Discrete Algebraic Structures

WiSe 2025/2026

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Research Group for Theoretical Computer Science

Overview of the course

Part 5: Algebra
- Cryptography
- Error correction
- Quantum Computing

Part 3: Combinatorics and counting
- Runtime analysis of algorithms
- Basics of probabilities

next 2 weeks

Part 4: basic number theory

Part 2: Graphs and relations
- Representing data and relationships between things
- Important basis for the rest of your studies (in particular "Algorithms and Data Structures")

Part 1: introduction to mathematics
- Sets, functions
- Basics in logic and proofs

- Runtime of algorithms:

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              for b in range(n)
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important for your Algorithms course in 2nd year

By comparison: combinatorial proofs

By computations

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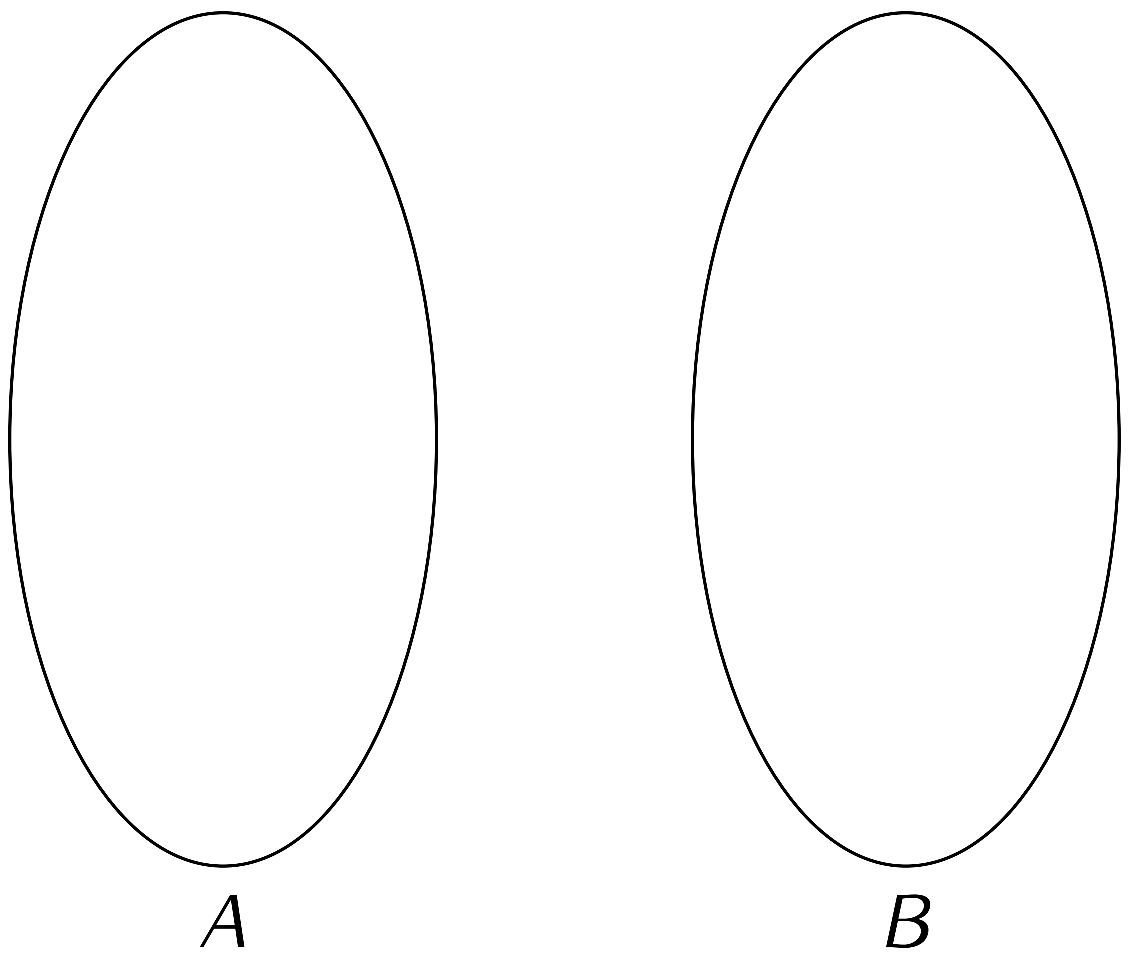
- Size of A already known
- Want to know the size of B



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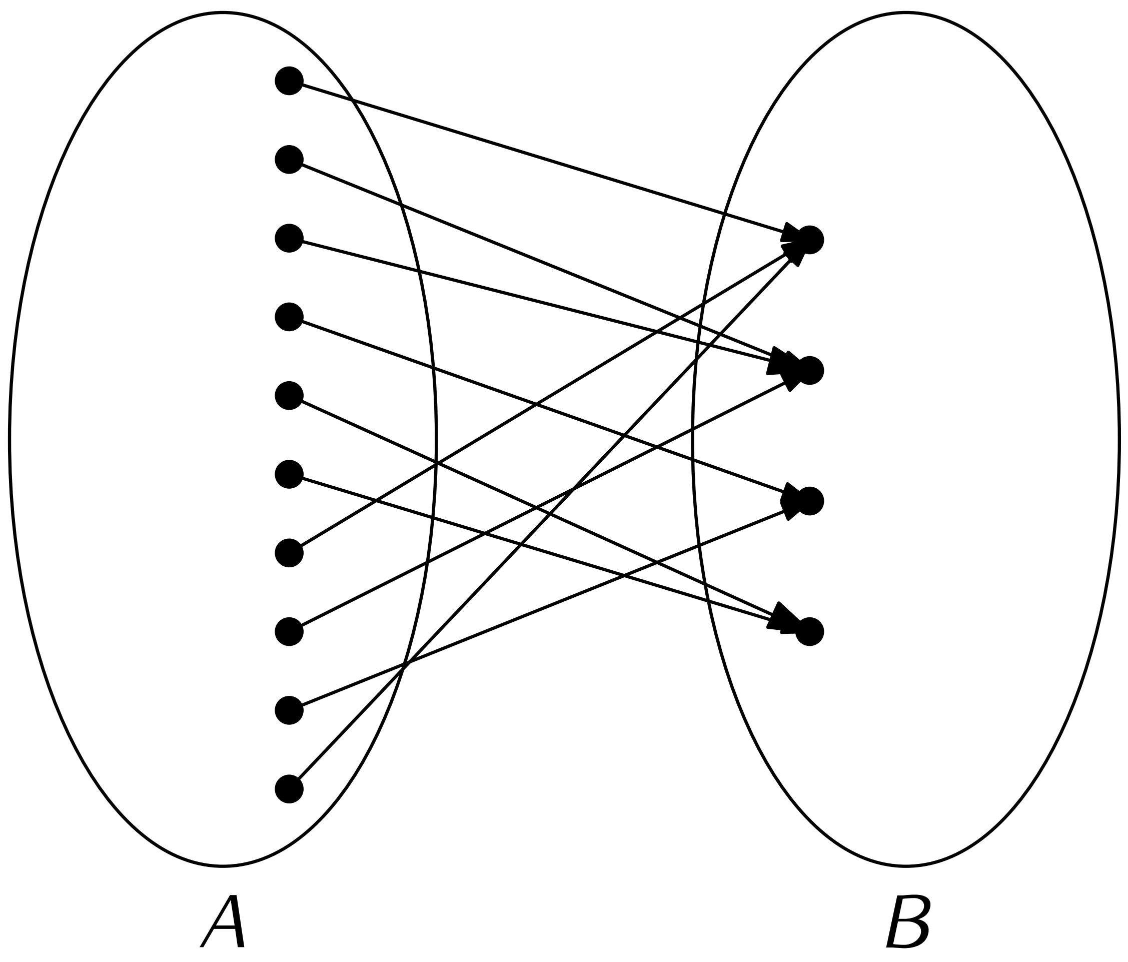

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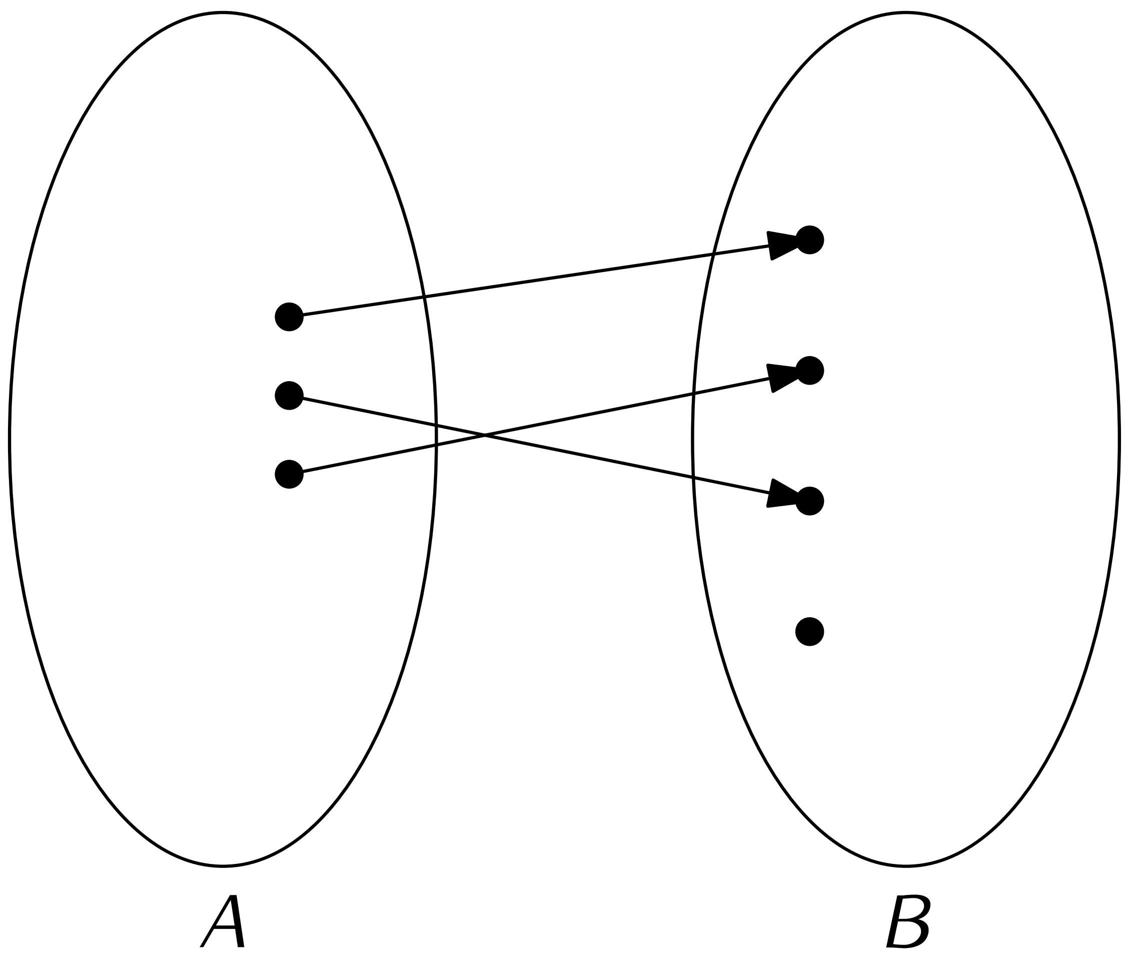

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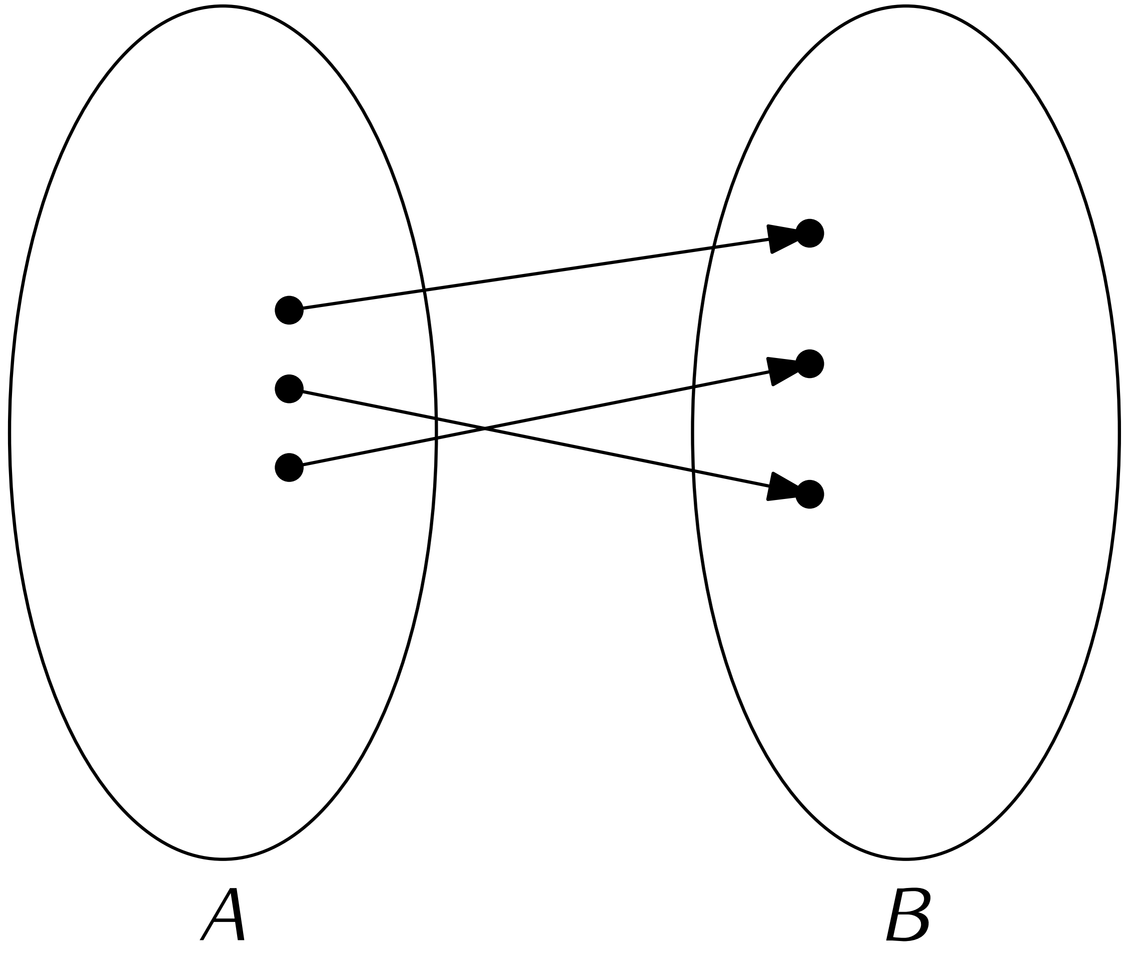

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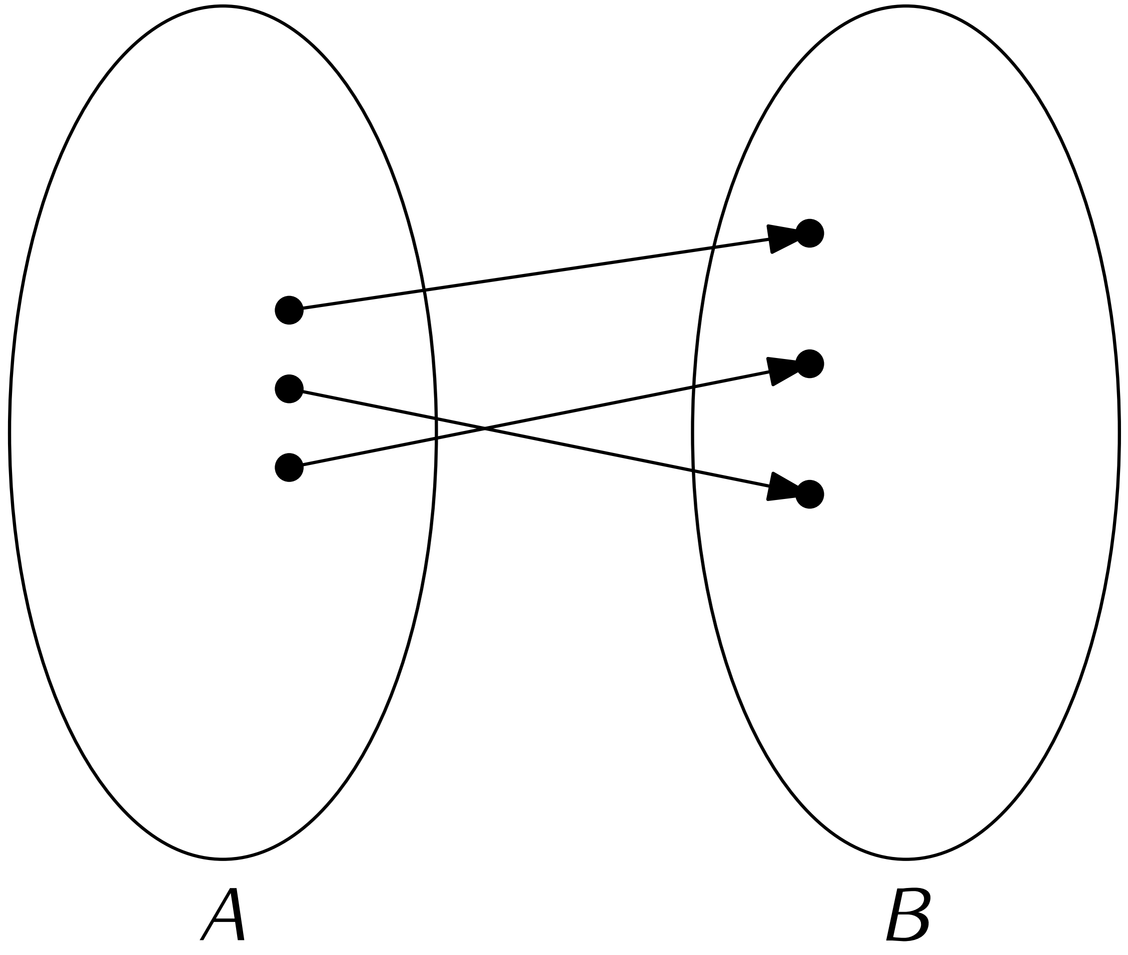

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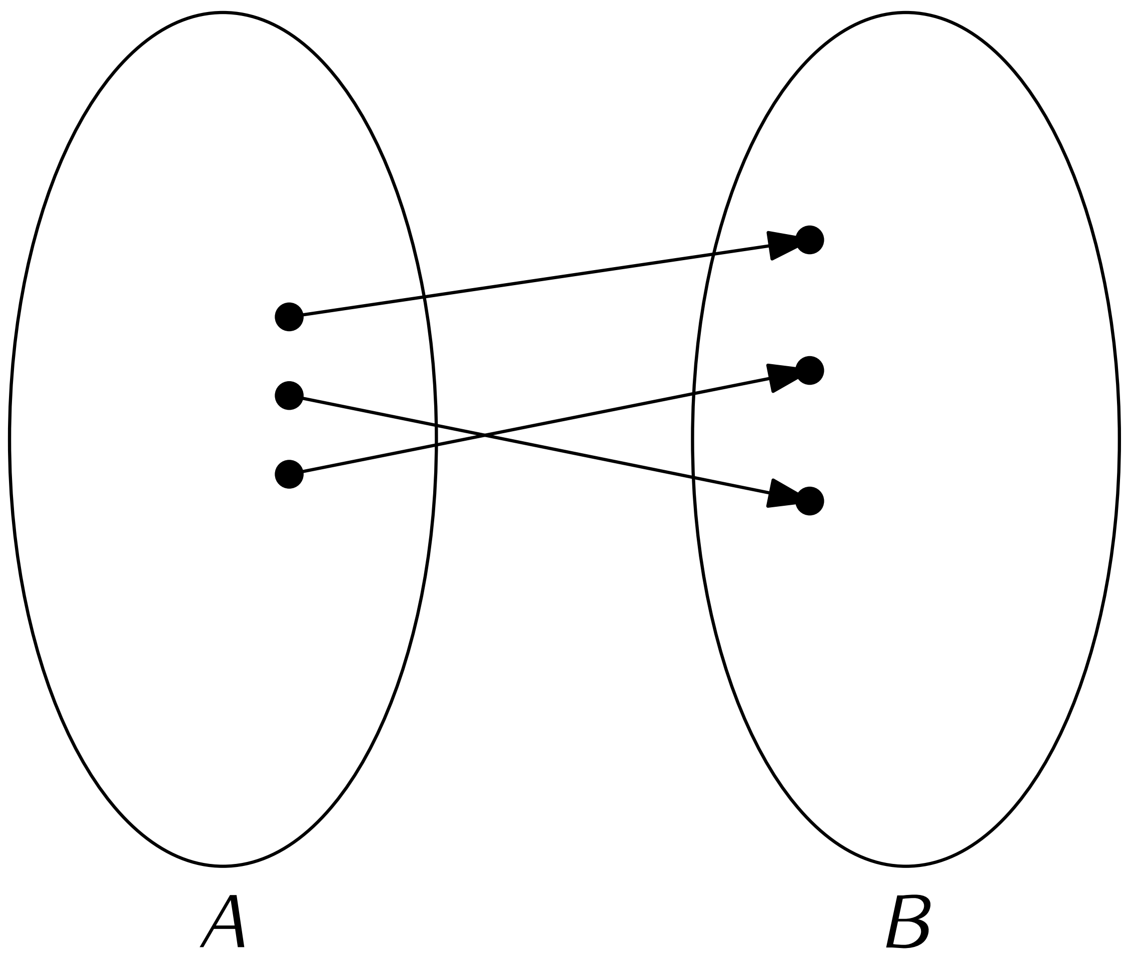

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	Order matters	Order does not matter
Replacement	n^k	$\frac{n!}{k!(n-k)!}$
No replacement	$n^{\underline{k}}$	$\binom{n+k-1}{n-1}$

Cardinalities

Notation. We write $|A|$ for the size of the set A.

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$$A = \{f(1), f(2), f(3), \dots\}$$

$$\boxed{\text{Countable set} = \text{can be represented on a computer}}$$

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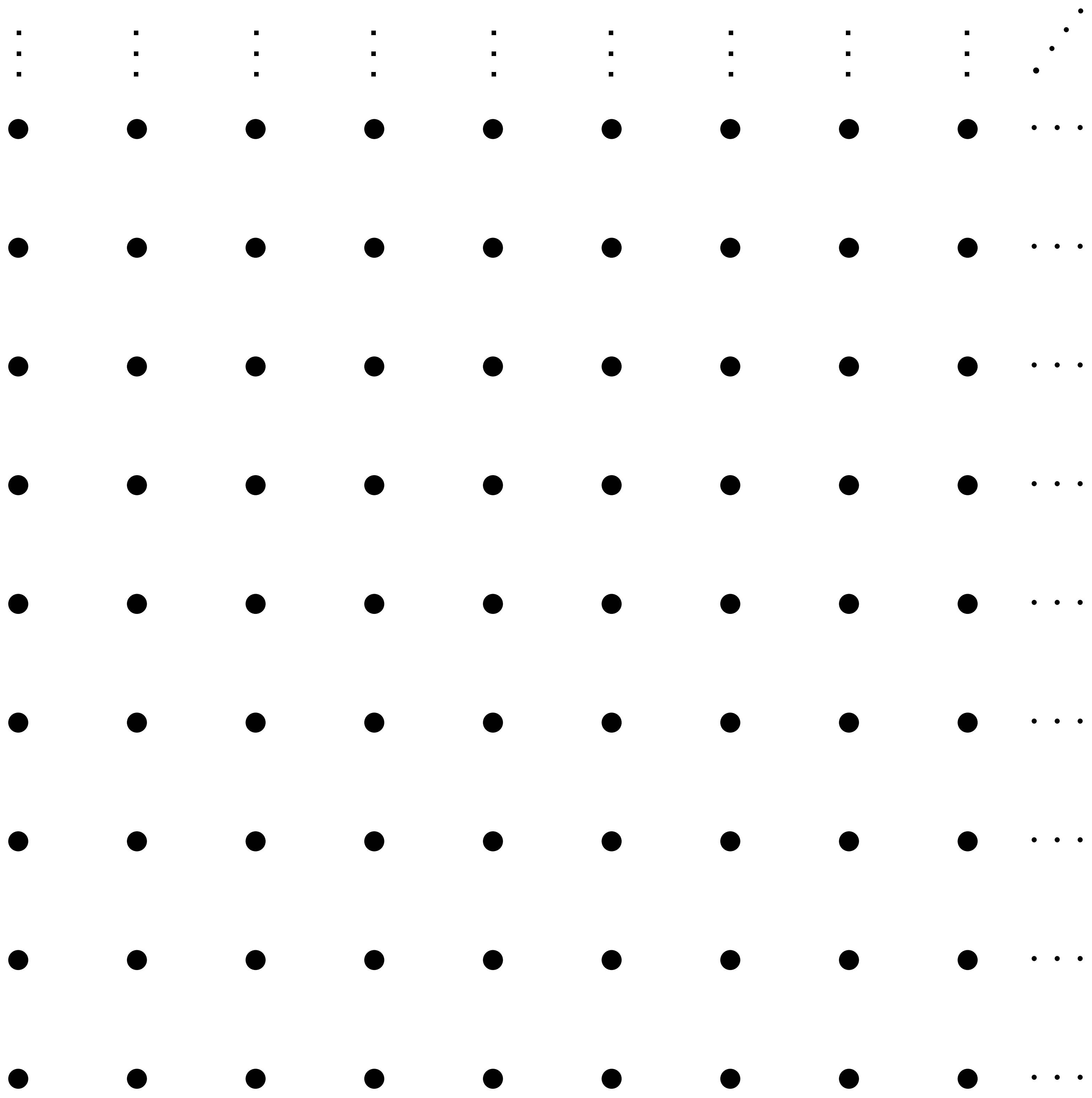
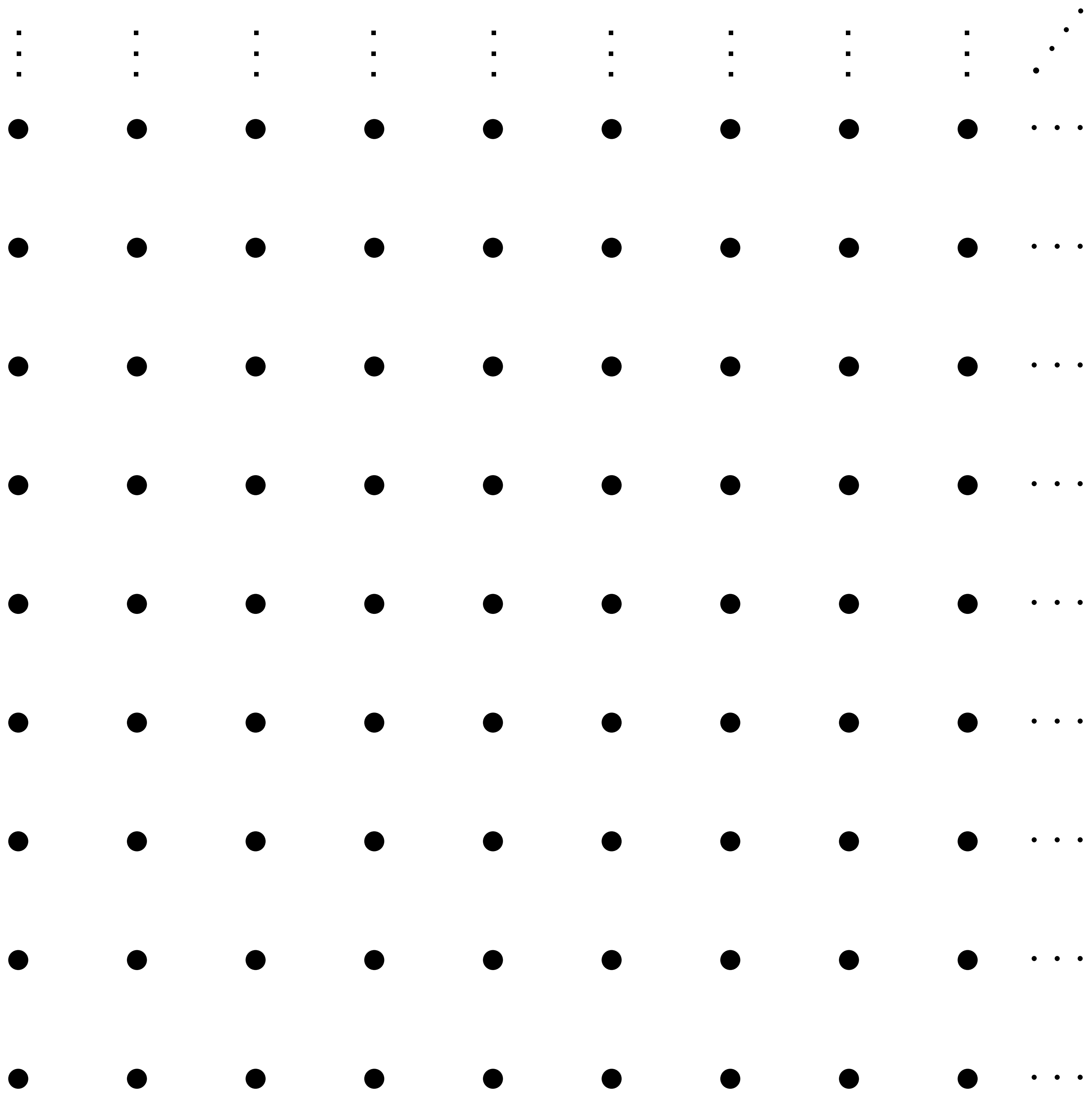
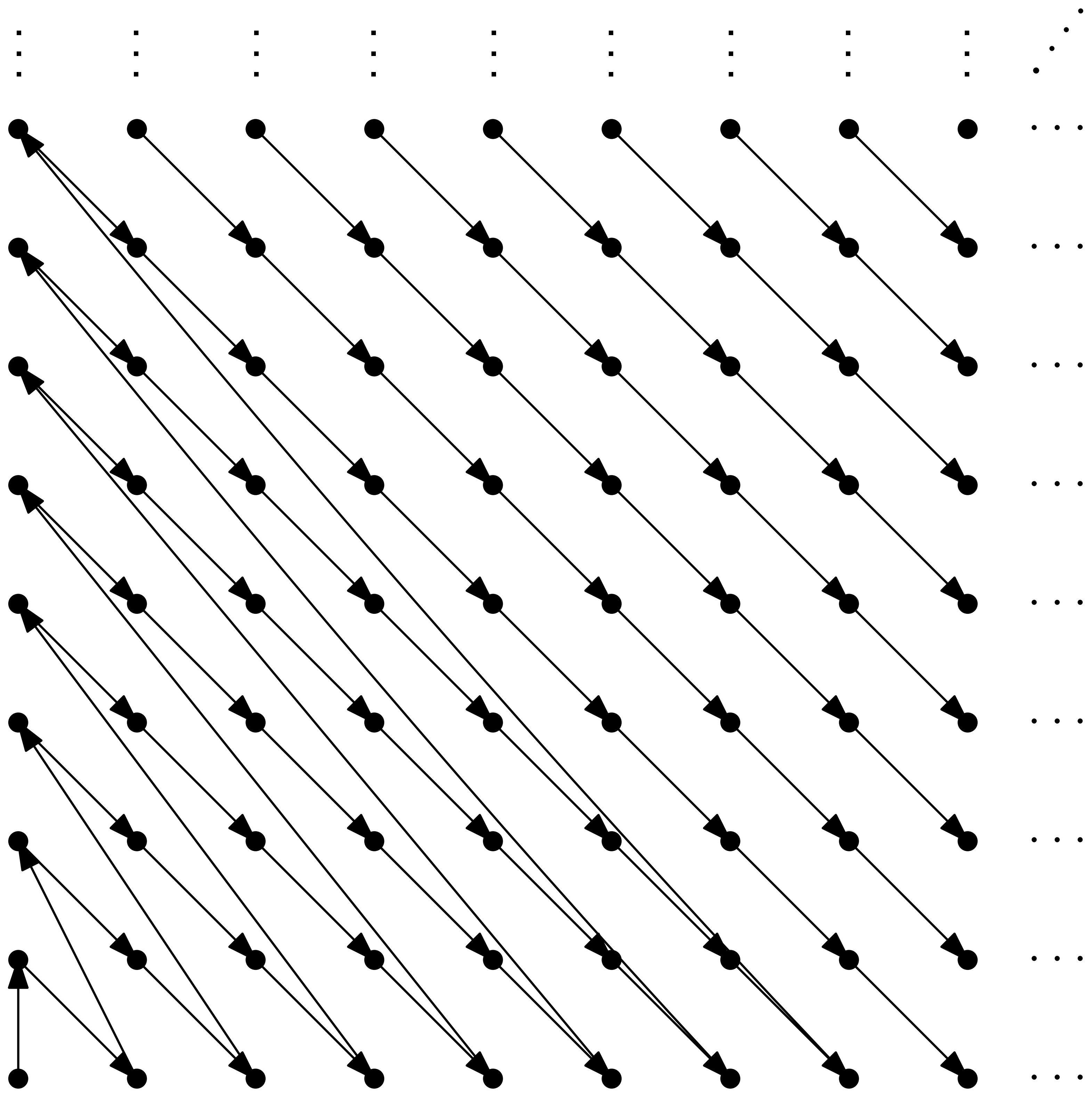


$\mathbb{N}^2$ is countable



$$f : \mathbb{N}^2 \to \mathbb{N}$$

$$f(n, m) =$$



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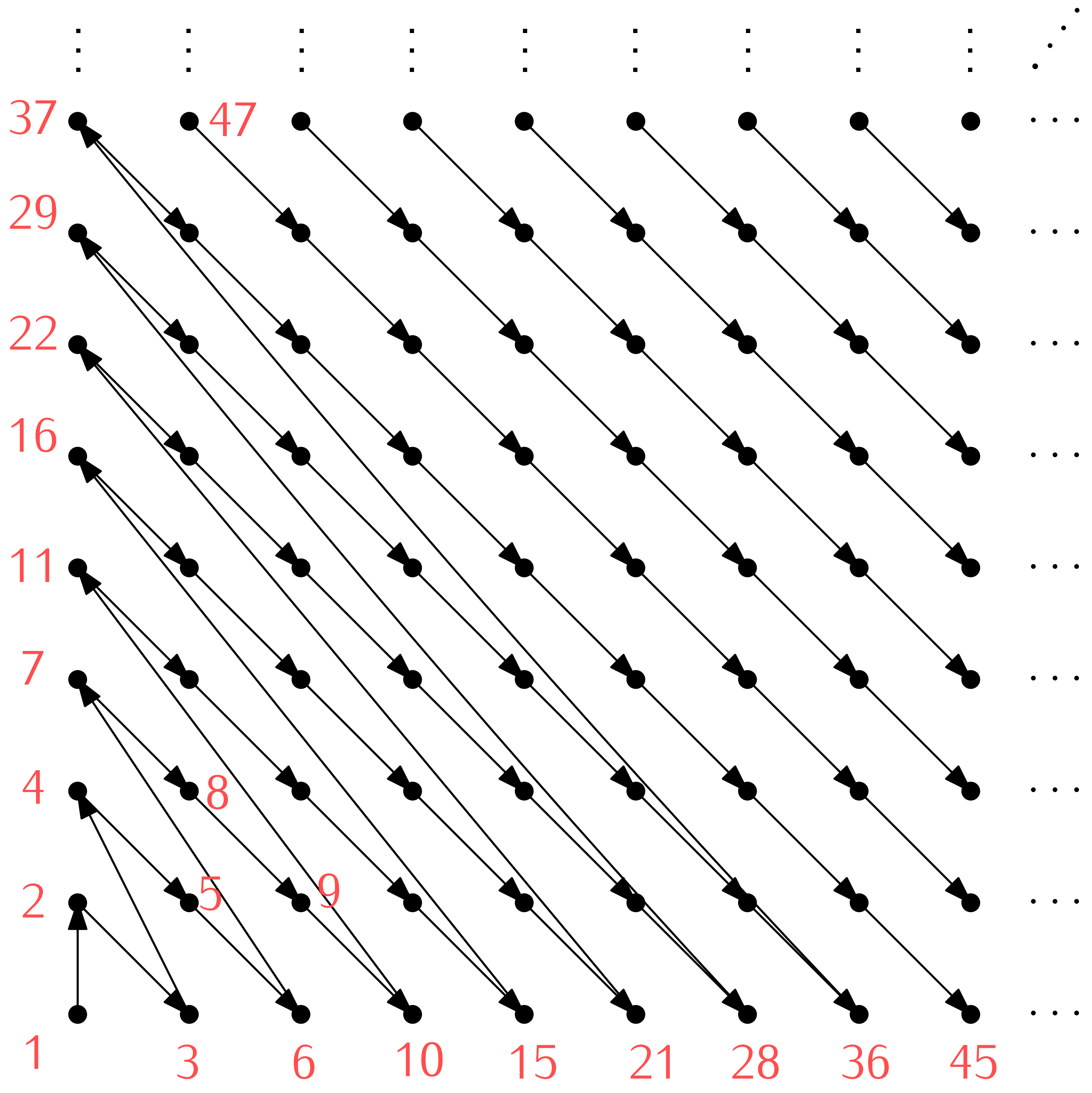
$f(n, m) = ((n + m)^2 - n - 3m + 2)/2$



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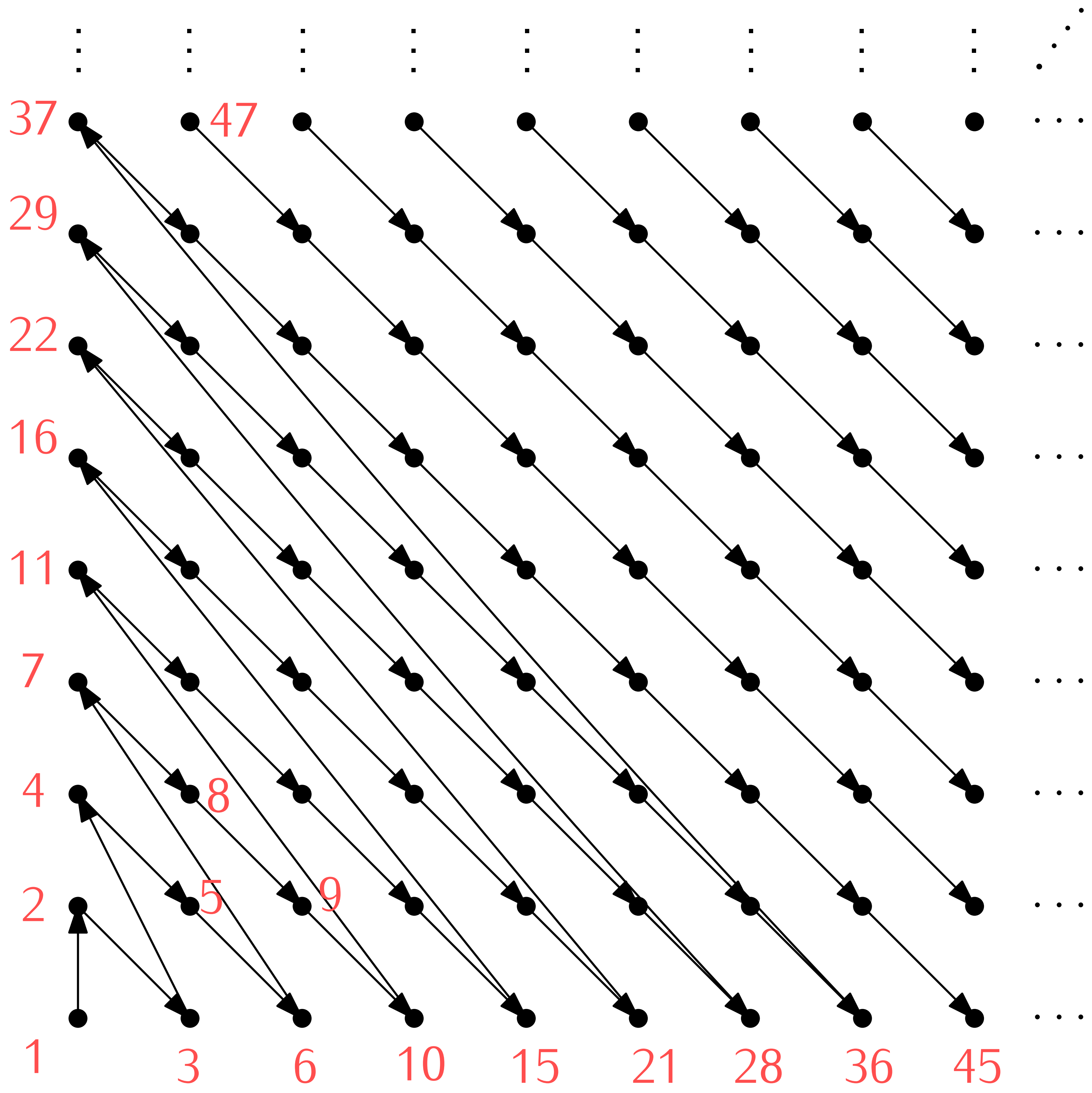
2	1	3
3	3	13
$\vdots$	$\vdots$	$\vdots$



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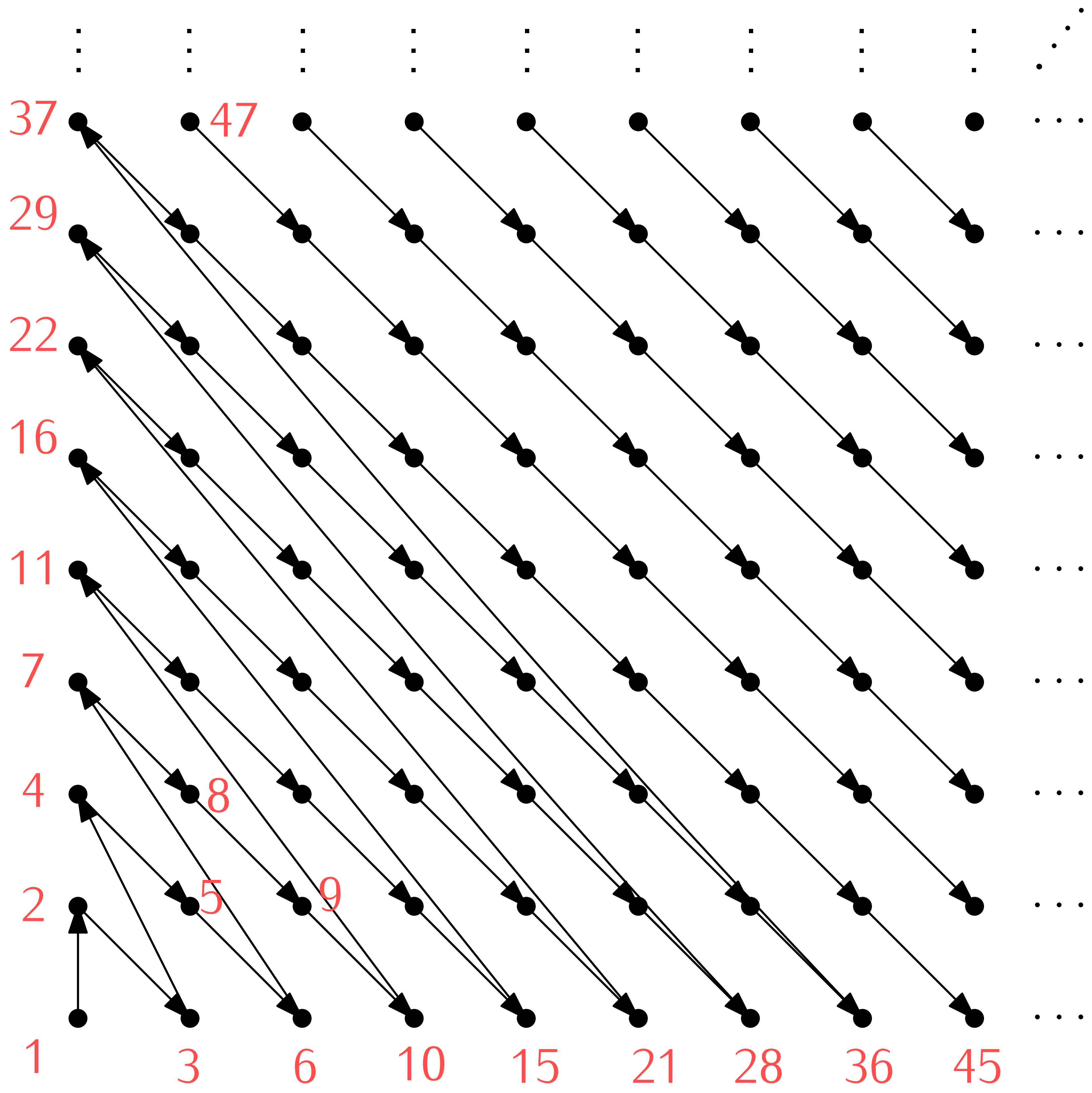

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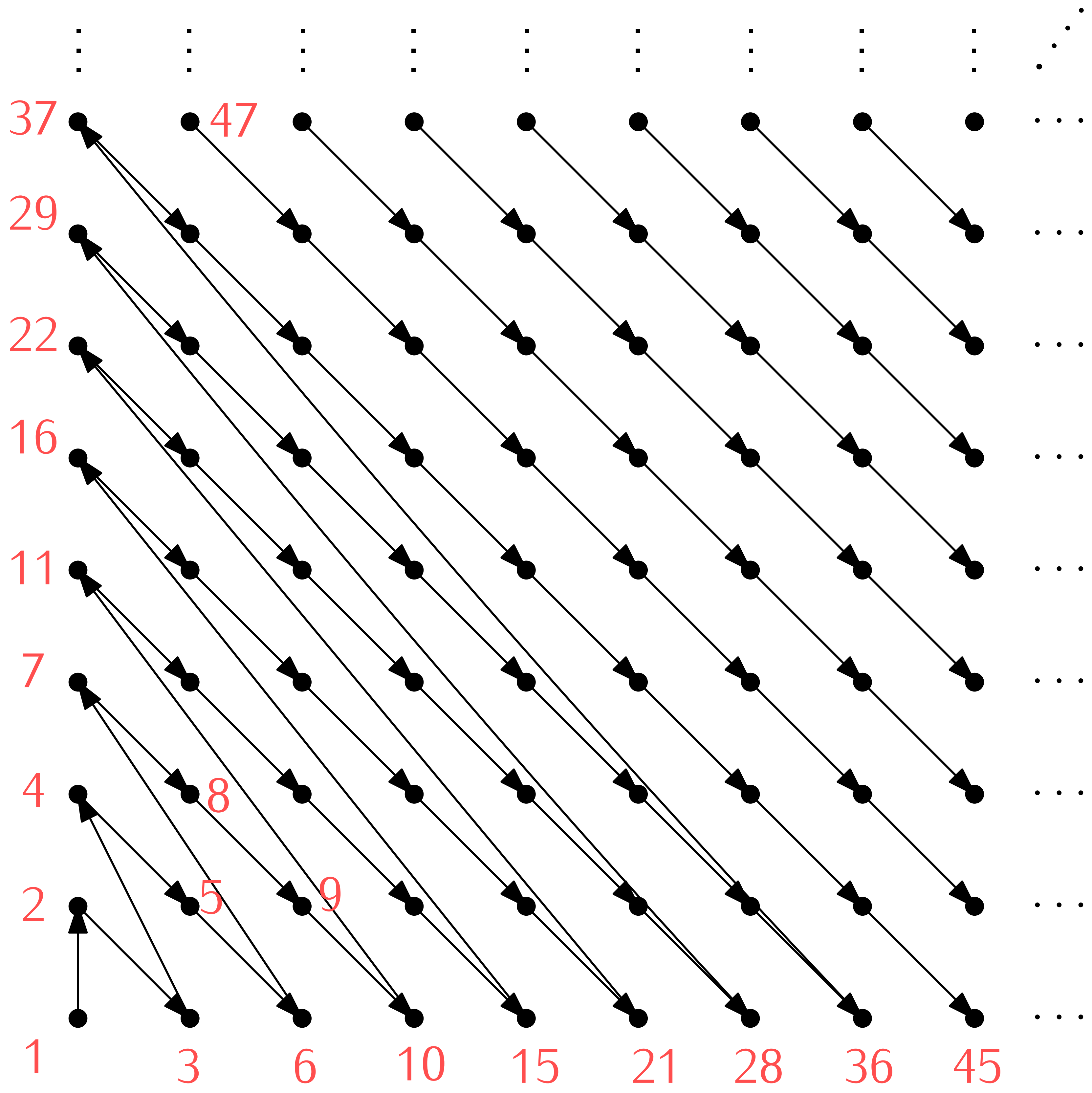

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b) Prove that f is surjective
c) Find $g : \mathbb{N} \to \mathbb{N}^2$ such that
- $g(f(n,m)) = (n,m)$ for all $n, m \in \mathbb{N}$
- $f(g(k)) = k$ for all $k \in \mathbb{N}$

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Proof. By contradiction: suppose $f\colon \mathbb{N} \to \mathbb{R}$ is surjective.

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- Since f is surjective, we must have $x = f(n)$ for some n.
- But x and $f(n)$ differ on the nth digit after decimal point. $\square$

Basic counting rules and combinatorial proofs

(Back to finite sets)

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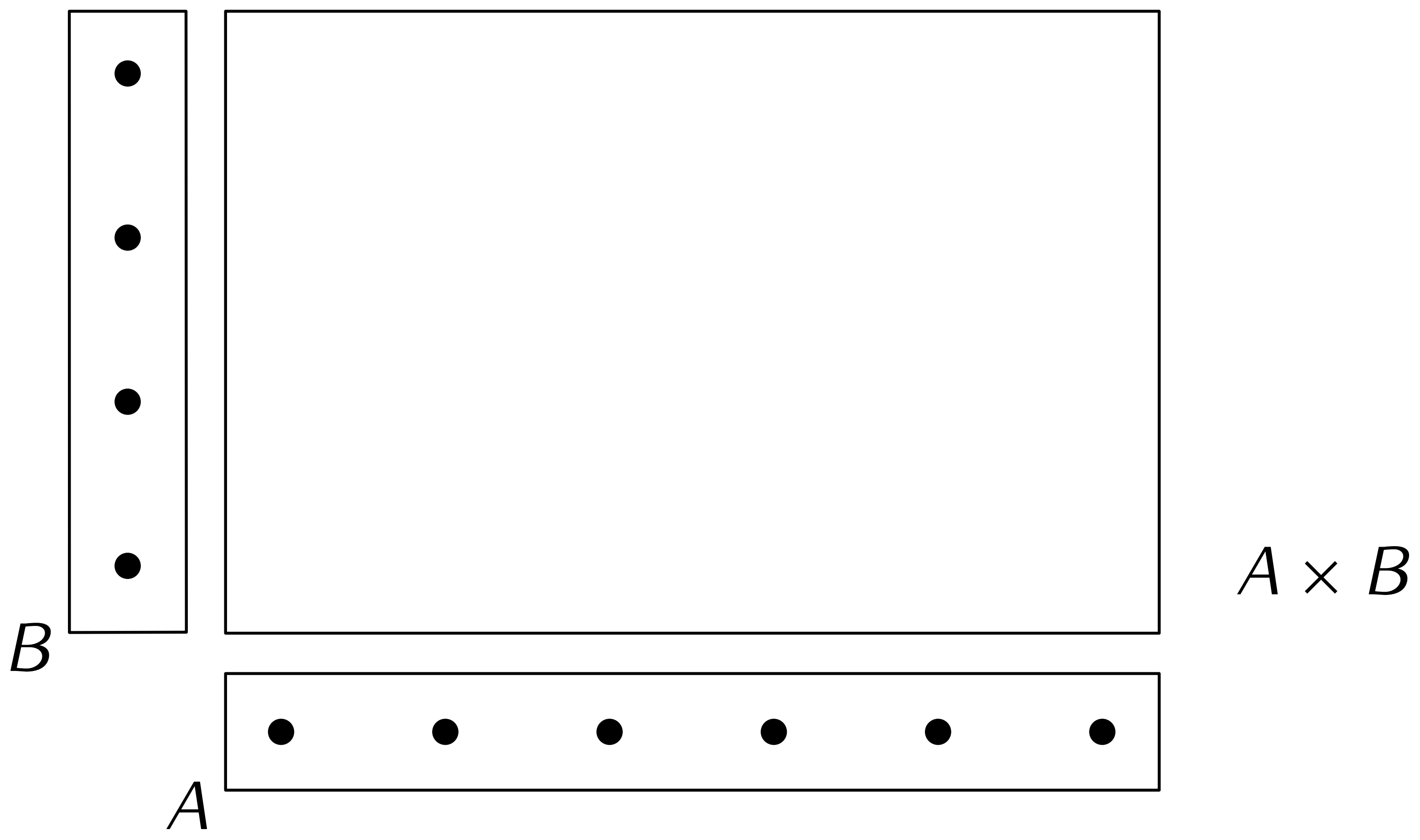

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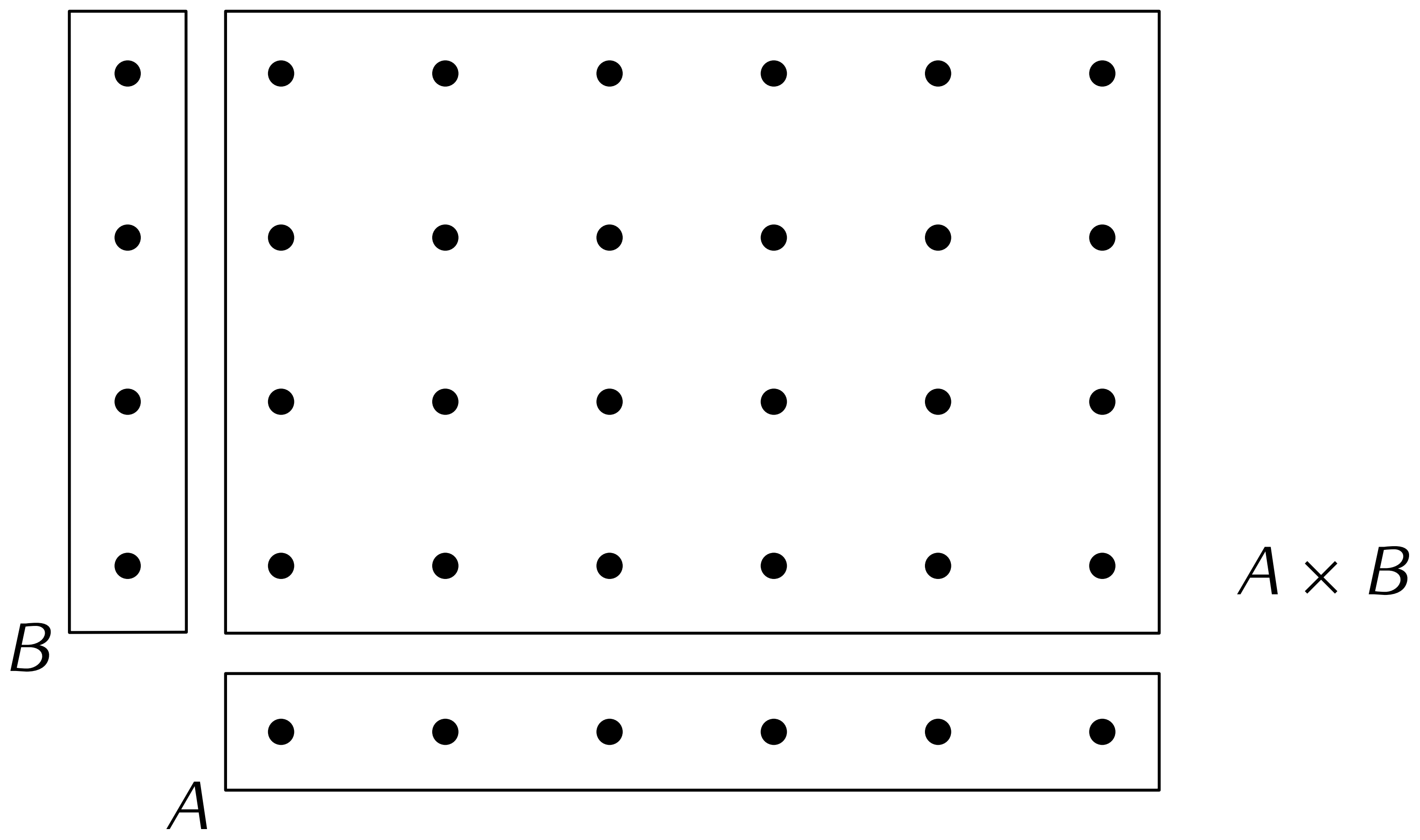

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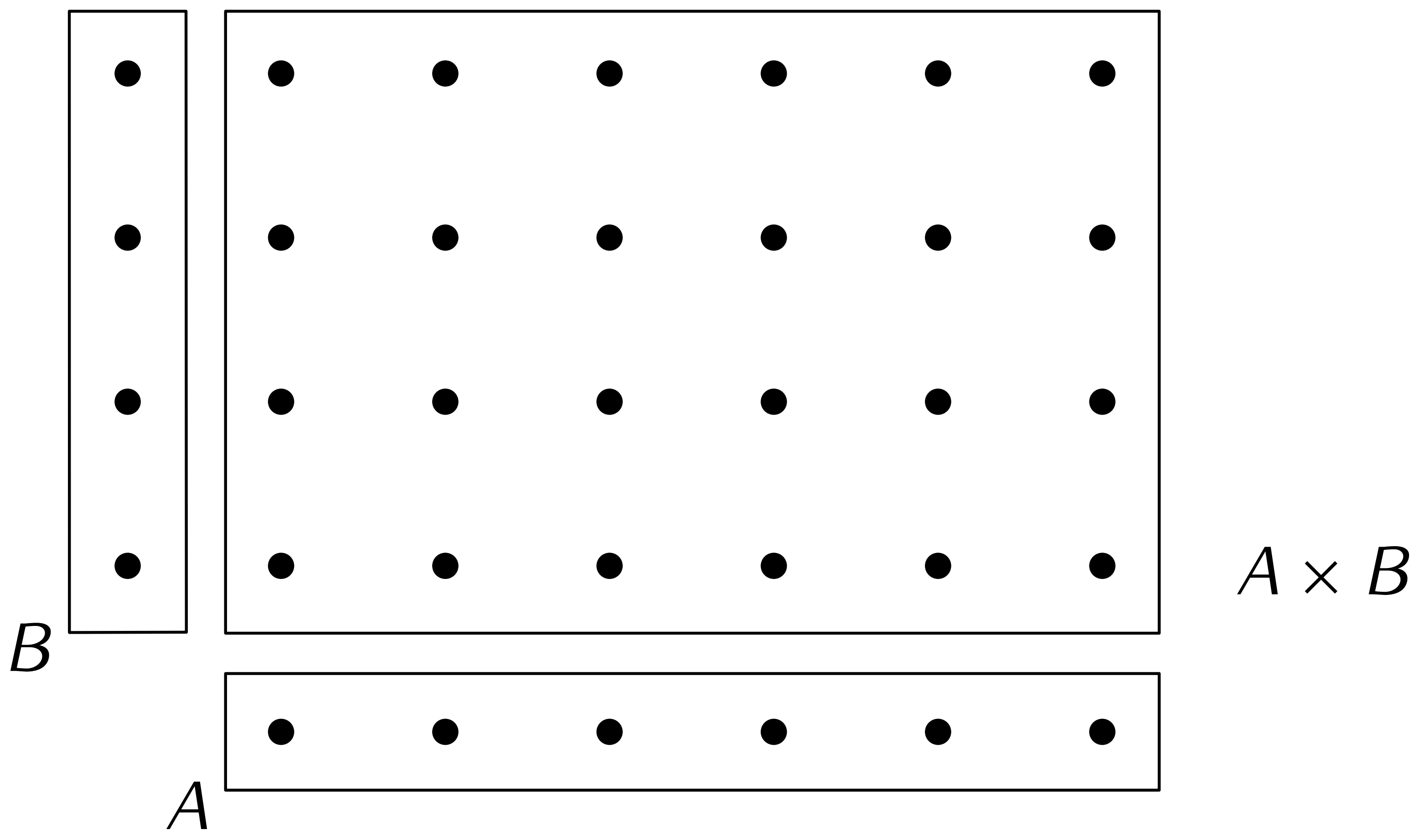

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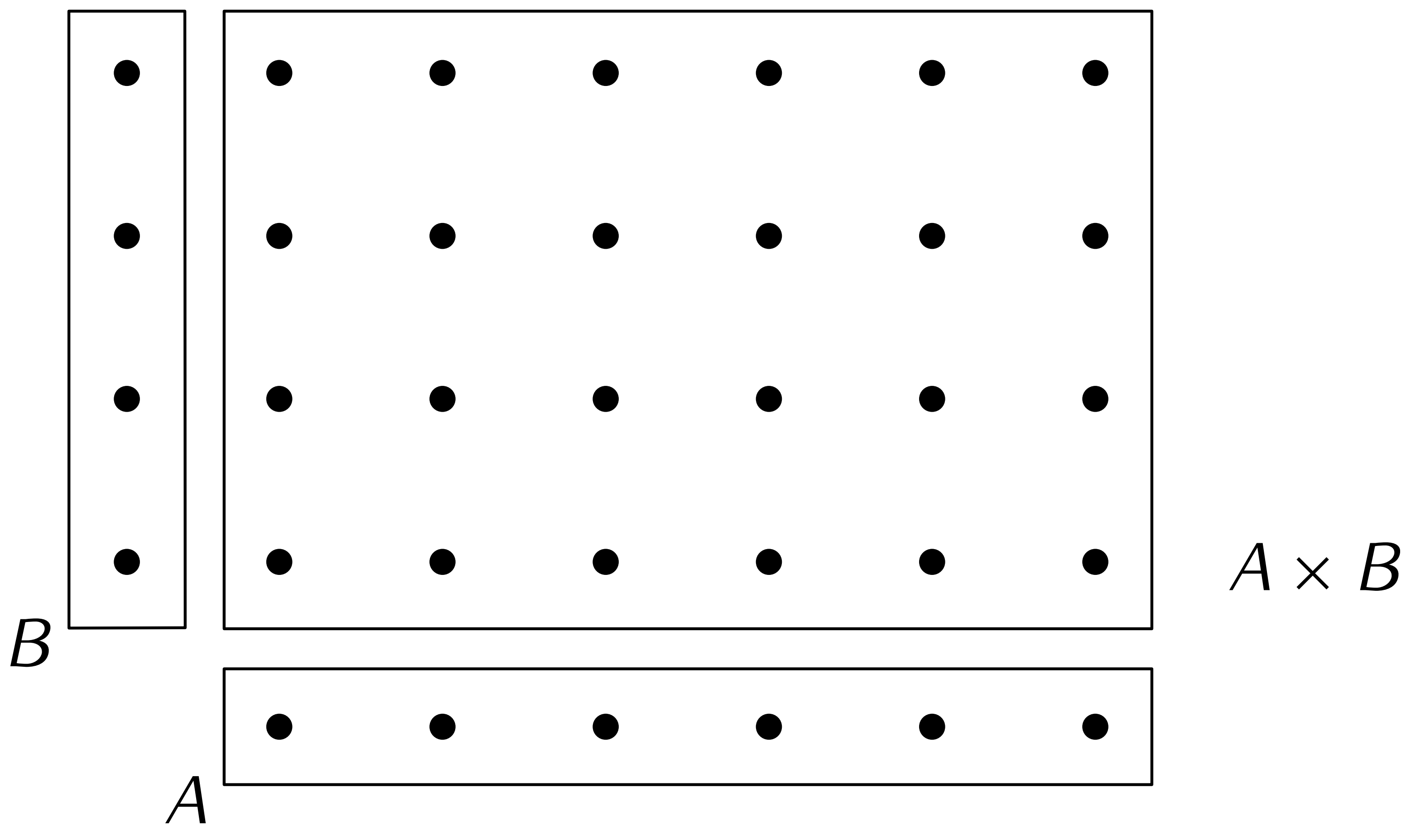

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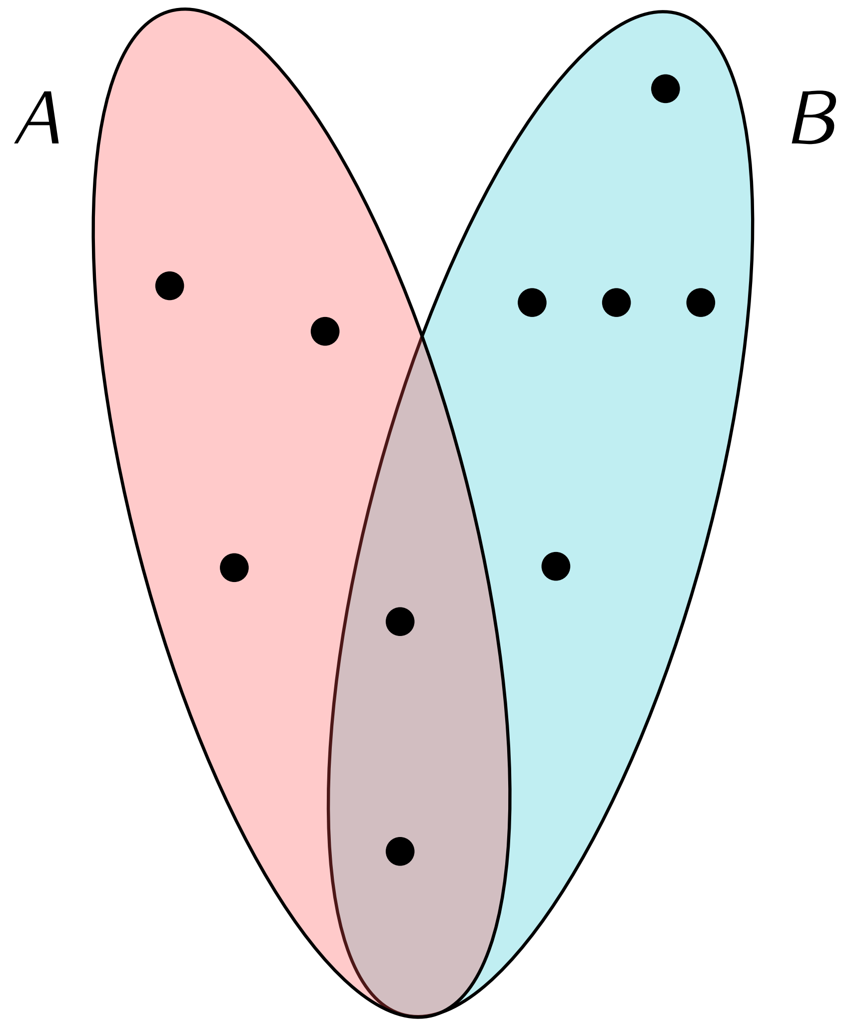

$$|A| =$$
$$|B| =$$
$$|A \cup B| =$$

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Theorem. Let A, B be finite sets. Then:

- $|A \times B| = |A| \times |B|$
- $|A \cup B| = |A| + |B| - |A \cap B|$.

In particular $|A \cup B| \leq |A| + |B|$ and $|A \cup B| = |A| + |B|$ if $A \cap B = \emptyset$.

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Examples. We obtain from the basic rules:

- Number of binary strings of length 2:
- Number of binary strings of length 3:
- Number of binary strings of length n:
- Number of binary strings of length $\leq n$:

Definition. Let $p \leq q$ and $a_p, \ldots, a_q$ in a set where $+$ makes sense. Then $\sum_{i=p}^{q} a_i = a_p + a_{p+1} + \cdots + a_q$.

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Similar notation exists for many other operators:

- $\bigcup_{i=p}^{q} A_i = A_p \cup A_{p+1} \cup \cdots \cup A_q$

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def union_symbol(A,p,q):
  result = set() # initialize to be the empty set
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  Empty product (when $q < p$) is 1.

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Theorem. Let A, B be finite sets. Let $f\colon A \to B$.

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- symbol $+$ or $\sum$: union of disjoint sets
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How do we find a function $f\colon A \to B$?
Draw a picture and work from there.

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direct translation of the equality!

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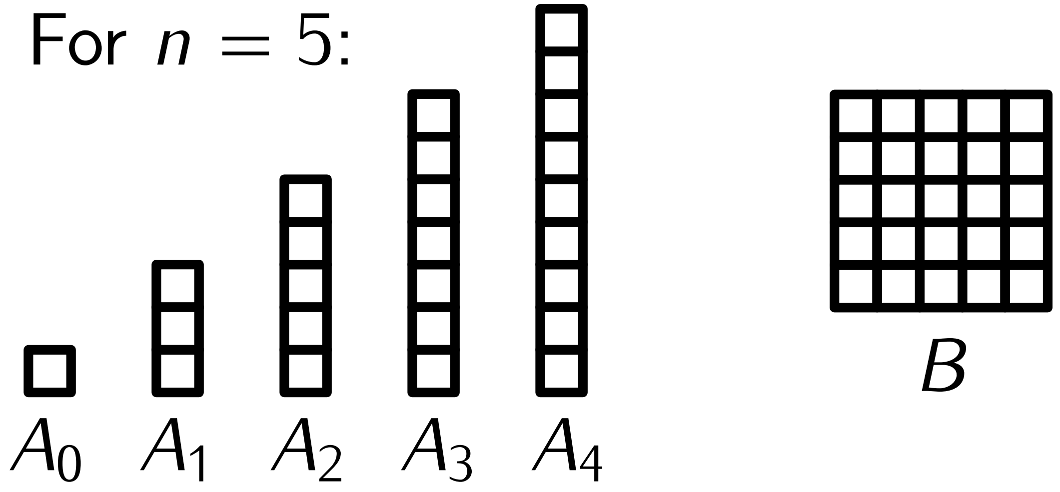

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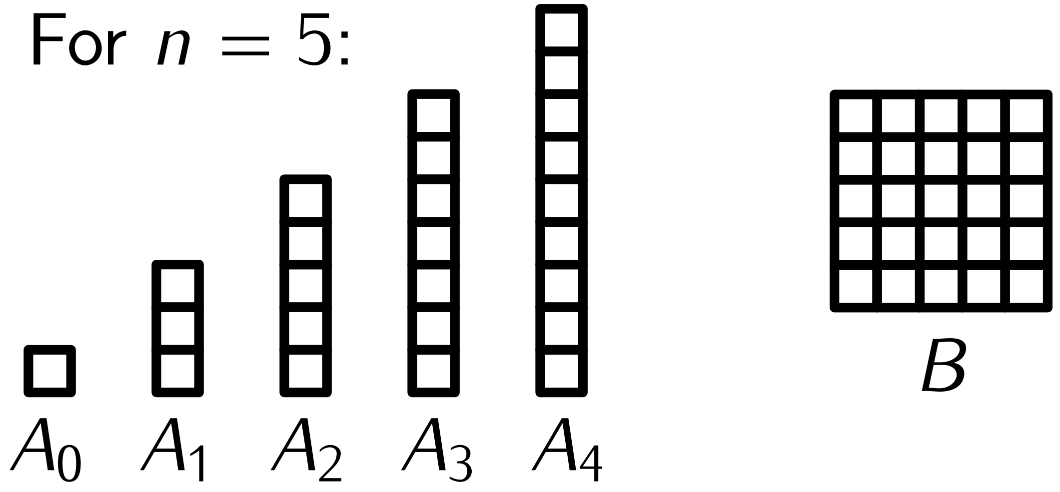

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Example of a combinatorial proof

Antoine Wiehe

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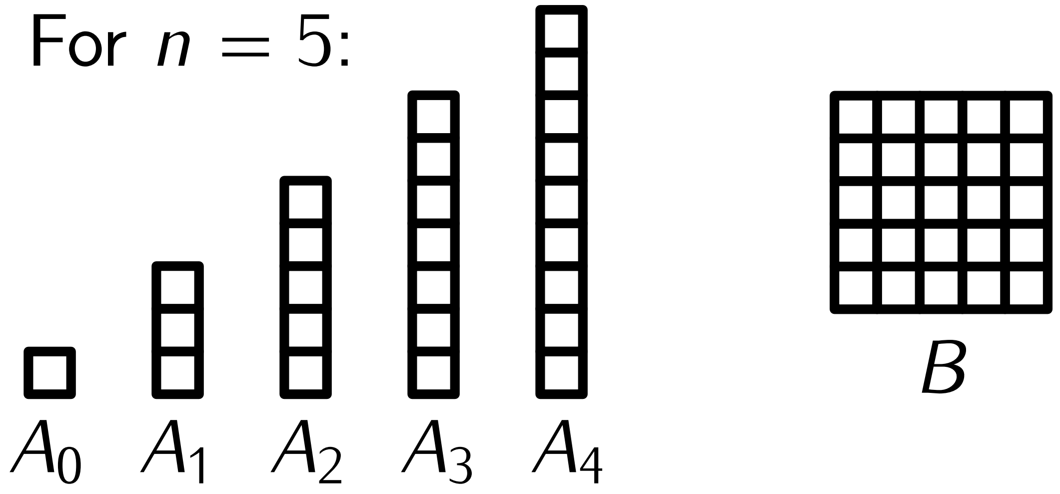

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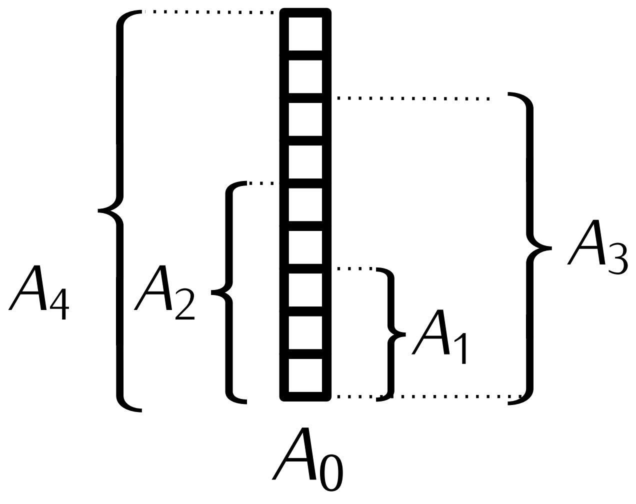

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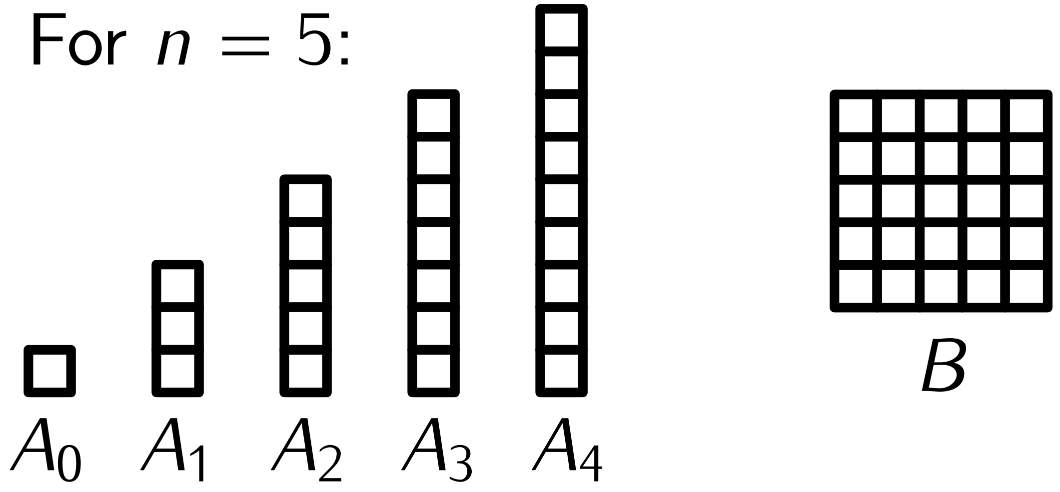

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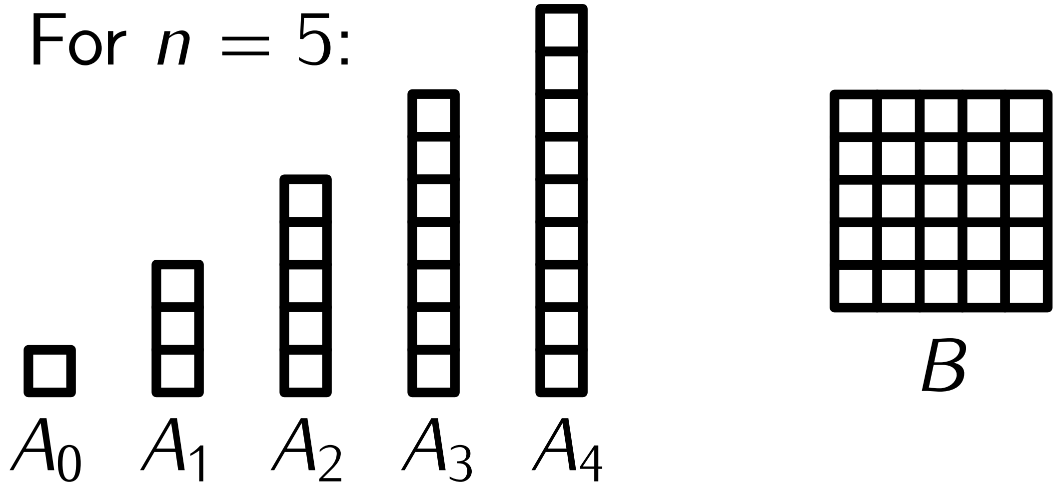

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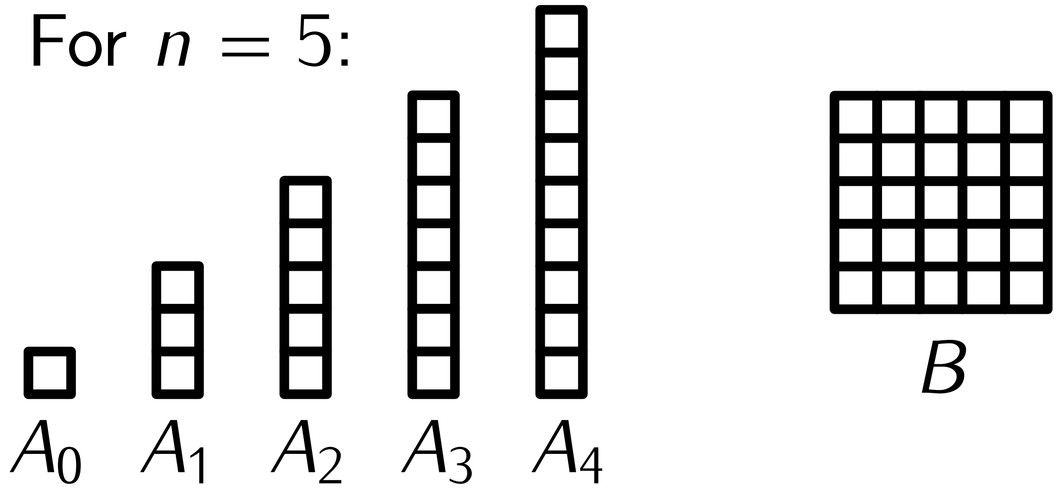

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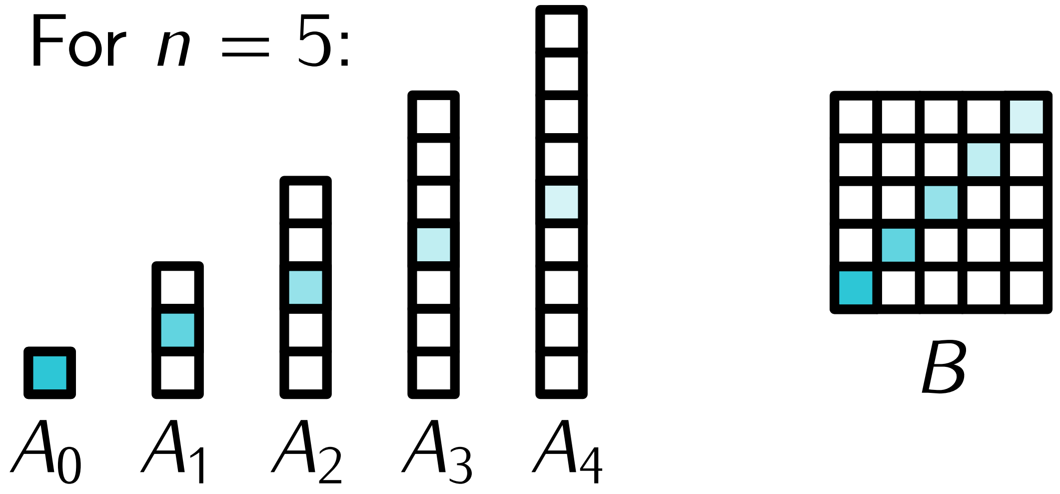

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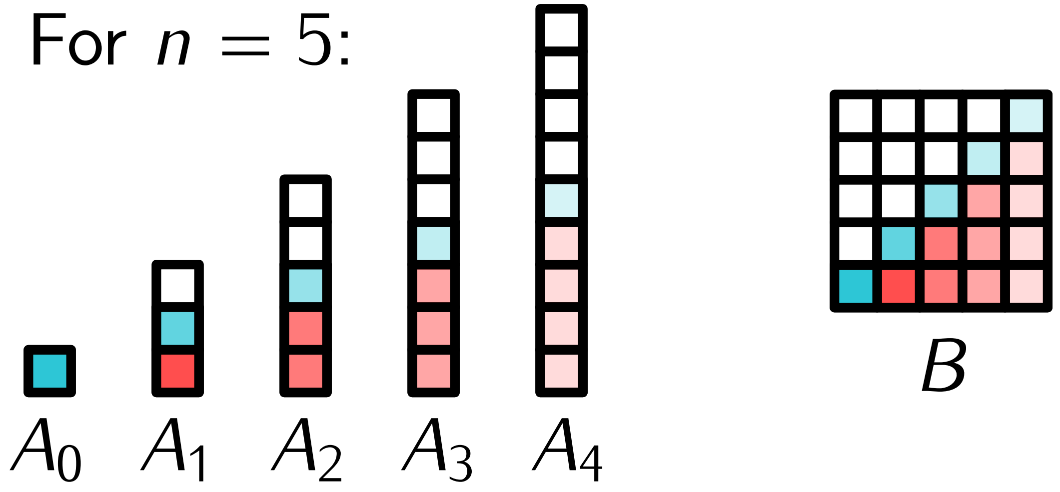

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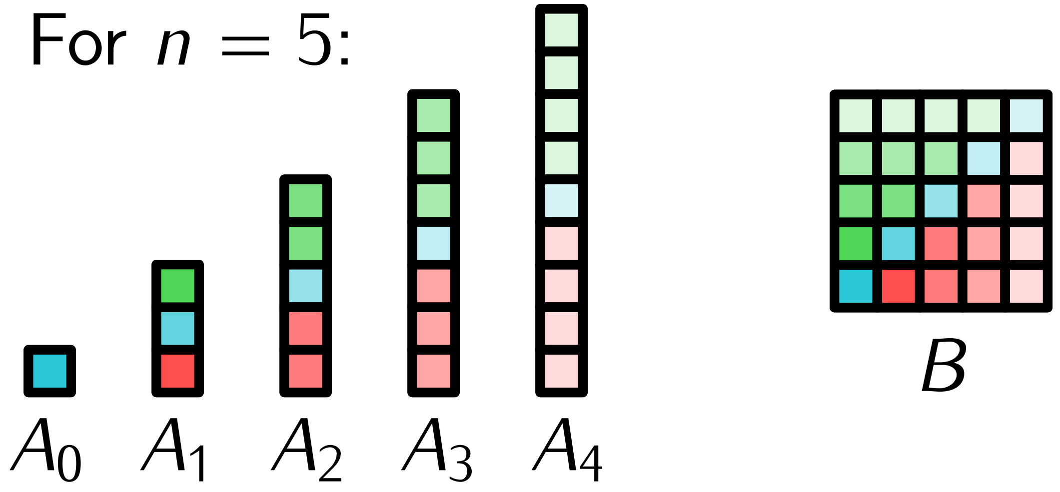

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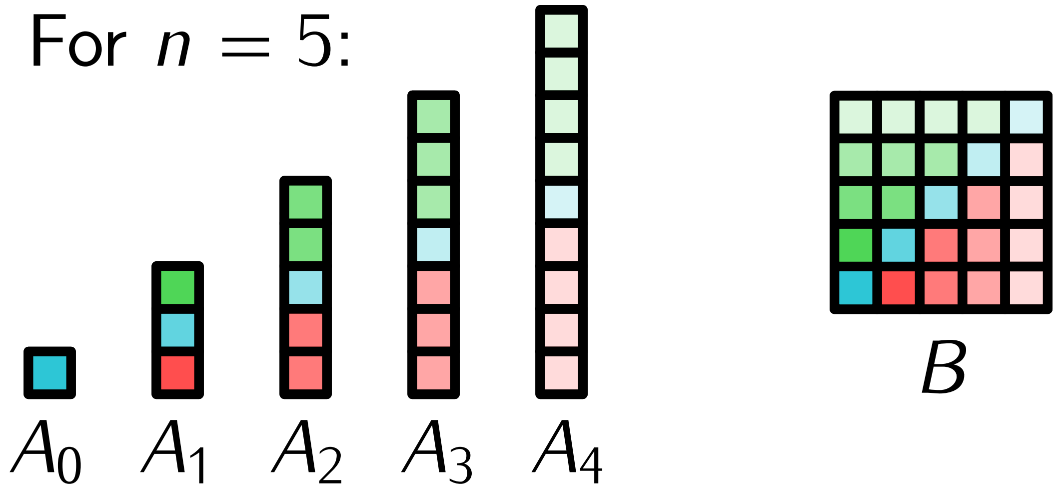

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Exercise: prove that this function is a bijection.

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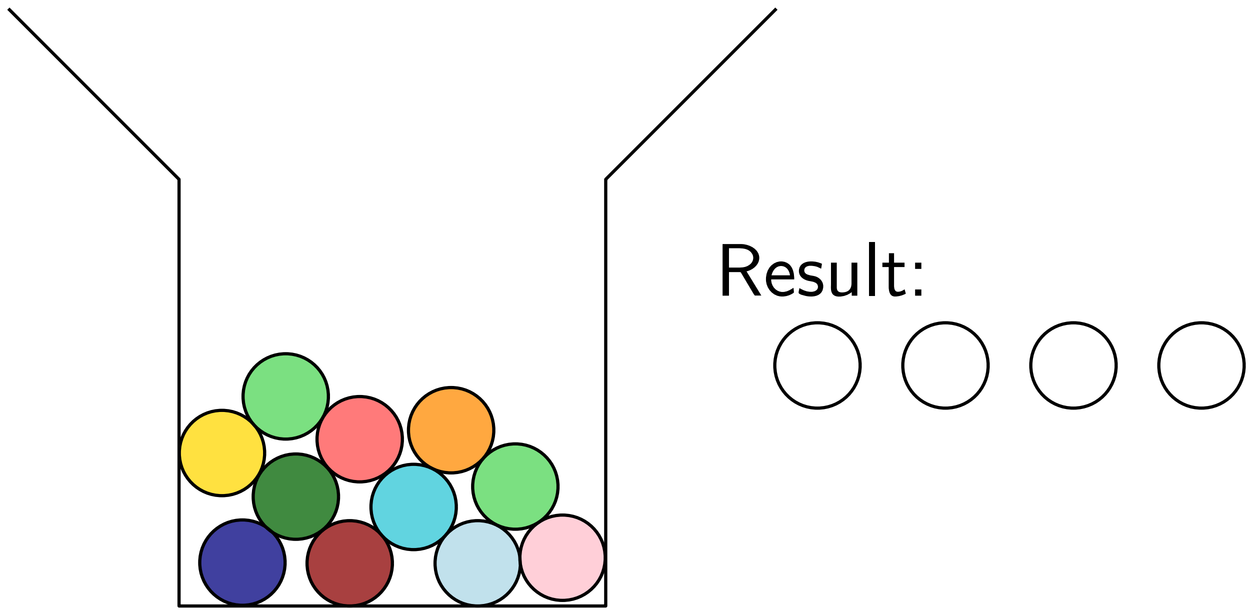
Drawing from a set

Many objects can be seen as the result of the following procedure:

1. We start with a container A that has n elements
2. We extract (= draw) elements $a_1, \ldots, a_k$ from A



Drawing = taking objects from a box

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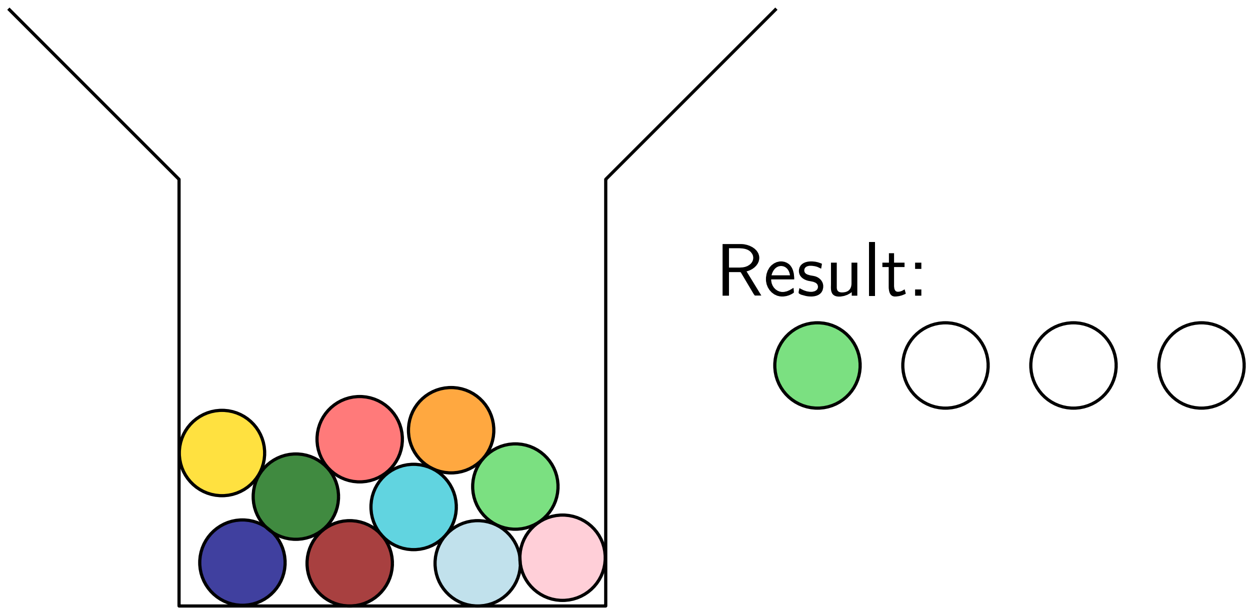

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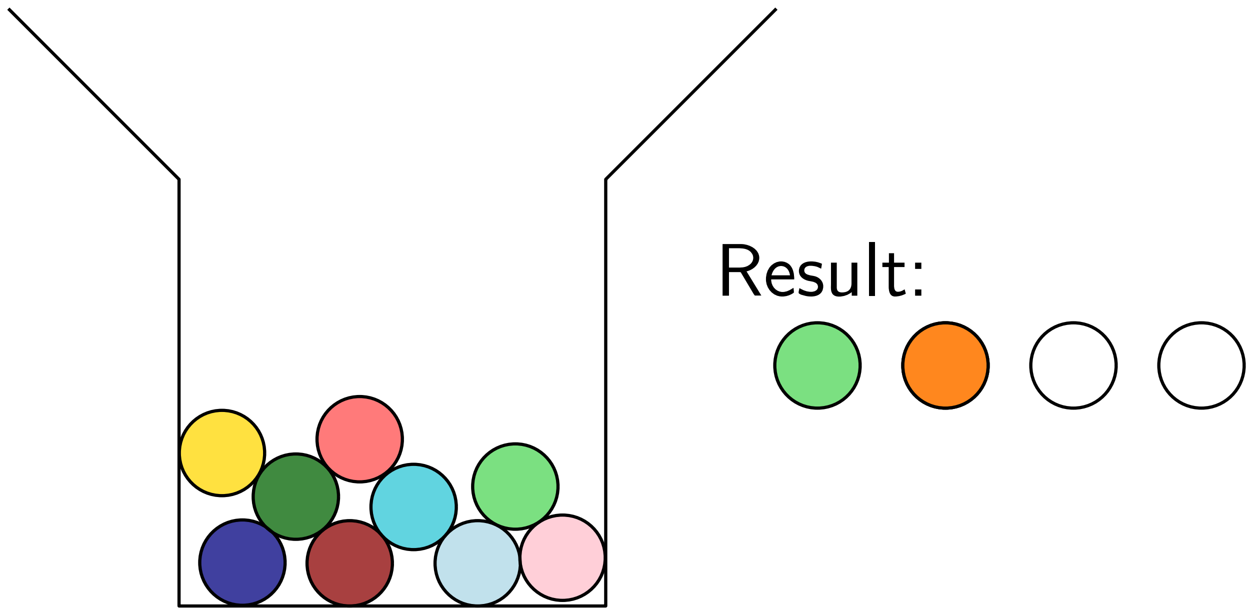

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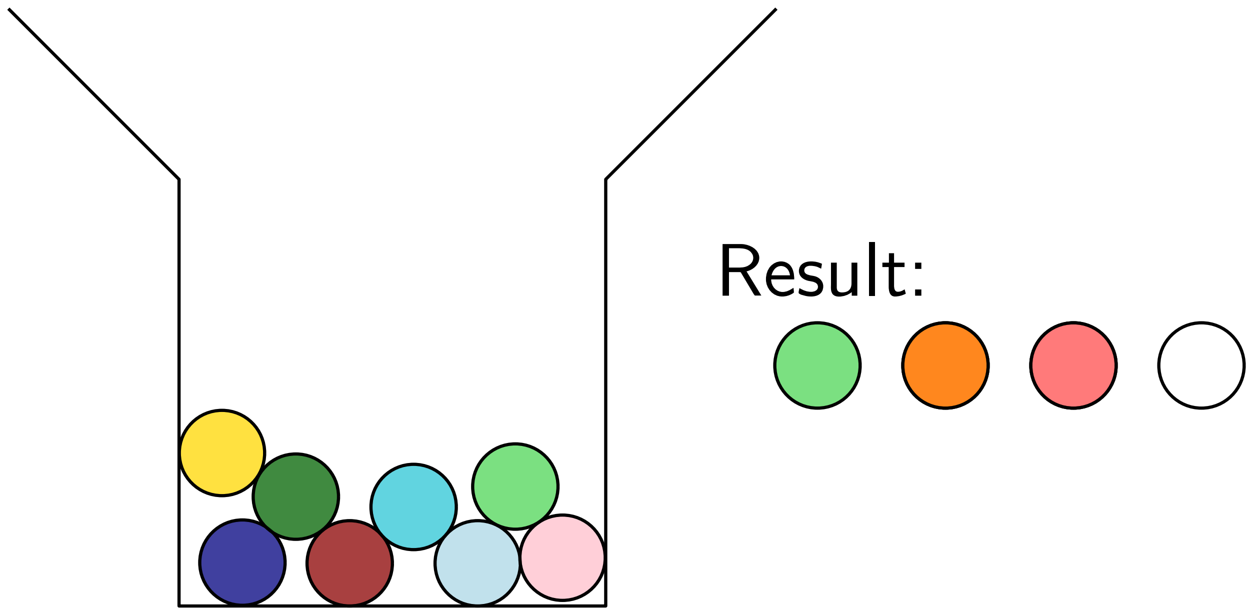

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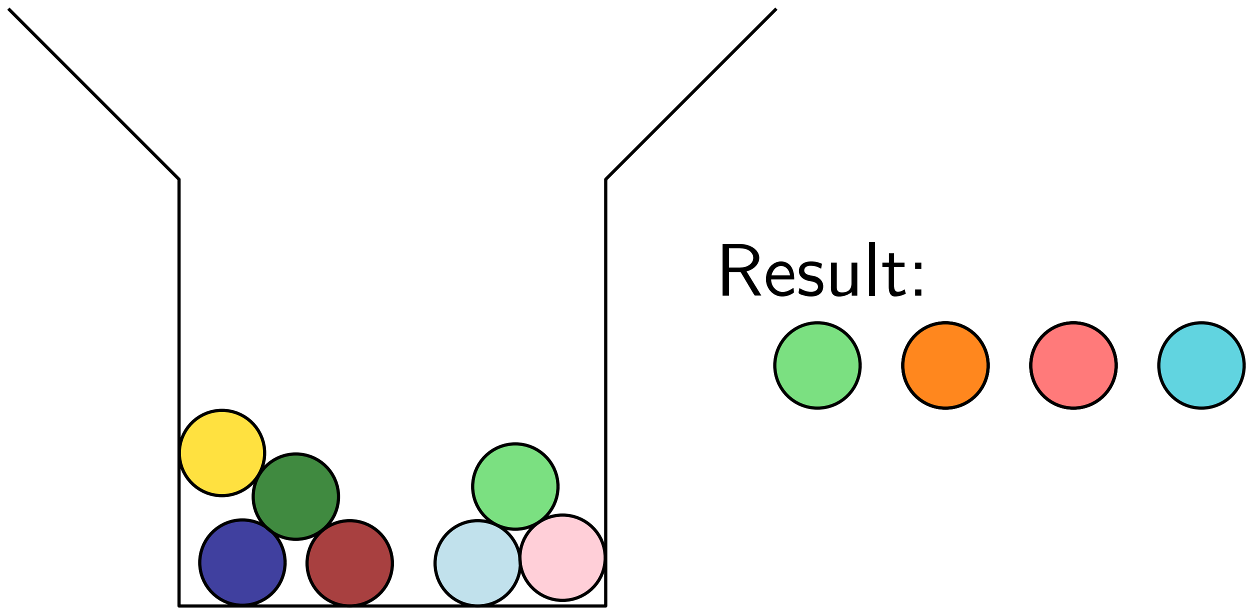

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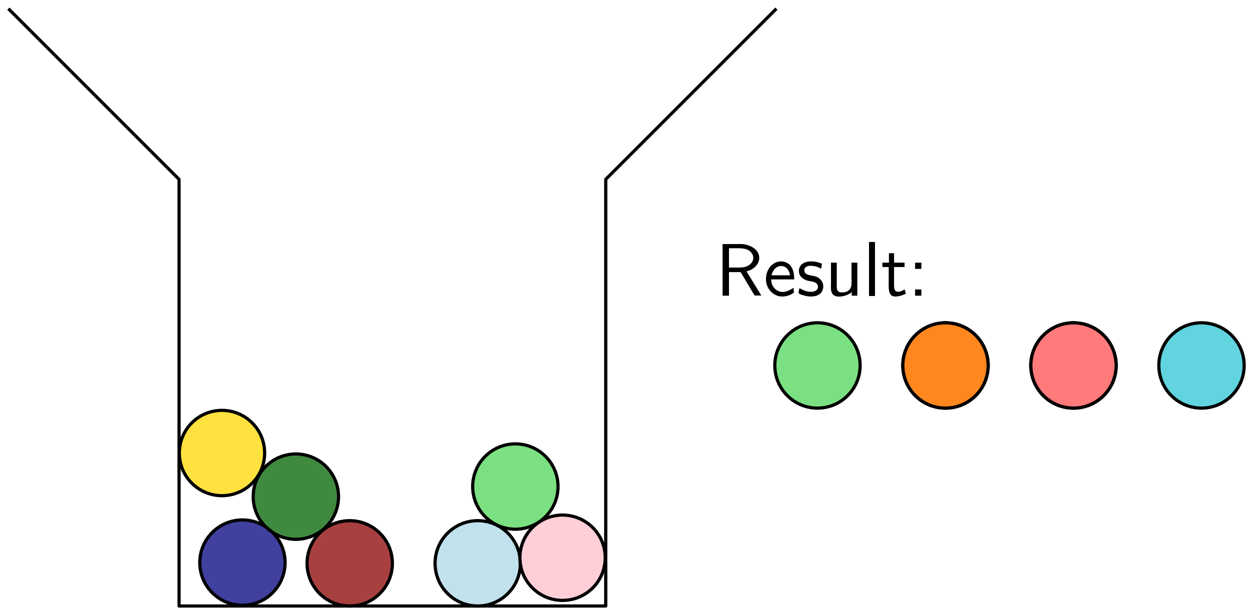

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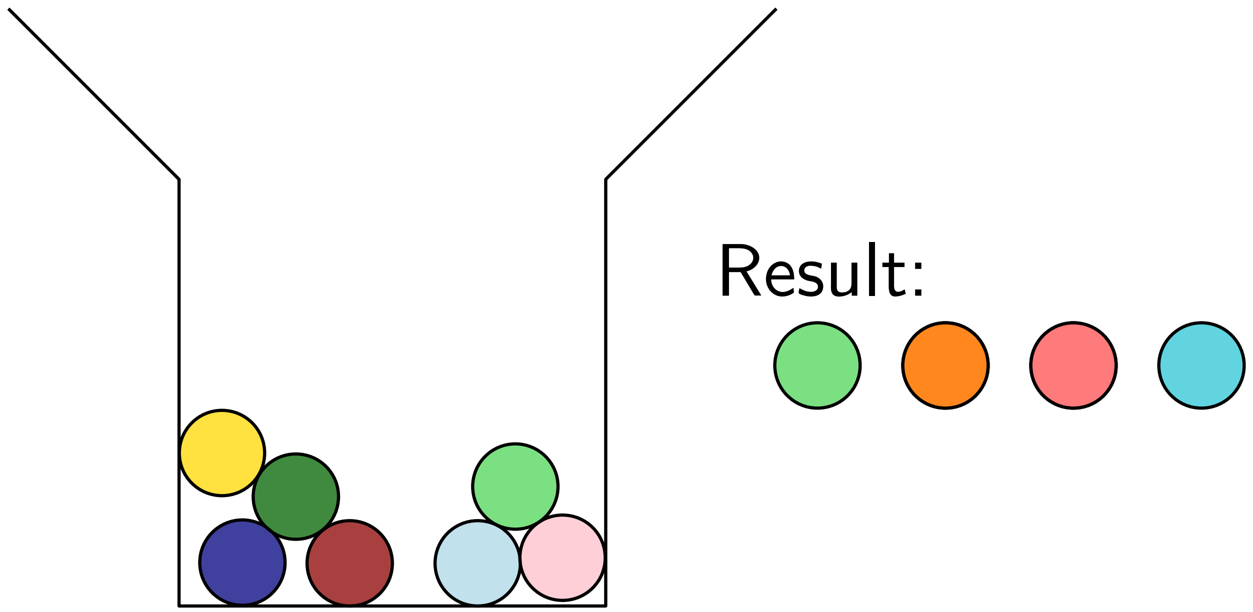

4 main variations:

- Do we care about the order in which the balls were taken?
- Do we put the ball back in the box after taking it?
  Drawing with or without replacement

Definition. Drawing with order and with replacement:

- A is a set of size n
- Pick an item from A, remember the result and put it back
- Repeat k times
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Set of possible results: A^k

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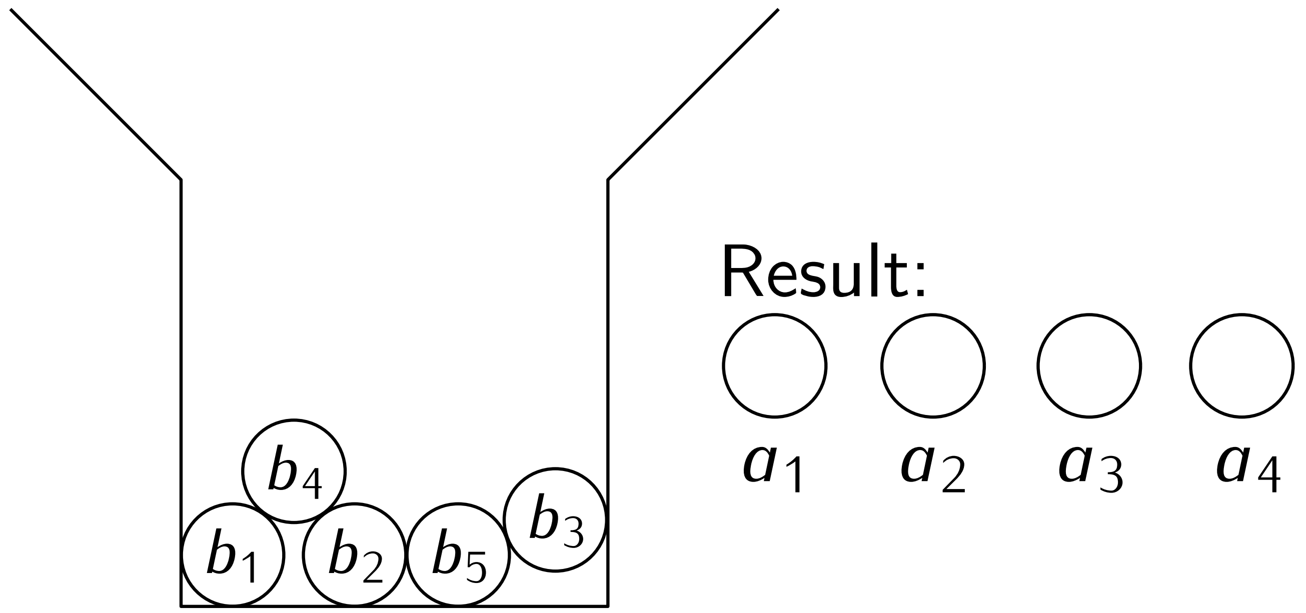
Application: counting functions $f : A \to B$



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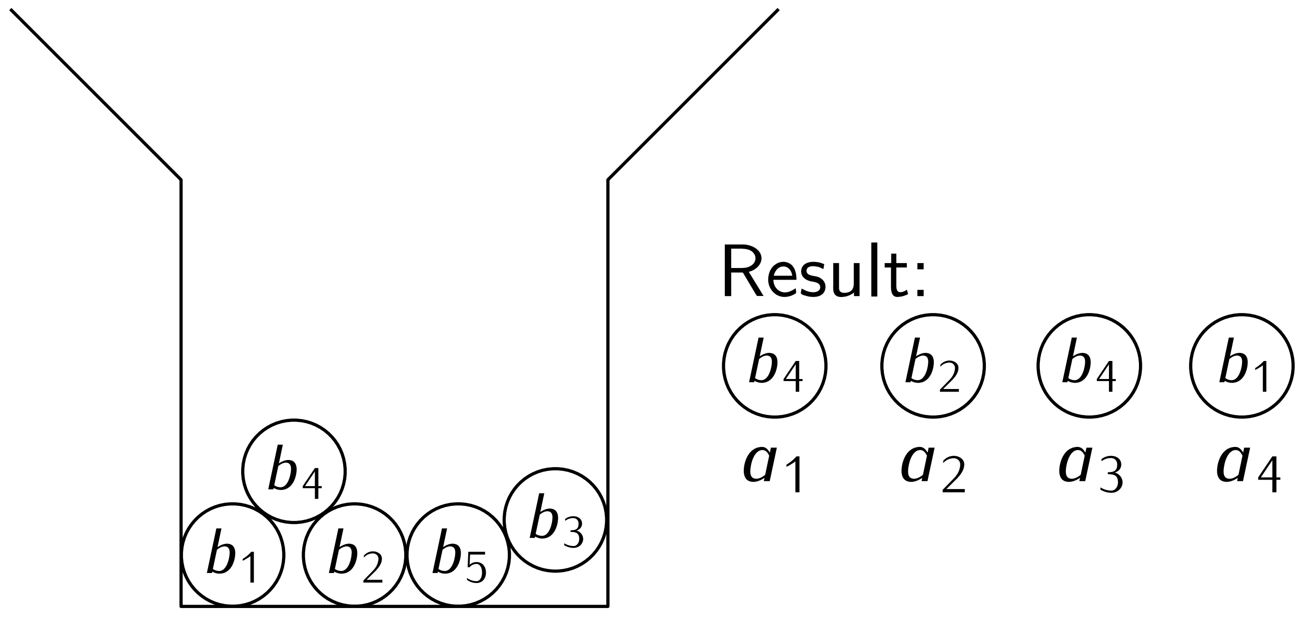

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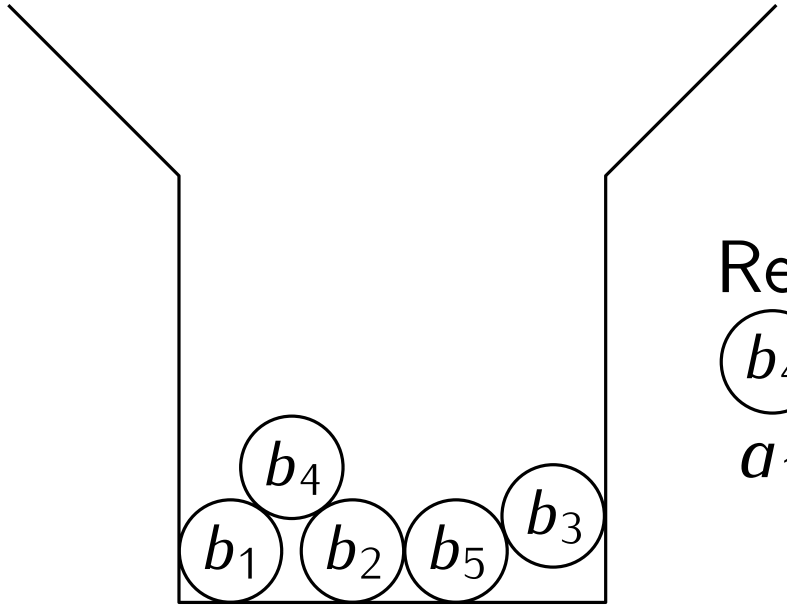

Result:

b_4	b_2	b_4	b_1
a_1	a_2	a_3	a_4

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How many binary strings with n bits are there?

- 2
- $2n$
- n^2
- 2^n

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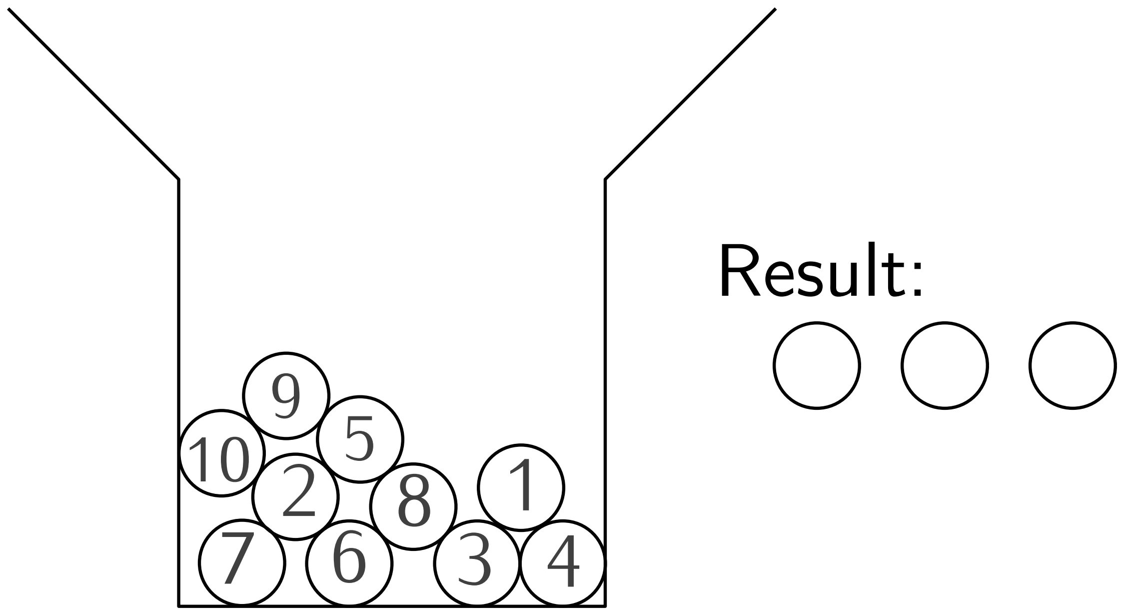

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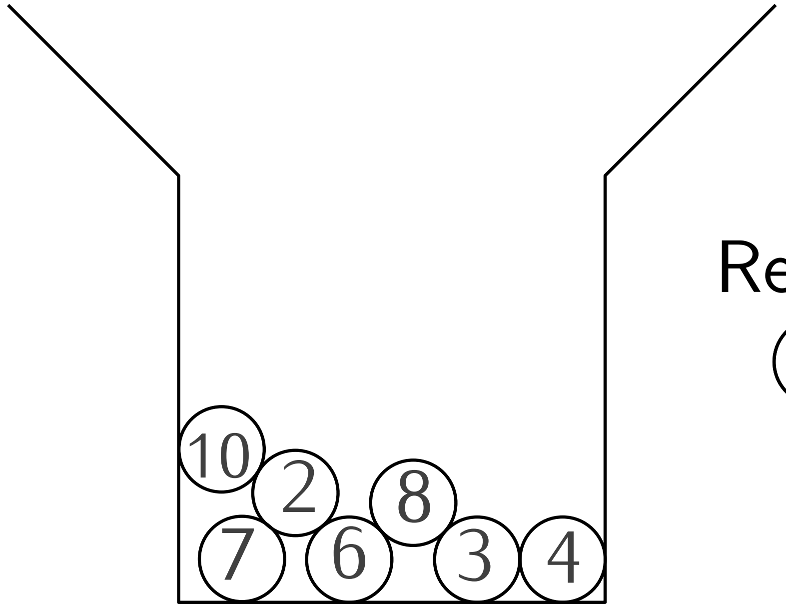

Result:

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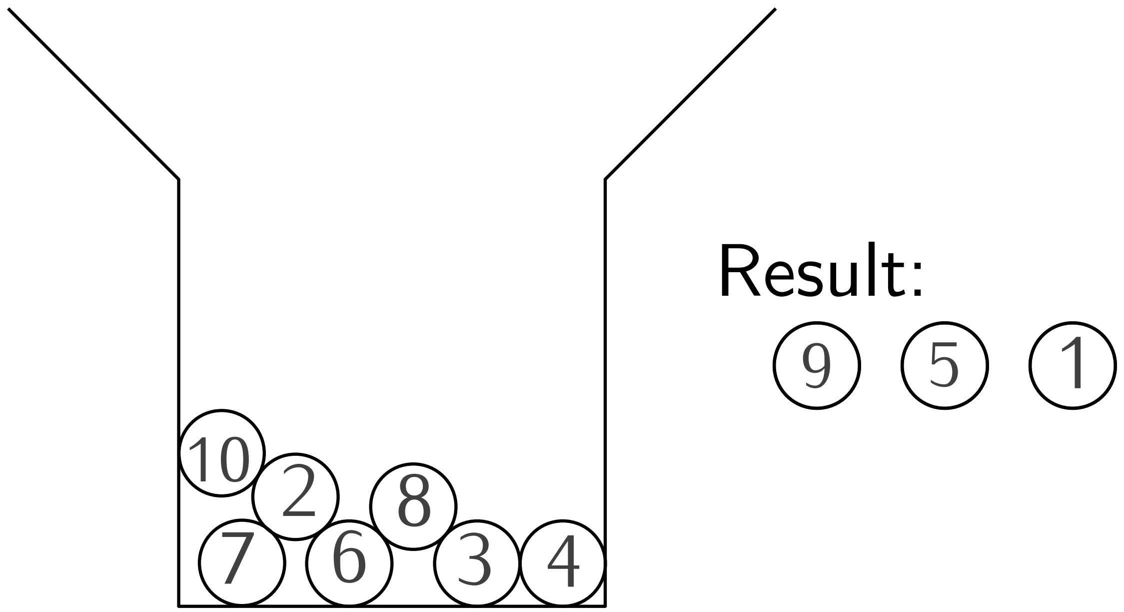

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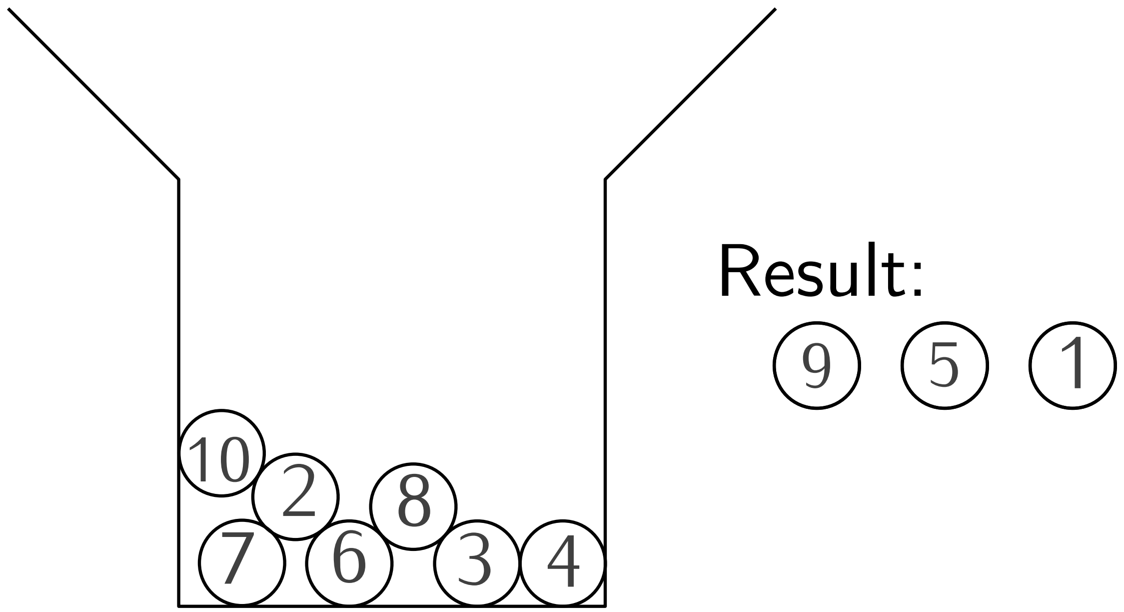

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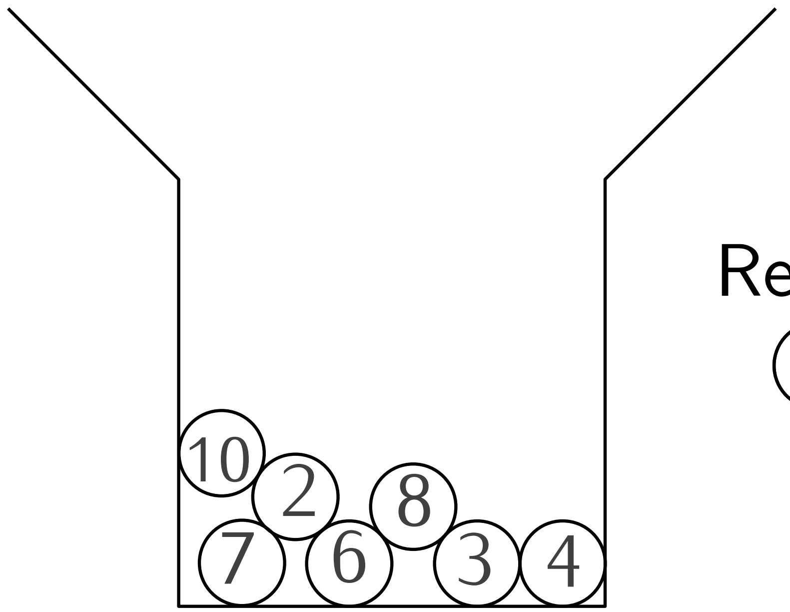
n choices for a_1

$n-1$ choices for a_2

$\ddots$

$n-k+1$ choices for a_k



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9 5 1

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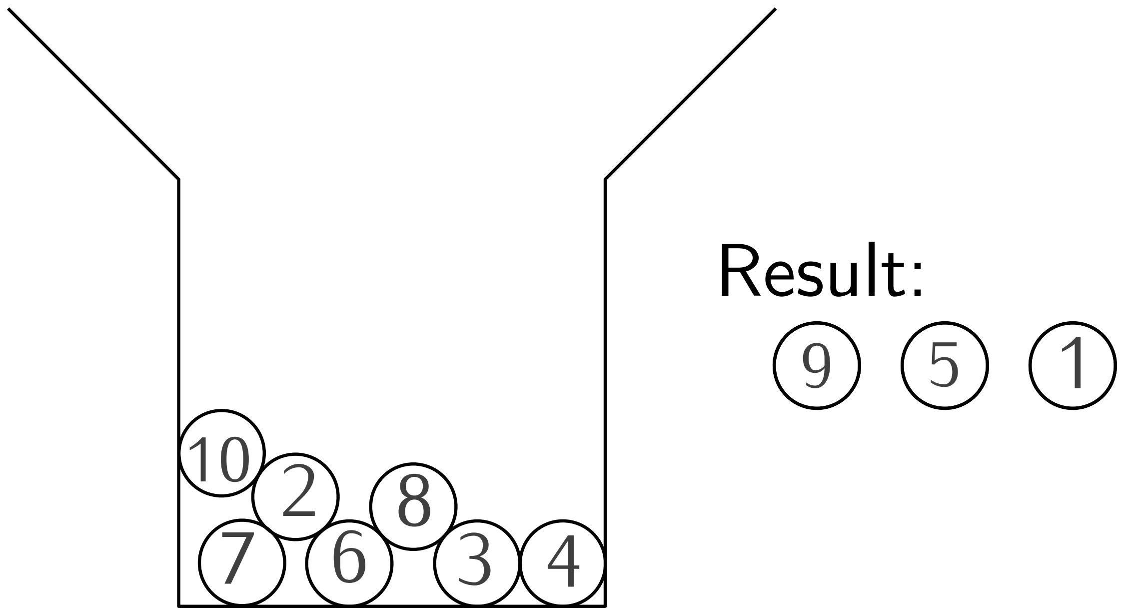

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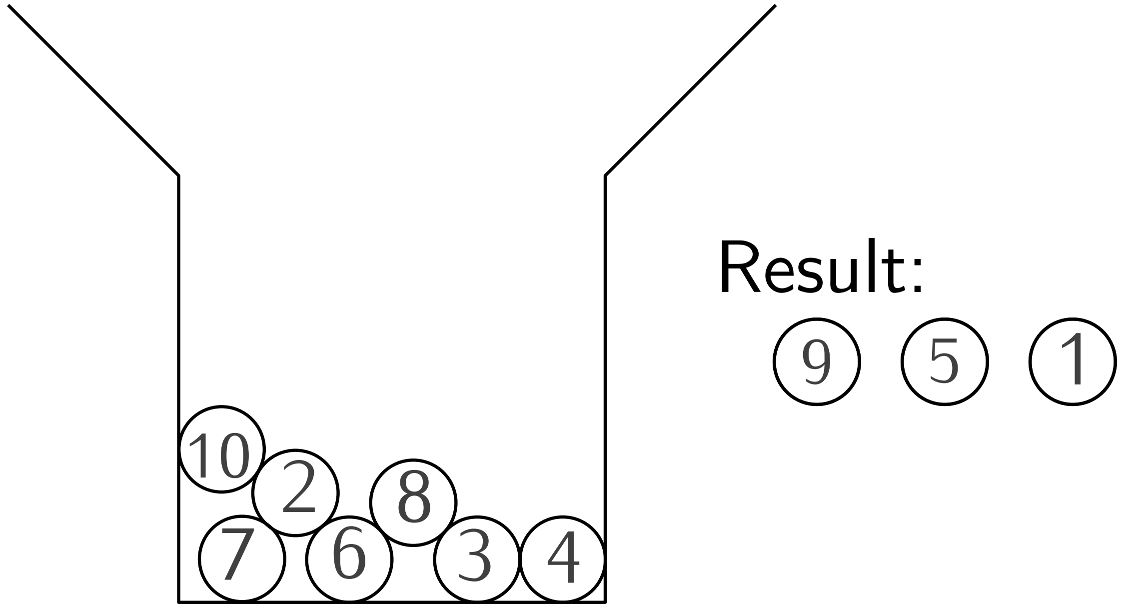

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How many ways are there to write all numbers $1, 2, \ldots, n$ in an arbitrary order?

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- $n^{\underline{n}}$
- $n^{\underline{2}}$
- n^n

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We often write $n!$ instead of $n^{\underline{n}}$, this is the factorial of n.

Definition. Drawing without order and without replacement:

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$|A|$ ↓

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's triangle:
Size of $\mathcal{P}_k(A)$

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\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
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\end{array}
\quad \Big\downarrow |A|
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Theorem. For all n and $k \in \{0, \ldots, n\}$: $\binom{n}{k} = \frac{n^k}{k!}$

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number of permutations of a set with k elements

number of subsets with k elements

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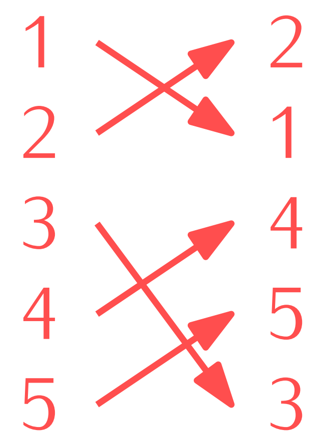

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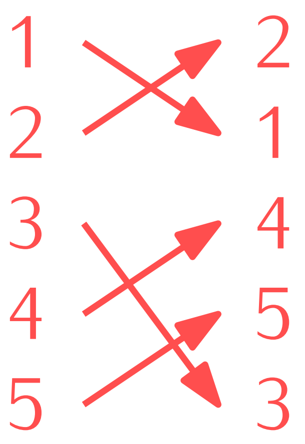

$\{1, 3, 4, 7, 9\}$

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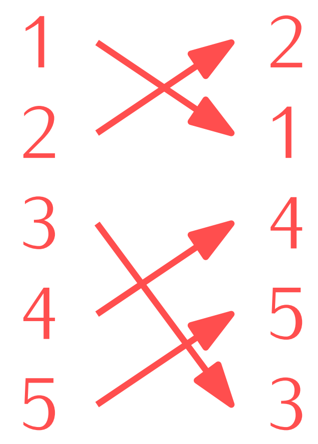

$\{1,3,4,7,9\} \mapsto (3,1,7,9,3)$

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$|A|$

				1				
		1		2		1		
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How many binary strings of length 2 are there that have exactly 2 times the bit 1?
- 2
- 10
- 20
- 30
- 40

Theorem. If A is finite, then $|\mathcal{P}(A)| = 2^{|A|}$.

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subsets of A

binary strings with exactly $n = |A|$ bits

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Define $f \colon \mathcal{P}(A) \to \{0,1\}^n$

$$S \mapsto (b_1, \ldots, b_i, \ldots, b_n)$$

1 if $i \in S$, 0 otherwise

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size of $\mathcal{P}(A)$, for any set A of size n

partition $\mathcal{P}(A)$ into $B_0, \dots, B_n$

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Proof. By the union rule: $\mathcal{P}(A) = \bigcup_{k=0}^{n} \mathcal{P}_k(A)$ and $\mathcal{P}_k(A) \cap \mathcal{P}_\ell(A) = \emptyset$ when $k \neq \ell$. $\square$

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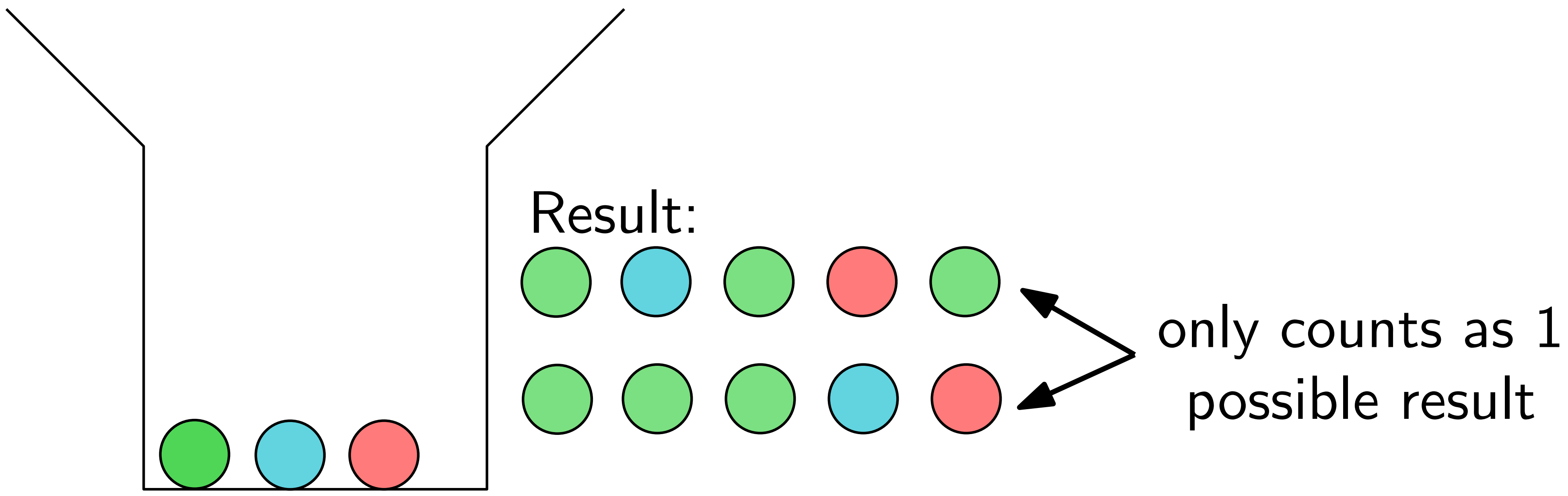

Definition. Drawing without order and with replacement:

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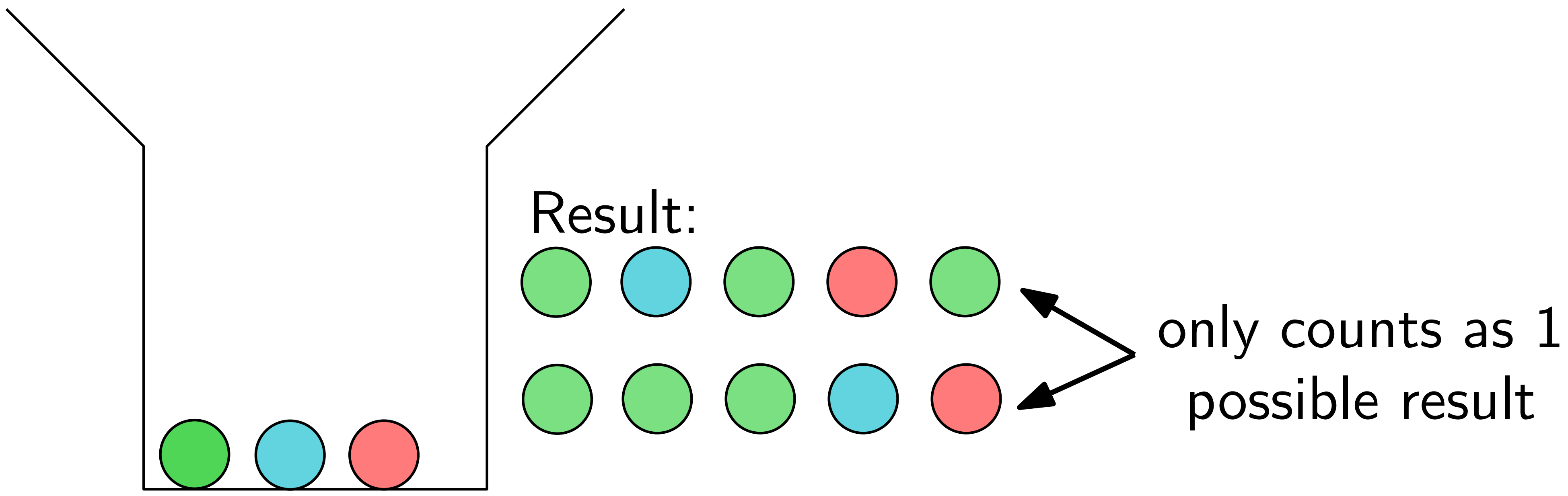
- Result is a multiset of size k (we forget which item was picked when)

A multiset is a “set” that can contain an object several times. We use $\{\{\dots\}\}$ to write multisets.



Definition. Drawing without order and with replacement:

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- Pick an item from A and put it back
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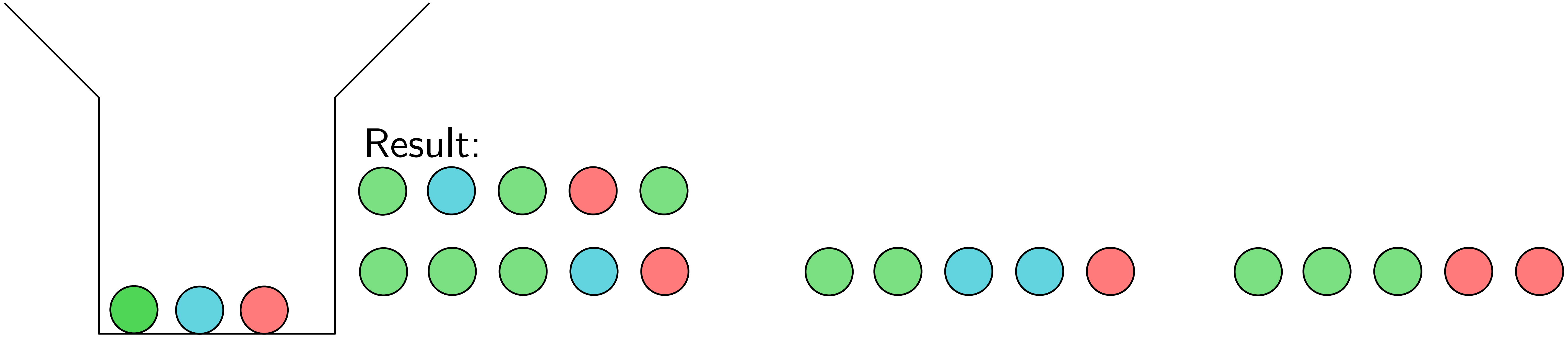

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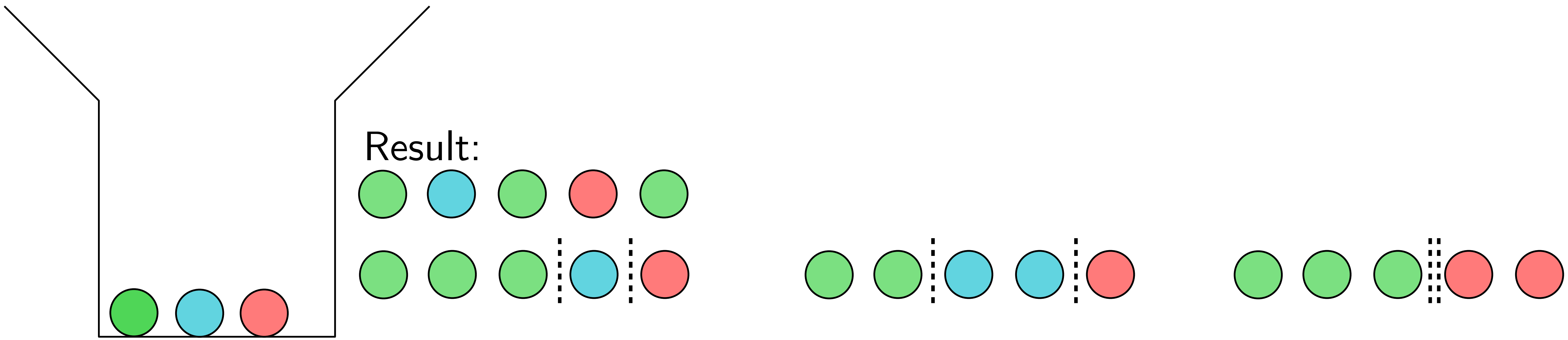

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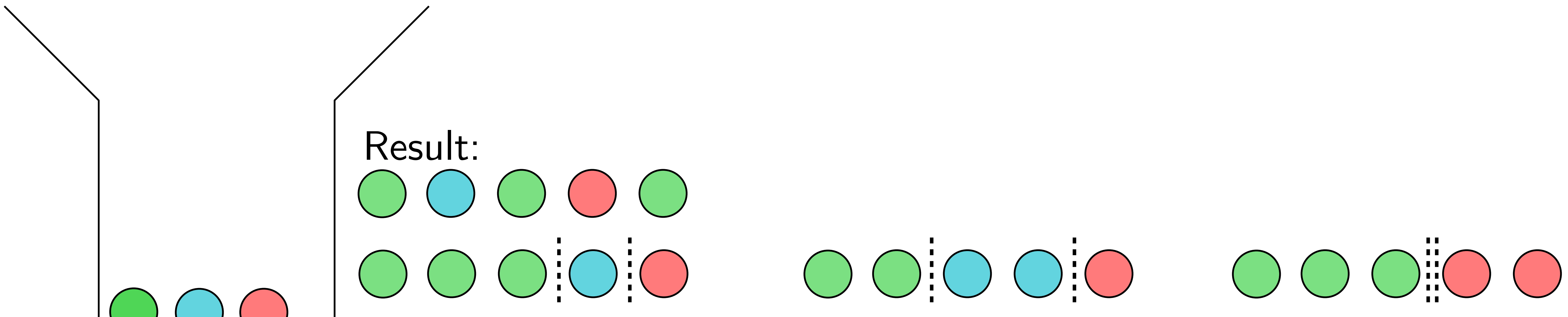

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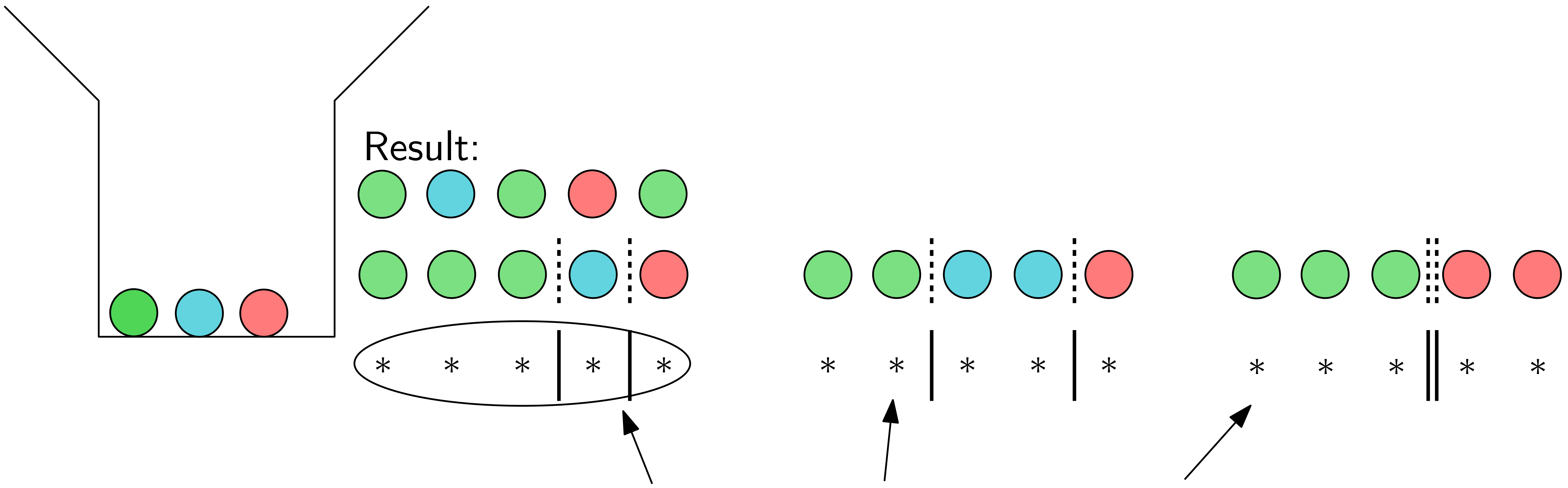

strings with symbols from the set $\{|, *\}$ and exactly bars

- Countable sets, uncountable sets
- Countable sets = what can be represented exactly on a computer
- Combinatorial proofs as a way to prove equalities/inequalities about numbers using functions

$$\begin{aligned} \text{Injection } A \to B &\quad\leftrightarrow\quad |A| \leq |B| \\ \text{Surjection } A \to B &\quad\leftrightarrow\quad |A| \geq |B| \\ \text{Bijection } A \to B &\quad\leftrightarrow\quad |A| = |B| \end{aligned}$$

- Drawing a tuple/(multi)set with/without replacement

  n = size of the set we are drawing from

  k = number of draws

	Order matters	Order does not matter
Replacement	n^k	$\frac{n!}{k!(n-k)!}$
No replacement	$n^{\underline{k}}$	$\binom{n+k-1}{n-1}$