

Discrete Algebraic Structures

Exercise 03

Exercise 1 (From logic to English):

Let p stand for the proposition “I bought a lottery ticket” and q for “I won the jackpot”. Express the following as natural English sentences:

- $\neg p$
- $p \vee q$
- $p \Rightarrow q$

You should do the three last items at home:

- $p \wedge q$
- $\neg p \Rightarrow \neg q$
- $\neg p \vee (p \wedge q)$

Exercise 2 (Satisfiable formulas, tautologies, and unsatisfiable formulas):

For each of the following propositions, construct a truth table and state whether the proposition is a tautology, satisfiable or unsatisfiable.

- $p \wedge \neg p$
- $(p \vee \neg q) \Rightarrow q$
- $(p \Rightarrow q) \wedge (p \wedge \neg q)$

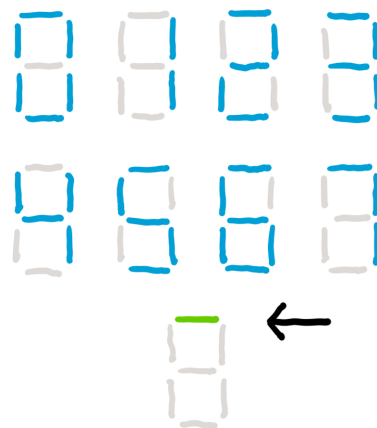
You should do the three last items at home:

- $p \vee \neg p$
- $(p \vee q) \Rightarrow (p \wedge q)$
- $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$

Exercise 3 (Clock display):

We store a number $n \in \{0, \dots, 7\}$ with three bits p, q, r as follows:

n	p	q	r
0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1



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Find a formula describing when the top segment of the display should light up, expressed in terms of p, q, r .

Exercise 4 (Negation of a quantified formula):

Let us define a formula ϕ starting from 3 predicates:

$$\phi := \forall x \in \mathbb{N}(\text{prime}(x) \Rightarrow (\exists y(x < y \wedge \neg \text{odd}(y))))$$

We ask you to write down the simplified version of $\neg\phi$. By simplified we mean a formula ψ which is equivalent to ϕ and such that the negated sub-formulas cannot be decomposed further.

Exercise 5 (Boolean Logic):

Exercise to do at home after the tutorials.

(a) Give a definition of the following boolean operators

- \oplus (“XOR”), the exclusive or,
- \uparrow (“NAND”), the not and operator.

x	y	$x \oplus y$
0	0	0
1	0	1
0	1	1
1	1	0

x	y	$x \uparrow y$
0	0	1
1	0	1
0	1	1
1	1	0

- (b) Show that every boolean formula is equivalent to one only using \uparrow .
- (c) Show that every boolean formula with variables from X is equivalent to one in *disjunctive normal form* (DNF), i.e. equivalent to a formula

$$\bigvee_{C \in \mathcal{C}} \bigwedge_{c \in C} c$$

where all $C \in \mathcal{C}$ are sets of literals (often called “clauses”), that is all $c \in C$ are of the form x or $\neg x$ for some variable x .

Exercise 6 (Does this prove the claim?):

Exercise to do at home after the tutorials.

In this exercise, we are interested in the following mathematical statement:

Let $a, b \in \mathbb{N}$. Suppose that ab is even. Then a is even or b is even.

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- (a) Translate this statement in first-order logic using a unary predicate $Even(-)$.

We ask four students to write a proof of that statement. Below are their proofs. We take for granted that all computations are correct and that the proofs are correctly written (miracles do happen). Remember that a number a is *even* if there exists k such that $a = 2k$ (i.e., a is divisible by 2), and a is *odd* otherwise, which means that a can be written as $a = 2k + 1$ for some k .

For each of the four proofs below:

- Write in first-order logic what is the statement that they prove
 - Determine whether this is logically a valid proof of the original statement.
- (b) Suppose that a and b are odd. Indeed, we have $a = 2k + 1$ and $b = 2m + 1$ for some integers k, m . Then

$$\begin{aligned} ab &= (2k + 1)(2m + 1) \\ &= 4km + 2k + 2m + 1 \\ &= 2(2km + k + m) + 1. \end{aligned}$$

Therefore, ab is odd.

- (c) Assume that a is even or b is even. Without loss of generality, say it is a that is even, so that we have $a = 2k$ for some integer k . Then $ab = (2kb) = 2(kb)$, and therefore ab is even.
- (d) Suppose that ab is even but a and b are both odd. Then we have $ab = 2n, a = 2k + 1, b = 2\ell + 1$ for some integers n, k, ℓ . Then

$$\begin{aligned} 2n &= (2k + 1)(2\ell + 1) \\ &= 4k\ell + 2(k + \ell) + 1 \end{aligned}$$

and dividing each side by 2, we get $n = 2k\ell + k + \ell + \frac{1}{2}$. This gives $\frac{1}{2} = n - 2k\ell - k - \ell$. Since the right hand side of the equality is an integer, and the left hand side is not, we obtain a contradiction.

- (e) Let ab be an even number, and a be an odd number. Let $ab = 2n$ and $a = 2k + 1$ for some integers n, k . Then $ab = (2k + 1)b$, so $2n = 2kb + b$, which we can rewrite as $b = 2n - 2kb = 2(n - kb)$. Thus, b must be even.

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You should do the three last items at home:

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- $\neg p \Rightarrow \neg q$
- $\neg p \vee (p \wedge q)$

$\neg p$: I didn't buy a lottery ticket

$p \vee q$: I bought a lottery ticket or won the jackpot

$p \Rightarrow q$: If I bought a lottery ticket, I won the jackpot

$p \wedge q$: I bought a lottery ticket and won the jackpot

$\neg p \Rightarrow \neg q$: If I didn't buy a lottery ticket, I didn't win the jackpot

$\neg p \vee (p \wedge q)$: I didn't buy a lottery ticket or I did buy one and won the lottery

Exercise 2 (Satisfiable formulas, tautologies, and unsatisfiable formulas):

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- $(p \Rightarrow q) \wedge (p \wedge \neg q)$

You should do the three last items at home:

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- $(p \vee q) \Rightarrow (p \wedge q)$
- $(p \Rightarrow q) \Rightarrow (q \Rightarrow p)$

$$p \vee \neg p$$

$$(p \vee \neg q) \Rightarrow q$$

p	$p \vee \neg p$
1	1
0	1

(tautology)

p	q	$(p \vee \neg q)$	$(p \vee \neg q) \Rightarrow q$
0	0	1	0
1	0	1	0
0	1	0	1
1	1	1	1

(satisfiable)

$$(p \Rightarrow q) \wedge (p \wedge \neg q)$$

p	q	$(p \Rightarrow q)$	$(p \wedge \neg q)$	$(p \Rightarrow q) \wedge (p \wedge \neg q)$
0	0	1	0	0
1	0	0	1	0
0	1	1	0	0
1	1	1	0	0

(unsatisfiable)