Discrete Algebraic Structures

WiSe 2025/2026

Prof. Dr. Antoine Wiehe
Research Group for Theoretical Computer Science

Meetings:

- Lectures: Monday 15:00-16:30
- Lecture hall exercises:
  - Wednesday 08:30-09:15
  - Recap of the lecture, exercises to practice the methods
- Tutorials: Starting this week.
  - Choose a group on Stud.IP (in Participants → Groups) and stick to that group.
  - Deeper look into the content of the lecture.
- All learning material available on StudIP: slides, lecture notes, exercise sheets, recordings of the lectures

Exam and Bonus points:

- Tests every week at the beginning of the tutorial
- Get up to 3 points per week ($3 \times 12 = 36$ points)
- Need ≥ 16 points to validate the *Studienleistung*
- Extra points are given as bonus for the exam (up to 10% of the grade)

Test points	Exam points	End result
< 16		module not validated
$16 + y$	x	$x + y$
36	80/200 (fail)	100/200 (pass)

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- Develop algorithms
- Solve problems
- Think and communicate with others about complex/abstract concepts

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"*Der Data Scientist arbeitet forschungsnah und ist somit immer auf dem neuesten Stand über neue Entwicklungen in dem sich rasant entwickelnden Feld.*"

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- Maths is necessary for keeping up with the state of the art
- English is as well (research papers are published in English, no matter where the research is done)

Two types of mathematics:

- Study of “continuous” things: real numbers, functions, quantities that can change smoothly
- Study of “discrete” things: natural numbers, things that can only take values changing in steps

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  So π is not a computer thing
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Turing award 2012

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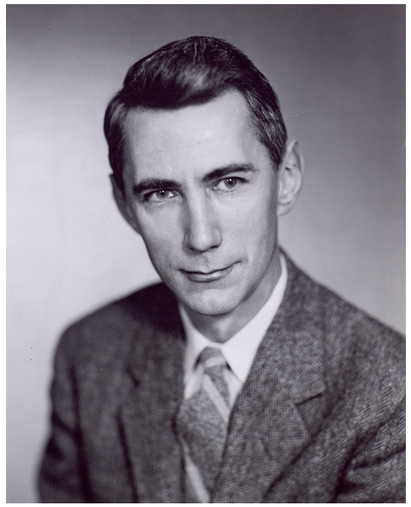

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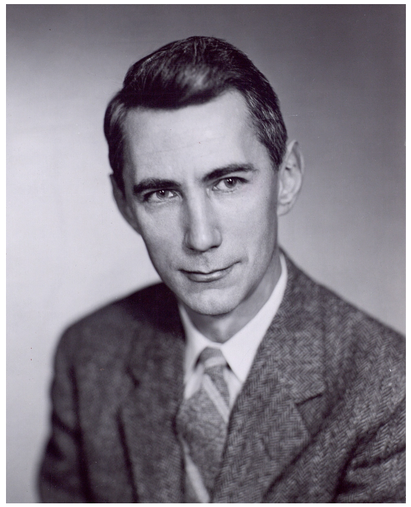

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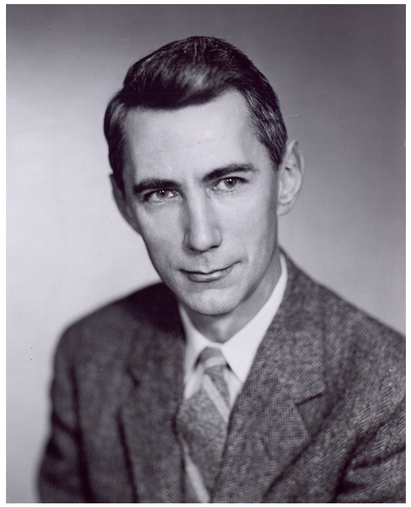

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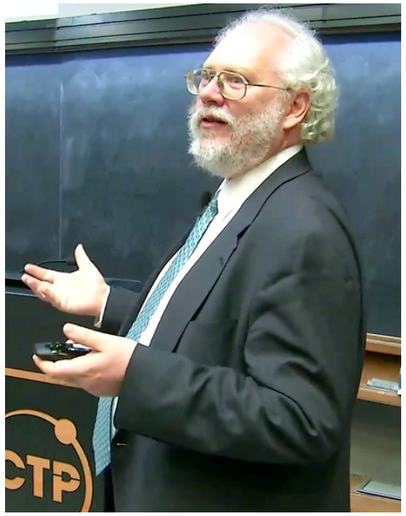

Peter Shor
Shannon award 2025

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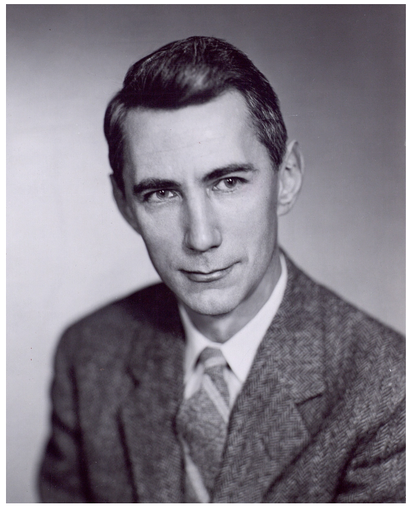

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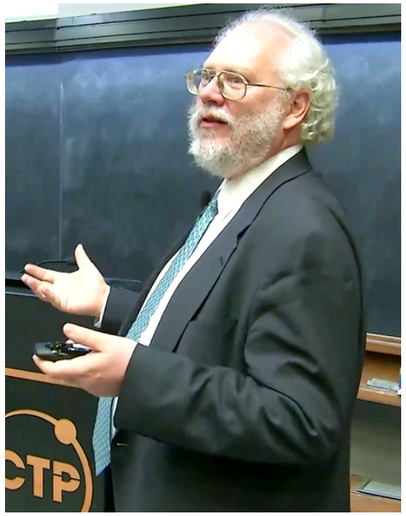

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- No mathematical knowledge is expected
- Critical thinking
- That's it

You:

- dedicate at least 8–10 hours per week to the course
  - Lectures: 1.5 hours
  - Lecture hall exercises: 0.75 hour
  - Tutorials: 1.5 hours
  - All the rest: read the material, do exercises, talk with your study partner
- find somebody to study with
- are active and use the material we give you: lecture notes, reference books, supplemental exercises
- will reach out to your tutors and use the consultation hours if you are falling behind

We (me + tutors):

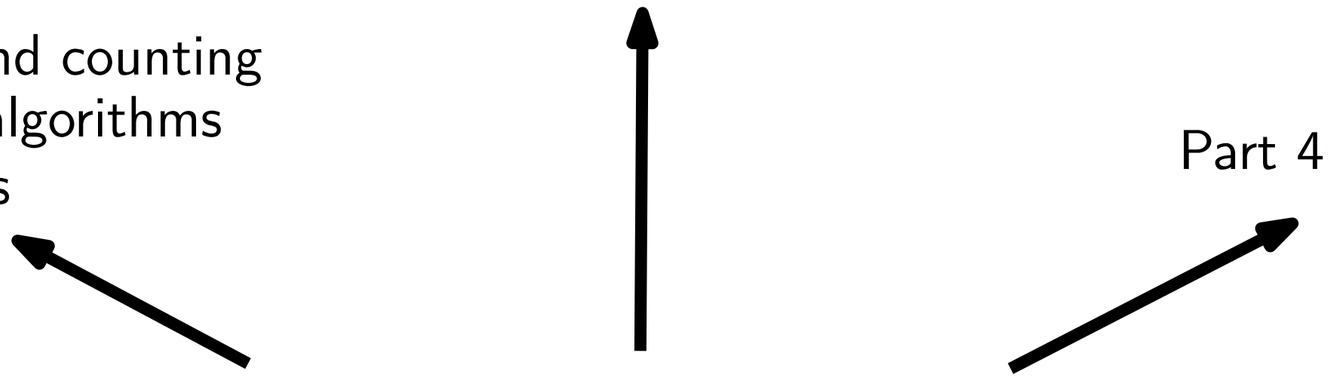
- prepare exercises and solutions for you
- are available for taking care of you if you have difficulties
- present on the forum to answer questions
- will do what we can so that you succeed

Common main objective: your success

Part 5: Algebra
- Cryptography
- Error correction
- Quantum Computing

Part 3: Combinatorics and counting
- Runtime analysis of algorithms
- Basics of probabilities

Part 4: basic number theory



Part 2: Graphs and relations
- Representing data and relationships between things
- Important basis for the rest of your studies (in particular "Algorithms and Data Structures")

Part 1: introduction to mathematics
- Sets
- Basics in logic and proofs

Many everyday things are built on mathematics:

- Google Maps
- Online banking
- QR codes
- Recommendations in Netflix/...
- Large Language Models
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Probably not the right attitude!

Part 1
Introduction to mathematics

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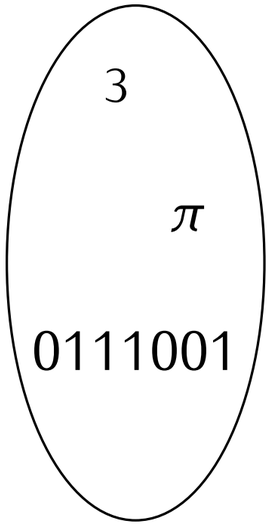
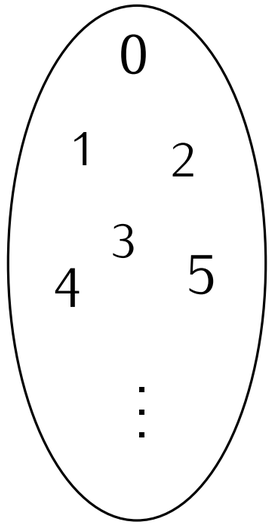


A set with 3 elements



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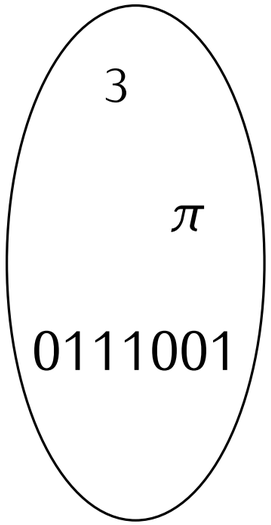

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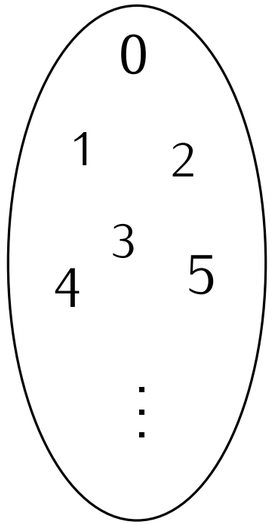

A set with infinitely many elements

Instead of drawing potatoes, we use the following notation:

Notation. $\{3, \pi, 0111001\}, \{0, 1, 2, 3, 4, 5, \dots\}$

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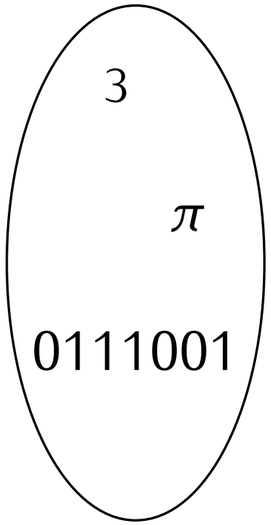

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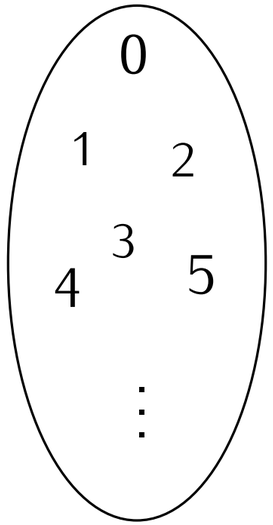

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This notation does not care about multiple elements and order:

$$\{3, 3, 1, 2\} = \{1, 2, 3\} = \{1, 1, 2, 2, 3, 3\}$$

What is your relationship to programming?

- don't know anything about it
- I know (at least a little) C
- I know (at least a little) Python
- I know (at least a little) another programming language

Definition. A set is an unordered bag containing distinct objects.

- $y \in S$: "y is in S", "y is an element of S", "S contains y"

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  Same as $S \subseteq T$ and $T \subseteq S$ being true at the same time!

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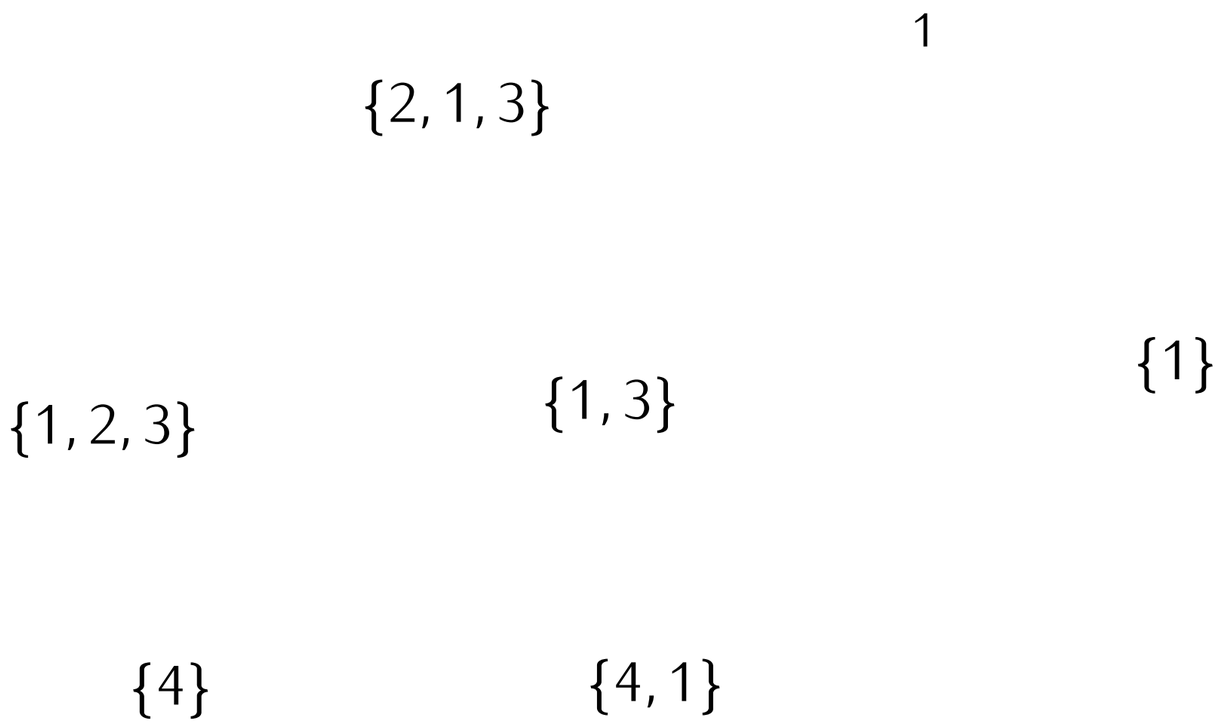
Remark. Note the following:
- For each y and S: either $y \in S$ or $y \notin S$ (any thing is either in or out of S)
- A set can have other sets as elements: $\{0, 1, \{0, 1\}, 2\}$ is a set with elements

Draw an arrow from x to y if $x \subseteq y$:



Notation. We use the following notation:

- $\mathbb{N}_0 \overset{\text{def}}{=} \{0, 1, 2, 3, \dots\}$ natural numbers
- $\mathbb{N} \overset{\text{def}}{=} \{1, 2, 3, \dots\}$
- $\mathbb{Z} \overset{\text{def}}{=} \{0, 1, -1, 2, -2, \dots\}$ integers
- $\mathbb{Q} \overset{\text{def}}{=} \{0, 1, \frac{1}{2}, -\frac{1}{5}, \frac{7}{12}, \dots\}$ rational numbers
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When a set S is already known, we can create a subset of it by selecting some elements:

$$\{x \in S \mid \text{selection criterion } P(x)\}$$

- $\{x \in \mathbb{N}_0 \mid x \text{ is even}\}$
- $\{x \in \mathbb{R} \mid x^2 = 1\}$
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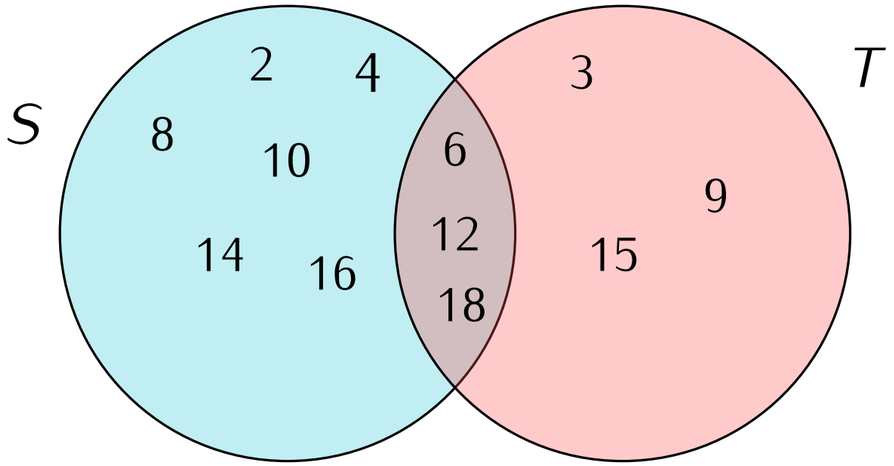
```
def setbuilder(S:set, P) -> set:
    T = set()
    # loop through all elements of S
    for x in S:
        # If x satisfies the criterion
        if P(x) == True:
            T.add(x) # Add it
    return T

{ x for x in S if P(x) } # Same thing
```

- $S \cup T$: things that are in at least one of S or T



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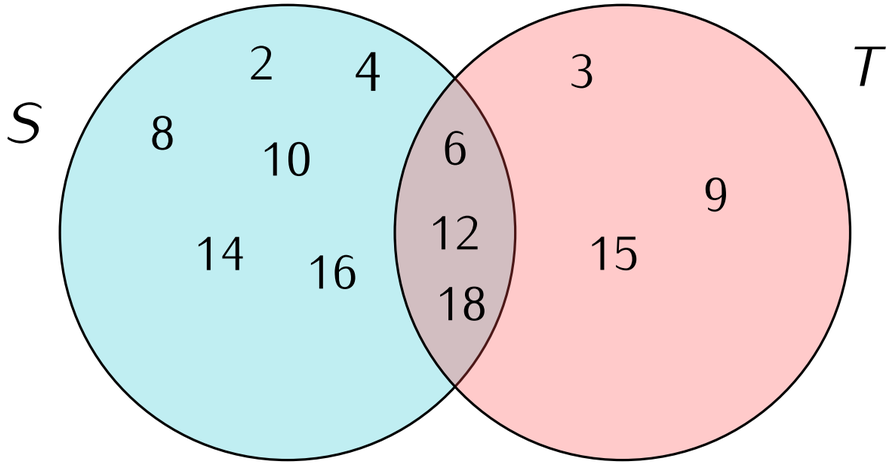

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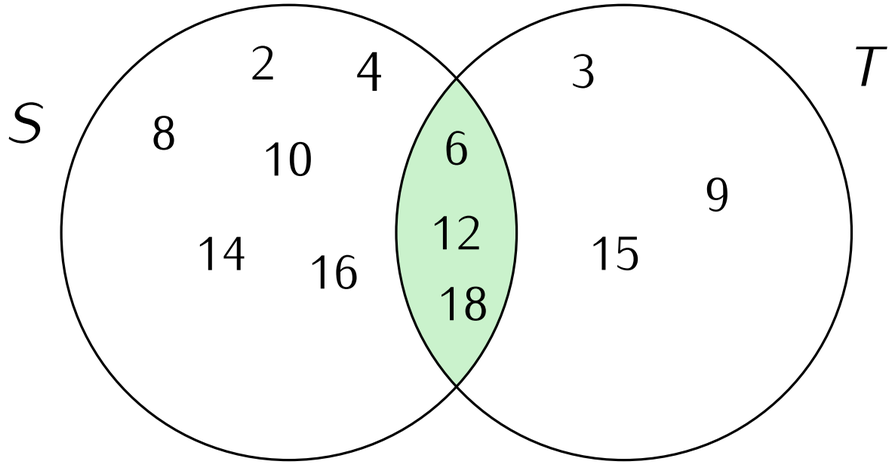

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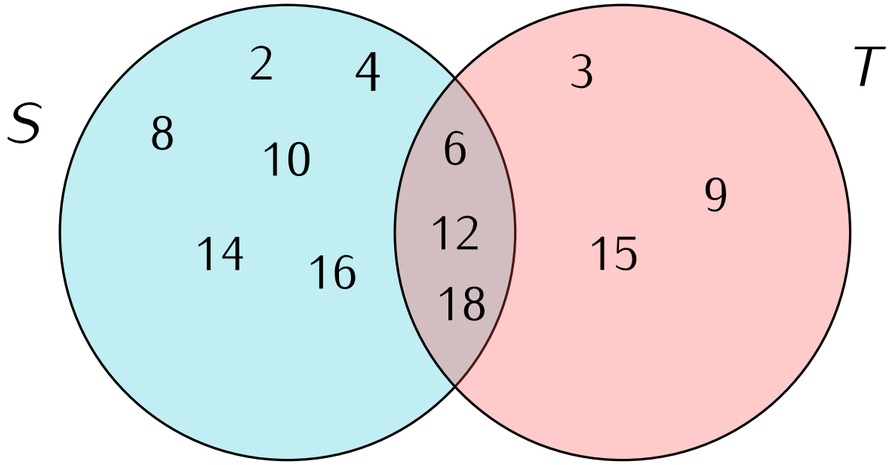

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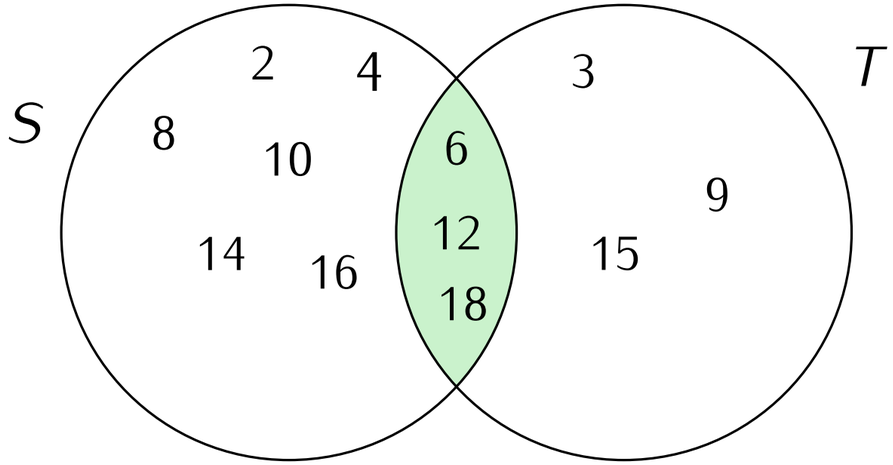

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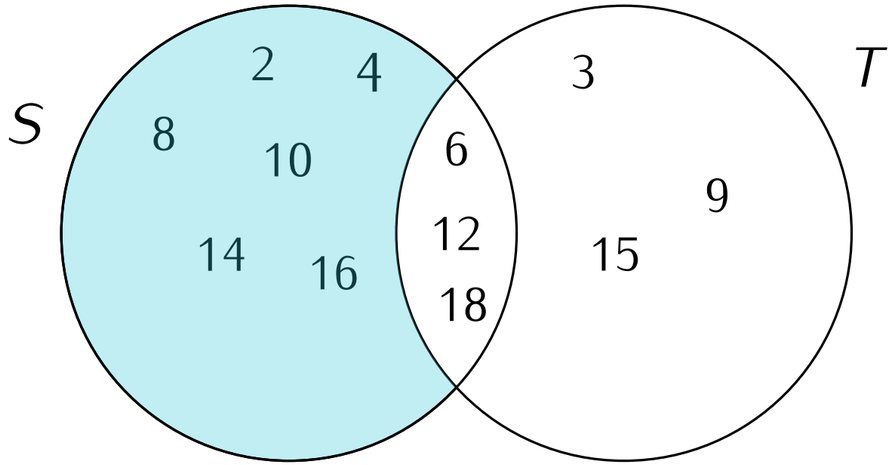

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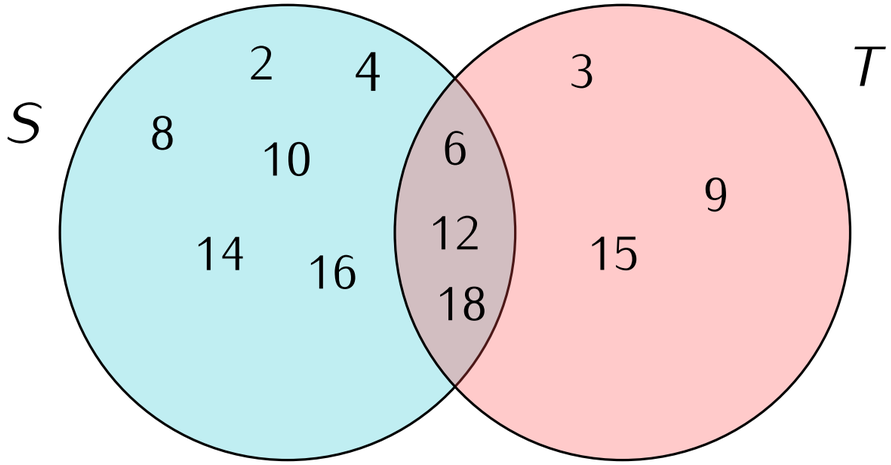

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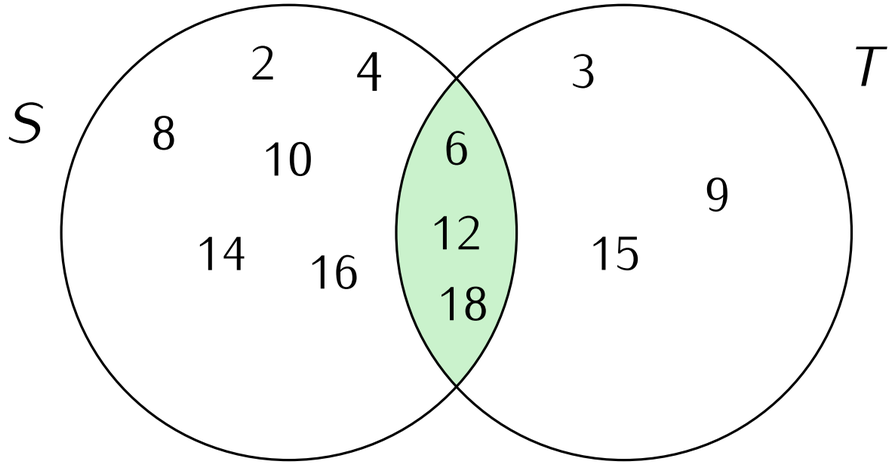

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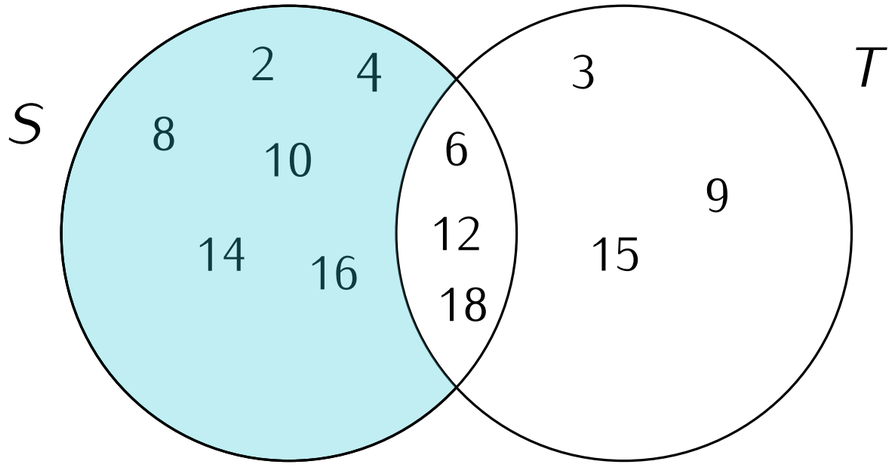

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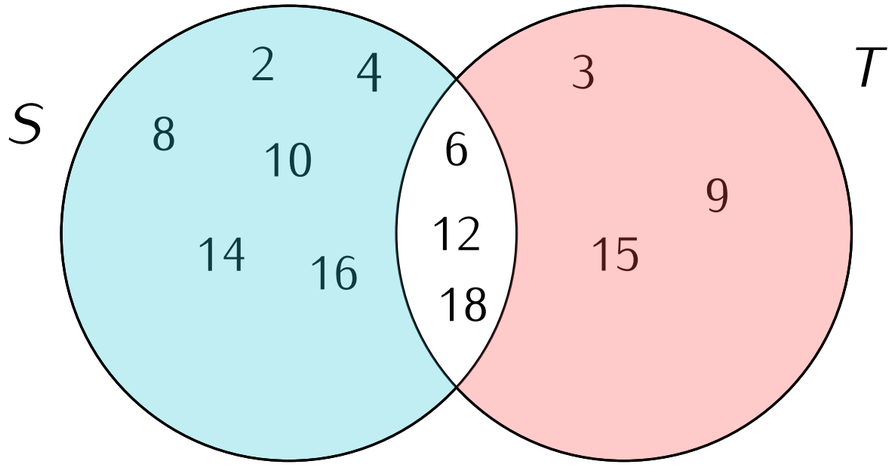

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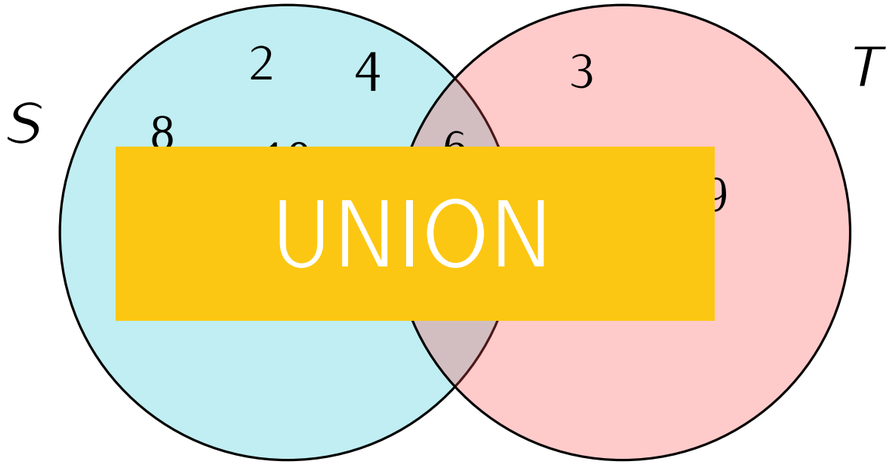

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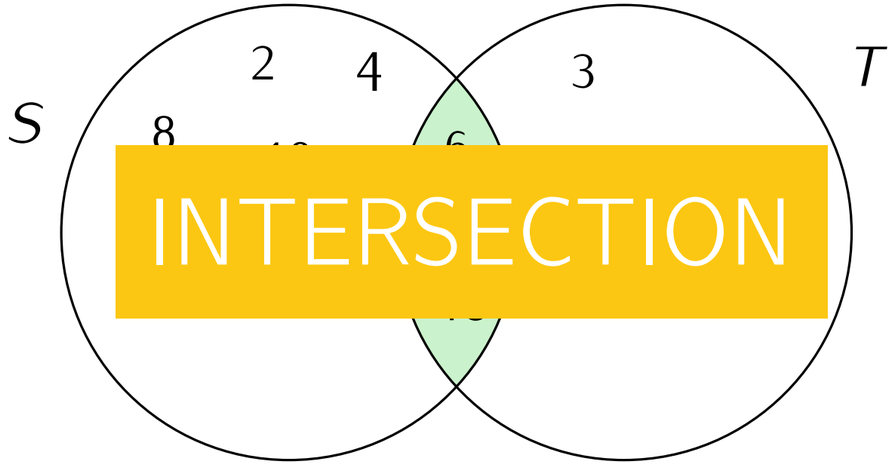

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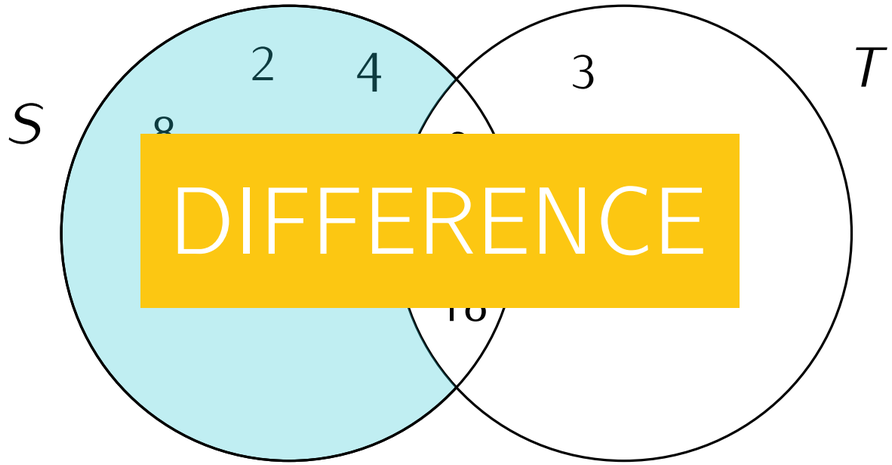

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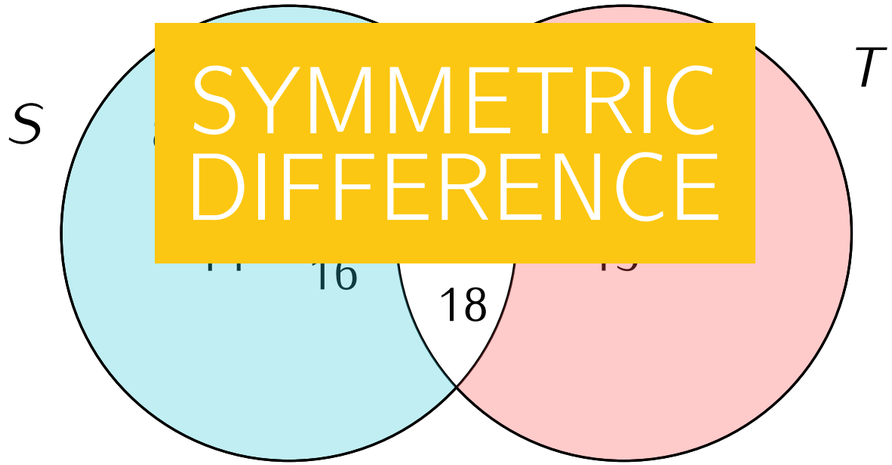

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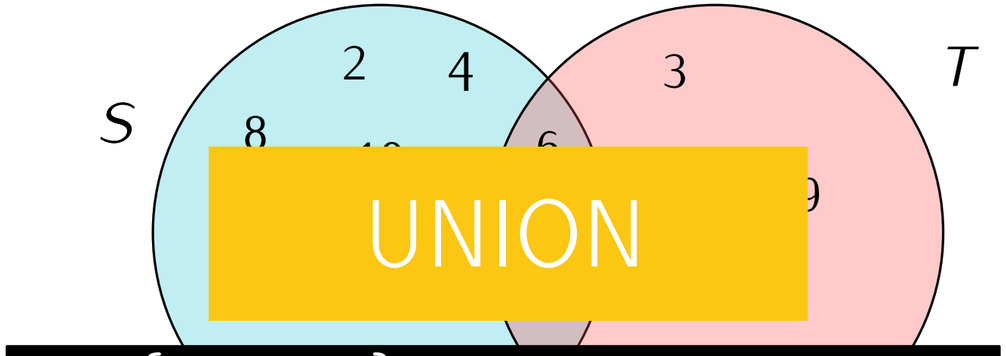
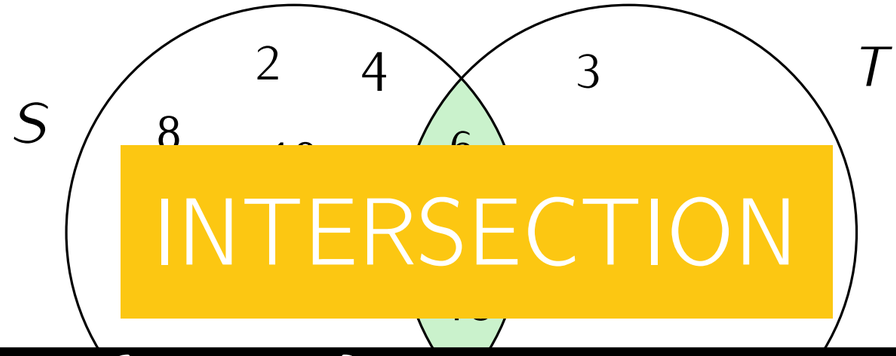


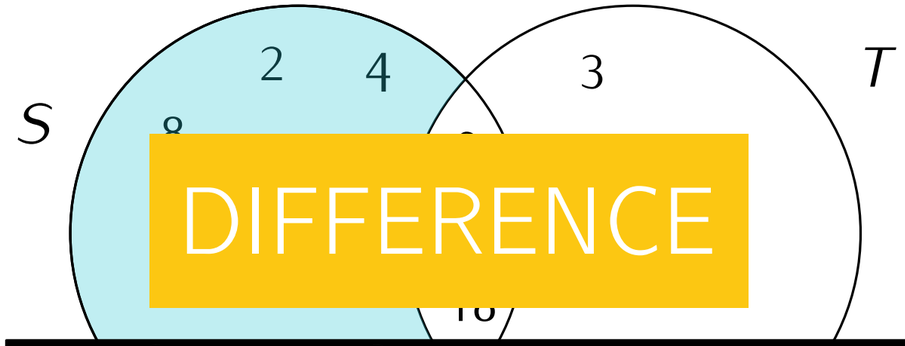

```
S = {3,4,5,6}
T = {2,3,7}
print(S.union(T)) # {3,4,5,6,2,7}
```

- $S \cap T$: things that are in both S and T




```
S = {3,4,5,6}
T = {2,3,7}
print(S.intersection(T)) # {3}
```

- $S \setminus T$: things in S but not in T




```
S = {3,4,5,6}
T = {2,3,7}
print(S.difference(T)) # {4,5,6}
```

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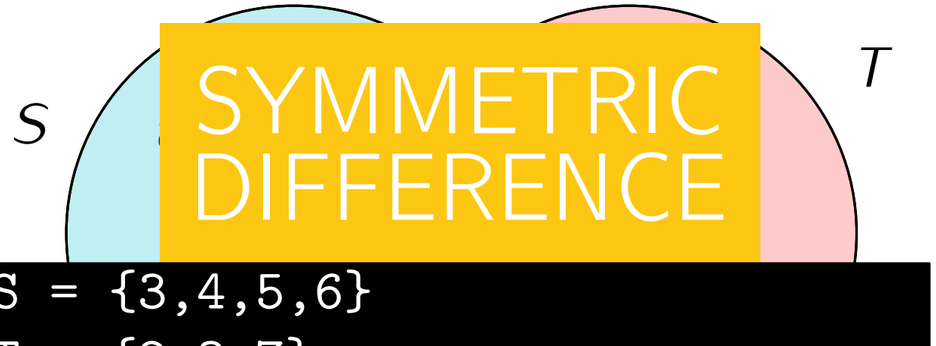


```
S = {3,4,5,6}
T = {2,3,7}
# {4,5,6,2,7}
print(S.symmetric_difference(T))
```

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- $S \cap T$: things that are in both S and T
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If $S = \{1, 2, \{1, 3\}\}$ and $T = \{1, 3\}$, what is $S \cap T$?

- $\{1, 2, 3\}$
- $\{1, \{1\}\}$
- $\{1, 3\}$
- $\{1\}$

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If $S = \{1, 2, \{1, 3\}\}$ and $T = \{1, 3\}$, what is $S \setminus T$?

- $\{1, 2, 3\}$
- $\{1, 2\}$
- $\{2, \{1, 3\}\}$
- $\{2\}$

- Common class/structure in programming: tuples
  Tuple: ordered list that can contain repetitions: $(1, 1, 2), (\text{'antoine'}, 20, 10), \ldots$
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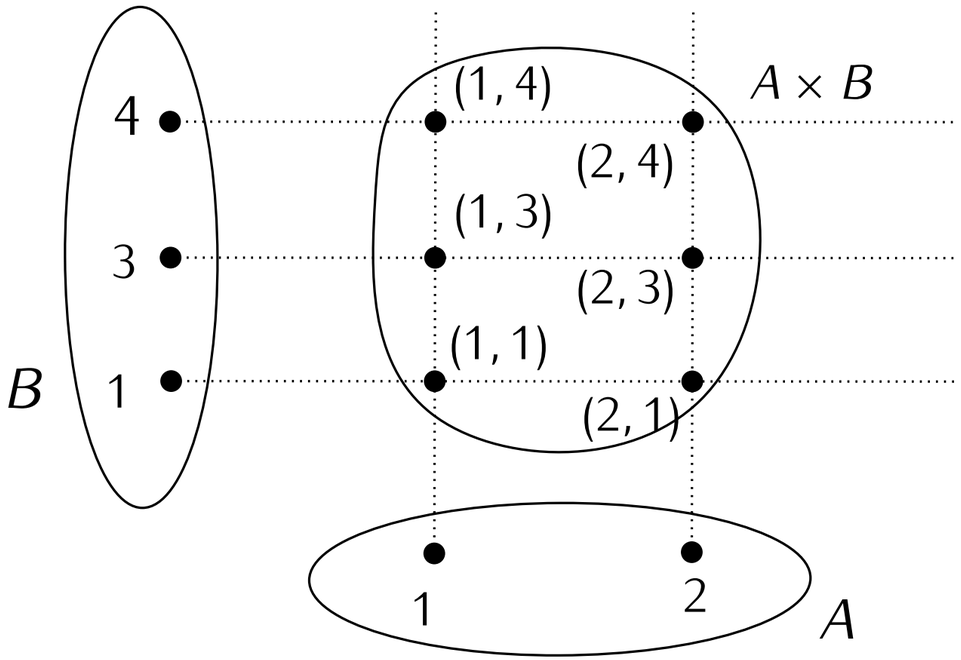
$A \times B = \{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4)\}$



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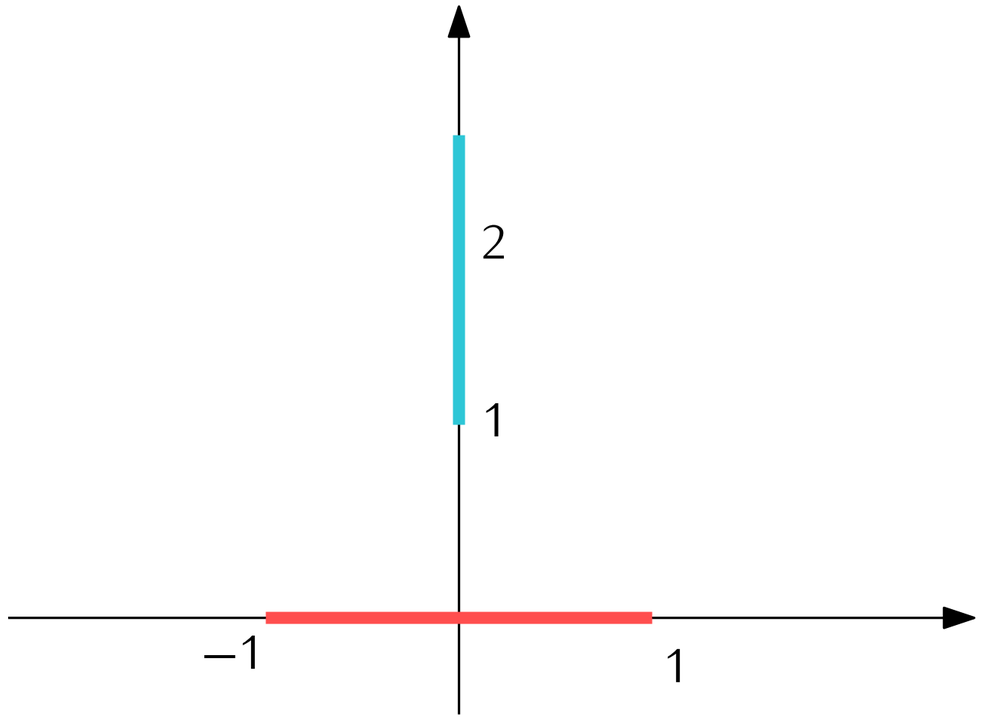

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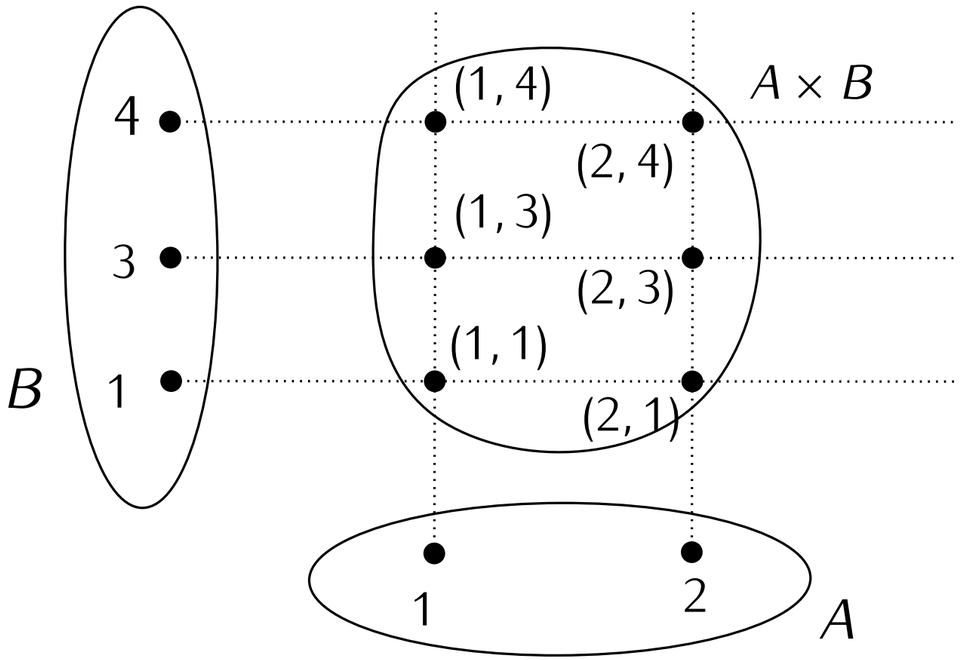

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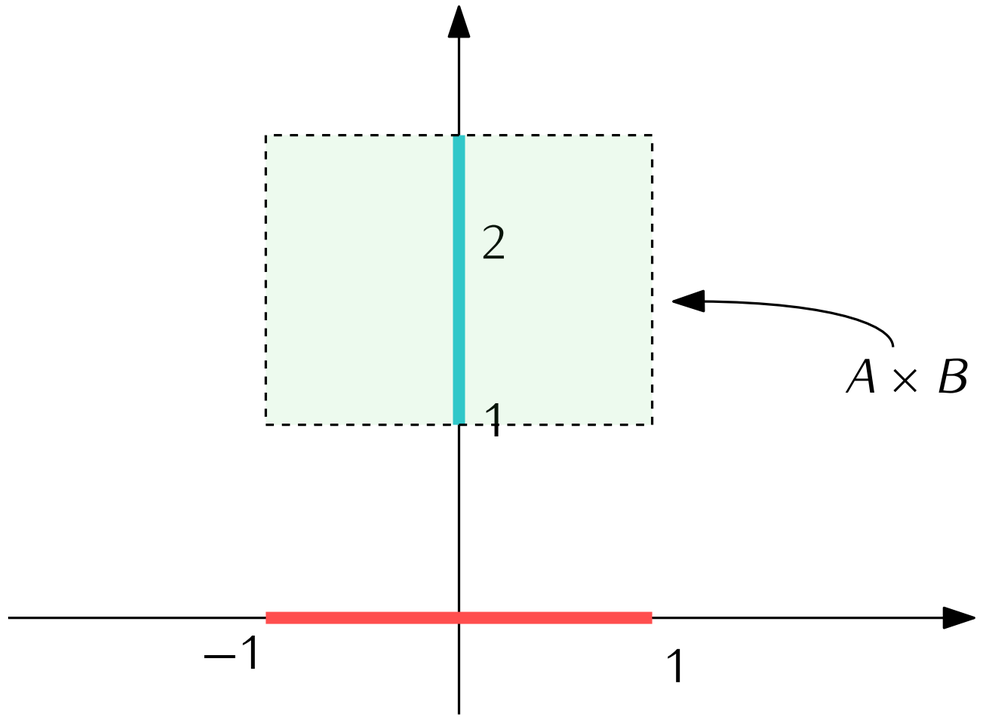

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- Binary strings with n bits = elements of $\{0, 1\}^n$
- Character strings = elements of $\{\text{'a'}, \text{'b'}, \ldots, \text{'z'}\}^n$
- Vectors in n dimensions = elements of $\mathbb{R}^n$

Definition. The powerset of S is the set $\mathcal{P}(S)$ that contains all the subsets of S.

So $A \in \mathcal{P}(S)$ is the same as $A \subseteq S$.
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```
from more-itertools import powerset

S = {1,2,3}
from A in powerset(S):
    print(A)
# ()
# (1,), (2,), (3,)
# (1,2), (1,3), (2,3)
# (1,2,3)
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Example. $\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

How many elements does $\mathcal{P}(\{e, \pi\})$ have?

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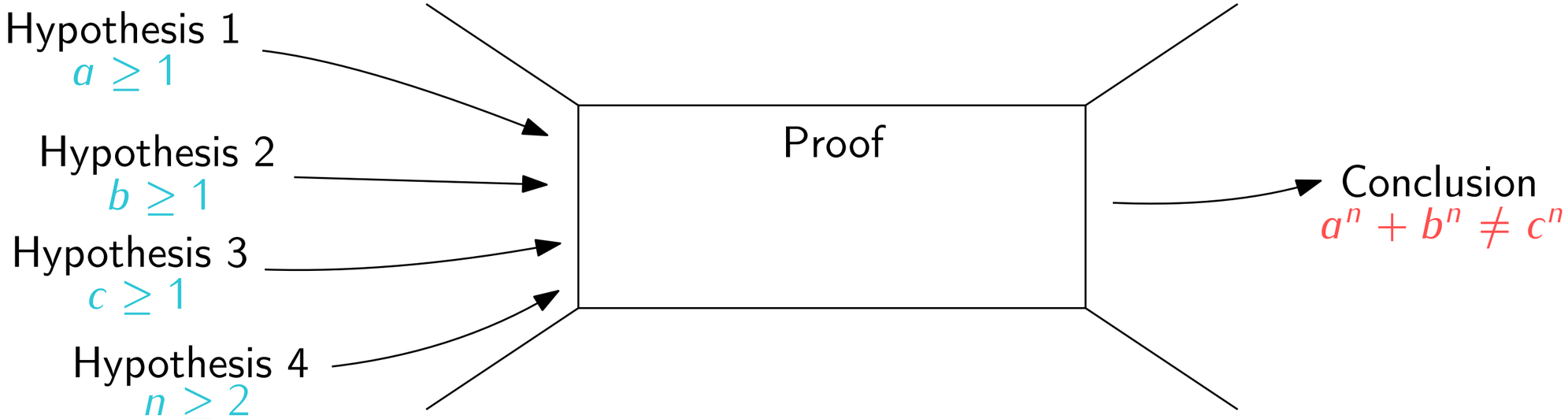
For all $a, b, c \geq 1$ and $n \geq 3$, we have $a^n + b^n \neq c^n$



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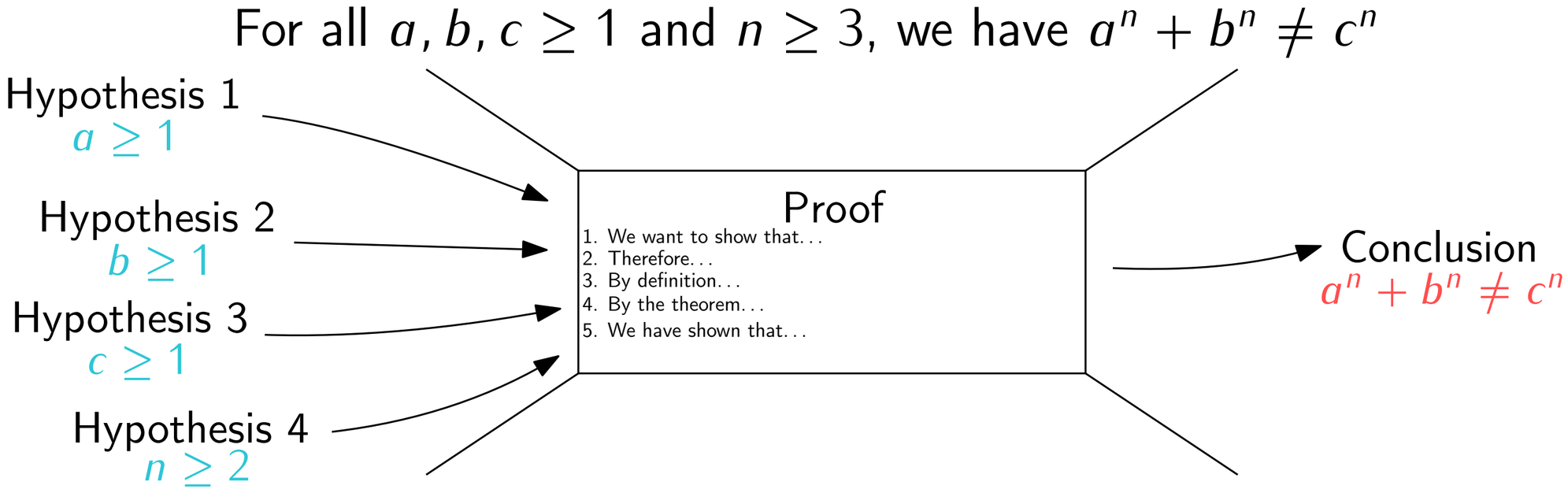

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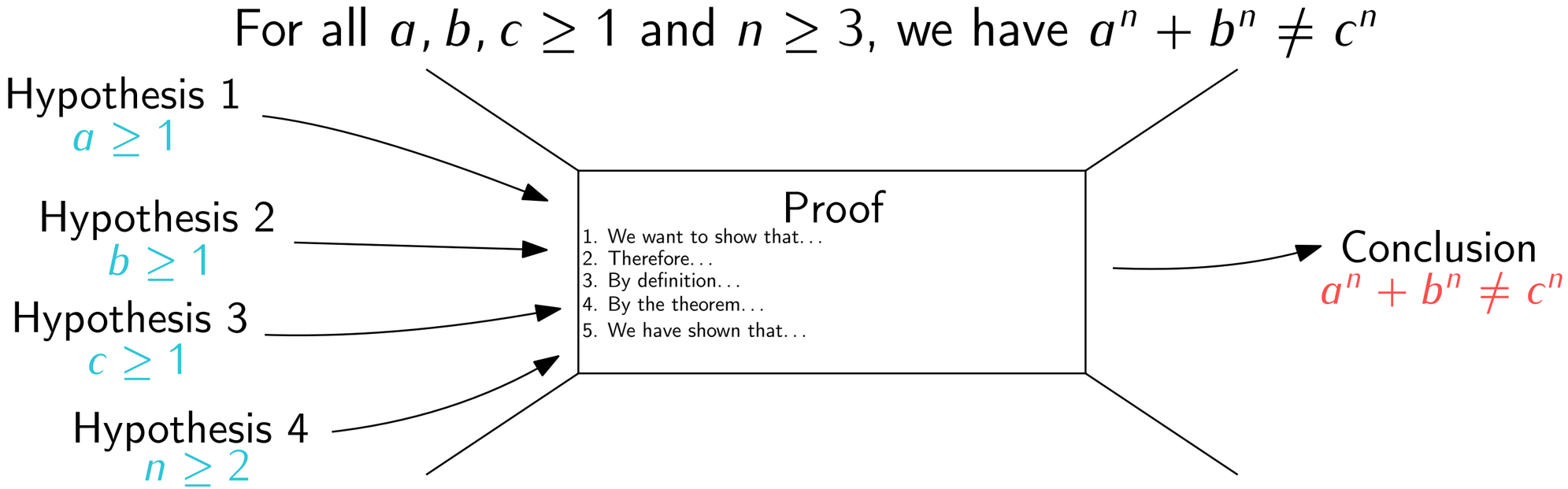

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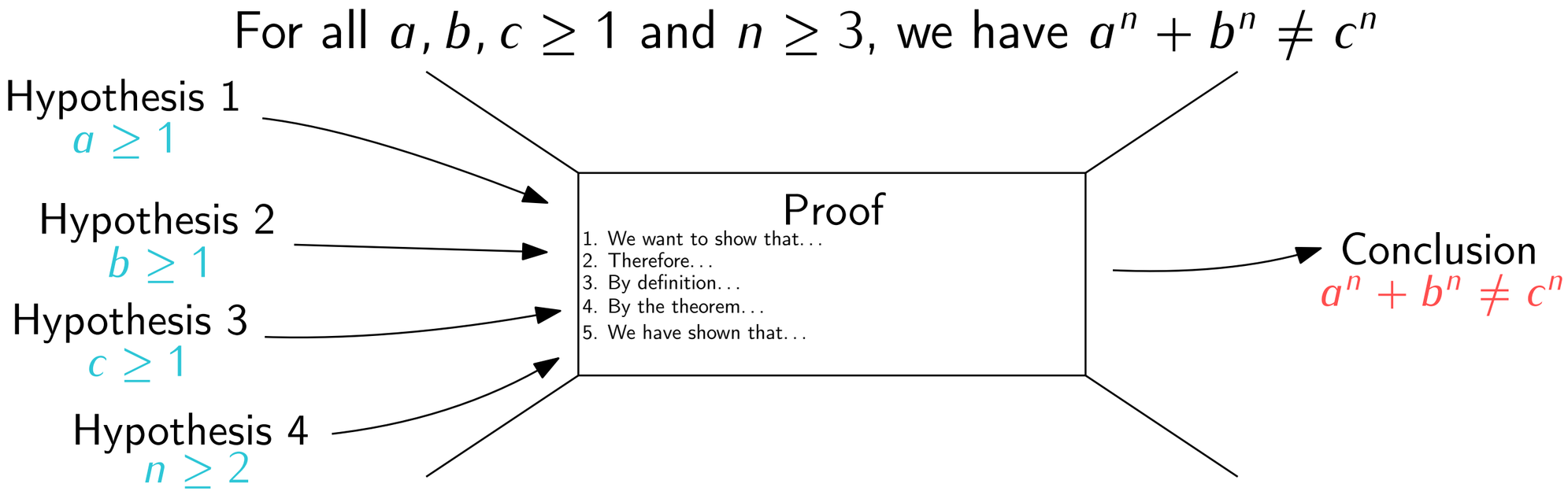

- A proof is a sequence of steps, like a cooking recipe. At each step:
  - Apply a definition
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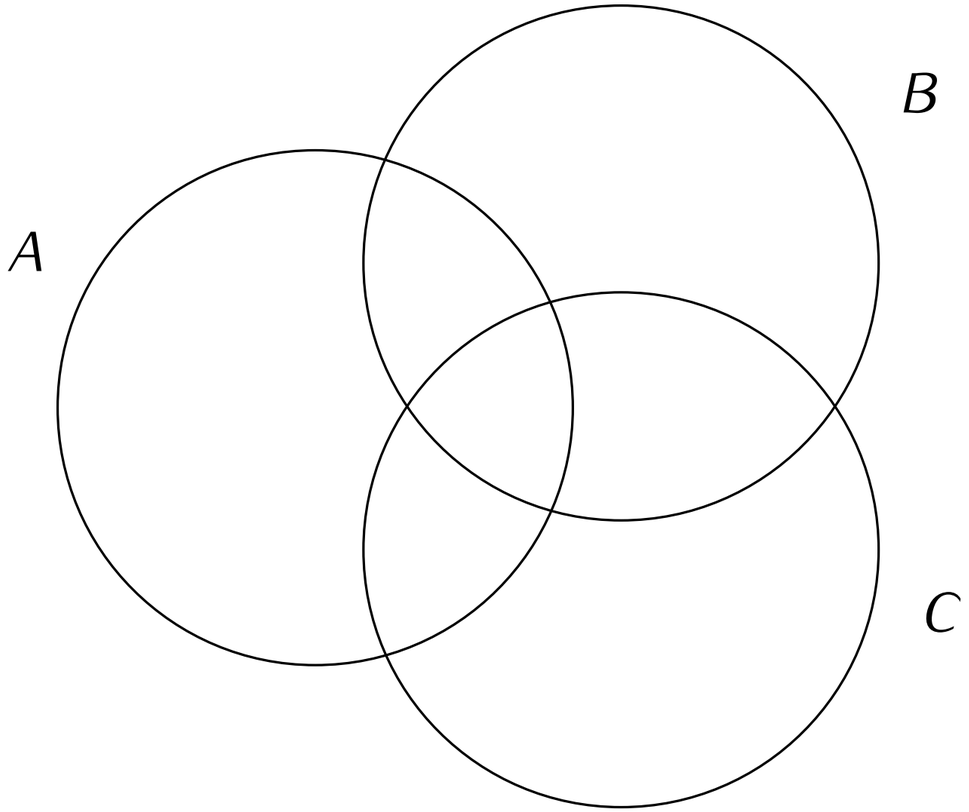
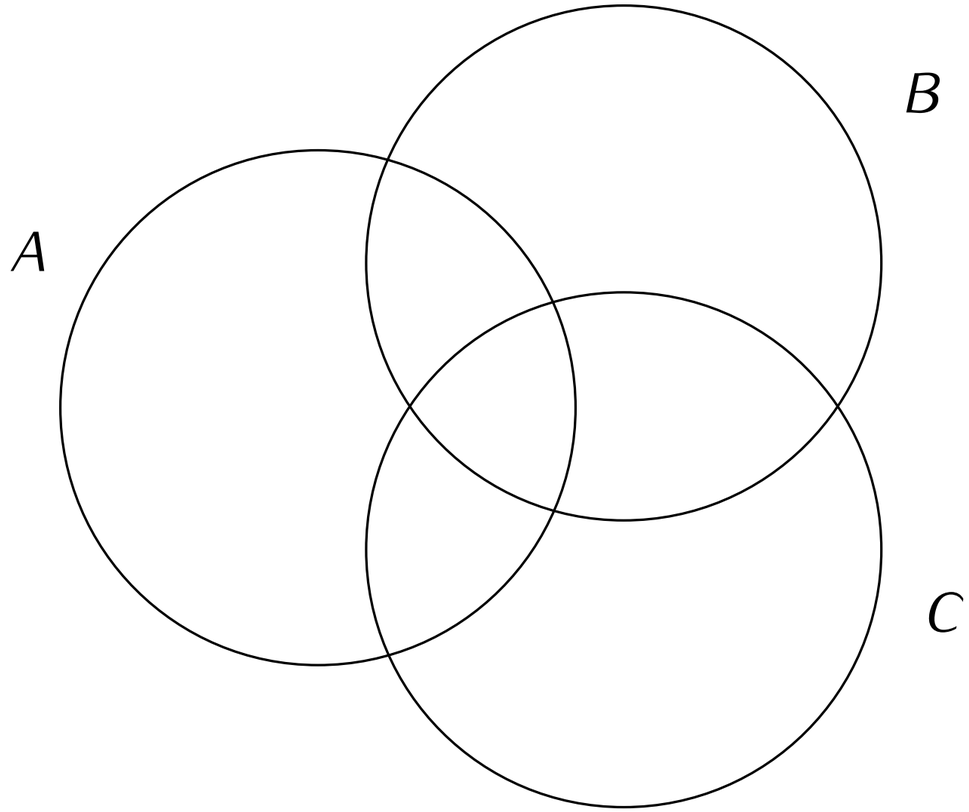
- No recipe to learn proofs other than practice
- An example is **not** a proof!

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.





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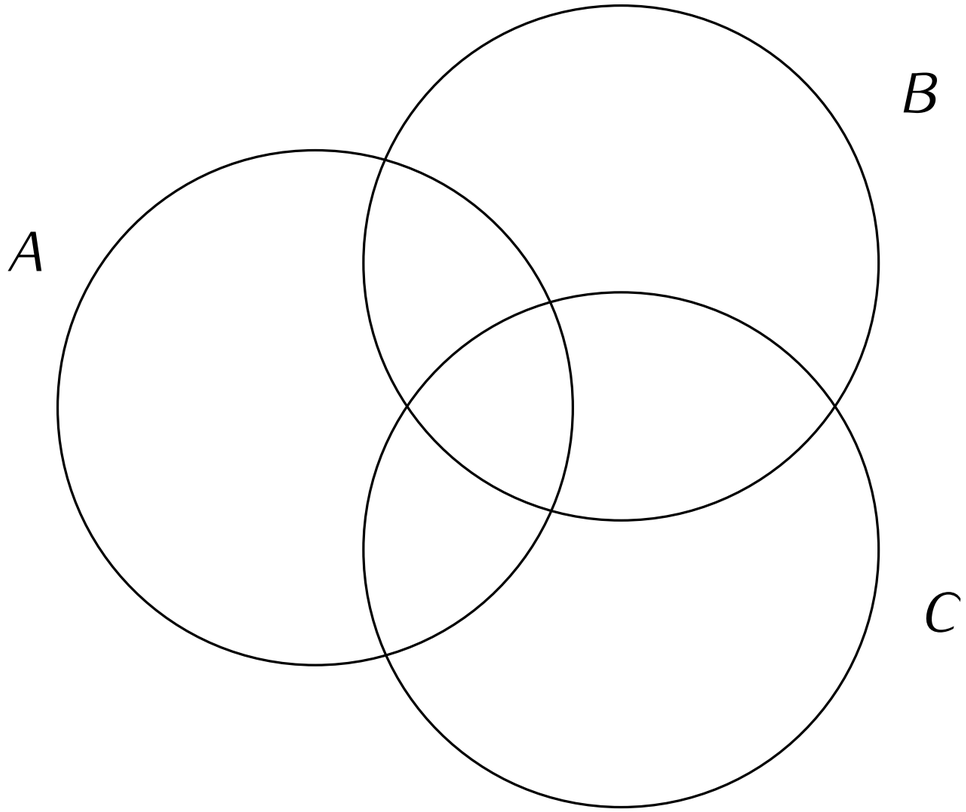

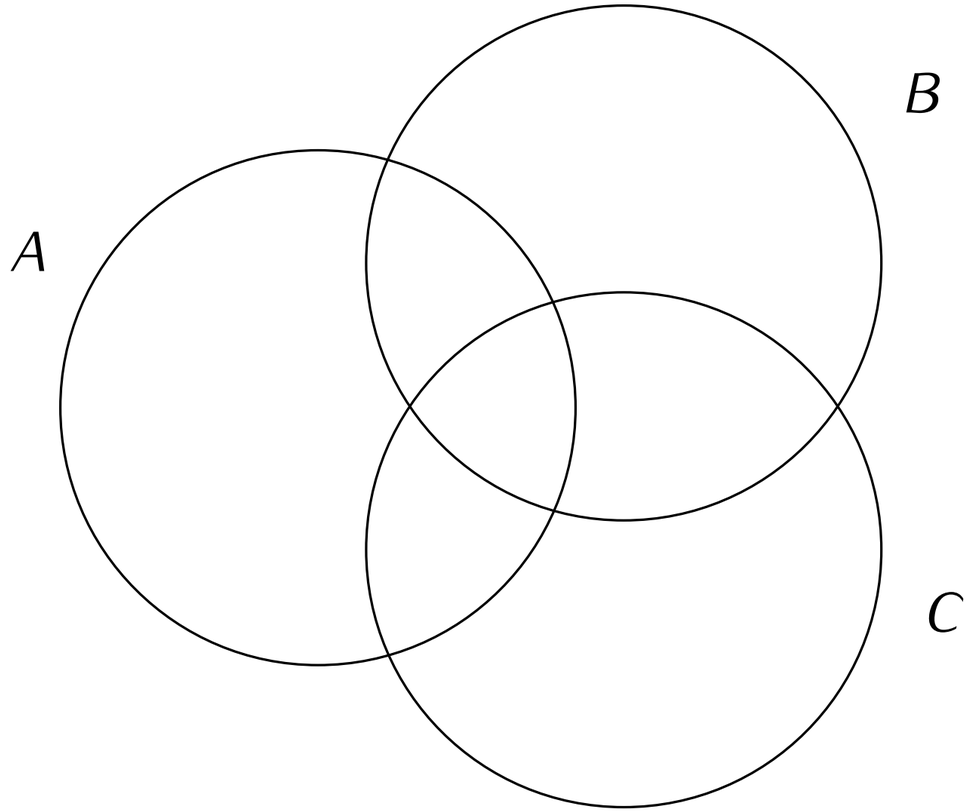

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- how can we be sure it does not depend on how we have drawn the potatoes?
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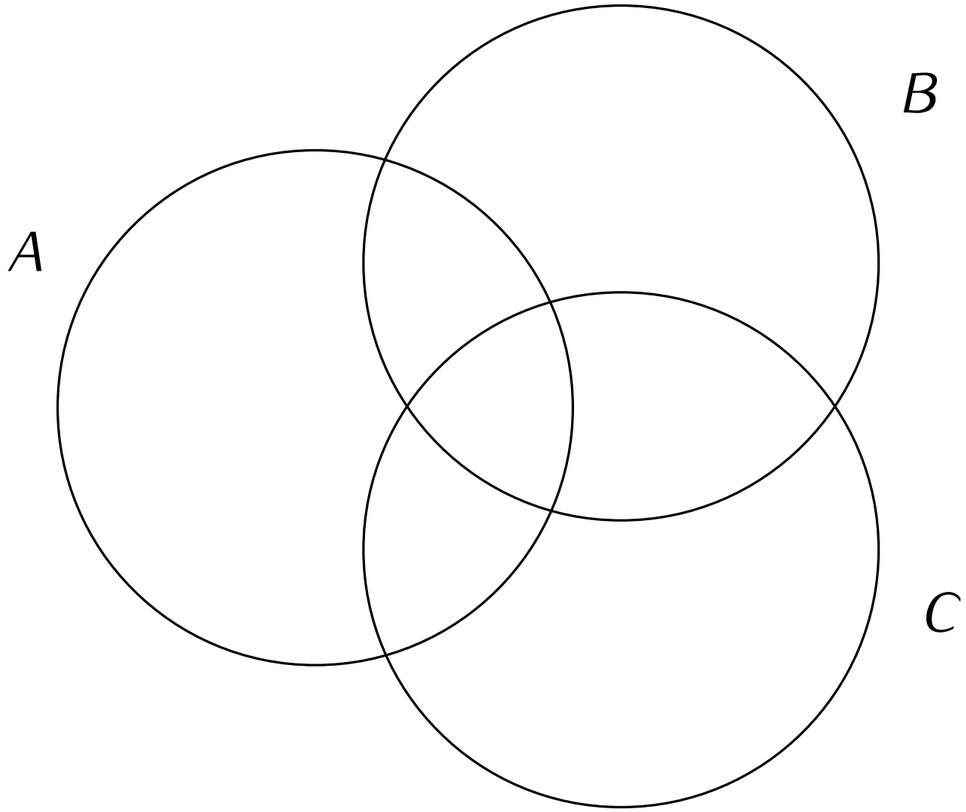

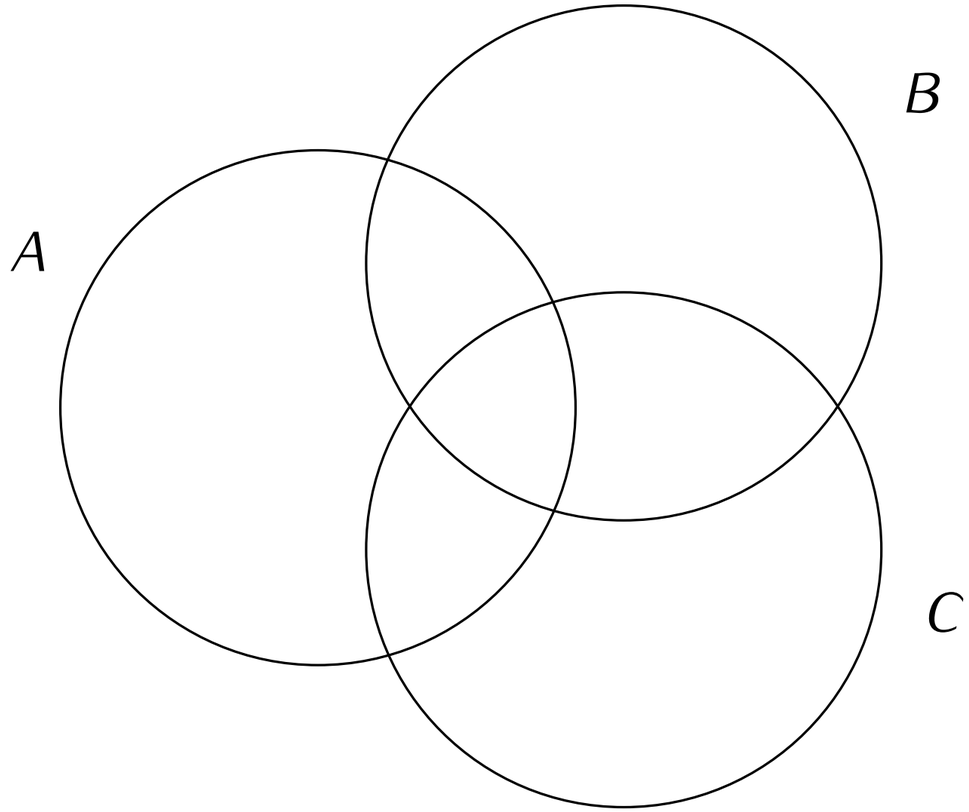

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$$A \cap (B_1 \cup \cdots \cup B_9) \subseteq (A \cap B_1) \cup \cdots \cup (A \cap B_9)$$

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Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Proof. (what you write down)

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set

Goal: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Proof. (what you write down)

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set

Goal: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

Proof. (what you write down)

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set

Goal: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.
  Goal is to prove $\ldots \subseteq \ldots$.
  We need to apply the definition.
  This generates a new hypothesis.

Proof. (what you write down)
1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set

Goal: if $x \in A \cap (B \cup C)$, then $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.
  Goal is to prove $\ldots \subseteq \ldots$.
  We need to apply the definition.
  This generates a new hypothesis.

Proof. (what you write down)
1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 4: $x \in A \cap (B \cup C)$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

Proof. (what you write down)
1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 4: $x \in A \cap (B \cup C)$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses
  Hyp. 4 is $x \in \ldots \cap \ldots$

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 5: $x \in A$
Hypothesis 6: $x \in B \cup C$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses
  Hyp. 4 is $x \in \ldots \cap \ldots$

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 5: $x \in A$
Hypothesis 6: $x \in B \cup C$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses
  Hyp. 6 is $x \in \ldots \cup \ldots$
  This means x is in at least one of the two sets
  There are 2 cases to deal with

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 5: $x \in A$
Hypothesis 7: $x \in B$ or $x \in C$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses
  Hyp. 6 is $x \in \ldots \cup \ldots$
  This means x is in at least one of the two sets
  There are 2 cases to deal with

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 5: $x \in A$
Hypothesis 8: $x \in B$

Goal: $x \in (A \cap B) \cup (A \cap C)$

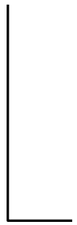
- Step 1: understand the goal.

- Step 2: understand the hypotheses

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

4a. Case 1: $x \in B$



Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 9: $x \in A \cap B$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

4a. Case 1: $x \in B$

Hyp. 5+8: $x \in A \cap B$

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 9: $x \in A \cap B$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

4a. Case 1: $x \in B$

> Hyp. 5+8: $x \in A \cap B$
>
> Hyp. 9 and def. of $\cup$: $x \in (A \cap B) \cup (A \cap C)$
>
> This is what we wanted to prove

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 5: $x \in A$
Hypothesis 7: $x \in B$ or $x \in C$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

4a. Case 1: $x \in B$

> Hyp. 5+8: $x \in A \cap B$
> Hyp. 9 and def. of $\cup$: $x \in (A \cap B) \cup (A \cap C)$
> This is what we wanted to prove

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 5: $x \in A$
Hypothesis 10: $x \in C$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

4a. Case 1: $x \in B$

- Hyp. 5+8: $x \in A \cap B$
- Hyp. 9 and def. of $\cup$: $x \in (A \cap B) \cup (A \cap C)$
- This is what we wanted to prove

4b. Case 2: $x \in C$

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 11: $x \in A \cap C$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

4a. Case 1: $x \in B$

- Hyp. 5+8: $x \in A \cap B$
- Hyp. 9 and def. of $\cup$: $x \in (A \cap B) \cup (A \cap C)$
- This is what we wanted to prove

4b. Case 2: $x \in C$

- Hyp. 5+10: $x \in A \cap C$

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 11: $x \in A \cap C$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.
- Step 2: understand the hypotheses

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

4a. Case 1: $x \in B$

> Hyp. 5+8: $x \in A \cap B$
> Hyp. 9 and def. of $\cup$: $x \in (A \cap B) \cup (A \cap C)$
> This is what we wanted to prove

4b. Case 2: $x \in C$

> Hyp. 5+10: $x \in A \cap C$
> Hyp. 11 and def. of $\cup$: $x \in (A \cap B) \cup (A \cap C)$
> This is what we wanted to prove

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Notes. (how you start thinking about the problem)

- Step 0: identify the hypotheses and the conclusion

Hypothesis 1: A is a set
Hypothesis 2: B is a set
Hypothesis 3: C is a set
Hypothesis 11: $x \in A \cap C$

Goal: $x \in (A \cap B) \cup (A \cap C)$

- Step 1: understand the goal.

- Step 2: understand the hypotheses

Proof. (what you write down)

1. Def: $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ means "every element of $A \cap (B \cup C)$ must be an element of $(A \cap B) \cup (A \cap C)$."
2. Def: $x \in A \cap (B \cup C)$ means "$x \in A$ **and** $x \in B \cup C$"
3. Def: $x \in B \cup C$ means "$x \in B$ **or** $x \in C$"

4a. Case 1: $x \in B$
- Hyp. 5+8: $x \in A \cap B$
- Hyp. 9 and def. of $\cup$: $x \in (A \cap B) \cup (A \cap C)$
- This is what we wanted to prove

4b. Case 2: $x \in C$
- Hyp. 5+10: $x \in A \cap C$
- Hyp. 11 and def. of $\cup$: $x \in (A \cap B) \cup (A \cap C)$
- This is what we wanted to prove

5. Done! □

Theorem. If A, B, C are sets, then $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Proof.

By definition of $\subseteq$, we need to prove that every element of $A \cap (B \cup C)$ is in $(A \cap B) \cup (A \cap C)$.
By definition of $\cap$, $x \in A$ **and** $x \in B \cup C$.
By definition of $\cup$, $x \in B$ **or** $x \in C$.

Case 1: $x \in B$

Since $x \in A$ and $x \in B$, we know $x \in A \cap B$.
By definition of $\cup$: $x \in (A \cap B) \cup (A \cap C)$.
This is what we wanted to prove.

Case 2: $x \in C$

Since $x \in A$ and $x \in C$, we know $x \in A \cap C$.
By definition of $\cup$: $x \in (A \cap B) \cup (A \cap C)$.
This is what we wanted to prove. $\square$

- do not only use examples: examples are good to get intuition, but examples are not a proof
- rephrase/decompose hypotheses and goal using the definitions
- in general, to prove $S \subseteq T$ you need to
  - start with "Assume $x \in S$." // "Sei $x \in S$"
  - blablabla...
  - "Therefore $x \in T$." // "Deshalb $x \in T$."

- do not only use examples: examples are good to get intuition, but examples are not a proof
- rephrase/decompose hypotheses and goal using the definitions
- in general, to prove $S \subseteq T$ you need to
  - start with "Assume $x \in S$." // "Sei $x \in S$"
  - blablabla...
  - "Therefore $x \in T$." // "Deshalb $x \in T$."

How to check whether your proof is correct:

- Every step should be understandable and require no further explanation.
- Give your proof to your study partner. They should be able to read and understand without any help (if they know the definitions).

- Definition of $\cap, \cup, \times, \setminus, \Delta, \mathcal{P}$
- How to prove $S \subseteq T$
- How to prove $S = T$
  (prove $S \subseteq T$ and $T \subseteq S$)