

Discrete Algebraic Structures

Sheet 1

Exercise 0 (Warmup):

Compute the following objects:

1. $(\{1, 2\} \cap \{2, 3\}) \cup \{2, 4\} = \{2, 4\}$
2. $(\{1, 2\} \cup \{2, 3\}) \setminus \{2, 4\} = \{1, 3\}$
3. $(\{1, 2\} \cap \{2, 3\}) \Delta \{2, 4\} = \{4\}$
4. $(\{1, 2\} \cup \{2, 3\}) \times \{2, 4\} = \{1, 2, 3\} \times \{2, 4\} = \{\{1, 2\}, \{1, 4\}, \{2, 2\}, \{2, 4\}, \{3, 2\}, \{3, 4\}\}$
5. $\mathcal{P}(\{1, 3, 4\}) = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}$

Exercise 1 (Sets):

- (a) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- (b) Prove that $A \cap (A \cup B) = A$.
- (c) Prove that if $B \subseteq A$, then $A \cap B = B$ and $A \cup B = A$.

Exercise 2 (De Morgan's law):

Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ using a written proof in the style of Exercise 1(a).

Exercise 3 (Distributivity of Cartesian Product):

Let A, B, C, D be sets.

1. Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$
2. Show that $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3. Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
4. Do we have $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$? If not, is one of both inclusions true?
5. Illustrate the results of this exercise using figures. You can use for example $A = [0, 2]$, $B = [1, 3]$, $C = [0, 3]$, $D = [1, 4]$.

Exercise 4 (Symmetric Difference):

This is a supplementary exercise that you should do at home, to get more practice after the tutorials.

- (a) Let $A \Delta B = C$. What is $A \Delta C$?
- (b) Prove that $A \Delta B = (A \setminus B) \cup (B \setminus A)$.
- (c) Show that the symmetric difference is associative, i.e., that $A \Delta (B \Delta C) = (A \Delta B) \Delta C$ is true for all sets A, B, C . Can you find a natural description of the elements that are in $A \Delta B \Delta C$?

0.

1. (This is needlessly complicated and relies on ungiven premises)

notes

0: $(A \cup B) \times C \stackrel{!}{=} (A \times C) \cup (B \times C)$ (Thesis)

Hyp. 1: A, B, C are sets

Hyp. 2: $x \in (A \cup B) \times C$

Hyp. 3: using $S = (A \cup B) \rightarrow S \times C$

Hyp. 4: $A \subseteq S$

Hyp. 5: $B \subseteq S$

Hyp. 6: $A \cup B \subseteq S$

Hyp. 6: $x \in (A \cap C) \times (B \cap C)$

proof

Def. 0: Thesis means, "Every tuple from $(A \cup B) \times C$ must correspond to a tuple from $(A \times C) \cup (B \times C)$ (and vice versa)"

Def. 1: For $S = (A \cup B)$ and $S \times C$, $S \times C$ can be written as a union between all of S 's subsets $\times C$

Def. 2: A and B are subsets of S and make up S . Any operation on S can be written as a union between that operation performed on all its subsets

Def. 3: Therefore: $(A \cup B) \times C = (A \times C) \cup (B \times C)$
q.e.d.

1. (again)

Thesis: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Splits into: 1. $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
2. $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

1. $A \cup (B \cap C) \subseteq \dots$

1.1. $x \in A \cup (B \cap C)$

1.2. $x \in A$ or $x \in (B \cap C)$

1.3. Assuming $x \in A$:

1.3.1. x must be in $(A \cup B)$ and $(A \cup C)$

1.3.2. x must be in $(A \cup B) \cap (A \cup C)$ (since it's in both operands)

1.4. Assuming $x \in (B \cap C)$

1.4.1. $x \in B$ and $x \in C$

1.4.2. Therefore: x must be part of $(A \cup B)$ and $(A \cup C)$

1.4.3. " \cup only extends sets" and "rewritten as operator": $x \in (A \cup B) \cap (A \cup C)$

2. $(A \cup B) \cap (A \cup C) \subseteq \dots$

2.1. $x \in (A \cup B) \cap (A \cup C)$

2.2. $x \in (A \cup B)$ and $x \in (A \cup C)$

2.3. Case: $x \in A$:

2.3.1. Entails $x \in A \cup (B \cap C)$ (\cup extension doesn't break $x \in A$)

2.4. Case: $x \notin A$:

2.4.1. $x \in B$ and $x \in C$

2.4.2. rewritten: $x \in (B \cap C)$

2.4.3. extend with arbitrary union (here: with A): $x \in A \cup (B \cap C)$

Q.E.D

7. (6)

Thesis: $A \cap (A \cup B) = A$

Split Theses:

1. $A \cap (A \cup B) \supseteq A$

1.1. $x \in A \cap (A \cup B)$

1.2. $x \in A$ and $x \in (A \cup B)$

(1.3. $x \in A \Leftrightarrow x \in B$

1.3.1. already filled by 1.2: $x \in A$) could be skipped

1.4. 1.1 and 1.2 ($x \in A$), Thesis 1 fulfilled

2. $A \supseteq A \cap (A \cup B)$

2.1. $x \in A$

2.2. Therefore: $x \in (A \cup B)$

2.3. Known: $x \in A$ and $x \in (A \cup B)$

2.3.1. Combination: $x \in A \cap (A \cup B)$ (2.1, 2.2)

Q.E.D.

(c)

Thesis: Given $B \subseteq A$, $A \cap B = B$ and $A \cup B = A$

Subtheses: $A \cap B \supseteq B$, $A \cap B \subseteq B$, $A \cup B \supseteq A$, $A \cup B \subseteq A$

1. $A \cap B \supseteq B$

1.1. $x \in (A \cap B)$

1.2. $x \in A$ and $x \in B$

1.2.1. Thesis 1 satisfied (1.1, 1.2), $x \in (A \cap B)$, $x \in B$

2. $B \supseteq A \cap B$

2.1. repeat: $B \subseteq A$

2.2. $x \in B$

2.3. from 2.1: $x \in A$

2.4. 2.2, 2.3; $x \in A \cap B$

2.4.1. Thesis 2 satisfied

3. $A \cup B \supseteq A$

3.1. rp.: $B \subseteq A$

3.2. $x \in A \cup B$

3.3. $x \in A$ or $x \in B$

3.3.1. case $x \in A$:

3.3.1.1. satisfies Th. 3 together with 3.2.

3.3.2. case $x \in B$:

3.3.2.1. from 3.1: $x \in A$

3.3.2.2. satisfies Th. 3 (w/ 3.2.)

4. $A \supseteq A \cup B$

4.1. rp.: $B \subseteq A$

4.2. $x \in A$

4.3. write 4.2. as: $x \in A \cup B$ (arbitrary union doesn't break membership)

4.3.1. Th. 4 satisfied (4.3, 4.2)

Satisfied Th. 1-4

Q.E.D

Exercise 2 (De Morgan's law):

Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ using a written proof in the style of Exercise 1(a).

Thesis: $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Suptheses: 1. $\dots \subseteq \dots$

2. $\dots \supseteq \dots$

1. $A \setminus (B \cap C) \supseteq (A \setminus B) \cup (A \setminus C)$

1.1. let $x \in A \setminus (B \cap C)$

1.2. $x \notin (B \cap C)$

1.2.1. $x \notin B$ or $x \notin C$

1.3. $x \in A$

1.4. from 1.2.1:

1.4.1. case 1: $x \notin B$:

1.4.1.2. $\rightarrow x \in A \setminus B$

1.4.2. case 2: $x \notin C$

1.4.2.1. $\rightarrow x \in A \setminus C$

1.5. 1.4.1.2 or 1.4.2.1 is true,
meaning: $x \in (A \setminus B) \cup (A \setminus C)$ (or meaning "or")
satisfies 1

2. $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$

2.1. let $x \in (A \setminus B) \cup (A \setminus C)$

2.2. $x \in (A \setminus B)$ or $x \in (A \setminus C)$

2.2.1. $x \in (A \setminus B)$

2.2.1.1. $x \in A$

2.2.1.2. $x \notin B$

2.2.2. $x \in (A \setminus C)$

2.2.2.1. $x \in A$

2.2.2.2. $x \notin C$

2.3. same def. from 2.2.2 and 2.2.1: $x \in A$

2.4. $x \notin B$ or $x \notin C$

2.5. $x \notin B \cap C$ (can't be part of both if \notin of at least one)

2.6. 2.5, 2.3.: $x \in A \setminus (B \cap C)$

Q.E.D.

Exercise 3 (Distributivity of Cartesian Product):

Let A, B, C, D be sets.

1. Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$
2. Show that $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3. Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
4. Do we have $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$? If not, is one of both inclusions true?
5. Illustrate the results of this exercise using figures. You can use for example $A = [0, 2]$, $B = [1, 3]$, $C = [0, 3]$, $D = [1, 4]$.

1) Thesis: $(A \cup B) \times C = (A \times C) \cup (B \times C)$

1. $(A \cup B) \times C \supseteq (A \times C) \cup (B \times C)$

1.1. $s \in (A \cup B) \times C$

1.2. $s = (x, y)$ (follows definition of cartesian product)

1.3. $y \in C$

1.4. $x \in A$ or $x \in B$

1.4.1. case $x \in A$: $s \in A \times C$

1.4.2. case $x \in B$: $s \in B \times C$

1.5. from 1.4: $s \in A \times C$ or $s \in B \times C$

1.5.1. $\Rightarrow s \in (A \times C) \cup (B \times C)$

supthesis 1 satisfied

2. $(A \times C) \cup (B \times C) \supseteq (A \cup B) \times C$

2.1. $(x, y) \in (A \times C) \cup (B \times C)$

2.2. $(x, y) \in (A \times C)$ or $(x, y) \in (B \times C)$

2.2.1. case $(x, y) \in (A \times C)$: $x \in A$ and $y \in C$

2.2.2. case $(x, y) \in (B \times C)$: $x \in B$ and $y \in C$

2.3. $\Rightarrow y \in C$

2.4. from 2.2. $\Rightarrow x \in A$ or $x \in B \Rightarrow x \in (A \cup B)$

2.5. $\Rightarrow (x, y) \in (A \cup B) \times C$ (since $(A \cup B)$ contributes and C y)

synthesis 2 satisfied

Q.E.D.

2) Thesis: $(A \cap B) \times C = (A \times C) \cap (B \times C)$

1. $(A \cap B) \times C \supseteq (A \times C) \cap (B \times C)$

1.1. $(x, y) \in (A \cap B) \times C$

1.2. $y \in C$

1.3. $x \in A$ and $x \in B$ ($x \in (A \cap B)$)

1.4. $\Rightarrow (x, y) \in A \times C$ and $(x, y) \in B \times C$ (permitted since x is in both A and B)

1.5. $\Rightarrow (x, y) \in (A \times C) \cap (B \times C)$

2. $(A \times C) \cap (B \times C) \supseteq (A \cap B) \times C$

2.1. let $(x, y) \in (A \times C) \cap (B \times C)$

2.2. $\Rightarrow (x, y) \in (A \times C)$ and $(x, y) \in (B \times C)$

2.3. $\Rightarrow y \in C$

2.4. cartesian product def.: $x \in A$ and $x \in B$

2.5. $(x, y) \in (A \cap B) \times C$ (from 2.4 and 2.3.)

Q.E.D.

3) Thesis: $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

1. $(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)$

1.1. let $(x, y) \in (A \cap B) \times (C \cap D)$

1.2. $x \in (A \cap B)$ and $y \in (C \cap D)$

1.3. $x \in A$ and $x \in B$

1.4. $y \in C$ and $y \in D$

1.5. $\Rightarrow (x, y) \in (A \times C)$ (following cartesian product def.)

1.6. $\Rightarrow (x, y) \in (B \times D)$

1.7. from 1.5 and 1.6: $(x, y) \in (A \times C) \cap (B \times D)$

$$2. (A \times C) \cap (B \times D) \subseteq (A \cap B) \times (C \cap D)$$

$$2.1. \text{ let } (x, y) \in (A \times C) \cap (B \times D)$$

$$2.2. (x, y) \in (A \times C) \text{ and } (x, y) \in (B \times D)$$

$$2.3. x \in A \text{ and } x \in B$$

$$2.4. y \in C \text{ and } y \in D$$

$$2.5. \Rightarrow x \in (A \cap B)$$

$$2.6. \Rightarrow y \in (C \cap D)$$

$$2.7. \Rightarrow (x, y) \in (A \cap B) \times (C \cap D) \text{ (following CP)}$$

Q.E.D.

$$4) \text{ Thesis: } (A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$$

$$1. (A \cup B) \times (C \cup D) \subseteq (A \times C) \cup (B \times D)$$

$$1.1. \text{ let } (x, y) \in (A \cup B) \times (C \cup D)$$

$$1.2. x \in (A \cup B), x \in A \text{ or } x \in B$$

$$1.3. y \in (C \cup D), y \in C \text{ or } y \in D$$

$$1.4. \text{ from 1.2 and 1.3}$$

1.4.1. cases:

$$(x, y) \in A \times C$$

$$(x, y) \in A \times D$$

$$(x, y) \in B \times C$$

$$(x, y) \in B \times D$$

$$1.4.2. \text{ combine: } (x, y) \in (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$$

thesis impossible without additional unions,
or $B \subseteq C$ and $C \subseteq D$

$$2. (A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$$

$$2.1. \text{ let } s \in (A \times C) \cup (B \times D) \text{ i } x, y \in s$$

$$2.2. s \in (A \times C) \text{ or } s \in (B \times D)$$

$$2.2.1. \Rightarrow x \in A \text{ or } x \in B \rightarrow x \in (A \cup B)$$

$$2.2.2. \Rightarrow y \in C \text{ or } y \in D \rightarrow y \in (C \cup D)$$

$$2.3. \Rightarrow (x, y) \in (A \cup B) \times (C \cup D)$$

original thesis disproven, but t2 holds

$$4) A = \{0, 2\}, B = \{1, 3\}, C = \{0, 3\}, D = \{1, 4\}$$

from 3):

$$T1: \{0, 1, 2, 3\} \times \{0, 1, 3, 4\} = \{(0, 0), (0, 1), (0, 3), (0, 4), (1, 0), (1, 1), (1, 3), (1, 4), (2, 0), (2, 1), (2, 3), (2, 4), (3, 0), (3, 1), (3, 3), (3, 4)\} = S$$

$$T2: (\{0, 2\} \times \{0, 3\}) \cup (\{1, 3\} \times \{1, 4\}) = \{(0, 0), (0, 3), (2, 0), (2, 3)\} \cup \{(1, 1), (1, 4), (3, 1), (3, 4)\} = E$$

as we can see, $S \neq E$, but $E \subseteq S$

Exercise 4 (Symmetric Difference):

This is a supplementary exercise that you should do at home, to get more practice after the tutorials.

- (a) Let $A \Delta B = C$. What is $A \Delta C$?
- (b) Prove that $A \Delta B = (A \setminus B) \cup (B \setminus A)$.
- (c) Show that the symmetric difference is associative, i.e., that $A \Delta (B \Delta C) = (A \Delta B) \Delta C$ is true for all sets A, B, C . Can you find a natural description of the elements that are in $A \Delta B \Delta C$?

a) $A \Delta B$ (defined as C), is the set of elements which are exclusive to either A or B , so not part of both.

b) Thesis: $A \Delta B = (A \setminus B) \cup (B \setminus A)$

$$1. A \Delta B \subseteq (A \setminus B) \cup (B \setminus A)$$

$$1.1. \text{ Let } x \in A \Delta B$$

$$1.2. \Rightarrow x \notin A \cap B$$

$$1.3. \Rightarrow x \in A \cup B$$

1.4. written out: x is exclusive to A or exclusive to B

1.5. cf. 1.3.:

1.5.1. case $x \in A$:

$$x \notin B$$

1.5.2. c. $x \in B$:

$$x \notin A$$

$$1.6. \Rightarrow x \in A \text{ but not } B \text{ or } x \in B \text{ but not } A$$

$$1.7. x \in (A \setminus B) \cup (B \setminus A) \quad \checkmark$$

$$2. (A \setminus B) \cup (B \setminus A) \subseteq A \Delta B$$

$$2.1. \text{ let } x \in (A \setminus B) \cup (B \setminus A)$$

$$2.2. \Rightarrow x \in (A \setminus B) \text{ or } x \in (B \setminus A):$$

$$2.2.1. \text{ c. } x \in (A \setminus B): x \in A \text{ and } x \notin B$$

$$2.2.2. \text{ c. } x \in (B \setminus A): x \in B \text{ and } x \notin A$$

2.3. Summary: x is an element of A and not of B OR x is an element of B and not of A

$\Rightarrow x$ is exclusive to A or B

2.4. 2.3. is def. of symmetric difference (between A and B), so:

$$x \in A \Delta B$$

Q.E.D

c) Thesis: $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

1. $A \Delta (B \Delta C) \subseteq (A \Delta B) \Delta C$

1.1. let $x \in A \Delta (B \Delta C)$

1.2. $(x \in A) \oplus (x \in B \Delta C)$

1.3. $(x \in A) \oplus (x \in B \oplus x \in C)$

1.4. XOR (\oplus) is associative per definition

(only one operand can be true. If any other operand across brace-boundaries evaluates true, the chain will still converge to false. This could be illustrated through a truth table)

1.5. $\Rightarrow (x \in A \oplus x \in B) \oplus x \in C$, which corresponds to:

$$x \in (A \Delta B) \Delta C$$

\uparrow
This is probably ps

2. $(A \Delta B) \Delta C \subseteq A \Delta (B \Delta C)$

2.1. let $x \in (A \Delta B) \Delta C$

2.2. $(x \in A \oplus x \in B) \oplus x \in C$

2.3. Apply def. of XOR associativity from 1:

$$x \in A \oplus x \in B \oplus x \in C$$

2.4. $\Rightarrow x \in A \Delta (B \Delta C)$

Q.E.D.

Definition: $A \Delta B \Delta C$ describes the set of all elements not shared between two sets out of A, B, C