begin{array}{rcrcr} 1x_1 & - & 4x_2 & = & 1 \\ 2x_1 & - & 2x_2 & = & 3 \end{array}$ (b) $\begin{array}{rcrcr} -3x_1 & + & 2x_2 & = & 5 \\ 9x_1 & - & 6x_2 & = & 13 \end{array}$

Solution of part (a):

- Compute II − 2I:

$$\begin{array}{rcrcr} 1x_1 & - & 4x_2 & = & 1 \\ 2x_1 & - & 2x_2 & = & 3 \end{array} \quad \Rightarrow \quad \begin{array}{rcrcr} 1x_1 & - & 4x_2 & = & 1 \\ & & 6x_2 & = & 1 \end{array}$$

- Solve second new equation:

$$6x_2 = 1 \quad \Rightarrow \quad x_2 = \frac{1}{6}.$$

- Insert into first equation:

$$x_1 = 1 + 4x_2 = 1 + \frac{4}{6} = \frac{10}{6} = \frac{5}{3} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{6} \end{pmatrix}.$$

Exercises

Solve the following systems with the method of elimination:

(a)
$$\begin{array}{rcrcr} 1x_1 & - & 4x_2 & = & 1 \\ 2x_1 & - & 2x_2 & = & 3 \end{array}$$

(b)
$$\begin{array}{rcrcr} -3x_1 & + & 2x_2 & = & 5 \\ 9x_1 & - & 6x_2 & = & 13 \end{array}$$

Solution of part (b):

Exercises

Solve the following systems with the method of elimination:

(a) $$\begin{array}{rcrcr} 1x_1 & - & 4x_2 & = & 1 \\ 2x_1 & - & 2x_2 & = & 3 \end{array}$$

(b) $$\begin{array}{rcrcr} -3x_1 & + & 2x_2 & = & 5 \\ 9x_1 & - & 6x_2 & = & 13 \end{array}$$

Solution of part (b):

- Compute $\text{II} + 3\text{I}$:

$$\begin{array}{rcrcr} -3x_1 & + & 2x_2 & = & 5 \\ 9x_1 & - & 6x_2 & = & 13 \end{array} \quad \Rightarrow \quad \begin{array}{rcrcr} -3x_1 & + & 2x_2 & = & 5 \\ & & 0x_2 & = & 28 \end{array}$$

- Solve second new equation:

$$0x_2 = 28 \quad \Rightarrow \quad \text{not solvable.}$$

- The linear system is not solvable.

Solving bigger systems of linear equations

Observations

- The set of solutions does not change if the ordering of the equations is changed.
- The set of solutions does not change if one equation is subtracted (multiple times) from the other equations.

Solving bigger systems of linear equations

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- The set of solutions does not change if the ordering of the equations is changed.
- The set of solutions does not change if one equation is subtracted (multiple times) from the other equations.

Recipe for calculations:

- Bring the system to a "triangle form" or "row echelon form".
- Afterwards determine the set of solutions.

Example

We determine the unique solution of the following system of linear equations:

$$\begin{array}{rcrcrcr} 1x_1 & + & 3x_2 & - & 3x_3 & = & 3 \\ 2x_1 & + & 7x_2 & - & 5x_3 & = & 4 \\ -1x_1 & + & 1x_2 & + & 9x_3 & = & -13 \end{array} \qquad \text{short,} \qquad \left(\begin{array}{rrr|r} 1 & 3 & -3 & 3 \\ 2 & 7 & -5 & 4 \\ -1 & 1 & 9 & -13 \end{array}\right).$$

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Solution:

- Compute upper triangular form:

$$\left(\begin{array}{rrr|r} 1 & 3 & -3 & 3 \\ 2 & 7 & -5 & 4 \\ -1 & 1 & 9 & -13 \end{array}\right) \begin{array}{l} \\ \mathsf{II} - 2 \cdot \mathsf{I} \\ \mathsf{III} + 1 \cdot \mathsf{I} \end{array} \quad \rightsquigarrow \quad \left(\begin{array}{rrr|r} 1 & 3 & -3 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 4 & 6 & -10 \end{array}\right)$$

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$$\left(\begin{array}{rrr|r} 1 & 3 & -3 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & -2 \end{array}\right)$$

- Solve from below:

$$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}.$$

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Exercise

Determine the unique solution of the following system of linear equations:

$$\begin{array}{rcrcrcr} 1x_1 & + & 3x_2 & - & 2x_3 & = & 5 \\ 3x_1 & + & 11x_2 & - & 5x_3 & = & 11 \\ 2x_1 & + & 2x_2 & - & 4x_3 & = & 14 \end{array} \qquad \text{short,} \qquad \left(\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 3 & 11 & -5 & 11 \\ 2 & 2 & -4 & 14 \end{array}\right).$$

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$$\left(\begin{array}{rrr|r} 1 & 3 & -2 & 5 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 2 & -4 \end{array}\right)$$

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$$\begin{pmatrix} 1 & 3 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ -4 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \quad & \quad & \quad \end{pmatrix}.$$

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An example with infinitely many solutions

We determine the solutions of the following system of linear equations:

$$\begin{array}{rcrcrcr} 1x_1 & - & 5x_2 & + & 4x_3 & = & -2 \\ 1x_1 & - & 4x_2 & + & 2x_3 & = & 1 \\ -2x_1 & + & 7x_2 & - & 2x_3 & = & -5 \end{array} \qquad \text{short,} \qquad \left(\begin{array}{rrr|r} 1 & -5 & 4 & -2 \\ 1 & -4 & 2 & 1 \\ -2 & 7 & -2 & -5 \end{array}\right).$$

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Solution:

- Compute upper triangular form:

$$\left(\begin{array}{rrr|r} 1 & -5 & 4 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

An example with infinitely many solutions

We determine the solutions of the following system of linear equations:

$$\begin{array}{rcrcrcr} 1x_1 & - & 5x_2 & + & 4x_3 & = & -2 \\ 1x_1 & - & 4x_2 & + & 2x_3 & = & 1 \\ -2x_1 & + & 7x_2 & - & 2x_3 & = & -5 \end{array} \qquad \text{short,} \qquad \left(\begin{array}{rrr|r} 1 & -5 & 4 & -2 \\ 1 & -4 & 2 & 1 \\ -2 & 7 & -2 & -5 \end{array}\right).$$

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- Solve from below:

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