Discrete Algebraic Structures

WiSe 2025/2026

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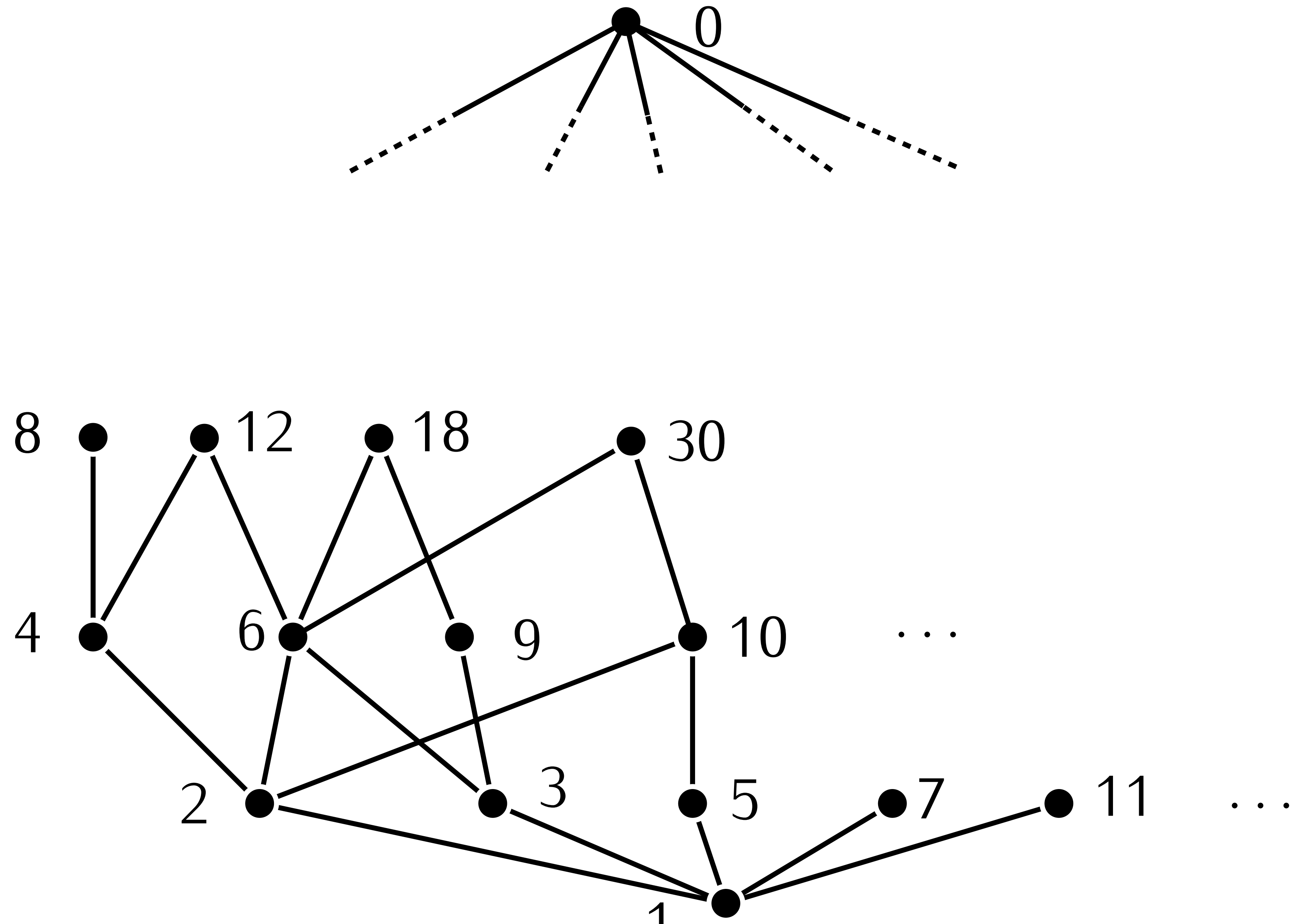
Research Group for Theoretical Computer Science


0
8
12
18
30
4
6
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10
...
2
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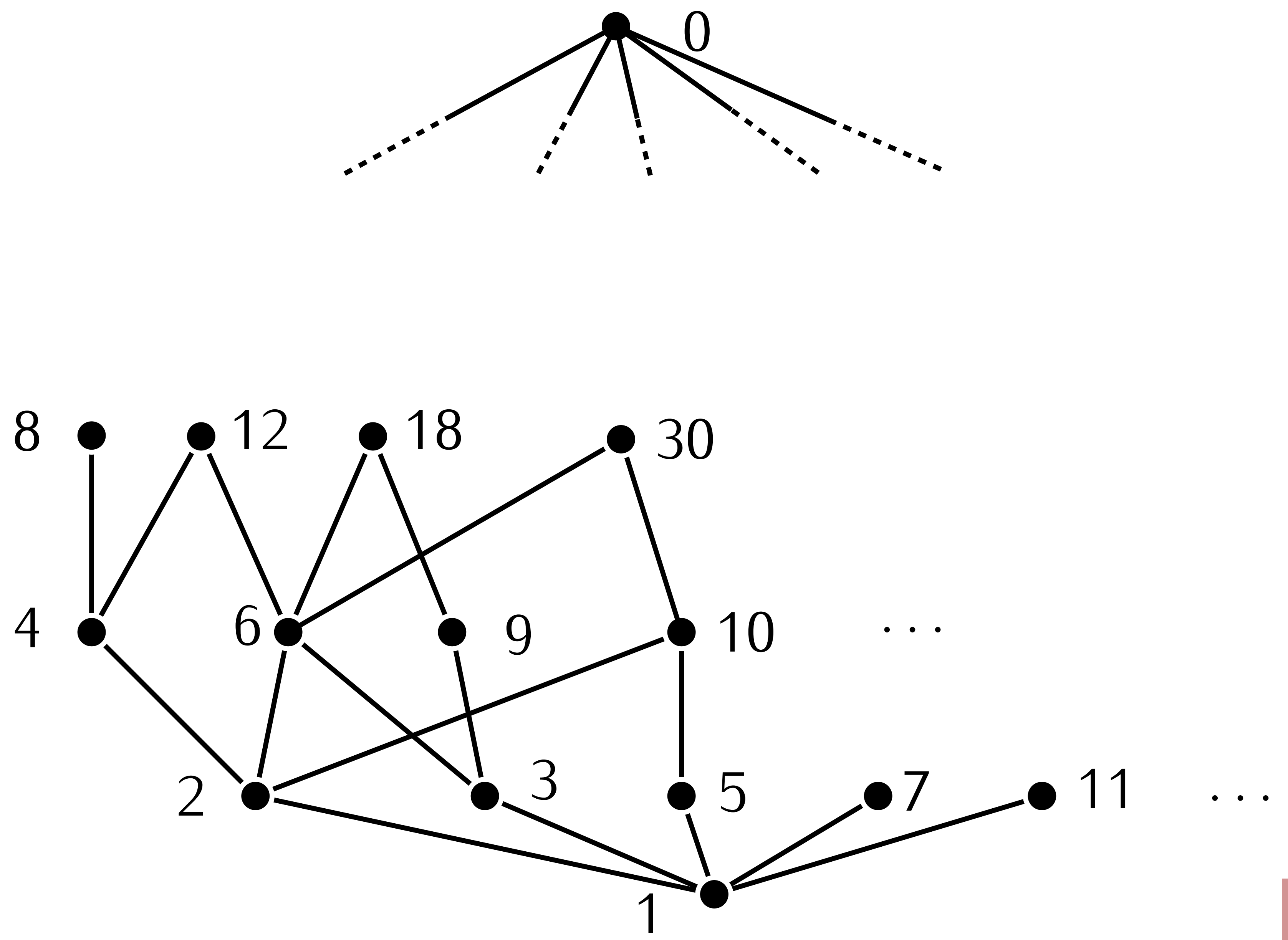
5
7
11
...
1

Definition. Let $S \subseteq \mathbb{N}_0$, and $d \in \mathbb{N}_0$. We say:

- d is a common divisor of S if for all $s \in S$, we have d divides s
- d is a greatest common divisor of S if it is a common divisor and for every common divisor d' of S, we have d' divides d

We write $a \wedge b$ for the greatest common divisor (gcd) of $\{a, b\}$.



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Proof by induction: `remainders` $= (r_0, r_1, \ldots, r_k, 0)$
For every $i \in \{0, \ldots, k\}$, d divides r_i
True for $i = 0, 1$ by assumption
For $i \geq 2$:

$$r_i = q \cdot r_{i+1} + r_{i+2}$$

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So $r_{i+2} = d \cdot (x - qy)$ is divisible by d. $\square$

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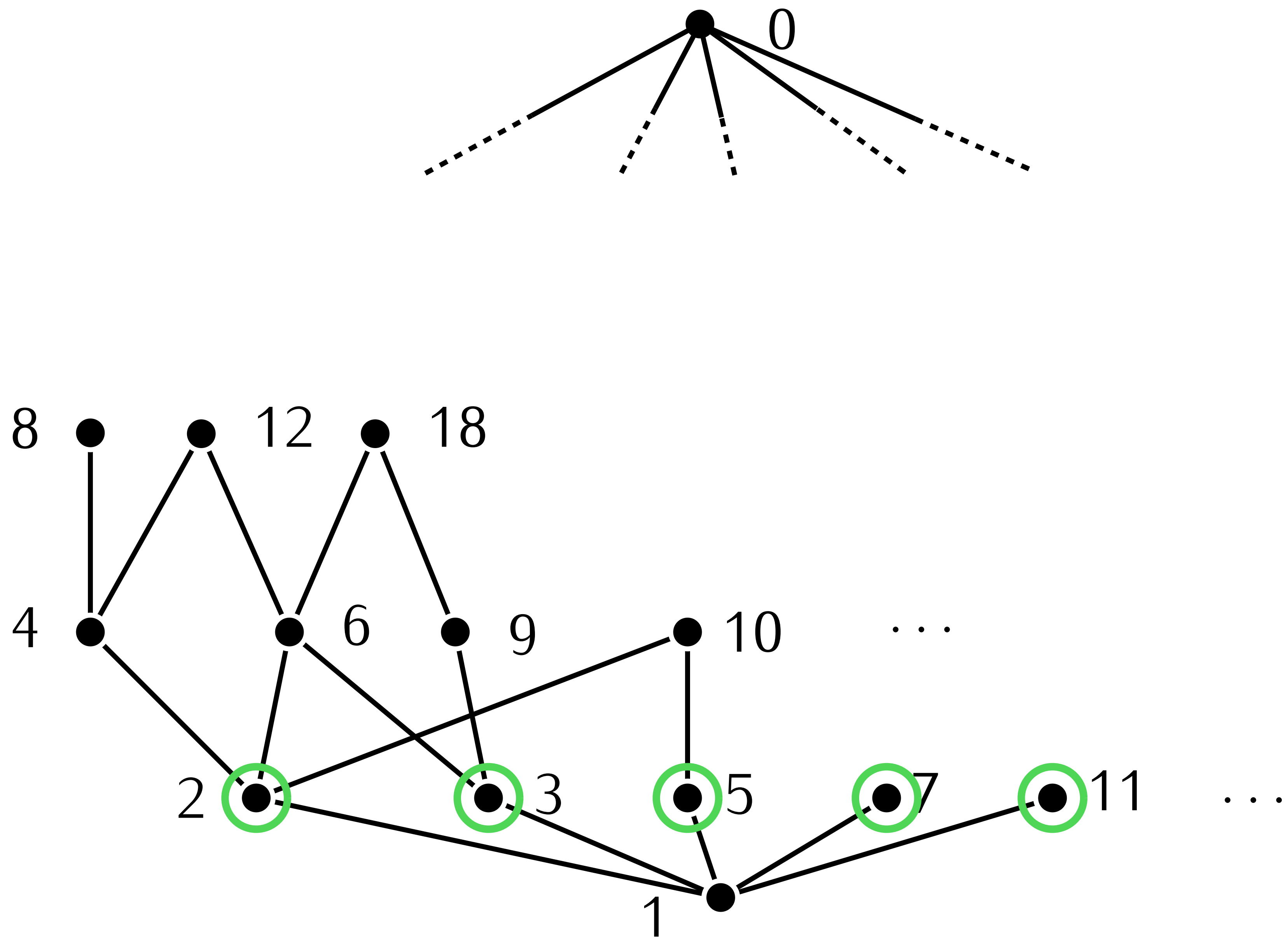

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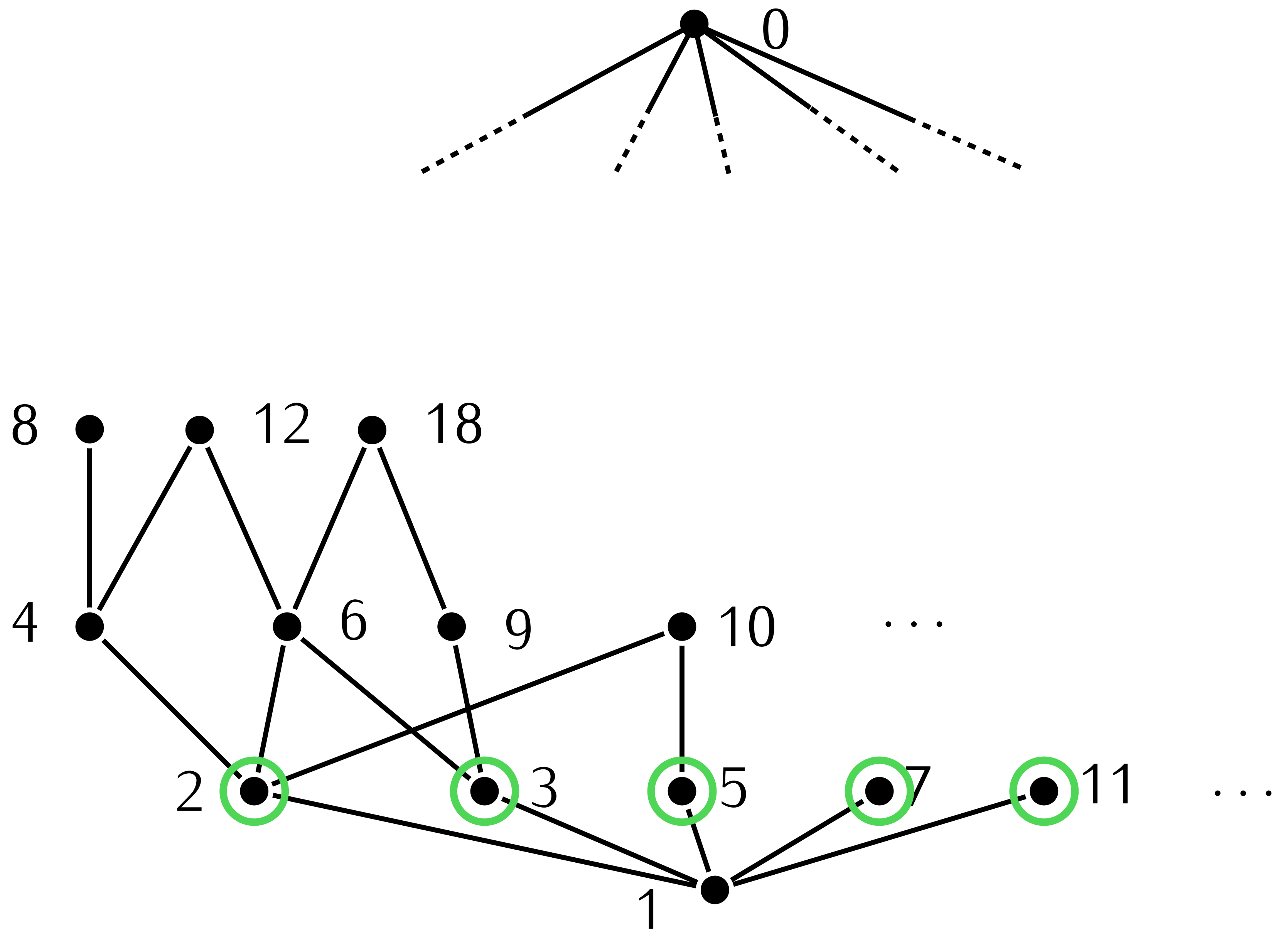

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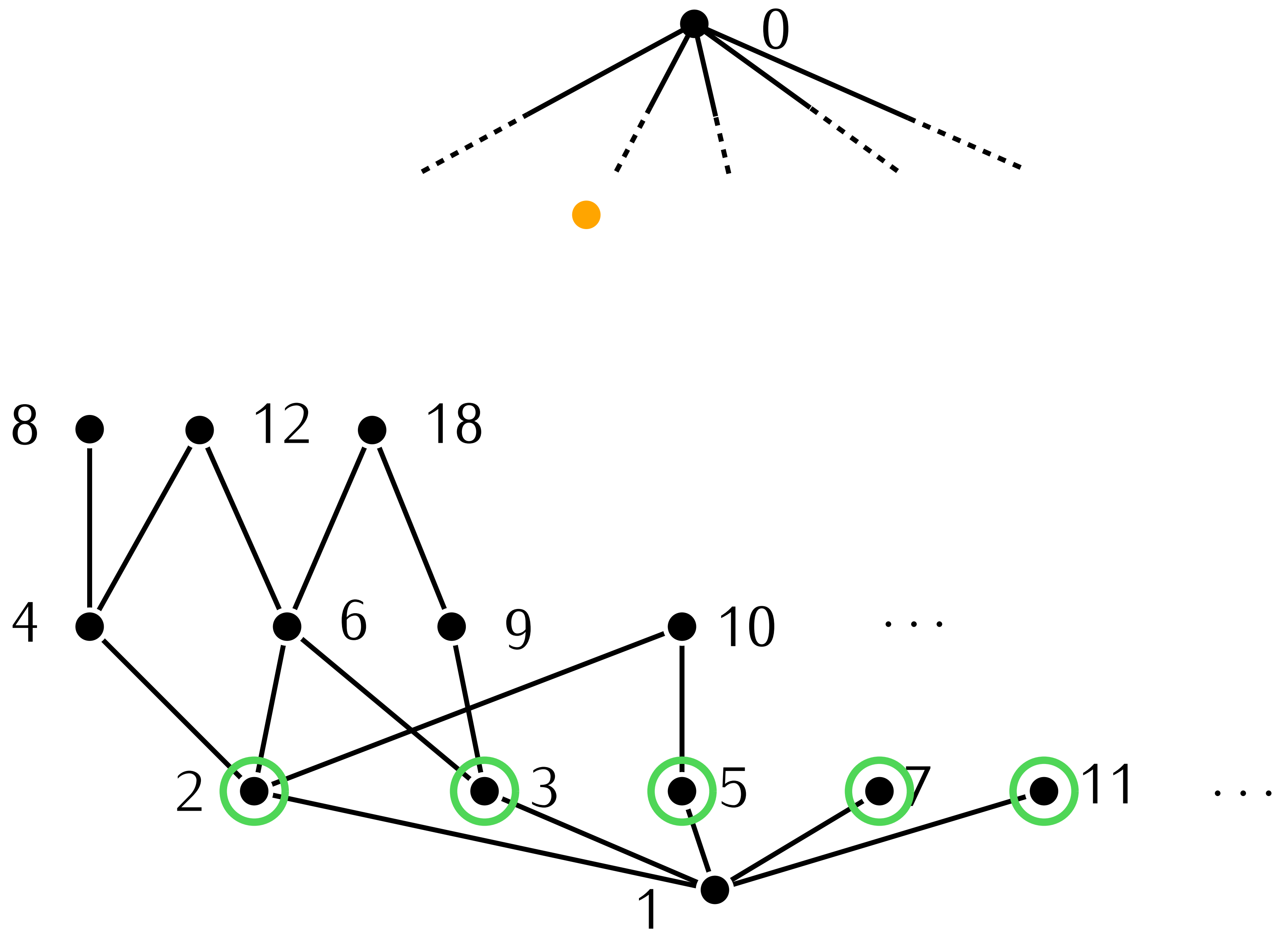

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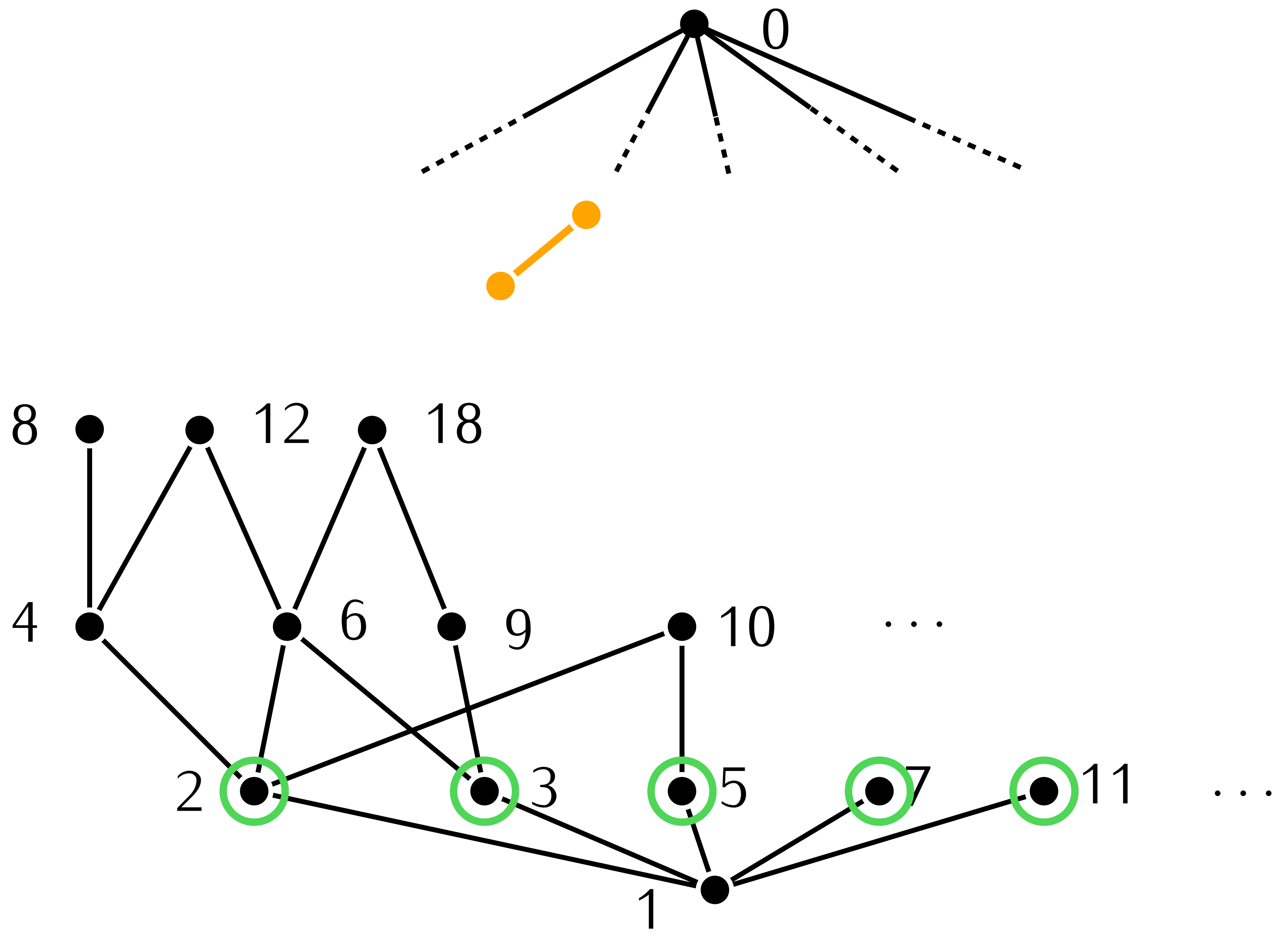

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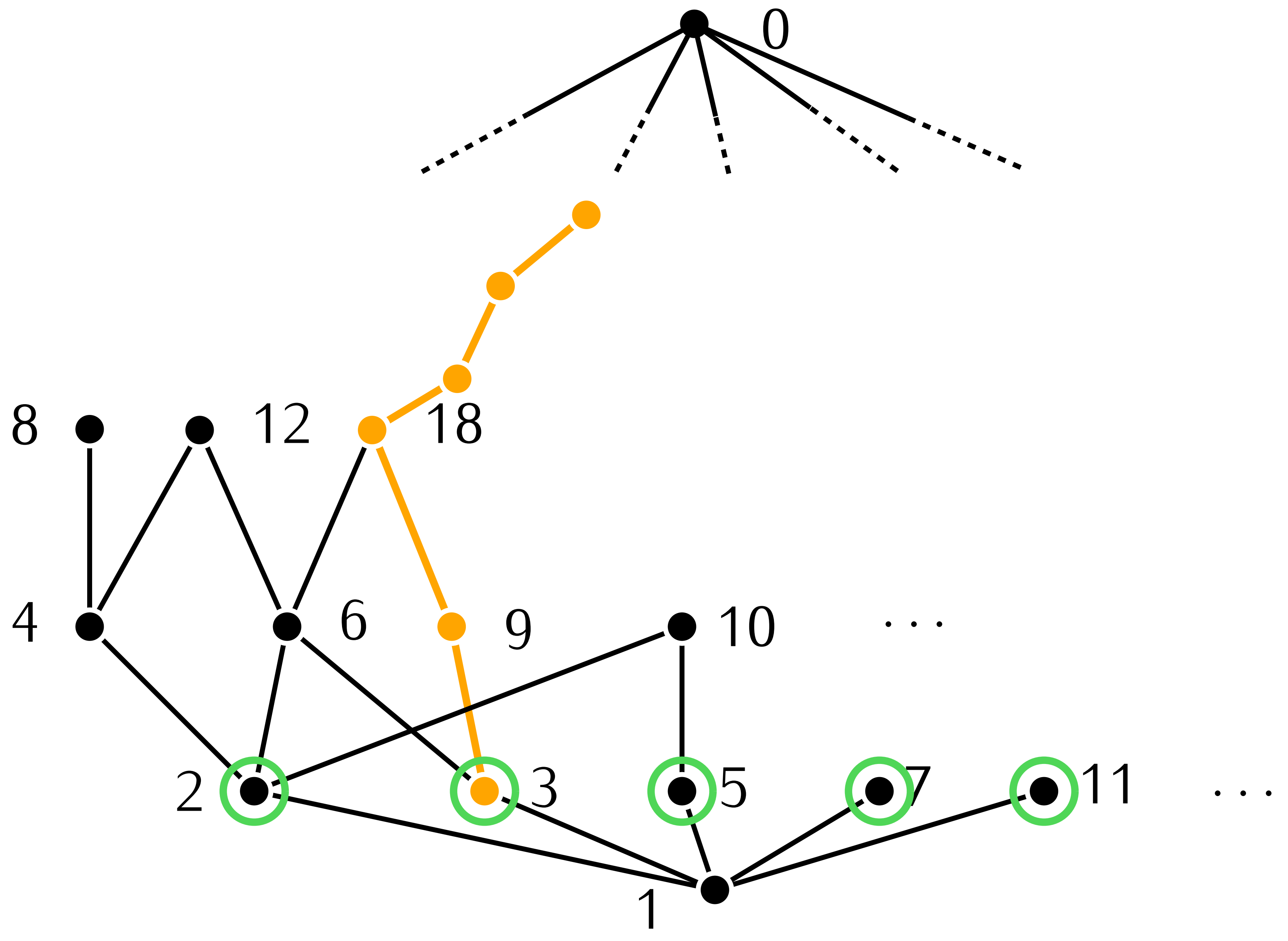

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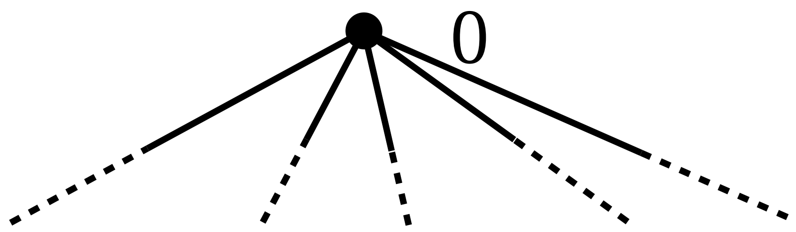

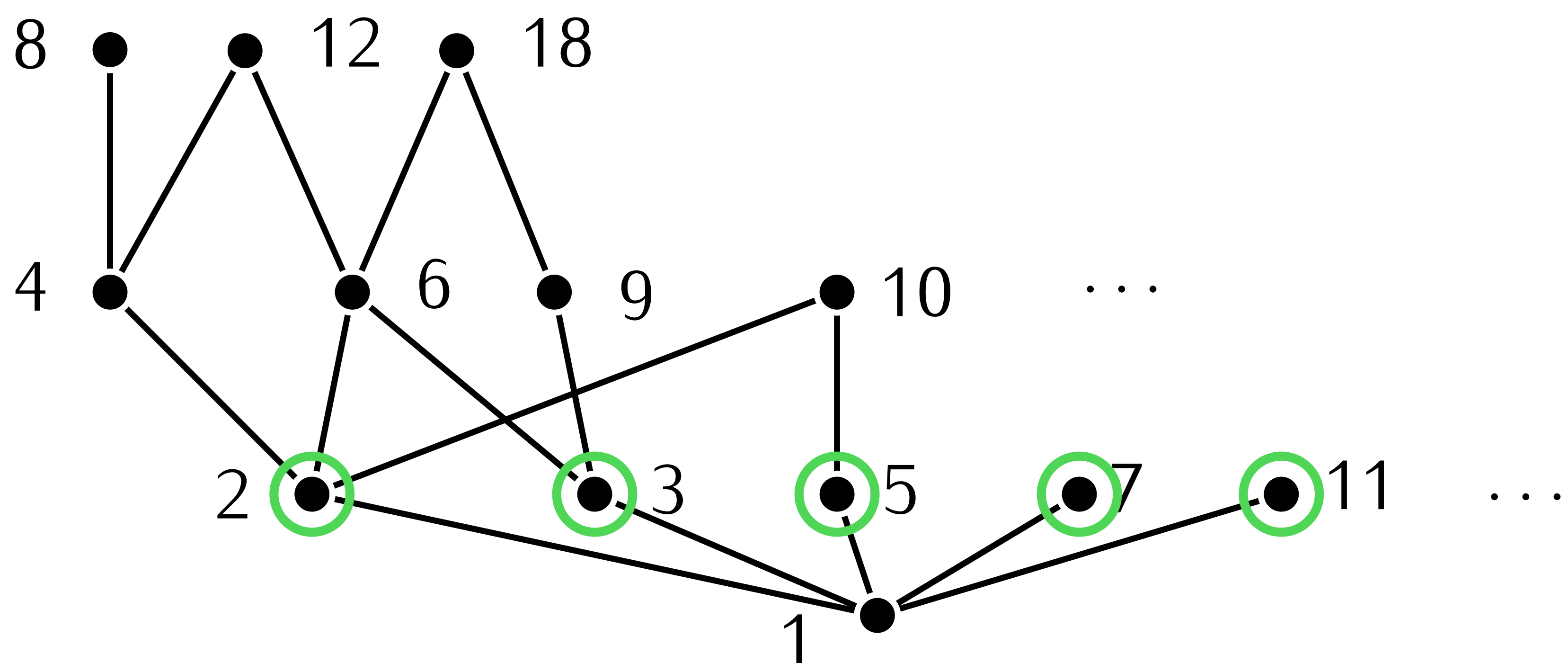

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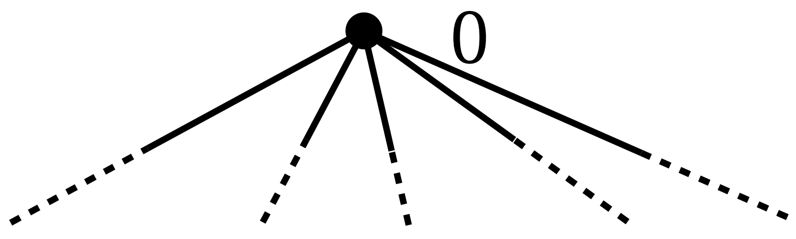

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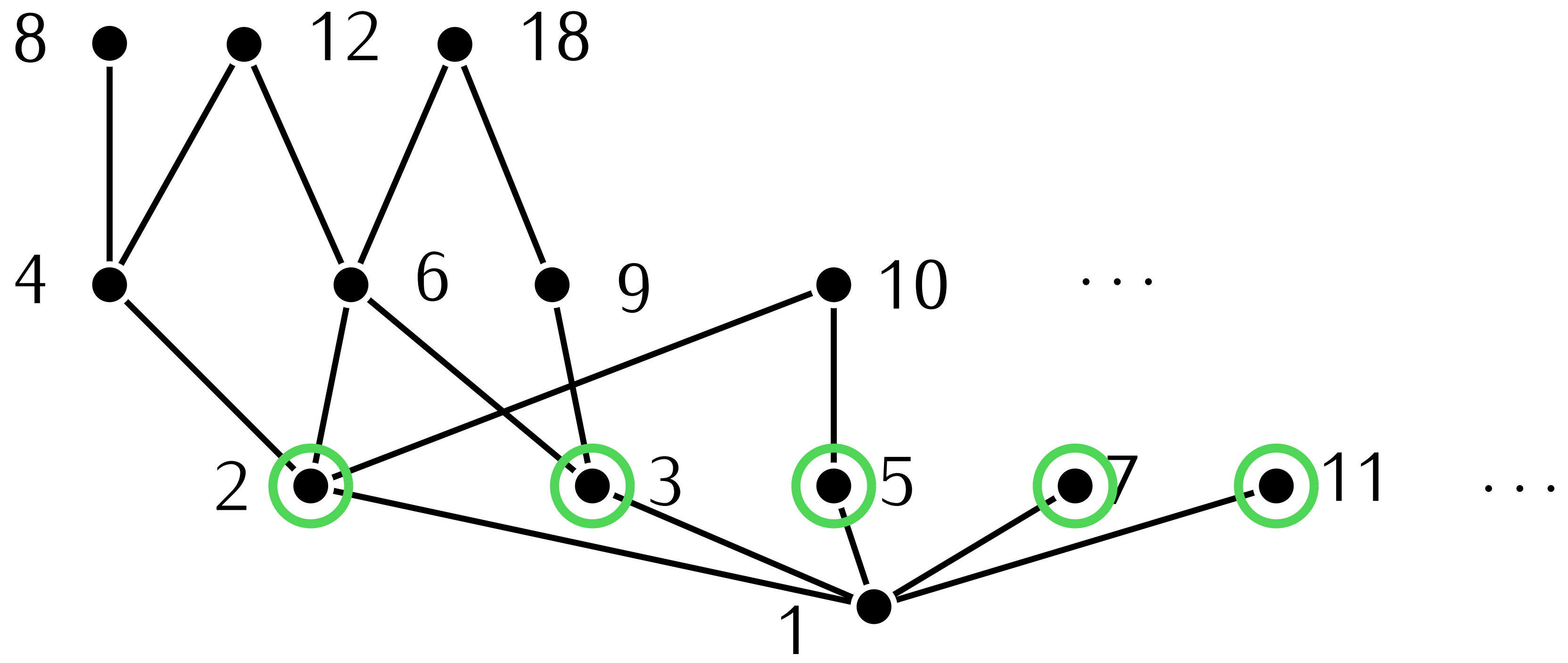

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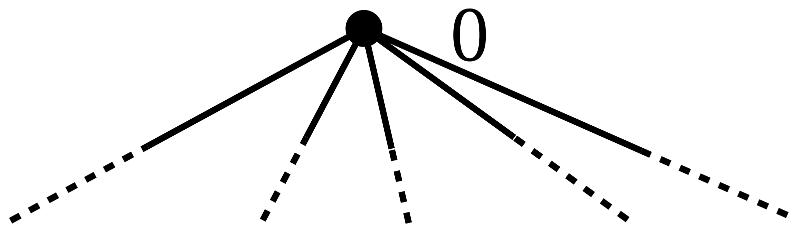

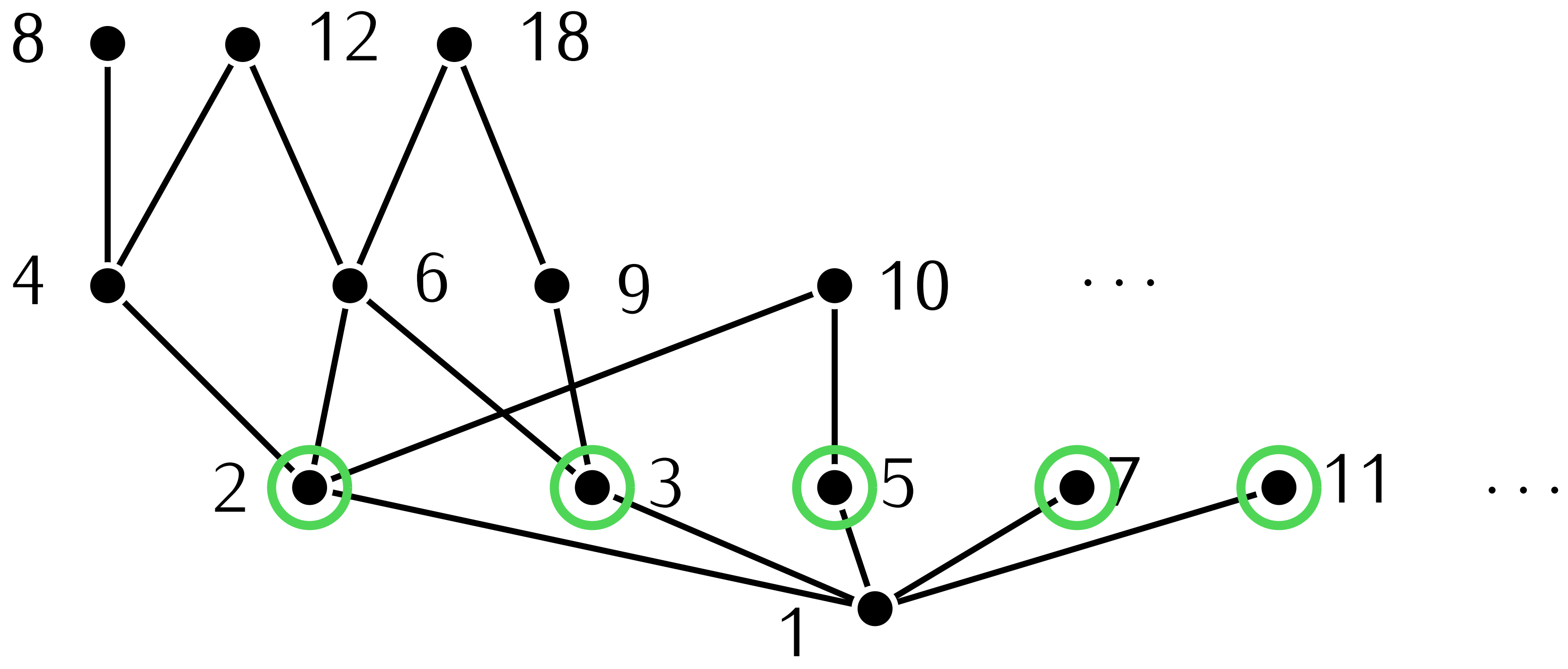

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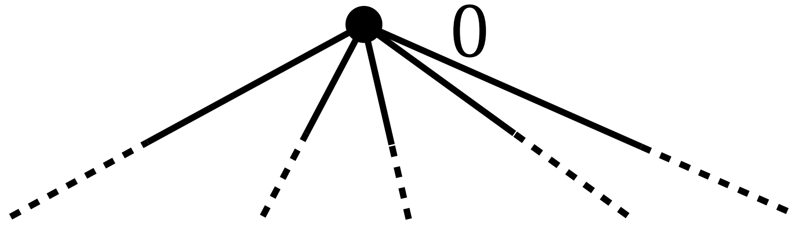

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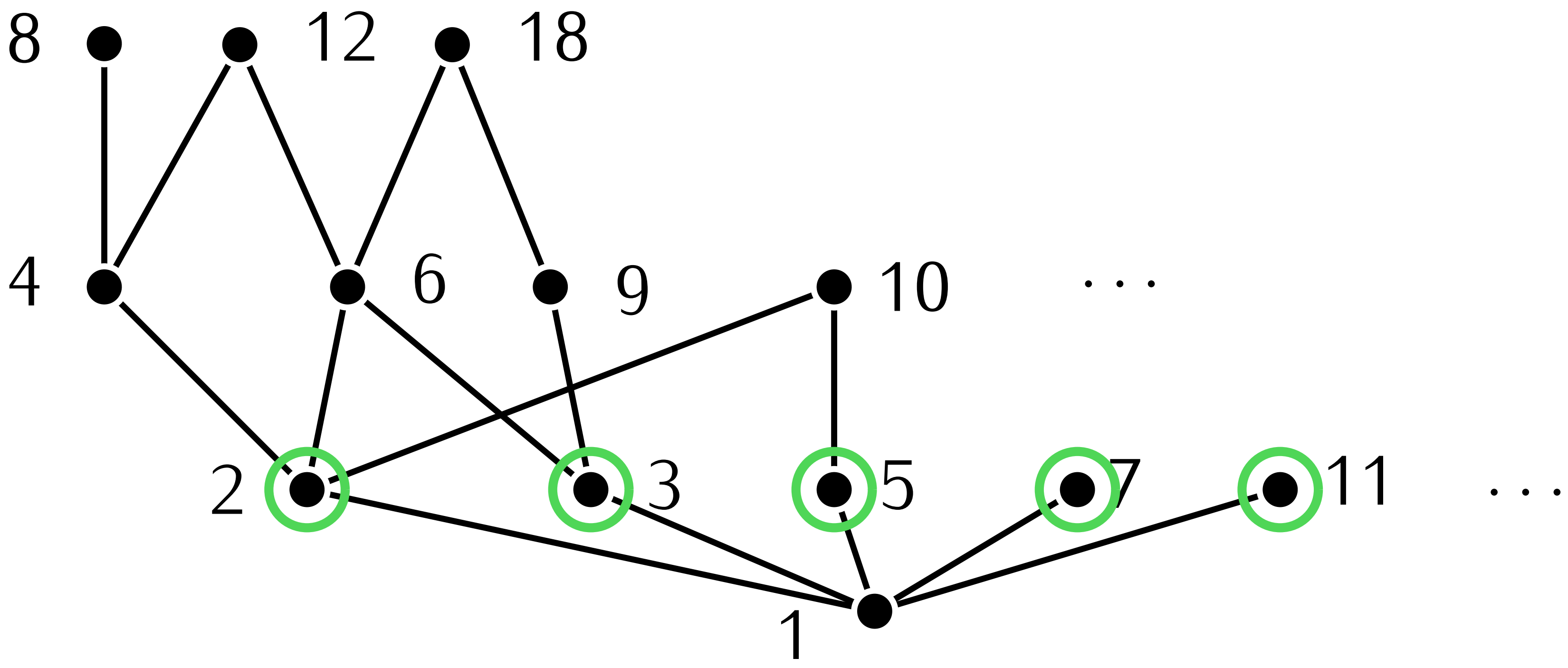

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- $p_1, \ldots, p_k$ cannot be a list of all the primes $\square$

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Theorem. Let $n \in \mathbb{N}$ be such that $n \geq 2$.
There exist prime numbers $p_1 < \cdots < p_k$ and $e_1, \ldots, e_k \in \mathbb{N}$ such that $n = p_1^{e_1} \cdots p_k^{e_k}$.
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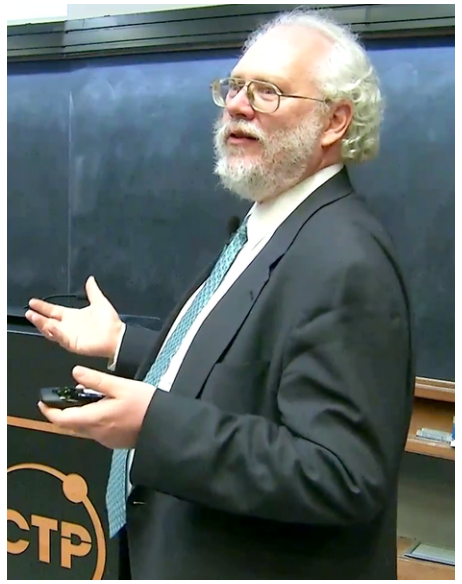
- It seems hard to compute $p_1, \ldots, p_k$ if given $n \in \mathbb{N}$
- **Theoretically,** there is a quantum algorithm that can do this and **break RSA**.

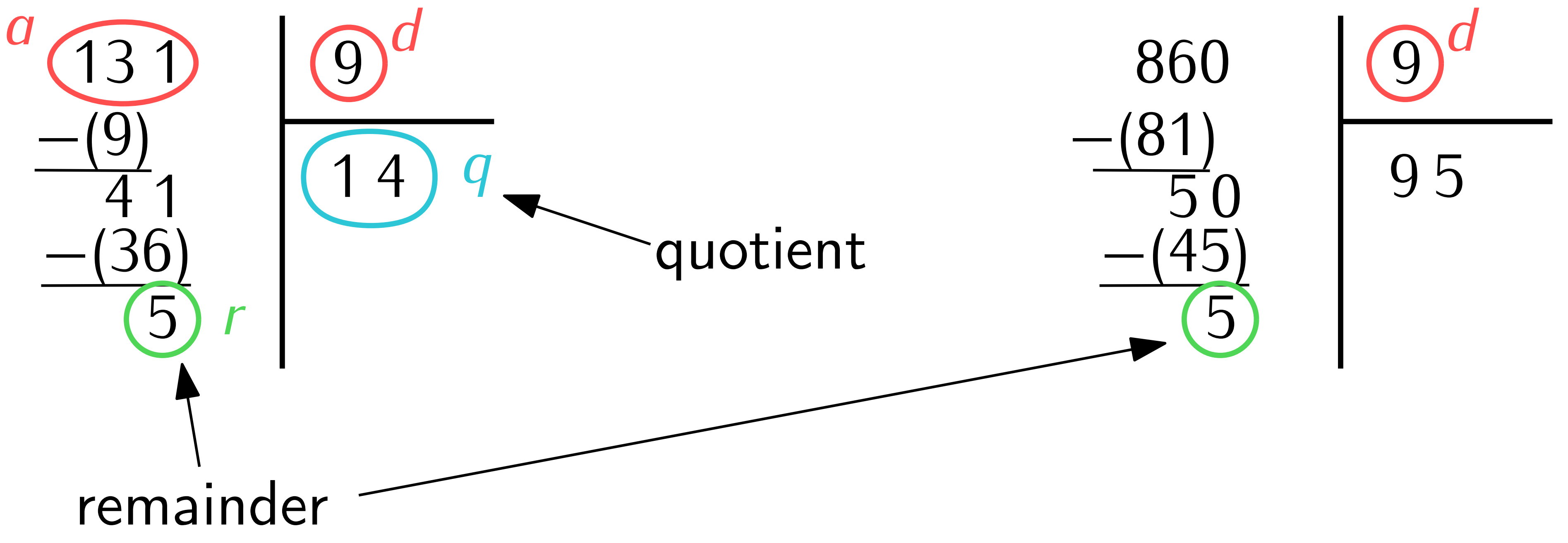


Peter Shor

Modular arithmetic

Theorem. For all $a, d \in \mathbb{Z}$ such that $a \neq 0$, there exists a unique pair $(q, r) \in \mathbb{Z}^2$ such that:

- $a = q \cdot d + r$
- $r \in \{0, \ldots, |d| - 1\}$



Almost the same as `divmod` in Python:

```
divmod(131,9) # (14,5)
divmod(10,-3) # (-4,-2)
```

- If remainder is 0: we say d divides a
  a is a multiple of d

Notation: $d \mid a$

- Define $b \equiv_d b'$ by "they have the same remainder in the division by d"
  $131 \equiv_9 860$

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Check if 711 is divisible by 72

Divide 1075 by 72: $1075 = 14 \times 72 + 67$
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Which of the following are true?

- $14 \equiv 6 \bmod 2$
- $3 \equiv 1 \bmod 4$
- $61 \equiv 5 \bmod 7$

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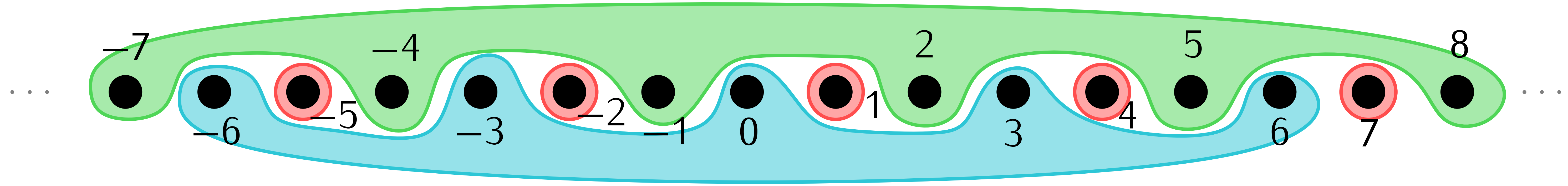
There are exactly d equivalence classes: $|\mathbb{Z}/d\mathbb{Z}| = d$



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Examples.
- $d = 12$: $\mathbb{Z}/d\mathbb{Z}$ is $\{[1]_{12}, [2]_{12}, \ldots, [12]_{12}\}$ hours on a clock
- $d = 7$: $\{[1]_7, [2]_7, \ldots, [7]_7\}$ days of the week
- $d = 2$: $\{[0]_2, [1]_2\}$ parity

Algebra: study of **operations** and **equations** on *stuff*

Number theory

Boolean algebra

Numbers: $+, \times, 1/x, 1, 0$

Matrices: $+, \times, M^{-1}, I, 0$

Sets: $\cap, \cup, \times, \Delta, \emptyset, \ldots$

Functions: $\circ, f^{-1}, \mathrm{Id}_A$

Booleans: $\wedge, \vee, \Rightarrow, \neg, \top, \bot$

Relations: $\circ, R^T, \mathrm{Id}, \cup, \cap, \times, \ldots$

Linear algebra

Relational algebra

Modular integers: $+, \times, [a]_d^{-1}, [1]_d, [0]_d$

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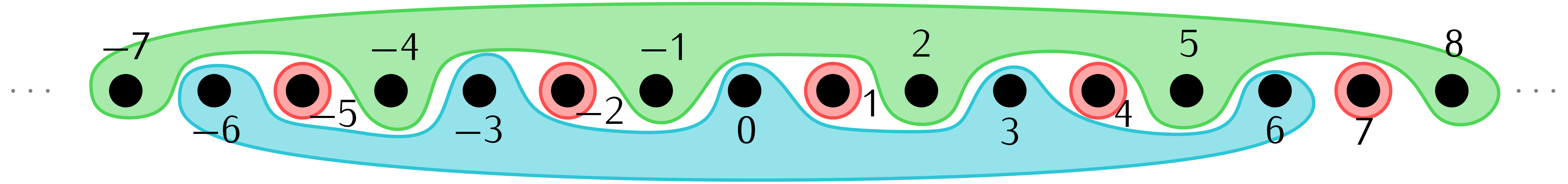

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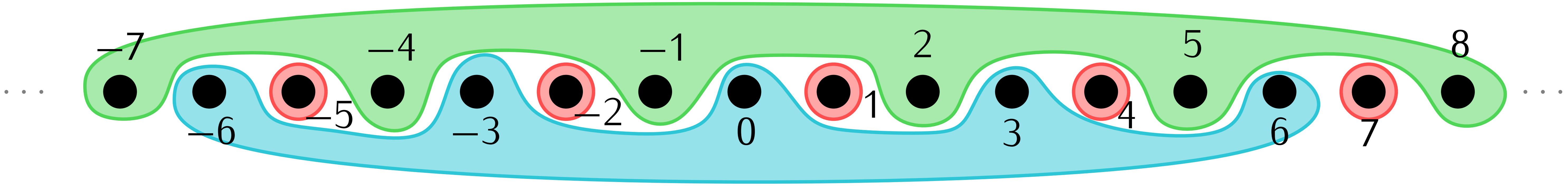

Note that this is a **definition** and it depends on d:

$\boxed{\{\dots,-1,2,5,\dots\}} + \boxed{\{\dots,-3,0,3,\dots\}} =$ $\boxed{\{\dots,-1,2,5,\dots\}} + \boxed{\{\dots,-2,1,4,\dots\}} =$

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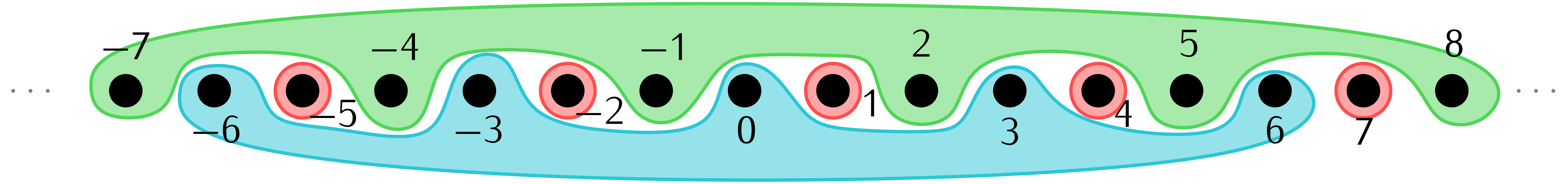

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The usual rules of addition and multiplication are true in modular arithmetic:

- $[a]_d + [b]_d = [b]_d + [a]_d$ and $[a]_d \times [b]_d = [b]_d \times [a]_d$
- $[0]_d + [a]_d = [a]_d$
- $[a]_d \times ([b]_d + [c]_d) = [a]_d \times [b]_d + [a]_d \times [c]_d$

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~~Which day of the week will it be in one month (January 15, 2026)?~~

Which day of the week was it two months ago (October 15, 2025)?

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Definition (Reminder (?) from your Math 1 course)**.** Let M be an $n \times n$ matrix. An inverse of N is a matrix such that $MN = I_n$.

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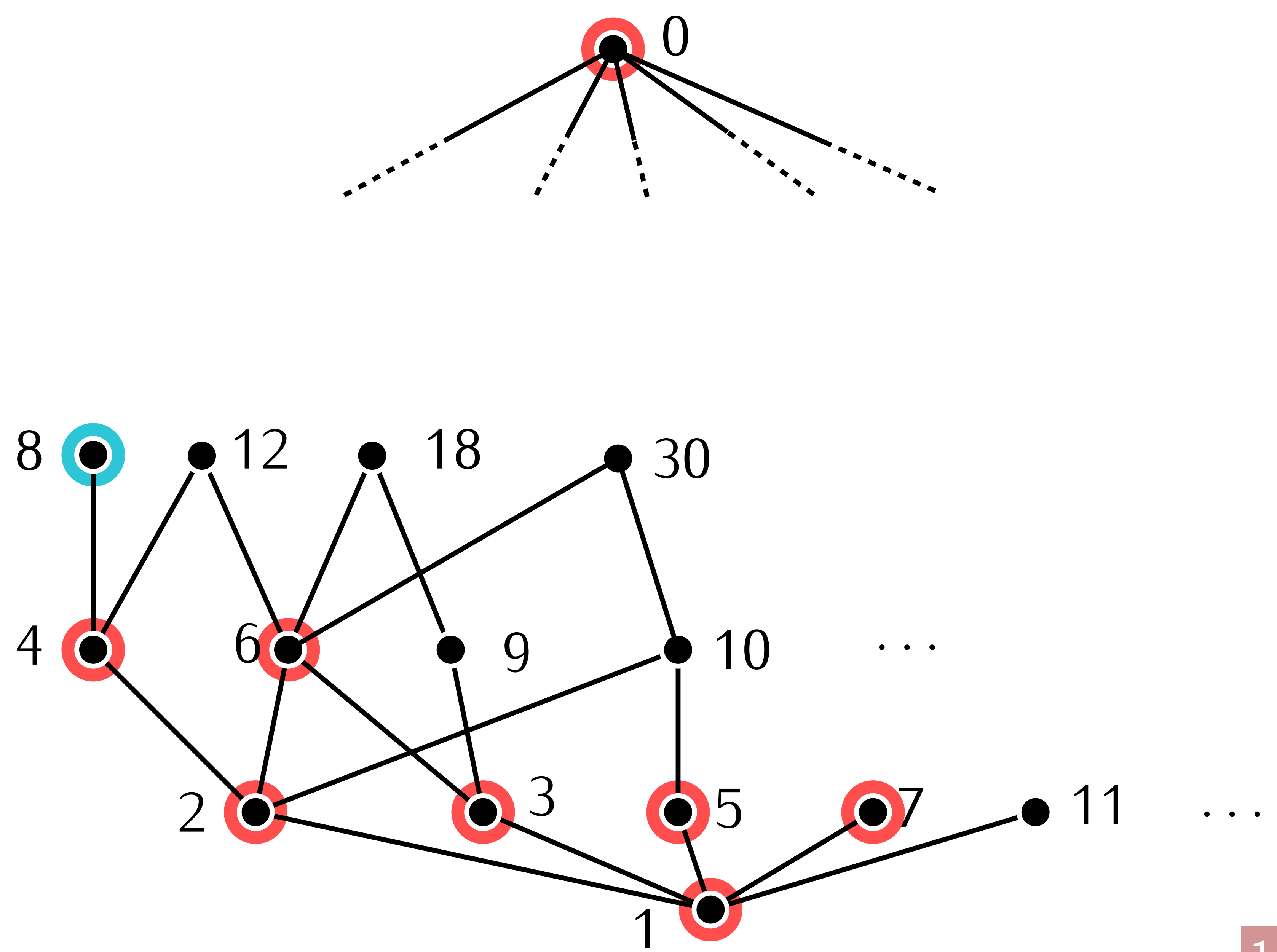
Which of the numbers in $\{0, \ldots, 7\}$ have an inverse modulo 8?



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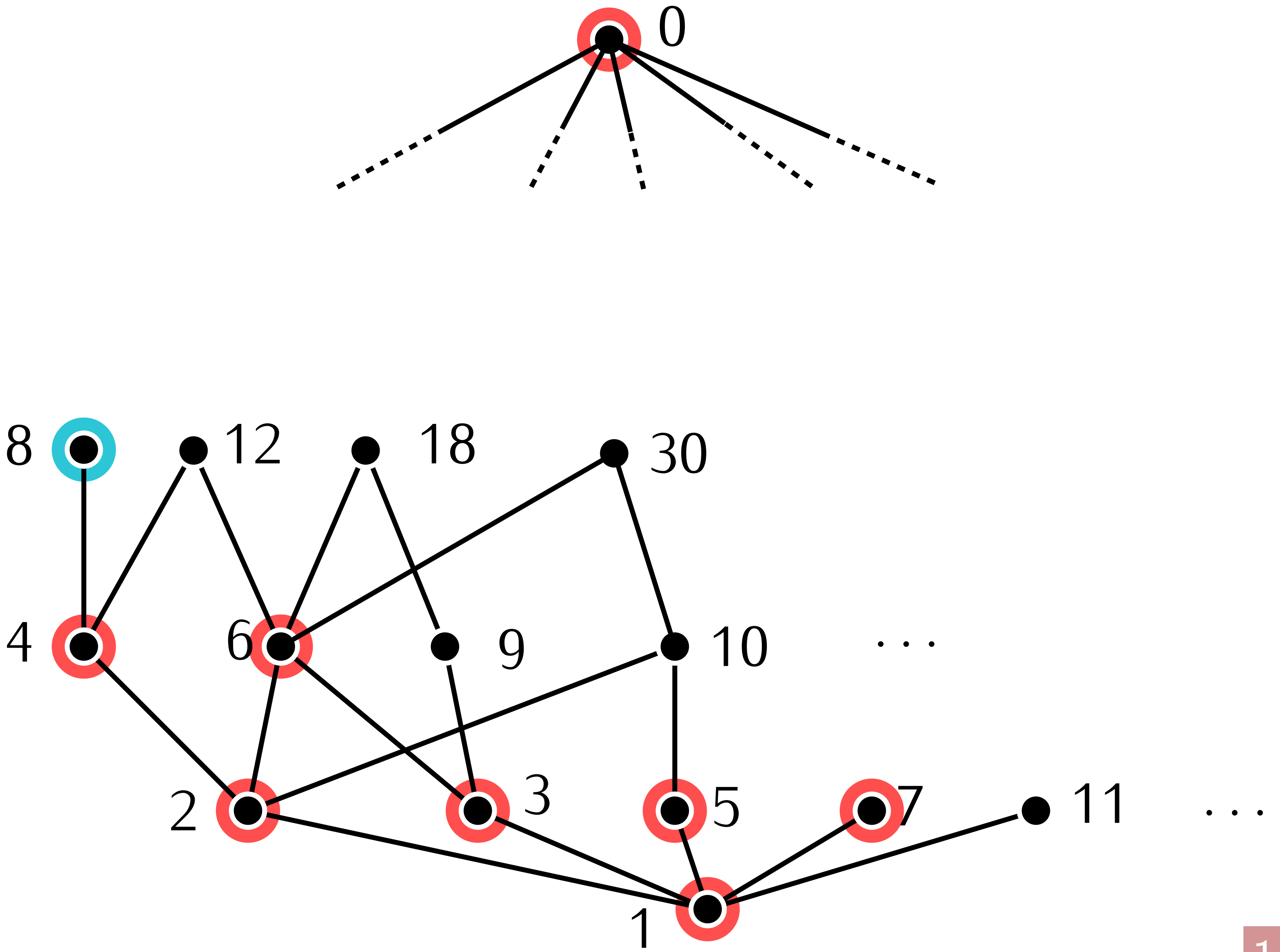

Definition. Let $n \geq 1$. Define $\varphi(n)$ to be the number of numbers in $\{0, \ldots, n-1\}$ that have an inverse modulo n.

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n	1	2	3	4	5	6	7	8	9
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What is $\varphi(60)$?

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- Define $f\colon \{0, \ldots, mn-1\} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ by $f(x) = ([x]_m, [x]_n)$

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There is exactly one such x in $\{0, \dots, mn-1\}$.

Proof of existence:

- Let u, v be the Bézout coefficients for m, n:
$$um + vn = 1$$
- Define $x = umb + vna$
- Divide x by mn with remainder to get a solution in $\{0, \dots, mn-1\}$

Proof of uniqueness:

- Define $f \colon \{0, \dots, mn-1\} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ by $f(x) = ([x]_m, [x]_n)$
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- This means
  if $[x]_m = [y]_m$ and $[x]_n = [y]_n$, then $x = y$.