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Institute of Mathematics

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Homework 6 for
Mathematics I (winter term 25/26)

Submit your solutions to **Problems 2 and 3** until Sunday, **December 07**, 11:59 pm at the latest, using **abGabi**. Only one of these problems will be chosen by us to be corrected and graded. We strongly suggest that you **submit in pairs**. Please state the names and matriculation numbers of both persons on your sumbissions and only submit once per group (the other person will still receive credit). Submission in larger groups is not permitted.

Problem 1 *Series*

Consider the sequence $(x_k)_{k\in\mathbb{N}}$ defined by

$$x_k = \begin{cases} \frac{1}{3^k} & \text{if } k \text{ is odd,} \\ \frac{1}{5^k} & \text{if } k \text{ is even.} \end{cases}$$

(a) Compute the values of the first five sequence elements.

(b) Prove that the series $\sum_{k=1}^{\infty} x_k$ converges absolutely by applying the root test.

(c) Show that the ratio test applied to $\sum_{k=1}^{\infty} x_k$ is inconclusive.

(d) Determine the limit of $\sum_{k=1}^{\infty} x_k$. (Hint: It might help to split the series into two geometric series).

Problem 2 *Series*

Determine whether the following series converge or diverge. In case of convergence, also check for absolute convergence.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k(k-1)}{k}$,

(b) $\sum_{k=1}^{\infty} 5q^{2k}$ with $q \in (-1, 1)$,

(c) $\sum_{k=1}^{\infty} \frac{k^3+1}{k^5+1}$,

(d) $\sum_{k=1}^{\infty} \left(\sqrt[k]{k} - 1\right)^k$.

Problem 3 *Kernel, rank, LES*

Define

$$\mathbf{A} := \begin{pmatrix} 0 & 1 & 0 & 2 \\ -2 & 0 & 4 & 2 \\ 1 & -3 & -2 & -7 \end{pmatrix}.$$

(a) Determine $\operatorname{Rank}(\mathbf{A})$ and $\operatorname{Ker}(\mathbf{A})$.

(b) Find a vector $\mathbf{b} \in \mathbb{R}^3$ such that the system $\mathbf{Ax} = \mathbf{b}$ has no solution, or explain why such a vector does not exist.

Problem 4 *Linear maps*

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$ be chosen such that for the linear functions $f_{\mathbf{A}}, f_{\mathbf{B}} : \mathbb{R}^2 \to \mathbb{R}^2$ with $f_{\mathbf{A}} : \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ and $f_{\mathbf{B}} : \mathbf{x} \mapsto \mathbf{B}\mathbf{x}$ the following holds:

$$\begin{pmatrix}1\\0\end{pmatrix} \overset{f_{\mathbf{A}}}{\mapsto} \begin{pmatrix}3\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \overset{f_{\mathbf{A}}}{\mapsto} \begin{pmatrix}1\\1\end{pmatrix} \qquad \text{resp.} \qquad \begin{pmatrix}2\\-4\end{pmatrix} \overset{f_{\mathbf{B}}}{\mapsto} \begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\2\end{pmatrix} \overset{f_{\mathbf{B}}}{\mapsto} \begin{pmatrix}3\\-2\end{pmatrix}.$$

Calculate $f_{\mathbf{A}}(\mathbf{u})$ and $f_{\mathbf{B}}(\mathbf{w})$ for

$$\mathbf{u} := \begin{pmatrix}1\\-2\end{pmatrix}, \quad \mathbf{w} := \begin{pmatrix}2\\1\end{pmatrix}.$$

Problem 2 *Series*

Determine whether the following series converge or diverge. In case of convergence, also check for absolute convergence.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k(k-1)}{k}$,

(b) $\sum_{k=1}^{\infty} 5q^{2k}$ with $q \in (-1,1)$,

(c) $\sum_{k=1}^{\infty} \frac{k^3+1}{k^5+1}$,

(d) $\sum_{k=1}^{\infty} \left(\sqrt[k]{k}-1\right)^k$.

(a)

check for null sequence:

$$\lim_{k\to\infty} \frac{(-1)^k(k-1)}{k} = \lim_{k\to\infty} (-1)^k \cdot \frac{k-1}{k}$$

$$= \lim_{k\to\infty} (-1)^k \cdot \frac{k}{k} \cdot \left(1-\frac{1}{k}\right)$$

$$= \lim_{k\to\infty} (-1)^k \cdot \lim_{k\to\infty}\left(1-\frac{1}{k}\right)$$

$$= DNE \cdot 1 = DNE$$

$\Rightarrow$ no sequence limit $l=0$, series cannot converge

(b)

$$5 \cdot \sum_{k=1}^{\infty} q^{2k} = 5 \cdot \sum_{k=1}^{\infty} q^k \cdot \sum_{k=1}^{\infty} q^k$$

by geometric sequence:

$$= 5 \cdot \left(\frac{1}{1-q}\right)^2 = 5 \cdot \frac{1}{(1-q)^2} = \frac{5}{1-2q+q^2}$$

$\Rightarrow$ series converges absolutely

(c)

$$\sum_{k=1}^{\infty} \frac{k^3+1}{k^5+1}$$

null sequence check:

$$\lim_{k\to\infty} \frac{k^3+1}{k^5+1} = \lim_{k\to\infty} \frac{k^3}{k^5} = \frac{1}{k^2} = 0$$

=> series can converge

divergent minorant check:

$t_k = \frac{k^3}{k^5} = \frac{1}{k^2}$ X k-power $\not\leq 1$ => divergence not proven

convergent majorant check:

$u_k = \frac{2k^3}{k^5} = \frac{2}{k^2}$ (u_k as majorant)

$\frac{k^3+1}{k^5+1} \leq \frac{2}{k^2}$ (for $k \geq 1$) ✓

$|u_k|$ is a known convergent majorant (p-series with $p > 1$)

=> series converges absolutely

(d)

root test:

$$\sqrt[k]{\left|\left(\sqrt[k]{k}-1\right)^k\right|} = \sqrt[k]{k}-1$$

$$\lim_{k\to\infty} \sqrt[k]{k}-1 < 1$$

=> series is absolutely convergent

3a) Rang(A)

$$A := \begin{pmatrix} 0 & 1 & 0 & 2 \\ -2 & 0 & 4 & 2 \\ 1 & -3 & -2 & -7 \end{pmatrix}$$

$$\hookrightarrow \begin{pmatrix} 1 & -3 & -2 & -7 \\ -2 & 0 & 4 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\begin{pmatrix} 1 & -3 & -2 & -7 \\ 0 & -6 & 0 & -12 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$R_2 \to R_2 / (-6)$

$$\begin{pmatrix} 1 & -3 & -2 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & -3 & -2 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\hookrightarrow$ Rang$(A) = 2$

Ker (A)

$x \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ with $Ax = 0$

$x_2 + 2x_4 = 0 \Rightarrow x_2 = -2x_4$

$x_1 - 3x_2 - 2x_3 - 7x_4 = 0$

$x_1 - 3(-2x_4) - 2x_3 - 7x_4 = 0$

$x_1 + 6x_4 - 2x_3 - 7x_4 = 0$

$x_1 - 2x_3 - x_4 = 0 \Rightarrow x_1 = 2x_3 + x_4$

$x_3 = y \quad x_4 = z$

$$x = \begin{pmatrix} 2y + z \\ -2z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

basis of Ker(A): $\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$

b)

$$A^T = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & -3 \\ 0 & 4 & -2 \\ 2 & 2 & -7 \end{pmatrix}$$

redukfion of A^T

$$A^T \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \\ 2 & 2 & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 1 \\ 0 & 4 & -2 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$b_1 - 3b_3 = 0 \Rightarrow b_1 = 3b_3$

$-2b_2 + b_3 = 0 \Rightarrow b_3 = 2b_2$

$b_2 = 1 \quad b_3 = 2 \quad b_1 = 6$

$\hookrightarrow b = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$ = Vektor b