Discrete Algebraic Structures

Sheet 1

Exercise 0 (Warmup):

Compute the following objects:

1. $(\{1,2\} \cap \{2,3\}) \cup \{2,4\}$ $= \{2,4\}$
2. $(\{1,2\} \cup \{2,3\}) \setminus \{2,4\}$ $= \{1,3\}$
3. $(\{1,2\} \cap \{2,3\}) \,\Delta\, \{2,4\}$ $= \{4\}$
4. $(\{1,2\} \cup \{2,3\}) \times \{2,4\}$ $= \{1,2,3\} \times \{2,4\} = \{\{1,2\}, \{1,4\}, \{2,2\}, \{2,4\}, \{3,2\}, \{3,4\}\}$
5. $\mathcal{P}(\{1,3,4\})$ $= \{1, \{1,3\}, \{1,4\}, 3, \{3,4\}, 4, \{1,3,4\}\}$

Exercise 1 (Sets):

(a) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) Prove that $A \cap (A \cup B) = A$.

(c) Prove that if $B \subseteq A$, then $A \cap B = B$ and $A \cup B = A$.

Exercise 2 (De Morgan's law):

Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ using a written proof in the style of Exercise 1(a).

Exercise 3 (Distributivity of Cartesian Product):

Let A, B, C, D be sets.

1. Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$
2. Show that $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3. Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
4. Do we have $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$? If not, is one of both inclusions true?
5. Illustrate the results of this exercise using figures. You can use for example $A = [0,2]$, $B = [1,3]$, $C = [0,3]$, $D = [1,4]$.

Exercise 4 (Symmetric Difference):

This is a supplementary exercise that you should do at home, to get more practice after the tutorials.

(a) Let $A \Delta B = C$. What is $A \Delta C$?

(b) Prove that $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

(c) Show that the symmetric difference is associative, i.e., that $A\Delta(B\Delta C) = (A\Delta B)\Delta C$ is true for all sets A, B, C. Can you find a natural description of the elements that are in $A \Delta B \Delta C$?

0.

1. (This is needlessly complicated and relies on ungiven premises)

notes

0: $(A\cup B)\times C \overset{!}{=} (A\times C)\cup(B\times C)$ | (Thesis)

Hyp. 1: A, B, C are Sets

Hyp. 2: $x\in(A\cup B)\times C$

Hyp. 3: using $S=(A\cup B) \to S\times C$

Hyp. 4: $A\subseteq S$

Hyp. 5: $B\subseteq S$

Hyp. 6: $A\cup B\subseteq S$

Hyp. 6: $x\in(A\cap C)\times(B\cap C)$

proof

Def. 0: Thesis means „Every tuple from $(A\cup B)\times C$ must correspond to a tuple from $(A\times C)\cup(B\times C)$ (and vice versa)

Def. 1: For $S=(A\cup B)$ and $S\times C$, $S\times C$ can be written as a union between all of S's subsets $\times C$

Def. 2: A and B are subsets of, and make up S. Any operation on S can be written as a union between that operation performed on all its subsets

Def. 3: Therefore: $(A\cup B)\times C=(A\times C)\cup(B\times C)$

q.e.d.

1. (again)

Thesis: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Splits into: 1: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

2. ... $\supseteq$ ...

1. $A \cup (B \cap C) \subseteq$ ...

1.1. $x \in A \cup (B \cap C)$

1.2. $x \in A$ or $x \in (B \cap C)$

1.3. Assuming $x \in A$:

1.3.1 x must be in $(A \cup B)$ and $(A \cup C)$

1.3.2. x must be in $(A \cup B) \cap (A \cup C)$ (since its in both operands)

1.4. Assuming $x \in (B \cap C)$

1.4.1. $x \in B$ and $x \in C$

1.4.2 Therefore: x must be part of $(A \cup B)$ and $(A \cup C)$ ($\cup$ only extends sets)

1.4.3 "and" rewritten as operator: $x \in (A \cup B) \cap (A \cup C)$

2. $(A \cup B) \cap (A \cup C) \supseteq$ ...

2.1. $x \in (A \cup B) \cap (A \cup C)$

2.2. $x \in (A \cup B)$ and $x \in (A \cup C)$

2.3. Case: $x \in A$:

2.3.1. Entails $x \in A \cup (B \cap C)$ ($\cup$ extension doesn't break $x \in A$)

2.4. Case: $x \notin A$:

2.4.1. $x \in B$ and $x \in C$

2.4.2. rewritten: $x \in (B \cap C)$

2.4.3. extend with arbitrary union (here: with A): $x \in A \cup (B \cap C)$

Q.E.D

1.(b)

Thesis: $A \cap (A \cup B) = A$

Split Theses:

1. $A \cap (A \cup B) \supseteq A$
   1.1. $x \in A \cap (A \cup B)$
   1.2. $x \in A$ and $x \in (A \cup B)$
   (1.3. $x \in A$ or $x \in B$
      1.3.1. already filled by 1.2: $x \in A$) could be skipped
   1.4. 1.1 and 1.2 ($x \in A$), Thesis 1 fulfilled
2. $A \supseteq A \cap (A \cup B)$
   2.1. $x \in A$
   2.2. Therefore: $x \in (A \cup B)$
   2.3. Known: $x \in A$ and $x \in (A \cup B)$
      2.3.1. Combination: $x \in A \cap (A \cup B)$ (2.1, 2.2)

Q.E.D.

(c)

Thesis: Given $B \subseteq A$, $A \cap B = B$ and $A \cup B = A$

Subtheses: $A \cap B \supseteq B$, $A \cap B \subseteq B$, $A \cup B \supseteq A$, $A \cup B \subseteq A$

1. $A \cap B \overset{!}{\subseteq} B$
  1.1. $x \in (A \cap B)$
  1.2. $x \in A$ and $x \in B$
    1.2.1. Thesis 1 satisfied (1.1, 1.2), $x \in (A \cap B)$, $x \in B$
2. $B \subseteq A \cap B$
  2.1. repeat: $B \subseteq A$
  2.2. $x \in B$
  2.3. from 2.1: $x \in A$
  2.4. 2.2, 2.3: $x \in A \cap B$
    2.4.1. Thesis 2 satisfied
3. $A \cup B \subseteq A$
  3.1. rp.: $B \subseteq A$
  3.2. $x \in A \cup B$
  3.3. $x \in A$ or $x \in B$
    3.3.1. case $x \in A$:
      3.3.1.1. satisfies Th. 3 together with 3.2.
    3.3.2. case $x \in B$:
      3.3.2.1. from 3.1.: $x \in A$
      3.3.2.2. satisfies Th. 3 (w/ 3.2.)
4. $A \subseteq A \cup B$
  4.1. rp.: $B \subseteq A$
  4.2. $x \in A$
  4.3. write 4.2. as: $x \in A \cup B$ (arbitrary union doesn't break membership)
    4.3.1. Th. 4. satisfied (4.3, 4.2)

Satisfied Th. 1–4

Q.E.D