Prep Course Mathematics

Elementary algebra

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1. Notations, basic arithmetic, algebraic manipulation
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   - Conventions, parentheses, and absolute value
   - Rules
   - Calculations with fractions
2. Elementary algebra
   - Binomial expansion
   - Powers with integer exponents
   - Roots
   - Powers with real exponents
   - Logarithms
   - Proportionality, cross-multiplication
   - Calculating percentages

Notations, basic arithmetic, algebraic manipulation

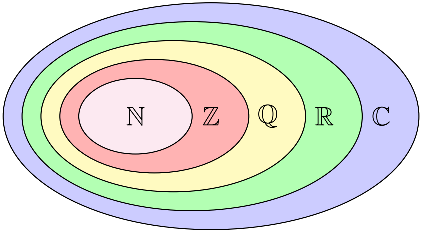
Numbers

You already know the following numbers:

- $\mathbb{N}$ (natural numbers)
- $\mathbb{Z}$ (integers)
- $\mathbb{Q}$ (rational numbers)
- $\mathbb{R}$ (real numbers)

Later we introduce:

- $\mathbb{C}$ (complex numbers)



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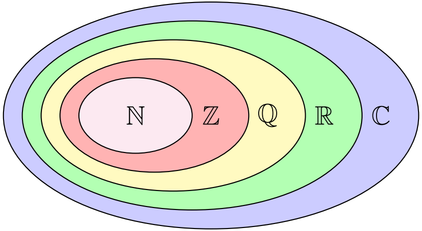

Through elementary operations (addition, subtraction, multiplication and division) two numbers yield another number.

Conventions

Order of operations:

- Parentheses first (from inside to out)
- Exponents first, then multiplication and division, then addition/subtraction
- from left to right

⚠ There are some cases where parentheses are implied by the notation:

- The fraction line denotes an operation that requires parentheses, e.g.,
  - $\dfrac{a \pm b}{c \pm d} = \dfrac{(a \pm b)}{(c \pm d)}$
- An exponent itself is always in parentheses, e.g.,
  - $a^{x \pm y} = a^{(x \pm y)}$
  - $a^{x \cdot y} = a^{(x \cdot y)}$
  - $a^{\frac{x}{y}} = a^{(\frac{x}{y})}$

Parentheses and absolute value

To override operator precedence, parentheses must be used.

Example:

$$2 \cdot 3 + 4 = 10, \qquad \text{but} \qquad 2 \cdot (3 + 4) = 2 \cdot 7 = 14.$$

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Exercise:

$$(2x + 3y) - (3x + 2y) =$$

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$$\begin{aligned} (2x + 3y) - (3x + 2y) &= 2x + 3y - 3x - 2y \\ &= -x + y. \end{aligned}$$

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For $x \in \mathbb{R}$: absolute value

$$|x| := \begin{cases} x & x \geqslant 0, \\ -x & x < 0. \end{cases}$$

Example:

$$\begin{aligned} |3| &= 3, \\ |-5| &= 5. \end{aligned}$$

Rules

Commutative laws: for $a, b \in \mathbb{R}$:

$$a + b = b + a,$$
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Distributive laws: for $a, b, c \in \mathbb{R}$:

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$
$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

Exercise

Simplify:

$$x \cdot 5 + 3 + 2 \cdot x \cdot 2 - (2 + 4) =$$

$$(x - 1)(2 - x) - (3x + x^2) + x \cdot 3 =$$

$$3x^2 + 2x - 2(x - 1)(-x - 3) =$$

$$-(-x - 1)(-2 - x)(-x - 3) =$$

Exercise

Simplify:

$$\begin{aligned} x \cdot 5 + 3 + 2 \cdot x \cdot 2 - (2+4) &= 5x + 3 + 4x - 6 \\ &= 9x - 3. \end{aligned}$$

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$$\begin{aligned} -(-x-1)(-2-x)(-x-3) &= -(2x + x^2 + 2 + x)(-x-3) \\ &= -(x^2 + 3x + 2)(-x-3) \\ &= -(-x^3 - 3x^2 - 3x^2 - 9x - 2x - 6) \\ &= -(-x^3 - 6x^2 - 11x - 6) \\ &= x^3 + 6x^2 + 11x + 6. \end{aligned}$$

Calculating with fractions

For $a, b, c, d \in \mathbb{R}$ (as long as denominator different from zero):

- Expansion/Simplification: $\dfrac{a}{b} = \dfrac{a \cdot c}{b \cdot c}$

  Example:

$$\frac{2x+4}{6x+6} = \frac{2(x+2)}{2(3x+3)} = \frac{x+2}{3x+3}.$$

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- Multiplication/Division: $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$ and $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$

  Example:

$$\frac{\frac{3x^2}{x+1}}{\frac{2x^2+2x}{(x+1)^2}} = \frac{3x^2}{x+1} \cdot \frac{(x+1)^2}{2x^2+2x} = \frac{3x^2(x+1)^2}{(x+1)(2x^2+2x)} = \frac{3x^2(x+1)}{(2x+2)x} = \frac{3x(x+1)}{2(x+1)} = \frac{3x}{2}.$$

Exercise

Calculate or simplify:

$$\frac{5}{6} - \frac{1}{3} \cdot \frac{3}{4} =$$

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}} =$$

$$\frac{\frac{1}{x} - \frac{1}{x^2}}{\frac{1}{x}} =$$

$$\left(\frac{x}{2} + \frac{2}{5}\right) \cdot \left(\frac{1}{\frac{x}{3}} - \frac{1}{x}\right) =$$

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$$\left(\frac{x}{2} + \frac{2}{5}\right) \cdot \left(\frac{1}{\frac{x}{3}} - \frac{1}{x}\right) = \left(\frac{5x}{10} + \frac{4}{10}\right) \cdot \left(\frac{3}{x} - \frac{1}{x}\right) = \frac{5x+4}{10} \cdot \frac{2}{x} = \frac{5x+4}{5x}.$$

Elementary algebra

Binomial expansion

For $a, b \in \mathbb{R}$:

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b)(a-b) = a^2 - b^2$

Example:

$$\begin{aligned} 32^2 = (30+2)^2 = 30^2 + 2 \cdot 30 \cdot 2 + 2^2 = 900 + 120 + 4 = 1024, \\ (3x-2)^2 = (3x)^2 - 2 \cdot 3x \cdot 2 + 2^2 = 9x^2 - 12x + 4, \\ (3x-4)(3x+4) = (3x)^2 - 4^2 = 9x^2 - 16. \end{aligned}$$

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For other powers: $(a+b)^n = a^n + na^{n-1}b^1 + \ldots + na^1b^{n-1} + b^n$

Pascal's triangle

$$\begin{array}{lccccccccc} n=0: & & & & & 1 & & & & \\ n=1: & & & & 1 & & 1 & & & \\ n=2: & & & 1 & & 2 & & 1 & & \\ n=3: & & 1 & & 3 & & 3 & & 1 & \\ n=4: & 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Powers with integer exponents

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For $a \in \mathbb{R}$ and $n \in \mathbb{Z}$:

$$a^n := \begin{cases} \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}, & n > 0, \\ 1, & n = 0, \\ \frac{1}{a^{-n}}, & n < 0. \end{cases}$$

Example:

$$3^{-2} = \frac{1}{3^2} = \frac{1}{3 \cdot 3} = \frac{1}{9}.$$

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Power laws: for $a, b \in \mathbb{R}$ and $m, n \in \mathbb{Z}$:

$$\begin{aligned} a^m \cdot a^n &= a^{m+n}, \\ \frac{a^m}{a^n} &= a^{m-n} \qquad \text{if } a \neq 0, \\ (a \cdot b)^n &= a^n \cdot b^n, \\ \left(\frac{a}{b}\right)^n &= \frac{a^n}{b^n} \qquad \text{if } b \neq 0, \\ (a^m)^n &= a^{m \cdot n}. \end{aligned}$$

Roots

For $a \geqslant 0$ and $n \in \mathbb{N}$: n-th root

$$\sqrt[n]{a} \geqslant 0 \qquad \text{the number satisfying} \qquad \left(\sqrt[n]{a}\right)^n = a.$$

Example:

$$\begin{aligned}\sqrt[3]{\frac{1}{27}} &= \frac{1}{3},\\ \sqrt[4]{4096} &= 8,\\ \sqrt[1]{5} &= 5.\end{aligned}$$

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Root laws: for $a, b \geqslant 0$ and $m, n \in \mathbb{N}$:

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b},$$
$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad \text{if } b \neq 0,$$
$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[m \cdot n]{a}.$$

Powers with real exponents

For $a > 0$ and $m \in \mathbb{Z}$, $n \in \mathbb{N}$:

$$a^{\frac{m}{n}} := \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

Roots are special powers: $\sqrt[n]{a} = a^{\frac{1}{n}}$.

Example:

$$8^{\frac{4}{3}} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16.$$

For $a > 0$ and $x \in \mathbb{R}$: via approximation with rational exponents: real power a^x.

Power laws: for $a, b > 0$ and $x, y \in \mathbb{R}$:

$$\begin{aligned} a^x \cdot a^y &= a^{x+y}, \\ \frac{a^x}{a^y} &= a^{x-y}, \\ (a \cdot b)^x &= a^x \cdot b^x, \\ \left(\frac{a}{b}\right)^x &= \frac{a^x}{b^x}, \\ (a^x)^y &= a^{x \cdot y}, \end{aligned}$$

Furthermore: $0^x = 0$ for $x > 0$.

Exercise

Calculate or simplify:

$$\sqrt{\frac{2}{3}}(\sqrt{6}-\sqrt{3}) =$$

$$\left(\sqrt{2}+\sqrt{3}\right)^2 =$$

$$\frac{2}{2-\sqrt{3}} =$$

$$\frac{2}{1+\sqrt{x}}+\frac{2\sqrt{x}}{1-\sqrt{x}} =$$

Exercise

Calculate or simplify:

$$\sqrt{\frac{2}{3}}(\sqrt{6}-\sqrt{3}) = \frac{\sqrt{2}\sqrt{2}\sqrt{3}}{\sqrt{3}} - \frac{\sqrt{2}\sqrt{3}}{\sqrt{3}} = 2-\sqrt{2}.$$

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$$\frac{2}{2-\sqrt{3}} = \frac{2(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{4+2\sqrt{3}}{4-3} = 4+2\sqrt{3}.$$

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$$\begin{aligned}\frac{2}{1+\sqrt{x}} + \frac{2\sqrt{x}}{1-\sqrt{x}} &= \frac{2(1-\sqrt{x})}{(1+\sqrt{x})(1-\sqrt{x})} + \frac{2\sqrt{x}(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})} \\ &= \frac{2-2\sqrt{x}+2\sqrt{x}+2\sqrt{x}^2}{1-x} = \frac{2+2x}{1-x}.\end{aligned}$$

Logarithms

For $a, b > 0$: logarithm of b for base a:

$$\log_a b \qquad \text{the number such that} \qquad a^{\log_a b} = b.$$

Example:

$$\log_5 25 = 2,$$
$$\log_2 16 = 4,$$
$$\log_4 \frac{1}{2} = -\frac{1}{2}.$$

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Important special case: $a = \mathrm{e} \approx 2.7182818285$, Euler number. Then $\log_e b = \ln b$.
Every logarithm of this form: $\log_a b = \frac{\ln b}{\ln a}$.

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logarithm laws: for $a > 0$ and $u, v > 0$, $x \in \mathbb{R}$:

$$\log_a(u \cdot v) = \log_a(u) + \log_a(v),$$
$$\log_a \frac{u}{v} = \log_a(u) - \log_a(v),$$
$$\log_a(u^x) = x \log_a(u).$$

Exercise

Determine:

$$\log_2 8 =$$

$$\log_3(27^2) =$$

$$\log_{10} 0.0001 =$$

$$\ln(\mathrm{e}^3 \cdot \mathrm{e}^{-5}) =$$

Exercise

Determine:

$$\log_2 8 = 3.$$

$$\log_3(27^2) = 2\log_3(27) = 2 \cdot 3 = 6.$$

$$\log_{10} 0.0001 = -4.$$

$$\ln(e^3 \cdot e^{-5}) = \ln(e^3) + \ln(e^{-5}) = 3 - 5 = -2.$$

Proportionality, cross-multiplication

y (directly) proportional to x if there exists $k \in \mathbb{R}$, $k \neq 0$ such that

$$y = kx.$$

Example: k cost per unit, x amount of units, y total cost

Cross-multiplication: 3 apples cost 1.50 EUR. Then 1 apple costs 0.50 EUR. Hence, 10 apples cost 5.00 EUR.

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y inverse proportional to x if there exists $k \in \mathbb{R}$, $k \neq 0$ such that

$$y = \frac{k}{x}.$$

Example: k required working time, x number of workers, y time needed

Cross-multiplication: 3 workers need 4 hours for one product. Then 1 worker needs 12 hours. Hence, 4 workers need 3 hours.

Calculating percentages

Fundamental identity:

$$\frac{\text{percentage}}{100\%} = \frac{\text{percentage value}}{\text{base value}}.$$

Example: investment of $10\,000$ EUR with 3.5% for one year. Interest payment:

$$\frac{3.5\%}{100\%} = \frac{\text{interest payment}}{10\,000 \text{ EUR}},$$

hence 350 EUR.

Exercise

1. Is y proportional to x?

x	2	3	5	8
y	6	9	14	23

2. Calculate:

$$2\% \text{ of } 20 =$$

$$200\% \text{ of } 15 =$$

$$250\% \text{ of } 20 =$$

Exercise

1. Is y proportional to x?

x	2	3	5	8
y	6	9	14	23

No, as $6 = 3 \cdot 2$, $9 = 3 \cdot 3$, $14 = 2.8 \cdot 5$, $23 = 2.875 \cdot 8$.

2. Calculate:

$$\begin{aligned} 2\% \ \text{ of } \ 20 &= 0.4, \\ 200\% \ \text{ of } \ 15 &= 30, \\ 250\% \ \text{ of } \ 20 &= 50. \end{aligned}$$