Mathematics 1 - Linear Algebra

Lecture 06 – §3.5 Linear combination, basis, dimension, . . .

Sabine Le Borne

TUHH

INSTITUTE OF MATHEMATICS

Solvability of systems of linear equations

Range of $\mathbf{A}$

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. The set

$$\mathrm{Ran}(\mathbf{A}) := \{\mathbf{A}\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

is called *(column) range* of $\mathbf{A}$.

$\mathrm{Ran}(\mathbf{A})$ is the set of all possible linear combinations of the columns of $\mathbf{A}$:

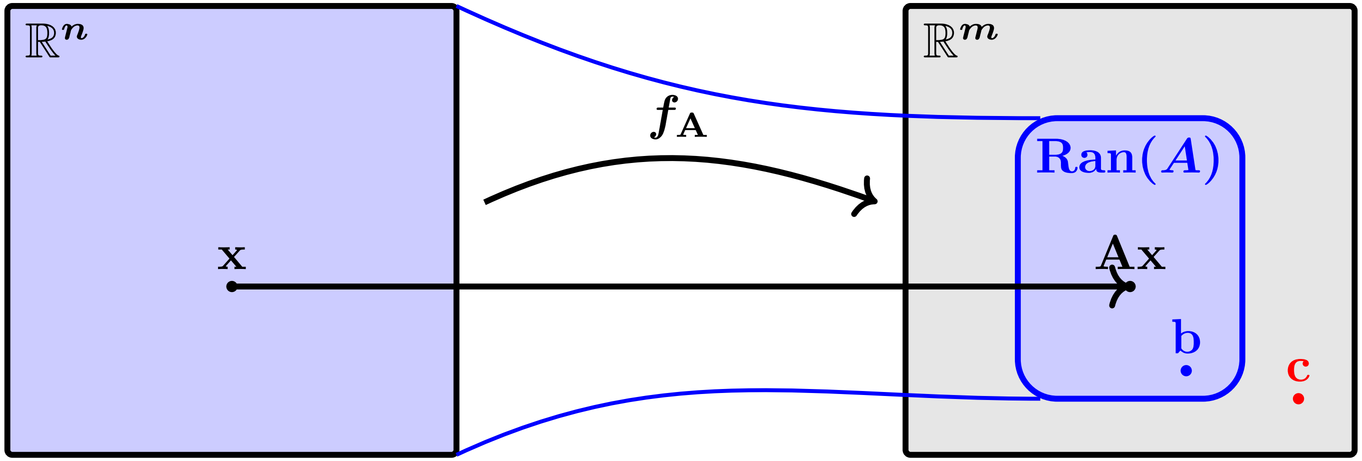
$$\begin{aligned}
\mathrm{Ran}(\mathbf{A}) &= \{\mathbf{A}\mathbf{x} \colon \mathbf{x} \in \mathbb{R}^n\} \\
&= \left\{ \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \ldots & \mathbf{a}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\} \\
&= \left\{ x_1 \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + \cdots + x_n \begin{pmatrix} | \\ \mathbf{a}_n \\ | \end{pmatrix} : x_1, \ldots, x_n \in \mathbb{R} \right\} \\
&= \{x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n \colon x_1, \ldots, x_n \in \mathbb{R}\} \subset \mathbb{R}^m.
\end{aligned}$$

Solvability of systems of linear equations

Theorem 3.28 (solvability and range)

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^{m}$. Then the following statements are equivalent:

(i) $\mathbf{A}\mathbf{x} = \mathbf{b}$ is solvable (i.e., it has at least one solution $\mathbf{x}$);
(ii) $\mathbf{b} \in \mathrm{Ran}(\mathbf{A})$;
(iii) $\mathbf{b}$ can be written as a linear combination of the columns of $\mathbf{A}$.



$$\mathbf{b} \in \mathbf{Ran}(\mathbf{A}) \quad \Rightarrow \quad \mathbf{A}\mathbf{x} = \mathbf{b} \text{ is solvable.}$$

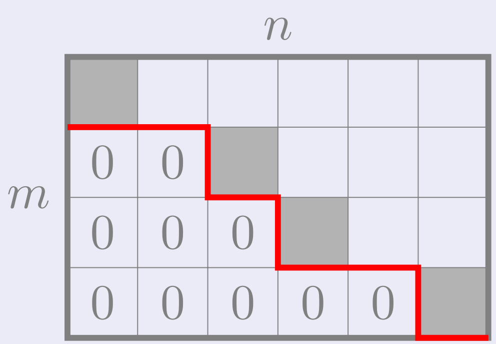
$$\mathbf{c} \notin \mathbf{Ran}(\mathbf{A}) \quad \Rightarrow \quad \mathbf{A}\mathbf{x} = \mathbf{c} \text{ is } \underline{\text{not}} \text{ solvable.}$$

Solvability of systems of linear equations

Theorem 3.30 (Unconditional solvability and surjectivity of $f_{\mathbf{A}}$)

For $\mathbf{A} \in \mathbb{R}^{m \times n}$, the following statements are equivalent:

(i) The equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has **at least** one solution $\mathbf{x}$ for **every** $\mathbf{b} \in \mathbb{R}^m$.
(ii) **All** $\mathbf{b} \in \mathbb{R}^m$ lie in $\mathrm{Ran}(\mathbf{A})$.
(iii) $\mathrm{Ran}(\mathbf{A}) = \mathbb{R}^m$.
(iv) $\mathrm{rank}(\mathbf{A}) = m \leqslant n$.
(v) When $\mathbf{A}$ is transformed into row echelon form $\mathbf{A}'$, then **each row** has a pivot element.
(vi) $f_{\mathbf{A}}$ is surjective.



Row echelon form $\mathbf{A}'$ of $\mathbf{A}$:

- Each row has a pivot element.
- There is no zero row in $\mathbf{A}'$.
- Hence $(0 \ \cdots \ 0 \mid c \neq 0)$ can never occur.

Proof. (i)⇔(ii)⇔(iii)⇔(vi) and (iv)⇔(v) are straightforward.
We next show (iv)⇒(i) and (i)⇒(v).

Solvability of systems of linear equations

Proof (continued).

(iv) $\operatorname{rank}(\mathbf{A}) = m \leqslant n \quad \Rightarrow \quad$ **(i)** $\mathbf{Ax} = \mathbf{b}$ has **at least** one solution $\mathbf{x}$ for **every** $\mathbf{b} \in \mathbb{R}^m$.

Let $\operatorname{rank}(\mathbf{A}) = m$. Then, for every $\mathbf{b} \in \mathbb{R}^m$ there holds $\operatorname{rank}(\mathbf{A}|\mathbf{b}) = m$ in view of $\operatorname{rank}(\mathbf{A}) \leqslant \operatorname{rank}(\mathbf{A}|\mathbf{b}) \leqslant m$ and hence $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b})$.
By Theorem 3.24a) there holds (i).

Recall Theorem 3.24a): $\mathbf{Ax} = \mathbf{b}$ solvable. $\Leftrightarrow \operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{A}|\mathbf{b})$.

(i) $\mathbf{Ax} = \mathbf{b}$ has **at least** one solution $\mathbf{x}$ for **every** $\mathbf{b} \in \mathbb{R}^m$. $\quad \Rightarrow \quad$ **(v)** When $\mathbf{A}$ is transformed into row echelon form $\mathbf{A}'$, then **each row** has a pivot element.

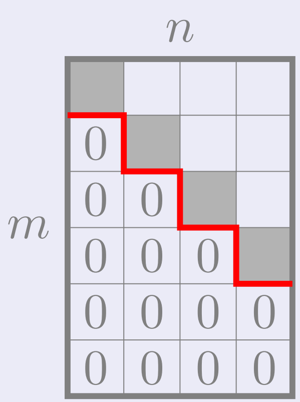
Contraposition: Let $\neg$(v).
There exists an elimination $\mathbf{A} \rightsquigarrow \mathbf{A}'$ to row echelon form $\mathbf{A}'$ with at least one zero row.
Hence $\mathbf{b} \in \mathbb{R}^m$ can be chosen such that $(\mathbf{A}'|\mathbf{b}')$ is of the form $(0 \ldots 0 \mid c)$ with $c \neq 0$.
By Theorem 3.19 it follows that $\mathbf{Ax} = \mathbf{b}$ is unsolvable.
Hence there holds $\neg$(i). $\square$

Solvability of systems of linear equations

Unique solvability (injectivity of $f_{\mathbf{A}}$)

For $\mathbf{A} \in \mathbb{R}^{m \times n}$ the following statements are equivalent:

(i) For **every** $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A}\mathbf{x} = \mathbf{b}$ has **at most** one solution $\mathbf{x}$.
(ii) $\mathbf{A}\mathbf{x} = \mathbf{o}$ has only the solution $\mathbf{x} = \mathbf{o}$.
(iii) $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{o}\}$.
(iv) $\mathrm{rank}(\mathbf{A}) = n \leqslant m$.
(v) When $\mathbf{A}$ is transformed to row echelon form $\mathbf{A}'$, then **every column** has a pivot element.
(vi) $f_{\mathbf{A}}$ is injective.



row echelon form $\mathbf{A}'$ of $\mathbf{A}$:

- Each column has a pivot element.
- All variables are dependent.
- There are no free variables.
- Hence there can never exist more than one solution.

Solvability of systems of linear equations

Unique solvability (injectivity of $f_{\mathbf{A}}$)

For $\mathbf{A} \in \mathbb{R}^{m \times n}$ the following statements are equivalent:

(i) For **every** $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{Ax} = \mathbf{b}$ has **at most** one solution $\mathbf{x}$.
(ii) $\mathbf{Ax} = \mathbf{o}$ has only the solution $\mathbf{x} = \mathbf{o}$.
(iii) $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{o}\}$.
(iv) $\mathrm{rank}(\mathbf{A}) = n \leqslant m$.
(v) When $\mathbf{A}$ is transformed to row echelon form $\mathbf{A}'$, then **every column** has a pivot element.
(vi) $f_{\mathbf{A}}$ is injective.

Proof

- **(i)⇔(iii):** This follows from $\mathcal{L} = \mathbf{v}_0 + \mathrm{Ker}(\mathbf{A})$, see Theorem 3.26d).
- (i)⇔(vi): This holds by the Definition 1.35 of injectivity.
- (ii)⇔(iii): This follows directly from the definition of $\mathrm{Ker}(\mathbf{A})$.
- **(iii)⇔(iv):** This follows by $\mathrm{Ker}(\mathbf{A}) = \{\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k : \lambda_1, \ldots, \lambda_k \in \mathbb{R}\} \cup \{\mathbf{o}\}$ (see Theorem 3.26c)) with $k = n - \mathrm{rank}(\mathbf{A})$.
- (iv)⇔(v): This follows by Definition 3.22 of the rank.

□

Solvability of systems of linear equations

Theorem 3.34 (Invertibility of $f_{\mathbf{A}}$)

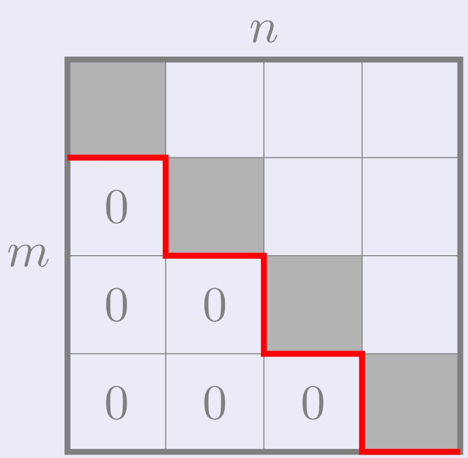
For $\mathbf{A} \in \mathbb{R}^{m \times n}$, the following statements are equivalent:

(i) $\mathbf{A}\mathbf{x} = \mathbf{b}$ is **uniquely** solvable **for all** $\mathbf{b} \in \mathbb{R}^m$.

(ii) $\operatorname{Ker}(\mathbf{A}) = \{\mathbf{o}\}$ and $\operatorname{Ran}(\mathbf{A}) = \mathbb{R}^m$.

(iii) $\operatorname{rank}(\mathbf{A}) = m = n$, i.e., $\mathbf{A}$ is square and has maximal rank.

(iv) $f_{\mathbf{A}}$ is surjective and injective, i.e., bijective (invertible.)



row echelon form $\mathbf{A}'$ of $\mathbf{A}$:

- Each row and every column have a pivot element.
- Hence the matrix must be square.
- There holds $\operatorname{rank}(\mathbf{A}) = m = n$.
- The row echelon form $\mathbf{A}'$ is even a triangular form.

Solvability of systems of linear equations

Theorem 3.35 (special case $m{=}n$: square matrices)

For square matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$, the following statements are equivalent:

(i) $\mathbf{A}\mathbf{x} = \mathbf{b}$ is solvable **for all** $\mathbf{b} \in \mathbb{R}^n$.
(ii) $\mathbf{A}\mathbf{x} = \mathbf{b}$ is **uniquely solvable** for some $\mathbf{b} \in \mathbb{R}^n$.
(iii) $\mathbf{A}\mathbf{x} = \mathbf{b}$ is **uniquely solvable** for **all** $\mathbf{b} \in \mathbb{R}^n$.
(iv) $\mathrm{Ker}(\mathbf{A}) = \{\mathbf{o}\}$.
(v) $\mathrm{Ran}(\mathbf{A}) = \mathbb{R}^n$.
(vi) $\mathrm{rank}(\mathbf{A}) = n$.
(vii) When $\mathbf{A}$ is transformed to row echelon form $\mathbf{A}'$, then each **row** has a pivot element.
(viii) When $\mathbf{A}$ is transformed to row echelon form $\mathbf{A}'$, then each **column** has a pivot element.
(ix) $f_{\mathbf{A}}$ is surjective.
(x) $f_{\mathbf{A}}$ is injective.
(xi) $f_{\mathbf{A}}$ is surjective and injective, i.e., bijective (invertible).

Proof In view of $m = n$, the statements $\mathrm{rank}(\mathbf{A}) = m$ and $\mathrm{rank}(\mathbf{A}) = n$ in Theorem 3.30 and Theorem 3.32 are equivalent. Hence all other statements of these theorems are equivalent as well. $\square$

Linear hull, span, subset

Definition 3.37 (linear hull, span, generating set)

For $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$, we call

$$V := \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_\mathrm{k}) \;:=\; \{\lambda_1 \mathbf{v}_1 + \ldots + \lambda_\mathrm{k} \mathbf{v}_\mathrm{k} \;:\; \lambda_1, \ldots, \lambda_\mathrm{k} \in \mathbb{R}\} \subset \mathbb{R}^\mathrm{n}$$

the *linear hull* or the *span* of $\mathbf{v}_1, \ldots, \mathbf{v}_k$.
We say that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *span the set V* or represent a *generating set* for V.
For the empty set we define $\operatorname{span}(\emptyset) = \{\mathbf{o}\}$.

Examples for linear hulls

a) $\operatorname{span}(\begin{pmatrix} 3 \\ 1 \end{pmatrix}) \subset \mathbb{R}^2$ is the line spanned by the vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ through the origin.

b) $\operatorname{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ is the entire plane $\mathbb{R}^2$. The same is true for $\operatorname{span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$.

c) $\operatorname{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$ is the so-called xy-plane in $\mathbb{R}^3$, $\operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)$ as well.

Linear hull, span, subset

Theorem 3.39 (properties of the linear hull)

Let $k \in \mathbb{N}$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and let $V := \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_\mathrm{k})$. Then there holds

a) For all $\mathbf{v} \in V$ and all $\alpha \in \mathbb{R}$, $\alpha\mathbf{v}$ is also in V.
In particular, the origin $\mathbf{o} = 0\mathbf{v}$ always lies in V.

b) For all $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{u} + \mathbf{v}$ also lies in V.

Proof Let $\mathbf{u}, \mathbf{v} \in V$ and $\alpha \in \mathbb{R}$ be arbitrary.

Then there exist $\lambda_1, \ldots, \lambda_k$ and $\mu_1, \ldots, \mu_k$ in $\mathbb{R}$ such that

$$\mathbf{u} = \lambda_1\mathbf{v}_1 + \ldots + \lambda_k\mathbf{v}_k \qquad \text{and} \qquad \mathbf{v} = \mu_1\mathbf{v}_1 + \ldots + \mu_k\mathbf{v}_k.$$

Hence there hold

$$\alpha\mathbf{u} = \alpha\lambda_1\mathbf{v}_1 + \ldots + \alpha\lambda_k\mathbf{v}_k \in V \qquad \text{and}$$
$$\mathbf{u} + \mathbf{v} = (\lambda_1 + \mu_1)\mathbf{v}_1 + \ldots + (\lambda_k + \mu_k)\mathbf{v}_k \in V.$$

$\square$

Definition 3.40 (vector space, subspace of $\mathbb{R}^n$)

The set $\mathbb{R}^n$ with the

- componentwise addition $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the
- scalar multiplication $\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

is a so-called *vector space*. It satisfies the following properties:

(1) $\forall \mathbf{v}, \mathbf{w} \in V : \quad \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (+ is commutative)

(2) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V : \quad \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (+ is associative)

(3) There exists a zero vector $\mathbf{o} \in V$ such that $\forall \mathbf{v} \in V$ there holds: $\mathbf{v} + \mathbf{o} = \mathbf{v}$.

(4) For every vector $\mathbf{v} \in V$ there exists a vector $-\mathbf{v} \in V$ with $\mathbf{v} + (-\mathbf{v}) = \mathbf{o}$.

(5) For the number $1 \in \mathbb{R}$ and every $\mathbf{v} \in V$ there holds: $1 \cdot \mathbf{v} = \mathbf{v}$.

(6) $\forall \alpha, \beta \in \mathbb{K} \quad \forall \mathbf{v} \in V : \quad \alpha \cdot (\beta \cdot \mathbf{v}) = (\alpha\beta) \cdot \mathbf{v}$ ($\cdot$ is assoziative)

(7) $\forall \alpha \in \mathbb{K} \quad \forall \mathbf{v}, \mathbf{w} \in V : \quad \alpha \cdot (\mathbf{v} + \mathbf{w}) = (\alpha \cdot \mathbf{v}) + (\alpha \cdot \mathbf{w})$ (distributive $\cdot +$)

(8) $\forall \alpha, \beta \in \mathbb{K} \quad \forall \mathbf{v} \in V : \quad (\alpha + \beta) \cdot \mathbf{v} = (\alpha \cdot \mathbf{v}) + (\beta \cdot \mathbf{v})$ (distributive $+ \cdot$)

A **nonempty** subset V of $\mathbb{R}^n$ with inherited addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{R} \times V \to V$ which has these eight properties is called a *subspace* of $\mathbb{R}^n$.

Linear hull, span, subset

Theorem 3.40 (subspace of $\mathbb{R}^n$)

A subset $V \subset \mathbb{R}^n$ which has the **three** properties

a) $\alpha \in \mathbb{R}, \mathbf{v} \in V \quad \Rightarrow \quad \alpha\mathbf{v} \in V,$

b) $\mathbf{u}, \mathbf{v} \in V \quad \Rightarrow \quad \mathbf{u} + \mathbf{v} \in V,$

c) $V \neq \emptyset,$

is a *subspace* of $\mathbb{R}^n$.

Examples for subspaces

- The simplest subspaces of $\mathbb{R}^n$ are $V = \{\mathbf{o}\}$ and $V = \mathbb{R}^n$.
- Each linear hull/span is a subspace.
- For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathrm{Ker}(\mathbf{A})$ is a subspace of $\mathbb{R}^n$ and $\mathrm{Ran}(\mathbf{A})$ is a subspace of $\mathbb{R}^m$.

1. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ as well as $f_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$.
Which of the following statements are equivalent?

a) $f_{\mathbf{A}}$ is injective.
b) $f_{\mathbf{A}}$ is surjective.
c) $f_{\mathbf{A}}$ is bijective.
d) $\operatorname{rank}(\mathbf{A}) = m$
e) $\operatorname{rank}(\mathbf{A}) = n$
f) $\operatorname{rank}(\mathbf{A}) = m = n$
g) $\mathbf{A}\mathbf{x} = \mathbf{o} \Rightarrow \mathbf{x} = \mathbf{0}$.
h) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} \Rightarrow \mathbf{x} = \mathbf{y}$.
i) $\forall \mathbf{b} \in \mathbb{R}^m\ \exists \mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}$.

True or false?

2. $\begin{pmatrix} 1 \\ 5 \end{pmatrix} \in \operatorname{span}\left(\begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}\right)$
3. $\begin{pmatrix} 1 \\ 5 \end{pmatrix} \in \operatorname{span}\left(\begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}\right)$
4. The following four subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$:

   $a)$ the line $g := \left\{\mathbf{x} \in \mathbb{R}^3 \colon \exists \alpha \in \mathbb{R} \text{ with } \mathbf{x} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$,

   $b)$ the plane through the origin $\mathbf{o}$ with normal vector $\mathbf{n} = \mathbf{e}_1 + \mathbf{e}_2$,

   $c)$ the sphere $K := \left\{\mathbf{x} \in \mathbb{R}^3 \colon \|\mathbf{x}\| \leqslant 1\right\}$,
   $d)$ $\mathcal{L} := \left\{\mathbf{x} \in \mathbb{R}^3 \colon \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$.

Linear (in-)dependence

Definition 3.41 (linear dependence and linear independence, family)

We denote a (here typically finite) sequence of elements of a set as a *family*.

We call a family $\mathcal{F} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ of vectors in $\mathbb{R}^n$ *linearly dependent* if one of the vectors in $\mathcal{F}$ can be removed without changing the linear hull, i.e., there exists an $i \in \{1, ..., k\}$ such that

$$\operatorname{span} \mathcal{F} \; = \; \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_{\mathrm{i}}\}).$$

Otherwise we call the family $\mathcal{F}$ *linearly independent*.

Examples for linear (in-)dependence

a) The family $(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix})$ is linearly dependent since any one of the three vectors can be removed: $\operatorname{span}(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}) = \operatorname{span}(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}) = \operatorname{span}(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}) = \operatorname{span}(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}) = \mathbb{R}^2$.

b) $(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix})$ is linearly dependent. $\begin{pmatrix}1\\1\end{pmatrix}$ or $\begin{pmatrix}2\\2\end{pmatrix}$ may be removed but not $\begin{pmatrix}1\\0\end{pmatrix}$.

c) Each family $\mathcal{F}$ that contains the zero vector $\mathbf{o}$ is linearly dependent. The zero vector $\mathbf{o}$ has no contribution to the linear hull: $\operatorname{span} \mathcal{F} = \operatorname{span}(\mathcal{F} \setminus \{\mathbf{o}\})$.

Linear (in-)dependence

Theorem 3.43 (linear dependence)

For a family $\mathcal{F} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ of vectors in $\mathbb{R}^n$, the following statements are equivalent:

(i) $\mathcal{F}$ is linearly dependent.

(ii) There exists an $i \in \{1, \ldots, k\}$ with $\operatorname{span} \mathcal{F} = \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$.

(iii) There exists an $i \in \{1, \ldots, k\}$ with $\mathbf{v}_i \in \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$.

(iv) There exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ which are not all zero but yield $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k = \mathbf{o}$.

In (ii) and (iii) one can use the same i. For this i, there holds $\lambda_i \neq 0$ in (iv).

Proof. (i)⇔(ii) follows from Definition 3.41.

(ii)⇒(iii): Let i be as in (ii). Then there holds $\mathbf{v}_i \in \operatorname{span} \mathcal{F} \overset{\text{(ii)}}{=} \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$ and hence (iii).

(iii)⇒(iv): Let i be as in (iii). Then there exist $\mu_j \in \mathbb{R}$ such that $\mathbf{v}_i = \sum_{j \in \{1,\ldots,k\} \setminus \{i\}} \mu_j \mathbf{v}_j$.

Moving $\mathbf{v}_i$ to the other side yields (iv):

$$\underbrace{\mu_1}_{\lambda_1} \mathbf{v}_1 + \ldots + \underbrace{\mu_{i-1}}_{\lambda_{i-1}} \mathbf{v}_{i-1} + \underbrace{(-1)}_{\lambda_i \neq 0} \mathbf{v}_i + \underbrace{\mu_{i+1}}_{\lambda_{i+1}} \mathbf{v}_{i+1} + \ldots + \underbrace{\mu_k}_{\lambda_k} \mathbf{v}_k = \mathbf{o}.$$

Linear (in-)dependence

Theorem 3.43 (linear dependence)

For a family $\mathcal{F} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ of vectors in $\mathbb{R}^n$, the following statements are equivalent:

(i) $\mathcal{F}$ is linearly dependent.
(ii) There exists an $i \in \{1, \dots, k\}$ with $\operatorname{span} \mathcal{F} = \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$.
(iii) There exists an $i \in \{1, \dots, k\}$ with $\mathbf{v}_i \in \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$.
(iv) There exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ which are not all zero but yield $\lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k = \mathbf{o}$.

In (ii) and (iii) one can use the same i. For this i, there holds $\lambda_i \neq 0$ in (iv).

Proof. (iv)⇒(ii): Let $\mathbf{o} = \lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k$ with $\lambda_i \neq 0$ for $i \in \{1, \dots, k\}$.
Show $\operatorname{span} \mathcal{F} \subset \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$ (the other inclusion is obvious).

Let $\mathbf{v} \in \operatorname{span} \mathcal{F}$. Then there exist $\mu_1, \dots, \mu_k \in \mathbb{R}$ such that

$$\mathbf{v} = \mu_1 \mathbf{v}_1 + \dots + \mu_k \mathbf{v}_k - \frac{\mu_i}{\lambda_i} \underbrace{\left(\lambda_1 \mathbf{v}_1 + \dots + \lambda_k \mathbf{v}_k\right)}_{=\mathbf{o}}$$

$$= (\mu_1 - \tfrac{\mu_i}{\lambda_i}\lambda_1)\mathbf{v}_1 + \dots + 0\mathbf{v}_i + \dots + (\mu_k - \tfrac{\mu_i}{\lambda_i}\lambda_k)\mathbf{v}_k \in \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\}).$$

□

Linear (in-)dependence, basis

Theorem 3.44 (linear independence)

For a family $\mathcal{F} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ of vectors in $\mathbb{R}^n$, the following statements are equivalent:

(i) $\mathcal{F}$ is linearly independent;

(ii) There exists no $i \in \{1, \ldots, k\}$ with $\operatorname{span} \mathcal{F} = \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$;

(ii') For all $i \in \{1, \ldots, k\}$ there holds $\operatorname{span} \mathcal{F} \neq \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$;

(iii) There exists no $i \in \{1, \ldots, k\}$ with $\mathbf{v}_i \in \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$;

(iii') For all $i \in \{1, \ldots, k\}$ there holds $\mathbf{v}_i \notin \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$;

(iv) The only coefficients $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ that satisfy $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k = \mathbf{o}$ are $\lambda_1 = \ldots = \lambda_k = 0$.

Proof. Negation of the statements in Theorem 3.43. □

Theorem 3.43 (linear dependence)

For a family $\mathcal{F} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ of vectors in $\mathbb{R}^n$, the following statements are equivalent:

(i) $\mathcal{F}$ is linearly dependent.

(ii) There exists an $i \in \{1, \ldots, k\}$ with $\operatorname{span} \mathcal{F} = \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$.

(iii) There exists an $i \in \{1, \ldots, k\}$ with $\mathbf{v}_i \in \operatorname{span}(\mathcal{F} \setminus \{\mathbf{v}_\mathrm{i}\})$.

(iv) There exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ which are not all zero but yield $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k = \mathbf{o}$.

Basis

Definition 3.47 (basis, basis vectors)

Let V be a subspace of $\mathbb{R}^n$. A family $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is called *basis* of V if

a) $V = \operatorname{span} \mathcal{B}$ and

b) $\mathcal{B}$ is linearly independent.

The vectors in a basis are called *basis vectors* of V.

Basis

Theorem & Definition 3.48 (unique coefficients thanks to basis)

Let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ be a basis of the subspace $V \subset \mathbb{R}^n$. Then every $\mathbf{v} \in V$ can be expressed as a linear combination $\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k$ with **uniquely determined** coefficients $\lambda_1, \dots, \lambda_k$. They are called the *coordinates* of $\mathbf{v}$ with respect to the basis $\mathcal{B}$.

Proof. Let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ be a basis of V. Thanks to property a) of Definition 3.47 (basis is a generating set), each $\mathbf{v} \in V$ can be represented as a linear combination $\mathbf{v} = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k$.

The uniqueness of the coefficients $\lambda_1, \dots, \lambda_k$ follows from property b) of Definition 3.47 (basis is linearly independent): Suppose there exist two representations of a vector $\mathbf{v} \in V$:

$$\lambda_1 \mathbf{v}_1 + \ldots + \lambda_k \mathbf{v}_k \;=\; \mathbf{v} \;=\; \mu_1 \mathbf{v}_1 + \ldots + \mu_k \mathbf{v}_k.$$

This is equivalent to

$$(\lambda_1 - \mu_1)\mathbf{v}_1 + \ldots + (\lambda_k - \mu_k)\mathbf{v}_k \;=\; \mathbf{o}.$$

In view of the linear independence of the family $\mathcal{B}$, Theorem 3.44 yields

$$\lambda_1 - \mu_1 = \ldots = \lambda_k - \mu_k = 0,$$

hence the two representations of $\mathbf{v}$ are identical. $\square$

Basis

Theorem 3.49 (all bases have the same number of elements)

Let V be a subspace of $\mathbb{R}^d$ and let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be a basis of V.

a) Every family $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ in V with $n > m$ is linearly dependent.
b) Every basis of V has exactly m elements.

Proof. a) Each $\mathbf{w}_i$ has unique coordinates $a_{1i}, \ldots, a_{mi}$ wrt. the basis $\mathcal{B}$, i.e., $\mathbf{w}_i = a_{1i}\mathbf{v}_1 + \ldots + a_{mi}\mathbf{v}_m$ for $i = 1, \ldots, n$. Let $x_1, \ldots, x_n \in \mathbb{R}$ be coefficients such that

$$\begin{aligned} \mathbf{o} &= x_1\mathbf{w}_1 + \ldots + x_n\mathbf{w}_n \\ &= x_1(a_{11}\mathbf{v}_1 + \ldots + a_{m1}\mathbf{v}_m) \ + \ \ldots \ + \ x_n(a_{1n}\mathbf{v}_1 + \ldots + a_{mn}\mathbf{v}_m) \\ &= (x_1a_{11} + \ldots + x_na_{1n})\mathbf{v}_1 \ + \ \ldots \ + \ (x_1a_{m1} + \ldots + x_na_{mn})\mathbf{v}_m. \end{aligned}$$

Due to the linear independence of $\mathcal{B}$, all terms in the parentheses are equal to zero:

$$\begin{array}{cccccc} x_1a_{11}+ & \ldots & +x_na_{1n} & = & 0, \\ & & & \vdots & \\ x_1a_{m1}+ & \ldots & +x_na_{mn} & = & 0, \end{array} \qquad \text{hence} \qquad \underbrace{\begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}}_{=:\mathbf{A}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Basis

Theorem 3.49 (all bases have the same number of elements)

Let V be a subspace of $\mathbb{R}^d$ and let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ be a basis of V.

a) Every family $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$ in V with $n > m$ is linearly dependent.
b) Every basis of V has exactly m elements.

Proof (continuation). a) Each $\mathbf{w}_i$ has unique coordinates $a_{1i}, \ldots, a_{mi}$ wrt. the basis $\mathcal{B}$, i.e., $\mathbf{w}_i = a_{1i}\mathbf{v}_1 + \ldots + a_{mi}\mathbf{v}_m$ for $i = 1, \ldots, n$. Let $x_1, \ldots, x_n \in \mathbb{R}$ be coefficients such that

$$\mathbf{o} = x_1\mathbf{w}_1 + \ldots + x_n\mathbf{w}_n. \qquad \underbrace{\begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}}_{=:\mathbf{A}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

There holds $\operatorname{rank}(\mathbf{A}) \leqslant m < n$. By Theorem 3.26c), $\operatorname{Ker}(\mathbf{A})$ is spanned by $k := n - \operatorname{rank}(\mathbf{A}) > 0$ vectors which are not $\mathbf{o}$. Any such vector in the kernel yields $x_1, \ldots, x_n$, not all zero, and hence linear dependence of the family $(\mathbf{w}_1, \ldots, \mathbf{w}_n)$.

b) Let $\mathcal{C} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ be another basis of V.
If $n > m$, then by a) $\mathcal{C}$ would be linearly dependent, hence not a basis.
If $m > n$, then by a) $\mathcal{B}$ would be linearly dependent, hence not a basis. Hence, $m = n$. □

Dimension

Definition 3.50 (dimension of a subspace)

Let V be a subspace of $\mathbb{R}^n$ and let $\mathcal{B}$ be any basis of V.
The number of elements of $\mathcal{B}$ is called <u>*dimension*</u> of V and is denoted by $\dim(V)$.
In the case $V = \{\mathbf{o}\}$, one sets $\dim(V) = 0$.

Examples for basis and dimension

a) $\mathcal{B} = (\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix})$ is a basis of $V = \mathbb{R}^2$. Hence $\dim(\mathbb{R}^2) = 2$.

$(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix})$ is also a basis, but not $(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix})$, $(\begin{pmatrix}1\\1\end{pmatrix})$, $(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix})$.

b) In general: The canonical unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ of $\mathbb{R}^n$ form a basis of $V = \mathbb{R}^n$:
There holds $\mathrm{span}(\mathbf{e}_1, \dots, \mathbf{e}_\mathrm{n}) = \mathbb{R}^\mathrm{n}$. The linear independence follows from

$$x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n = \mathbf{o} \qquad \Rightarrow \qquad \begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} = \mathbf{o} \qquad \Rightarrow \qquad x_1 = \ldots = x_n = 0.$$

Hence $\dim(\mathbb{R}^n) = n$.

Basis

Theorem 3.56 (basis test, without proof)

Let V be a subspace of $\mathbb{R}^n$ with $\dim(V) = k$ and let $\mathcal{F} = (\mathbf{v}_1, \ldots, \mathbf{v}_k)$ be a family of k vectors in V. Then there holds:

a) If $\mathcal{F}$ is linearly independent, then $\mathcal{F}$ is a basis of V.
b) If $\mathcal{F}$ is a generating set for V, then $\mathcal{F}$ is a basis of V.

Hence if the number of elements matches the dimension, then only one of the two conditions in the definition of a basis has to be checked.