

# Discrete Algebraic Structures

## Exercise 03

### Exercise 1 (From logic to English):

Let  $p$  stand for the proposition “I bought a lottery ticket” and  $q$  for “I won the jackpot”. Express the following as natural English sentences:

- $\neg p$
- $p \vee q$
- $p \Rightarrow q$

*You should do the three last items at home:*

- $p \wedge q$
- $\neg p \Rightarrow \neg q$
- $\neg p \vee (p \wedge q)$

### Exercise 2 (Satisfiable formulas, tautologies, and unsatisfiable formulas):

For each of the following propositions, construct a truth table and state whether the proposition is a tautology, satisfiable or unsatisfiable.

- $p \wedge \neg p$
- $(p \vee \neg q) \Rightarrow q$
- $(p \Rightarrow q) \wedge (p \wedge \neg q)$

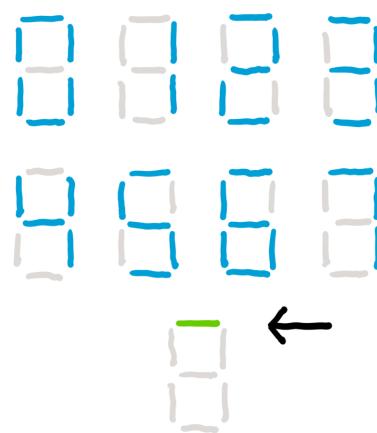
*You should do the three last items at home:*

- $p \vee \neg p$
- $(p \vee q) \Rightarrow (p \wedge q)$
- $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$

### Exercise 3 (Clock display):

We store a number  $n \in \{0, \dots, 7\}$  with three bits  $p, q, r$  as follows:

$n$	$p$	$q$	$r$
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1



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Find a formula describing when the top segment of the display should light up, expressed in terms of  $p, q, r$ .

### **Exercise 4 (Negation of a quantified formula):**

Let us define a formula  $\phi$  starting from 3 predicates:

$$\phi := \forall x \in \mathbb{N}(\text{prime}(x) \Rightarrow (\exists y(x < y \wedge \neg \text{odd}(y))))$$

We ask you to write down the simplified version of  $\neg\phi$ . By simplified we mean a formula  $\psi$  which is equivalent to  $\phi$  and such that the negated sub-formulas cannot be decomposed further.

### **Exercise 5 (Boolean Logic):**

*Exercise to do at home after the tutorials.*

(a) Give a definition of the following boolean operators

- $\oplus$  (“XOR”), the exclusive or,
- $\uparrow$  (“NAND”), the not and operator.

$x$	$y$	$x \oplus y$
0	0	0
1	0	1
0	1	1
1	1	0

$x$	$y$	$x \uparrow y$
0	0	1
1	0	1
0	1	1
1	1	0

- (b) Show that every boolean formula is equivalent to one only using  $\uparrow$ .
- (c) Show that every boolean formula with variables from  $X$  is equivalent to one in *disjunctive normal form* (DNF), i.e. equivalent to a formula

$$\bigvee_{C \in \mathcal{C}} \bigwedge_{c \in C} c$$

where all  $C \in \mathcal{C}$  are sets of literals (often called “clauses”), that is all  $c \in C$  are of the form  $x$  or  $\neg x$  for some variable  $x$ .

### **Exercise 6 (Does this prove the claim?):**

*Exercise to do at home after the tutorials.*

In this exercise, we are interested in the following mathematical statement:

Let  $a, b \in \mathbb{N}$ . Suppose that  $ab$  is even. Then  $a$  is even or  $b$  is even.

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- (a) Translate this statement in first-order logic using a unary predicate  $Even(-)$ .

We ask four students to write a proof of that statement. Below are their proofs. We take for granted that all computations are correct and that the proofs are correctly written (miracles do happen). Remember that a number  $a$  is *even* if there exists  $k$  such that  $a = 2k$  (i.e.,  $a$  is divisible by 2), and  $a$  is *odd* otherwise, which means that  $a$  can be written as  $a = 2k + 1$  for some  $k$ .

**For each of the four proofs below:**

- Write in first-order logic what is the statement that they prove
- Determine whether this is logically a valid proof of the original statement.

- (b) Suppose that  $a$  and  $b$  are odd. Indeed, we have  $a = 2k + 1$  and  $b = 2m + 1$  for some integers  $k, m$ . Then

$$\begin{aligned} ab &= (2k + 1)(2m + 1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1. \end{aligned}$$

Therefore,  $ab$  is odd.

- (c) Assume that  $a$  is even or  $b$  is even. Without loss of generality, say it is  $a$  that is even, so that we have  $a = 2k$  for some integer  $k$ . Then  $ab = (2kb) = 2(kb)$ , and therefore  $ab$  is even.  
 (d) Suppose that  $ab$  is even but  $a$  and  $b$  are both odd. Then we have  $ab = 2n, a = 2k+1, b = 2\ell+1$  for some integers  $n, k, \ell$ . Then

$$\begin{aligned} 2n &= (2k + 1)(2\ell + 1) \\ &= 4k\ell + 2(k + \ell) + 1 \end{aligned}$$

and dividing each side by 2, we get  $n = 2k\ell + k + \ell + \frac{1}{2}$ . This gives  $\frac{1}{2} = n - 2k\ell - k - \ell$ . Since the right hand side of the equality is an integer, and the left hand side is not, we obtain a contradiction.

- (e) Let  $ab$  be an even number, and  $a$  be an odd number. Let  $ab = 2n$  and  $a = 2k + 1$  for some integers  $n, k$ . Then  $ab = (2k + 1)b$ , so  $2n = 2kb + b$ , which we can rewrite as  $b = 2n - 2kb = 2(n - kb)$ . Thus,  $b$  must be even.

**Exercise 1 (From logic to English):**

Let  $p$  stand for the proposition "I bought a lottery ticket" and  $q$  for "I won the jackpot". Express the following as natural English sentences:

- $\neg p$
- $p \vee q$
- $p \Rightarrow q$

You should do the three last items at home:

- $p \wedge q$
- $\neg p \Rightarrow \neg q$
- $\neg p \vee (p \wedge q)$

$\neg p$ : I didn't buy a lottery ticket

$p \vee q$ : I bought a lottery ticket or won the jackpot

$p \Rightarrow q$ : If I bought a lottery ticket, I won the jackpot

$p \wedge q$ : I bought a lottery ticket and won the jackpot

$\neg p \Rightarrow \neg q$ : If I didn't buy a lottery ticket, I didn't win the jackpot

$\neg p \vee (p \wedge q)$ : I didn't buy a lottery ticket or I did buy one and won the lottery

### Exercise 2 (Satisfiable formulas, tautologies, and unsatisfiable formulas):

For each of the following propositions, construct a truth table and state whether the proposition is a tautology, satisfiable or unsatisfiable.

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- $(p \vee \neg q) \Rightarrow q$
- $(p \Rightarrow q) \wedge (p \wedge \neg q)$

You should do the three last items at home:

- $p \vee \neg p$
- $(p \vee q) \Rightarrow (p \wedge q)$
- $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$

$p \vee \neg p$			$(p \vee \neg q) \Rightarrow q$		
$p$	$\neg p$	$p \vee \neg p$	$p$	$\neg q$	$(p \vee \neg q) \Rightarrow q$
1	0	1	0	1	1
0	1	1	1	0	0
(tautology)					
1	1	1	1	1	1
					(satisfiable)

$$(p \Rightarrow q) \wedge (p \wedge \neg q)$$

$p, q$			$(p \Rightarrow q) \wedge (p \wedge \neg q)$		
$p$	$q$	$p \Rightarrow q$	$p$	$\neg q$	$(p \Rightarrow q) \wedge (p \wedge \neg q)$
0	0	1	0	1	0
1	0	0	1	1	0
0	1	1	0	0	0
1	1	1	0	0	0
(unsatisfiable)					