Mathematics 1 - Linear Algebra

Lecture 04 – §3.1 - §3.3 Linear systems of equations and matrices

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Systems of linear equations

Example for a system of linear equations

The main building A of TUHH has a total of 13 classrooms with each 12, 18, 24 or 32 seats. There is only one room of each of the two smaller sizes. All larger rooms with 24 or 32 seats taken together have seats for 296 students. How many of the rooms have exactly 24 seats?

Solution:

x_1 : number of rooms with 12 seats

x_2 : number of rooms with 18 seats

x_3 : number of rooms with 24 seats

x_4 : number of rooms with 32 seats

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 13 \\ x_1 &= 1 \\ x_2 &= 1 \\ 24x_3 + 32x_4 &= 296 \end{aligned}$$

The first three equations yield $x_3 + x_4 = 11$, hence $x_4 = 11 - x_3$.
Division of the fourth equation by 8 yields $3x_3 + 4x_4 = 37$.
Substitution of x_4 finally yields

$$3x_3 + 4x_4 = 37 \iff 3x_3 + 4(11 - x_3) = 37 \iff 3x_3 + 44 - 4x_3 = 37 \iff x_3 = 7.$$

Hence there are 7 classrooms with exactly 24 seats.

Systems of linear equations

In the example: 4 equations for 4 unknowns (number of rooms of different sizes).

In general: m equations for n unknowns of the form

$$\text{constant} \cdot x_1 \; + \; \text{constant} \cdot x_2 \; + \; \cdots \; + \; \text{constant} \cdot x_n \; = \; \text{constant}$$

Questions:

1. Does a solution exist?
2. If yes, is it unique, or are there more solutions?
3. How can solutions be computed?

System of linear equations

Definition 3.1 (System of linear equations, LES)

Let $m, n \in \mathbb{N}$. A *system of linear equations* (LES) in the variables $x_1, \ldots, x_n$ is of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

with a_{ij} and b_i being given (usually real) numbers.

An assignment of values for $x_1, \ldots, x_n$ such that all equations are satisfied is called a solution of this system of equations. Such a solution is written as a vector.

Systems of linear equations

Rowwise view

- Each row yields an equation.
- **Special case $n = 2$:** Each equation describes a line in $\mathbb{R}^2$. A solution of the LES satisfies all m equations, hence lies on all m lines, hence is an intersection point of the m lines.
- **Special case $n = 3$:** Each equation describes a plane in $\mathbb{R}^3$. A solution of the LES satisfies all m equations, hence lies on all m planes, hence lies in the intersection of the m planes.
- For $n = 2$ or $n = 3$, we may have no, exactly one or an infinite number of solutions.

Systems of linear equations

Columnwise view: Write the coefficients of the unknown x_i as a vector:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

$$x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Linear combination with the unknowns as coefficients

Once again apples and pears

Reminder:

$$\text{apple:} \quad \begin{array}{c} \begin{pmatrix} 12 \\ 5 \end{pmatrix} \end{array} \begin{array}{l} \scriptstyle\text{VitC} \\ \scriptstyle\text{Mg} \end{array}, \qquad \text{pear:} \quad \begin{array}{c} \begin{pmatrix} 5 \\ 8 \end{pmatrix} \end{array} \begin{array}{l} \scriptstyle\text{VitC} \\ \scriptstyle\text{Mg} \end{array}$$

Question: How many apples and pears do I have to eat to take in exactly 22mg of vitamin C and 21mg of magnesium?

$$x_1 \begin{pmatrix} 12 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 22 \\ 21 \end{pmatrix}$$

$$\begin{aligned} 12x_1 + 5x_2 &= 22 \\ 5x_1 + 8x_2 &= 21 \end{aligned}$$

(Solution: $x_1 = 1$, $x_2 = 2$.)

Matrices

Linear system of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

in matrix-vector notation

$$
\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\mathbf{A}}
\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}}
=
\underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\mathbf{b}}
$$

Even shorter: $\mathbf{Ax} = \mathbf{b}$

Apple-pear example

$$
\begin{aligned}
12x_1 + 5x_2 &= 22 \\
5x_1 + 8x_2 &= 21
\end{aligned}
$$

$$
\begin{pmatrix} 12 & 5 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 22 \\ 21 \end{pmatrix}
$$

Matrices

Definition 3.2 (matrix)

Let $m, n \in \mathbb{N}$ and let

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \text{with all} \qquad a_{ij} \in \mathbb{R}.$$

m denotes the ***number of rows*** (or ***height***) of $\mathbf{A}$,
n denotes the ***number of columns*** (or ***width***) of $\mathbf{A}$.

We call $\mathbf{A}$ an $m \times n$ matrix and write $\mathbf{A} \in \mathbb{R}^{m \times n}$.
$\mathbf{A}$ is called a ***square matrix*** if $m = n$.

For $i = 1, \ldots, m$: The $1 \times n$ matrix $\begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}$ is called i-th ***row*** of $\mathbf{A}$.

For $j = 1, \ldots, n$: The $m \times 1$ matrix $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ is called j-th ***column*** of $\mathbf{A}$.

The entries $a_{11}, a_{22}, \ldots, a_{kk}$ with $k = \min(m, n)$ form the ***main diagonal*** of $\mathbf{A}$.

Matrices

once again our example (see apples and pears)

$$\begin{aligned} 12x_1 + 5x_2 &= 22 \\ 5x_1 + 8x_2 &= 21 \end{aligned}$$

$$\underbrace{\begin{pmatrix} 12 & 5 \\ 5 & 8 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 22 \\ 21 \end{pmatrix}}_{\mathbf{b}}$$

$$\underbrace{\mathbf{A}\mathbf{x}}_{\text{matrix} \cdot \text{vector}} = \mathbf{b}$$

Definition 3.3 (matrix · vector = vector)

Let $m, n \in \mathbb{N}$ and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n} \qquad \text{and} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

The product $\mathbf{A} \cdot \mathbf{x}$ (also $\mathbf{A}\mathbf{x}$) is defined as the vector

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{A}\mathbf{x} := \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} \in \mathbb{R}^m.$$

Matrices

Examples (Matrix-vector multiplication)

$$\begin{pmatrix} 12 & 5 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} 12 & 5 \\ 5 & 8 \\ 17 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} 12 & 5 & 7 \\ 5 & 8 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} 12 & 5 & 7 \\ 5 & 8 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{pmatrix} 12x_1 + 5x_2 \\ 5x_1 + 8x_2 \end{pmatrix} \qquad \begin{pmatrix} 12x_1 + 5x_2 \\ 5x_1 + 8x_2 \\ 17x_1 + 13x_2 \end{pmatrix} \qquad \begin{pmatrix} 12x_1 + 5x_2 + 7x_? \\ 5x_1 + 8x_2 + 1x_? \end{pmatrix} \qquad \begin{pmatrix} 12x_1 + 5x_2 + 7x_3 \\ 5x_1 + 8x_2 + 1x_3 \end{pmatrix}$$

undefined!

Attention: The width of $\mathbf{A}$ must equal the height of $\mathbf{x}$ for $\mathbf{Ax}$ to be defined.

$$\mathbf{A} \in \mathbb{R}^{m \times n}, \ \mathbf{x} \in \mathbb{R}^{n} \implies \mathbf{Ax} \in \mathbb{R}^{m}$$

Matrices

Rowwise and columnwise view of matrix-vector multiplication

Rowwise view The i-th component of $\mathbf{Ax}$ is the scalar product of the i-th row of $\mathbf{A}$ and $\mathbf{x}$:

$$\mathbf{Ax} = \begin{pmatrix} \text{---}\ \mathbf{a}_1\ \text{---} \\ \cdots \\ \text{---}\ \mathbf{a}_m\ \text{---} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{x} \rangle \\ \vdots \\ \langle \mathbf{a}_m, \mathbf{x} \rangle \end{pmatrix}$$

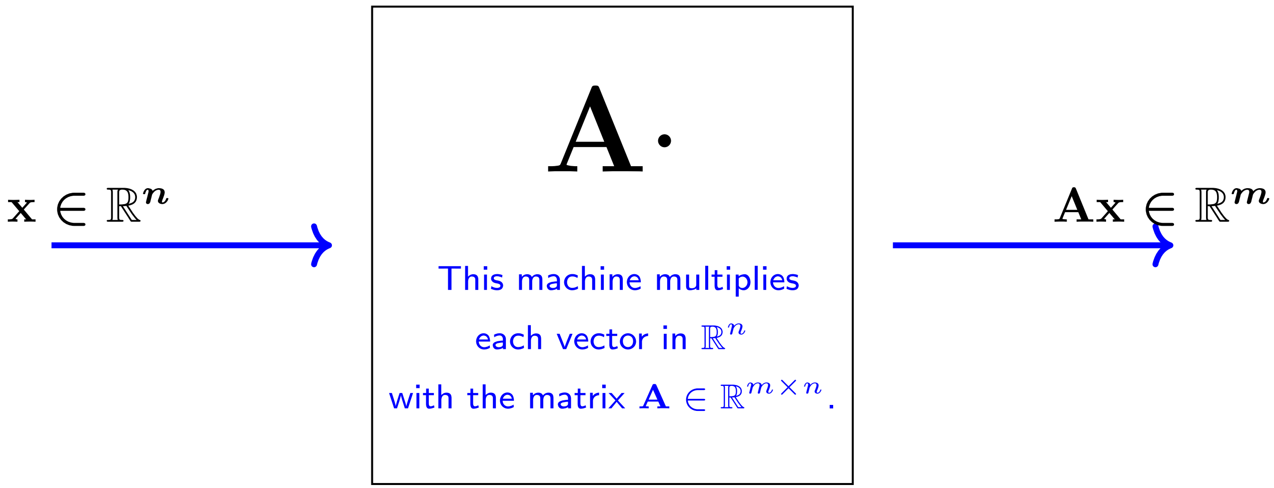
Columnwise view: $\mathbf{Ax}$ is a linear combination of the columns of $\mathbf{A}$:

$$\mathbf{Ax} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \ldots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + \cdots + x_n \begin{pmatrix} | \\ \mathbf{a}_n \\ | \end{pmatrix}$$

Example: We compute $\mathbf{Ax}$ in these two different ways:

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \cdot (-1) + 2 \cdot 0 + 3 \cdot 1 \\ 4 \cdot (-1) + 5 \cdot 0 + 6 \cdot 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \\ &= -1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \end{aligned}$$

Matrices: View as a mapping



The function $f_{\mathbf{A}}$ multiplies vectors with the matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$

$$f_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m, \qquad \text{where} \qquad f_{\mathbf{A}} : \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$$

System of linear equations

- Does $\mathbf{A}\mathbf{x} = \mathbf{b}$ have a solution?
- If yes, then solve $\mathbf{A}\mathbf{x} = \mathbf{b}$.

View as a mapping

- Is $\mathbf{b}$ in the range of $f_{\mathbf{A}}$?
- If yes, then find the inverse image $f_{\mathbf{A}}^{-1}(\mathbf{b})$.

Systems of linear equations

Some special systems of linear equations/mappings

1. <u>*Zero matrix*</u> $\mathbf{0} \in \mathbb{R}^{m\times n}$ ($m \times n$ matrix, all entries are zero)

$$\mathbf{0x} = \mathbf{o} \in \mathbb{R}^m \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

   $f_\mathbf{0}$ is the zero mapping $f_\mathbf{0} : \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{o}$.

2. <u>*Identity matrix*</u> $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ ($n \times n$ matrix, all diagonal entries are one, all others zero):

$$\mathbf{I} := \mathbf{I}_n := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}. \qquad \text{There holds } \mathbf{Ix} = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

   $f_\mathbf{I}$ is the identity mapping $f_\mathbf{I} = \mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n,\ \mathbf{x} \mapsto \mathbf{x}$.

3. <u>*Reflection w.r.t. the x_2 axis*</u> of $\mathbb{R}^2$: If $\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then $f_\mathbf{D} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$.

4. <u>*Reflection w.r.t. the x_1 axis*</u> of $\mathbb{R}^2$: If $\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then $f_\mathbf{E} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$.

Systems of linear equations

Some special systems of linear equations/mappings

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let

$$f_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^2,\ \mathbf{x} \mapsto \mathbf{A}\mathbf{x} \text{ for } \mathbf{A} \in \{\mathbf{0}, \mathbf{I}, \mathbf{D}, \mathbf{E}\}.$$

1. Is $f_{\mathbf{A}}$ injective? $\mathbf{0}$ no, $\mathbf{I}$ yes, $\mathbf{D}$ yes, $\mathbf{E}$ yes
2. Is $f_{\mathbf{A}}$ surjective? $\mathbf{0}$ no, $\mathbf{I}$ yes, $\mathbf{D}$ yes, $\mathbf{E}$ yes
3. What is the inverse image $f_{\mathbf{A}}^{-1}(\mathbf{b})$ for a $\mathbf{b} \in \mathbb{R}^2$? What are the solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$?

$$f_{\mathbf{0}}^{-1}(\mathbf{b}) = \begin{cases} \mathbb{R}^2 & : \quad \mathbf{b} = \mathbf{o} \\ \emptyset & : \quad \mathbf{b} \neq \mathbf{o} \end{cases} \qquad f_{\mathbf{I}}^{-1}(\mathbf{b}) = \mathbf{b}$$

$$f_{\mathbf{D}}^{-1}\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = \begin{pmatrix} -b_1 \\ b_2 \end{pmatrix} \qquad f_{\mathbf{E}}^{-1}\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = \begin{pmatrix} b_1 \\ -b_2 \end{pmatrix}$$

4. How about more general systems of linear equations?

True or false?

1. $\begin{aligned} 3x_1 + x_3 &= 7 \\ 2x_1 + x_2 &= 0 \end{aligned}$ is a linear system of equations.

2. $\begin{aligned} 3x_1 + \sin x_3 &= 7 \\ 2x_1 + x_2 &= 0 \end{aligned}$ is a linear system of equations.

3. $\begin{aligned} 3x_1 + x_3 &= \sin 7 \\ 2x_1 + x_2 &= 0 \end{aligned}$ is a linear system of equations.

4. $\begin{pmatrix} 2 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

5. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix}$. Then $f_{\mathbf{A}}$ is injective.

6. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix}$. Then $f_{\mathbf{A}}$ is surjective.

7. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix}$. Then $f_{\mathbf{A}}^{-1}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

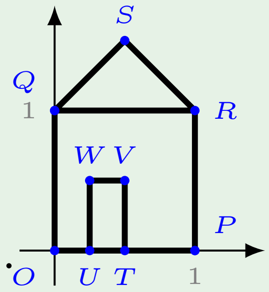
Linearity

Our house H

Let $H \subset \mathbb{R}^2$ be the house with position vectors to the marked points:

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e_1},$$

$$\mathbf{w} = \begin{pmatrix} 0.25 \\ 0.50 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{e_2}, \quad \mathbf{s} = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



What happens to the house H under the mapping $f_{\mathbf{A}} : H \to \mathbb{R}^2, \ \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$?

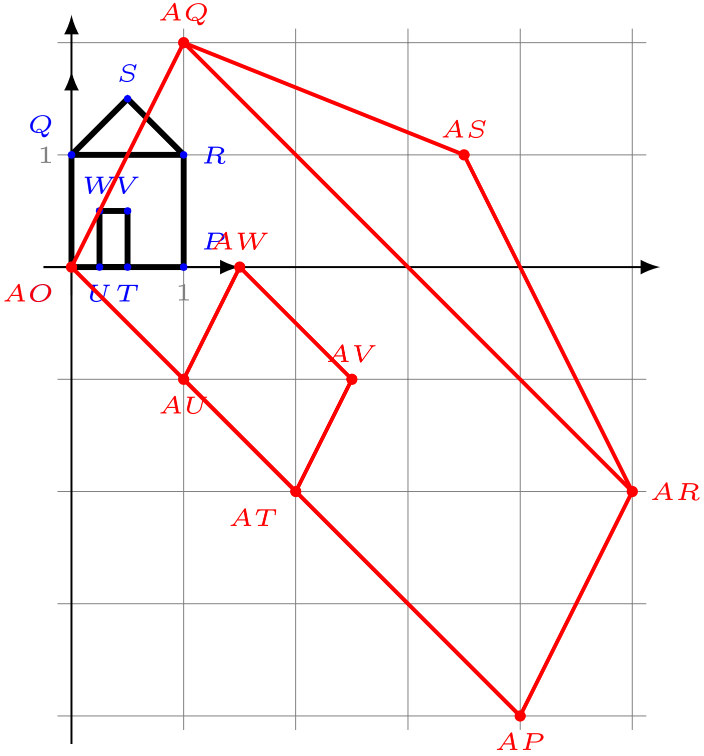
$$H' := f_{\mathbf{A}}(H) = \{\mathbf{A}\mathbf{x} : \mathbf{x} \in H\}$$

Four questions regarding the new house H'

1. Does H' still look like a house?
2. Are the edges slanted and/or warped?
3. How large is H'?
4. Where is the door?

Linearity

Some preliminary considerations



$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{pmatrix} := \begin{pmatrix} 4 & 1 \\ -4 & 2 \end{pmatrix}$$

$$\mathbf{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \mathbf{Ao} = 0\mathbf{a}_1 + 0\mathbf{a}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} 0.25 \\ 0 \end{pmatrix}, \qquad \mathbf{Au} = 0.25\mathbf{a}_1 + 0\mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathbf{t} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}, \qquad \mathbf{At} = 0.5\mathbf{a}_1 + 0\mathbf{a}_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$\mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{Ap} = 1\mathbf{a}_1 + 0\mathbf{a}_2 = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{Ar} = 1\mathbf{a}_1 + 1\mathbf{a}_2 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

The house $H' = \{\mathbf{Ax} : \mathbf{x} \in H\}$ is deformed, but straight line segments remain straight.

Linearity

Theorem 3.9 (Properties of $f_\mathbf{A}$ or matrix-vector multiplication, resp.)

Let $m, n \in \mathbb{N}$, $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $f_\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$.

a) For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ there holds

$$f_\mathbf{A}(\mathbf{x}+\mathbf{y}) = f_\mathbf{A}(\mathbf{x}) + f_\mathbf{A}(\mathbf{y}), \quad \text{i.e.} \quad \mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}. \tag{$+$}$$

first add, then map = first map, then add

b) For all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ there holds

$$f_\mathbf{A}(\alpha\mathbf{x}) = \alpha f_\mathbf{A}(\mathbf{x}), \quad \text{i.e.} \quad \mathbf{A}(\alpha\mathbf{x}) = \alpha\mathbf{A}\mathbf{x}. \tag{$\cdot$}$$

first stretch, then map = first map, then stretch

Linearity

Proof. Let $\alpha \in \mathbb{R},\ m, n \in \mathbb{N}$, $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$. Then there hold

$$
\begin{aligned}
f_{\mathbf{A}}(\mathbf{x}+\mathbf{y}) = \mathbf{A}(\mathbf{x}+\mathbf{y}) &= \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \\
&= (x_1+y_1)\mathbf{a}_1 + \cdots + (x_n+y_n)\mathbf{a}_n \\
&= x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n + y_1\mathbf{a}_1 + \cdots + y_n\mathbf{a}_n \\
&= \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \\
&= \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y} = f_{\mathbf{A}}(\mathbf{x}) + f_{\mathbf{A}}(\mathbf{y}),
\end{aligned}
$$

$$
f_{\mathbf{A}}(\alpha\mathbf{x}) = \mathbf{A}(\alpha\mathbf{x}) = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix} = \cdots = \alpha\mathbf{A}\mathbf{x} = \alpha f_{\mathbf{A}}(\mathbf{x}).
$$

$\square$

Linearity

Theorem 3.9 (Properties of $f_{\mathbf{A}}$ or matrix-vector multiplication, resp.)

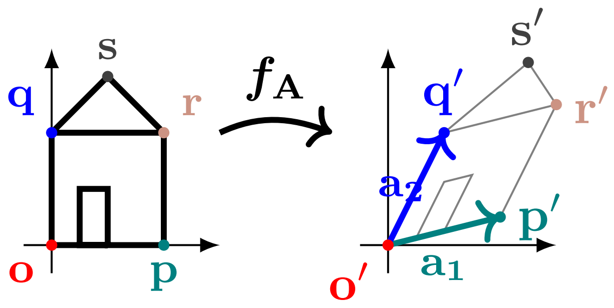
Let $m, n \in \mathbb{N}$, $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $f_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$.
Then there hold for all $\alpha \in \mathbb{R}$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$f_{\mathbf{A}}(\mathbf{x}+\mathbf{y}) = f_{\mathbf{A}}(\mathbf{x}) + f_{\mathbf{A}}(\mathbf{y}), \qquad \text{i.e.} \quad \mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y},$$

$$f_{\mathbf{A}}(\alpha\mathbf{x}) = \alpha f_{\mathbf{A}}(\mathbf{x}), \qquad \text{i.e.} \quad \mathbf{A}(\alpha\mathbf{x}) = \alpha\mathbf{A}\mathbf{x}.$$

Implication for our house

$$\underbrace{g = \{\mathbf{p} + \alpha\mathbf{v} : \alpha \in \mathbb{R}\}}_{\text{a line}} \implies g' := \mathbf{A}g := \{\mathbf{A}(\mathbf{p}+\alpha\mathbf{v}) : \alpha \in \mathbb{R}\} = \underbrace{\{\overbrace{\mathbf{A}\mathbf{p}}^{\text{new point}} + \alpha \overbrace{\mathbf{A}\mathbf{v})}^{\text{new direction}} : \alpha \in \mathbb{R}\}}_{\text{is still a line}}$$



$$\mathbf{p} = \mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \mathbf{q} = \mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix}x_1\\x_2\end{pmatrix} = x_1\mathbf{p} + x_2\mathbf{q}$$

$$\mathbf{x}' := \mathbf{A}\mathbf{x} = \mathbf{A}(x_1\mathbf{p} + x_2\mathbf{q}) = x_1\mathbf{A}\mathbf{p} + x_2\mathbf{A}\mathbf{q} = x_1\mathbf{p}' + x_2\mathbf{q}'$$

$\mathbf{x}' = f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is known as soon as $\mathbf{p}'$ and $\mathbf{q}'$ are known.

Linearity

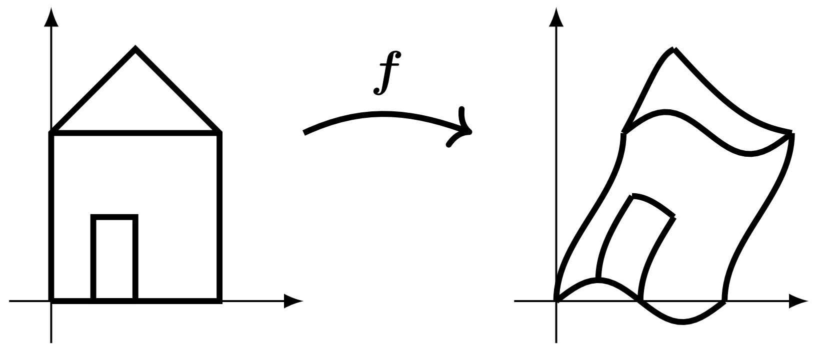
Definition 3.11 (additive / homogeneous / linear mapping)

$$\text{A mapping } f : \mathbb{R}^n \to \mathbb{R}^m \text{ which } \begin{cases} \text{satisfies } (+) \text{ is called } \underline{\textit{additive}}, \\ \text{satisfies } (\,\cdot\,) \text{ is called } \underline{\textit{homogeneous}}, \\ \text{satisfies } (+) \text{ and } (\,\cdot\,) \text{ is called } \underline{\textit{linear}}. \end{cases}$$

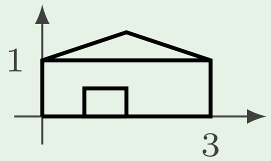
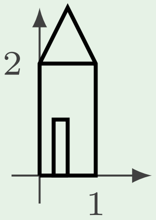
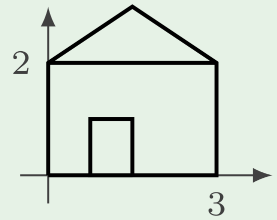
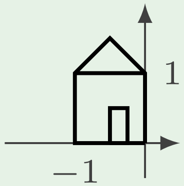
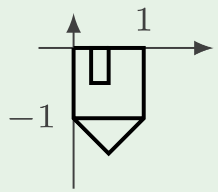
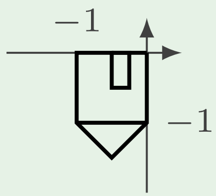
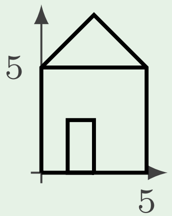
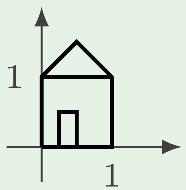
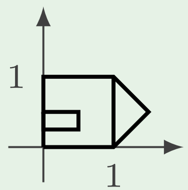
The mapping $f_{\mathbf{A}}$ (matrix-vector multiplication) is an example for a **linear mapping**.

A **nonlinear mapping** is for example

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - \frac{1}{5}(\cos(\pi y) - 1) \\ y + \frac{1}{8}\sin(2\pi x) \end{pmatrix}.$$

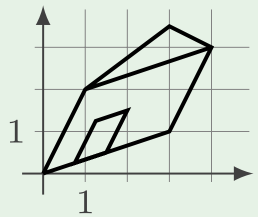


Good houses, bad houses

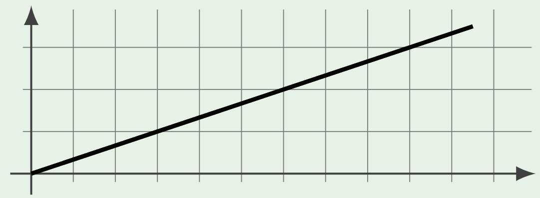
$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ 	$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ 	$\mathbf{C} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ 
$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ 	$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 	$\mathbf{F} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ 
$\mathbf{G} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ $H' = 5H$ 	$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $H' = H$ $f_I = \mathsf{id}$ 	$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ 

Good houses, bad houses

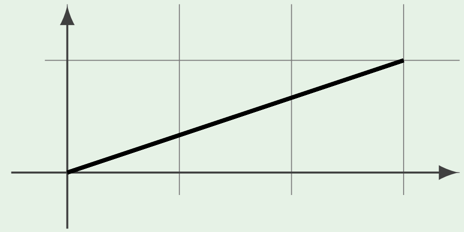
$$\mathbf{K} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$



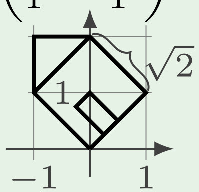
$$\mathbf{L} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$$



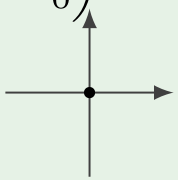
$$\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$$



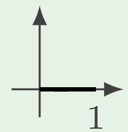
$$\mathbf{N} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$



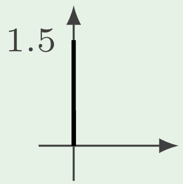
$$\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$



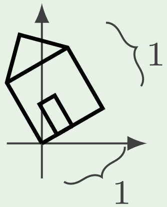
$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$



$$\mathbf{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



$$\mathbf{R} = \begin{pmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{pmatrix}$$



$$\mathbf{S} = \begin{pmatrix} -1 & -1 \\ -3 & 3 \end{pmatrix}$$