Exercise 03

Exercise 1 (From logic to English):

Let p stand for the proposition "I bought a lottery ticket" and q for "I won the jackpot". Express the following as natural English sentences:

- ¬p
- p ∨ q
- p ⇒ q

You should do the three last items at home:

- p ∧ q
- ¬p ⇒ ¬q
- ¬p ∨ (p ∧ q)

Exercise 2 (Satisfiable formulas, tautologies, and unsatisfiable formulas):

For each of the following propositions, construct a truth table and state whether the proposition is a tautology, satisfiable or unsatisfiable.

- p ∧ ¬p
- (p ∨ ¬q) ⇒ q
- (p ⇒ q) ∧ (p ∧ ¬q)

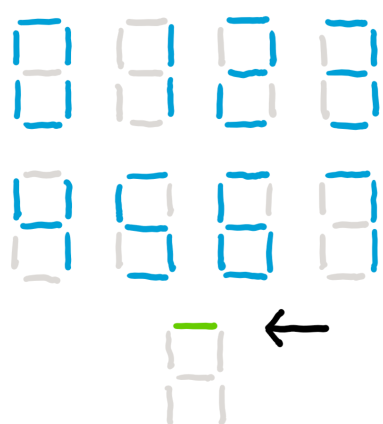
You should do the three last items at home:

- p ∨ ¬p
- (p ∨ q) ⇒ (p ∧ q)
- (p ⇒ q) ⇒ (q ⇒ p)

Exercise 3 (Clock display):

We store a number $n \in \{0, \ldots, 7\}$ with three bits p, q, r as follows:

n	p	q	r
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1



Find a formula describing when the top segment of the display should light up, expressed in terms of p, q, r.

Exercise 4 (Negation of a quantified formula):

Let us define a formula ϕ starting from 3 predicates:

$$\phi := \forall x \in \mathbb{N}(\text{prime}(x) \Rightarrow (\exists y(x < y \wedge \neg\,\text{odd}(y))))$$

We ask you to write down the simplified version of $\neg\phi$. By simplified we mean a formula ψ which is equivalent to ϕ and such that the negated sub-formulas cannot be decomposed further.

Exercise 5 (Boolean Logic):

Exercise to do at home after the tutorials.

(a) Give a definition of the following boolean operators

- $\bigoplus$ ("XOR"), the exclusive or,
- $\uparrow$ ("NAND"), the not and operator.

x	y	$x \bigoplus y$
0	0	0
1	0	1
0	1	1
1	1	0

x	y	$x \uparrow y$
0	0	1
1	0	1
0	1	1
1	1	0

(b) Show that every boolean formula is equivalent to one only using $\uparrow$.

(c) Show that every boolean formula with variables from X is equivalent to one in *disjunctive normal form* (DNF), i.e. equivalent to a formula

$$\bigvee_{C \in \mathcal{C}} \bigwedge_{c \in C} c$$

where all $C \in \mathcal{C}$ are sets of literals (often called "clauses"), that is all $c \in C$ are of the form x or $\neg x$ for some variable x.

Exercise 6 (Does this prove the claim?):

Exercise to do at home after the tutorials.

In this exercise, we are interested in the following mathematical statement:

Let $a, b \in \mathbb{N}$. Suppose that ab is even. Then a is even or b is even.

(a) Translate this statement in first-order logic using a unary predicate $Even(-)$.

We ask four students to write a proof of that statement. Below are their proofs. We take for granted that all computations are correct and that the proofs are correctly written (miracles do happen). Remember that a number a is *even* if there exists k such that $a = 2k$ (i.e., a is divisible by 2), and a is *odd* otherwise, which means that a can be written as $a = 2k + 1$ for some k.

For each of the four proofs below:

- Write in first-order logic what is the statement that they prove
- Determine whether this is logically a valid proof of the original statement.

(b) Suppose that a and b are odd. Indeed, we have $a = 2k+1$ and $b = 2m+1$ for some integers k, m. Then

$$\begin{aligned} ab &= (2k+1)(2m+1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1. \end{aligned}$$

Therefore, ab is odd.

(c) Assume that a is even or b is even. Without loss of generality, say it is a that is even, so that we have $a = 2k$ for some integer k. Then $ab = (2kb) = 2(kb)$, and therefore ab is even.

(d) Suppose that ab is even but a and b are both odd. Then we have $ab = 2n, a = 2k+1, b = 2\ell+1$ for some integers n, k, ℓ. Then

$$\begin{aligned} 2n &= (2k+1)(2\ell+1) \\ &= 4k\ell + 2(k+\ell) + 1 \end{aligned}$$

and dividing each side by 2, we get $n = 2k\ell + k + \ell + \frac{1}{2}$. This gives $\frac{1}{2} = n - 2k\ell - k - \ell$. Since the right hand side of the equality is an integer, and the left hand side is not, we obtain a contradiction.

(e) Let ab be an even number, and a be an odd number. Let $ab = 2n$ and $a = 2k + 1$ for some integers n, k. Then $ab = (2k+1)b$, so $2n = 2kb + b$, which we can rewrite as $b = 2n - 2kb = 2(n - kb)$. Thus, b must be even.