Discrete Algebraic Structures

Exercise 4

Exercise 1 (Graphs):

(a) Write down the relation represented by the picture below. Is it:

- reflexive?
- symmetric?
- transitive?
- antisymmetric?
- antireflexive?

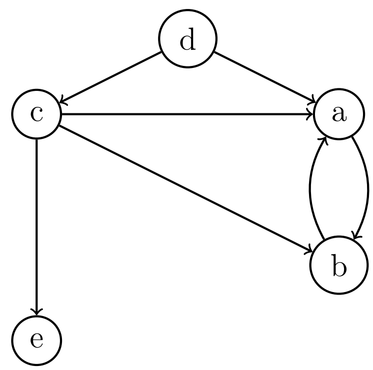
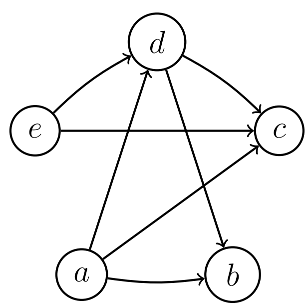
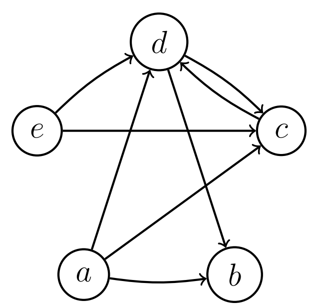


Figure 1: A directed graph

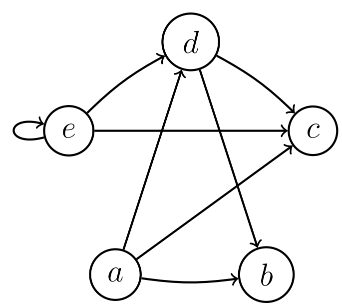
(b) Do the same for the following relations.
Do the first 3 during the tutorial, and the 3 others at home.



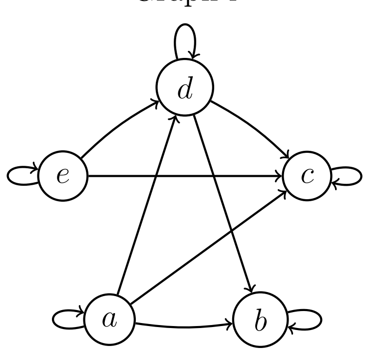
Graph i



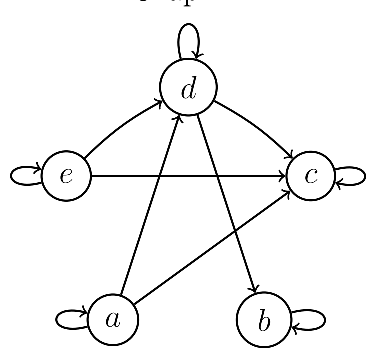
Graph ii



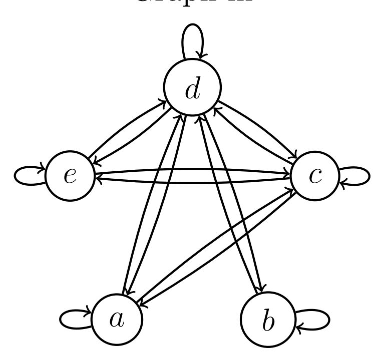
Graph iii



Graph iv



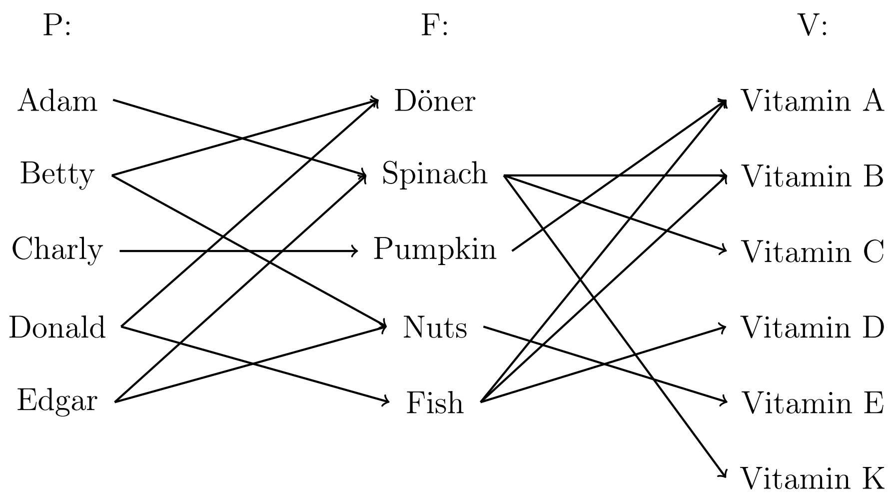
Graph v



Graph vi

Exercise 2 (Composition of Relations):

(a) Let P, F, V sets, and $E \subseteq P \times F$, $N \subseteq F \times V$ binary relations as given by the picture. Compute $N \circ E$.



(b) Compute $R_1 \circ R_2$ and $R_2 \circ R_1$ where $R_1 = \{(2,0),(2,1),(1,0),(0,1)\}$ and $R_2 = \{(1,0),(0,1),(1,2),(0,0)\}$.

(c) Let $D = \{(d_1, d_2) \in \mathbb{Z}^2 \mid |d_2 - d_1| = 1\}$, compute $D \circ D$.

(d) Let $f\colon A \to B$ and $g\colon B \to C$. Write F resp. G for f resp. g seen as a relation. Show $G \circ F = g \circ f$ (in other words, relation composition applied to functions reduces to function composition).

Exercise 3 (Closure Properties):

To do at home.

Let $R, T \subseteq A \times A$ for some set A and assume $R \subseteq T$. Show that the following two statements hold.

(a) If T is reflexive, then it contains the reflexive closure of R.

(b) If T is symmetric, then it contains the symmetric closure of R.

Hint[1]

Exercise 4 (Powers of a Relation):

To do at home.

In this exercise, $R \subseteq A^2$ is a reflexive relation.

(a) Show that $R \subseteq R \circ R$.

(b) Show that $R \circ R$ is reflexive.

[1]Use that Q reflexive $\Leftrightarrow \mathrm{Id}_A \subseteq Q$ and Q symmetric $\Leftrightarrow Q = Q^\top$.

(c) Suppose that A is finite. For $n \in \mathbb{N}$, define R^n as $R \circ R \circ \cdots \circ R$, repeated n times. In words, this is the relation containing all pairs $(a, b) \in A^2$ such that there exists a path of length n from a to b. Show that there exists $n \in \mathbb{N}$ such that R^n is transitive. [2]

(d) Let now $R = \{(a, b) \in \mathbb{N}^2 \mid |a - b| \leq 1\}$. Show that for all $n \in \mathbb{N}$, R^n is *not* transitive.

[2] Hint: Use the fact that Q transitive $\Leftrightarrow Q \circ Q \subseteq Q$