Prep Course Mathematics

Lecture 7 – Analytic geometry

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TUHH

INSTITUTE OF MATHEMATICS

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1. Analytic geometry
   - Representing lines
   - Positional relationships
   - Inner product, cross product
   - Orthogonality
   - Representing planes
   - Positional relationships
   - Reminder: system of linear equations

Analytic geometry

Representing lines in $\mathbb{R}^2$

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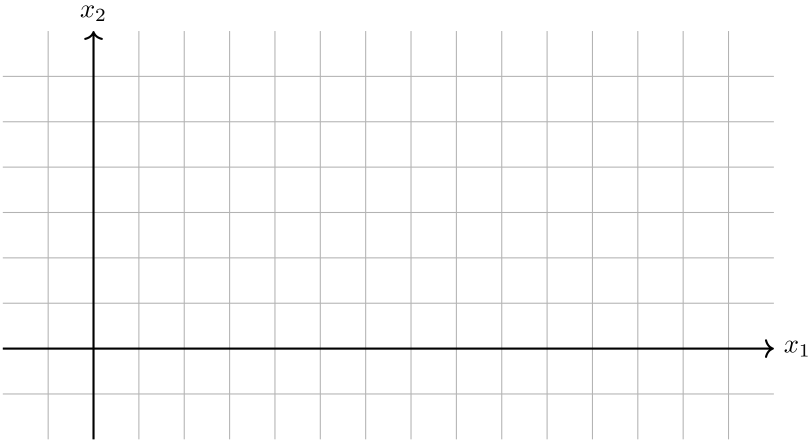

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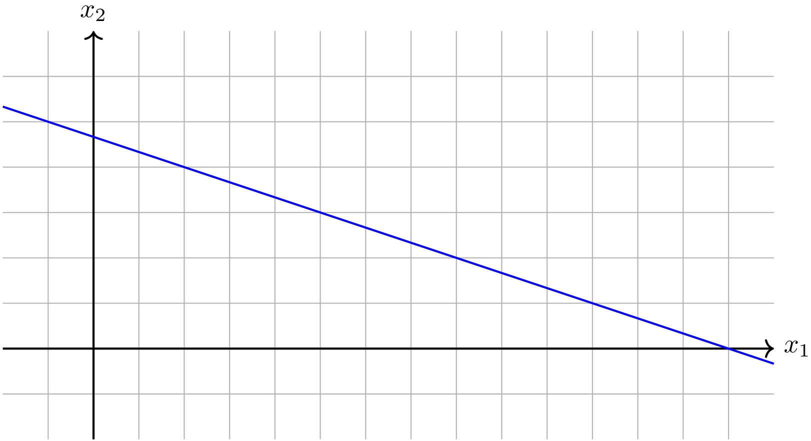

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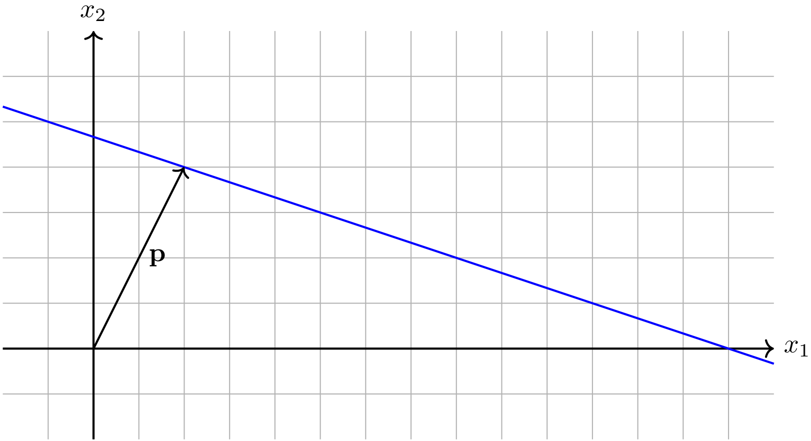

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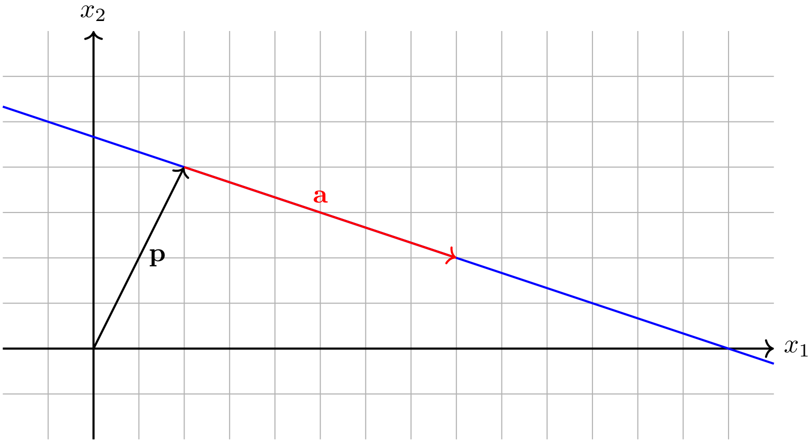

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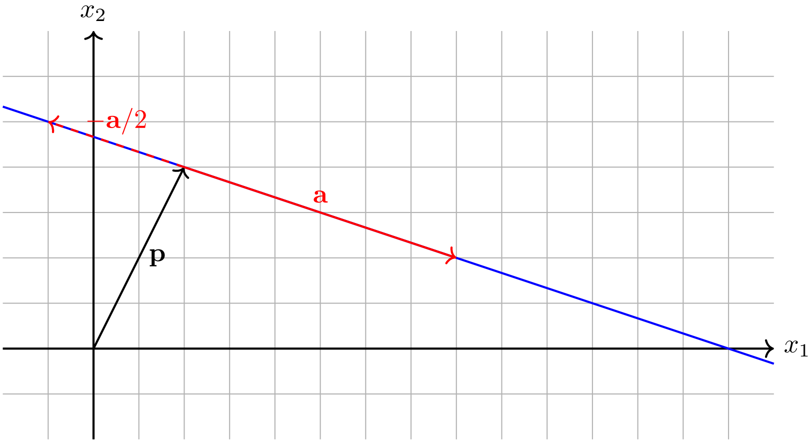

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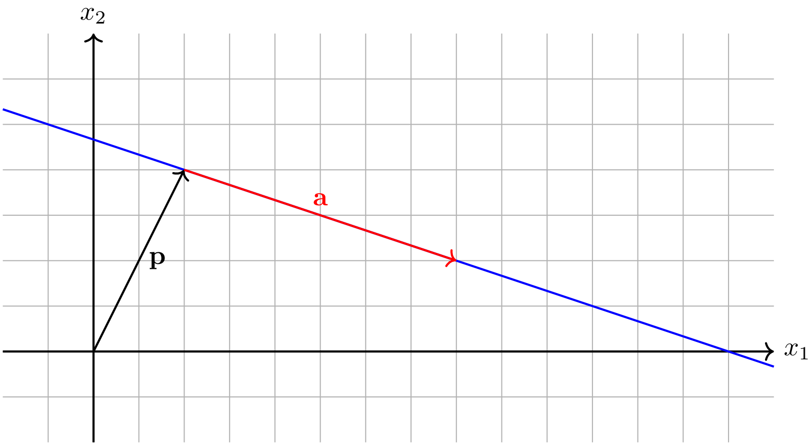

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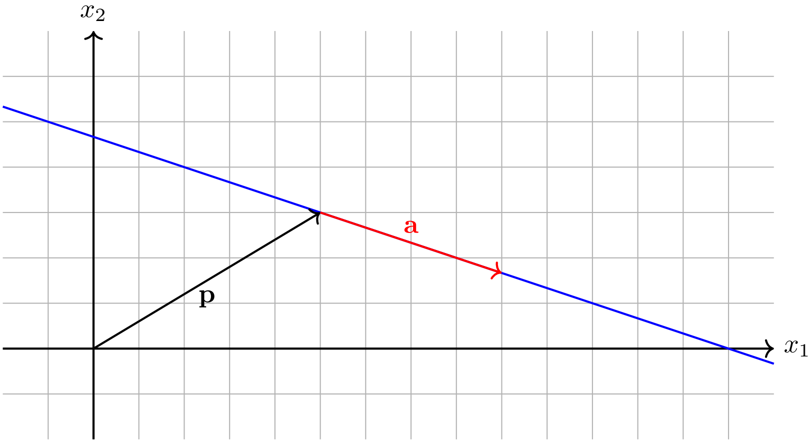

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Idea: a line is uniquely determined if we know a point and the direction of the line.

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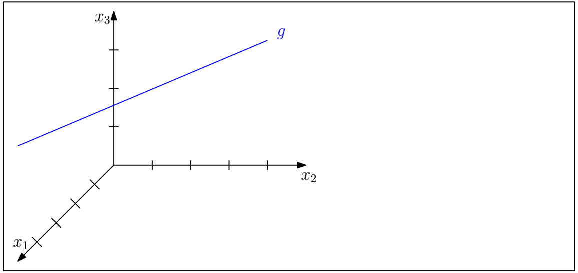

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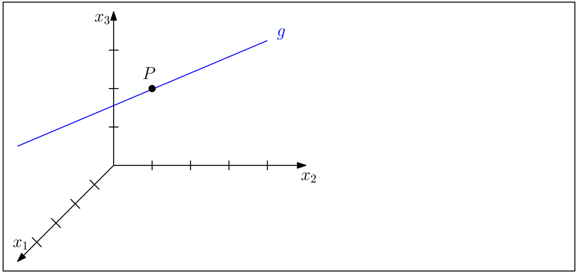

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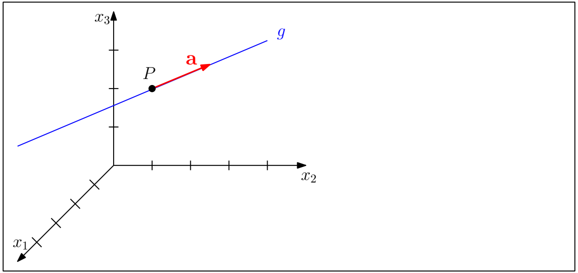

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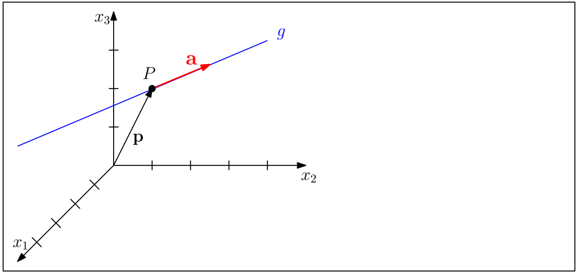

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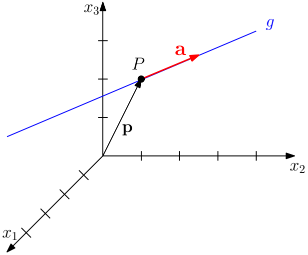

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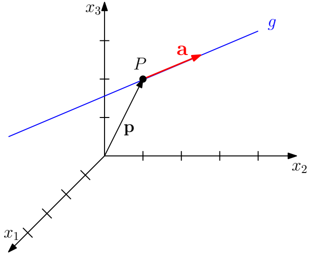

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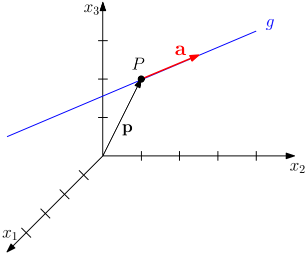

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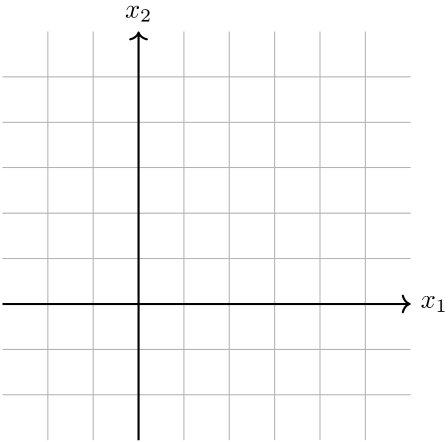
Hence, every point on g can be represented by a position vector of the form $\mathbf{p}+\lambda\mathbf{a}$

Parameter form of a line

Every line in $\mathbb{R}^2$ and $\mathbb{R}^3$ (and $\mathbb{R}^n$) can be described in the so-called parameter form:

$$g = \{\mathbf{p} + \lambda\mathbf{a} : \lambda \in \mathbb{R}\} = \mathbf{p} + \mathrm{Span}(\mathbf{a}).$$

Examples



$$g_1 = \left\{ \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

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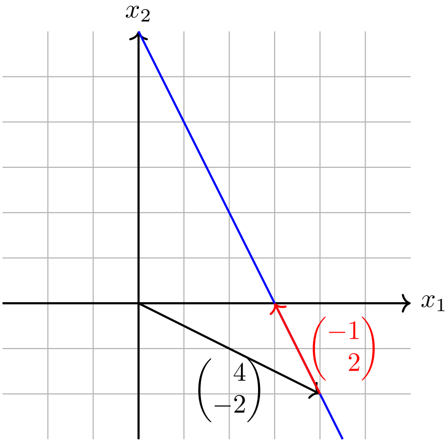

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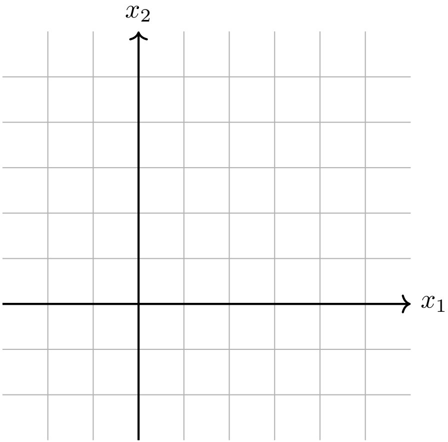

$$g_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \mathsf{Span}\left(\begin{pmatrix} 0 \\ -2 \end{pmatrix}\right)$$

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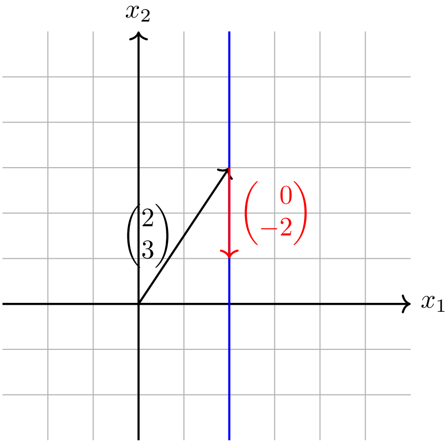

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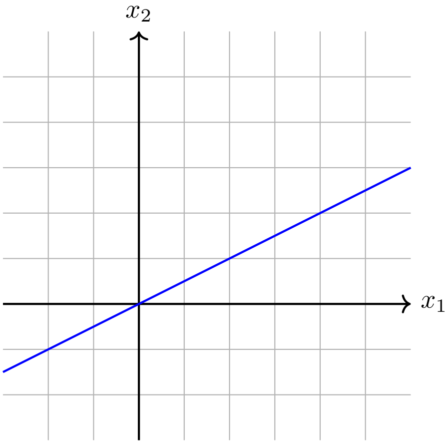

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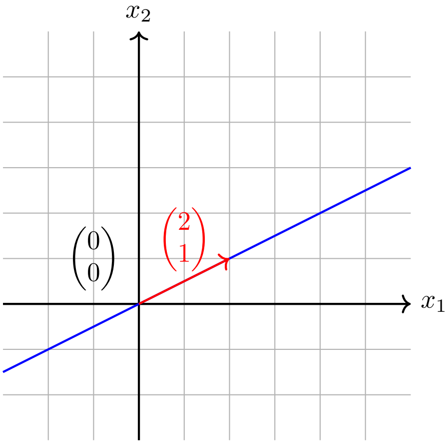
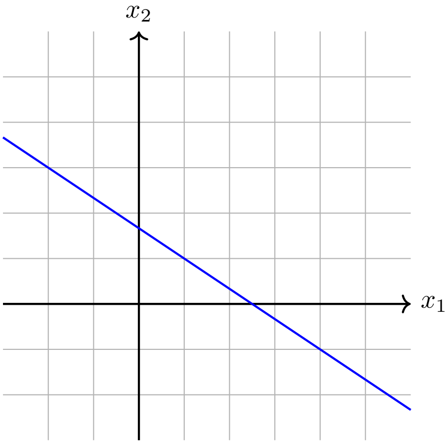


$$g_3 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

Examples



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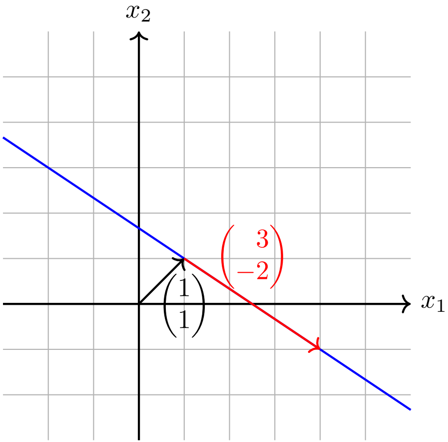

$$g_4 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

Exercises

(a) Describe the line g_1 which goes through the points

$$A := (0, 1, 2) \quad \text{and} \quad B := (-2, 7, 10)$$

with the help of a parameter form.

(b) Describe the line g_2 which goes through the points

$$C := (5, 4, -4) \quad \text{and} \quad D := (-1, -5, -1)$$

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(c) Extra question: do the lines g_1 and g_2 have an intersection point S? If yes, determine the point.

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$$g_1 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

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$$g_1 = \left\{ \begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

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This is a linear system of equations:

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} -2 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix} + \lambda_2 \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \left(\begin{array}{cc|c} -2 & 6 & 5 \\ 6 & 9 & 3 \\ 8 & -3 & -6 \end{array}\right)$$

We solve via elimination:

$$\left(\begin{array}{cc|c} -2 & 6 & 5 \\ 6 & 9 & 3 \\ 8 & -3 & -6 \end{array}\right) \begin{array}{l} \\ \mathsf{II} + 3 \cdot \mathsf{I} \\ \mathsf{III} + 4 \cdot \mathsf{I} \end{array} \rightsquigarrow \left(\begin{array}{cc|c} -2 & 6 & 5 \\ 0 & 27 & 18 \\ 0 & 21 & 14 \end{array}\right) \begin{array}{l} \\ \\ \mathsf{III} - \frac{7}{9} \cdot \mathsf{II} \end{array} \rightsquigarrow \left(\begin{array}{cc|c} -2 & 6 & 5 \\ 0 & 27 & 18 \\ 0 & 0 & 0 \end{array}\right)$$

The solution exists, λ_2 suffices:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \text{exists} \\ \frac{2}{3} \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \quad S = (1, -2, -2).$$

Positional relationships

Determining positional relationships for lines

Let two lines g_1 and g_2 be given by a parameter form:

$$g_1 = \{\mathbf{p}_1 + \lambda_1 \mathbf{r}_1 : \lambda_1 \in \mathbb{R}\}, \quad g_2 = \{\mathbf{p}_2 + \lambda_2 \mathbf{r}_2 : \lambda_2 \in \mathbb{R}\}.$$

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In order to determine the positional relationship we use the following system of linear equations:

$$\mathbf{p}_1 + \lambda_1 \mathbf{r}_1 = \mathbf{p}_2 + \lambda_2 \mathbf{r}_2 \quad \Rightarrow \quad \lambda_1 \mathbf{r}_1 - \lambda_2 \mathbf{r}_2 = \mathbf{p}_2 - \mathbf{p}_1 \quad \Rightarrow \quad \left(\mathbf{r}_1 \quad -\mathbf{r}_2 \;\middle|\; \mathbf{p}_2 - \mathbf{p}_1\right).$$

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There are four cases:

exactly one solution	intersection point (insert λ_1 or λ_2 into the parameter forms)
infinitely many solutions	identical lines
no solution (λ_1, λ_2) & $\mathbf{r}_1$ and $\mathbf{r}_2$ are multiples of each other	lines are parallel but not identical
no solution (λ_1, λ_2) & $\mathbf{r}_1$ and $\mathbf{r}_2$ are no multiples	skew lines

Inner product, norm, and angle

Inner product, norm, and angle

Standard inner product (Defintion)

Consider any vectors

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

in $\mathbb{R}^n$.

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in $\mathbb{R}^n$.Then the (standard) inner product of $\mathbf{v}$ and $\mathbf{w}$ is defined as follows:

$$\langle \mathbf{v}, \mathbf{w} \rangle := v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k.$$

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In particular, the length $\|\mathbf{v}\|$ of $\mathbf{v}$ (also called norm) can be written as follows:

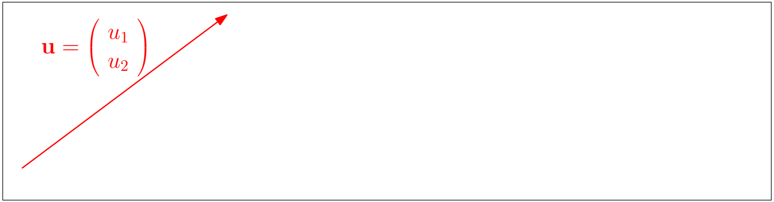
$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Inner product, norm, and angle

Length, distance, angle

Consider any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, let $\langle \cdot, \cdot \rangle$ be the standard inner product and let $\| \cdot \|$ denote the norm. Then

- the length of the vector $\mathbf{v}$ is given by $\|\mathbf{v}\|$,



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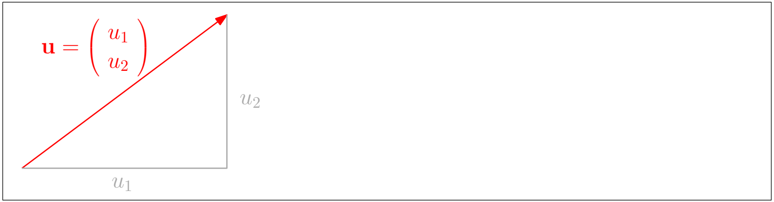

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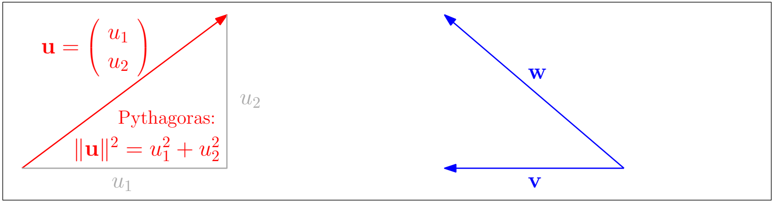
- the length of the vector $\mathbf{v}$ is given by $\|\mathbf{v}\|$,
- the angle α between $\mathbf{v}$ and $\mathbf{w}$ is given by $\cos(\alpha) = \dfrac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|}$,



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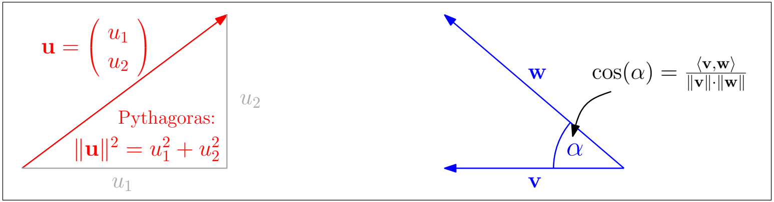

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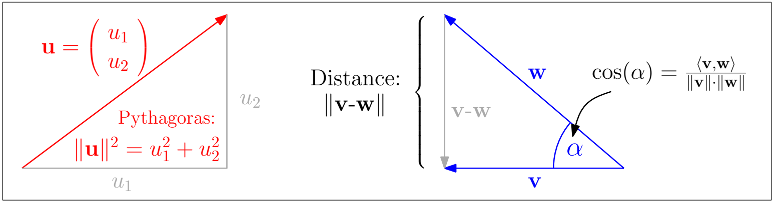
- the length of the vector $\mathbf{v}$ is given by $\|\mathbf{v}\|$,
- the angle α between $\mathbf{v}$ and $\mathbf{w}$ is given by $\cos(\alpha) = \dfrac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|}$,
- the distance of $\mathbf{v}$ and $\mathbf{w}$ is given by $\|\mathbf{v} - \mathbf{w}\|$.



Inner product, norm, and angle

Exercise: consider

$$\mathbf{v} := \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} := \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(a) Determine the lengths of these vectors.

(b) Determine the angle between $\mathbf{v}$ and $\mathbf{w}$.

(c) Extra problem: for which value $x \in \mathbb{R}$ is the distance between

$$\mathbf{u}_x := \begin{pmatrix} x \\ 1 - x \end{pmatrix}$$

and $\mathbf{v}$ the smallest?

Inner product, norm, and angle

Solution: consider

$$\mathbf{v} := \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} := \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(a) Lengths:

$$\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| = \qquad\qquad \left\| \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\| =$$

(b) Angle:

$$\cos(\alpha) = \frac{\left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\|}$$

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$$\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| = \sqrt{2^2 + 1^2} = \sqrt{5}, \quad \left\| \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\| = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

(b) Angle:

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(b) Angle:

$$\cos(\alpha) = \frac{\left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\|} = \frac{2 \cdot 1 + 1 \cdot 3}{\sqrt{5} \cdot \sqrt{10}} = \frac{\sqrt{25}}{\sqrt{50}} = \frac{1}{\sqrt{2}}, \quad \alpha = \frac{\pi}{4}.$$

Inner product, norm, and angle

Solution: consider

$$\mathbf{v} := \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} := \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

(a) Lengths:

$$\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| = \sqrt{2^2 + 1^2} = \sqrt{5}, \quad \left\| \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\| = \sqrt{1^2 + 3^2} = \sqrt{10}.$$

(b) Angle:

$$\cos(\alpha) = \frac{\left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\|} = \frac{2 \cdot 1 + 1 \cdot 3}{\sqrt{5} \cdot \sqrt{10}} = \frac{\sqrt{25}}{\sqrt{50}} = \frac{1}{\sqrt{2}}, \quad \alpha = \frac{\pi}{4}.$$

Inner product, norm, and angle

Solution: consider for $x \in \mathbb{R}$

$$\mathbf{v} := \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_x := \begin{pmatrix} x \\ 1-x \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - x \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(c) Minimal distance:

$$\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} x \\ 1-x \end{pmatrix} \right\| =$$

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$$\min_{x \in \mathbb{R}} \left((2 - x)^2 + x^2 \right) = \min_{x \in \mathbb{R}} \left(4 - 4x + 2x^2 \right), \quad -4 + 4x = 0 \quad \Rightarrow \quad x = 1.$$

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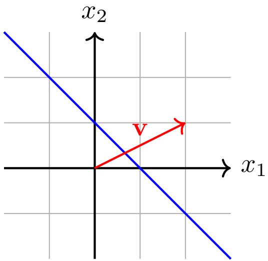

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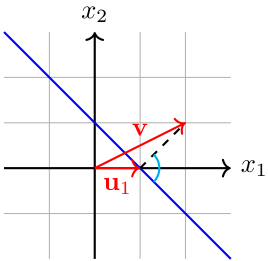

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Exercise: consider a triangle with the corner points

$$A := (-1, 0, 1), \quad B := (1, 2, 3), \quad \text{and} \quad C := (-2, 2, 0).$$

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$$\mathbf{v}_{BC} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}.$$

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$$\langle \mathbf{v}_{AB}, \mathbf{v}_{AC} \rangle = -2 + 4 - 2 = 0, \qquad \langle \mathbf{v}_{AB}, \mathbf{v}_{BC} \rangle = -6 + 0 - 6 = -12,$$
$$\langle \mathbf{v}_{AC}, \mathbf{v}_{BC} \rangle = 3 + 0 + 3 = 6.$$

The cross product in $\mathbb{R}^3$

Cross product (Definition)

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^3$. Then the cross product of $\mathbf{v}$ and $\mathbf{w}$ is defined as follows:

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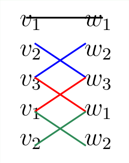
Mnemonic



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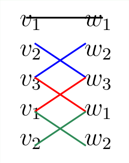

$\rightsquigarrow$ write the vectors next to each other,
below them add the first two components again

$\rightsquigarrow$ delete the first row

$\rightsquigarrow$ determine the entries of $\mathbf{v} \times \mathbf{w}$ using the drawn crosses:

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

The cross product in $\mathbb{R}^3$

Examples:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} =$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}3\\2\\1\end{pmatrix} =$$

$$\mathbf{v} \times \mathbf{w} = \qquad\qquad = -\mathbf{w} \times \mathbf{v}.$$

$$\begin{pmatrix}1\\4\\9\end{pmatrix} \times \begin{pmatrix}1\\4\\9\end{pmatrix} =$$

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$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix}v_2w_3 - v_3w_2\\ v_3w_1 - v_1w_3\\ v_1w_2 - v_2w_1\end{pmatrix} = \begin{pmatrix}-w_2v_3 + w_3v_2\\ -w_3v_1 + w_1v_3\\ -w_1v_2 + w_2v_1\end{pmatrix} = -\mathbf{w} \times \mathbf{v}.$$

$$\begin{pmatrix}1\\4\\9\end{pmatrix} \times \begin{pmatrix}1\\4\\9\end{pmatrix} = \begin{pmatrix}4\cdot 9 - 9\cdot 4\\ 9\cdot 1 - 1\cdot 9\\ 1\cdot 4 - 4\cdot 1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}, \qquad \mathbf{v} \times \mathbf{v} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

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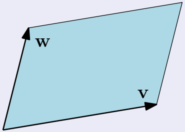

Exercise

(a) Calculate
$$\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}.$$

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$$\mathbf{v} := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} := \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

(i) Find a vector which is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.
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(i) $$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 - 1 \cdot 0 \\ 1 \cdot (-1) - 1 \cdot 2 \\ 1 \cdot 0 - 2 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix},$$

(ii) $$\left\| \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \right\| = \sqrt{4^2 + (-3)^2 + 2^2} = \sqrt{29},$$

(iii) $$\frac{1}{\sqrt{29}} \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.$$

Representing planes: parameter form

Idea: a plane is determined uniquely by a point and two vectors that do not lie along one line.

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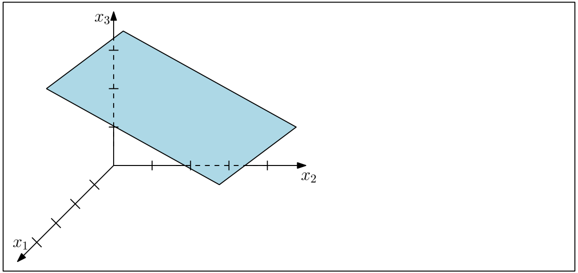

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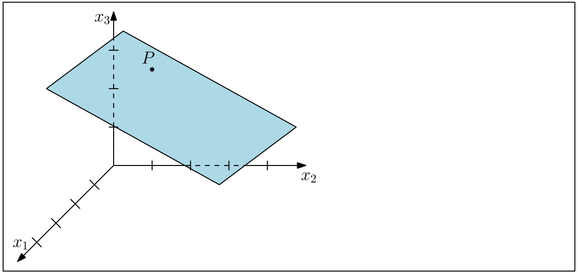

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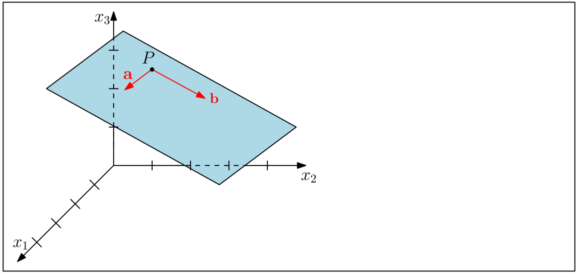

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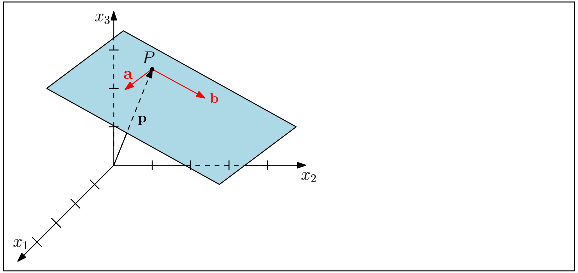

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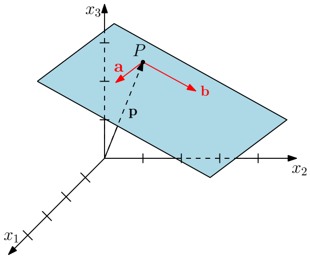
Idea: a plane is determined uniquely by a point and two vectors that do not lie along one line.



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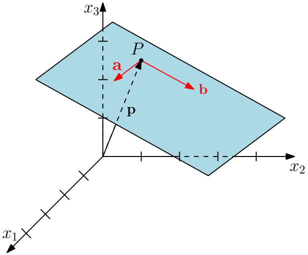

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In $\mathbb{R}^3$ there exists a unique plane E which contains the points $R := (1,1,1)$, $S := (1,3,2)$, and $T := (-1,4,3)$. We determine a parameter form of E.

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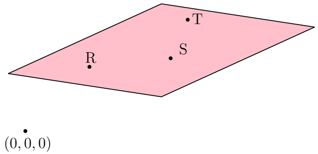

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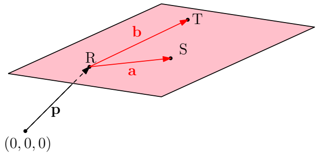

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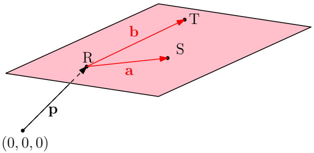

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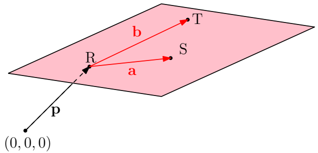

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Exercise

(a) Find a parameter form of the plane E which contains the following points:

$$A := (6, 3, 0), \quad B := (-3, 10, 2), \quad \text{and} \quad C := (5, 3, 3).$$

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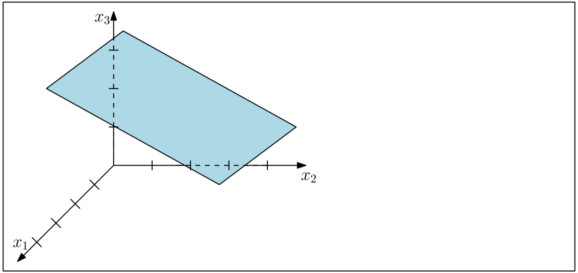

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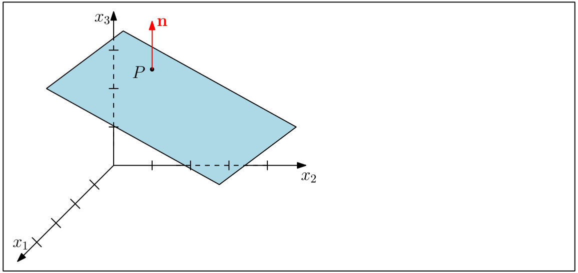

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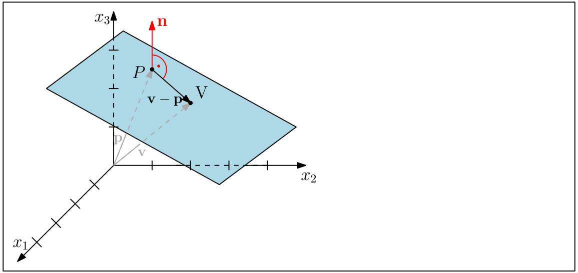

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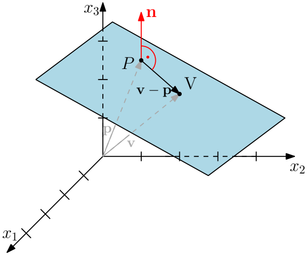

Let $\mathbf{n}$ be a vector that is orthogonal to E.

For every point V on E the vector **v-p** between P and V is orthogonal to $\mathbf{n}$.

Hence, it holds that $\langle \mathbf{v}\text{-}\mathbf{p}, \mathbf{n} \rangle = 0$ for every position vector $\mathbf{v}$ of a point on E.

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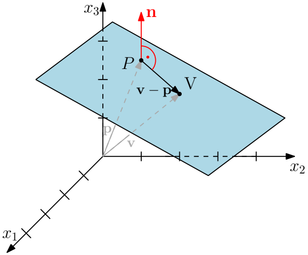

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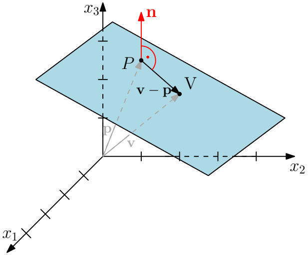

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The vector $\mathbf{n} \neq \mathbf{o}$ is called normal vector of E, i.e. it is orthogonal to E.

Representing planes: coordinate form

Coordinate form of a plane

Every plane E in $\mathbb{R}^3$ can be written in the so-called coordinate form:

$$E = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : a_1 x_1 + a_2 x_2 + a_3 x_3 = d \right\}, \quad a_1, a_2, a_3, d \in \mathbb{R}.$$

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Example

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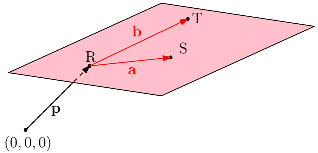

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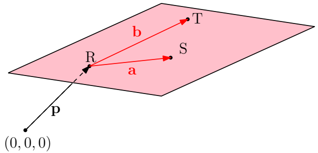

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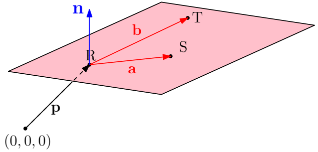

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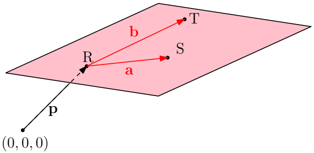

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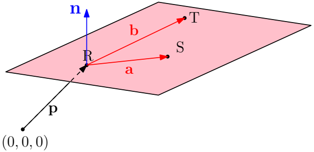

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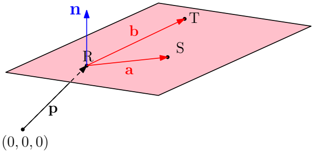

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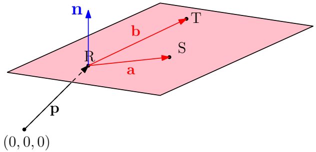

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To obtain a coordinate form, we substitute $\mathbf{v} = \mathbf{x}$ into the equation of the normal form:

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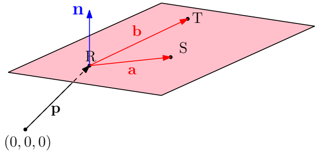

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Example

Hence we have the following representations of the plane E:

- Parameter form:

$$E = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\},$$

- Normal form:

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- Coordinate form:

$$E = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - 2x_2 + 4x_3 = 3 \right\}.$$

Exercise

Find a coordinate form for each of the following planes:

(i) $$E_1 := \left\{ \mathbf{v} \in \mathbb{R}^3 : \left\langle \mathbf{v} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle = 0 \right\}$$

(ii) $$E_2 := \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$$

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$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle = x_2 + 2x_3 = 2 = \left\langle \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle$$

(ii) $E_2 := \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix},$$

Exercise

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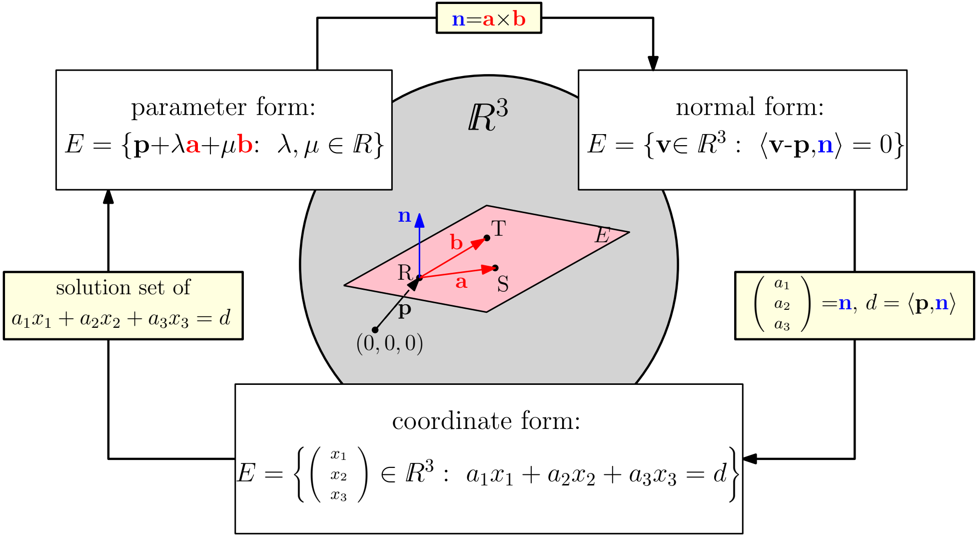
Overview: representations



Hesse normal form

Hesse normal form (definition)

A normal form $\{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}$ of a plane is called Hesse normal form (HNF) if $\mathbf{n}$ has length 1 ($\|\mathbf{n}\| = 1$).

Hesse normal form

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Normal form $\rightsquigarrow$ HNF: scale normal vector to unit length.

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Normal form $\rightsquigarrow$ HNF: scale normal vector to unit length.

Example: consider the plane

$$E := \left\{ \mathbf{v} \in \mathbb{R}^3 : \left\langle \mathbf{v} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\rangle = 0 \right\}$$

in normal form.

Hesse normal form

Hesse normal form (definition)

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in normal form. A Hesse normal form is

$$E = \left\{ \mathbf{v} \in \mathbb{R}^3 : \left\langle \mathbf{v} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} \right\rangle = 0 \right\}.$$

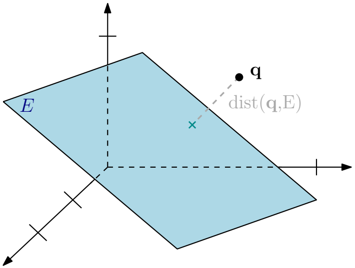
Hesse normal form and distance

Hesse normal form (definition)

A normal form $\{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}$ of a plane is called Hesse normal form (HNF) if $\mathbf{n}$ has length 1 ($\|\mathbf{n}\| = 1$).

Distance point/plane

Let a point (position vector) $\mathbf{q}$ and a plane $E = \{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v} - \mathbf{p}, \mathbf{n} \rangle = 0\}$ in Hesse normal form be given. Then the distance between $\mathbf{q}$ and E equals $|\langle \mathbf{q} - \mathbf{p}, \mathbf{n} \rangle|$.



Hesse normal form and distance

Exercise: Determine the distance between the point $\mathbf{q}$ and the plane E given by

$$\mathbf{q} := \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \quad E := \left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}.$$

Solution:

Hesse normal form and distance

Exercise: Determine the distance between the point $\mathbf{q}$ and the plane E given by

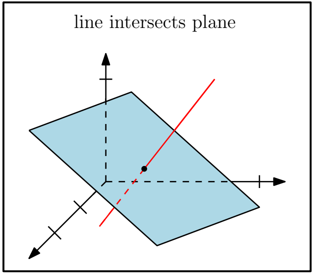
$$\mathbf{q} := \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}, \quad E := \left\{ \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} : \lambda, \mu \in \mathbb{R} \right\}.$$

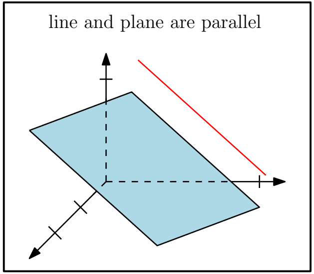
Solution:

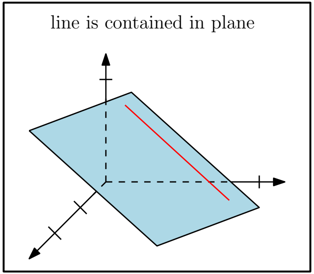
$$\mathbf{n} = \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}}{\left\| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \right\|} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad \operatorname{dist}(\mathbf{q}, E) = \left| \left\langle \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \frac{1}{5} \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rangle \right|$$

$$= \frac{4 \cdot 3 + 2 \cdot 4 - 2 \cdot 0}{5} = \frac{20}{5} = 4.$$

Positional relationship: line and plane







Positional relationship: line and plane

Scheme: Positional relationship between line and plane

Let a line g be given in parameter form

$$g = \{\mathbf{p} + \lambda \mathbf{a} : \lambda \in \mathbb{R}\}$$

and let a plane E be given in coordinate form.

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Then one may put the 3 components of the general vector $\mathbf{p} + \lambda \mathbf{a}$ of the line g into the equation of the coordinate form of E.

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- the equation has no solution

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- the equation has no solution: then g and E have no common point.

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- the equation has no solution: then g and E have no common point.
- the equation has a unique solution λ

Positional relationship: line and plane

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- the equation has a unique solution λ: then there is an intersection point. Its position vector can be determined by substituting the solution λ into the general vector $\mathbf{p} + \lambda\mathbf{a}$.

Positional relationship: line and plane

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- the equation has infinitely many solutions

Positional relationship: line and plane

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- the equation has a unique solution λ: then there is an intersection point. Its position vector can be determined by substituting the solution λ into the general vector $\mathbf{p} + \lambda\mathbf{a}$.
- the equation has infinitely many solutions: g is contained in E.

Positional relationship: line and plane

Example: We check whether the following line g and the following plane E have an intersection point:

$$g = \left\{ \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\} \quad \text{and} \quad E = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : 2x_1 - x_2 - 2x_3 = 1 \right\}.$$

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Solution: we insert the generic position vector of g into the equation of the plane,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 + \lambda \\ -3 - 2\lambda \\ \lambda \end{pmatrix} \quad \rightsquigarrow \quad \begin{array}{c} 2 \cdot (2 + \lambda) - (-3 - 2\lambda) - 2 \cdot (\lambda) = 1, \\ \Leftrightarrow \quad 7 + 2\lambda = 1. \end{array}$$

Positional relationship: line and plane

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This equation has the unique solution $\lambda = -3$.

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This equation has the unique solution $\lambda = -3$. The corresponding intersection point is

$$S = (-1, 3, -3), \quad \text{since} \quad \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}.$$

Positional relationship: line and plane

Exercise: determine the positional relationship of the following plane E and line g:

$$E := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : 3x_1 - 4x_2 + 2x_2 = 12 \right\}, \qquad g := \left\{ \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ 10 \end{pmatrix} : \lambda \in \mathbb{R} \right\}.$$

Solution:

Positional relationship: line and plane

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Solution: we look for a point of intersection,

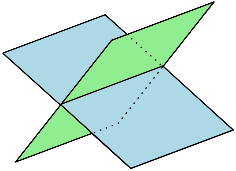
$$3(6+4\lambda) - 4(2+5\lambda) + 2(1+10\lambda) = 12 \quad \Leftrightarrow \quad 12 + 12\lambda = 12 \quad \Leftrightarrow \quad \lambda = 0.$$

The unique point of intersection is defined by $\mathbf{p}$,

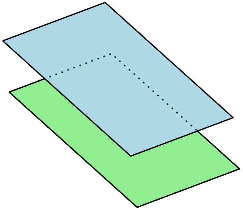
$$\mathbf{p} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}, \quad 3 \cdot 6 - 4 \cdot 2 + 2 \cdot 1 = 18 - 8 + 2 = 12.$$

Positional relationship: two planes

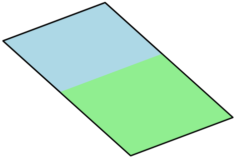
planes intersect in a line



planes are parallel



planes are identical



Positional relationship: two planes

Scheme: positional relationship of two planes

Assume that two planes E_1 and E_2 are given by coordinate forms.

Positional relationship: two planes

Scheme: positional relationship of two planes

Assume that two planes E_1 and E_2 are given by coordinate forms. Then each common point (x_1, x_2, x_3) must satisfy both coordinate equations which leads to a LES of the form

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= d, \\ b_1x_1 + b_2x_2 + b_3x_3 &= e. \end{aligned}$$

Positional relationship: two planes

Scheme: positional relationship of two planes

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There are three cases:

Positional relationship: two planes

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There are three cases:

- the LES has no solution

Positional relationship: two planes

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There are three cases:

- the LES has no solution: then E_1 and E_2 have no common point (parallel).

Positional relationship: two planes

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There are three cases:

- the LES has no solution: then E_1 and E_2 have no common point (parallel).
- the LES has solutions, and both equations are not multiples of each other

Positional relationship: two planes

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There are three cases:

- the LES has no solution: then E_1 and E_2 have no common point (parallel).
- the LES has solutions, and both equations are not multiples of each other: then there is an intersection line which equals the solution set of the LES.

Positional relationship: two planes

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- the LES has solutions, and one equation is a multiple of the other equation

Positional relationship: two planes

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There are three cases:

- the LES has no solution: then E_1 and E_2 have no common point (parallel).
- the LES has solutions, and both equations are not multiples of each other: then there is an intersection line which equals the solution set of the LES.
- the LES has solutions, and one equation is a multiple of the other equation: then the planes are identical.

Reminder: system of linear equations

System of linear equations

Let $m, n \in \mathbb{N}$. A system of linear equations (LES) in the variables $x_1, x_2, \ldots, x_n$ is of the form

$$\begin{aligned} a_{1,1}\,x_1 + a_{1,2}\,x_2 + \cdots + a_{1,n}\,x_n &= b_1, \\ a_{2,1}\,x_1 + a_{2,2}\,x_2 + \cdots + a_{2,n}\,x_n &= b_2, \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m. \end{aligned}$$

with $a_{i,j}$ and b_i being (usually real) numbers. An assignment of values for $x_1, \ldots, x_n$ such that all equations are satisfied is called a solution of this system of equations. Such a solution is written as a vector.

LES: looking at rows

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Equations describe:

- lines ($\mathbb{R}^2$),
- planes ($\mathbb{R}^3$),
- hyperplanes ($\mathbb{R}^n$).

Solution set is their intersection.

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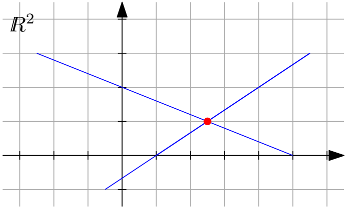

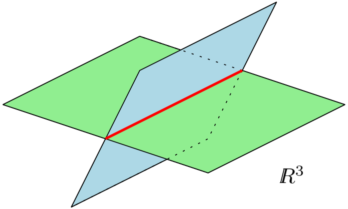

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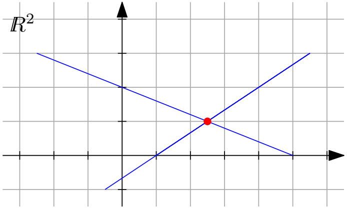

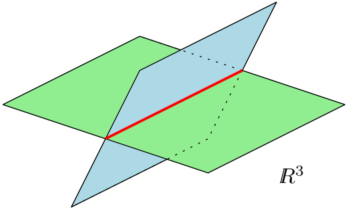

solving LES $\widehat{=}$ finding intersection of hyperplanes

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solving LES $\widehat{=}$ finding linear combination

LES: laziness

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$\Downarrow$ introduce notation

$$\boxed{\mathbf{Ax} = \mathbf{b}}$$

LES: laziness

Real matrix

Let $m, n \in \mathbb{N}$. Then

$$\mathbf{A} := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \quad \text{with every } a_{i,j} \in \mathbb{R}$$

is called a real matrix. The set of all such matrices is denoted by $\mathbb{R}^{m \times n}$. The integer m is the number of rows, the integer n is the number of columns of the matrix $\mathbf{A}$.

Matrix-vector-product

Definition

Let a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{x} \in \mathbb{R}^n$ be given,

$$\mathbf{A} = \begin{pmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ | & & | \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

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- The number of columns of $\mathbf{A}$ must equal the number of components of $\mathbf{x}$.
- The product $\mathbf{Ax}$ is a linear combination of the columns of $\mathbf{A}$.