Prep Course Mathematics

Equations and inequalities

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Content

Equations

Equations and equivalence transformations

Equivalence transformations modify equations without altering their solutions.

Important equivalence transformations:

- swapping sides: $a = b$ if and only if $b = a$
- addition/subtraction of $c \in \mathbb{R}$: $a = b$ if and only if $a \pm c = b \pm c$
- multiplication with or division by $c \neq 0$: $a = b$ if and only if $ac = bc$ if and only if $\frac{a}{c} = \frac{b}{c}$

⚠ Taking powers or roots are **not** equivalence transformations.

Solving equations

By subtracting the terms on one side, every equation in one unknown x can equivalently be written in the form $f(x) = 0$.

Example:

$$3x + 4 = 5 \quad \text{if and only if} \quad 3x - 1 = 0,$$

hence $f(x) = 0$ with $f(x) := 3x - 1$.

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hence $f(x) = 0$ with $f(x) := 3x - 1$.

Hence: solutions of equations are exactly zeros of f.

Linear equations

For $a, b \in \mathbb{R}$: linear equation

$$ax = b.$$

Solution:

- $a \neq 0$: exactly one: $x = \frac{b}{a}$
- $a = 0$, $b \neq 0$: no solution
- $a = 0$, $b = 0$: infinitely many: $x \in \mathbb{R}$

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Example:

$$5x = 3.$$

Then

$$x = \frac{3}{5}.$$

Quadratic equations

For $a, b, c \in \mathbb{R}$, $a \neq 0$: quadratic equation

$$ax^2 + bx + c = 0.$$

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Since $a \neq 0$:

$$x^2 + \underbrace{\frac{b}{a}}_{=:p} x + \underbrace{\frac{c}{a}}_{=:q} = 0.$$

Solution: $D := \left(\frac{p}{2}\right)^2 - q$

- $D > 0$: exactly two: $x_\pm = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$
- $D = 0$: exactly one: $x = -\frac{p}{2}$
- $D < 0$: no solution

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- $D < 0$: no solution

Example: $2x^2 - 2x - 12 = 0.$
Then

$$x_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 6} = \frac{1 \pm 5}{2}.$$

Exercise

Solve:

- $-3x = 6$.

- $-2x^2 + 10x - 12 = 0$.

Exercise

Solve:

- $-3x = 6$.
  Solution: $x = -2$.

- $-2x^2 + 10x - 12 = 0$.
  Divide by -2: $x^2 - 5x + 6 = 0$.
  Hence
  $$x_\pm = \frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 6} = \frac{5}{2} \pm \sqrt{\frac{1}{4}} = \frac{5 \pm 1}{2}.$$
  Solutions: $x = 2$ or $x = 3$.

Polynomial equations

How to solve equations of higher order, e.g. $x^5 - x - 1 = 0$?

Techniques, which may help sometimes:

- Factoring out: Example: $x^3 + 2x^2 + x = 0$
- **Substitution:** Example: $x^4 - 10x^2 + 9 = 0$

Remark: there are (complicated) formulas for polynomial equations up to order four. From order five onwards, no such formulas exist.

Polynomial equations

How to solve equations of higher order, e.g. $x^5 - x - 1 = 0$?

Techniques, which may help sometimes:

- Factoring out: Example: $x^3 + 2x^2 + x = 0$
  Then
  $$0 = x^3 + 2x + x = x(x^2 + 2x + 1) = x(x+1)^2.$$
  Solutions: $x = 0$, $x = -1$
- **Substitution:** Example: $x^4 - 10x^2 + 9 = 0$
  Then with $z := x^2$
  $$0 = x^4 - 10x^2 + 9 = z^2 - 10z + 9.$$
  Solutions: $z = 1$, $z = 9$, hence $x = -1$, $x = 1$, $x = -3$, $x = 3$.

Remark: there are (complicated) formulas for polynomial equations up to order four. From order five onwards, no such formulas exist.

Exercise

Solve:

- $3(-x^3 + 5) = -9 - 6x^3$.

- $2x^4 - 8 = 0$.

Exercise

Solve:

- $3(-x^3 + 5) = -9 - 6x^3$.
  Then
  $$-3x^3 + 15 = -9 - 6x^3,$$
  hence
  $$x^3 = -8.$$
  Solution: $x = -2$.
- $2x^4 - 8 = 0$.
  With $z := x^2$:
  $$2z^2 - 8 = 0,$$
  hence
  $$z^2 = 4,$$
  and therefore
  $$z_\pm = \pm 2.$$
  Solutions: $x_\pm = \pm\sqrt{2}$.

Radical equations

In radical equations, the variable appears under one (or more) roots, and possibly outside of roots as well.

Method to solve radical equations:

1. Isolate a root under which the variable appears.
2. Take squares on both sides (this might enlarge the solution set).
3. Repeat the first two steps until all roots with variable have been eliminated.
4. Solve resulting equation.
5. Check all solution candidates to eliminate false solutions.

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5. Check all solution candidates to eliminate false solutions.

Example: $\sqrt{x+7} = x+1$. Then by taking squares

$$x + 7 = (x+1)^2 = x^2 + 2x + 1,$$

hence

$$x^2 + x - 6 = 0, \qquad x_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 6} = \frac{-1 \pm 5}{2}.$$

Check:

- $x_+ = 2$: $\sqrt{x_+ + 7} = \sqrt{9} = 3 = x_+ + 1$,
- $x_- = -3$: $\sqrt{x_- + 7} = \sqrt{4} = 2 \neq -2 = x_- + 1$.

Exercise

Solve: $\sqrt{x+2} = \sqrt{x} + \sqrt{4x+1}$.

Exercise

Solve: $\sqrt{x+2} = \sqrt{x} + \sqrt{4x+1}$.

By squaring:

$$x + 2 = \left(\sqrt{x} + \sqrt{4x+1}\right)^2 = x + 2\sqrt{x}\sqrt{4x+1} + 4x + 1.$$

Hence

$$2\sqrt{x}\sqrt{4x+1} = -4x + 1.$$

Squaring again:

$$4x(4x+1) = (-4x+1)^2 = 16x^2 - 8x + 1,$$

thus

$$16x^2 + 4x = 16x^2 - 8x + 1, \qquad 12x = 1.$$

Solutions: $x = \frac{1}{12}$.
Check:

$$\sqrt{\frac{1}{12} + 2} = \frac{5}{\sqrt{12}}, \qquad \sqrt{\frac{1}{12}} + \sqrt{4 \cdot \frac{1}{12} + 1} = \frac{1+4}{\sqrt{12}}.$$

Exponential equations

In exponential equations, the variable appears as exponent in one or more powers.

Methods to solve exponential equations:

- comparing exponents: Example: $7^{3-x} = 7^x$.

- take logarithms: Example: $2 \cdot 3^{x+1} = 18$.

Exponential equations

In exponential equations, the variable appears as exponent in one or more powers.

Methods to solve exponential equations:

- comparing exponents: Example: $7^{3-x} = 7^x$.
  Then
  $$3 - x = x,$$
  hence $2x = 3$.
  Solution: $x = \frac{3}{2}$.
- take logarithms: Example: $2 \cdot 3^{x+1} = 18$.
  Then
  $$3^{x+1} = 9,$$
  hence $(x+1)\ln 3 = \ln 3^{x+1} = \ln 9$.
  Solution: $x = \frac{\ln 9}{\ln 3} - 1 = 1$.

Exercise

Solve: $2^x \cdot 3^x = 4^{x+2}$.

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Taking logarithms:

$$x \ln 2 + x \ln 3 = \ln(2^x) + \ln(3^x) = \ln(2^x \cdot 3^x) = \ln(4^{x+2}) = (x+2) \ln 4,$$

hence

$$(\ln 2 + \ln 3 - \ln 4)x = 2 \ln 4.$$

Solution:

$$x = \frac{2 \ln 4}{\ln 2 + \ln 3 - \ln 4} = \frac{\ln 16}{\ln \frac{2 \cdot 3}{4}} \approx 6.8380.$$

Solving equations with absolute values

If absolute values appear in an equation, they can be eliminated by case-by-case analysis.

Example: $|x+5| = 7$.

- case 1: $x+5 \geqslant 0$. Then $x+5=7$.
  Solution: $x=2$.
- case 2: $x+5<0$. Then $-(x+5)=7$, hence $-x-5=7$.
  Solution: $x=-12$.

Thus:

- case 1: $x \geqslant -5$, $x=2$.
- case 2: $x<-5$, $x=-12$.

Solutions: $x=2$ or $x=-12$.

Exercise

Solve: $4|x+2| = -2x+1$.

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Solve: $4|x+2| = -2x+1$.

- case 1: $x+2 \geqslant 0$. Then
$$4(x+2) = -2x+1,$$
hence
$$6x = -7.$$
Solution: $x = -\frac{7}{6}$.
- case 2: $x+2 < 0$. Then
$$-4(x+2) = -2x+1,$$
hence
$$-2x = 9.$$
Solution: $x = -\frac{9}{2}$.

Thus:

- case 1: $x \geqslant -2$, $x = -\frac{7}{6}$.
- case 2: $x < -2$, $x = -\frac{9}{2}$.

Solutions: $x = -\frac{7}{6}$ or $x = -\frac{9}{2}$.

Inequalities

Inequalities: equivalence transformations

Inequalities are written using the comparison relations $<$, $\leqslant$, $>$ and $\geqslant$.

Important equivalence transformations (using the example $<$):

- swapping sides flips the comparison relation: $a < b$ if and only if $b > a$
- addition/subtraction of $c \in \mathbb{R}$: $a < b$ if and only if $a \pm c < b \pm c$
- multiplication with or division by $c > 0$: $a < b$ if and only if $ac < bc$ if and only if $\frac{a}{c} < \frac{b}{c}$
- multiplication with or division by $c < 0$: $a < b$ if and only if $ac > bc$ if and only if $\frac{a}{c} > \frac{b}{c}$

The types of equations studied also appear as types of inequalities.

Solving inequalities

2 typical methods:

- Use equivalence transformations to isolate the variable.
  Example: $2x + 3 > 7$.
  Then $2x > 4$, hence $x > 2$.

Solving inequalities

2 typical methods:

- Use equivalence transformations to isolate the variable.
  Example: $2x + 3 > 7$.
  Then $2x > 4$, hence $x > 2$.
- Solve the associated equation and then check values in between the solutions.
  Example: $x^2 + 2x - 1 < 2$.
  First, $x^2 + 2x - 3 = 0$, hence $x_\pm = -1 \pm \sqrt{1+3} = -1 \pm 2$.
  - $x < -3$: e.g. $x = -4$:
    $$(-4)^2 + 2 \cdot (-4) - 3 = 5 > 0.$$
  - $-3 < x < 1$: e.g. $x = 0$:
    $$0^2 + 2 \cdot 0 - 3 = -3 < 0.$$
  - $x > 1$: e.g. $x = 2$:
    $$2^2 + 2 \cdot 2 - 3 = 5 > 0.$$

  solutions: $-3 < x < 1$.

Exercise

Solve $x^3 > 2x^2 - x$.

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Solve $x^3 > 2x^2 - x$.

First, $x^3 = 2x^2 - x$, hence $x(x^2 - 2x + 1) = 0$. Therefore, $x = 0$ or $x^2 - 2x + 1 = 0$, thus $x_\pm = 1 \pm \sqrt{1 - 1} = 1$.

Exercise

Solve $x^3 > 2x^2 - x$.

First, $x^3 = 2x^2 - x$, hence $x(x^2 - 2x + 1) = 0$. Therefore, $x = 0$ or $x^2 - 2x + 1 = 0$, thus $x_\pm = 1 \pm \sqrt{1-1} = 1$.

- $x < 0$: e.g. $x = -1$:

$$(-1)^3 = -1, \qquad 2(-1)^2 - (-1) = 2 + 1 = 3.$$

- $0 < x < 1$: e.g. $x = \frac{1}{2}$:

$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}, \qquad 2\left(\frac{1}{2}\right)^2 - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0.$$

- $x > 1$: e.g. $x = 2$:

$$2^3 = 8, \qquad 2 \cdot 2^2 - 2 = 8 - 2 = 6.$$

Exercise

Solve $x^3 > 2x^2 - x$.

First, $x^3 = 2x^2 - x$, hence $x(x^2 - 2x + 1) = 0$. Therefore, $x = 0$ or $x^2 - 2x + 1 = 0$, thus $x_\pm = 1 \pm \sqrt{1 - 1} = 1$.

- $x < 0$: e.g. $x = -1$:

$$(-1)^3 = -1, \qquad 2(-1)^2 - (-1) = 2 + 1 = 3.$$

- $0 < x < 1$: e.g. $x = \frac{1}{2}$:

$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}, \qquad 2\left(\frac{1}{2}\right)^2 - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0.$$

- $x > 1$: e.g. $x = 2$:

$$2^3 = 8, \qquad 2 \cdot 2^2 - 2 = 8 - 2 = 6.$$

Solutions: $0 < x < 1$ or $x > 1$.

Solving inequalities with absolute values

Example: $|x+3|+|x+4|-9<0$.

- case 1: $x+3 \geqslant 0$. Then $x+3+|x+4|-9<0$.
  - case a: $x+4 \geqslant 0$. Then $x+3+x+4-9<0$, hence $2x-2<0$.
    Solutions: $x<1$.
  - case b: $x+4<0$. Then $x+3-(x+4)-9<0$, hence $-10<0$.
    Solutions: $x \in \mathbb{R}$.
- case 2: $x+3<0$. Then $-(x+3)+|x+4|-9<0$.
  - case a: $x+4 \geqslant 0$. Then $-(x+3)+x+4-9<0$, hence $-8<0$.
    Solutions: $x \in \mathbb{R}$.
  - case b: $x+4<0$. Then $-(x+3)-(x+4)-9<0$, hence $-2x-16<0$.
    Solutions: $x>-8$.

Therefore:

- case 1a: $x \geqslant -3$, $x \geqslant -4$ and $x<1$, hence $-3 \leqslant x<1$.
- case 1b: $x \geqslant -3$, $x<-4$ and $x \in \mathbb{R}$, hence no solutions.
- case 2a: $x<-3$, $x \geqslant -4$ and $x \in \mathbb{R}$, hence $-4 \leqslant x<-3$.
- case 2b: $x<-3$, $x<-4$ and $x>-8$, hence $-8<x<-4$.

solutions: $-8<x<1$.

Exercise

Solve $|x+1|+5<|2x-4|$.

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- case 1: $x+1 \geqslant 0$. Then $x+1+5<|2x-4|$.
  - case a: $2x-4 \geqslant 0$. Then $x+1+5<2x-4$, hence $6<x-4$.
    Solutions: $x>10$.
  - case b: $2x-4<0$. Then $x+1+5<-(2x-4)$, hence $3x+6<4$.
    Solutions: $x<-\frac{2}{3}$.
- case 2: $x+1<0$. Then $-(x+1)+5<|2x-4|$.
  - case a: $2x-4 \geqslant 0$. Then $-(x+1)+5<2x-4$, hence $4<3x-4$.
    Solutions: $x>\frac{8}{3}$.
  - case b: $2x-4<0$. Then $-(x+1)+5<-(2x-4)$, hence $x+4<4$.
    Solutions: $x<0$.

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    Solutions: $x>10$.
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    Solutions: $x>\frac{8}{3}$.
  - case b: $2x-4<0$. Then $-(x+1)+5<-(2x-4)$, hence $x+4<4$.
    Solutions: $x<0$.

Therefore:

- case 1a: $x\geqslant -1$, $x\geqslant 2$ and $x>10$, hence $x>10$.
- case 1b: $x\geqslant -1$, $x<2$ and $x<-\frac{2}{3}$, hence $-1\leqslant x<-\frac{2}{3}$.
- case 2a: $x<-1$, $x\geqslant 2$ and $x>\frac{8}{3}$, hence no solutions.
- case 2b: $x<-1$, $x<2$ and $x<0$, hence $x<-1$.

Exercise

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  - case a: $2x-4 \geqslant 0$. Then $x+1+5<2x-4$, hence $6<x-4$.
    Solutions: $x>10$.
  - case b: $2x-4<0$. Then $x+1+5<-(2x-4)$, hence $3x+6<4$.
    Solutions: $x<-\frac{2}{3}$.
- case 2: $x+1<0$. Then $-(x+1)+5<|2x-4|$.
  - case a: $2x-4 \geqslant 0$. Then $-(x+1)+5<2x-4$, hence $4<3x-4$.
    Solutions: $x>\frac{8}{3}$.
  - case b: $2x-4<0$. Then $-(x+1)+5<-(2x-4)$, hence $x+4<4$.
    Solutions: $x<0$.

Therefore:

- case 1a: $x \geqslant -1$, $x \geqslant 2$ and $x>10$, hence $x>10$.
- case 1b: $x \geqslant -1$, $x<2$ and $x<-\frac{2}{3}$, hence $-1 \leqslant x<-\frac{2}{3}$.
- case 2a: $x<-1$, $x \geqslant 2$ and $x>\frac{8}{3}$, hence no solutions.
- case 2b: $x<-1$, $x<2$ and $x<0$, hence $x<-1$.

solutions: $x<-\frac{2}{3}$ or $x>10$.