Mathematics 1 - Linear Algebra

Lecture 10 – §5.1 Affine subspaces, §5.2 Distances, §5.3 scalar product and norm

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Affine subspaces

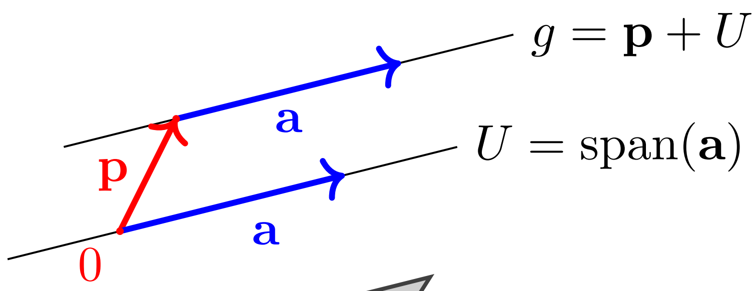
Lines and planes in $\mathbb{R}^3$ are special cases of affine subspaces of $\mathbb{R}^n$.

Definition 5.1 (affine subspace)

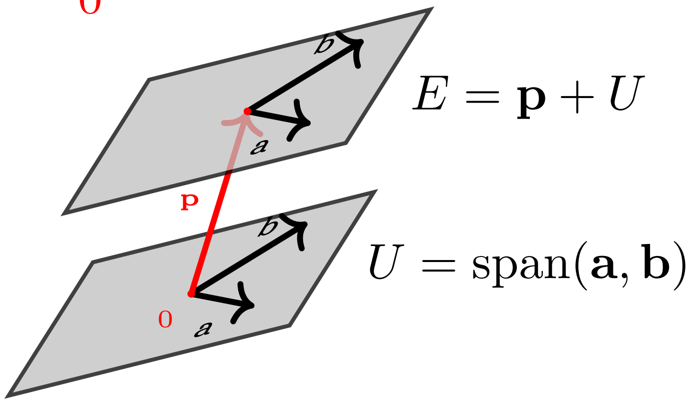
Let U be a subspace of $\mathbb{R}^n$ (hence closed wrt vector addition and scalar multiplication), and let $\mathbf{p} \in \mathbb{R}^n$. Then an *affine subspace* of $\mathbb{R}^n$ is defined by

$$\mathbf{p} + U := \{\mathbf{p} + \mathbf{u} : \mathbf{u} \in U\}.$$

$$g = \{\mathbf{p} + \lambda\mathbf{a} : \lambda \in \mathbb{R}\} = \mathbf{p} + \underbrace{\operatorname{span}(\mathbf{a})}_{U}$$



$$E = \{\mathbf{p} + \lambda\mathbf{a} + \mu\mathbf{b} : \lambda, \mu \in \mathbb{R}\} = \mathbf{p} + \underbrace{\operatorname{span}(\mathbf{a}, \mathbf{b})}_{U}$$



Affine subspaces

Theorem 5.2

a) Subspaces of $\mathbb{R}^n$ are closed wrt linear combinations.

b) **Affine** subspaces of $\mathbb{R}^n$ are closed wrt **affine** combinations.

Proof. a) This follows directly from the definition of a subspace.

b) Let $\mathbf{p} \in \mathbb{R}^n$, U be a subspace of $\mathbb{R}^n$, $\mathbf{u}_1, \ldots, \mathbf{u}_k \in U$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ mit $\alpha_1 + \ldots + \alpha_k = 1$. Then there holds

$$\alpha_1 \underbrace{(\mathbf{p} + \mathbf{u}_1)}_{\in\, \mathbf{p}+U} + \ldots + \alpha_k \underbrace{(\mathbf{p} + \mathbf{u}_k)}_{\in\, \mathbf{p}+U} = \underbrace{(\alpha_1 + \ldots + \alpha_k)}_{=1} \mathbf{p} + \underbrace{\alpha_1 \mathbf{u}_1 + \ldots + \alpha_k \mathbf{u}_k}_{\in U} \in \mathbf{p} + U.$$

Hence an affine combination of $\mathbf{p} + \mathbf{u}_1, \ldots, \mathbf{p} + \mathbf{u}_k \in \mathbf{p} + U$ is always in $\mathbf{p} + U$. $\square$

Properties of the scalar product and the norm

Theorem 5.4 (properties of the standard scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^n$)

The standard scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \ldots + x_n y_n \quad \text{for vectors } \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \text{ in } \mathbb{R}^n$$

has the following properties: For all $\mathbf{x}, \mathbf{x}', \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ there holds

a) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ for all $\mathbf{x} \neq \mathbf{o}$, and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ for $\mathbf{x} = \mathbf{o}$,

b) $\langle \mathbf{x} + \mathbf{x}', \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}', \mathbf{y} \rangle$,

c) $\langle \alpha \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$,

d) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

Proof. Straightforward computations. $\square$

Properties of the scalar product and the norm

Theorem 5.5 (properties of the norm $\|\cdot\|$ in $\mathbb{R}^n$)

The norm

$$\|\mathbf{x}\| = \sqrt{|x_1|^2 + \ldots + |x_n|^2} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \quad \text{for vectors } \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ of } \mathbb{R}^n$$

has the following properties: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ there holds

a) $\|\mathbf{x}\| > 0$ for all $\mathbf{x} \neq \mathbf{o}$, and $\|\mathbf{x}\| = 0$ for $\mathbf{x} = \mathbf{o}$,
b) $\|\alpha \mathbf{x}\| = |\alpha| \; \|\mathbf{x}\|$,
c) $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$.

Proof. a) This follows directly from Theorem 5.4a).

b) There holds $\|\alpha \mathbf{x}\| = \sqrt{\langle \alpha \mathbf{x}, \alpha \mathbf{x} \rangle} = \sqrt{\alpha^2 \langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\alpha^2}\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = |\alpha| \|\mathbf{x}\|$.

c) For convenience we initially consider squares:

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle \\ &\leqslant \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2|\langle \mathbf{x}, \mathbf{y} \rangle| \overset{\text{Theorem 5.6}}{\leqslant} \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\| \, \|\mathbf{y}\| = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2. \quad \square \end{aligned}$$

Properties of the scalar product and the norm

Theorem 5.6 (Cauchy-Schwarz inequality)

For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ there holds
$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \|\mathbf{x}\| \, \|\mathbf{y}\|.$$
Equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are multiples of each other.

Proof. In the case $\mathbf{y} = \mathbf{o}$ the inequality obviously holds for all $\mathbf{x} \in \mathbb{R}^n$.

Hence let $\mathbf{y} \neq \mathbf{o}$. For all $\mathbf{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ there holds
$$0 \leqslant \langle \mathbf{x} - \alpha\mathbf{y}, \mathbf{x} - \alpha\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - 2\alpha\langle \mathbf{x}, \mathbf{y} \rangle + \alpha^2 \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 - 2\alpha\langle \mathbf{x}, \mathbf{y} \rangle + \alpha^2 \|\mathbf{y}\|^2.$$
Now set $\alpha = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2}$. Then it follows that
$$0 \leqslant \|\mathbf{x}\|^2 - 2\frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{y}\|^2} + \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{y}\|^2} = \|\mathbf{x}\|^2 - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\|\mathbf{y}\|^2},$$
which implies $\langle \mathbf{x}, \mathbf{y} \rangle^2 \leqslant \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$ and hence $|\langle \mathbf{x}, \mathbf{y} \rangle| \leqslant \|\mathbf{x}\| \, \|\mathbf{y}\|$.

Equality holds if and only if $\mathbf{x} = \alpha\mathbf{y}$.

□

Distance, scalar product and orthogonality

Lines in $\mathbb{R}^2$: parameter form $\leftrightarrow$ normal form

$$\overbrace{\{\mathbf{p}+\lambda\mathbf{a} \,:\, \lambda\in\mathbb{R}\}}^{\text{parameter form}} \;=\; \mathbf{p}+\operatorname{span}(\mathbf{a}) \;=\; g \;=\; \overbrace{\{\mathbf{v}\in\mathbb{R}^2 \,:\, \langle\mathbf{n},\mathbf{v}-\mathbf{p}\rangle = 0\}}^{\text{normal form}}$$

The direction vector $\mathbf{a}\in\mathbb{R}^2$ and the normal vector $\mathbf{n}\in\mathbb{R}^2$ are orthogonal to each other.
The location vector $\mathbf{p}$ may be the same in both representations.

Example

Let two lines g, h and the point P be given:

$$g = \left\{ \begin{pmatrix}1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\0\end{pmatrix} \,:\, \lambda\in\mathbb{R} \right\},\quad h = \left\{ \mathbf{v}\in\mathbb{R}^2 \,:\, \left\langle \begin{pmatrix}1\\-1\end{pmatrix}, \mathbf{v} - \begin{pmatrix}-2\\0\end{pmatrix} \right\rangle = 0 \right\},\quad P = (-1,1).$$

1. Normal form of g: $g = \left\{ \mathbf{v}\in\mathbb{R}^2 \,:\, \left\langle \begin{pmatrix}0\\1\end{pmatrix}, \mathbf{v} - \begin{pmatrix}1\\2\end{pmatrix} \right\rangle = 0 \right\}$
2. Parameter form of h: $h = \left\{ \begin{pmatrix}-2\\0\end{pmatrix} + \mu\begin{pmatrix}1\\1\end{pmatrix} \,:\, \mu\in\mathbb{R} \right\}$
3. Intersection of g, h: Set parameter forms equal and solve for λ, μ.
4. Does P lie on g or h? Insert position vector of P into normal form.

Distance, scalar product and orthogonality

Planes in $\mathbb{R}^3$: parameter form $\leftrightarrow$ normal form

$$\overbrace{\{\mathbf{p}+\lambda\mathbf{a}+\mu\mathbf{b} : \lambda,\mu\in\mathbb{R}\}}^{\text{parameter form}} = \mathbf{p}+\operatorname{span}(\mathbf{a},\mathbf{b}) = \mathrm{E} = \overbrace{\{\mathbf{v}\in\mathbb{R}^3 : \langle\mathbf{n},\mathbf{v}-\mathbf{p}\rangle = 0\}}^{\text{normal form}}$$

If the vectors $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$ are given, then one chooses the normal vector $\mathbf{n}\in\mathbb{R}^3$ orthogonal to both, e.g., by using the cross product $\mathbf{n} := \mathbf{a}\times\mathbf{b}$. $\mathbf{p}$ may be the same in both representations.

If $\mathbf{n} = \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$ and $\mathbf{p}\in\mathbb{R}^3$ are given, then the parameter form of E is the solution of

$\mathcal{L} = \left\{\mathbf{v} = \begin{pmatrix}x\\y\\z\end{pmatrix} : \underbrace{\alpha x+\beta y+\gamma z}_{\langle\mathbf{n},\mathbf{v}\rangle} = \langle\mathbf{n},\mathbf{p}\rangle\right\}$ computed with the Gauss algorithm.

$$E = \left\{\mathbf{v}\in\mathbb{R}^3 : \left\langle\begin{pmatrix}1\\0\\-1\end{pmatrix},\mathbf{v}-\begin{pmatrix}2\\1\\1\end{pmatrix}\right\rangle = 0\right\} = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 : x-z=1\right\}$$
$$= \left\{\begin{pmatrix}1+z\\y\\z\end{pmatrix} : y,z\in\mathbb{R}\right\} = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}+y\begin{pmatrix}0\\1\\0\end{pmatrix}+z\begin{pmatrix}1\\0\\1\end{pmatrix} : y,z\in\mathbb{R}\right\}$$

Distance, scalar product and orthogonality

Definition 5.3a) (Hesse normal form, HNF)

The normal form

$$\{\mathbf{v} : \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}$$

of a line $g \subset \mathbb{R}^2$ or plane $E \subset \mathbb{R}^3$ with normal vector $\mathbf{n}$ and included point (location vector) $\mathbf{p}$ is called *Hesse normal form (HNF)* if there holds $\|\mathbf{n}\| = 1$.

Definition 5.3b) (distance $\mathrm{dist}(\cdot,\cdot)$) for affine subspaces

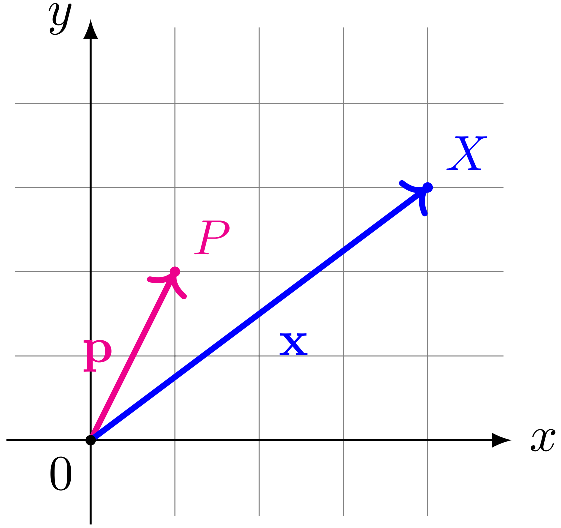
For a point (location vector) $\mathbf{v}$ and affine subspaces S, T in $\mathbb{R}^n$ we define

$$\mathrm{dist}(\mathbf{v}, T) := \min_{\mathbf{t} \in T} \|\mathbf{v} - \mathbf{t}\| \quad \text{and} \quad \mathrm{dist}(S, T) := \min_{\mathbf{s} \in S} \mathrm{dist}(\mathbf{s}, T) = \min_{\mathbf{s} \in S} \min_{\mathbf{t} \in T} \|\mathbf{s} - \mathbf{t}\|$$

for the (shortest) distance between $\mathbf{v}$ and T or between S and T, resp..

Distance, scalar product and orthogonality

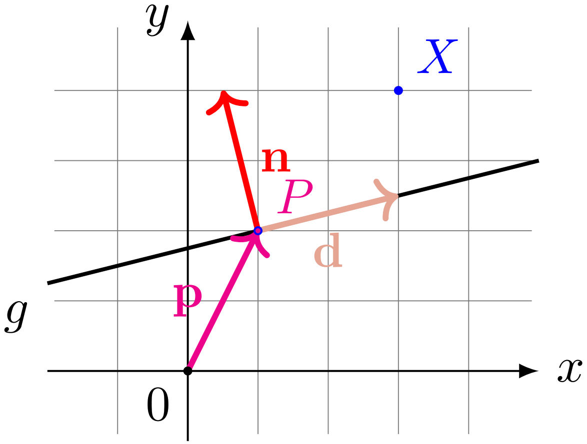
Preliminary considerations for distance computation and orthogonality



$$\operatorname{dist}(P, X) := \|\mathbf{x} - \mathbf{p}\|$$

with location vectors $\mathbf{p}, \mathbf{x}$ for points P, X and norm

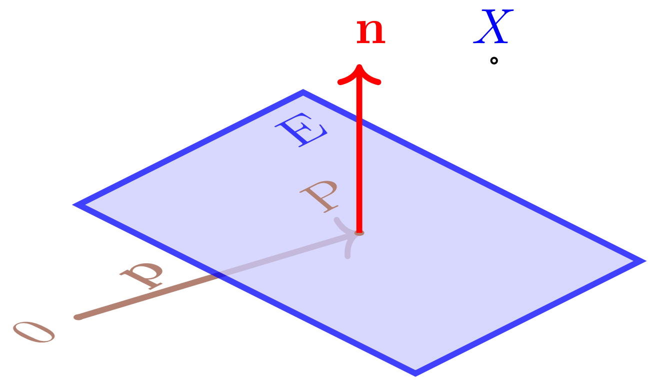
$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$



$$\operatorname{dist}(X, g) := \min_{V \in g} \{\|\mathbf{x} - \mathbf{v}\|\}$$

with location vectors $\mathbf{v}, \mathbf{x}$ for points V, X and line

$$\begin{aligned} g &= \{\mathbf{v} = \mathbf{p} + \alpha \mathbf{d} \colon \alpha \in \mathbb{R}\} \\ &= \{\mathbf{v} \in \mathbb{R}^2 \colon \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}. \end{aligned}$$



$$\operatorname{dist}(X, E) := \min_{V \in E} \{\|\mathbf{x} - \mathbf{v}\|\}$$

with location vectors $\mathbf{v}, \mathbf{x}$ for points V, X and plane

$$E = \{\mathbf{v} \in \mathbb{R}^3 \colon \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}.$$

Distance, scalar product and orthogonality

Relationship between scalar product and angle/orthogonality

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ with $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$.

Then there exist $\alpha, \beta \in \mathbb{R}$ such that

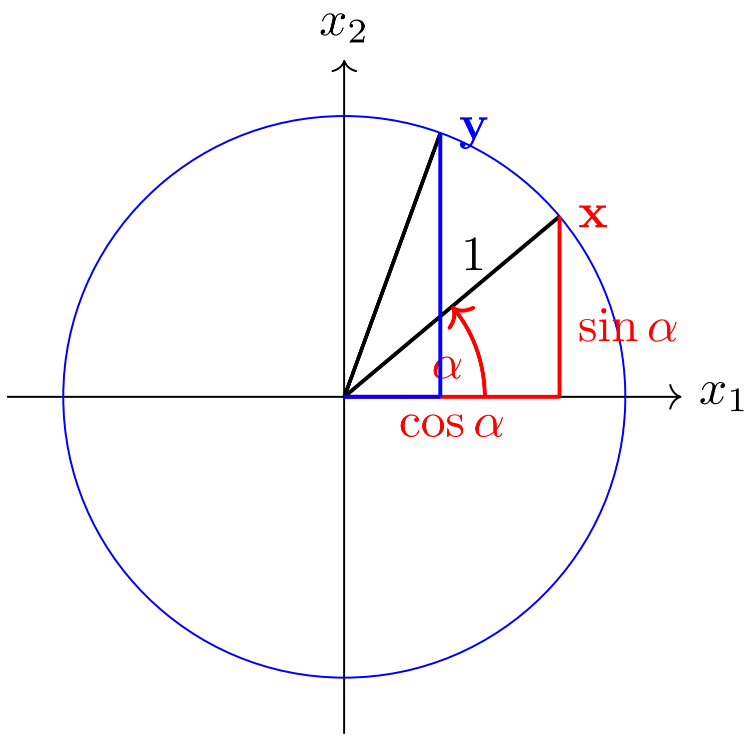
$$\mathbf{x} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}.$$

The angle sum identity for cosine yields

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \cos(\beta - \alpha) = \cos(\angle(\mathbf{x}, \mathbf{y})), \end{aligned}$$

hence

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \quad \Longleftrightarrow \quad \beta - \alpha = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \quad \text{(right angle)}.$$



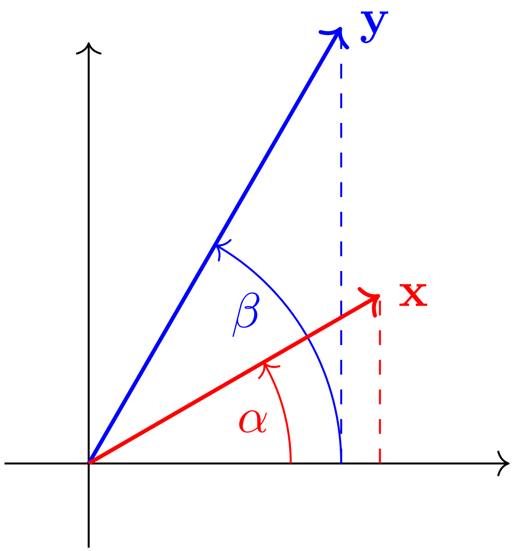
Distance, scalar product and orthogonality

Relationship between scalar product and angle/orthogonality

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{o}\}$ of arbitrary length there holds

$$\mathbf{x} = \|\mathbf{x}\| \cdot \underbrace{\frac{1}{\|\mathbf{x}\|}\mathbf{x}}_{\text{vector of length } 1} \quad , \quad \mathbf{y} = \|\mathbf{y}\| \cdot \underbrace{\frac{1}{\|\mathbf{y}\|}\mathbf{y}}_{\text{vector of length } 1} \quad ,$$

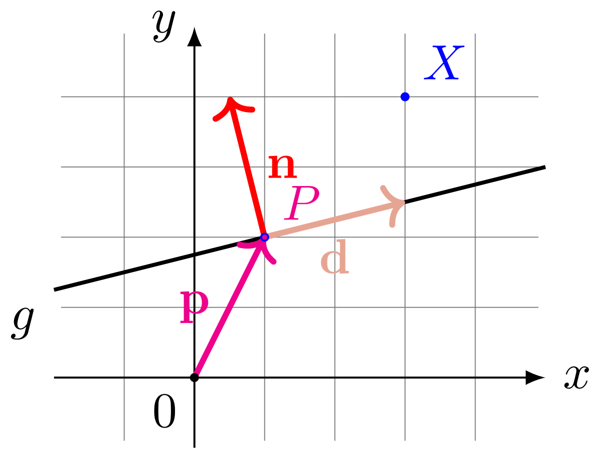
hence
$$\cos(\angle(\mathbf{x}, \mathbf{y})) = \cos(\angle(\tfrac{1}{\|\mathbf{x}\|}\mathbf{x}, \tfrac{1}{\|\mathbf{y}\|}\mathbf{y}))$$
$$= \left\langle \tfrac{1}{\|\mathbf{x}\|}\mathbf{x}, \tfrac{1}{\|\mathbf{y}\|}\mathbf{y} \right\rangle = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|\|\mathbf{y}\|} \in [-1, 1].$$



Accordingly, we define the **angle** $\alpha = \angle(\mathbf{x}, \mathbf{y})$ between two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{o}\}$:

$$\alpha := \arccos\left(\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|}\right) \in [0, \pi].$$

Distance, scalar product and orthogonality



$$\operatorname{dist}(X,g) := \min_{V\in g}\{\|\mathbf{x}-\mathbf{v}\|\}$$

with location vectors $\mathbf{v},\mathbf{x}$ for points V, X and line

$$\begin{aligned} g &= \{\mathbf{v} = \mathbf{p} + \alpha\mathbf{d} \colon \alpha \in \mathbb{R}\} \\ &= \{\mathbf{v} \in \mathbb{R}^2 \colon \langle \mathbf{v}-\mathbf{p}, \mathbf{n}\rangle = 0\}. \end{aligned}$$

Computation of $\operatorname{dist}(X,g)$: Consider

$$\mathbf{v}_{PX} = \alpha\mathbf{d} + \beta\mathbf{n} = \mathbf{v}_{PQ} + \mathbf{v}_{QX}.$$

The point Q is (still) unknown, we are looking for

$$\operatorname{dist}(X,g) = \|\mathbf{v}_{QX}\|.$$

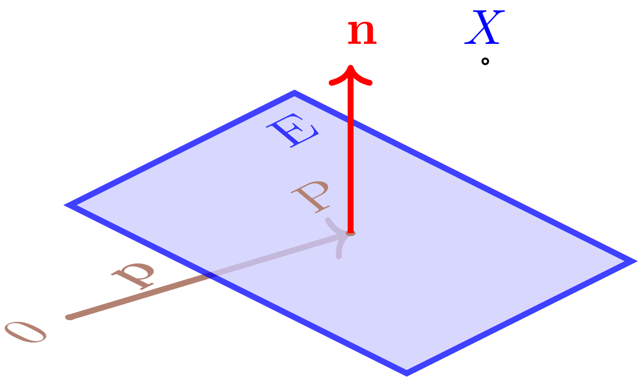
Forming the scalar product on both sides with $\mathbf{n}$:

$$\begin{aligned} \langle \mathbf{v}_{PX}, \mathbf{n}\rangle &= \langle \alpha\mathbf{d} + \beta\mathbf{n}, \mathbf{n}\rangle \\ &= \alpha \underbrace{\langle \mathbf{d}, \mathbf{n}\rangle}_{=0} + \beta \langle \mathbf{n}, \mathbf{n}\rangle = \beta\|\mathbf{n}\|^2 \end{aligned}$$

$$\Longrightarrow \quad \operatorname{dist}(X,g) = \|\mathbf{v}_{QX}\| = \|\beta\mathbf{n}\| = \frac{|\langle \mathbf{v}_{PX}, \mathbf{n}\rangle|}{\|\mathbf{n}\|}$$

$$\boxed{\operatorname{dist}(X,g) = \left|\left\langle \mathbf{x}-\mathbf{p}, \tfrac{1}{\|\mathbf{n}\|}\mathbf{n}\right\rangle\right|}$$

Distance, scalar product and orthogonality



$$\operatorname{dist}(X,E) := \min_{V\in E}\{\|\mathbf{x}-\mathbf{v}\|\}$$

with location vectors $\mathbf{v},\mathbf{x}$ for points V, X and plane

$$E = \{\mathbf{v}\in\mathbb{R}^3\colon\ \langle \mathbf{v}-\mathbf{p},\mathbf{n}\rangle = 0\}.$$

Computation of $\operatorname{dist}(X,E)$: Consider

$$\mathbf{v}_{PX} = \mathbf{v}_{PQ} + \mathbf{v}_{QX}$$

with $Q\in E$ and $\mathbf{v}_{PQ}\perp\mathbf{v}_{QX} = \beta\mathbf{n}$.
The point Q is (still) unknown, we are looking for

$$\operatorname{dist}(X,E) = \|\mathbf{v}_{QX}\|.$$

Forming the scalar product on both sides with $\mathbf{n}$:

$$\begin{aligned}\langle\mathbf{v}_{PX},\mathbf{n}\rangle &= \langle\mathbf{v}_{PQ}+\beta\mathbf{n},\mathbf{n}\rangle\\ &= \underbrace{\langle\mathbf{v}_{PQ},\mathbf{n}\rangle}_{=0} + \beta\,\langle\mathbf{n},\mathbf{n}\rangle = \beta\|\mathbf{n}\|^2\end{aligned}$$

$$\Longrightarrow\quad \operatorname{dist}(X,E) = \|\mathbf{v}_{QX}\| = \|\beta\mathbf{n}\| = \frac{|\langle\mathbf{v}_{PX},\mathbf{n}\rangle|}{\|\mathbf{n}\|}$$

$$\boxed{\operatorname{dist}(X,E) = \left|\left\langle\mathbf{x}-\mathbf{p},\tfrac{1}{\|\mathbf{n}\|}\mathbf{n}\right\rangle\right|}$$

Distance, scalar product and orthogonality

Distances between point, line $g_{\mathbf{p},\mathbf{d}}$ and plane $E_{\mathbf{p},\mathbf{n}}$ in $\mathbb{R}^3$ (with $\|\mathbf{n}\| = 1$)

- **point/point:** $\mathrm{dist}(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$
- **point/plane:** $\mathrm{dist}(\mathbf{v}, E_{\mathbf{p},\mathbf{n}}) = |\langle \mathbf{v} - \mathbf{p}, \mathbf{n}\rangle|$
- **line/plane:**

$$\mathrm{dist}(g_{\mathbf{p},\mathbf{d}}, E_{\mathbf{q},\mathbf{n}}) = \begin{cases} \mathrm{dist}(\mathbf{p}, E_{\mathbf{q},\mathbf{n}}) & : \quad g_{\mathbf{p},\mathbf{d}} \text{ is parallel to } E_{\mathbf{q},\mathbf{n}}, \\ 0 & : \quad \text{else } (g_{\mathbf{p},\mathbf{d}} \text{ and } E_{\mathbf{q},\mathbf{n}} \text{ intersect}). \end{cases}$$

  $g_{\mathbf{p},\mathbf{d}}$ is parallel to $E_{\mathbf{q},\mathbf{n}}$ when there holds $\langle \mathbf{d}, \mathbf{n}\rangle = 0$.

- **plane/plane:**

$$\mathrm{dist}(E_{\mathbf{p}_1,\mathbf{n}_1}, E_{\mathbf{p}_2,\mathbf{n}_2}) = \begin{cases} \mathrm{dist}(\mathbf{p}_1, E_{\mathbf{p}_2,\mathbf{n}_2}) & : \quad E_{\mathbf{p}_1,\mathbf{n}_1} \text{ is parallel to} E_{\mathbf{p}_2,\mathbf{n}_2}, \\ 0 & : \quad \text{else } (E_{\mathbf{p}_1,\mathbf{n}_1} \text{ and } E_{\mathbf{p}_2,\mathbf{n}_2} \text{ intersect}). \end{cases}$$

  $E_{\mathbf{p}_1,\mathbf{n}_1}$ and $E_{\mathbf{p}_2,\mathbf{n}_2}$ are parallel if their normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are multiples of each other.

Distance, scalar product and orthogonality

Distance between two lines in $\mathbb{R}^3$

- **line/line:** Let two lines in $\mathbb{R}^3$ be given:

$$g_1 = g_{\mathbf{p},\mathbf{a}} = \mathbf{p} + \operatorname{span}(\mathbf{a}) \quad \text{and} \quad \mathrm{g}_2 = \mathrm{g}_{\mathbf{q},\mathbf{b}} = \mathbf{q} + \operatorname{span}(\mathbf{b})$$

  - 1. Case: If $\mathbf{a}$ and $\mathbf{b}$ are parallel (i.e., multiples of each other), then g_1 and g_2 are parallel. Let $E_{\mathbf{p},\mathbf{n}}$ be the plane through $\mathbf{p}$ with normal vector $\mathbf{n} := \frac{1}{\|\mathbf{a}\|}\mathbf{a}$.
Let the intersection point of $E_{\mathbf{p},\mathbf{n}}$ and g_2 be denoted by $\mathbf{p}'$ (Computation: Substitute the parameter form of g_2 into the normal form of $E_{\mathbf{p},\mathbf{a}}$). Then there holds

$$\operatorname{dist}(g_1, g_2) = \operatorname{dist}(\mathbf{p}, \mathbf{p}') = \|\mathbf{p} - \mathbf{p}'\|.$$

  - 2. Case: If $\mathbf{a}$ and $\mathbf{b}$ (and hence g_1 and g_2) are not parallel ("skew"), then $E_{\mathbf{q},\mathbf{n}}$ with $\mathbf{n} := \frac{1}{\|\mathbf{a}\times\mathbf{b}\|}\mathbf{a} \times \mathbf{b}$ is the plane that contains g_2 and is parallel to g_1. Then there holds

$$\operatorname{dist}(g_1, g_2) = \operatorname{dist}(\mathbf{p}, E_{\mathbf{q},\mathbf{n}}).$$

Distance, scalar product and orthogonality

Distance between a point $\mathbf{p}$ and a line $g_{\mathbf{q},\mathbf{a}}$ in $\mathbb{R}^3$

- **point/line:** Let $\mathbf{p}$ be a point and $g_{\mathbf{q},\mathbf{a}} = \mathbf{q} + \mathrm{span}(\mathbf{a})$ a line in $\mathbb{R}^3$.

  Let $E_{\mathbf{p},\mathbf{n}}$ with $\mathbf{n} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$, and let $\mathbf{p}'$ be the intersection of $E_{\mathbf{p},\mathbf{n}}$ and $g_{\mathbf{q},\mathbf{a}}$. Then there holds

$$\mathrm{dist}(\mathbf{p}, g_{\mathbf{q},\mathbf{a}}) = \mathrm{dist}(\mathbf{p}, \mathbf{p}').$$

True or false?

1. The lines

$$g = \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} \ : \ \lambda \in \mathbb{R} \right\} \text{ and } h = \left\{ \mathbf{v} \in \mathbb{R}^2 \ : \ \langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \mathbf{v} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle = 0 \right\}$$

are disjoint (i.e., have no point of intersection).

2. For vectors

$$\mathbf{p} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

the distance between the line $g_{\mathbf{p},\mathbf{d}}$ through $\mathbf{p}$ with direction $\mathbf{d}$ and the plane $E_{\mathbf{q},\mathbf{n}}$ through $\mathbf{q}$ and with normal vector $\mathbf{n}$ is $\operatorname{dist}(g_{\mathbf{p},\mathbf{d}}, E_{\mathbf{q},\mathbf{n}}) = 1$.