

# Discrete Algebraic Structures

## Sheet 1

### Exercise 0 (Warmup):

Compute the following objects:

1.  $(\{1, 2\} \cap \{2, 3\}) \cup \{2, 4\} = \{2, 4\}$
2.  $(\{1, 2\} \cup \{2, 3\}) \setminus \{2, 4\} = \{1, 3\}$
3.  $(\{1, 2\} \cap \{2, 3\}) \Delta \{2, 4\} = \{4\}$
4.  $(\{1, 2\} \cup \{2, 3\}) \times \{2, 4\} = \{\{1, 2\}, \{1, 4\}, \{2, 2\}, \{2, 4\}, \{3, 2\}, \{3, 4\}\}$
5.  $\mathcal{P}(\{1, 3, 4\}) = \{\emptyset, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}$

### Exercise 1 (Sets):

- (a) Prove that  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .
- (b) Prove that  $A \cap (A \cup B) = A$ .
- (c) Prove that if  $B \subseteq A$ , then  $A \cap B = B$  and  $A \cup B = A$ .

### Exercise 2 (De Morgan's law):

Prove that  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$  using a written proof in the style of Exercise 1(a).

### Exercise 3 (Distributivity of Cartesian Product):

Let  $A, B, C, D$  be sets.

1. Show that  $(A \cup B) \times C = (A \times C) \cup (B \times C)$
2. Show that  $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3. Show that  $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
4. Do we have  $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$ ? If not, is one of both inclusions true?
5. Illustrate the results of this exercise using figures. You can use for example  $A = [0, 2]$ ,  $B = [1, 3]$ ,  $C = [0, 3]$ ,  $D = [1, 4]$ .

### Exercise 4 (Symmetric Difference):

*This is a supplementary exercise that you should do at home, to get more practice after the tutorials.*

- (a) Let  $A \Delta B = C$ . What is  $A \Delta C$ ?
- (b) Prove that  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ .
- (c) Show that the symmetric difference is associative, i.e., that  $A \Delta (B \Delta C) = (A \Delta B) \Delta C$  is true for all sets  $A, B, C$ . Can you find a natural description of the elements that are in  $A \Delta B \Delta C$ ?

O.

1. (This is needlessly complicated and relies on ungiven  
1 premises)

notes

proof

$$0: (A \cup B) \times C \stackrel{!}{=} (A \times C) \cup (B \times C) \quad (\text{thesis})$$

Hyp. 1: A, B, C are sets

Hyp. 2:  $x \in (A \cup B) \times C$

Hyp. 3: using  $S = (A \cup B) \rightarrow S \times C$

Hyp. 4:  $A \subseteq S$

Hyp. 5:  $B \subseteq S$

Hyp. 6:  $A \cup B \subseteq S$

Hyp. 6:  $x \in (A \cap C) \times (B \cap C)$

Def. 0: This means, Every tuple from  $(A \cup B) \times C$  must correspond to a tuple from  $(A \times C) \cup (B \times C)$  (and vice versa)

Def. 1: For  $S = (A \cup B)$  and  $S \times C$ ,  $S \times C$  can be written as a union between all of S's subsets  $\times C$

Def. 2: A and B are subsets of, and make up S. Any operation on S can be written as a union between that operation performed on all its subsets

Def. 3: Therefore:  $(A \cup B) \times C = (A \times C) \cup (B \times C)$   
q.e.d.

1. (again)

Thesis:  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Splits into: 1:  $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

2. ... 2 ...

1.  $A \cup (B \cap C) \subseteq \dots$

1.1.  $x \in A \cup (B \cap C)$

1.2.  $x \in A \text{ or } x \in (B \cap C)$

1.3. Assuming  $x \in A$ :

1.3.1.  $x$  must be in  $(A \cup B)$  and  $(A \cup C)$

1.3.2.  $x$  must be in  $(A \cup B) \cap (A \cup C)$  (since it's in both operands)

1.4. Assuming  $x \in (B \cap C)$

1.4.1.  $x \in B$  and  $x \in C$

1.4.2. Therefore:  $x$  must be part of  $(A \cup B)$  and  $(A \cup C)$

1.4.3. "and" re-written as operator:  $x \in (A \cup B) \cap (A \cup C)$

2.  $(A \cup B) \cap (A \cup C) \subseteq \dots$

2.1.  $x \in (A \cup B) \cap (A \cup C)$

2.2.  $x \in (A \cup B) \text{ and } x \in (A \cup C)$

2.3. Case:  $x \in A$ :

2.3.1. It fails  $x \in A \cup (B \cap C)$  ( $\vee$  extension doesn't break  
 $x \in A$ )

2.4. Case:  $x \notin A$ :

2.4.1.  $x \in B$  and  $x \in C$

2.4.2. re-written:  $x \in (B \cap C)$

2.4.3. extend with arbitrary union (here: with  $A$ ):  $x \in A \cup (B \cap C)$

Q. E. D.

7. (6)

Thesis:  $A \cap (A \cup B) = A$

Split Theses:

1.  $A \cap (A \cup B) \supseteq A$

1.1.  $x \in A \cap (A \cup B)$

1.2.  $x \in A$  and  $x \in (A \cup B)$

(1.3.  $x \in A \subseteq x \in B$

(1.3.1. already filled by 1.2:  $x \in A$ ) could be skipped

1.4. 1.1 and 1.2 ( $x \in A$ ), Thesis 1 fulfilled

2.  $A \supseteq A \cap (A \cup B)$

2.1.  $x \in A$

2.2. Therefore:  $x \in (A \cup B)$

2.3. Known:  $x \in A$  and  $x \in (A \cup B)$

2.3.1. Combination:  $x \in A \cap (A \cup B)$  (2.1, 2.2)

Q.E.D.

(c)

Theorem: Given  $B \subseteq A$ ,  $A \cap B = B$  and  $A \cup B = A$

Subtheses:  $A \cap B \supseteq B$ ,  $A \cap B \subseteq B$ ,  $A \cup B = A$ ,  $A \cup B \subseteq A$

1.  $A \cap B \supseteq B$

1.1.  $x \in (A \cap B)$

1.2.  $x \in A$  and  $x \in B$

1.2.1. Thesis 1 satisfied (1.1, 1.2),  $x \in (A \cap B)$ ,  $x \in B$

2.  $B \supseteq A \cap B$

2.1. repeat:  $B \subseteq A$

2.2.  $x \in B$

2.3. from 2.2:  $x \in A$

2.4. 2.2, 2.3;  $x \in A \cap B$

2.4.1. Thesis 2 satisfied

3.  $A \cup B = A$

3.1. r/p.:  $B \subseteq A$

3.2.  $x \in A \cup B$

3.3.  $x \in A$  or  $x \in B$

3.3.1. case  $x \in A$ :

3.3.1.1. satisfies Th. 3 together with 3.2.

3.3.2. case  $x \in B$ :

3.3.2.1. From 3.2:  $x \in A$

3.3.2.2. satisfies Th. 3 (w/ 3.2.)

4.  $A = A \cup B$

4.1. r/p.:  $B \subseteq A$

4.2.  $x \in A$

4.3. write 4.2. as:  $x \in A \cup B$  (arbitrary union doesn't break membership)  
4.3.1. Th. 4. satisfied (4.3, 4.2)

Satisfied Th. 1-4

Q.E.D

**Exercise 2 (De Morgan's law):**

Prove that  $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$  using a written proof in the style of Exercise 1(a).

$$\text{Thesis: } A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

Suppositions: 1. ...  $\subseteq \dots$

2. ...  $\supseteq \dots$

$$1. A \setminus (B \cap C) \supseteq (A \setminus B) \cup (A \setminus C)$$

1.1. let  $x \in A \setminus (B \cap C)$

1.2.  $x \notin (B \cap C)$

1.2.1.  $x \notin B \text{ or } x \notin C$

1.3.  $x \in A$

1.4. from 1.2.1.:

1.4.1. case 1:  $x \notin B$ :

1.4.1.2.  $\rightarrow x \in A \setminus B$

1.4.1.2. case 2:  $x \notin C$

1.4.1.2.1.  $\rightarrow x \in A \setminus C$

1.5. 1.4.1.2.  $\text{or}$  1.4.1.2.1. is true  
meaning:  $x \in (A \setminus B) \cup (A \setminus C)$  ( $\cup$  meaning "or")  
satisfies 1

$$2. (A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$$

2.1. let  $x \in (A \setminus B) \cup (A \setminus C)$

2.2.  $x \in (A \setminus B) \text{ or } x \in (A \setminus C)$

2.2.1.  $x \in (A \setminus B)$

2.2.1.1.  $x \in A$

2.2.1.2.  $x \notin B$

2.2.2.  $x \in (A \setminus C)$

2.2.2.1.  $x \in A$

2.2.2.2.  $x \notin C$

2.3. same def. from 2.2.2 and 2.2.1:  $x \in A$

2.4.  $x \notin B \text{ or } x \notin C$

2. S.  $x \notin B \cap C$  (can't be part of both if  $\notin$  of at least one)

2.6. 2.5, 2.3.:  $x \in A \setminus (B \cap C)$

Q.E.D.

**Exercise 3 (Distributivity of Cartesian Product):**

Let  $A, B, C, D$  be sets.

1. Show that  $(A \cup B) \times C = (A \times C) \cup (B \times C)$
2. Show that  $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3. Show that  $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
4. Do we have  $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$ ? If not, is one of both inclusions true?
5. Illustrate the results of this exercise using figures. You can use for example  $A = [0, 2]$ ,  $B = [1, 3]$ ,  $C = [0, 3]$ ,  $D = [1, 4]$ .

1) Thesis:  $(A \cup B) \times C = (A \times C) \cup (B \times C)$

1.  $(A \cup B) \times C \supseteq (A \times C) \cup (B \times C)$

1.1.  $s \in (A \cup B) \times C$

1.2.  $s = (x, y)$  (follows definition of cartesian product)

1.3.  $y \in C$

1.4.  $x \in A$  or  $x \in B$

1.4.1. case  $x \in A$ :  $s \in A \times C$

1.4.2. case  $x \in B$ :  $s \in B \times C$

1.5. from 1.4:  $s \in A \times C$  or  $s \in B \times C$

1.5.1.  $\Rightarrow s \in (A \times C) \cup (B \times C)$

Subthesis 1 satisfied

2.  $(A \times C) \cup (B \times C) \supseteq (A \cup B) \times C$

2.1.  $(x, y) \in (A \times C) \cup (B \times C)$

2.2.  $(x, y) \in (A \times C)$  or  $x \in (B \times C)$

2.2.1. case  $(x, y) \in (A \times C)$ :  $x \in A$  and  $y \in C$

2.2.2. case  $(x, y) \in (B \times C)$ :  $x \in B$  and  $y \in C$

2.3.  $\Rightarrow y \in C$

2.4. from 2.2.  $\Rightarrow x \in A$  or  $x \in B \Rightarrow x \in (A \cup B)$

2.5.  $\Rightarrow (x, y) \in (A \cup B) \times C$  (since  $(A \cup B)$  contributes and  $C$   $y$ )

synthesis 2 satisfied

Q.E.D.

2) Thesis:  $(A \cap B) \times C = (A \times C) \cap (B \times C)$

1.  $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$

1.1.  $(x, y) \in (A \cap B) \times C$

1.2.  $y \in C$

1.3.  $x \in A$  and  $x \in B$  ( $x \in (A \cap B)$ )

1.4.  $\Rightarrow (x, y) \in A \times C$  and  $(x, y) \in B \times C$  (permitted since  $x$  is in both  $A$  and  $B$ )

1.5.  $\Rightarrow (x, y) \in (A \times C) \cap (B \times C)$

2.  $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$

2.1. let  $(x, y) \in (A \times C) \cap (B \times C)$

2.2.  $\Rightarrow (x, y) \in (A \times C)$  and  $(x, y) \in (B \times C)$

2.3.  $\Rightarrow y \in C$

2.4. cartesian product def.:  $x \in A$  and  $x \in B$

2.5.  $(x, y) \in (A \cap B) \times C$  (from 2.4 and 2.3.)

Q.E.D.

3) Thesis:  $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

1.  $(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)$

1.1. let  $(x, y) \in (A \cap B) \times (C \cap D)$

1.2.  $x \in (A \cap B)$  and  $y \in (C \cap D)$

1.3.  $x \in A$  and  $x \in B$

1.4.  $y \in C$  and  $y \in D$

1.5.  $\Rightarrow (x, y) \in (A \times C)$  (following cartesian product def.)

1.6.  $\Rightarrow (x, y) \in (B \times D)$

1.7. from 1.5 and 1.6:  $(x, y) \in (A \times C) \cap (B \times D)$

$$2. (A \times C) \cap (B \times D) \supseteq (A \cap B) \times (C \cap D)$$

2.1. let  $(x, y) \in (A \times C) \cap (B \times D)$

2.2.  $(x, y) \in (A \times C)$  and  $(x, y) \in (B \times D)$

2.3.  $x \in A$  and  $x \in B$

2.4.  $y \in C$  and  $y \in D$

2.5.  $\Rightarrow x \in (A \cap B)$

2.6.  $\Rightarrow y \in (C \cap D)$

2.7.  $\Rightarrow (x, y) \in (A \cap B) \times (C \cap D)$  (following CP)

Q.E.D.

$$4) \text{ Thesis: } (A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$$

$$1. (A \cup B) \times (C \cup D) \subseteq (A \times C) \cup (B \times D)$$

1.1. let  $(x, y) \in (A \cup B) \times (C \cup D)$

1.2.  $x \in (A \cup B)$ ,  $x \in A$  or  $x \in B$

1.3.  $y \in (C \cup D)$ ,  $y \in C$  or  $y \in D$

1.4. from 1.2 and 1.3

1.4.1. cases:

$$(x, y) \in A \times C$$

$$(x, y) \in A \times D$$

$$(x, y) \in B \times C$$

$$(x, y) \in B \times D$$

1.4.2. combine:  $(x, y) \in (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D)$

If thesis is impossible without additional unions,  
or  $D \subseteq C$  and  $C \subseteq D$

$$2. (A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$$

2.1. let  $s \in (A \times C) \cup (B \times D)$ ;  $x, y \in s$

2.2.  $s \in (A \times C)$  or  $s \in (B \times D)$

2.2.1.  $\Rightarrow x \in A$  or  $x \in B \rightarrow x \in (A \cup B)$

2.2.2.  $\Rightarrow y \in C$  or  $y \in D \rightarrow y \in (C \cup D)$

2.3.  $\Rightarrow (x, y) \in (A \cup B) \times (C \cup D)$

original thesis disproved, but t2 holds

$$④ A = \{0, 2\}, B = \{1, 3\}, C = \{0, 3\}, D = \{1, 4\}$$

From 3):

$$T1: \{0, 1, 2, 3\} \times \{0, 1, 3, 4\} = \{(0, 0), (0, 1), (0, 3), (0, 4), (1, 0), (1, 1), (1, 3), (1, 4), (2, 0), (2, 1), (2, 3), (2, 4), (3, 0), (3, 1)\}$$

$$T2: (\{0, 2\} \times \{0, 3\}) \cup (\{1, 3\} \times \{1, 4\}) = \{(0, 0), (0, 3), (1, 0), (1, 3)\} \cup \{(1, 1), (1, 4), (3, 1), (3, 4)\} = 5$$
$$= \{(0, 0), (0, 3), (2, 0), (2, 3), (1, 1), (1, 4), (3, 1), (3, 4)\} = 8$$

as we can see,  $S \neq T$ , but  $T \subseteq S$

#### Exercise 4 (Symmetric Difference):

This is a supplementary exercise that you should do at home, to get more practice after the tutorials.

- Let  $A \Delta B = C$ . What is  $A \Delta C$ ?
- Prove that  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ .
- Show that the symmetric difference is associative, i.e., that  $A \Delta (B \Delta C) = (A \Delta B) \Delta C$  is true for all sets  $A, B, C$ . Can you find a natural description of the elements that are in  $A \Delta B \Delta C$ ?

a)  $A \Delta B$  (defined as  $C$ ), is the set of elements which are exclusive to either  $A$  or  $B$ , so not part of both.

D Thesis:  $A \Delta B = (A \setminus B) \cup (B \setminus A)$

1.  $A \Delta B \subseteq (A \setminus B) \cup (B \setminus A)$

1.1. Let  $x \in A \Delta B$

1.2.  $\Rightarrow x \notin A \cap B$

1.3.  $\Rightarrow x \in A \cup B$

1.4. written out:  $x$  is exclusive to  $A$  or exclusive to  $B$

1.5. ct. 1.3.:

1.5.1. case  $x \in A$ :

$$x \notin B$$

1.5.2. c.  $x \in B$ :

$$x \notin A$$

1.6.  $\Rightarrow x \in A$  but not  $B$  or  $x \in B$  but not  $A$

1.7.  $x \in (A \setminus B) \cup (B \setminus A)$  ✓

2.  $(A \setminus B) \cup (B \setminus A) \supseteq A \Delta B$

2.1. let  $x \in (A \setminus B) \cup (B \setminus A)$

2.2.  $\Rightarrow x \in (A \setminus B)$  or  $x \in (B \setminus A)$ :

2.2.1. c.  $x \in (A \setminus B)$ :  $x \in A$  and  $x \notin B$

2.2.2. -  $x \in (B \setminus A)$ :  $x \in B$  and  $x \notin A$

2.3. summary:  $x$  is an element of  $A$  and not of  $B$  OR  $x$  is an element of  $B$  and not of  $A$

$\Rightarrow x$  is exclusive to  $A$  or  $B$

2.4. 2.3. is def. of symmetric difference (between  $A$  and  $B$ ), so:

$$x \in A \Delta B \quad \text{Q.E.D}$$

c) Thesis:  $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

1.  $A \Delta (B \Delta C) \subseteq (A \Delta B) \Delta C$

1.1. Let  $x \in A \Delta (B \Delta C)$

1.2.  $(x \in A) \oplus (x \in B \Delta C)$

1.3.  $(x \in A) \oplus (x \in B \oplus x \in C)$

1.4. XOR ( $\oplus$ ) is associative per definition

(only one operand can be true. If any other - placed across, brace-boundaries evaluates true, the chain will still converge to false. This could be illustrated through a truth table)

1.5.  $\Rightarrow (x \in A \oplus x \in B) \oplus x \in C$ , which corresponds to:

$x \in (A \Delta B) \Delta C$

This is probably ps  
↑

2.  $(A \Delta B) \Delta C \subseteq A \Delta (B \Delta C)$

2.1. Let  $x \in (A \Delta B) \Delta C$

2.2.  $(x \in A \oplus x \in B) \oplus x \in C$

2.3. Apply def. of XOR associativity from 1:

$x \in A \oplus x \in B \oplus x \in C$

2.4.  $\Rightarrow x \in A \Delta (B \Delta C)$

Q.E.D.

Definition:  $A \Delta B \Delta C$  describes the set of all elements not shared between two sets out of  $A, B, C$