Mathematics 1 - Linear Algebra

Lecture 09 – §4 Determinants

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Determinants

The determinant is a certain function $\boxed{\det : \mathbb{R}^{n\times n} \to \mathbb{R}.}$

The result $\det(\mathbf{A}) \in \mathbb{R}$ of the determinant function $\det$ is called the determinant of $\mathbf{A}$.

Preview for $n = 2$

Define $\det : \mathbb{R}^{2\times 2} \to \mathbb{R},\ \det\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) := a_{11}a_{22} - a_{12}a_{21},$

Example: $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2.$

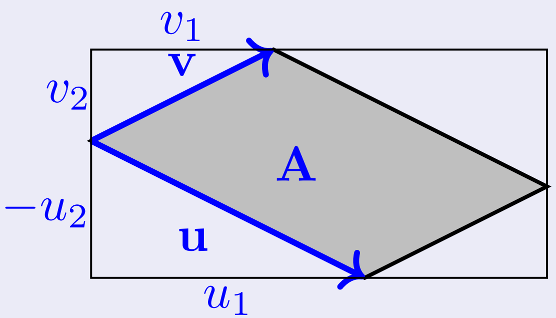
Questions:

- Where do determinants come from, where do we need them?
  e.g. for the computation of areas, volumes
- What are the properties of the determinant function?
  preferably those that we expect for areas and volumes
- How does the determinant function look like for general $n \in \mathbb{N}$?
  There is only one function that satisfies all the required properties but several ways/formulas to compute it (all yielding the same result).

Determinants: the two-dimensional case

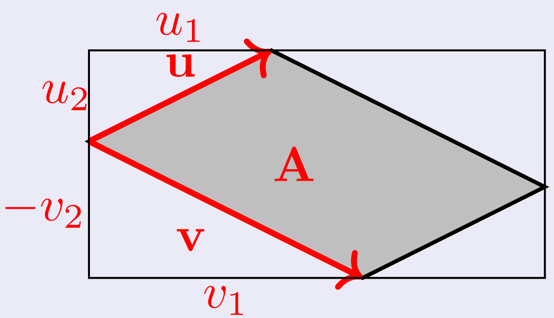
Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$.

How can we compute the area of the parallelogram spanned by vectors $\mathbf{u}$ and $\mathbf{v}$?



$$\begin{aligned} \mathbf{A} &= (u_1 + v_1) \cdot (v_2 - u_2) - v_1 v_2 - u_1(-u_2) \\ &= u_1 v_2 - u_1 u_2 + v_1 v_2 - v_1 u_2 - v_1 v_2 + u_1 u_2 \\ &= u_1 v_2 - v_1 u_2 = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}. \end{aligned}$$

If we swap $\mathbf{u}$ and $\mathbf{v}$, then we compute analogously



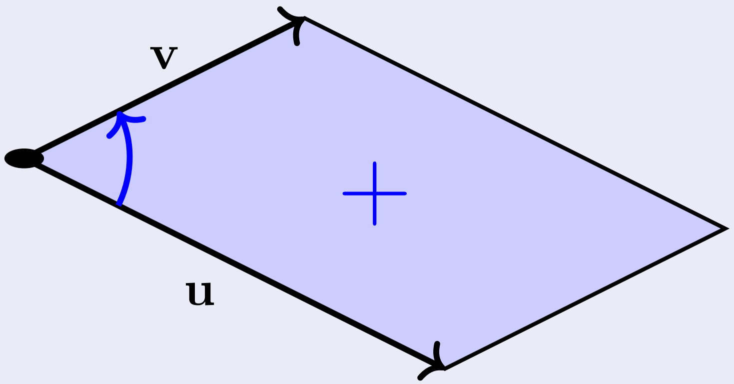
$$\begin{aligned} \mathbf{A} &= v_1 u_2 - u_1 v_2 = \det \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix} \\ &= -(u_1 v_2 - v_1 u_2) = -\det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \end{aligned}$$

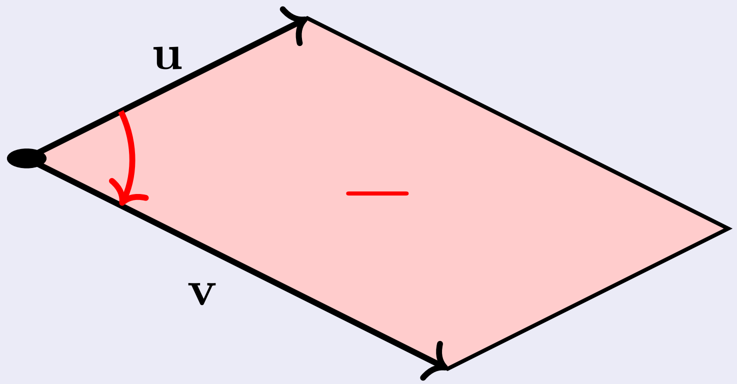
$$\implies \mathbf{A} = \mathbf{A} = \left|\det(\mathbf{u}\ \mathbf{v})\right| = \left|\det(\mathbf{v}\ \mathbf{u})\right|.$$

Determinants: the two-dimensional case

We define $\mathrm{area}(\mathbf{u}, \mathbf{v})$ to be the area of the parallelogram with a sign, namely

+ if one traverses the parallelogram from $\mathbf{u}$ to $\mathbf{v}$ in the mathematically positive sense (counterclockwise), and
– if one rotates in the mathematically negative sense (clockwise).





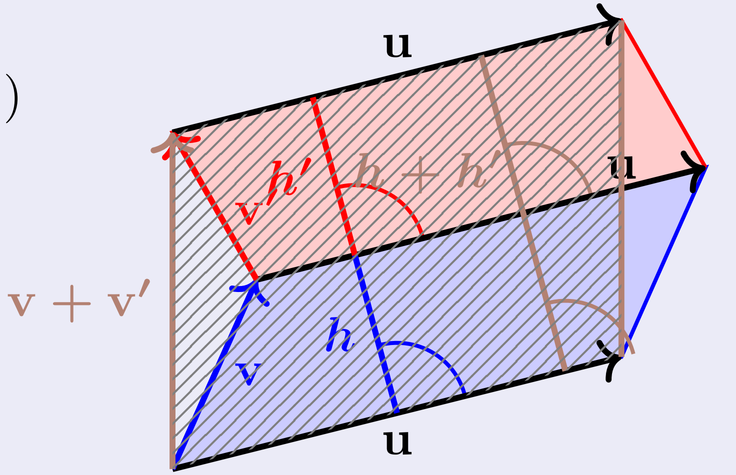
$0 < \mathrm{area}(\mathbf{u}, \mathbf{v}) = -\mathrm{area}(\mathbf{v}, \mathbf{u})$ $\quad$ $0 > \mathrm{area}(\mathbf{u}, \mathbf{v}) = -\mathrm{area}(\mathbf{v}, \mathbf{u})$

Determinants: the two-dimensional case

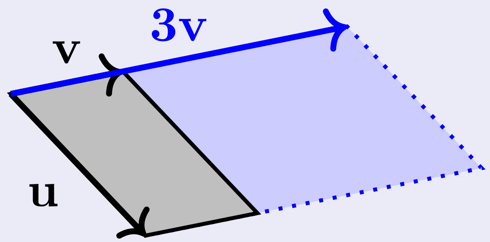
Four properties that $\mathrm{area}(\mathbf{u},\mathbf{v})$ should satisfy

For all $\mathbf{u}, \mathbf{v}, \mathbf{v}' \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ shall hold:

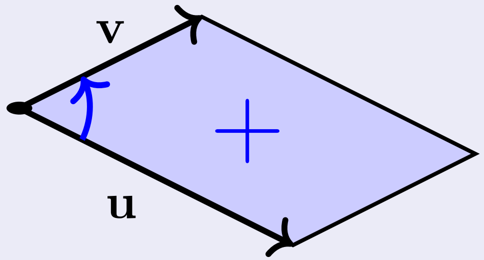
(1) $\mathrm{area}(\mathbf{u}, \mathbf{v} + \mathbf{v}') = \mathrm{area}(\mathbf{u}, \mathbf{v}) + \mathrm{area}(\mathbf{u}, \mathbf{v}')$

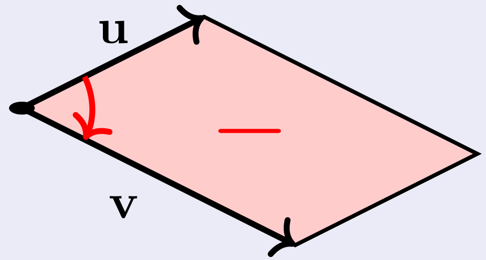


(2) $\mathrm{area}(\mathbf{u}, \alpha\mathbf{v}) = \alpha \, \mathrm{area}(\mathbf{u}, \mathbf{v})$



(3) $\mathrm{area}(\mathbf{u}, \mathbf{v}) = -\mathrm{area}(\mathbf{v}, \mathbf{u})$





(4) The unit square ($\mathbf{u} = \mathbf{e}_1, \mathbf{v} = \mathbf{e}_2$) has the area 1: $\mathrm{area}(\mathbf{e}_1, \mathbf{e}_2) = 1$.

Determinants: the two-dimensional case

Theorem 4.3 ((1) and (2) also hold analogously in the first component)

Let all $\mathbf{u}, \mathbf{u}', \mathbf{v} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ satisfy

(1) $\operatorname{area}(\mathbf{u}, \mathbf{v} + \mathbf{v}') = \operatorname{area}(\mathbf{u}, \mathbf{v}) + \operatorname{area}(\mathbf{u}, \mathbf{v}')$,

(2) $\operatorname{area}(\mathbf{u}, \alpha\mathbf{v}) = \alpha \operatorname{area}(\mathbf{u}, \mathbf{v})$,

(3) $\operatorname{area}(\mathbf{u}, \mathbf{v}) = -\operatorname{area}(\mathbf{v}, \mathbf{u})$.

Then there also holds

(1′) $\operatorname{area}(\mathbf{u} + \mathbf{u}', \mathbf{v}) = \operatorname{area}(\mathbf{u}, \mathbf{v}) + \operatorname{area}(\mathbf{u}', \mathbf{v})$,

(2′) $\operatorname{area}(\alpha\mathbf{u}, \mathbf{v}) = \alpha \operatorname{area}(\mathbf{u}, \mathbf{v})$.

Proof. Let $\mathbf{u}, \mathbf{u}', \mathbf{v} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ be arbitrary. Then there hold

$$\begin{aligned}\operatorname{area}(\mathbf{u} + \mathbf{u}', \mathbf{v}) &\overset{(3)}{=} -\operatorname{area}(\mathbf{v}, \mathbf{u} + \mathbf{u}') \overset{(1)}{=} -(\operatorname{area}(\mathbf{v}, \mathbf{u}) + \operatorname{area}(\mathbf{v}, \mathbf{u}')) \\ &\overset{(3)}{=} -(-\operatorname{area}(\mathbf{u}, \mathbf{v}) - \operatorname{area}(\mathbf{u}', \mathbf{v})) = \operatorname{area}(\mathbf{u}, \mathbf{v}) + \operatorname{area}(\mathbf{u}', \mathbf{v})\end{aligned}$$

and

$$\operatorname{area}(\alpha\mathbf{u}, \mathbf{v}) \overset{(3)}{=} -\operatorname{area}(\mathbf{v}, \alpha\mathbf{u}) \overset{(2)}{=} -\alpha \operatorname{area}(\mathbf{v}, \mathbf{u}) \overset{(3)}{=} \alpha \operatorname{area}(\mathbf{u}, \mathbf{v}). \quad \square$$

Determinants: the two-dimensional case

Theorem 4.6 (parallel vectors span an area of size zero)

For all $\mathbf{u} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ there holds $\mathrm{area}(\mathbf{u}, \alpha\mathbf{u}) = 0$.

Proof. (3) implies $\mathrm{area}(\mathbf{u}, \mathbf{u}) = -\mathrm{area}(\mathbf{u}, \mathbf{u})$ which leads to $\mathrm{area}(\mathbf{u}, \mathbf{u}) = 0$.
Now (2) leads to $\mathrm{area}(\mathbf{u}, \alpha\mathbf{u}) = \alpha\, \mathrm{area}(\mathbf{u}, \mathbf{u}) = \alpha\, 0 = 0$.

$\square$

Theorem 4.5 (formula for parallelogram area follows uniquely from (1) – (4))

Let there hold

(1) $\mathrm{area}(\mathbf{u}, \mathbf{v}+\mathbf{w}) = \mathrm{area}(\mathbf{u}, \mathbf{v}) + \mathrm{area}(\mathbf{u}, \mathbf{w})$,

(2) $\mathrm{area}(\mathbf{u}, \alpha\mathbf{v}) = \alpha\mathrm{area}(\mathbf{u}, \mathbf{v})$,

(3) $\mathrm{area}(\mathbf{u}, \mathbf{v}) = -\mathrm{area}(\mathbf{v}, \mathbf{u})$,

(4) $\mathrm{area}(\mathbf{e}_1, \mathbf{e}_2) = 1$.

Then there holds for all $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$

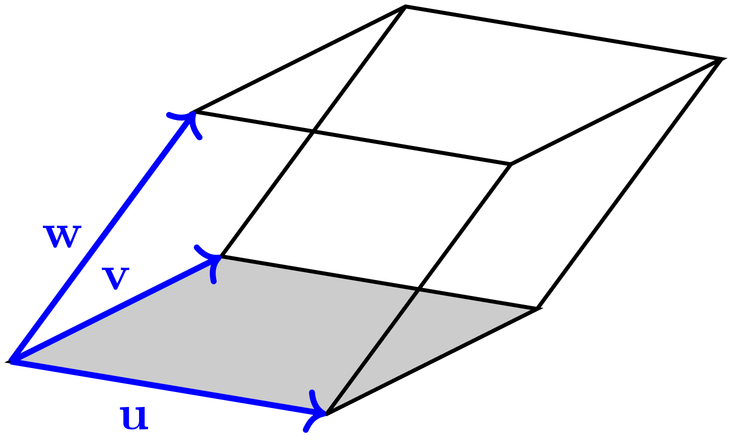
$$\boxed{\mathrm{area}(\mathbf{u}, \mathbf{v}) = +u_1 v_2 - u_2 v_1.}$$

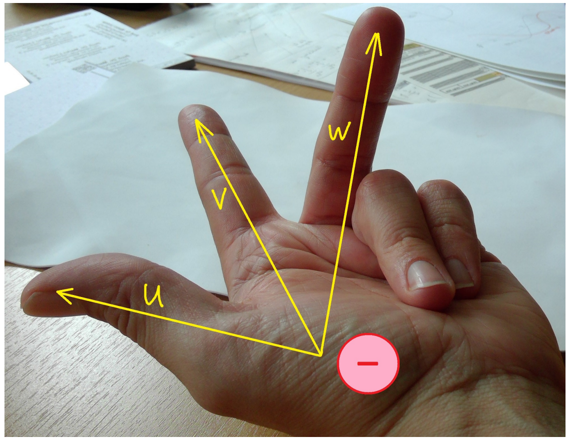
$$\boxed{\begin{array}{cc} u_1 & v_1 \\ u_2 & v_2 \end{array}}$$

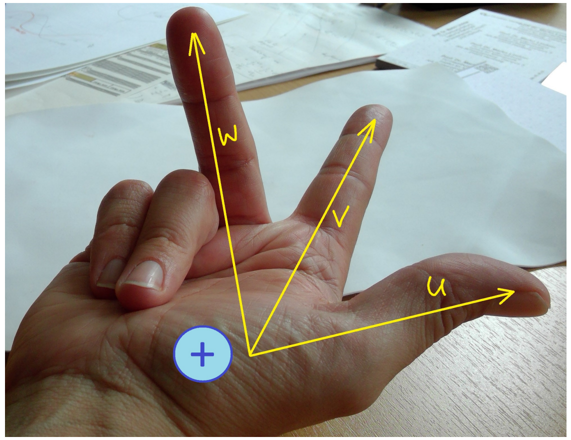
Proof. Besides (1), (2), (3), (4) we may also use (1') and (2'):

$$\begin{aligned}
\mathrm{area}(\mathbf{u}, \mathbf{v}) &= \mathrm{area}\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \mathbf{v}\right) \overset{(1')}{=} \mathrm{area}\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \mathbf{v}\right) + \mathrm{area}\left(\begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \mathbf{v}\right) \\
&\overset{(1)}{=} \underbrace{\mathrm{area}\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}\right)}_{=0 \text{ (Theorem 4.6)}} + \mathrm{area}\left(\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}\right) + \mathrm{area}\left(\begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}\right) + 0 \\
&\overset{(2),(2')}{=} u_1 v_2 \underbrace{\mathrm{area}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}_{=1, \text{ due to } (4)} + u_2 v_1 \underbrace{\mathrm{area}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}_{=-1, \text{ due to } (3),(4)} = u_1 v_2 - u_2 v_1. \qquad \square
\end{aligned}$$

Determinants: the three-dimensional case







We define the volume of the parallelepiped with a sign as $\mathrm{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w})$, in particular

+ if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the order of thumb, index finger and bent middle finger of the right hand (hence forming a so-called right-handed trihedron)
− if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the order of thumb, index finger and bent middle finger of the left hand (hence forming a so-called left-handed trihedron).

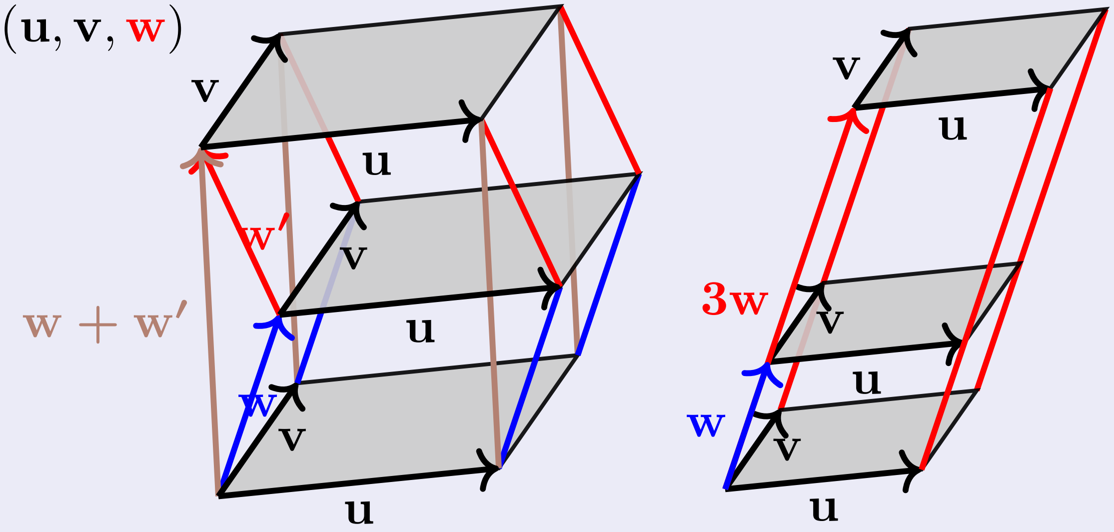
As in the two-dimensional case, we will formulate properties of $\mathrm{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ which lead to a formula for its computation.

Determinants: the three-dimensional case

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{w}' \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ there shall hold:

(1) $\text{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w} + \mathbf{w}') = \text{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w}) + \text{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w}')$

(2) $\text{volume}(\mathbf{u}, \mathbf{v}, \alpha\, \mathbf{w}) = \alpha\, \text{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w})$



(3) If two vectors are swapped, then the sign changes, i.e.,

$$-\text{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \text{volume}(\mathbf{v}, \mathbf{u}, \mathbf{w}) = \text{volume}(\mathbf{w}, \mathbf{v}, \mathbf{u}) = \text{volume}(\mathbf{u}, \mathbf{w}, \mathbf{v}).$$

(4) The unit cube has volume 1: $\quad \text{volume}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1.$

In view of (3), the properties (1) and (2) hold analogously for $\mathbf{u}$ and $\mathbf{v}$.

Determinants: the three-dimensional case

Theorem 4.6 (linearly dependent systems yield volume equal to zero)

For arbitrary $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ there holds:

a) If two of the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are the same, then there holds $\mathrm{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$.

b) If the family $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ in $\mathbb{R}^3$ is linearly dependent, then there holds $\mathrm{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$.

Proof. a) If two of the three vectors are the same, then $d := \mathrm{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ does not change when these two are swapped. In view of (3) we have $d = -d$, hence $d = 0$ is the only option.

b) One of the three vectors can be expressed as a linear combination of the other two. W.l.o.g. let $\mathbf{w}$ be this vector. Then there exist $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$. There holds

$$\begin{aligned} \mathrm{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \mathrm{volume}(\mathbf{u}, \mathbf{v}, \alpha\mathbf{u} + \beta\mathbf{v}) \\ &= \alpha \underbrace{\mathrm{volume}(\mathbf{u}, \mathbf{v}, \mathbf{u})}_{=0, \text{ see a)}} + \beta \underbrace{\mathrm{volume}(\mathbf{u}, \mathbf{v}, \mathbf{v})}_{=0, \text{ see a)}} = 0. \end{aligned}$$

$\square$

Determinants: the three-dimensional case

As in the two-dimensional case we obtain the volume formula from the properties (1) – (4):

$$
\begin{aligned}
\operatorname{volume}(\mathbf{u},\mathbf{v},\mathbf{w}) &= \operatorname{volume}(\underbrace{u_1\mathbf{e}_1+u_2\mathbf{e}_2+u_3\mathbf{e}_3}_{\mathbf{u}},\underbrace{v_1\mathbf{e}_1+v_2\mathbf{e}_2+v_3\mathbf{e}_3}_{\mathbf{v}},\underbrace{w_1\mathbf{e}_1+w_2\mathbf{e}_2+w_3\mathbf{e}_3}_{\mathbf{w}})\\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} u_i v_j w_k \operatorname{volume}(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k)\\
&= u_1v_2w_3\underbrace{\operatorname{volume}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)}_{=1,\text{ see (4)}} + u_2v_1w_3\underbrace{\operatorname{volume}(\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_3)}_{=-1\text{ (1 swap)}}\\
&\quad + u_3v_2w_1\underbrace{\operatorname{volume}(\mathbf{e}_3,\mathbf{e}_2,\mathbf{e}_1)}_{=-1\text{ (1 swap)}} + u_1v_3w_2\underbrace{\operatorname{volume}(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_2)}_{=-1\text{ (1 swap)}}\\
&\quad + u_3v_1w_2\underbrace{\operatorname{volume}(\mathbf{e}_3,\mathbf{e}_1,\mathbf{e}_2)}_{=1\text{ (2 swaps)}} + u_2v_3w_1\underbrace{\operatorname{volume}(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_1)}_{=1\text{ (2 swaps)}}\\
&= u_1v_2w_3 - u_2v_1w_3 - u_3v_2w_1 - u_1v_3w_2 + u_3v_1w_2 + u_2v_3w_1.
\end{aligned}
$$

Determinants: the general case

Definition 4.7 (determinant)

Let $n \in \mathbb{N}$, let $\mathbf{a}_1, \dots, \mathbf{a}_n, \mathbf{a}'_n \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$. A function

$$\det : \mathbb{R}^{n \times n} \to \mathbb{R}$$

is called ***determinant function*** and $\det(\mathbf{A})$ the ***determinant*** of $\mathbf{A} \in \mathbb{R}^{n \times n}$ if it satisfies the following properties:

(1) $$\det \begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n + \mathbf{a}'_n \\ | & & | & | \end{pmatrix} = \det \begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & & | & | \end{pmatrix} + \det \begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}'_n \\ | & & | & | \end{pmatrix}.$$

(2) $$\det \begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \alpha\mathbf{a}_n \\ | & & | & | \end{pmatrix} = \alpha \det \begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & & | & | \end{pmatrix}.$$

(3) If $\mathbf{A}'$ results from $\mathbf{A}$ by swapping two columns, then there holds $\det(\mathbf{A}') = -\det(\mathbf{A})$.

(4) $$\det \begin{pmatrix} | & & | \\ \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ | & & | \end{pmatrix} = 1, \quad \text{in short: } \det(\mathbf{I}_n) = 1.$$

Determinants: the general case

The properties (1)–(4) once again define a unique function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$.

For $n = 1, 2, 3$, we computed it already:

For $u \in \mathbb{R}^1$: $\qquad \det(u) = u$

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$: $\qquad \det\begin{pmatrix} | & | \\ \mathbf{u} & \mathbf{v} \\ | & | \end{pmatrix} = \text{area}(\mathbf{u}, \mathbf{v}) = +u_1v_2 - u_2v_1,$

For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$: $\quad \det\begin{pmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{pmatrix} = \text{volume}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{array}{l} +u_1v_2w_3 + v_1w_2u_3 + w_1u_2v_3 \\ -u_3v_2w_1 - v_3w_2u_1 - w_3u_2v_1. \end{array}$

Determinants: the general case

Corollary 4.8 (properties of the determinant)

Let $n \in \mathbb{N}$ and let $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^n$ be the columns of $\mathbf{A} \in \mathbb{R}^{n \times n}$. Then there holds

a) $\det(\cdot)$ is linear wrt every column (additive & homogeneous), i.e.,

$$(1) \quad \det\begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n + \mathbf{a}_n' \\ | & & | & | \end{pmatrix} = \det\begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & & | & | \end{pmatrix} + \det\begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n' \\ | & & | & | \end{pmatrix}.$$

$$(2) \quad \det\begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \alpha\mathbf{a}_n \\ | & & | & | \end{pmatrix} = \alpha \det\begin{pmatrix} | & & | & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_n \\ | & & | & | \end{pmatrix}$$

hold not only in the last but in all columns.

b) For all $\alpha \in \mathbb{R}$ there holds $\det(\alpha\,\mathbf{A}) = \alpha^n \det(\mathbf{A})$.

c) If two columns of $\mathbf{A}$ are equal, then $\det(\mathbf{A}) = 0$.

Proof. Analogous to the case $n = 3$. □

Determinants: the general case

We derive a formula to compute $\det \mathbf{A}$ analogously to the two- and three-dimensional cases:

- Write each column of $\mathbf{A}$ as a linear combination of the unit vectors:

$$\mathbf{A} := \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \dots & \mathbf{a}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\begin{matrix}|\\ \mathbf{e}_1\\ |\end{matrix} + \ldots + a_{n1}\begin{matrix}|\\ \mathbf{e}_n\\ |\end{matrix} & \cdots & a_{1n}\begin{matrix}|\\ \mathbf{e}_1\\ |\end{matrix} + \ldots + a_{nn}\begin{matrix}|\\ \mathbf{e}_n\\ |\end{matrix} \end{pmatrix}$$

- Applying (1), (2) to each column yields

$$\det(\mathbf{A}) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} \underbrace{\det(\mathbf{e}_{i_1} \ \mathbf{e}_{i_2} \ \cdots \ \mathbf{e}_{i_n})}_{\in\{0,-1,1\}}$$

Determinants: the general case

Examples:

$$\det(\mathbf{e}_1\ \mathbf{e}_3\ \mathbf{e}_2) = (-1)\det(\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) = -1,$$
$$\det(\mathbf{e}_2\ \mathbf{e}_3\ \mathbf{e}_1) = -\det(\mathbf{e}_1\ \mathbf{e}_3\ \mathbf{e}_2) = (-1)^2\det(\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3) = 1.$$

There holds

$$\det(\mathbf{e}_{i_1}\ \mathbf{e}_{i_2}\ \cdots\ \mathbf{e}_{i_n}) = \begin{cases} 0 & : \quad \exists j \neq k : e_j = e_k, \quad \text{(two identical columns)} \\ (-1)^{\#(i_1,\dots,i_n)} & : \quad \text{else}, \end{cases} ,$$

where

$$\#(i_1,\dots,i_n) := \begin{array}{l}\text{(minimal) number of pairwise exchanges to transition} \\ \text{from } (i_1,\dots,i_n) \text{ to } (1,2,\dots,n).\end{array}$$

Examples: $\#(1,3,2) = 1,\ \#(2,3,1) = 2$

Determinants: the general case

Determinant formula No. 1: Permutation formula by Leibniz

$$\det(\mathbf{A}) = \sum_{(i_1,\ldots,i_n)\in P_n} a_{i_1,1}a_{i_2,2}\cdots a_{i_n,n}(-1)^{\#(i_1,\ldots,i_n)}.$$

Here P_n denotes the set of all index tuples $(i_1,\ldots,i_n)$ with pairwise distinct i_j (permutations):

$$\begin{aligned} P_n &= \{(i_1,\ldots,i_n)\ :\ i_1,\ldots,\ i_n \in \{1,\ldots,n\}, i_j \neq\ i_k \text{ for all } 1 \leqslant j,k \leqslant n\} \\ &= \text{set of all } n! \text{ permutations/orderings of } \{1,2,\ldots,n\}. \end{aligned}$$

$$\begin{aligned} &P_1 = \{(1)\}, \\ &P_2 = \{(1,2),(2,1)\}, \\ &P_3 = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}, \\ &P_4: \quad 4\cdot 6 = 24 \text{ permutations}, \\ &P_5: \quad 5\cdot 24 = 120 \text{ permutations}. \end{aligned}$$

Starting with $n=4$, the Leibniz formula becomes very work intensive ($n!$ terms).

Determinants: the general case

Determinant Formula No. 2: Determinant with Gauss (see Theorem 4.9)

If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is transformed to upper triangular form using row- (or column) operations,

$$\mathbf{A} \quad \rightsquigarrow \quad \mathbf{B} = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ 0 & b_{2,2} & & b_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{n,n} \end{pmatrix},$$

then there holds

$$\det(\mathbf{A}) = (-1)^m \det(\mathbf{B}) = (-1)^m \, b_{1,1} b_{2,2} \cdots b_{n,n},$$

where $m \in \mathbb{N}_0$ is the number of (column and) row exchanges during "$\rightsquigarrow$".

Example for Determinant Formula No. 1 and No. 2

Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ 2 & 3 & 2 \end{pmatrix}$.

Determinant Formula No. 1 (Leibniz), $\det(\mathbf{A}) = \sum_{(i_1,\dots,i_n)\in P_n} a_{i_1,1} \cdots a_{i_n,n} (-1)^{\#(i_1,\dots,i_n)}$:

$$\begin{aligned}
\det(\mathbf{A}) = \quad & a_{11}a_{22}a_{33}(-1)^{\#(1,2,3)} + a_{11}a_{32}a_{23}(-1)^{\#(1,3,2)} + a_{21}a_{12}a_{33}(-1)^{\#(2,1,3)} \\
& + a_{21}a_{32}a_{13}(-1)^{\#(2,3,1)} + a_{31}a_{12}a_{23}(-1)^{\#(3,1,2)} + a_{31}a_{22}a_{13}(-1)^{\#(3,2,1)} \\
= & \, 1 \cdot 1 \cdot 2 \cdot (-1)^0 + 1 \cdot 3 \cdot 0 \cdot (-1)^1 + 1 \cdot 2 \cdot 2 \cdot (-1)^1 \\
& + 1 \cdot 3 \cdot (-3) \cdot (-1)^2 + 2 \cdot 2 \cdot 0 \cdot (-1)^2 + 2 \cdot 1 \cdot (-3) \cdot (-1)^1 \\
= & \, 2 + 0 - 4 - 9 + 0 + 6 = -5
\end{aligned}$$

Determinant Formula No. 2 (Gauss):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ 2 & 3 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 3 \\ 0 & -1 & 8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{pmatrix} =: \mathbf{B}$$

$$\implies \quad \det(\mathbf{A}) = (-1)^0 \det(\mathbf{B}) = 1 \cdot (-1) \cdot 5 = -5$$

True or false?

1. $\det \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = 2$

2. $\det \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} = 0$

3. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

4. $\det \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 0 & 1 & -2 \end{pmatrix} = -2$

5. $\det \mathbf{A} = 0 \quad \Longleftrightarrow \quad \mathbf{A} = \mathbf{O}$

Determinants: the general case

Determinant formula No. 2 follows from

Theorem 4.9 (Determinant under row or column Gauss elimination)

Let $n \in \mathbb{N}$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a square matrix. Then there holds:

a) $\det(\mathbf{A}^\mathsf{T}) = \det(\mathbf{A})$.

b) The Gaussian elimination steps
   - subtract a multiple of a row from another one
   - swap two rows,

   and analogously applied to columns, do not change the absolute value of the determinant; the sign changes with each swapping of two rows or columns.

c) The determinant of a triangular matrix is the product of the diagonal elements.

Sketch of the proof. a) Each summand in Determinant Formula No. 1 selects one entry per row and column of $\mathbf{A}$ and multiplies their product with $(-1)^{\#(i_1,\dots,i_n)}$.
The same summand also occurs in Formula No. 1 for $\det(\mathbf{A}^\mathsf{T})$; the permutation is now the inverse of the above one with the same number of pairwise exchanges.

Determinants: the general case

Theorem 4.9 (Determinant under row or column Gauss elimination)

Let $n \in \mathbb{N}$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ a square matrix. Then there holds:

b) The Gaussian elimination steps
   - subtract a multiple of a row from another one
   - swap two rows,

   and analogously applied to columns, do not change the absolute value of the determinant; the sign changes with each swapping of rows or columns.

c) The determinant of a triangular matrix is the product of the diagonal elements.

Sketch of the proof.

b) If we replace the j-th column $\mathbf{a}_j$ by $\mathbf{a}_j - \alpha \mathbf{a}_i$ with $i \neq j$, then there holds

$$\det(\cdots\ \mathbf{a}_i\ \cdots\ \underbrace{\mathbf{a}_j - \alpha\mathbf{a}_i}_{j\text{-th column}}\ \cdots) = \det\underbrace{(\cdots\ \mathbf{a}_i\ \cdots\ \mathbf{a}_j\ \cdots)}_{\mathbf{A}} - \alpha\ \underbrace{\det(\cdots\ \mathbf{a}_i\ \cdots\ \mathbf{a}_i\ \cdots)}_{0} = \det(\mathbf{A}).$$

Due to Def. 4.7 (3), each pairwise column exchange leads to a multiplication with (-1). In view of a) ($\det \mathbf{A} = \det \mathbf{A}^\mathsf{T}$), the same statements also hold for row operations.

Determinants: the general case

Theorem 4.9 (Determinant under row or column Gauss elimination)

Let $n \in \mathbb{N}$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ a square matrix. Then there holds:

c) The determinant of a triangular matrix is the product of the diagonal elements.

Sketch of the proof.

c) In view of a) ($\det \mathbf{A} = \det \mathbf{A}^\mathsf{T}$) it is sufficient to consider an upper triangular matrix.

Each summand in Determinant Formula No. 1 selects one entry per row and column of $\mathbf{A}$ and forms their product.

The only summand that does not select any entries of the lower triangular part (which by assumption contains only zero entries) is the one with $(i_1, ..., i_n) = (1, ..., n)$. It hence is equal to $a_{1,1}a_{2,2}\cdots a_{n,n}(-1)^0$, the product of the diagonal entries.

□

Determinants: the general case

Theorem 4.10 (determinants of block triangular matrices)

Let $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{C} \in \mathbb{R}^{m\times m}$ be two square matrices (of possibly different sizes) and let $\mathbf{B} \in \mathbb{R}^{n\times m}$. Then there holds

$$\det \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix} = \det(\mathbf{A}) \det(\mathbf{C}).$$

Determinants: the general case

Theorem 4.12 (Determinant Formula No. 3: Laplace expansion)

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ and let $\mathbf{A}_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix that remains if one deletes the i-th row and j-th column from $\mathbf{A}$. Then there holds for any $i \in \{1, \ldots, n\}$

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij}).$$

This is called the <u>*expansion of* $\det(\mathbf{A})$ *with respect to the* i*-th row.*</u>

Analogously there holds for all $j \in \{1, \ldots, n\}$

$$\det(\mathbf{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij}).$$

This is called the <u>*expansion of* $\det(\mathbf{A})$ *with respect to the* j*-th column.*</u>

For a 1×1 matrix there holds $\det(a) = a$.

Determinants: the general case

Expansion wrt the i-th row	Expansion wrt the j-th column
$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij})$	$\det(\mathbf{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij})$

Example for Determinant Formula No. 3 (Laplace expansion)

Expansion wrt the second row:

$$\det \begin{pmatrix} 1 & 2 & -3 \\ \boxed{1} & \boxed{1} & \boxed{0} \\ 2 & 3 & 2 \end{pmatrix} = (-1)^3 \boxed{1} \det \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} + (-1)^4 \boxed{1} \det \begin{pmatrix} 1 & -3 \\ 2 & 2 \end{pmatrix} + (-1)^5 \boxed{0} (\ldots)$$

$$= -(2 \cdot 2 - 3 \cdot (-3)) + 1 \cdot 2 - 2 \cdot (-3) = -13 + 8 = -5$$

Expansion wrt the third column:

$$\det \begin{pmatrix} 1 & 2 & \boxed{-3} \\ 1 & 1 & \boxed{0} \\ 2 & 3 & \boxed{2} \end{pmatrix} = \boxed{-3} \det \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} + \boxed{2} \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -3 \cdot 1 + 2 \cdot (-1) = -5$$

Determinants: the general case

Expansion wrt the i-th row	Expansion wrt the j-th column
$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij})$	$\det(\mathbf{A}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(\mathbf{A}_{ij})$

Sketch of the proof. The proof can be derived from the determinant formula

$$\det(\mathbf{A}) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} a_{i_1,1} a_{i_2,2} \cdots a_{i_n,n} \underbrace{\det(\mathbf{e}_{i_1} \ \mathbf{e}_{i_2} \ \cdots \ \mathbf{e}_{i_n})}_{\in\{0,-1,1\}}.$$

If one selects the entry of column j and row $i = i_j$, i.e., $a_{i_j,j}$, then all the other entries of the respective summand must come from a column other than j and a row other than i. Hence one only considers permutations of the remaining rows and columns which leads to $\det(\mathbf{A}_{ij})$. $\square$

Advice: Choose a row or column with many zeros for the Laplace expansion.

Determinants: the general case

Complexity of the determinant formulas

	n	2	3	4	5	$\cdots$	9	10	$\cdots$	20
Gauss	$n^3/3$	3	9	21	42	$\cdots$	243	333	$\cdots$	2667
Leibniz/Laplace	$n!$	2	6	24	120	$\cdots$	362880	3628800	$\cdots$	$2.4 \cdot 10^{18}$

Gauss wins starting with $n = 4$, the advantage keeps growing as the matrix size increases. Laplace may be advantageous if $\mathbf{A}$ has many zero entries.

Determinants: the general case

Theorem 4.15 (Determinant of a matrix product, without proof)

For all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ there holds

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B}).$$

Corollary: If the matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is regular, then there holds

$$1 = \det(\mathbf{I}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A}) \det(\mathbf{A}^{-1}) \quad \Longrightarrow \quad \det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}.$$

Application for Theorem 4.15

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Then $\mathrm{area}(\mathbf{u}, \mathbf{v}) = \det(\begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix})$ is the area of the parallelogram spanned by $\mathbf{u}, \mathbf{v}$ (with a sign). Let $\mathbf{B} \in \mathbb{R}^{2 \times 2}$ with the mapping $f_{\mathbf{B}} : \mathbb{R}^2 \to \mathbb{R}^2,\ \mathbf{x} \mapsto \mathbf{Bx}$.
The vectors $\mathbf{Bu}, \mathbf{Bv}$ again span a parallelogram. Its area (with a sign) is

$$\det(\begin{pmatrix} \mathbf{Bu} & \mathbf{Bv} \end{pmatrix}) = \det(\mathbf{B} \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}) = \det(\mathbf{B}) \det(\begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}).$$

The area of the original parallelogram has changed by the factor $\det(\mathbf{B})$.

Determinants: the general case

Application for Theorem 4.15 (continued)

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}$. Then $\det(\mathbf{A})$ is the volume of the parallelepiped spanned by the column vectors of $\mathbf{A}$. The columns of $f_{\mathbf{B}}(\mathbf{A}) := \mathbf{BA}$ span a parallelepiped with volume

$$\underbrace{\det(\mathbf{BA})}_{\text{new volume}} = \det(\mathbf{B}) \underbrace{\det(\mathbf{A})}_{\text{original volume}} .$$

$\det(\mathbf{A})$ = Factor for area/volume change under application of $f_{\mathbf{A}}$

$$\text{figure } F \quad \overset{f_{\mathbf{A}}}{\longmapsto} \quad \text{figure } F' = f_{\mathbf{A}}(F)$$

$$\xrightarrow{\text{area/volume} \cdot \det(\mathbf{A})}$$

Determinants: the general case

Theorem 4.14 (Characterization of a regular matrix)

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent:

(i) $\det(\mathbf{A}) \neq 0$.
(ii) The columns of $\mathbf{A}$ form a linearly independent family.
(iii) The rows of $\mathbf{A}$ form a linearly independent family.
(iv) $\operatorname{rank}(\mathbf{A}) = n$.
(v) $\mathbf{A}$ is invertible.
(vi) $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.

All (further) statements of Theorem 3.35 can be added to this list.

Sketch of a proof. The equivalence of statements (ii)-(vi) follows with Theorem 3.35.

(i) $\Rightarrow$ (ii): (Contraposition) We assume that (ii) is false, i.e., the columns of $\mathbf{A}$ are linearly dependent. Hence one column can be written as a linear combination of the other columns. Using properties (1),(2) of the determinant and Corollary 4.8c) leads to $\det(\mathbf{A}) = 0$, i.e, (i) is false.

(v)$\Rightarrow$(i): If $\mathbf{A}$ is invertible, then $1 = \det(\mathbf{I}) = \det(\mathbf{A}\mathbf{A}^{-1}) = \det(\mathbf{A})\det(\mathbf{A}^{-1})$, hence $\det(\mathbf{A}) \neq 0$. $\square$

Determinants: the general case

The determinant can be used for the solution of square systems of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$:

Theorem 4.17 (Cramer's rule)

$$\vdots$$

Attention: Cramer's rule is significantly more work intensive than the Gauss algorithm and hence should not be used.