Sheet 1

Exercise 0 (Warmup):

Compute the following objects:

1. $(\{1,2\} \cap \{2,3\}) \cup \{2,4\}$ = {2,4}
2. $(\{1,2\} \cup \{2,3\}) \setminus \{2,4\}$ = {1,3}
3. $(\{1,2\} \cap \{2,3\}) \,\Delta\, \{2,4\}$ = {4}
4. $(\{1,2\} \cup \{2,3\}) \times \{2,4\}$ = {1,2,3} × {2,4} = {{1,2}, {1,4}, {2,2}, {2,4}, {3,2}, {3,4}}
5. $\mathcal{P}(\{1,3,4\})$ = {1, {1,3}, {1,4}, 3, {3,4}, 4, {1,3,4}}

Exercise 1 (Sets):

(a) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) Prove that $A \cap (A \cup B) = A$.

(c) Prove that if $B \subseteq A$, then $A \cap B = B$ and $A \cup B = A$.

Exercise 2 (De Morgan's law):

Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ using a written proof in the style of Exercise 1(a).

Exercise 3 (Distributivity of Cartesian Product):

Let A, B, C, D be sets.

1. Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$
2. Show that $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3. Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
4. Do we have $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$? If not, is one of both inclusions true?
5. Illustrate the results of this exercise using figures. You can use for example $A = [0,2]$, $B = [1,3]$, $C = [0,3]$, $D = [1,4]$.

Exercise 4 (Symmetric Difference):

This is a supplementary exercise that you should do at home, to get more practice after the tutorials.

(a) Let $A \Delta B = C$. What is $A \Delta C$?

(b) Prove that $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

(c) Show that the symmetric difference is associative, i.e., that $A\Delta(B\Delta C) = (A\Delta B)\Delta C$ is true for all sets A, B, C. Can you find a natural description of the elements that are in $A\Delta B\Delta C$?

0.

1. (This is needlessly complicated and relies on ungiven premises)

notes	proof
0: $(A \cup B) \times C \overset{!}{=} (A \times C) \cup (B \times C)$ | (Thesis)	Def. 0: Thesis means „Every tuple from $(A \cup B) \times C$ must correspond to a tuple from $(A \times C) \cup (B \times C)$ (and vice versa)
Hyp. 1: A, B, C are Sets	
Hyp. 2: $x \in (A \cup B) \times C$	
Hyp. 3: using $S = (A \cup B) \rightarrow S \times C$	
Hyp. 4: $A \subseteq S$	Def. 1: For $S = (A \cup B)$ and $S \times C$, $S \times C$ can be written as a union between all of S's subsets $\times C$
Hyp. 5: $B \subseteq S$	
Hyp. 6: $A \cup B \subseteq S$	Def. 2: A and B are subsets of, and make up S. Any operation on S can be written as a union between that operation performed on all its subsets
Hyp. 6: $x \in (A \cap C) \times (B \cap C)$	Def. 3: Therefore: $(A \cup B) \times C = (A \times C) \cup (B \times C)$
	q.e.d.

1. (again)

Thesis: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Splits into: 1: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

2. ... $\supseteq$ ...

1. $A \cup (B \cap C) \subseteq$ ...
   1.1. $x \in A \cup (B \cap C)$
   1.2. $x \in A$ or $x \in (B \cap C)$
   1.3. Assuming $x \in A$:
      1.3.1. x must be in $(A \cup B)$ and $(A \cup C)$
      1.3.2. x must be in $(A \cup B) \cap (A \cup C)$ (since its in both operands)
   1.4. Assuming $x \in (B \cap C)$
      1.4.1. $x \in B$ and $x \in C$
      1.4.2 Therefore: x must be part of $(A \cup B)$ and $(A \cup C)$ ($\cup$ only extends sets)
      1.4.3 "and" rewritten as operator: $x \in (A \cup B) \cap (A \cup C)$
2. $(A \cup B) \cap (A \cup C) \supseteq$ ...
   2.1. $x \in (A \cup B) \cap (A \cup C)$
   2.2. $x \in (A \cup B)$ and $x \in (A \cup C)$
   2.3. Case: $x \in A$:
      2.3.1. Entails $x \in A \cup (B \cap C)$ ($\cup$ extension doesn't break $x \in A$)
   2.4. Case: $x \notin A$:
      2.4.1. $x \in B$ and $x \in C$
      2.4.2. rewritten: $x \in (B \cap C)$
      2.4.3. extend with arbitrary union (here: with A): $x \in A \cup (B \cap C)$

Q.E.D

1.(b)

Thesis: $A \cap (A \cup B) = A$

Split Theses:

1. $A \cap (A \cup B) \supseteq A$
   1.1. $x \in A \cap (A \cup B)$
   1.2. $x \in A$ and $x \in (A \cup B)$
   (1.3. $x \in A$ or $x \in B$
      1.3.1. already filled by 1.2: $x \in A$) could be skipped
   1.4. 1.1 and 1.2 ($x \in A$), Thesis 1 fulfilled
2. $A \supseteq A \cap (A \cup B)$
   2.1. $x \in A$
   2.2. Therefore: $x \in (A \cup B)$
   2.3. Known: $x \in A$ and $x \in (A \cup B)$
      2.3.1. Combination: $x \in A \cap (A \cup B)$ (2.1, 2.2)

Q.E.D.

(c)

Thesis: Given $B \subseteq A$, $A \cap B = B$ and $A \cup B = A$

Subtheses: $A \cap B \supseteq B$, $A \cap B \subseteq B$, $A \cup B \supseteq A$, $A \cup B \subseteq A$

1. $A \cap B \supseteq B$
   1.1. $x \in (A \cap B)$
   1.2. $x \in A$ and $x \in B$
      1.2.1. Thesis 1 satisfied (1.1, 1.2), $x \in (A \cap B)$, $x \in B$
2. $B \supseteq A \cap B$
   2.1. repeat: $B \subseteq A$
   2.2. $x \in B$
   2.3. from 2.1: $x \in A$
   2.4. 2.2, 2.3; $x \in A \cap B$
      2.4.1. Thesis 2 satisfied
3. $A \cup B \supseteq A$
   3.1. rp: $B \subseteq A$
   3.2. $x \in A \cup B$
   3.3. $x \in A$ or $x \in B$
      3.3.1. case $x \in A$:
         3.3.1.1. satisfies Th. 3 together with 3.2.
      3.3.2. case $x \in B$:
         3.3.2.1. from 3.1.: $x \in A$
         3.3.2.2. satisfies Th. 3 (w/ 3.2.)
4. $A \supseteq A \cup B$
   4.1. rp: $B \subseteq A$
   4.2. $x \in A$
   4.3. write 4.2. as: $x \in A \cup B$ (arbitrary union doesn't break membership)
      4.3.1. Th. 4. satisfied (4.3, 4.2)

Satisfied Th. 1–4

Q.E.D

Exercise 2 (De Morgan's law):
Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ using a written proof in the style of Exercise 1(a).

Thesis: $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
Subtheses: 1. $\ldots \subseteq \ldots$
2. $\ldots \supseteq \ldots$

1. $A \setminus (B \cap C) \supseteq (A \setminus B) \cup (A \setminus C)$
   1.1. let $x \in A \setminus (B \cap C)$
   1.2. $x \notin (B \cap C)$
      1.2.1. $x \notin B$ or $x \notin C$
   1.3. $x \in A$
   1.4. from 1.2.1.:
      1.4.1. case 1: $x \notin B$:
         1.4.1.2. $\to x \in A \setminus B$
      1.4.2. case 2: $x \notin C$
         1.4.2.1. $\to x \in A \setminus C$
   1.5. 1.4.1.2 or 1.4.2.1 is true, meaning: $x \in (A \setminus B) \cup (A \setminus C)$ ($\cup$ meaning "or") satisfies 1
2. $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C)$
   2.1. let $x \in (A \setminus B) \cup (A \setminus C)$
   2.2. $x \in (A \setminus B)$ or $x \in (A \setminus C)$
      2.2.1. $x \in (A \setminus B)$
         2.2.1.1. $x \in A$
         2.2.1.2. $x \notin B$
      2.2.2. $x \in (A \setminus C)$
         2.2.2.1. $x \in A$
         2.2.2.2. $x \notin C$
   2.3. same def. from 2.2.2 and 2.2.1: $x \in A$
   2.4. $x \notin B$ or $x \notin C$

2.5. $x \notin B \cap C$ (can't be part of both if $\notin$ of at least one)

2.6. 2.5, 2.3.: $x \in A \setminus (B \cap C)$

Q.E.D.

Exercise 3 (Distributivity of Cartesian Product):
Let A, B, C, D be sets.

1. Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$
2. Show that $(A \cap B) \times C = (A \times C) \cap (B \times C)$
3. Show that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
4. Do we have $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$? If not, is one of both inclusions true?
5. Illustrate the results of this exercise using figures. You can use for example $A = [0, 2]$, $B = [1, 3]$, $C = [0, 3]$, $D = [1, 4]$.

1) Thesis: $(A \cup B) \times C = (A \times C) \cup (B \times C)$

1. $(A \cup B) \times C \supseteq (A \times C) \cup (B \times C)$

1.1. $s \in (A \cup B) \times C$

1.2. $s = (x, y)$ (follows definition of cartesian product)

1.3. $y \in C$

1.4. $x \in A$ or $x \in B$

1.4.1. case $x \in A$: $s \in A \times C$

1.4.2. case $x \in B$: $s \in B \times C$

1.5. from 1.4: $s \in A \times C$ or $s \in B \times C$

1.5.1. $\Rightarrow s \in (A \times C) \cup (B \times C)$

subthesis 1 satisfied

2. $(A \times C) \cup (B \times C) \supseteq (A \cup B) \times C$

2.1. $(x, y) \in (A \times C) \cup (B \times C)$

2.2. $(x, y) \in (A \times C)$ or $x \in (B \times C)$

2.2.1. case $(x, y) \in (A \times C)$: $x \in A$ and $y \in C$

2.2.2. case $(x, y) \in (B \times C)$: $x \in B$ and $y \in C$

2.3. $\Rightarrow y \in C$

2.4. from 2.2. $\Rightarrow x \in A$ or $x \in B \Rightarrow x \in (A \cup B)$

2.5. $\Rightarrow (x, y) \in (A \cup B) \times C$ (since $(A \cup B)$ contributes and C y)

supthesis 2 satisfied
Q.E.D.

2) Thesis: $(A \cap B) \times C = (A \times C) \cap (B \times C)$

1. $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$
   1.1. $(x, y) \in (A \cap B) \times C$
   1.2. $y \in C$
   1.3. $x \in A$ and $x \in B$ $(x \in (A \cap B))$
   1.4. $\Rightarrow (x, y) \in A \times C$ and $(x, y) \in B \times C$ (permitted since x is in both A and B)
   1.5. $\Rightarrow (x, y) \in (A \times C) \cap (A \times C)$
2. $(A \times C) \cap (B \times C) \supseteq (A \cap B) \times C$
   2.1. Let $(x, y) \in (A \times C) \cap (B \times C)$
   2.2. $\Rightarrow (x, y) \in (A \times C)$ and $(x, y) \in (B \times C)$
   2.3. $\Rightarrow y \in C$
   2.4. cartesian product def.: $x \in A$ and $x \in B$
   2.5. $(x, y) \in (A \cap B) \times C$ (from 2.4 and 2.3.)

   Q.E.D.

3) Thesis: $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

1. $(A \cap B) \times (C \cap D) \subseteq (A \times C) \cap (B \times D)$
   1.1. let $(x, y) \in (A \cap B) \times (C \cap D)$
   1.2. $x \in (A \cap B)$ and $y \in (C \cap D)$
   1.3. $x \in A$ and $x \in B$
   1.4. $y \in C$ and $y \in D$
   1.5. $\Rightarrow (x, y) \in (A \times C)$ (following cartesian product def.)
   1.6. $\Rightarrow (x, y) \in (B \times D)$
   1.7. from 1.5 and 1.6: $(x, y) \in (A \times C) \cap (B \times D)$

2. $(A\times C)\cap(B\times D) \supseteq (A\cap B)\times(C\cap D)$

2.1. let $(x,y)\in(A\times C)\cap(B\times D)$

2.2. $(x,y)\in(A\times C)$ and $(x,y)\in(B\times D)$

2.3. $x\in A$ and $x\in B$

2.4. $y\in C$ and $y\in D$

2.5. $\Rightarrow x\in(A\cap B)$

2.6. $\Rightarrow y\in(C\cap D)$

2.7. $\Rightarrow (x,y)\in(A\cap B)\times(C\cap D)$ (following CP)

Q.E.D.

4) Thesis: $(A\cup B)\times(C\cup D)=(A\times C)\cup(B\times D)$

1. $(A\cup B)\times(C\cup D)\subseteq(A\times C)\cup(B\times D)$

 1.1. let $(x,y)\in(A\cup B)\times(C\cup D)$

 1.2. $x\in(A\cup B)$, $x\in A$ or $x\in B$

 1.3. $y\in(C\cup D)$, $y\in C$ or $y\in D$

 1.4. from 1.2 and 1.3

  1.4.1. cases:

   $(x,y)\in A\times C$

   $(x,y)\in A\times D$

   $(x,y)\in B\times C$

   $(x,y)\in B\times D$

  1.4.2. combine: $(x,y)\in(A\times C)\cup(A\times D)\cup(B\times C)\cup(B\times D)$

  ↯ thesis impossible without additional unions, or ($D\subseteq C$ and $C\subseteq D$)

2. $(A\times C)\cup(B\times D)\subseteq(A\cup B)\times(C\cup D)$

 2.1. let $s\in(A\times C)\cup(B\times D)$; $x,y\in s$

 2.2. $s\in(A\times C)$ or $s\in(B\times D)$

  2.2.1. $\Rightarrow x\in A$ or $x\in B\rightarrow x\in(A\cup B)$

  2.2.2. $\Rightarrow y\in C$ or $y\in D\rightarrow y\in(C\cup D)$

 2.3. $\Rightarrow (x,y)\in(A\cup B)\times(C\cup D)$

original thesis disproven, but T2 holds

4) $A = \{0,2\}$, $B = \{1,3\}$, $C = \{0,3\}$, $D = \{1,4\}$

from 3):

T1: $\{0,1,2,3\} \times \{0,1,3,4\} = \{(0,0),(0,1),(0,3),(0,4),(1,0),(1,1),(1,3),$
$(1,4),(2,0),(2,1),(2,3),(2,4),(3,0),(3,1),$
$(3,3),(3,4)\} = S$

T2: $(\{0,2\} \times \{0,3\}) \cup (\{1,3\} \times \{1,4\}) = \{(0,0),(0,3),(2,0),(2,3)\}$
$\cup \{(1,1),(1,4),(3,1),(3,4)\}$
$= \{(0,0),(0,3),(2,0),(2,3),$
$(1,1),(1,4),(3,1),(3,4)\} = t$

as we can see, $s \neq t$, but $t \subseteq s$