Analysis 1

A course for engineering students.

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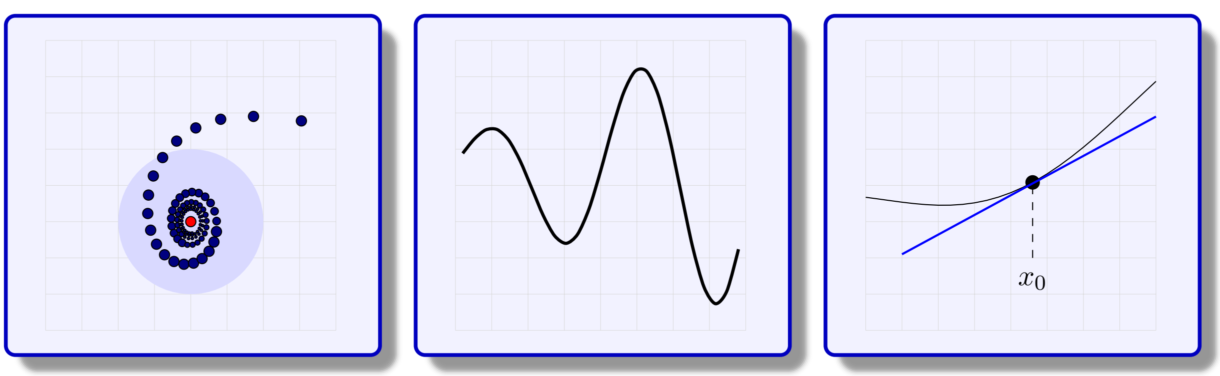

Limits, continuity, derivatives, and all the rest of it.

Version of October 17, 2025

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Hamburg, October 17, 2025 M.L.

Contents

Chapter 1

Foundations

For this we refer to the first chapter of the lecture notes of your “Linear Algebra I” course.

Chapter 2

Sequences

Germany started recording daily temperatures in 1881. For Hamburg this record might start like this:

time	day 1	day 2	day 3	day 4	day 5	day 6	$\cdots$
temperature in °C	14.3	12.6	10.8	10.9	17.5	15.3	$\cdots$

Written down consecutively, we have this sequence of numbers:

$$(x_1, x_2, x_3, x_4, x_5, x_6, \cdots) \;=\; (14.3,\ 12.6,\ 10.8,\ 10.9,\ 17.5,\ 15.3,\ \cdots).$$

Instead of saying a number sequence is a function on the day-based time axis since 1881, we rather say the following:

Definition 2.1. Sequence or number sequence

A sequence (or number sequence) is a function $x : \mathbb{N} \to \mathbb{R}$.
For $n \in \mathbb{N}$ we write x_n instead of $x(n)$ for the nth member of the sequence.
Instead of nth member, we will also say nth term or nth entry of the sequence.
When we mean the whole sequence, we write $(x_n)_{n\in\mathbb{N}}$ or $(x_n)_{n=1}^{\infty}$ or just $(x_n)_n$.

We know that it is unimportant which name we give a function. Still, we prefer calling sequences x, y or z.

An essential feature of sequences is that they never end.
(Let's keep fingers crossed for Hamburg and its weather...)

Instead of writing down all(?!) the numbers x_n explicitly, we try to give a formula for x_n whenever we can.

Example 2.2. some first simple sequences

a) *Let* x_n *be given by* $\frac{1}{n}$ *for each* $n \in \mathbb{N}$*, so that the sequence starts* $x_1 = \frac{1}{1}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{1}{3}$, $x_4 = \frac{1}{4}$, $\ldots$. *We abbreviate this by saying* $(x_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$.

b) *Look at* $(y_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, \ldots)$. *This time we have*

$$y_n \;=\; \begin{cases} -1, & n \text{ odd}, \\ 1, & n \text{ even}, \end{cases} \qquad n \in \mathbb{N}.$$

c) *Let* $z_n = z_{n-1} + 2n - 1$ *for* $n = 2, 3, \ldots$ *and let* $z_1 = 1$.

Nerd box 2.3. explicit vs recursive description

In example c) we speak of a recursive description (the members of $(z_n)_n$ *are given in terms of other members of* $(z_n)_n$*), while the description in a) and b) is referred to as explicit. (*x_n *and* y_n *can be computed directly in terms of* n*.)*
Can you give an explicit formula for z_n *from c)?*

2.1 Convergent sequences

We speak of convergence of a sequence if the members x_n get closer to the so-called limit as n gets larger. The formally correct definition comes below.

Since we stop writing $x : \mathbb{N} \to \mathbb{R}$ for the sequence $(x_n)_{n\in\mathbb{N}}$ after Definition 2.1, the notation x as a single letter is free again. We will use it for the limit, which is a number $x \in \mathbb{R}$.

Definition 2.4. Convergence and limit of a sequence

We say that a sequence $(x_n)_{n\in\mathbb{N}}$ converges *to a number* $x \in \mathbb{R}$ *if, for every* $\varepsilon > 0$*, there is an* $n_0 \in \mathbb{N}$ *such that*

$$|x_n - x| < \varepsilon \quad \text{for all} \quad n \geq n_0$$

In this case x *is called the* limit *of the sequence* $(x_n)_{n\in\mathbb{N}}$*, and we write*

$$x = \lim_{n\to\infty} x_n \qquad \text{or} \qquad x_n \to x \qquad \text{for} \qquad n \to \infty.$$

A sequence with a limit is called convergent*; otherwise it is called* divergent*.*

The essence of the definition again, now with all the quantors:

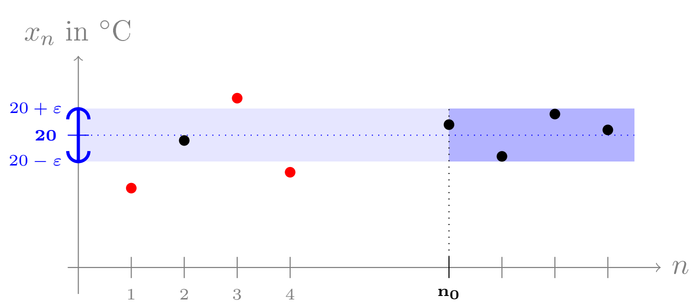
Convergence without words

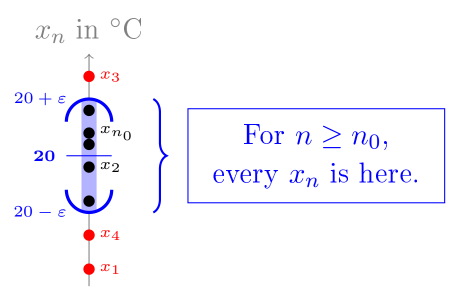
$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N}: \quad \forall n \geq n_0: \quad |x_n - x| < \varepsilon$$

This sounds more complicated than it is. Some more attempts (with many more words):

- $|x_n - x| < \varepsilon$ means $x_n \in (x - \varepsilon, x + \varepsilon)$,
- this holds for all x_n, except perhaps $x_1, x_2, \ldots, x_{n_0-1}$,
- i.e., for all sequence members x_n up to finitely many exceptions,
- the whole thing has to be true for arbitrarily small precisions $\varepsilon > 0$,
- n_0 (or the number of exceptions) may depend on ε.

So if someone claims the temperature in Hamburg converges to 20 °C then, after a certain day n_0, the temperature would have to enter the interval $(19, 21)$ – and never leave that interval again. A little later it would even enter the interval $(19.9, 20.1)$, and so on.





Example 2.5. some convergent and some divergent sequences

a) *$\frac{1}{n} \to 0$ as $n \to \infty$: For $\varepsilon = 0.1$, $n_0 = 11$ is enough, since $|\frac{1}{n} - 0| < \frac{1}{10} = 0.1$ for all $n \geq 11$. For $\varepsilon = 0.01$ we already need $n_0 = 101$, and in general, n_0 has to be chosen a bit larger than $\frac{1}{\varepsilon}$.*

b) *$(-1)^n \not\to 1$ since already for $\varepsilon = \frac{1}{2}$ the sequence keeps leaving the interval $(1 - \varepsilon, 1 + \varepsilon) = (0.5, 1.5)$. This sequence doesn't converge to any other real number either. But this is something we will show a little bit further down.*

c) *Constant sequences are clearly convergent: If $x_n = 5$ for all $n \in \mathbb{N}$ then $x_n \in (5 - \varepsilon, 5 + \varepsilon)$ for all $n \geq 1$, no matter how small the $\varepsilon > 0$. Hence, $x_n \to 5$.*

For a more fluent language and argument, let us introduce the following handy notation:

Definition 2.6. short notation $x \overset{\varepsilon}{\approx} y$

For two numbers $x, y \in \mathbb{R}$ and an $\varepsilon > 0$, let us write

$$x \overset{\varepsilon}{\approx} y \qquad \textit{if} \qquad |x-y| < \varepsilon.$$

The following important rule holds

$$x \overset{\varepsilon}{\approx} y \overset{\delta}{\approx} z \qquad \Longrightarrow \qquad x \overset{\varepsilon+\delta}{\approx} z \tag{2.1}$$

for all $x, y, z \in \mathbb{R}$ and $\varepsilon, \delta > 0$, which corresponds to the so-called

triangle inequality

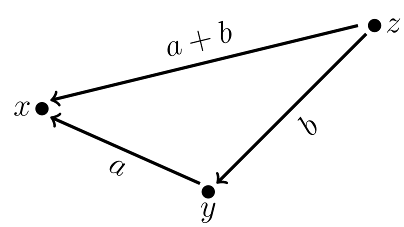
$$|x-z| \;\leq\; |x-y| + |y-z|, \qquad x, y, z \in \mathbb{R},$$

in other words,

$$|a+b| \;\leq\; |a| + |b|, \qquad a, b \in \mathbb{R}. \tag{2.2}$$

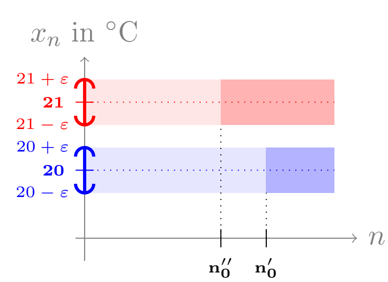
(Put $a = x - y$ and $b = y - z$, so that $a + b = x - y + y - z = x - z$.)

It is named like this since the inequality extends to vectors x, y, z in the plane (and beyond), where $|x-y|$, $|y-z|$ and $|x-z|$, resp. $|a|$, $|b|$ and $|a+b|$, are the lengths of the sides of a triangle with corners x, y, z. In that situation it is obvious that the sum of two side lengths cannot be smaller than the length of the third side.



With our short notation from Definition 2.6, $x_n \to x$ translates into $x_n \overset{\varepsilon}{\approx} x$ holding sooner or later (meaning: for all $n \geq$ some n_0), for any prescribed precision $\varepsilon > 0$.

When a sequence converges to 20, it cannot also converge to 21, because, if it did, then x_n were contained in both $(20-\varepsilon, 20+\varepsilon)$ and $(21-\varepsilon, 21+\varepsilon)$ for sufficiently large n. But already for $\varepsilon = \frac{1}{2}$ there is no number in the intersection of both intervals. Or, arguing even swifter:
$20 \overset{\varepsilon}{\approx} x_n \overset{\varepsilon}{\approx} 21 \;\Rightarrow\; 20 \overset{2\varepsilon}{\approx} 21$, which is only possible for $\varepsilon > \frac{1}{2}$.



Proposition 2.7. The limit is unique

A convergent sequence can only have one limit.

To be precise, it is only now that the notation "$\lim x_n$" is legitimate.

Proof: Let's just copy the argument from above: Suppose $x_n \to x$ and $x_n \to y$, while $x \neq y$. Then, for every $\varepsilon > 0$, including $\varepsilon = \frac{1}{2}|x-y|$, there is a $n_0' \in \mathbb{N}$ with $x_n \in (x-\varepsilon, x+\varepsilon)$ for all $n \geq n_0'$ and a $n_0'' \in \mathbb{N}$ with $x_n \in (y-\varepsilon, y+\varepsilon)$ for all $n \geq n_0''$.
But then, for all $n \geq n_0 := \max(n_0', n_0'')$, x_n is contained in both intervals, which is impossible by the choice of ε since $(x-\varepsilon, x+\varepsilon) \cap (y-\varepsilon, y+\varepsilon) = \varnothing$. Contradiction! ↯ ■

2.2 Convergence and boundedness

We will now get to know a minimum requirement for convergence.

Definition 2.8. bounded sequence

A sequence $(x_n)_{n\in\mathbb{N}}$ is called

- *bounded from below, if there is an $a \in \mathbb{R}$ with $a \leq x_n$ for all $n \in \mathbb{N}$,*
- *bounded from above, if there is a $b \in \mathbb{R}$ with $x_n \leq b$ for all $n \in \mathbb{N}$,*
- *bounded, if it is bounded from below and above, i.e., if there exist $a, b \in \mathbb{R}$ with $x_n \in [a, b]$ for all $n \in \mathbb{N}$. In particular, then holds $|x_n| \leq c$ for all $n \in \mathbb{N}$, where $c = \max(|a|, |b|)$,*
- *unbounded if it is not bounded.*

Such numbers a and b are then called lower bounds resp. upper bounds on $(x_n)_{n\in\mathbb{N}}$.

The enclosing interval for bounded sequences can always be chosen symmetric to zero since $[a, b] \subset [-c, c]$ with c as above. The condition $|x_n| \leq c$ then means $x_n \in [-c, c]$.

Example 2.9. bounded and unbounded sequences

a) *The sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n = (-1)^n$ for $n \in \mathbb{N}$ is bounded since*

$$|x_n| = |(-1)^n| = 1 \leq 1, \quad \text{i.e.} \quad x_n \in [-1, 1] \quad \text{for all } n \in \mathbb{N}.$$

b) *The sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n = n$ for $n \in \mathbb{N}$ is bounded from below (by $a = 1$) but not from above since every number $b \in \mathbb{R}$ will be exceeded by y_n once $n > b$. The sequence is hence unbounded.*

We will now see that boundedness is necessary (it is a minimal requirement) for convergence. Unbounded sequences like $(y_n)_{n\in\mathbb{N}}$ from b) cannot be convergent.

Proposition 2.10. convergence $\implies$ boundedness

Every convergent sequence is bounded.

Proof: Let $x_n \to x$ and $\varepsilon > 0$ be arbitrary. Then there exists a $n_0 \in \mathbb{N}$ with $x_n \in (x-\varepsilon, x+\varepsilon)$ for all $n \geq n_0$. Among the finitely many other members, $x_1, \ldots, x_{n_0-1}$, there is a smallest, x_k, and a largest, x_l, with certain indices $k, l \in \{1, \ldots, n_0 - 1\}$. Then the interval $[a, b]$ with $a = \min(x_k, x-\varepsilon)$ and $b = \max(x_l, x+\varepsilon)$ will contain every x_n with $n \in \mathbb{N}$. ■

⚠ Attention! The implication goes only one way!

The reverse implication of Proposition 2.10 does not hold. Not every bounded sequence converges, see e.g. $((-1)^n)_{n\in\mathbb{N}}$ from Example 2.9 a) and Example 2.5 b).

Next we look at sequences that eventually exceed every number $M \in \mathbb{R}$ without ever getting below again – or the other way round.

Definition 2.11. $\lim x_n = \infty$

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence. If, for every $M \in \mathbb{R}$, there exists a $n_0 \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq n_0$, then we write

$$x_n \to +\infty \qquad \text{or} \qquad \lim_{n\to\infty} x_n = +\infty.$$

If, for every $M \in \mathbb{R}$, there exists a $n_0 \in \mathbb{N}$ such that $x_n \leq M$ for all $n \geq n_0$, then we write

$$x_n \to -\infty \qquad \text{or} \qquad \lim_{n\to\infty} x_n = -\infty.$$

Such sequences are unbounded and hence divergent by Proposition 2.10! That's why we refrain from talking about "convergence to ∞" or similar. We will however use the handy notations from Definition 2.11 – "by abuse of notation", so to say.

Nerd box 2.12. Neighbourhoods of $+\infty$ and $-\infty$

If we really wanted to accept $x_n \to -\infty$ or $x_n \to +\infty$ as convergence, we would first have to add the two desired limits as elements to the real numbers and then extend the notion of an ε-neighbourhood to the new members:
What the interval $(x-\varepsilon, x+\varepsilon)$ is for good old real numbers, $x \in \mathbb{R}$, namely, the set where convergent sequences eventually go and never leave again, has to be replaced by something like $(\frac{1}{\varepsilon}, +\infty]$ in case $x = +\infty$. Similarly for $x = -\infty$. Since these new ε-neighbourhoods are unbounded (in the usual sense), we lose Proposition 2.10 – as was desired.

The concept of boundedness can actually be dropped in that new sense since the new numbers $-\infty$ and $+\infty$ will always work as lower resp. upper bounds on any sequence. Next, one had to extend the usual operations $+, -, \cdot, :$ to the new number set $\mathbb{R} \cup \{-\infty, +\infty\}$ and thereby tackle lots of other problems. We decide that this is not worth the hassle.

Proposition 2.13. $x_n \to \infty \implies \frac{1}{x_n} \to 0$

a) *If $x_n \to +\infty$ or $x_n \to -\infty$ as in Definition 2.11 then $\frac{1}{x_n} \to 0$.*
b) *If x_n does not change its sign then also the reverse inplication holds.*

Proof: We show the claim for $x_n \to +\infty$. The case $x_n \to -\infty$ works by analogy. Let $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ be such that $x_n > M := \frac{1}{\varepsilon}$ for all $n \geq n_0$. Then $\frac{1}{x_n} \in (0, \varepsilon)$ for all $n \geq n_0$, i.e. $\frac{1}{x_n} \to 0$. From $\frac{1}{x_n} \in (0, \varepsilon)$ we conclude $x_n \in (M, +\infty)$. ∎

2.3 Working with null sequences

Definition 2.14. **null sequence**

A convergent sequence with limit zero is called a <u>*null sequence*</u>.

The next section will simplify quite a bit when we know some rules about null sequences:

Lemma 2.15. **null sequence + null sequence = null sequence**

If $x_n \to 0$ and $y_n \to 0$ then $x_n + y_n \to 0$.

A lemma is, in our way of using the word, a useful little helper of the bigger propositions and theorems that are often a bit more center stage.

Proof: In order to show that $x_n + y_n \to 0$, we take an arbitrary $\varepsilon > 0$ and then we have to find a $n_0 \in \mathbb{N}$ such that $x_n + y_n \in (0 - \varepsilon, 0 + \varepsilon)$ for all $n \geq n_0$.

By $x_n \to 0$ and $y_n \to 0$ there are $n_0', n_0'' \in \mathbb{N}$, such that $x_n \in (0 - \frac{\varepsilon}{2}, 0 + \frac{\varepsilon}{2})$ for all $n \geq n_0'$ and $y_n \in (0 - \frac{\varepsilon}{2}, 0 + \frac{\varepsilon}{2})$ for all $n \geq n_0''$. (Since $\frac{\varepsilon}{2}$ is only just another $\varepsilon' > 0$.)

For $n \geq n_0 := \max(n_0', n_0'')$ it then follows that $x_n + y_n \in (0 - \varepsilon, 0 + \varepsilon)$. ∎

In the last line of the proof we used that

$$a \in (-\alpha, +\alpha) \text{ and } b \in (-\beta, +\beta) \text{ always imply } a + b \in (-\alpha - \beta, \alpha + \beta).$$

This is (by looking at all different sign combinations of a and b) easy to check. The same statement can also be phrased like this:

$$\text{From } |a| < \alpha \text{ and } |b| < \beta \text{ it follows that } |a+b| < \alpha + \beta.$$

This however is by the (equally simple to check) triangle inequality (2.2).

Lemma 2.16. convergent sequence = constant + null sequence

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence and let $x \in \mathbb{R}$ be a real number. Then

$$x_n \to x \quad \Longleftrightarrow \quad x_n = x + z_n \text{ for all } n \in \mathbb{N} \text{ with a null sequence } (z_n)_{n\in\mathbb{N}}.$$

Proof: For $\varepsilon > 0$ and $n \in \mathbb{N}$, it holds that $x_n \overset{\varepsilon}{\approx} x$ if and only if $x_n - x \overset{\varepsilon}{\approx} 0$. If the first holds for all $n \geq n_0$ then so does the second. Hence: $x_n \to x \iff z_n := x_n - x \to 0$. ∎

Lemma 2.17. bounded sequence · null sequence = null sequence

If $(x_n)_n$ is bounded and $(z_n)_n$ is a null sequence then $(x_n{\cdot}z_n)_n$ is again a null sequence.

Proof: Let $\varepsilon > 0$ be arbitrary. We have to show that there is a $n_0 \in \mathbb{N}$ such that $|x_n z_n - 0| < \varepsilon$ for all $n \geq n_0$.

Fix $c > 0$ such that $|x_n| \leq c$ for all $n \in \mathbb{N}$, and let $n_0 \in \mathbb{N}$ be such that $|z_n - 0| < \frac{\varepsilon}{c}$ for all $n \geq n_0$. Then it holds that $|x_n z_n - 0| = |x_n z_n| = |x_n|\,|z_n| < c \cdot \frac{\varepsilon}{c} = \varepsilon$ for all $n \geq n_0$. ∎

Together with Proposition 2.10 we conclude:

Corollary 2.18. convergent sequence · null sequence = null sequence

If $(x_n)_n$ converges and $(z_n)_n$ is a null sequence then $(x_n \cdot z_n)_n$ is a null sequence.

Proof: By Proposition 2.10, $(x_n)_n$ is bounded, and it remains to apply Lemma 2.17. ∎

2.4 $\lim$ commutes with $+, -, \cdot$ and $:$

We connect two sequences with the basic operations, $+, -, \cdot, :$, by applying them to the members of the sequences (which are real numbers):

$$(x_n)_{n\in\mathbb{N}} \circledcirc (y_n)_{n\in\mathbb{N}} \quad := \quad (x_n \circledcirc y_n)_{n\in\mathbb{N}} \qquad \text{for} \qquad \circledcirc \in \{+, -, \cdot, :\}.$$

The “new” operation on the left hand side is defined in terms of the “old” operation between the real numbers on the right.

The next proposition says that passing to the limit commutes with e.g. addition: Whether we first add two sequences and then pass to the limit or we first pass to the limits and add them afterwards – the result is the same. The same is true for $-, \cdot$ und $:$.

Proposition 2.19. lim **commutes with** $+, -, \cdot$ **and** $:$

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be convergent sequences with $x_n \to x$ and $y_n \to y$.

a) *Then also the sequences $(x_n)_n + (y_n)_n$ and $(x_n)_n - (y_n)_n$ converge, where*

$$x_n \pm y_n \;\to\; x \pm y, \qquad i.e. \qquad \lim_{n\to\infty}(x_n \pm y_n) \;=\; (\lim_{n\to\infty} x_n) \pm (\lim_{n\to\infty} y_n).$$

b) *For every constant $c \in \mathbb{R}$, also $c \cdot (x_n)_n := (cx_n)_n$ converges, where*

$$cx_n \;\to\; cx, \qquad i.e. \qquad \lim_{n\to\infty}(cx_n) \;= c \lim_{n\to\infty} x_n.$$

c) *Also the sequence $(x_n)_n \cdot (y_n)_n$ is convergent, and it holds that*

$$x_n \cdot y_n \;\to\; x \cdot y, \qquad i.e. \qquad \lim_{n\to\infty}(x_n \cdot y_n) \;=\; (\lim_{n\to\infty} x_n) \cdot (\lim_{n\to\infty} y_n).$$

d) *If $y \neq 0$ then also the sequence $(\frac{x_n}{y_n})_n$ is convergent, and*

$$\frac{x_n}{y_n} \;\to\; \frac{x}{y}, \qquad i.e. \qquad \lim_{n\to\infty}\frac{x_n}{y_n} \;=\; \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}.$$

Proof: This is looking like a lot of work. Let's try to be economic here:
We start by showing the $+$ part of **a)** and then **c)**. With this shown, we also have **b)** and the $-$ part of **a)** sorted since **b)** is a special case of **c)** and since $x_n - y_n = x_n + (-1) \cdot y_n$. At the end, we still have to show **d)**.

Take home message:
Try to reduce new problems to previously solved (or soon to be solved) problems. Whenever you can!

a) In accordance with Lemma 2.16 we write $x_n = x + z_n$ and $y_n = y + z'_n$ with two null sequences $(z_n)_n$ and $(z'_n)_n$. Then it holds for all $n \in \mathbb{N}$ that

$$x_n + y_n \;=\; (x + z_n) + (y + z'_n) \;=\; (x + y) + (z_n + z'_n).$$

By Lemma 2.15, $(z_n + z'_n)_n$ is a null sequence, and by Lemma 2.16, using the reverse direction this time, $(x_n + y_n)_n$ converges to $x + y$.

c) By Lemma 2.16, we again write $x_n = x + z_n$ and $y_n = y + z'_n$ with two null sequences $(z_n)_n$ and $(z'_n)_n$. Then we conclude, for all $n \in \mathbb{N}$,

$$x_n \cdot y_n \;=\; (x + z_n) \cdot (y + z'_n) \;=\; (x \cdot y) + (x \cdot z'_n) + (z_n \cdot y) + (z_n \cdot z'_n).$$

By Lemma 2.17, $(x \cdot z'_n)_n$ and $(z_n \cdot y)_n$ are null sequences. By Corollary 2.18, also $(z_n \cdot z'_n)_n$ is a null sequence, and by Lemma 2.15, the sum of these three null sequences is a null sequence again. By Lemma 2.16 (backwards again), it follows that $x_n y_n \to xy$.

d) By $\frac{x_n}{y_n} = x_n \cdot \frac{1}{y_n}$ and **c)**, it is enough to show $\frac{1}{y_n} \to \frac{1}{y}$. Therefore, let $y \neq 0$ and $\varepsilon = \frac{|y|}{2} > 0$, such that 0 is surely (with distance ε) outside of the interval $I = (y - \varepsilon, y + \varepsilon)$. For a certain $n_0 \in \mathbb{N}$, all y_n with $n \geq n_0$ are located in I and are hence nonzero. It holds that

$$\frac{1}{y_n} - \frac{1}{y} = \frac{y - y_n}{y_n y} = (y - y_n) \cdot \frac{1}{y} \cdot \frac{1}{y_n}. \tag{2.3}$$

By $y_n \in I$, it holds that $|y_n| \geq \varepsilon = \frac{|y|}{2}$ and hence $|\frac{1}{y_n}| \leq \frac{2}{|y|}$. Consequently, and by Lemma 2.16, the right-hand side of (2.3) is of the form "null sequence · bounded sequence", so that Lemma 2.17 and 2.16 finish the proof: $\frac{1}{y_n} \to \frac{1}{y}$ and hence $\frac{x_n}{y_n} = x_n \cdot \frac{1}{y_n} \to x \cdot \frac{1}{y} = \frac{x}{y}$. ■

Example 2.20. limits – now a bit more complicated

a) *We start with*

$$x_n = \frac{1}{n^3} = \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \to 0 \cdot 0 \cdot 0 = 0.$$

b) *Consequently,*

$$y_n = \frac{n^2 - 7n + 4}{2n^3 + n - 5} = \frac{\cancel{n^2}}{n^{\cancel{3}}} \cdot \frac{1 - \frac{7}{n} + \frac{4}{n^2}}{2 + \frac{1}{n^2} - \frac{5}{n^3}} \to 0 \cdot \frac{1 - 7 \cdot 0 + 4 \cdot 0}{2 + 0 - 5 \cdot 0} = 0 \cdot \frac{1}{2} = 0.$$

2.5 The relation $\leq$ extends to the limit

Proposition 2.21. $\leq$ extends to the limit

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be convergent sequences with $x_n \to x$ und $y_n \to y$. If $x_n \leq y_n$ for all $n \in \mathbb{N}$ then also $x \leq y$ holds. In short:

$$x_n \leq y_n \quad \text{for all } n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

Since changes to finitely many members do not change the limit, the claim, $\lim x_n \leq \lim y_n$, still holds when $x_n \leq y_n$ is only true for all $n \geq n_0$.

Proof: By Proposition 2.19 a), it suffices to show that, for every convergent sequence $(z_n)_n$, the implication

$$z_n \geq 0 \quad \text{for all } n \in \mathbb{N} \qquad \Longrightarrow \qquad \lim_{n\to\infty} z_n \geq 0 \tag{2.4}$$

holds and then to apply it to $z_n = y_n - x_n$. So let $z_n \geq 0$ for all $n \in \mathbb{N}$.
Suppose, $z := \lim z_n < 0$. Then, for $\varepsilon := -\frac{z}{2} > 0$, there exists $n_0 \in \mathbb{N}$, such that $z_n \in (z - \varepsilon, z + \varepsilon)$ for all $n \geq n_0$. By $z + \varepsilon = \frac{z}{2} < 0$, this interval is entirely negative, so that $z_n < 0$ for all $n \geq n_0$. Contradiction! ↯ It follows that (2.4) holds. ■

⚠ Attention! The relation < does not extend to the limit!

Example: Although $0 < \frac{1}{n}$ *for all* $n \in \mathbb{N}$, *we have* $\lim 0 \not< \lim \frac{1}{n}$ *for the limits.*

2.6 The sandwich lemma and some of its applications

The following lemma runs under many names: sandwich lemma, squeeze lemma and police lemma are just some of them. It is probably our most valuable tool in the hunt for convergence and limits.

Lemma 2.22. sandwich lemma

Let $x \in \mathbb{R}$ *and let* $(x_n)_{n\in\mathbb{N}}$ *and* $(z_n)_{n\in\mathbb{N}}$ *be convergent sequences with* $x_n \to x$ *and* $z_n \to x$. *If* $x_n \leq y_n \leq z_n$ *for all* $n \in \mathbb{N}$ *then also* $(y_n)_{n\in\mathbb{N}}$ *converges, where* $y_n \to x$.

Proof: Let $\varepsilon > 0$ be arbitrary. By $x_n \to x$ and $z_n \to x$, there are $n_0', n_0'' \in \mathbb{N}$, such that $x_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \geq n_0'$ and $z_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \geq n_0''$. For $n \geq n_0 := \max(n_0', n_0'')$, we hence have both, $x_n, z_n \in (x - \varepsilon, x + \varepsilon)$. Since y_n is always between x_n and z_n, it must also be located in $(x - \varepsilon, x + \varepsilon)$ for all $n \geq n_0$. Hence, $y_n \to x$. ■

Let's get to applications right away:

Example 2.23. Juggling with n-th roots

a) $\sqrt[n]{5} \to 1$, *precisely:* $\sqrt[n]{c} \to 1$ *for all* $c > 0$.
b) *Even* $\sqrt[n]{n} \to 1$ *and* $\sqrt[n]{n^3} \to 1$.
c) $\sqrt[n]{7^n + 4^n} \to 7$, *precisely:* $\sqrt[n]{a^n + b^n} \to \max(a, b)$ *for all* $a, b \geq 0$.

Proof: **a)** Let $x_n := \sqrt[n]{5} - 1$. We show that $x_n \to 0$. By the binomial formula,

$$5 = (1 + x_n)^n = \binom{n}{0} + \binom{n}{1} x_n + \binom{n}{2} x_n^2 + \cdots + \binom{n}{n} x_n^n \geq n x_n$$

since $x_n > 0$. It follows that $0 < x_n \leq \frac{5}{n}$, and, by the sandwich lemma, $x_n \to 0$. The proof for $c > 1$ works exactly like the one for $c = 5$. For $c = 1$, we have $\sqrt[n]{c} \equiv 1$, and for $c \in (0, 1)$, we argue via $\sqrt[n]{c} = 1/\sqrt[n]{1/c}$ with $1/c > 1$ as above.

b) In analogy, let $y_n := \sqrt[n]{n} - 1$. We show that $y_n \to 0$. By the binomial formula, one has

$$n = (1+y_n)^n = \binom{n}{0} + \binom{n}{1} y_n + \binom{n}{2} y_n^2 + \cdots + \binom{n}{n} y_n^n \geq \frac{n(n-1)}{2} y_n^2$$

since $y_n > 0$. It follows that $0 < y_n^2 \leq \frac{2}{n-1}$, so that $y_n^2 \to 0$ (sandwich lemma) and $y_n \to 0$. It follows that $\sqrt[n]{n} \to 1$ and also $\sqrt[n]{n^3} = \sqrt[n]{n} \cdot \sqrt[n]{n} \cdot \sqrt[n]{n} \to 1 \cdot 1 \cdot 1 = 1$.

c)

$$\sqrt[n]{7^n + 4^n} = \sqrt[n]{7^n \left(1 + \frac{4^n}{7^n}\right)} = 7 \cdot \sqrt[n]{1 + \left(\frac{4}{7}\right)^n} \to 7$$

by the sandwich lemma since

$$1 \leq \sqrt[n]{1 + \left(\frac{4}{7}\right)^n} \leq \sqrt[n]{2} \to 1. \qquad \blacksquare$$

Next, we look at three sequences that all grow, nonstop. Precisely, we mean the sequences $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$ with their members equal to

$$x_n = n^a, \qquad y_n = b^n, \qquad z_n = n!, \qquad n \in \mathbb{N},$$

where $a, b \in \mathbb{N}$ are fixed, with $b \geq 2$, and $n! = 1 \cdot 2 \cdot \ldots \cdot n$ is the factorial function.

We all agree that (in the sense of Definition 2.11):

$$x_n \to +\infty, \qquad y_n \to +\infty \qquad \text{and} \qquad z_n \to +\infty,$$

while we are wondering which of the three sequences is fastest in their race to $+\infty$.

To find out, we look at pairwise quotients, arguing that, e.g. n^5 grows faster than n^2 since even their ratio, $\frac{n^5}{n^2} = n^3 \to +\infty$ or, equivalently, since $\frac{n^2}{n^5} = \frac{1}{n^3} \to 0$.

Proposition 2.24. Exponential function b^n beats power function n^a

Let $a, b \in \mathbb{N}$ with $b \geq 2$. Then

$$\frac{n^a}{b^n} \to 0 \qquad \textit{i.e.} \qquad \frac{b^n}{n^a} \to +\infty \qquad \textit{for} \qquad n \to \infty.$$

The following idea is not quite easy to have but straightforward to verify:

Proof: Put $c := b - 1 > 0$. Then, for all $n \geq 2a$,

$$b^n = (1+c)^n = \binom{n}{0} + \binom{n}{1} c^1 + \cdots + \binom{n}{n} c^n \geq \binom{n}{a+1} c^{a+1},$$

where

$$\binom{n}{a+1} = \frac{n\cdot(n-1)\cdot\ldots\cdot(n-a)}{(a+1)!} \geq \frac{(n/2)^{a+1}}{(a+1)!}, \quad \text{since} \quad n, n-1, \ldots, n-a \geq \frac{n}{2}.$$

Consequently,

$$b^n \geq \frac{n^{a+1}c^{a+1}}{2^{a+1}(a+1)!}, \quad \text{whence} \quad \frac{n^a}{b^n} \leq \frac{n^a}{n^{a+1}}\cdot\frac{2^{a+1}(a+1)!}{c^{a+1}} = \frac{1}{n}\cdot\frac{2^{a+1}(a+1)!}{c^{a+1}} \to 0. \quad \blacksquare$$

Proposition 2.25. Factorial $n!$ beats exponential function b^n

Let $b \geq 2$ be integer. Then it holds that

$$\frac{b^n}{n!} \to 0 \qquad i.e. \qquad \frac{n!}{b^n} \to +\infty \qquad \textit{for} \qquad n \to \infty.$$

Proof: For $n > 2b$, we argue as follows:

$$\frac{b^n}{n!} = \frac{b\cdot b\cdot\ldots\cdot b}{1\cdot 2\cdot\ldots\cdot n} = \frac{b^{2b}}{(2b)!}\cdot\underbrace{\frac{b}{2b+1}}_{\leq\frac{1}{2}}\cdot\ldots\cdot\underbrace{\frac{b}{n}}_{\leq\frac{1}{2}} \leq \frac{b^{2b}}{(2b)!}\cdot\frac{1}{2^{n-2b}} \to 0. \qquad \blacksquare$$

Nerd box 2.26. And by how much is $n!$ faster than b^n?

The so-called Stirling formula shows that $n!$ grows as fast as $\sqrt{n}\cdot(\frac{n}{e})^n$. Precisely, the quotient of the two converges to $\frac{1}{\sqrt{2\pi}}$, where $e \approx 2.71828\ldots$ denotes the basis of the natural exponential function that we will meet in Section 6.4.

It seems pretty redundant to still check $x_n = n^a$ versus $z_n = n!$. Indeed,

$$\frac{x_n}{z_n} = \frac{x_n}{y_n}\cdot\frac{y_n}{z_n} \to 0\cdot 0 = 0.$$

2.7 Cauchy sequences

That $(x_n)_n$ with $x_n = (-1)^n$ is a divergent sequence, was mentioned in Example 2.5 b) already but so far we postponed the proof. And this is understandable because: Who wants to check that $x_n \not\to x$, for every single $x \in \mathbb{R}$? We will now find a clever argument.

For unbounded sequences we already have such a clever argument: By Proposition 2.10, every convergent sequence is bounded, so that unbounded sequences surely diverge. This kind of argument via a necessary condition for convergence saves a lot of work. We need something similar for our example $(x_n)_n$ with $x_n = (-1)^n$.

So let $(x_n)_n$ be convergent, say $x_n \to x$. Then we have that, for every $\varepsilon > 0$, sooner or later, i.e., for all n bigger than some n_0: $x_n \overset{\varepsilon}{\approx} x$.

For $m, n \geq n_0$ we hence conclude $x_m \overset{\varepsilon}{\approx} x \overset{\varepsilon}{\approx} x_n$ and therefore, $x_m \overset{2\varepsilon}{\approx} x_n$.

Lesson learnt: The members of a convergent sequence do not only get increasingly close to a (so far unknown) limit x but also to each other. Sequences with the latter property will get their own name:

Definition 2.27. Cauchy sequence **(Augustin-Louis Cauchy, 1789–1857)**

A sequence $(x_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if, for every $\varepsilon > 0$, there is a $n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq n_0$.

With quantors instead of words:

Cauchy sequence without words

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} : \quad \forall m, n \geq n_0 : \quad |x_m - x_n| < \varepsilon$$

A popular short notation is

$$x_m - x_n \;\to\; 0 \qquad \text{for} \qquad m, n \to \infty.$$

Right before Definition 2.27, we learnt this (noting that 2ε is just another ε):

Proposition 2.28. convergent sequence $\Longrightarrow$ Cauchy sequence

Every convergent sequence is a Cauchy sequence.

Remarkably, in the setting of real numbers, also the reverse implication holds:

Proposition 2.29. In $\mathbb{R}$, we also have the reverse implication.

Every Cauchy sequence of real numbers is convergent.

Of course, when we say "sequence" in this course, we *always* mean a sequence of real numbers. Still, we purposely make that distinction between Propositions 2.28 and 2.29 to emphasize which implication holds in arbitrary so-called metric spaces, while the other implication fails in number spaces like the rational numbers, $\mathbb{Q}$.

Number spaces, where the reverse implication holds (i.e., where all Cauchy sequences converge), are called complete. The completeness of $\mathbb{R}$ is hardwired into the definition of $\mathbb{R}$, which is one of the most difficult topics in each Analysis course for maths students. Let's skip that issue here.

Corollary 2.30. In $\mathbb{R}$, it holds that: convergent sequence = Cauchy sequence

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Let us close the full circle: The sequence $(x_n)_n$ with $x_n = (-1)^n$ is obviously not a Cauchy sequence (and hence not convergent) since $x_m - x_n \not\to 0$ for $m, n \to \infty$. In particular, $|x_{n+1} - x_n| = 2$ for all $n \in \mathbb{N}$.

2.8 Monotonicity and convergence

Definition 2.31. monotonicity

A sequence $(x_n)_n$ is called

- *monotonically increasing, if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$,*
- *strictly increasing, if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$,*
- *monotonically decreasing, if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$,*
- *strictly decreasing, if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$,*
- *monotonic, if it is monotonically increasing or decreasing, and*
- *strictly monotonic, if it is strictly increasing or decreasing.*

Instead of "increasing" and "decreasing" we also say "growing" and "falling".

As a second ingredient, recall the notions of boundedness from Definition 2.8. Bringing the two together, we get one of the two super tools (besides the sandwich lemma) of the current chapter:

Proposition 2.32. monotonicity & boundedness $\implies$ convergence

a) *A monotonically growing sequence that is bounded from above is convergent.*
b) *A monotonically falling sequence that is bounded from below is convergent.*
c) *A monotonic and bounded sequence is convergent.*

Proof: **a)** Let $(x_n)_n$ be monotonically growing and bounded from above. Suppose, $(x_n)_n$ were divergent.

By Proposition 2.29, $(x_n)_n$ is not a Cauchy sequence, i.e.,

$$\exists \varepsilon > 0 \quad \forall n_0 \in \mathbb{N}: \quad \exists m, n \geq n_0: \quad x_m \overset{\varepsilon}{\not\approx} x_n. \tag{2.5}$$

By the monotonicity of $(x_n)_n$, we can assume that $m = n_0$. Let us denote one of these n by n_1. By $n_1 \geq n_0$ and monotonicity again, $x_{n_1} \geq x_{n_0} + \varepsilon$ holds. If we repeat the argument starting at (2.5) with n_1 instead of n_0, we conclude the existence of a

$n_2 \geq n_1$ with $x_{n_2} \geq x_{n_1} + \varepsilon \geq x_{n_0} + 2\varepsilon$. Iteration of this argument shows that some x_n get arbitrarily large, contradicting the boundedness from above. $\lightning$
So actually, $(x_n)_n$ is convergent.

b) Apply part **a)** to the sequence $(-x_n)_n$.

c) Bounded sequences are bounded from above and below. Monotonic sequences are growing or falling. So one of the cases **a)** or **b)** applies. ∎

Example 2.33. Our Proposition 2.32 in action

a) *Let $x_n = q^n$ with $q \in (0,1)$. Then $x_{n+1} = q \cdot x_n < x_n$ and $x_n > 0$ for all $n \in \mathbb{N}$. By Proposition 2.32 b), $(x_n)_n$ converges. Let $x := \lim x_n$. A little trick:*

$$x = \lim x_n = \lim x_{n+1} = \lim(q \cdot x_n) = q \cdot \lim x_n = q \cdot x,$$

so that $0 = x - q \cdot x = (1-q) \cdot x$. By $1 - q \neq 0$, only $x = 0$ remains.

b) *Let $y_n = \sqrt[n]{5}$. We know from Example 2.23 a), that $y_n \to 1$. Another proof: By $y_n^n = 5 > 1$, we have $y_n > 1$. If y_{n+1} were $\geq y_n$ then $5 = y_{n+1}^{n+1} \geq y_n^{n+1} = y_n \cdot y_n^n = y_n \cdot 5 > 5$. Contradiction! $\lightning$ Hence, $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$. By Proposition 2.32 b), $(y_n)_n$ is convergent. Let $y := \lim y_n$. A little trick:*

$$y = \lim y_n = \lim y_{2n} = \lim \sqrt{y_n} = \sqrt{\lim y_n} = \sqrt{y}.$$

So $y^2 = y$, whence $y(y-1) = 0$, leaving us with $y = 0$ and $y = 1$. From $y_n > 1$ follows $y \geq 1$, so that the limit is $y = 1$.

PS: The trick in **b)** *uses a so-called subsequence and Proposition 2.40 from below.*

The limit $\lim x_n$ from Proposition 2.32 plays another special role for the sequence (x_n):

Proposition 2.34. smallest upper and largest lower bound

a) *Let $(x_n)_n$ be monotonically growing and bounded from above. The limit $x = \lim_{n\to\infty} x_n$ from Proposition 2.32 is then the smallest upper bound on the sequence $(x_n)_n$.*

b) *Let $(x_n)_n$ be monotonically falling and bounded from below. The limit $x = \lim_{n\to\infty} x_n$ from Proposition 2.32 is then the largest lower bound on the sequence $(x_n)_n$.*

Proof: **a)** The limit x is an upper bound: x is not exceeded by any x_n since, suppose $x_n = x + \delta$ with $\delta > 0$ for some $n \in \mathbb{N}$, then, by monotonicity, $x_m \geq x + \delta$ for all $m \geq n$ and hence, $x = \lim_m x_m \geq x + \delta$ by Proposition 2.21.

x is the smallest upper bound: Suppose $b < x$ were another upper bound on (x_n). By $x_n \to x$, there is, for all $\varepsilon > 0$, some $n \in \mathbb{N}$ with $x_n \in (x - \varepsilon, x + \varepsilon)$. This also holds for $\varepsilon := x - b$. But then is $x_n > b$, and b is not an upper bound on (x_n). $\lightning$

b) The second case is completely symmetric to the first. ∎

From here it is not far to showing that every bounded sequence (also if non-monotonic) has a smallest upper and a largest lower bound.

Definition 2.35. infimum, supremum, minumum and maximum of a sequence

Let $(x_n)_n$ be a bounded sequence.

- *The largest lower bound of the sequence is called its infimum.*
- *If the infimum is equal to a sequence member x_n then it is also called minimum.*
- *The smallest upper bound of the sequence is called its supremum.*
- *If the supremum is equal to a sequence member x_n then it is also called maximum.*

The corresponding notations are $\inf x_n$, $\min x_n$, $\sup x_n$ and $\max x_n$.

For unbounded sequences $(x_n)_n$ with no lower bound, we put $\inf x_n := -\infty$, and if it has no upper bound then let $\sup x_n := +\infty$.

Time for examples:

Example 2.36. infimum, supremum, minumum and maximum

a) *The sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n = \frac{n-1}{n} = 1 - \frac{1}{n}$ is bounded. The infimum is $x_1 = 0$ (and hence also the minimum), and the supremum of this monotonically growing sequence is $\lim x_n = 1$. Since 1 is not a sequence member x_n, the supremum is not a maximum.*

b) *The sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n = \sqrt{n}$ is just bounded from below. Infimum (and at the same time minimum) is $y_1 = 1$. Since there is no upper bound, we have $\sup y_n = +\infty$, which is clearly not a maximum.*

Also in this section we got very close to the topic of completeness of $\mathbb{R}$:

Nerd box 2.37. Here we go again: completeness of $\mathbb{R}$

Via Proposition 2.29, we've been using the completeness of $\mathbb{R}$ in the proof of Proposition 2.32. The existence of the supremum and the infimum of a bounded sequence is indeed equivalent to the completeness of $\mathbb{R}$. In $\mathbb{Q}$, the sequence $x_n = \lfloor 10^n\,\pi \rfloor\, 10^{-n}$ with the integer round-down function $\lfloor\cdot\rfloor$ is bounded from above but there is no smallest upper bound: The set of all upper bounds in $\mathbb{Q}$ is $\{q \in \mathbb{Q} : q \geq \pi\}$, which indeed has no smallest element.

An alternative proof of Proposition 2.32 a), when Proposition 2.29 is not available but instead the existence of supremum and infimum are guaranteed:

Let $x := \sup x_n$ and take some $\varepsilon > 0$. Since $x - \varepsilon$ is not an upper bound (x is the smallest), there is a $n_0 \in \mathbb{N}$ such that $x_{n_0} \in (x - \varepsilon, x]$. By monotonicity, all x_n with $n \geq n_0$ are located in $[x_{n_0}, x] \subset (x - \varepsilon, x] \subset (x - \varepsilon, x + \varepsilon)$. ■

2.9 Subsequences and their limits

Definition 2.38. subsequence

The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is called a <u>subsequence</u> *of the sequence $(x_n)_{n\in\mathbb{N}}$ if $(n_1, n_2, n_3, \dots)$ is a strictly growing sequence of natural numbers, $n_1 < n_2 < n_3 < \dots$.*

If, for example, $(n_k)_{k\in\mathbb{N}}$ is the sequence of prime numbers, $(2, 3, 5, 7, 11, \dots)$, then, out of the original sequence $(x_1, x_2, x_3, \dots)$ of real numbers, we are extracting the subsequence $(x_2, x_3, x_5, x_7, x_{11}, \dots)$, e.g., the Hamburg air temperatures on the prime number days.

Important to keep in mind:

- subsequences are still infinitely long,
- they are derived by omitting members from the original sequence,
- as long as there are still infinitely many left.
- Reordering is not allowed.
- Also the original sequence is a subsequence of itself. (No omissions.)

Example 2.39. subsequences

Let the sequence $(x_n)_n = ((-1)^n)_{n\in\mathbb{N}} = (-1, +1, -1, +1, \dots)$ be given.

a) *The subsequence $(x_{n_k})_k$ with $n_k = 2k$ is the constant sequence $(+1, +1, +1, \dots)$.*
b) *The subsequence $(x_{n_k})_k$ with $n_k = 2k+1$ is the constant sequence $(-1, -1, -1, \dots)$.*
c) *The subsequence $(x_{n_k})_k$ with $n_k = k$ is the original sequence.*
d) *The subsequence $(x_{n_k})_k$ with $n_k = 3k$ is equal to the original sequence.*

The following statement is hardly surprising:

Proposition 2.40. Subsequences of convergent sequences are convergent

If $x_n \to x$ then also every subsequence of $(x_n)_n$ converges to x.

Proof: Let $\varepsilon > 0$ be arbitrary. By $x_n \to x$ it follows that $x_n \overset{\varepsilon}{\approx} x$ for all $n \in \mathbb{N}$ up to finitely many exceptions. This property is preserved if we omit sequence members. ■

We know from Section 2.2 that convergent sequences are bounded but that bounded sequences need not coverge. They have, however, always a convergent subsequence:

Proposition 2.41. of Bolzano-Weierstraß — Bernard Bolzano 1781–1848, Karl Weierstraß 1815–1897

Every bounded sequence has a convergent subsequence.

Proof: Let $(x_n)_n$ be a bounded sequence and $[a_1, b_1]$ an interval containing all x_n. If we cut the interval in two halfs, one of them contains infinitely many x_n. Denote that half by $[a_2, b_2]$. Cutting $[a_2, b_2]$ again in halfs and continuing like this, we get a sequence of nested intervals $[a_k, b_k]$, with $k \in \mathbb{N}$, each containing infinitely many x_n.

The left endpoints a_k form a monotonically growing sequence with upper bound b_1, converging, by Proposition 2.32 a). Say, $a_k \to a$. The interval lengths $b_k - a_k = \frac{1}{2^{k-1}}(b_1 - a_1)$ converge to zero, so that also $b_k \to a$. Now pick natural numbers $n_1 < n_2 < n_3 < \dots$ with $x_{n_k} \in [a_k, b_k]$ and conclude $x_{n_k} \to a$ by the sandwich lemma. ■

Looking back, this seems actually pretty obvious: Infinitely many points sitting in a bounded interval – they have to accumulate somewhere.

Definition 2.42. partial limit

The limit of a convergent subsequence of $(x_n)_n$ is called a partial limit *(or sometimes an* accumulation point*) of $(x_n)_n$.*

Example 2.43. partial limits

- **a)** *If $(x_n)_n$ is convergent with $x_n \to x$ then it has just one partial limit: x.*
- **b)** *Our sequence $((-1)^n)_n$ does have the two partial limits -1 and $+1$.*
- **c)** *The sequence $(\sin(n\frac{\pi}{2}))_n$ does have the three partial limits -1, 0 and $+1$.*

It is easy to see that also the reverse implication holds in Example a).

Nerd box 2.44. limes superior and limes inferior

The set S of all partial limits of a bounded sequence $(x_n)_n$ is

- *nonempty (Proposition 2.41, Bolzano-Weierstraß),*
- *bounded (Proposition 2.21) and*
- *closed (see Definition 4.8 below).*

That's why the set S has a largest and a smallest element, that we denote by $\limsup x_n$ and $\liminf x_n$, respectively. By Proposition 2.40 and the reverse direction of Example 2.43 a), it follows that $x_n \to x \iff \liminf x_n = \limsup x_n = x$.

Summary[1]

- Sequences are functions $\mathbb{N} \to \mathbb{R}$, thought of as infinite vectors $(x_1, x_2, \dots)$ of real numbers. We write $(x_n)_{n\in\mathbb{N}}$ or just $(x_n)_n$.
- A sequence $(x_n)_n$ is said to converge to $x \in \mathbb{R}$, written as $x_n \to x$, if, for every precision $\varepsilon > 0$, it holds that $x_n \overset{\varepsilon}{\approx} x$ for all $n \in \mathbb{N}$ up to finitely many exceptions.
- Convergent sequences are bounded – but not the other way round.
- lim commutes with $+, -, \cdot$ and $:$.
- If $x_n \leq y_n$ then $\lim x_n \leq \lim y_n$.
- From $x_n \leq y_n \leq z_n$ and $x_n \to y \leftarrow z_n$ it follows that $y_n \to y$.
- Convergent sequences are Cauchy sequences, i.e., $x_m - x_n \to 0$ as $m, n \to \infty$.
- Monotonic and bounded sequences are convergent.
  - In this case, the limit is either the infimum or the supremum of the sequence.
- Every bounded sequence has convergent subsequences.
  - If every one of them has the same limit then the original sequence converges.
  - And vice versa.

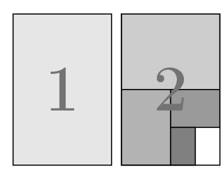
[1]Disclaimer: At the end of each chapter, we have a very short summary like this. Things that are missing here are however important. (Otherwise they would not be in the notes, and these would be just one page per chapter.)

Chapter 3

Series

Granny is not feeling great so I'll go and visit her. If I bring two pieces of cake, that will make her feel better. She looks happy but she doesn't even touch her piece of cake. (She must feel really bad today.) My piece was delicious. It's long finished by now. I could have more... I ask Granny if she's sure about not having any bit of cake. And then, being polite, I have half of her piece. And a bit later half of what's left. Then again half of the rest. And so on, until nothing is left but crumbles.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \stackrel{?}{=} 2$$



It looks like I had both pieces, after all.
Then I guess I'll have all the wine, too.

This infinite process of summation seems to lead to a finite result. Is it really 2?

- More than two pieces have never been on the table.
- But I didn't have much less than two.
- The amount of crumbles on the plate converges to zero as I keep eating.

In general: Take a sequence $(x_n)_n$ and start adding up:

$$\underbrace{\underbrace{\underbrace{x_1}_{s_1} + x_2}_{s_2} + x_3}_{s_3} + x_4 + \dots$$

The interim results, $s_1, s_2, s_3, \dots$, form a sequence that is to be studied for convergence. In our cake example, apparently, $s_m \to 2$ as $m \to \infty$.

Definition 3.1. series, partial sums

For a given sequence, $(x_n)_{n\in\mathbb{N}}$, the number

$$s_m \;:=\; \sum_{n=1}^{m} x_n, \qquad m \in \mathbb{N}$$

is called the m-th partial sum of $(x_n)_{n\in\mathbb{N}}$.
The sequence $(s_m)_{m\in\mathbb{N}}$ is called the series with the members x_n.
We say that the series converges to $s \in \mathbb{R}$ if $s_m \to s$, and we write

$$\sum_{n=1}^{\infty} x_n \;=\; s.$$

Remarks:

- Instead of "interim results" we say partial sums. The rest was quite right.
- The symbol $\sum_{n=1}^{\infty} x_n$ is used for the series, $(s_m)_{m\in\mathbb{N}}$, and for its limit.
- In the sense of Definition 2.11, we write $\sum_{n=1}^{\infty} x_n = +\infty$ if $s_m \to +\infty$.
- Sometimes it makes life easier to start summation at $n = 0$, not 1. Or possibly at 7.

Attention! series $\neq$ sequence

The words "series" and "sequence" often mean the same in everyday life. But, careful: not in mathematics!

3.1 Some first examples and criteria

Example 3.2. the geometric series

a) *Our cake example:*

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \;=\; \sum_{n=0}^{\infty} \frac{1}{2^n} \;=\; \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \;=\; 2,$$

as we will show in a minute.

b) *A so-called geometric series arises, when we fix some $q \in (-1, 1)$ and form the series $\sum_{n=0}^{\infty} q^n$. For $m \in \mathbb{N}$, let s_m denote the m-th partial sum:*

$$s_m \;=\; 1 + q + q^2 + \dots + q^m.$$

Multiplying this equality by q, we get

$$q \cdot s_m = q + q^2 + q^3 + \cdots + q^{m+1}.$$

Subtracting the second equality from the first, we have

$$s_m - qs_m = 1 - q^{m+1} \qquad \text{and hence} \qquad s_m = \frac{1-q^{m+1}}{1-q}.$$

By Example 2.33 a), $|q^m| = |q|^m \to 0$ and therefore $q^m \to 0$, which is why

$$s_m \to \frac{1}{1-q} = \sum_{n=0}^{\infty} q^n.$$

In our cake example **a)** *with $q = \frac{1}{2}$, we thus confirm that $s_m \to \frac{1}{1-\frac{1}{2}} = 2$.*

c) *For $q = 1$, we get $s_m = 1 + \cdots + 1 = m + 1 \to +\infty$. Divergence!*

d) *For $q = -1$, we get $1 - 1 + 1 - 1 + - \ldots$ Where is this converging to?*

- *Answer 1:* $s = (1-1) + (1-1) + \ldots = 0 + 0 + \ldots = 0.$
- *Answer 2:* $s = 1 + (-1+1) + (-1+1) + \ldots = 1 + 0 + 0 + \ldots = 1.$
- *Answer 3:* $s = 1 - (1 - 1 + 1 - 1 + - \ldots) = 1 - s \Rightarrow 2s = 1 \Rightarrow s = \frac{1}{2}.$

Solution: All wrong. The series does not converge at all, by Proposition 3.3.

Attention! Don't do this at home!

When convergence of the series is not yet clear, manipulations as in Example 3.2 **d)** *can go arbitrarily wrong.*

If someone had guaranteed us the convergence of b) beforehand, be could have argued in b) as sporty as in d):

$$s = \sum_{n=0}^{\infty} q^n \quad \Longrightarrow \quad qs = \sum_{n=0}^{\infty} q^{n+1} \quad \Longrightarrow \quad s - qs = q^0 = 1 \quad \Longrightarrow \quad s = \frac{1}{1-q}.$$

With this kind of argument, you can, e.g., see really fast why $0.999\ldots = 1$ since

$$\begin{aligned} s &:= 0.999\ldots && | \cdot 10 \\ 10s &= 9.999\ldots && | \text{ subtract first equality from here} \\ 9s &= 9, \quad \text{so that} && \underline{\underline{s = 1}}. \end{aligned}$$

But first, the convergence of the series (in this case: $s = 9 \cdot \sum_{n=1}^{\infty} 0.1^n$) has to be ensured.

Proposition 3.3. necessary convergence condition for series

If the series $\sum_{n=1}^{\infty} x_n$ *converges then* $x_n \to 0$.

Proof: Let $s_m = \sum_{n=1}^{m} x_n \to s$. Then $x_n = s_n - s_{n-1} \to s - s = 0$. ■

This proposition is almost always used via contraposition:

$$\text{If } x_n \not\to 0 \text{ then } \sum_{n=1}^{\infty} x_n \text{ just cannot converge.}$$

And this is also the solution for Examples 3.2 c) and d) as well as for all $q \in \mathbb{R}$ with $|q| \geq 1$. Is the condition $x_n \to 0$, besides being necessary, also sufficient? No:

Example 3.4. the harmonic series

This time we study the so-called <u>*harmonic series*</u>*, that is*

$$\sum_{n=1}^{\infty} \frac{1}{n} \;=\; \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

The members, $\frac{1}{n}$*, form a null sequence, so that convergence and divergence are still both possible. By the following neat arrangement,*

$$\frac{1}{1} + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{> 2\cdot\frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{> 4\cdot\frac{1}{8} = \frac{1}{2}} + \underbrace{\frac{1}{9} + \dots + \frac{1}{16}}_{> 8\cdot\frac{1}{16} = \frac{1}{2}} + \dots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

we can see that the partial sums s_m *will eventually exceed every* $M \in \mathbb{R}$ *and grow monotonically, so that* $s_m \to +\infty$.

For null sequences $(x_n)_n$, it all depends on how fast they decay. $x_n = \frac{1}{2^n}$ is fast enough, while $\frac{1}{n}$ isn't. Where is the transition happening? Let us look at two more examples:

Example 3.5. denominator of second order

a) *We switch from* $x_n = \frac{1}{n}$ *to a term with denominator of second order:*

$$x_n \;=\; \frac{1}{n(n+1)} \;=\; \frac{1}{n} - \frac{1}{n+1}.$$

By the clever transformation in the last step, we conclude, for every $m \in \mathbb{N}$*:*

$$s_m \;=\; x_1 + \ldots + x_m \;=\; \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + - \dots + \frac{1}{m} - \frac{1}{m+1} \;=\; 1 - \frac{1}{m+1} \;\to\; 1.$$

b) *Also for* $x_n = \frac{1}{n^2}$, *we get convergence since*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \sum_{n=2}^{\infty} \frac{1}{n^2} = 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)^2}$$

and by $0 \leq \frac{1}{(k+1)^2} < \frac{1}{k(k+1)}$. *Consequently,* s_m *grows monotonically and is bounded from above by* $1+$ *the limit from* **a)**. *By Proposition 2.32, it converges.*

Nerd box 3.6. And where exactly do we switch from divergence to convergence?

For a series of the form $\sum_{n=1}^{\infty} \frac{1}{n^s}$, *the transition happens at* $s = 1$: *For* $s \leq 1$, *one has divergence, for* $s > 1$, *it's already convergence. The proof of this statement uses the same grouping technique as in Example 3.4, also see Nerd box 3.19 below.*

PS: The limit in Example 3.5 b), by the way, is $\frac{\pi^2}{6}$. *(↗ Basler problem)*

The final argument in Example 3.5 b) is often useful. Let us isolate it here:

Proposition 3.7. convergence = boundedness, if $x_n \geq 0$

If $x_n \geq 0$ *for all* $n \in \mathbb{N}$, *then:*

$$\sum_{n=1}^{\infty} x_n \text{ converges} \quad \Longleftrightarrow \quad (s_m)_{m \in \mathbb{N}} \text{ is bounded.}$$

Proof: The direction $\boxed{\Rightarrow}$ holds by Proposition 2.10. By $x_n \geq 0$, $(s_m)_{m\in\mathbb{N}}$ grows monotonically. Then $\boxed{\Leftarrow}$ follows by Proposition 2.32. ■

In contrast to the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$, that diverges, as seen in Example 3.4, the so-called *alternating harmonic series*

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots$$

is convergent (: to $\ln 2$), by the following result:

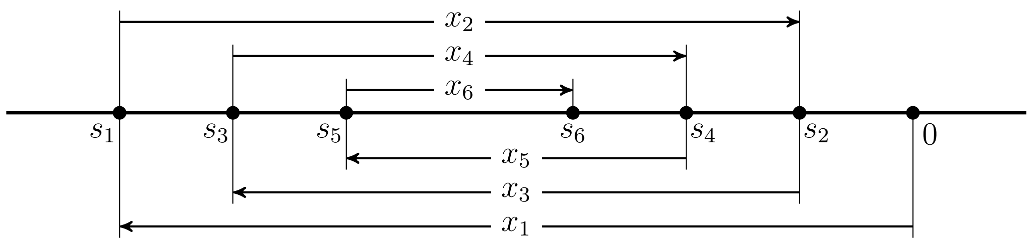
Proposition 3.8. Leibniz rule **(Gottfried Wilhelm Leibniz 1646–1716)**

Let $(x_n)_{n\in\mathbb{N}}$ *be a monotonically falling null sequence. Then the alternating series*

$$\sum_{n=1}^{\infty} (-1)^n x_n = -x_1 + x_2 - x_3 + x_4 - + \ldots$$

converges.

Proof: Look at the partial sums $s_1, s_2, s_3, \ldots$:



$$\Rightarrow \quad \text{nested intervals} \quad \underbrace{[s_1, s_2]}_{x_2} \supset \underbrace{[s_3, s_4]}_{x_4} \supset \underbrace{[s_5, s_6]}_{x_6} \supset \cdots$$
$$\text{with lengths} \quad x_2, \; x_4, \; x_6, \; \ldots \to 0,$$

By the so-called principle of nested intervals (: yet another equivalent way of imposing the completeness of $\mathbb{R}$), there is exactly one point $s \in \mathbb{R}$ in the intersection of all these intervals, and it holds that $s_m \to s$. ■

The sketch even shows the error bound, $|s_m - s| \le x_{m+1}$, at any point in time, $m \in \mathbb{N}$.

Also for series, Corollary 2.30 about the convergence of Cauchy sequences can be used:

$$\begin{aligned} \sum_{n=1}^{\infty} x_n \text{ converges} &\iff (s_m)_{m\in\mathbb{N}} \text{ converges} \overset{\text{F. 2.30}}{\iff} (s_m)_{m\in\mathbb{N}} \text{ is a Cauchy sequence} \\ &\iff \forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N}: \; \forall m, m' \ge n_0: \quad |s_{m'} - s_m| < \varepsilon, \\ &\iff \forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N}: \; \forall m, m' \ge n_0: \quad \left| \sum_{n=m+1}^{m'} x_n \right| < \varepsilon. \end{aligned} \tag{3.1}$$

The latter is called the *Cauchy criterion for series*.

3.2 Absolute convergence

Definition 3.9. absolute and condionional convergence of a series

A series $\sum_{n=1}^{\infty} x_n$ *is called absolutely convergent if even the series* $\sum_{n=1}^{\infty} |x_n|$ *converges.*
A convergent but not absolutely convergent series is called conditionally convergent.

Example 3.10. absolute and conditional convergence

a) *The geometric series (our cake example),* $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$*, is absolutely convergent.*

b) *The alternating harmonic series,* $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + - \ldots$, *is convergent, not absolutely convergent though, – hence conditionally convergent.*

By Proposition 3.7, a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, if the series (i.e. the sequence of partial sums) $\sum_{n=1}^{\infty} |x_n|$ is bounded.

As the wording "absolutely convergent" already suggests, one has:

Proposition 3.11. absolute convergence $\Longrightarrow$ convergence

Every absolutely convergent series is convergent.

Proof: Let $\varepsilon > 0$ be arbitrary. By the Cauchy criterion, (3.1), for $\sum_{n=1}^{\infty} |x_n|$, one has

$$\exists n_0 \in \mathbb{N}: \ \forall m, m' \geq n_0 : \ \varepsilon \ > \left| \sum_{n=m+1}^{m'} |x_n| \right| \ = \ \sum_{n=m+1}^{m'} |x_n| \ \geq \ \left| \sum_{n=m+1}^{m'} x_n \right|$$

$$\Longrightarrow \quad \sum_{n=1}^{\infty} x_n \text{ converges by the Cauchy criterion.} \quad \blacksquare$$

Hereby, the inequality

$$\sum_{n=1}^{m} |x_n| \ \geq \ \left| \sum_{n=1}^{m} x_n \right|$$

transfers to the limits as $m \to \infty$.

Example 3.10 b) shows that the reverse implication in Proposition 3.11 need not hold – so conditionally convergent series do exist.

Proposition 3.12. sums and multiples of (absolutely) convergent series

Let $\sum\limits_{n=1}^{\infty} x_n$ *and* $\sum\limits_{n=1}^{\infty} y_n$ *be two (absolutely) convergent series, and let* $c \in \mathbb{R}$ *be constant. Then also* $\sum\limits_{n=1}^{\infty} (x_n \pm y_n)$ *and* $\sum\limits_{n=1}^{\infty} (cx_n)$ *are (absolutely) convergent, and it holds that*

$$\sum_{n=1}^{\infty} (x_n \pm y_n) = \sum_{n=1}^{\infty} x_n \pm \sum_{n=1}^{\infty} y_n \qquad \textit{as well as} \qquad \sum_{n=1}^{\infty} (cx_n) = c \sum_{n=1}^{\infty} x_n, \qquad \textit{and}$$

$$\sum_{n=1}^{\infty} |x_n \pm y_n| \leq \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \qquad \textit{as well as} \qquad \sum_{n=1}^{\infty} |cx_n| = |c| \sum_{n=1}^{\infty} |x_n|.$$

Proof: For the corresponding partial sums, $\sum_{n=1}^{m} \ldots$, we have the formulas as claimed. By assumption, the limits, as $m \to \infty$, exist on the right-hand side, so that also the left-hand sides converge. For $\sum_{n=1}^{\infty} |x_n \pm y_n|$, we additionally involve Proposition 3.7. $\blacksquare$

3.3 Criteria by comparison with other series

In Examples 3.4 and 3.5, we already used bounds of unknown partial sums by known ones in order to prove convergence or divergence of the series. Here is the proper statement:

Proposition 3.13. direct comparison tests

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences. If $|x_n| \le |y_n|$ for all $n \in \mathbb{N}$, then

a) $\sum_{n=1}^{\infty} |y_n|$ *converges* $\implies$ $\sum_{n=1}^{\infty} |x_n|$ *converges* $\implies$ $\sum_{n=1}^{\infty} x_n$ *converges.*

b) $\sum_{n=1}^{\infty} x_n$ *diverges* $\implies$ $\sum_{n=1}^{\infty} |x_n|$ *diverges* $\implies$ $\sum_{n=1}^{\infty} |y_n|$ *diverges.*

Again, we allow finitely many exceptions from $|x_n| \le |y_n|$, i.e., $\forall n \ge n_0$ will suffice.

Proof: **b)** is the contraposition of **a)**.
a) The second implication $\Rightarrow$ holds by Proposition 3.11. It remains to prove the first: Let $\varepsilon > 0$ be arbitrary. By the Cauchy criterion for $\sum |y_n|$, there is a $n_0 \in \mathbb{N}$ such that

$$\forall m, m' \ge n_0 : \quad \varepsilon > \left| \sum_{n=m+1}^{m'} |y_n| \right| = \sum_{n=m+1}^{m'} |y_n| \ge \sum_{n=m+1}^{m'} |x_n| = \left| \sum_{n=m+1}^{m'} |x_n| \right|.$$

$\implies \quad \sum_{n=1}^{\infty} |x_n|$ converges, again by the Cauchy criterion. ■

Example 3.14. dominated convergence and its contraposition

a) *The series* $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ *converges since*

$$\frac{n!}{n^n} = \frac{1\cdot 2}{n\cdot n}\cdot\frac{3\cdots n}{n\cdots n} \le \frac{2}{n^2} \quad \text{for all } n \in \mathbb{N}$$

and since, by Example 3.5 b), $\sum_{n=1}^{\infty} \frac{2}{n^2}$ *converges (dominated convergence).*

b) *The series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$ *diverges since*

$$\frac{1}{\sqrt{n(n+1)}} > \frac{1}{n+1} \quad \text{for all } n \in \mathbb{N}$$

and since, by Example 3.4, $\sum_{n=1}^{\infty} \frac{1}{n+1} = \sum_{n=2}^{\infty} \frac{1}{n}$ *diverges.*

In the case of the series

$$400 + 200 + 100 + 50 + 25 + \ldots,$$

we can almost directly roll back to our cake example: Obviously, $x_n = 400 \cdot (\frac{1}{2})^n$ with $n = 0, 1, 2, \ldots$, whence the series converges. The hint comes by observing that $\frac{x_{n+1}}{x_n} = \frac{1}{2}$. Alternatively, we could have argued via $\sqrt[n]{x_n} = \sqrt[n]{400}\,\frac{1}{2} \to \frac{1}{2}$. Both are standard tests:

Proposition 3.15. the ratio test

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with at most finitely many members of the kind $x_n = 0$.

a) *If $\exists n_0 \in \mathbb{N}, q < 1 : \forall n \geq n_0 : \left|\frac{x_{n+1}}{x_n}\right| \leq q$, then $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent.*

b) *If $\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \left|\frac{x_{n+1}}{x_n}\right| \geq 1$, then $\sum\limits_{n=1}^{\infty} x_n$ diverges.*

Proof: **a)** From $\forall n \geq n_0 : \left|\frac{x_{n+1}}{x_n}\right| \leq q < 1$, it follows that

$$\forall n \geq n_0 : \quad |x_n| = \underbrace{\left|\frac{x_n}{x_{n-1}}\right|}_{\leq q} \cdot \underbrace{\left|\frac{x_{n-1}}{x_{n-2}}\right|}_{\leq q} \cdot \ldots \cdot \underbrace{\left|\frac{x_{n_0+1}}{x_{n_0}}\right|}_{\leq q} \cdot |x_{n_0}| \ \leq \ q^{n-n_0} \cdot |x_{n_0}|,$$

which implies the convergence of $\sum_{n=n_0}^{\infty} |x_n|$ via domination by the convergent series

$$\sum_{n=n_0}^{\infty} q^{n-n_0} |x_{n_0}| = \frac{|x_{n_0}|}{q^{n_0}} \cdot \sum_{n=n_0}^{\infty} q^n \quad \text{(geometric series with } q \in [0,1)\text{)}.$$

b) By $|x_{n+1}| \geq |x_n|$ for $n \geq n_0$, we have $x_n \not\to 0$, whence $\sum_{n=1}^{\infty} x_n$ diverges. ■

Proposition 3.16. the root test

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence.

a) *If $\exists n_0 \in \mathbb{N}, q < 1 : \forall n \geq n_0 : \sqrt[n]{|x_n|} \leq q$, then $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent.*

b) *If $\exists n_0 \in \mathbb{N} : \forall n \geq n_0 : \sqrt[n]{|x_n|} \geq 1$, then $\sum\limits_{n=1}^{\infty} x_n$ diverges.*

Proof: **a)** By $|x_n| \leq q^n$ for $n \geq n_0$, the series $\sum\limits_{n=n_0}^{\infty} q^n$ dominates our series $\sum\limits_{n=n_0}^{\infty} |x_n|$.

b) By $|x_n| \geq 1$ for $n \geq n_0$, $(x_n)_n$ is not a null sequence, whence $\sum_{n=1}^{\infty} x_n$ diverges. ■

Example 3.17. Both tests have their pro's and con's

a) *Let $a \in \mathbb{R}$ be arbitrary. $\sum\limits_{n=0}^{\infty} \frac{a^n}{n!}$ is absolutely convergent (where we put $0! := 1$) since, by the ratio test,*

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}}\right| = \frac{|a|}{n+1} \to 0 \quad \text{for } n \to \infty,$$

i.e. $\exists n_0 : \ \forall n \geq n_0 : \ \left|\frac{x_{n+1}}{x_n}\right| \leq \frac{1}{2} < 1$.
Note that the root test is a lot clumsier to use here.

b) *Let* $x_n = \begin{cases} \frac{1}{2^n} & , \ n \text{ even}, \\ \frac{1}{3^n} & , \ n \text{ odd}. \end{cases} \qquad \Longrightarrow \qquad \sqrt[n]{|x_n|} \leq \frac{1}{2} < 1,$ *(root test)*

$$\Longrightarrow \quad \sum_{n=1}^{\infty} x_n = \frac{1}{3^1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^5} + \dots \quad \text{converges (absolutely).}$$

Note that

$$\left|\frac{x_{n+1}}{x_n}\right| = \begin{cases} \left(\frac{3}{2}\right)^n \cdot \frac{1}{2} & > & 1, \ n \text{ odd}, \\ \left(\frac{2}{3}\right)^n \cdot \frac{1}{3} & < & 1, \ n \text{ even}, \end{cases}$$

so that the ratio test is unable to conclude anything!

Nerd box 3.18. ratio and root test via $\liminf$ **&** $\limsup$

When we use $\liminf$ *and* $\limsup$ *from Nerd box 2.44, we can reformulate both the ratio and root test in an almost equivalent way:*

$\div$ **a)** *If $\limsup \left|\frac{x_{n+1}}{x_n}\right| < 1$, then $\sum_{n=1}^{\infty} x_n$ converges absolutely.*
$\div$ **b)** *If $\liminf \left|\frac{x_{n+1}}{x_n}\right| > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.*

$\sqrt{\cdot}$ **a)** *If $\limsup \sqrt[n]{|x_n|} < 1$, then $\sum_{n=1}^{\infty} x_n$ converges absolutely.*
$\sqrt{\cdot}$ **b)** *If $\limsup \sqrt[n]{|x_n|} > 1$, then $\sum_{n=1}^{\infty} x_n$ diverges.*

- *The asymmetry in* $\liminf$ *vs* $\limsup$ *is not a typo; it's like it is written here.*
- *Nothing can be said if $\liminf \left|\frac{x_{n+1}}{x_n}\right| \leq 1 \leq \limsup \left|\frac{x_{n+1}}{x_n}\right|$,*
- *neither if $\limsup \sqrt[n]{|x_n|} = 1$.*

One can show that the root test is more powerful than the ratio test, in the following sense:

- When the ratio test signals convergence, then the root test does that, too.
- This is not true the other way round, as Example 3.17 b) shows.

- However, the ratio test if often easier to apply (if applicable).
- In the example $\sum_{n=1}^{\infty} \frac{1}{n^s}$ however, they are both clueless.

Nerd box 3.19. Cauchy's condensation test

The grouping strategy from Example 3.4 along the powers of two, $n = 2^k$, is easily generalised: If $(x_n)_n$ is a monotonically falling null sequence then

$$\sum_{n=1}^{\infty} x_n \text{ converges} \qquad \Longleftrightarrow \qquad \sum_{k=0}^{\infty} 2^k \cdot x_{2^k} \text{ converges.}$$

And this is how to crack the nut from Nerd box 3.6, where all our other tests fail.

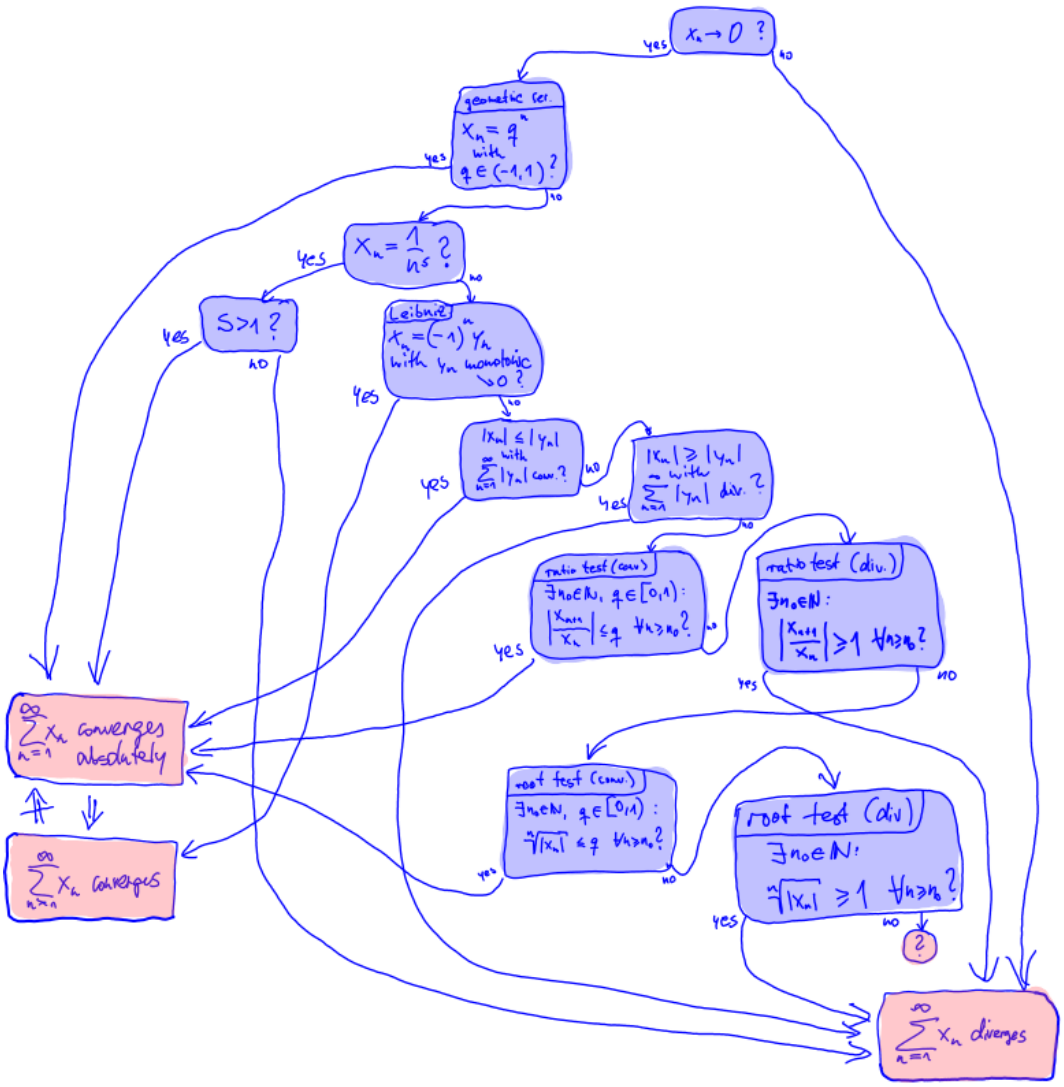
Summary

- One takes a sequence $(x_1, x_2, x_3, \dots)$ and starts adding its members:

$$\underbrace{\underbrace{\underbrace{x_1}_{s_1} + x_2}_{s_2} + x_3}_{s_3} + x_4 + \dots$$

- A series is the sequence $(s_m)_{m=1}^{\infty}$ of partial sums, $s_m = \sum_{n=1}^{m} x_n$.
  It is denoted by the symbol $\sum_{n=1}^{\infty} x_n$.
- The geometric series with $|q| < 1$ is a standard example of convergent series.
- For the convergence of the series $\sum_{n=1}^{\infty} x_n$ it is necessary that its members x_n form a null sequence. Otherwise there is no chance for convergence.

  - And is $x_n \to 0$ also enough (sufficient) for the convergence of the series $\sum_{n=1}^{\infty} x_n$? No, for this, one needs x_n to converge fast enough to 0.
  - The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, for example, while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
  - Precisely: In the special case where $x_n = \frac{1}{n^s}$ for some $s \in \mathbb{R}$, one has convergence of the series $\sum_{n=1}^{\infty} x_n$ if $s > 1$ and divergence otherwise.
  - In general cases, we suggest the following tests:



very short roundup

The topic of series essentially comes with two questions:

a) *Is my series convergent?*

We now know a number of tricks for this one.

b) *If yes, what's the limit?*

This question is much tougher, and we know much less.

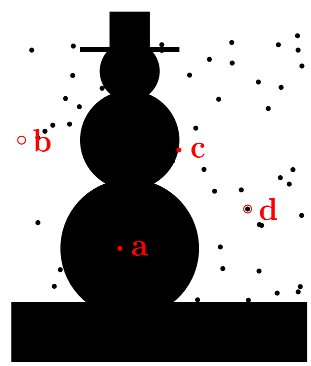
Chapter 4

Topology of the real line

Having discussed sequences and series, we now come to so-called topological properties of sets. Although all our sets will be subsets of $\mathbb{R}$, for the following illustration, let's look at a set in $\mathbb{R}^2$:

In the figure on the right, the points of a particular set $A \subset \mathbb{R}^2$ are shown in black. Then

a ... interior point of A,
b ... exterior point of A,
c ... boundary point of A,
d ... isolated point of A.



Next we explain these and further kinds of points in more details.

4.1 Some important kinds of points

For our classification of points we still need one more notion:

Definition 4.1. ε**-neighbourhood**

For $x \in \mathbb{R}$ and $\varepsilon > 0$, we refer to the set

$$U_\varepsilon(x) \ := \ (x-\varepsilon, x+\varepsilon) \ = \ \{y \in \mathbb{R} : |x-y| < \varepsilon\}$$

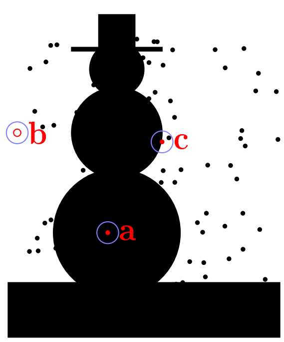
as the $\underline{\varepsilon\text{-neighbourhood}}$ of x.

On different length scales $\varepsilon > 0$, we now look at the ε-neighbourhood $U_\varepsilon(x)$ of a point x and decide about the kind of x, depending on how much of $U_\varepsilon(x)$ belongs to A or to its complement, A^c:

Definition 4.2. interior, exterior and boundary points

Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. The point x is called

- *an interior point of A, if $U_\varepsilon(x) \cap A = U_\varepsilon(x)$ for some $\varepsilon > 0$,*
  i.e., there is an $\varepsilon > 0$, such that $U_\varepsilon(x)$ is entirely in A,
  in particular, we then have $x \in A$,
- *exterior point of A, if $U_\varepsilon(x) \cap A = \varnothing$ for some $\varepsilon > 0$,*
  i.e., there is an $\varepsilon > 0$, such that $U_\varepsilon(x)$ is entirely outside of A,
  i.e., x is an interior point (and hence an element) of the complement, A^c,
- *boundary point of A, if none of the two cases above applies,*
  that is, when in every $U_\varepsilon(x)$ with $\varepsilon > 0$, there are both points from A and from A^c.



Back to our snowman:

- The point a has an entirely black ε-neighbourhood and is therefore an interior point of A.
- The point b has an entirely white ε-neighbourhood and is therefore an exterior point of A.
- For point c, every ε-neighbourhood is black/white mixed. Therefore, c is a boundary point of A.

Definition 4.3. interior, exterior and closure of A

Let $A \subset \mathbb{R}$ be a set.

- $\operatorname{int} A$ *be the set of all interior points of A, and it is called the interior of A,*
- $\operatorname{ext} A$ *be the set of all exterior point of A, and it is called the exterior of A,*
- ∂A *be the set of all boundary points of A, and it is called the boundary of A.*
- *The set $\operatorname{clos} A := \operatorname{int} A \mathbin{\dot\cup} \partial A$ is called the closure of A.*

$$\text{I.e.,} \quad x \in \operatorname{clos} A \iff x \notin \operatorname{ext} A \iff \neg(\exists \varepsilon > 0 : U_\varepsilon(x) \cap A = \varnothing)$$
$$\iff \forall \varepsilon > 0 : U_\varepsilon(x) \cap A \neq \varnothing \iff \exists x_1, x_2, \ldots \in A : x_n \to x.$$

Here $B \mathbin{\dot\cup} C$ (disjoint union) stands for $B \cup C$ with the extra info that $B \cap C = \varnothing$.

Example 4.4. some simple examples on the real axis

For $A = [0, 1)$ *and* $B = [1, 2] \cup \{3\}$, *we have that*

a) $\operatorname{int} A = (0, 1)$, $\operatorname{ext} A = (-\infty, 0) \cup (1, +\infty)$, $\partial A = \{0, 1\}$ *and* $\operatorname{clos} A = [0, 1]$.

b) $\operatorname{int} B = (1, 2)$, $\operatorname{ext} B = (-\infty, 1) \cup (2, 3) \cup (3, +\infty)$, $\partial B = \{1, 2, 3\}$, $\operatorname{clos} B = B$.

Immediately by Definitions 4.2 and 4.3, we conclude:

Corollary 4.5. connections between interior, exterior and boundary

One has

$$\operatorname{int} A \cup \operatorname{ext} A \cup \partial A = \mathbb{R}$$

as well as

$$\operatorname{int} A = A \setminus \partial A, \qquad \operatorname{ext} A = \operatorname{int}(A^c), \qquad \partial A = \partial(A^c).$$

We immediately conclude some futher formulas for the closure:

Corollary 4.6. Formulas for the closure

$$\begin{aligned} \operatorname{clos} A &= \operatorname{int} A \cup \partial A &&= A \cup \partial A, \\ \operatorname{clos} A &= (\operatorname{ext} A)^c &&= (\operatorname{int}(A^c))^c. \end{aligned}$$

We could study further kinds of points but those introduced so far are enough for us.

Nerd box 4.7. Isolated points and limit points

A point $x \in A \subset \mathbb{R}$ *is called*

- *an* *isolated point* *of* A, *if* $U_\varepsilon(x) \cap A = \{x\}$ *for some* $\varepsilon > 0$,
- *a* *limit point* *of* A, *if* $U_\varepsilon(x) \cap A$ *has infinitely many points for all* $\varepsilon > 0$.

Isolated points (like d *in the snow man) are special boundary points.*
x *is a limit point of* A *if and only if* $x \in \operatorname{clos}(A \setminus \{x\})$.
The limit points of A *are the closure of* A *without the isolated points.*

Attention! accumulation point $\neq$ limit point

The sequence $(1, 1, 1, \dots)$ *has the accumulation point (a.k.a. partial limit)* 1.
The set $\{1, 1, 1, \dots\} = \{1\}$ *however, has no accumulation points at all.*
We prefer "partial limit" over "accumulation point" to avoid confusion.

4.2 Open and closed sets

Definition 4.8. open set, closed set

A set $A \subset \mathbb{R}$ is called

- *an open set if every point $x \in A$ is an interior point of A,*
  i.e., one has $A = \operatorname{int} A$, so that $A \cap \partial A = \varnothing$,
  i.e., the set A does not contain any of its boundary points $x \in \partial A$,
- *a closed set if its complement, A^c, is open,*
  i.e., $A^c = \operatorname{int}(A^c) = \operatorname{ext} A$,
  and hence $A = (\operatorname{ext} A)^c = \operatorname{clos} A$, so that $\partial A \subset A$,
  i.e., the set A contains all of its boundary points, $x \in \partial A$.
  By $\operatorname{clos} A \subset A$, from $x_1, x_2, \ldots \in A$ and $x_n \to x$ it always follows that $x \in A$.

⚠ Attention! Open and closed are not the opposites of each other!

Both sets $A = \mathbb{R}$ and $B = \varnothing$ are open and closed at the same time.
In contrast, many sets, like $C = [0, 1)$ or $D = (2, 3]$, are neither open nor closed.

We continue with examples, including proofs of the last claims.

Example 4.9. open and/or closed sets

a) *So-called "open intervals", $A = (a, b)$, are indeed open sets: Every point $x \in A$ still has an ε-neighbourhood with $U_\varepsilon(x) \subset A$. E.g., take $\varepsilon := \min(x-a, b-x)$.*
Shorter argument: The boundary, $\partial A = \{a, b\}$, is entirely outside of A.

b) *So-called "closed intervals", $A = [a, b]$, are indeed closed sets:*
The complement, $A^c = (-\infty, a) \cup (b, +\infty)$, is open (arguing as in a).
Or in short: The boundary, $\partial A = \{a, b\}$, is entirely contained in A.

c) *So-called "half-open intervals", $A = (a, b]$, are neither open nor closed: They contain their boundary, $\partial A = \{a, b\}$, neither completely nor not at all.*
Argued differently: A is not open since $x = b$ has no neighbourhood $U_\varepsilon(x) \subset A$;
A is not closed since $x_n = a + \frac{1}{n} \in A$ converges to $a \notin A$.

d) *The set $A = \mathbb{R}$ is open since, for every $x \in \mathbb{R}$, one has $U_\varepsilon(x) \subset \mathbb{R}$ with $\varepsilon := 1$.*

e) *The set $A = \varnothing$ is open. Who contradicts, has to find an $x \in \varnothing$ (↯) that is not an interior point of A.*

f) *The set $A = \mathbb{R}$ is closed since its complement, $A^c = \varnothing$, is open.*

g) *The set $A = \varnothing$ is closed since its complement, $A^c = \mathbb{R}$, is open.*

It is not difficult to show that

- $\operatorname{int} A$ is open, precisely: the largest open set B with $B \subset A$,
- $\operatorname{clos} A$ is closed, precisely: the smallest closed set C with $A \subset C$
- and that ∂A is closed.

In Example 4.9 b), we claimed that the union of two open intervals is open. The following proposition finalizes this observation and more:

Proposition 4.10. Union of open sets and intersection of closed sets

a) *The union of arbitrarily many open sets is open.*

b) *The intersection of arbitrarily many closed sets is closed.*

Proof: **a)** Let I be an index set, A_i be open for all $i \in I$, and let $A = \cup_{i \in I} A_i$. If $x \in A$ then $x \in A_i$ for some $i \in I$. Since A_i is open, there is an $\varepsilon > 0$ with $U_\varepsilon(x) \subset A_i \subset A$. So x is an interior point of A. Since $x \in A$ was arbitrary, A is open.

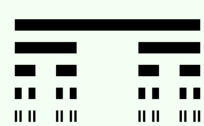
b) Let A_i be closed for all $i \in I$. Then all A_i^c are open and, by a), also $\cup_{i \in I} A_i^c = (\cap_{i \in I} A_i)^c$ is open (De Morgan). Consequently, $\cap_{i \in I} A_i$ is closed. ∎

Attention! Do not confuse!

The intersection of infinitely many open sets need not be open. E.g., $\cap_{i \in \mathbb{N}} (-\frac{1}{i}, \frac{1}{i}) = \{0\}$ *is not open. For finitely many open sets, also the intersection is open. Similarly for the union of closed sets.*

Example 4.11. Cantor set **(Georg Cantor, 1845–1918)**

Starting from $A_1 = [0,1]$, *we construct* A_{n+1} *in a set sequence* $A_1, A_2, \ldots$ *by deleting the open middle third from every interval in* A_n. *The* <u>*Cantor set*</u> *is then the intersection* $A := \cap_{n \in \mathbb{N}} A_n$.



We show that A is closed. This is best seen via the complement:
A_{n+1}^c *arises from* A_n^c *via union with certain open intervals (the open middle thirds of all intervals in* A_n*). Since* $A_1^c = (-\infty, 0) \cup (1, +\infty)$ *is open, it follows, inductively, by Proposition 4.10 a), that every* A_n^c *is open – and so is their union,* $\cup_{n \in \mathbb{N}} A_n^c$. *But the latter is* A^c *(De Morgan). Hence, A is closed.*

4.3 Bounded sets

In analogy to bounded sequences and Definition 2.8, we now continue with sets:

Definition 4.12. bounded sets and their bounds

A set $A \subset \mathbb{R}$ is called

- *bounded from below if there is an $a \in \mathbb{R}$ with $a \leq x$ for all $x \in A$,*
- *bounded from above if there is a $b \in \mathbb{R}$ with $x \leq b$ for all $x \in A$,*
- *bounded if it is bounded from below and above, i.e., if there are numbers $a, b \in \mathbb{R}$ such that $A \subset [a, b]$. In particular, one then has $|x| \leq c$ for all $x \in A$ with $c = \max(|a|, |b|)$,*
- *unbounded if it is not bounded.*

Such numbers a and b are then called lower bound, resp. upper bound, on A.

Due to the completeness of $\mathbb{R}$ (see Nerd box 2.37), every bounded set $A \subset \mathbb{R}$ has a largest lower bounds and a smallest upper bound in $\mathbb{R}$:

Definition 4.13. infimum, supremum, minimum and maximum of sets

Let $A \subset \mathbb{R}$ be a set that is bounded from below or above (or both).

- *The largest lower bound of the set is its infimum.*
- *If the infimum is itself an element $x \in A$, then we also call it the minimum.*
- *The smallest upper bound of the set is its supremum.*
- *If the supremum is itself an element $x \in A$, then we also call it the maximum.*

The corresponding notations are $\inf A$, $\min A$, $\sup A$ and $\max A$.

For unbounded sets A without a lower bound, we set $\inf A := -\infty$, and if it is without upper bound, then put $\sup A := +\infty$.

Example 4.14. $\inf, \sup$, $\min$ **and** $\max$

a) *If $A = (2, 5]$ then $\inf A = 2$ and $\sup A = 5 = \max A$. $\min A$ does not exist.*

b) *The set $B = \{x \in \mathbb{R} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2})$ is bounded. One has $\inf B = -\sqrt{2}$ and $\sup B = \sqrt{2}$. Minimum and maximum do not exist.*

c) *The set $C = \{x \in \mathbb{Q} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ is bounded. It has, however, no largest lower bound and no smallest upper bound in $\mathbb{Q}$. Their existence is a special feature (equivalent to completeness) of $\mathbb{R}$ (Nerd box 2.37).*

4.4 Compact sets

Finite sets always have a minimum and a maximum, and they also have other quantities that make working with them just a bit more convenient. Compact sets behave, in many aspects, like finite sets.

Definition 4.15. compact set

A set $A \subset \mathbb{R}$ is called compact if every sequence from A has a convergent subsequence whose limit is again in A.

Example 4.16. compact sets

- **a)** *$A = \{1, 2, 3\}$, as well as any other finite set, is compact. Indeed: Every sequence from A contains infinitely often the 1 or the 2 or the 3. The corresponding subsequence is constant, hence convergent, and its limit is in A.*
- **b)** *The interval $A = [3, 4]$ is compact. Indeed: Every sequence from A is bounded, hence, by Proposition 2.41 (Bolzano-Weierstraß), it has a convergent subsequence. Since A is closed, the limit of that subsequence is still in A.*
- **c)** *The argument from* **b)** *actually proves a lot more:*
  If $A \subset \mathbb{R}$ is bounded and closed, then A is compact.
- **d)** *The empty set is compact. Who contradicts, has to find a sequence $(x_n)_n$ with $x_n \in \varnothing$ ($\lightning$), that is without a convergent subsequence with limit in $\varnothing$.*
- **e)** *$\mathbb{N}$ is not compact: $(n)_n$ has no convergent subsequence with limit in $\mathbb{N}$.*
- **f)** *$(0, 1]$ is not compact: $(\frac{1}{n})_n$ has no convergent subsequence with limit in $(0, 1]$.*
- **g)** *The Cantor set from Example 4.11 is closed (see there) and bounded (it is contained in $[0, 1]$), whence it is compact, by* **c)**.

Also the reverse direction holds in the implication of Example 4.16 c):

Proposition 4.17. by Heine-Borel

Eduard Heine 1821–1881
Emile Borel 1871–1956

For $A \subset \mathbb{R}$, one has: A is compact $\iff$ A is bounded and closed.

Proof: $\boxed{\Leftarrow}$ holds by Example 4.16 c). $\boxed{\Rightarrow}$: Let A be compact and let $(x_n)_n$ be a convergent sequence from A. By compactness, $(x_n)_n$ has a convergent subsequence with its limit $x \in A$. Consequently, also $\lim x_n = x \in A$, i.e., A is closed. If A were not bounded, there were a sequence $(x_n)_n$ in A with $|x_n| > n$. But this sequence has no bounded, and hence no convergent, subsequence – contradicting the compactness of A. ■

Now compactness of a set $A \subset \mathbb{R}$ is much easier checked than by Definition 4.15.

Nerd box 4.18. Here we go again: completeness of $\mathbb{R}$

The direction $\boxed{\Rightarrow}$ *holds in arbitrary so-called metric spaces. The direction* $\boxed{\Leftarrow}$*, i.e. Example 4.16 c), relies on Proposition 2.41 and hence on Propositions 2.32 and 2.29, bringinng us back to the completeness of* $\mathbb{R}$*. This direction can be extended to* $\mathbb{R}^n$*.*

Our start into this section was that compact sets in $\mathbb{R}$, just like finite sets, should have a minimum and a maximum. Let us go the full circle now:

Proposition 4.19. Compact sets possess a minimum and a maximum

Nonempty compact sets $A \subset \mathbb{R}$ *possess a minimum and a maximum.*

Proof: Let $A \subset \mathbb{R}$ be compact. By Proposition 4.17, A is bounded and closed. By the boundedness, $a := \inf A$ and $b := \sup A$ exist. Since b is the smallest upper bound, one has, for every $n \in \mathbb{N}$, an $x_n \in A$ with $b - \frac{1}{n} < x_n \leq b$. It follows $x_n \to b$ (Sandwich-Lemma) and hence $b \in A$ (closedness of A), i.e. $b = \max A$. Analogously, $a = \min A$. ■

Summary

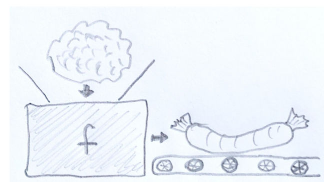
- With respect to a given set $A \subset \mathbb{R}$, the number line $\mathbb{R}$ splits into three disjoint subsets: the interior, the exterior and the boundary of A.
- When we switch from A to A^c (e.g. black $\leftrightarrow$ white for the snowman), then interior and exterior swap places, and the boundary stays the same.
- Put together, interior and boundary make up the closure of A.
- A set is called open if it is equal to its interior, i.e., if it contains none of its boundary points.
- A set is called closed if it is equal to its closure, i.e., if it contains all of its boundary points.
- Sets that contain part but not all of their boundary are neither open nor closed.
- $\varnothing$ and $\mathbb{R}$ are both open and closed at the same time.
- A is open $\iff$ A^c is closed.
- The union of arbitrarily many and the intersection of finitely many open sets is open.
- The intersection of arbitrarily many and the union of finitely many closed sets is closed.
- A set is called bounded if it fits into an interval $[a, b]$ of finite length.
- A set A is called compact if every sequence from A has a convergent subsequence with limit in A.

- Finite sets are compact.
- $A \subset \mathbb{R}$ is compact $\iff$ A is bounded and closed.

Chapter 5

Continuity

With exactly 100 g filling, the sausage reaches the desired length of 13 cm. We may deviate a little from this ideal case:



- Desired outcome: 13 ± 0.5 cm length
- Allowed input: $100 \pm \boxed{?}$ g filling

Continuity of f at $x_0 = 100$ means that $f(x) \approx f(100)$ if $x \approx 100$. More precisely: Any desired approximation quality of the result, $f(x) \approx f(100)$, is achieved if the approximation of the input variable, $x \approx 100$, is sufficiently good.

5.1 Definition: Limit and continuity

In this chapter we analyse functions $f : D \to \mathbb{R}$ with domain $D \subset \mathbb{R}$.

Reminder: $x_0 \in \mathbb{R}$ lies in the closure of D, in short: $x_0 \in \operatorname{clos} D$, if there are points from D in every neighbourhood of x_0, i. e. there is a sequence $x_1, x_2, \ldots \in D$ with $x_n \to x_0$.

So let $f : D \to \mathbb{R}$ be a function with domain $D \subset \mathbb{R}$.

Definition 5.1. limit of a function at a point

$y \in \mathbb{R}$ is the limit of f in $x_0 \in \operatorname{clos} D$, if

$$f(x_n) \to y \text{ for all sequences } (x_n)_n \text{ in } D \text{ with } x_n \to x_0. \tag{5.1}$$

Notation: $y = \lim\limits_{\substack{x \to x_0 \\ x \in D}} f(x)$ *or* $y = \lim\limits_{x \to x_0} f(x)$ *or* $f(x) \to y$ *as* $x \to x_0$

It holds that $\operatorname{clos} D \supset D$. If $x_0 \in D$, the constant sequence $x_n = x_0 \in D$ can also be selected for the approximation $x_n \to x_0$ by points $x_1, x_2, \ldots \in D$.

Proposition 5.2. limit at $x_0 \in D$

If $x_0 \in D$ and $\lim\limits_{x \to x_0} f(x)$ exist, then $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Proof: Assume $y := \lim_{x \to x_0} f(x)$. $f(x_n) \to y$ holds for all sequences $(x_n)_n$ in D with $x_n \to x_0$, especially for the constant sequence $x_n = x_0$. Therefore, it is also true that $f(x_0) \to y$, that is, $y = f(x_0)$. ■

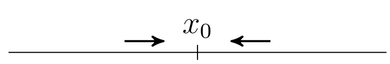
Nerd box 5.3. $\lim$ commutes with f

In the case of $x_0 \in D$ if the limit of f exists in x_0 , then you have

$$x_n \to x_0 \quad \Longrightarrow \quad f(x_n) \to f(x_0) \qquad \textit{resp.} \qquad \lim_{n \to \infty} f(x_n) \;=\; f(\lim_{n \to \infty} x_n).$$

Similar to Proposition 2.19 read this as: $\lim$ *commutes with f (at least nearby x_0).*

To bring a little order into the chaos, <u>all</u> possible approximation sequences $(x_n)_n$ in D to $x_0 \in \mathbb{R}$, we distinguish at least between approximation from the left and from the right:



For each of the two directions, the approximation of x_0 is of course still possible along various sequences $(x_n)_n$. For $x_0 \in \mathbb{R}$ and $D \subset \mathbb{R}$ we write

$$D_{<x_0} \;:=\; D \cap (-\infty, x_0) \qquad \text{and} \qquad D_{>x_0} \;:=\; D \cap (x_0, +\infty).$$

Definition 5.4. one-sided limits

for $f : D \to \mathbb{R}$ and $x_0 \in \operatorname{clos} D$ as well as $y \in \mathbb{R}$ we write

$$y = \lim_{x \nearrow x_0} f(x) \quad \textit{or} \quad y = \lim_{x \to x_0 - 0} f(x) \quad \textit{or} \quad y = f(x_0 - 0),$$

if $x_0 \in \operatorname{clos} D_{<x_0}$ and $f(x_n) \to y$ for all sequences $(x_n)_n$ in $D_{<x_0}$ with $x_n \to x_0$, as well as

$$y = \lim_{x \searrow x_0} f(x) \quad \textit{or} \quad y = \lim_{x \to x_0 + 0} f(x) \quad \textit{or} \quad y = f(x_0 + 0),$$

if $x_0 \in \operatorname{clos} D_{>x_0}$ and $f(x_n) \to y$ for all sequences $(x_n)_n$ in $D_{>x_0}$ with $x_n \to x_0$.
y is then called <u>left-sided limit</u> or <u>right-sided limit</u> of f in x_0.

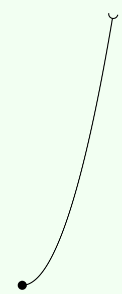
Note that this time x_0 itself is not allowed to be part of the approximating sequence $(x_n)_n$.

Example 5.5. limits of functions in points

a) *Let $f(x) = 5$ for all $x \in D = \mathbb{R}$ (constant function) and $x_0 \in \mathbb{R}$. From $x_n \to x_0$ it follows that $f(x_n) = 5 \to 5$, which means $\lim_{x \to x_0} f(x)$ exists and is equal to 5.*

b) *Let $f(x) = 2x + 1$ for all $x \in D = \mathbb{R}$ and choose an arbitrary $x_0 \in \mathbb{R}$. From $x_n \to x_0$ follows $f(x_n) = 2x_n + 1 \to 2x_0 + 1 = f(x_0)$, which means $\lim_{x \to x_0} f(x)$ exists and is equal to $f(x_0)$.*

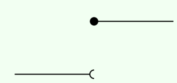
c) *Let $f(x) = x^2$ for all $x \in D = [0, 3) \subset \mathbb{R}$ and let $x_0 = 3 \in \operatorname{clos} D$. We show that the limit of f exists in x_0 and is equal to 9. Either apply Proposition 2.19 c) or recalculate:*



$$|f(x_n) - 9| = |x_n^2 - 9| = |x_n - 3| \cdot |x_n + 3| \leq |x_n - 3| \cdot 6 \to 0$$

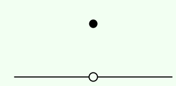
and therefore $f(x_n) \to 9$ as $x_n \to 3$. In $x_0 = 3$ no right-sided limit value can be formed since $D_{>3} = \varnothing$.

d) *Set* $f(x) = \begin{cases} 1\,, & x \geq 0 \\ 0\,, & x < 0 \end{cases}$ *with $D = \mathbb{R}$ and set $x_0 = 0$.*



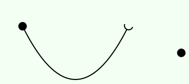
Then the one-sided limit values $\lim_{x \nearrow 0} f(x) = 0$ and $\lim_{x \searrow 0} f(x) = 1$ exist. However, the limit $\lim_{x \to 0} f(x)$ does not exist because, for example the sequences $(x_n)_n = (\frac{1}{n})_n$ and $(x_n‘)_n = (-\frac{1}{n})_n$ both converge to $x_0 = 0$ but $f(x_n) \to 1$, while $f(x_n') \to 0$, i.e. (5.1) does not apply.

e) *Set* $f(x) = \begin{cases} 0\,, & x \neq 0 \\ 1\,, & x = 0 \end{cases}$ *with $D = \mathbb{R}$ and set $x_0 = 0$.*



Then the one-sided limits $\lim_{x \nearrow 0} f(x) = 0$ and $\lim_{x \searrow 0} f(x) = 0$ exist. However, the limit $\lim_{x \to 0} f(x)$ does not exist, since $f(0) = 1$, i.e. the constant sequence $(x_n)_n = (0)_n$ leads to $f(x_n) = 1 \to 1$. (5.1) therefore does not apply.

f) *Set* $f(x) = \begin{cases} x^2, & x \in [-1, 1) \\ \frac{1}{2}\,, & x = 2 \end{cases}$ *with $D = [-1, 1) \cup \{2\}$.*

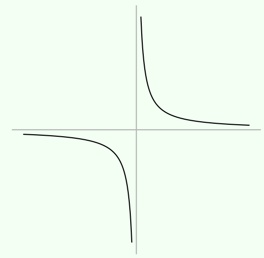


For $x_0 \in [-1, 1)$, the limit $\lim_{x \to x_0} f(x)$ exists and is equal to $f(x_0) = x_0^2$.
At the point $x_0 = 1$, the limit and the left-sided limit exist, but not the right-sided limit (see c).
At the (isolated  *) point $x_0 = 2$ there is neither a left-sided nor a right-sided limit, but the usual limit. Due to the lack of other sequences with $x_n \to 2$, (5.1) is reduced to the constant sequence $(x_n)_n = (2)_n$, which is why $\lim_{x \to 2} f(x) = f(2) = \frac{1}{2}$.*

g) *Let* $f(x) = \frac{1}{x}$ *be for all* $x \in D = \mathbb{R} \setminus \{0\}$.
The limit of f *exists in every* $x_0 \in D$, *because*

$$x_n \to x_0 \quad \Longrightarrow \quad f(x_n) = \frac{1}{x_n} \to \frac{1}{x_0} = f(x_0)$$

with respect to Proposition 2.19 d). In $x_0 = 0 \in \operatorname{clos} D$ *neither the one-sided nor the usual limits exist.*



We extend the statements $f(x) \to y$ as $x \to x_0$ and $\lim_{x \to x_0} f(x) = y$ as well as their one-sided variants $\lim_{x \nearrow x_0} f(x) = y$ and $\lim_{x \searrow x_0} f(x) = y$ to the cases $x_0 = \pm\infty$ and $y = \pm\infty$ by interpreting $x_n \to x_0$ and $f(x_n) \to y$ in the sense of Definition 2.11 .

Example 5.6. infinite limit or limit at infinity

Proposition 2.13 provides:

a) $\frac{1}{x} \to 0$ *as* $x \to +\infty$, *because* $\frac{1}{x_n} \to 0$ *for every sequence* $(x_n)_n$ *with* $x_n \to +\infty$,

b) $\frac{1}{x} \to +\infty$ *as* $x \searrow 0$, *because* $\frac{1}{x_n} \to +\infty$ *for every sequence* $(x_n)_n$ *with* $x_n > 0$ *and* $x_n \to 0$,

c) $\frac{1}{x} \to -\infty$ *as* $x \nearrow 0$, *because* $\frac{1}{x_n} \to -\infty$ *for every sequence* $(x_n)_n$ *with* $x_n < 0$ *and* $x_n \to 0$, $\lim_{x \to 0} \frac{1}{x}$ *does nor exist, because* $\lim_{x \searrow 0} \frac{1}{x} \neq \lim_{x \nearrow 0} \frac{1}{x}$.

Both are also possible at the same time:

d) $\sqrt{x} \to +\infty$ *as* $x \to +\infty$, *because* $\sqrt{x} > M > 0$ *for all* $x > M^2$.

For $f : D \subset \mathbb{R} \to \mathbb{R}$ and points x_0 that can be approximated from both sides, left and right, by points from D, i.e. $x_0 \in \operatorname{clos} D_{<x_0} \cap \operatorname{clos} D_{>x_0}$, the following proposition applies:

Proposition 5.7. limit and one-sided limits

a) *If the limit* $y = \lim\limits_{x \to x_0} f(x)$ *of* f *exists in* x_0, *then the left-sided and right-sided limits also exist and are the same, both equal to* y.

b) *In the case of* $x_0 \notin D$, *the reverse also applies.*

c) *In the case of* $x_0 \in D$, *the inverse applies if, in addition*

$$\lim_{x \nearrow x_0} f(x) \;=\; \lim_{x \searrow x_0} f(x) \;=\; f(x_0).$$

Proof: Part **a)** and **b)** follow directly from Definitions 5.1 and 5.4.

c) Since Definition 5.1 includes the case $x_n = x_0 \in D$ but Definition 5.4 does not, the constant sequence $(x_n)_n = (x_0)_n$, must also be considered here (cf. Example 5.5 e). ∎

Definition 5.8. continuity

Let $x_0 \in \operatorname{clos} D$ and let $\lim\limits_{x\to x_0} f(x) =: y$ be existent.

- *If $x_0 \in D$, then $y = f(x_0)$ and f is continuous in x_0.*
- *If $x_0 \notin D$, then we call f continuously extendable to x_0, and the function*

$$g : D \cup \{x_0\} \to \mathbb{R} \quad \text{with} \quad g(x) = \begin{cases} f(x), & x \in D, \\ y, & x = x_0 \end{cases}$$

is called continuous extension of f to x_0.

If $f : D \to \mathbb{R}$ is continuous at every point $x_0 \in D$ then we call f continuous (on D).

In Example 5.5 we therefore have continuity for a) and b) and a continuous extension for c), while for d) there is only so-called right-sided continuity and for e) not even this:

Definition 5.9. one-sided continuity

Let $x_0 \in D$. We call a function $f : D \to \mathbb{R}$

- *left-continuous at x_0, if $\lim\limits_{x\nearrow x_0} f(x) = f(x_0)$,*
- *right-continuous at x_0, if $\lim\limits_{x\searrow x_0} f(x) = f(x_0)$,*

provided the one-sided limits can be applied.

The functions from Example 5.5 f) and g) are continuous at every point of D and therefore continuous functions – although we cannot draw their graphs in one uninterrupted line.

5.2 ε-δ characterisation and continuity

Definition 5.1 is often impractical for verifying continuity or the existence of $\lim_{x\to x_0} f(x)$ as it is impossible to examine all sequences $(x_n)_n$ with $x_n \to x_0$. Rescue is close:

Proposition 5.10. ε-δ- criterion for continuity

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and $x_0 \in \operatorname{clos} D$.
Then $y = \lim\limits_{x\to x_0} f(x)$ (this is (5.1)) holds if and only if

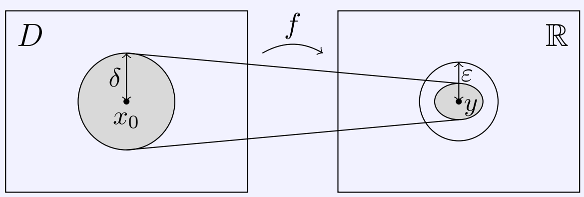
$$\forall \varepsilon > 0\ \exists \delta > 0 : \quad \underbrace{\forall x \in U_\delta(x_0) \cap D : \ f(x) \in U_\varepsilon(y)}_{\textit{Kurz:}\quad f(U_\delta(x_0)\cap D)\ \subset\ U_\varepsilon(y)}. \tag{5.2}$$

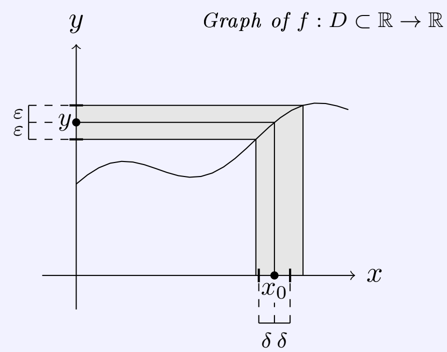
If $x_0 \in D$, then both statements are equivalent to: f is continuous in x_0.

Proof: (5.2) $\Rightarrow$ (5.1): $(x_n)_n$ be a sequence in D with $x_n \to x_0$ and set $\varepsilon > 0$ arbitrary. From (5.2) we know that $\exists \delta > 0 : f(U_\delta(x_0) \cap D) \subset U_\varepsilon(y)$. Because of $x_n \to x_0$, a n_0 exists, so that $\forall n \geq n_0 : x_n \in U_\delta(x_0)$. Because also $x_n \in D$, it follows from (5.2), $\forall n \geq n_0 : f(x_n) \in U_\varepsilon(y)$. Because $\varepsilon > 0$ was chosen arbitrarily, it follows that $f(x_n) \to y$. Furthermore, since the sequence $(x_n)_n$ is arbitrary in D with $x_n \to x_0$, (5.1) follows.

$\neg$(5.2) $\Rightarrow$ $\neg$(5.1): $\exists \varepsilon > 0 \, \forall \delta > 0 \, \exists x \in U_\delta(x_0) \cap D : f(x) \notin U_\varepsilon(y)$. So for $\delta = \frac{1}{1}, \frac{1}{2}, \ldots$ there are $x_1, x_2, \ldots$ with $x_n \in U_{\frac{1}{n}}(x_0) \cap D$ and $f(x_n) \notin U_\varepsilon(y)$. Hence, $x_n \to x_0$ but $f(x_n) \not\to y$. ■

Graphical illustration of the ε-δ criterion





The permissible tolerance ε for the output $f(x)$ can be guaranteed by limiting the input to a suitable δ-environment of x_0.

If $x_0 \in D$, where $y = f(x_0)$, this pretty much brings us back to the sausage machine:

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \quad x \overset{\delta}{\approx} x_0 \quad \Longrightarrow \quad f(x) \overset{\varepsilon}{\approx} f(x_0) \tag{5.3}$$

Often $D = \mathbb{R}$ will apply. Then every $x_0 \in \mathbb{R}$ is an element of D and the question about the existence of $\lim\limits_{x \to x_0} f(x)$ is the question about the continuity of f in x_0, i.e. (5.1)$\Leftrightarrow$(5.2)$\Leftrightarrow$(5.3). Here are two more important characterisations of continuity on $D = \mathbb{R}$:

Proposition 5.11. further characterisations of continuity

The following statements are equivalent for $f : \mathbb{R} \to \mathbb{R}$.

(i) *f is continuous (on all of $\mathbb{R}$).*
(ii) $\forall x_0 \in \mathbb{R} \; \forall$ *sequences* $(x_n)_n$ *in* $\mathbb{R}$ *with* $x_n \to x_0$: $f(x_n) \to f(x_0)$.
(iii) $\forall x_0 \in \mathbb{R} \; \forall \varepsilon > 0 \; \exists \delta > 0 : f(U_\delta(x_0)) \subset U_\varepsilon(f(x_0))$.
(iv) *For every open set $V \subset \mathbb{R}$, the preimage $f^{-1}(V)$ is open.*
(v) *For every closed set $V \subset \mathbb{R}$, the preimage $f^{-1}(V)$ is closed.*

Proof: The equivalence of (i), (ii) and (iii) are already shown.

$(iii) \Rightarrow (iv)$: Let $V \subset \mathbb{R}$ be open and let $U := f^{-1}(V)$. To be shown: U is open. Let $x_0 \in U$ be arbitrary, i.e., $f(x_0) \in V$. V is open $\Rightarrow \exists \varepsilon > 0 : U_\varepsilon(f(x_0)) \subset V$. $(iii) \Rightarrow \exists \delta > 0 :$

$f(U_\delta(x_0)) \subset U_\varepsilon(f(x_0)) \subset V$. $\Rightarrow U_\delta(x_0) \subset f^{-1}(V) = U \Rightarrow x_0 \in \operatorname{int}(U) \;\Rightarrow U$ is open.

$(iv) \Rightarrow (iii)$: Choose $x_0 \in X$ and $\varepsilon > 0$ arbitrarily. Because $V := U_\varepsilon(f(x_0)) \subset Y$ is open, it follows from (iv) that $U := f^{-1}(V)$ is open in X. From $f(x_0) \in V$ we conclude $x_0 \in U$. $\Rightarrow \exists \delta > 0 : U_\delta(x_0) \subset U = f^{-1}(V) \Rightarrow \; f(U_\delta(x_0)) \subset V = U_\varepsilon(f(x_0))$.

$(iv) \Leftrightarrow (v)$: We use that $f^{-1}(V^c) = \left[f^{-1}(V)\right]^c$ and that V is closed $\iff$ V^c is open. ∎

Attention! preimage, not image

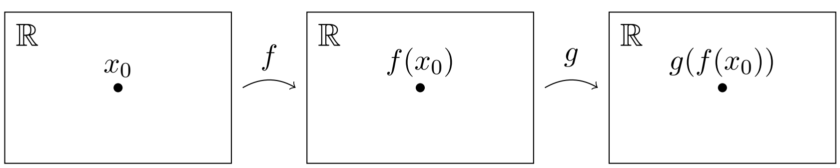
The image of an open set with respect to a continuous function does not have to be open. For the constant function $f(x) = 5$, for example, $f(\,(0,1)\,) = \{5\}$ is not open.

Nerd box 5.12. ε-δ-characterisation for infinite limits

When $x_0 = \pm\infty$, $y = \pm\infty$ or both, the statement $\lim_{x\to x_0} f(x) = y$ (as well as its one-sided variants with $x \nearrow x_0$ and $x \searrow x_0$) again corresponds to our ε-δ-characterisation with the environments of $\pm\infty$ from our Nerd-Box 2.12.

5.3 Combinations of continuous functions

The first kind of combination that we have in mind is the composition of functions.



Proposition 5.13. continous followed by continuous stays continuous

If $f : \mathbb{R} \to \mathbb{R}$ is continuous in x_0 and $g : \mathbb{R} \to \mathbb{R}$ is continuous in $f(x_0)$, then $g \circ f$ is continuous in x_0.

Proof: Let $x_n \to x_0$. f cont. $\Rightarrow$ $f(x_n) \to f(x_0)$. g cont. $\Rightarrow$ $g(f(x_n)) \to g(f(x_0))$. ∎

Because that went so quickly, here is an alternative proof (for continuity on all $\mathbb{R}$):

Proof: $V \subset \mathbb{R}$ be open. g cont. $\Rightarrow g^{-1}(V)$ is open. f cont. $\Rightarrow f^{-1}(g^{-1}(V))$ is open. ∎

Example 5.14. Composition of continuous functions

The functions f and g with $f(x) = 2x + 1$ and $g(x) = \frac{1}{x}$ are continous on their respective domains, $\mathbb{R}$ resp. $\mathbb{R} \setminus \{0\}$, see Example 5.5. Therefore, $f \circ g$ and $g \circ f$, precisely, $(f \circ g)(x) = \frac{2}{x} + 1$ and $(g \circ f)(x) = \frac{1}{2x+1}$, are also continuous where they are defined.

We also think of the combination of two functions $f, g : D \subset \mathbb{R} \to \mathbb{R}$ in terms of the basic arithmetic operations $+$, $-$, $\cdot$ and $:$. So if $\text{\textcircled{op}}$ stands for one of these four operations, $\text{\textcircled{op}} \in \{+, -, \cdot, :\}$, then let the function $f \,\text{\textcircled{op}}\, g : \mathbb{R} \to \mathbb{R}$ be pointwise defined via

$$(f \,\text{\textcircled{op}}\, g)(x) \ := \ f(x) \,\text{\textcircled{op}}\, g(x), \qquad x \in D,$$

where the known arithmetic operations between two real numbers are meant on the right, and, for division, $g(x) \neq 0$ must apply.

Proposition 5.15. **Sum, product & quotient of continuous functions are continuous.**

For continuous $f, g : \mathbb{R} \to \mathbb{R}$ in $x_0 \in \mathbb{R}$, also $f + g$ and $f \cdot g$ are continuous in x_0. If furthermore $g(x_0) \neq 0$, then $\frac{f}{g}$ is defined nearby x_0 and is also continuous in x_0.

Proof: Let $(x_n)_n$ be an arbitrary sequence with $x_n \to x_0$. Then $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$ apply. By the rules for limits of sequences (Proposition 2.19),

$$(f + g)(x_n) = f(x_n) + g(x_n) \to f(x_0) + g(x_0) = (f + g)(x_0) \quad \text{and similarly for } f \cdot g.$$

If every neighbourhood $U_{\frac{1}{n}}(x_0)$ contained a zero z_n of g, then $g(x_0) = \lim g(z_n) = 0$ would apply. $\lightning$ So there is a neighbourhood $U_\varepsilon(x_0)$, where g has no zero. For all $n \geq n_0$, $x_n \in U_\varepsilon(x_0)$ applies, so that $g(x_n) \neq 0$. With Proposition 2.19 it follows that $\left(\frac{f}{g}\right)(x_n) = \frac{f(x_n)}{g(x_n)} \to \frac{f(x_0)}{g(x_0)} = \left(\frac{f}{g}\right)(x_0)$. ∎

Nerd box 5.16. Actually, that was to be expected.

We had already established in Nerd-Box 5.3 that continuity of f is equivalent to f commuting with lim. *Since, according to Proposition 2.19,* lim *also commutes with $+, -, \cdot, :$, it is clear that* lim *commutes with $f \,\text{\textcircled{op}}\, g$ if it commutes with f and g and if $\text{\textcircled{op}} \in \{+, -, \cdot, :\}$*

Example 5.17. Composition of continuous functions via $+, -, \cdot, :$

a) *Like $f : x \mapsto 5$ from Example 5.5 a), $f : x \mapsto c$ is continuous for every $c \in \mathbb{R}$.*

b) *As for $g : x \mapsto 2x+1$ from Example 5.5 b), we see that $g : x \mapsto x$ is continuous.*

c) *So-called polynomials, i.e. sums and products of $f : x \mapsto c$ and $g : x \mapsto x$ from*

a) *and* **b)**, *such as* $p : x \mapsto 3x^2 - 5x + 2$, *are therefore continuous on* $\mathbb{R}$.

d) *Quotients of two polynomials, e.g.* $q : x \mapsto \frac{3x^2-5x+2}{7x^4+x^2-2x+6}$, *are continuous wherever they are defined, i.e., where the polynomial in the denominator is nonzero.*

5.4 Continuity on compact sets

We start with a nice preservation rule:

Proposition 5.18. Continuous functions map compact sets to compact sets

If $f : D \to \mathbb{R}$ *is continuous and* $D \subset \mathbb{R}$ *is compact, then* $f(D) \subset \mathbb{R}$ *is compact.*

Proof: $(y_n)_n$ be any sequence in $f(D)$. Then $\exists x_1, x_2, \ldots \in D$ with $y_n = f(x_n)$, $n \in \mathbb{N}$. D is compact $\Rightarrow$ $\exists$ subsequence $(x_{n_k})_k$ with $x_{n_k} \to x \in D$ as $k \to \infty$. f is continuous $\Rightarrow$ $y_{n_k} = f(x_{n_k}) \to f(x) \in f(D)$. $\Rightarrow$ $(y_n)_n$ has a conv. subsequence with limit in $f(D)$. ■

We mainly talk about compact sets because of the following proposition:

Proposition 5.19. Existence of minimum and maximum

Let $D \subset \mathbb{R}$ *be compact and non-empty. A continuous function* $f : D \to \mathbb{R}$ *attains both its maximum and its minimum in* D.

Proof: By Proposition 5.18, $f(D)$ is compact and, by Proposition 4.19, has a minimum α and a maximum β. Thus, there exist $x_\alpha, x_\beta \in D$ such that $\alpha = f(x_\alpha) \le f(x) \le f(x_\beta) = \beta \;\; \forall x \in D$. ■

A typical case in the above theorem is $D = [a, b]$ with $a, b \in \mathbb{R}$.

Nerd box 5.20. continuous, more continuous, most continuous

Continuous functions on compact sets are actually uniformly continuous; *this means*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x_0 \in \mathbb{R} : \quad f(U_\delta(x_0)) \subset U_\varepsilon(f(x_0)).$$

In contrast to (iii) from Proposition 5.11 (i.e., continuity), a single δ *works for all* x_0. *Even better:* f *is called* Lipschitz continuous *with* $L > 0$, *if* $\delta := \frac{\varepsilon}{L}$ *can be chosen.*

5.5 The intermediate value theorem

Proposition 5.21. Intermediate value theorem (IVT)

Let $f : [a,b] \to \mathbb{R}$ be continuous.

a) *If $f(a)$, $f(b)$ have different signs, there exists an $x \in (a,b)$ such that $f(x) = 0$.*

b) *Every y between $f(a)$ and $f(b)$ is assumed as $y = f(x)$ with $x \in [a,b]$*

The proof is constructive. The described algorithm is called *bisection*.

Proof: **a)** Let $a_1 := a$, $b_1 := b$ and $z_1 := \frac{a_1+b_1}{2}$. If $f(z_1) = 0$, then $x = z_1$. Otherwise: In one of the subintervals $[a_1, z_1]$ and $[z_1, b_1]$, f again has different signs at the endpoints. We call this interval $[a_2, b_2]$ and continue as above, with $[a_2, b_2]$ instead of $[a_1, b_1]$. This results in a sequence of nested intervals $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$.

The sequences $(a_n)_n$ and $(b_n)_n$ are monotonic and bounded, thus they converge by Proposition 2.32. Since $b_n - a_n = \frac{b-a}{2^{n-1}} \to 0$, it follows that $\lim a_n = \lim b_n =: x \in [a,b]$. Due to the continuity of f, we have $f(a_n) \to f(x)$ and $f(b_n) \to f(x)$. Since all $f(a_n)$ have one sign and all $f(b_n)$ the other, it follows from Proposition 2.21 that $f(x) \geq 0$ and $f(x) \leq 0$ simultaneously, so $f(x) = 0$.

b) The function g with $g(x) := f(x) - y$ is continuous, with $g(a)$ and $g(b)$ having different signs. By part **a)**, there exists an $x \in [a,b]$ with $g(x) = 0$. Thus, $y = f(x)$. ■

In what follows we refer to "interval" as any open, closed, or half-open interval – bounded or unbounded. Here is our next preservation rule:

Proposition 5.22. Continuous functions map intervals to intervals

If $D \subset \mathbb{R}$ is an interval and $f : D \to \mathbb{R}$ is continuous then $f(D)$ is also an interval.

Proof: Let $y_1, y_2 \in f(D)$ be arbitrary. If the interval (y_1, y_2) is also in $f(D)$, then $f(D)$ is an interval. Let $x_1, x_2 \in D$ with $y_1 = f(x_1)$ and $y_2 = f(x_2)$, and let $y \in (y_1, y_2)$ be arbitrary. Since D is an interval, $[x_1, x_2] \subset D$ or $[x_2, x_1] \subset D$. By Proposition 5.21 b), there exists an x between x_1 and x_2, thus $x \in D$, with $y = f(x) \in f(D)$. Therefore, $(y_1, y_2) \subset f(D)$ follows. ■

Example 5.23. on the intermediate value theorem

The function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is continuous. For $x \in [-1,2]$, $f(x)$ continuously covers the interval $[f(-1), f(2)] = [1,4]$ – actually even $[0,4]$. For the latter, one must segment $[-1,2]$ into the so-called monotonicity intervals of f: decreasing on $[-1,0]$, then increasing on $[0,2]$, and apply the IVT separately.

Monotonicity is also one of the key concepts for the next section.

5.6 Continuity and the inverse function

For continuous functions, invertibility is closely related to monotonicity. Therefore:

Definition 5.24. Monotonicity of functions

A function $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}$ is called

- *monotonically increasing, if* $\forall x_1, x_2 \in D: \ x_1 < x_2 \ \Rightarrow \ f(x_1) \le f(x_2)$
- *strictly monotonically increasing, if* $\forall x_1, x_2 \in D: \ x_1 < x_2 \ \Rightarrow \ f(x_1) < f(x_2)$,
- *monotonically decreasing, if* $\forall x_1, x_2 \in D: \ x_1 < x_2 \ \Rightarrow \ f(x_1) \ge f(x_2)$,
- *strictly monotonically decreasing, if* $\forall x_1, x_2 \in D: \ x_1 < x_2 \ \Rightarrow \ f(x_1) > f(x_2)$,
- *(strictly) monotonic, if it is (strictly) monotonically increasing or decreasing.*

And here is the announced theorem:

Proposition 5.25. Monotonicity vs. Invertibility

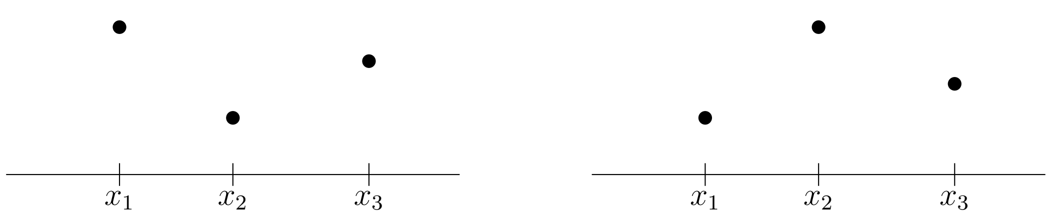
Let $D \subset \mathbb{R}$ be an interval and $f : D \to \mathbb{R}$ be continuous.

a) f *is injective* $\iff$ f *is strictly monotonic.*

b) f *strictly monotonic* $\Longrightarrow$ $f^{-1} : f(D) \to D$ *strictly monotonic and continuous.*

Proof: **a)** $\boxed{\Leftarrow}$ With strict monotonicity, we have: $x_1 \neq x_2 \ \Rightarrow \ f(x_1) \neq f(x_2)$.

$\boxed{\Rightarrow}$ Suppose f is not strictly monotonic. Then $\exists x_1 < x_2 < x_3$ in D such that $f(x_1) \ge f(x_2) \le f(x_3)$ or $f(x_1) \le f(x_2) \ge f(x_3)$.



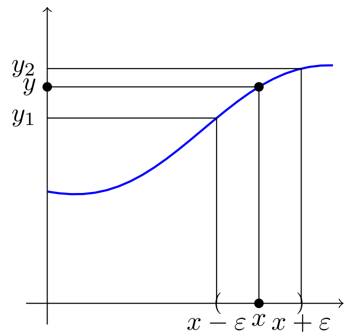
If (as in the figures) $f(x_3)$ lies between $f(x_1)$ and $f(x_2)$, then there is, by continuity and the intermediate value theorem, an $x \in [x_1, x_2] \subset D$ with $f(x) = f(x_3)$.

If $f(x_1)$ lies between $f(x_2)$ and $f(x_3)$, there is an $x \in [x_2, x_3] \subset D$ with $f(x) = f(x_1)$.

In both cases, f is not injective.

b) Let f be strictly monotonically increasing (analogously for decreasing) and let $Y := f(D)$. Due to **a)**, $f : D \to Y$ is even bijective. To be shown: f^{-1} is strictly monotonically increasing. Let $y_1 < y_2 \in Y$, i.e., $y_1 = f(x_1)$, $y_2 = f(x_2)$ with $x_1, x_2 \in D$. If $x_1 \geq x_2$, then $y_1 = f(x_1) \geq f(x_2) = y_2$ ↯. Thus, $x_1 < x_2$, i.e., $f^{-1}(y_1) < f^{-1}(y_2)$.

It remains to show: f^{-1} is continuous: Let $y \in Y$. By Proposition 5.22, $Y = f(D)$ is also an interval. Suppose y is an interior point. Due to strict monotonicity, then $x = f^{-1}(y)$ is also an interior point of the interval D. Let $\varepsilon > 0$ be arbitrary. It suffices to consider the ε with $x \pm \varepsilon \in D$. We set $y_1 := f(x - \varepsilon) < y$ and $y_2 := f(x + \varepsilon) > y$, as well as $\delta := \min(y - y_1, y_2 - y)$.



Then the following holds (again due to the monotonicity of f^{-1})

$$f^{-1}((y - \delta, y + \delta)) \quad \subset \quad f^{-1}((y_1, y_2)) \quad = \quad (x - \varepsilon, x + \varepsilon),$$

hence, f^{-1} is continuous (ε-δ criterion). For boundary points $y \in Y$, the argument is analogous. ■

We will see plenty of application examples in the next chapter.

Summary

- We write $f(x) \to y$ as $x \to x_0$ or $y = \lim_{x \to x_0} f(x)$ if $f(x_n) \to y$ for every sequence $(x_n)_n$ in the domain D of f with $x_n \to x_0$.
- Equivalently:
$$\forall \varepsilon > 0 \; \exists \delta > 0 : \quad x \overset{\delta}{\approx} x_0 \quad \Longrightarrow \quad f(x) \overset{\varepsilon}{\approx} y$$
- If $x_0 \in D$, i.e., if $f(x_0)$ is defined, then $\lim_{x \to x_0} f(x)$ is equal to $f(x_0)$, provided the limit exists. f is then called continuous at x_0.
- f is called continuous in a set $D \subset \mathbb{R}$ if it is continuous at every point $x_0 \in D$.
- This is equivalent to saying that the preimage of every open set is open.
- ...and also that the preimage of every closed set is closed.
- We also examine left-sided and right-sided limits and continuity.
- The combinations of continuous functions using $+, -, \cdot, :$ and $\circ$ are continuous.
- Continuous functions map compact sets onto compact sets.
- If D is compact, then any continuous function $f : D \to \mathbb{R}$ assumes its minimum and maximum in D.

- Continuous functions $f : [a, b] \to \mathbb{R}$ assume every value between $f(a)$ and $f(b)$.
- For continuous functions on intervals, injectivity is the same as strict monotonicity.
- The inverse function f^{-1} of such a function is also continuous.

Chapter 6

Elementary functions

With the available tools that we encountered, we can now explore many old and new functions together with their inverses:

6.1 Monomials and roots

Definition 6.1. Monomial

Let $n \in \mathbb{N}_0$. The function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^n$ is called (the n-th) monomial.

For $n \in \mathbb{N}$ it is easy to verify that

$$\lim_{x \to -\infty} x^n = \begin{cases} -\infty, & n \text{ odd}, \\ +\infty, & n \text{ even}, \end{cases} \quad \text{and} \quad \lim_{x \to +\infty} x^n = +\infty. \tag{6.1}$$

Proposition 6.2. about monomials

Let $n \in \mathbb{N}$ and $f(x) = x^n$ the n-th monomial. Then f is

a) *continuous on $\mathbb{R}$,*

b) *strictly monotonically increasing on the interval $[0, \infty)$, and*

c) *bijective as a restricted mapping $f : [0, \infty) \to [0, \infty)$.*

Proof: **a)** f is the n-fold product of the continuous function $g(x) = x$ with itself.

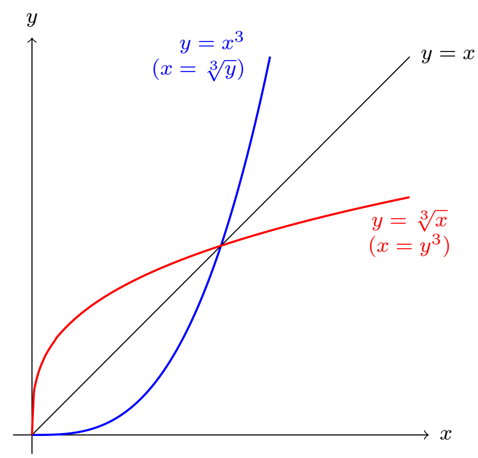
b) Let $x_1, x_2 > 0$ with $x_1 < x_2$. Multiplying this inequality by x_1 and x_2, respectively, yields $x_1^2 < x_1 x_2$ and $x_1 x_2 < x_2^2$, and thus $x_1^2 < x_2^2$. Multiplying the auxiliary inequalities by x_1 or x_2 yields $x_1^3 < x_1^2 x_2$, $x_1^2 x_2 < x_1 x_2^2$ and $x_1 x_2^2 < x_2^3$, therefore $x_1^3 < x_2^3$, etc.

c) Injectivity follows from **a), b)** and Proposition 5.25 a). Surjectivity follows from **a)**, $f(0) = 0$, $\lim_{x\to+\infty} f(x) = +\infty$ and Proposition 5.22. ∎

By Proposition 6.2 c), there exists the inverse function $f^{-1} : [0, +\infty) \to [0, +\infty)$ of the n-th monomial $f(x) = x^n$, i.e.

$$f^{-1} : \; x^n \;\mapsto\; x \quad \text{in short,} \quad f^{-1}(y) \;=\; \sqrt[n]{y},$$

which we encountered earlier in the root test. According to Proposition 5.25 b), the n-th root is a strictly monotonically increasing and continuous function from $[0, +\infty)$ to $[0, +\infty)$. Its graph is the reflection of the graph of the corresponding monomial f across the line $y = x$.



6.2 Polynomials and rational functions

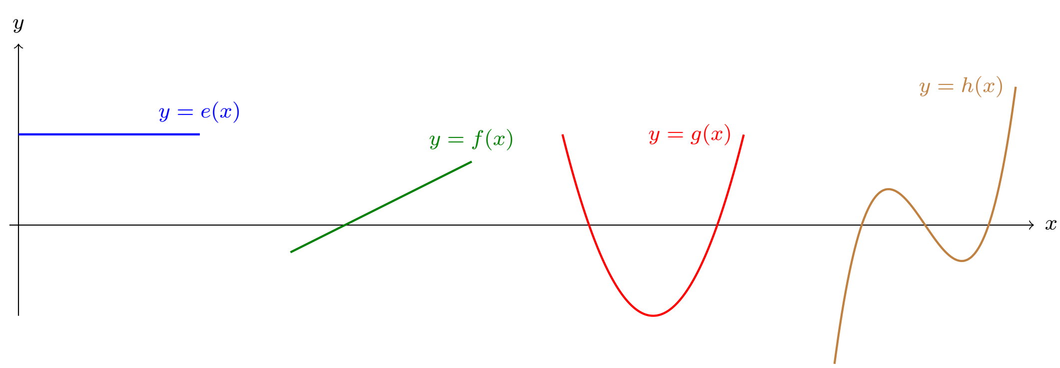
Definition 6.3. Polynomial, Degree

A polynomial is a (finite) linear combination of monomials, i.e., a sum

$$p(x) \;=\; a_n x^n \;+\; a_{n-1}x^{n-1} \;+\; \ldots \;+\; a_1 x^1 \;+\; a_0$$

with fixed coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$, understood as a function $p : \mathbb{R} \to \mathbb{R}$. The highest power of x occurring (above: n, if $a_n \neq 0$) is called the degree of p.

Polynomials are continuous on the entire real line $\mathbb{R}$, as Example 5.17 c) illustrates.



from left to right: Graphs of four polynomials, e, f, g and h, of degree $0, 1, 2$, respectively, 3.

The four polynomials above each have as many zeros as their degrees. They could have fewer zeros (consider $g(x)+5$ or $h(x)+3$); however, the following proposition always holds:

Proposition 6.4. Fundamental Theorem of Algebra (light)

Let p be a polynomial of degree $n \in \mathbb{N}$. Then one has

a) *p has at most n (real) zeros.*

b) *If n is odd, then p has at least one (real) zero.*

Proof: **a)** The actual Fundamental Theorem of Algebra (by C. F. Gauß) states that p has exactly n zeros – in the domain of the *complex* numbers $\mathbb{C}$, which you will soon encounter in Linear Algebra. However, $\mathbb{R} \subset \mathbb{C}$.

b) Let $a_n > 0$ ($a_n < 0$: similar). It is easy to see that $\lim_{x\to\pm\infty} p(x) = \lim_{\pm\infty} a_n x^n = \pm\infty$ by (6.1) since n is odd. By the IVT (Proposition 5.21), $p(c) = 0$ for some $c \in \mathbb{R}$. ∎

For polynomial degrees $n = 1$ and $n = 2$, we know the formulas

$$x_0 \;=\; -\frac{a_0}{a_1} \qquad \text{and} \qquad x_{1,2} \;=\; -\frac{b_1}{2b_2} \pm \sqrt{\left(\frac{b_1}{2b_2}\right)^2 - \frac{b_0}{b_2}}$$

for the polynomial zeros, i.e., the solutions of

$$0 \;=\; a_1 x + a_0 \qquad \text{and} \qquad 0 \;=\; b_2 x^2 + b_1 x + b_0 \;=\; b_2 \cdot \left(x^2 + \frac{b_1}{b_2} x + \frac{b_0}{b_2}\right).$$

For polynomial degrees $n = 3$ and $n = 4$, there are also solution formulas (N. Tartaglia, G. Cardano) using $+, -, \cdot, :$ and roots.

Nerd box 6.5. Birth of complex numbers and then also that of algebra

- *You obtain all the n solutions of the fundamental theorem by also allowing roots of negative numbers (thus using complex numbers).*
- *For $n \geq 5$, there are generally no solution formulas for $p(x) = 0$ using $+, -, \cdot, :$ and roots anymore (N.H. Abel, P. Ruffini)! This theorem and its emergence are considered by many as the birth of Algebra.*

Even without a solution formula, the zeros can be determined one by one. Let us first practice the so-called polynomial division:

Example 6.6. Polynomial division and zeros

Let, for example, $p(x) = 2x^3 - 6x^2 - 18x - 10$. Computation shows that $x_0 = 5$ is a zero of p, meaning $p(5) = 0$. Division by $x - 5$ (so-called polynomial division) works *like written division from school (grade 4):*

$$
\begin{array}{rrrrl}
(2x^3 & -6x^2 & -18x & -10 &) : (x-5) \;=\; 2x^2+4x+2 \\
2x^3 & -10x^2 & & & \\
\hline
 & 4x^2 & -18x & & \\
 & 4x^2 & -20x & & \\
\hline
 & & 2x & -10 & \\
 & & 2x & -10 & \\
\hline
 & & \text{remainder:} & \underline{\underline{0}} &
\end{array}
$$

Nerd box 6.7. It practically is the same

Our division problem $5238 : 27$ *from grade 4 literally corresponds to the polynomial division* $(5x^3 + 2x^2 + 3x + 8) : (2x + 7)$ *at the point* $x = 10$.

The division was exact. In Proposition 6.8 below we will see that this is by $p(5) = 0$. *The result,* $q(x) = 2x^2 + 4x + 2$, *is now of degree 2 (quadratic); this is something we are more familiar with. In this case, the 1st binomial formula helps directly:*

$$q(x) \;=\; 2x^2 + 4x + 2 \;=\; 2 \cdot (x^2 + 2x + 1) \;=\; 2 \cdot (x + 1)^2.$$

Thus,

$$
\begin{aligned}
p(x) \;&=\; (x-5)\cdot q(x) \;=\; (x-5)\cdot 2\cdot(x+1)\cdot(x+1) \\
&=\; 2\cdot(x-5)\cdot(x-(-1))\cdot(x-(-1)).
\end{aligned}
$$

The zreos of p *are thus* $x_0 = 5$, $x_1 = -1$ *und* $x_2 = -1$. *The factor* 2 *is the same that was already in front of the highest monomial,* x^3, *in* p.

Multiplying everything out, we end up, as a check, back at $p(x) = 2x^3 - 6x^2 - 18x - 10$. *In particular,* $2 \cdot 5 \cdot (-1) \cdot (-1) = 10$, *thus* $x_0 \cdot x_1 \cdot x_2 = \pm\frac{a_0}{2}$ *with* $a_0 = -10$ *being the coefficient of the 0-th monomial of* p *(F. Vieta).*
Assuming the exercise creators only use integer zeros, the initially guessed zero $x_0 = 5$ *wasn't such a spectacular guess after all.*

Proposition 6.8. Factorization into linear factors

A polynomial p *has a number* $x_0 \in \mathbb{R}$ *as a zero if and only if it contains the factor* $x - x_0$, *i.e.,* $p(x) = (x - x_0) \cdot q(x)$ *for another polynomial* q.

Proof: If you divide $p(x)$ by, e.g., $x - 5$, a constant c will be the remainder, i.e., $p(x) = (x - 5) \cdot q(x) + c$ with a polynomial q and a $c \in \mathbb{R}$. But then $p(5) = 0 \cdot q(5) + c$ and, hence,

$c = p(5)$: Thus, the division is exact if and only if 5 is a zero of p. The general proof is analogous, with x_0 instead of 5. ■

Hence: divide out the linear factor

Once a zero, x_0, of p is found, $p(x) : (x - x_0)$ is a polynomial of degree one lower. Its zeros are the remaining zeros of p.

This approach provides another proof by induction for Proposition 6.4 a).

Definition 6.9. rational function

If p and q are polynomials, the function

$$f = \frac{p}{q}, \qquad \textit{meaning} \qquad f(x) = \frac{p(x)}{q(x)},$$

is called a rational function. The domain of f is $D = \mathbb{R} \setminus N$, where N is the set of zeros of q.

By Example 5.17 d), rational functions are continuous on their domain. The gaps x_0 in the domain of $\frac{p}{q}$, where $q(x_0) = 0$ but $p(x_0) \neq 0$, are so-called poles. Precisely:

Definition 6.10. pole

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and $x_0 \in \operatorname{clos}(D) \setminus D$. The point x_0 is called a pole of f, if

$$\lim_{x \nearrow x_0} f(x) = -\infty \textit{ or } \lim_{x \nearrow x_0} f(x) = +\infty \textit{ or } \lim_{x \searrow x_0} f(x) = -\infty \textit{ or } \lim_{x \searrow x_0} f(x) = +\infty.$$

Example 6.11. Pole or not?

The function $f : x \mapsto \frac{x-3}{x^2-9} = \frac{x-3}{(x-3)(x+3)}$ has a pole at $x_0 = -3$, since even both

$$\lim_{x \nearrow -3} f(x) = -\infty \qquad \textit{and} \qquad \lim_{x \searrow -3} f(x) = +\infty.$$

The gap in the domain at $x_0 = 3$ is however not a pole since $\lim_{x\to 3} f(x) = \frac{1}{6}$.

6.3 Power series

In sloppy terms: power series are polynomials of degree ∞. More precisely:

Definition 6.12. power series

For a sequence $(a_n)_n$ in $\mathbb{R}$ and $x \in \mathbb{R}$, a series of the form

$$\sum_{n=0}^{\infty} a_n\, x^n \;=\; a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \dots$$

is called a power series. The set of all $x \in \mathbb{R}$, for which the above series converges, is its region of convergence.

Example 6.13. power series

a) *geometric series, i.e., $a_n \equiv 1$: $\sum_{n=0}^{\infty} 1 \cdot x^n$ converges for $x \in (-1, 1)$.*

b) *Exponential series, $a_n = \frac{1}{n!}$: $\sum_{n=0}^{\infty} \frac{1}{n!}\, x^n$ converges for all $x \in \mathbb{R}$,*

since $x_n := a_n x^n$ implies $|\frac{x_{n+1}}{x_n}| = \frac{|x^{n+1}|\, n!}{(n+1)!\, |x^n|} = \frac{|x|}{n+1} \to 0$ for all $x \in \mathbb{R}$.

c) *$a_n = n!$: $\sum_{n=0}^{\infty} n!\, x^n$ only converges when $x = 0$, because for $x_n := a_n x^n$, it holds that $|\frac{x_{n+1}}{x_n}| = \frac{(n+1)!|x^{n+1}|}{n!|x^n|} = (n+1)|x| \to \infty \geq 1$, as $x \neq 0$.*

The following holds for the region of convergence of a power series:

Lemma 6.14. power series and absolute convergence

If the power series $\sum_{n=0}^{\infty} a_n\, x^n$ converges at a point $x_0 \neq 0$, then it converges absolutely for every $x \in \mathbb{R}$ with $|x| < |x_0|$.

Proof: $\sum_{n=0}^{\infty} a_n x_0^n$ converges $\Longrightarrow$ $(a_n x_0^n)_n$ is a null sequence $\Longrightarrow$ $(a_n x_0^n)_n$ is bounded.
Let $M := \sup_{n \in \mathbb{N}_0} |a_n x_0^n|$ and let $q = |\frac{x}{x_0}| < 1$ for fixed x with $|x| < |x_0|$. Because

$$|a_n\, x^n| \;=\; |a_n\, x_0^n| \cdot |\frac{x}{x_0}|^n \;\leq\; M \cdot q^n,$$

$M \cdot \sum_{n=0}^{\infty} q^n$ is a convergent majorant for $\sum_{n=0}^{\infty} |a_n x^n|$. $\Rightarrow$ $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. ∎

Let us therefore seek to maximize $|x|$ with x in the region of convergence:

Definition 6.15. radius of convergence

The number $R \in \mathbb{R} \cup \{+\infty\}$ *with*

$$R = \sup\left\{ |x| \;:\; x \in \mathbb{R}, \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}$$

is called the radius of convergence *of the power series* $\sum\limits_{n=0}^{\infty} a_n x^n$.

It holds that

Proposition 6.16. location and convergence

The power series $\sum\limits_{n=0}^{\infty} a_n\, x^n$ *with radius of convergence* R *is*

a) *absolutely convergent for all* $x \in (-R, R)$ *and*

b) *divergent for all* $x \notin [-R, R]$.

No statement can be made for $x \in \{-R, R\}$.

Proof: **a)** Let $x \in (-R, R)$ and let $|x_0|$ be such that $|x| < |x_0| < R$ and $\sum_n a_n x_0^n$ converges (↗ Definition 6.15). Then $\sum_n a_n x^n$ is absolutely convergent by Lemma 6.14.

b) If $\sum_n a_n x^n$ converges for an $x \in \mathbb{R}$ with $|x| > R$, this would contradict Def. 6.15. ■

Thus, the region of convergence D is always an interval of the form $(-R, R)$, $(-R, R]$, $[-R, R)$, or $[-R, R]$ with $R \geq 0$, including the case $D = \mathbb{R}$ for $R = +\infty$.

- In the interior, $\operatorname{int} D = (-R, R)$, there is even absolute convergence,
- in the exterior, $\operatorname{ext} D = [-R, R]^c$, there is divergence,
- and on the boundary, $\partial D = \{-R, R\}$, anything can happen (↗ Example 6.18).

The ratio test and the root test provide formulas for the radius of convergence R:

Proposition 6.17. formulas for the radius of convergence

For the convergence radius R *of the power series* $\sum_{n=0}^{\infty} a_n\, x^n$, *the following applies:*

a) *If* $q := \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ *exists, then* $R = \frac{1}{q}$.

b) *If* $L := \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ *exists, then* $R = \frac{1}{L}$

c) *In any case,* $\ell := \limsup \sqrt[n]{|a_n|}$ *exists (see Nerd-Box 2.44 for* $\limsup$*), and* $R = \frac{1}{\ell}$ *(the so-called* Cauchy-Hadamard formula*) holds.*

Here we interpret $\frac{1}{0}$ as $+\infty$ and $\frac{1}{+\infty}$ as 0.

Proof: **a)** According to the ratio test, Proposition 3.15, convergence occurs if

$$1 \;>\; \lim_{n\to\infty}\left|\frac{a_{n+1}\,x^{n+1}}{a_n\,x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\cdot|x| = q\cdot|x|, \quad \text{d.h.,i.e.,} \quad |x| < \frac{1}{q},$$

and divergence occurs if $1 < \lim\cdots = q\cdot|x|$, so $|x| > \frac{1}{q}$. Thus (by Definition 6.15 or Proposition 6.16), the radius of convergence R is equal to $\frac{1}{q}$.

c) According to the root test (in the version in Nerd-Box 3.18), there is absolute convergence of $\sum_n a_n x^n$ if $1 > \limsup \sqrt[n]{|a_n x^n|} = \limsup \sqrt[n]{|a_n|}\cdot|x| = \ell\cdot|x|$, i.e., $|x| < \frac{1}{\ell}$, and divergence if $1 < \limsup \sqrt[n]{|a_n x^n|} = \ell\cdot|x|$, i.e., $|x| > \frac{1}{\ell}$. Hence, $R = \frac{1}{\ell}$.

b) If $L = \lim \sqrt[n]{|a_n|}$ exists, then $L = \ell$ and according to **c)**, $R = \frac{1}{\ell} = \frac{1}{L}$. ■

Example 6.18. possible behavior on the boundary of the convergence region

a) *The geometric series $\sum_n x^n$ has a region of convergence equal to $(-1,1)$, i.e., $R = 1$ according to Example 6.13a) or $L = \lim \sqrt[n]{1} = 1$ or $q = \lim \frac{1}{1} = 1$. For neither boundary point, $x = \pm 1$, does convergence occur, as then $x^n \not\to 0$.*

b) *Let $a_n = \frac{1}{n}$, i.e., the series is $\sum_{n=1}^{\infty}\frac{x^n}{n}$. The boundary points are again $x = \pm 1$:*

$$\left.\begin{array}{rcl} x=-1: & \text{Convergence (Leibniz)} & \Longrightarrow\; R\ge 1, \\ x=1: & \text{Divergence (harmonic series)} & \Longrightarrow\; R\le 1, \end{array}\right\} \Longrightarrow R = 1.$$

The latter is confirmed by $q = \lim \frac{1/(n+1)}{1/n} = 1$ and $L = \lim \sqrt[n]{\frac{1}{n}} = 1$.

c) *Lastly, let $a_n = \frac{1}{n^2}$, i.e., the series is $\sum_{n=1}^{\infty}\frac{x^n}{n^2}$. Again, $R = 1$, because $q = \lim \frac{1/(n+1)^2}{1/n^2} = 1$ and $L = \lim \sqrt[n]{\frac{1}{n^2}} = 1$. This time, there is convergence (even absolute) for both boundary points, $x = \pm 1$: $\sum_{n=1}^{\infty}\left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{|x|^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$.*

Nerd box 6.19. Disc of convergence?

The concept of power series also comes to full bloom when one admits complex instead of real numbers, x. The above propositions then continue to hold – only that $|x| < R$ describes a disc around the origin and not just the interval $(-R,R)\subset\mathbb{R}$. The region of convergence is then also called the <u>disc of convergence</u>*. The term* <u>radius of convergence</u> *fits even better.*

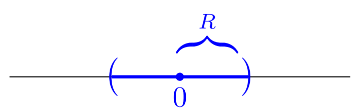
Hence, within the region of convergence $D \subset \mathbb{R}$ of the series,

$$f(x) \;:=\; \sum_{n=0}^{\infty} a_n\, x^n$$

is computable and can be considered as a function $f : D \to \mathbb{R}$.

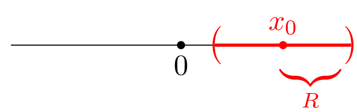
In fact, we have so far only considered particular power series:

$$\sum_{n=0}^{\infty} a_n\, x^n \quad \Longrightarrow \quad \begin{array}{c}\text{region of convergence}\\ U_R(0) = (-R, R)\\ \text{(possibly plus boundary points)}\end{array}$$



Shifting by $x_0 \in \mathbb{R}$, i.e. switching from x to $y := x - x_0$, results in

$$\sum_{n=0}^{\infty} a_n(\underbrace{x - x_0}_{y})^n \quad \Longrightarrow \quad \begin{array}{c}\text{region of convergence}\\ U_R(x_0) = (x_0 - R, x_0 + R)\\ \text{(possibly plus boundary points)}\end{array}$$



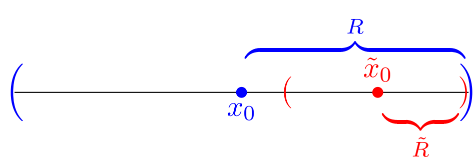
Example 6.20. shifted geometric series

$$\sum_{n=0}^{\infty}(x-2)^n \;=\; \frac{1}{1-(x-2)} \;=\; \frac{1}{3-x}, \quad \textit{where } |x-2| < 1, \textit{ i.e., } x \in U_1(2) = (1,3).$$

Definition 6.21. center of a power series

A series of the form $\sum\limits_{n=0}^{\infty} a_n\,(x - x_0)^n$ *is referred to as a* <u>*power series centered at*</u> x_0*, and* x_0 *is called the* <u>*center*</u> *(or the* <u>*point of expansion*</u>*) of that power series.*

Within the convergence interval $(x_0 - R, x_0 + R)$, one may also choose any new center, $\tilde{x}_0$. When <u>*moving the center*</u>, new coefficients, $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \ldots,$ and a new convergence radius $\tilde{R}$ arise:



Example 6.22. moving the center of a geometric series

$$\sum_{n=0}^{\infty} \frac{1}{3^n} x^n = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{1}{1-\frac{x}{3}} = \frac{3}{3-x} = \frac{3}{1-(x-2)} = \sum_{n=0}^{\infty} 3(x-2)^n,$$

where first $|\frac{x}{3}| < 1$, *i.e.*, $x \in (-3,3)$ *and later* $|x-2| < 1$, *i.e.*, $x \in (1,3)$.

Proposition 6.23. continuity of a power series

A power series $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ *with radius of convergence* R *is continuous on the interval* $U_R(x_0) = (x_0 - R, x_0 + R)$.

Proof: We will see later in Chapter 7 that f is even differentiable on the interval $(x_0 - R, x_0 + R)$, and that continuity follows from it. ■

6.4 Exponential function and logarithm

The concept of power series opens the door to entirely new functions:

Definition 6.24. the exponential function

The function $f : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

is called the <u>exponential function</u> *and is denoted by* exp.

According to Example 3.17 a) and 6.13 b), the convergence domain of exp is indeed all of $\mathbb{R}$. In this context, x^0 and $0!$ are both to be understood as 1. The constant

$$e := \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718281\ldots$$

is the famous <u>*Euler's number*</u>.

Proposition 6.25. exp turns + into ·

For all $x, y \in \mathbb{R}$, *we have:*

a) $\exp(x+y) = (\exp x) \cdot (\exp y)$.

b) $\exp(-x) = \frac{1}{\exp x}$, *in particular*, $\exp x \neq 0$ *for all* $x \in \mathbb{R}$.

Proof: **a)** We have not yet considered products of series. Here we have

$$(\exp x)\cdot(\exp y) \;=\; \sum_{i=0}^{\infty}\frac{x^i}{i!}\cdot\sum_{j=0}^{\infty}\frac{y^j}{j!} \;=\; \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i}{i!}\frac{y^j}{j!}.$$

By enumerating the region $(i,j)\in\{0,1,2,\dots\}^2$ along "diagonals" with $i+j=k$ for $k=0,1,2,\dots$, one arrives at

$$(\exp x)\cdot(\exp y) \;=\; \sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{x^i}{i!}\frac{y^{k-i}}{(k-i)!} \;=\; \sum_{k=0}^{\infty}\frac{1}{k!}\underbrace{\sum_{i=0}^{k}\binom{k}{i}x^i y^{k-i}}_{(x+y)^k} \;=\; \exp(x+y).$$

In almost every step, we use the *absolute* convergence of the exponential series.

b) Because of part **a)**, for all $x\in\mathbb{R}$, we have

$$1 \;=\; \exp(0) \;=\; \exp(x-x) = (\exp x)\cdot(\exp(-x)).$$

■

Proposition 6.26. monotonicity and bijectivity of the function $\exp:\mathbb{R}\to(0,+\infty)$

For the exponential function $\exp:\mathbb{R}\to\mathbb{R}$*, we have:*

a) $\forall x\in\mathbb{R}:\ \exp x>0.$

b) $\lim\limits_{x\to+\infty}\exp x=+\infty.$

c) $\lim\limits_{x\to-\infty}\exp x=0.$

d) $\exp$ *is strictly monotonically increasing.*

e) $\exp$ *is continuous.*

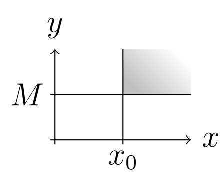
f) $\exp:\mathbb{R}\to(0,+\infty)$ *is bijective.*

Proof: **a)** For $x=0$, $\exp x=1>0$. Now let $x>0$ be arbitrary. Then we have

$$\exp x \;=\; 1+\frac{x}{1!}+\underbrace{\frac{x^2}{2!}+\frac{x^3}{3!}+\dots}_{>0} \;>\; 1+x \qquad (*)$$

and hence, $\exp x>1+x>1>0$. For $x<0$, we have $\exp x=\frac{1}{\exp(-x)}>0$, since $-x>0$.

b) We must show that $\forall M\in\mathbb{R}\ \exists x_0\in\mathbb{R}:\ \forall x>x_0:\ \ \exp x>M.$



Let $M\in\mathbb{R}$. According to $(*)$, we have $\forall x>x_0:=M:\ \exp x>1+x>1+M>M.$

c) For $x \to -\infty$, $\exp(-x) \to +\infty$ and hence $\exp x = \frac{1}{\exp(-x)} \to 0$ by Proposition 2.13.

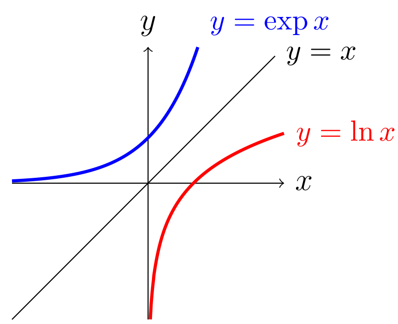
d) For $\delta > 0$, we have $\exp(x+\delta) = (\exp x)(\exp \delta) > (\exp x)(1+\delta) > \exp x$ by $(*)$.

e) Continuity follows directly from Proposition 6.23.

f) Injectivity follows from **d)**, **e)**, and Proposition 5.25 a). For surjectivity: According to Proposition 5.22, $I := \exp(\mathbb{R})$ is an interval. By **a)**, $I \subset (0, +\infty)$. Due to **b)** and **c)**, $\forall M, \varepsilon > 0 : (\varepsilon, M) \subset I$. Thus, $I = (0, +\infty)$, i.e., $\exp(\mathbb{R}) = (0, +\infty)$. ■

Definition 6.27. the natural logarithm

The inverse function $\exp^{-1} : (0, +\infty) \to \mathbb{R}$ *is called the natural logarithm and is denoted by* $\ln$.



According to Proposition 5.25, the natural logarithm, as a function $\ln : (0, +\infty) \to \mathbb{R}$, is strictly monotonically increasing and continuous.

Since exp turns addition into multiplication, it is the other way round with ln:

Corollary 6.28. $\ln$ turns $\cdot$ into $+$

$$\ln(x \cdot y) \;=\; \ln x \;+\; \ln y \qquad \textit{for all} \qquad x, y > 0.$$

Proof: Let $x, y > 0$ be arbitrary. Then we have

$$\exp(\ln(x \cdot y)) \;=\; x \cdot y \;=\; \exp(\ln x) \cdot \exp(\ln y) \;=\; \exp(\ln x + \ln y).$$

Applying $\exp^{-1}$, thus ln, on both sides yields the claim. ■

By repeated application of Corollary 6.28 (or from $\exp(ny) = (\exp y)^n$), the following formula is obtained for all $x > 0$ and $n \in \mathbb{N}$:

$$\ln(x^n) \;=\; n \cdot \ln x. \tag{6.2}$$

Definition 6.29. exponential function $\exp_a$ with any $a > 0$

Let $a > 0$. *We define* $\exp_a : \mathbb{R} \to (0, +\infty)$ *as*

$$\exp_a(x) \;:=\; \exp(x \cdot \ln a).$$

Between us: Actually, we would like to introduce $\exp_a(x)$ as a^x.

If we already knew about potency laws and also knew that $\exp x = e^x$ holds, we could write $a = \exp(\ln a) = e^{\ln a}$ and then conclude that

$$a^x \;=\; (e^{\ln a})^x \;=\; e^{x \cdot \ln a} \;=\; \exp(x \cdot \ln a).$$

But since we don't know any of these yet, we start the story from the other end.

The more general exponential function $\exp_a$ inherits the following property from exp:

Proposition 6.30. $\exp_a$ **turns** $+$ **into** $\cdot$

For all $x, y \in \mathbb{R}$, the following holds: $\exp_a(x + y) = (\exp_a x) \cdot (\exp_a y)$.

Proof: This follows from the corresponding rule for exp:

$$\begin{aligned}\exp_a(x+y) &= \exp((x+y)\ln a) = \exp(x\ln a + y\ln a)\\ &= \exp(x\ln a)\cdot\exp(y\ln a) = (\exp_a x)\cdot(\exp_a y) \;\blacksquare\end{aligned}$$

Proposition 6.31.

Let $a > 0$. The formula $\exp_a(x) = a^x$ holds

a) *for all $x \in \mathbb{N}$,*

b) *even for all $x \in \mathbb{Z}$,*

c) *and even for all $x \in \mathbb{Q}$.*

Proof: **a)** For all $n \in \mathbb{N}$, we have $\exp_a(n) = \exp(n \cdot \ln a) = \exp(\ln a^n) = a^n$.

b) Moreover, $\exp_a(-n) = \exp(-n \cdot \ln a) = \frac{1}{\exp(n \ln a)} = \frac{1}{a^n} = a^{-n}$ for all $n \in \mathbb{N}$ and $\exp_a(0) = \exp(0 \cdot \ln a) = \exp 0 = 1 = a^0$.

c) Now let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then $\frac{m}{n} \in \mathbb{Q}$, and we have

$$\left(\exp_a\left(\frac{m}{n}\right)\right)^n \;=\; \left(\exp\left(\frac{m}{n}\cdot\ln a\right)\right)^n \;=\; \exp\left(n\cdot\frac{m}{n}\cdot\ln a\right) \;=\; \exp_a(m) \;=\; a^m.$$

It remains to apply $\sqrt[n]{\ }$ on both sides. $\blacksquare$

The above proposition motivates the following notation

$$\exp_a(x) \;=:\; a^x \qquad \textit{for all} \qquad x \in \mathbb{R}.$$

In particular, $e^x = \exp(x \cdot \underbrace{\ln e}_{1}) = \exp x$ with Euler's number $e = \exp 1$.

Naturally, this also defines the power function

$$f(x) \ := \ x^b \ = \ \exp_x(b) \ = \ \exp(b \cdot \ln x), \qquad x > 0$$

for any $b \in \mathbb{R}$ as a function $f : (0, +\infty) \to (0, +\infty)$. Now, the statements of Proposition 2.24 regarding the race of the functions $n \mapsto n^a$ and $n \mapsto b^n$ towards $+\infty$ translate from integer to real parameters $a > 0$ and $b > 1$.

Proposition 6.32. potency laws

Let $a, b > 0$ and $x, y \in \mathbb{R}$. Then the potency laws are:

$$a^{x+y} = a^x \cdot a^y, \qquad (a^x)^y = a^{x \cdot y}, \qquad a^x \cdot b^x = (a \cdot b)^x, \qquad \left(\frac{1}{a}\right)^x = a^{-x}.$$

Proof: The first formula is Proposition 6.30. The other formulas are seen as follows:

$$(a^x)^y = \exp\left(y \cdot \ln\left(a^x\right)\right) = \exp(xy \ln a) = a^{xy}$$

$$a^x b^x = \exp(x \ln a) \cdot \exp(x \ln b) = \exp(x \cdot (\ln a + \ln b)) = \exp(x \cdot \ln(ab)) = (ab)^x$$

$$\left(\frac{1}{a}\right)^x = \exp\left(x \cdot \ln \frac{1}{a}\right) \overset{\frac{1}{a}=a^{-1}}{=} \exp(x \cdot (-1) \cdot \ln a) = \exp(-x \cdot \ln a) = a^{-x}$$

■

Proposition 6.33. monotonicity and invertibility of $\exp_a$

Let $a > 0$. For the function $\exp_a : \mathbb{R} \to (0, +\infty)$, the following holds:

- **a)** $\exp_a$ *is continuous.*
- **b)** $\exp_a$ *is* $\begin{cases} \textit{strictly increasing} & \textit{for } a > 1, \\ \textit{strictly decreasing} & \textit{for } 0 < a < 1, \\ \textit{constantly equal to } 1 & \textit{for } a = 1. \end{cases}$
- **c)** $\exp_a(\mathbb{R}) = (0, +\infty)$, *if $a \neq 1$.*
- **d)** $\exp_a : \mathbb{R} \to (0, +\infty)$ *is bijective if $a \neq 1$.*

Proof: **a)** The product and composition of continuous functions are continuous.

b) $\ln a$ is > 0 for $a > 1$, < 0 for $0 < a < 1$, and $= 0$ for $a = 1$. The outer function exp is also strictly increasing.

c) Since $\ln a \neq 0$, $x \mapsto x \cdot \ln a$ is a bijection $\mathbb{R} \to \mathbb{R}$ and $\exp_a(\mathbb{R}) = \exp(\mathbb{R}) = (0, +\infty)$.

d) Injectivity follows from **a)**, **b)**, and Proposition 5.25 a), while surjectivity follows from **c)**.

■

Definition 6.34. logarithm to the base $a > 0$

Let $a > 0$ but $a \neq 1$. The inverse function $\exp_a^{-1} : (0, +\infty) \to \mathbb{R}$ is called the logarithm to the base a and is denoted by $\log_a$.

According to Proposition 5.25, $\log_a$ is strictly increasing and continuous.

Corollary 6.35. Also $\log_a$ turns $\cdot$ into $+$

Let $a > 0$. Then we have

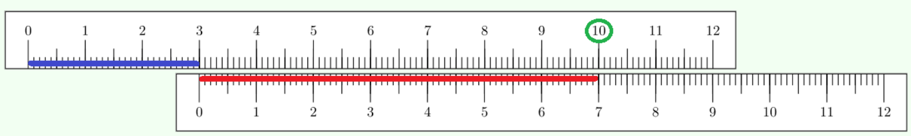
$$\log_a(x \cdot y) \;=\; \log_a x \;+\; \log_a y \qquad \textit{for all} \qquad x, y > 0.$$

Proof: This follows directly from Proposition 6.30 and $\log_a = \exp_a^{-1}$. ∎

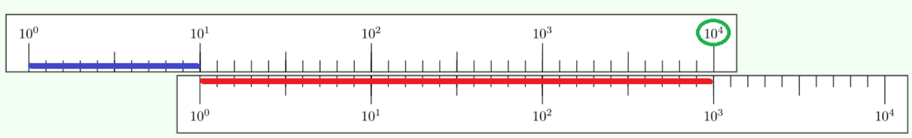
Here is an immediate application:

Example 6.36. Slide rulers: The pocket calculator from 1650 to 1970.

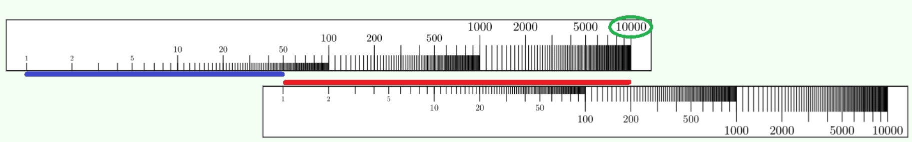
Addition can easily be performed with two rulers. $3 + 7 = 10$*:*



If one changes the scale on both rulers from linear to logarithmic, then the above alignment results in adding the logarithms, and thus multiplying the displayed numbers. In this example, $1 + 3 = 4$ *becomes* $10^1 \cdot 10^3 = 10^4$*:*



On the "proper" slide ruler, the scales are labeled with consecutive values instead of powers of ten, but otherwise, the two rulers remain the same. The rest works as before. In the next example, you can see that $50 \cdot 200 = 10.000$*:*



If one can then, for example, by placing a linear scale beside it, double logarithmic distances or divide tem by five, one accordingly obtains the square or the fifth root of the initial number.

Speaking of pocket calculators: The following Proposition 6.37 explains why it's sufficient to have only one log button on calculators. Up to a constant factor, there is essentially only one logarithm function:

Proposition 6.37. $\log_a x$ is just $c \cdot \ln x$ with the right constant c

Let $a > 0$ but $a \neq 1$. Then, for all $x \in \mathbb{R}$, we have

$$\log_a x \;=\; \frac{\ln x}{\ln a}.$$

Proof: $x = \exp_a(y) = a^y$ holds if and only if $y = \exp_a^{-1}(x) = \log_a x$. It follows that $x = a^y = \exp(y \cdot \ln a)$, hence $\ln x = \ln \exp(y \cdot \ln a) = y \cdot \ln a$, proving the claim. ∎

In short: The rescaling $\frac{1}{\ln a}$ cancels the factor $\ln a$ from Definition 6.29 of $\exp_a(x)$.

6.5 Trigonometric functions and their inverses

Sine and cosine are also defined using power series:

Definition 6.38. sine and cosine

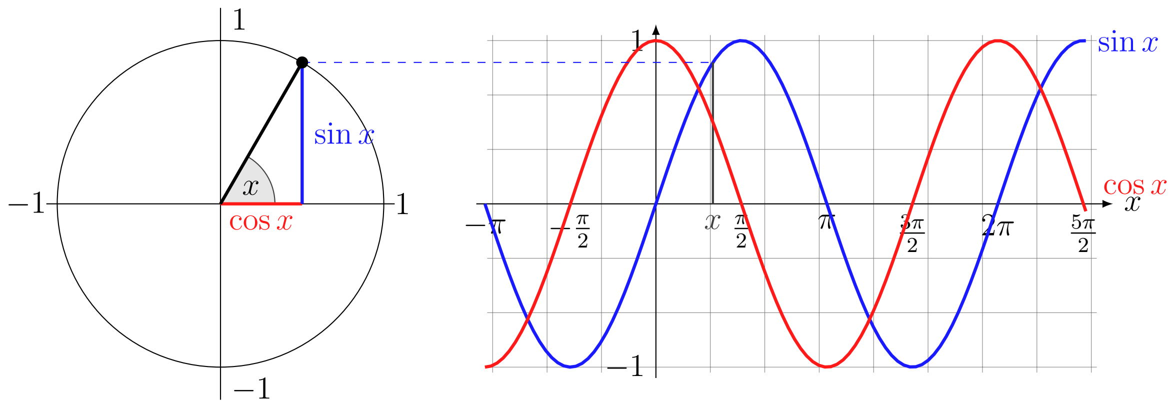
For $x \in \mathbb{R}$, we define

$$\cos x \;:=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \ldots$$

and

$$\sin x \;:=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \;=\; \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - + \ldots$$

As with exp, one can see that in both cases the radius of convergence is indeed $R = +\infty$. Being power series, sin and cos are again continuous functions. These result in the same functions sin and cos, which were formerly defined geometrically:



Nerd box 6.39. Complex numbers would greatly simplify the following

With the complex exponential function $z \mapsto e^z$ and $\mathrm{i}^2 = -1$, we have

$$\cos x = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2}, \qquad \sin x = \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{2\mathrm{i}} \qquad \text{and} \qquad e^{\mathrm{i}x} = \cos x + \mathrm{i} \cdot \sin x,$$

and the following proposition can easily be proven. (Thus, we omit the proof.)

Proposition 6.40. trigonometric identities, symmetry and pythagorean identity

For all $x, y \in \mathbb{R}$, the following formulas hold:

a) $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$,
$\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$,

b) $\cos(-x) = \cos x, \quad \sin(-x) = -\sin x$,

c) $(\cos x)^2 + (\sin x)^2 = 1$.

With $a := x + y$ and $b := x - y$, we get $x = \frac{a+b}{2}$ and $y = \frac{a-b}{2}$ as well as

$$\left.\begin{aligned} \cos a - \cos b &= -2\sin x \sin y = -2\sin\tfrac{a+b}{2}\sin\tfrac{a-b}{2}, \\ \sin a - \sin b &= 2\cos x \sin y = 2\cos\tfrac{a+b}{2}\sin\tfrac{a-b}{2}. \end{aligned}\right\} \tag{6.3}$$

If we can introduce sin and cos without geometry, then we can also introduce π:

Proposition & Definition 6.41. This is how we define π

The function $\cos$ *has exactly one zero in* $[0, 2]$*, let's call it* p.
Now define $\pi := 2p$*, i.e.* $p = \frac{\pi}{2}$*. We then have* $\cos\frac{\pi}{2} = 0$ *and* $\sin\frac{\pi}{2} = 1$.

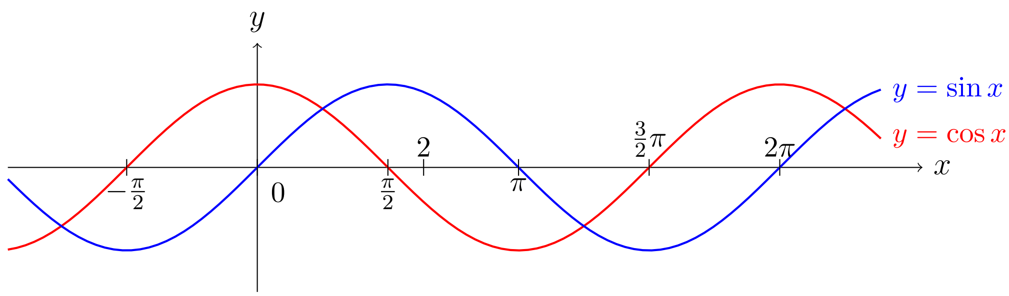
Proof: With the proof technique of the Leibniz rule, one shows $\cos 0 > 0 > \cos 2$. By the intermediate value theorem, cos has at least one zero in $[0, 2]$. With (6.3), one shows that cos is monotonically decreasing on $[0, 2]$ and thus has at most one zero in $[0, 2]$. ∎

Along with the addition theorems for cos and sin, it follows:

Corollary 6.42. **$\sin$ and $\cos$ under shifts by multiples of $\frac{\pi}{2}$**

For all $x \in \mathbb{R}$, the following hold:

$$\begin{array}{rclrclrcl} \cos(x+\frac{\pi}{2}) &=& -\sin x, & \cos(x+\pi) &=& -\cos x, & \cos(x+2\pi) &=& \cos x, \\ \sin(x+\frac{\pi}{2}) &=& \cos x, & \sin(x+\pi) &=& -\sin x, & \sin(x+2\pi) &=& \sin x. \end{array}$$



After a shift by 2π, sin and cos start again from the beginning. The more technical expression is the following: sin and cos are periodic functions with a period of 2π.

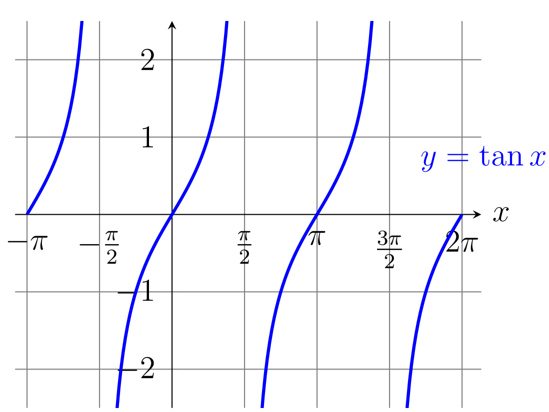
Definition 6.43. **periodic function**

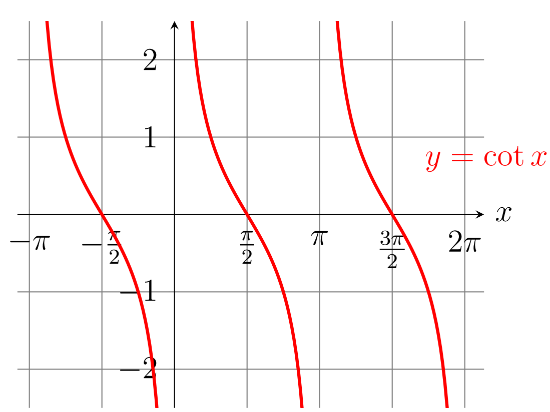
A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic with period $P \in \mathbb{R}$, if for all $x \in \mathbb{R}$ it holds that $f(x+P) = f(x)$.

Whenever possible (i.e. denominator $\neq 0$), we define

$$\tan x \;:=\; \frac{\sin x}{\cos x} \qquad \textit{and} \qquad \cot x \;:=\; \frac{\cos x}{\sin x}, \qquad x \in \mathbb{R}.$$

By definition, tan and cot are also 2π-periodic, indeed only π-periodic.

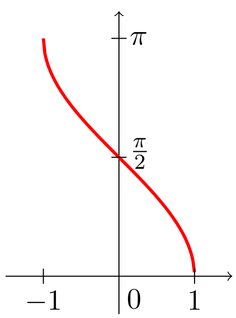




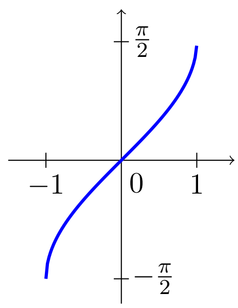
From (6.3) and Corollary 6.42, we obtain:

- cos is strictly decreasing on $[0, \pi] \to [-1, 1]$,
- sin is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$,
- $\tan x = \frac{\sin x}{\cos x}$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

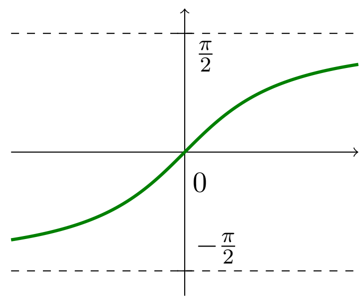
Moreover, the functions with the above interval pairs are surjective. Injectivity again follows from strict monotonicity, continuity, and Proposition 5.25 a). Therefore, bijectivity exists in these intervals, i.e., there are strictly monotonic continuous inverse functions:



$\arccos : [-1, 1] \to [0, \pi]$
arcus cosine



$\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$
arcus sine



$\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$
arcus tangens

With the chosen monotonicity intervals of cos, sin, tan (i.e., the ranges of arccos, arcsin, arctan), one speaks of the so-called *principal branches* of arccos, arcsin, and arctan.

Summary

- Monomials $x \mapsto x^n$ with $n \in \mathbb{N}$ are continuous, strictly increasing on $[0, +\infty)$ and bijective $[0, +\infty) \to [0, +\infty)$.
- The corresponding inverse function $[0, +\infty) \to [0, +\infty)$ with $x \mapsto \sqrt[n]{x}$ is hence also continuous, strictly increasing and bijective.
- Polynomials
  - are linear combinations of monomials.
  - They are continuous on all of $\mathbb{R}$,
  - they have at most as many real zeros as their degree n is.
  - The zeros are easy to find if $n \in \{1, 2\}$,
  - for $n \geq 3$, one first has to guess a zero x_0 and then divide by $(x - x_0)$.
- Power series
  - are of the form $\sum_{n=0}^{\infty} a_n(x - x_0)^n$,
  - they converge absolutely in $U_R(x_0) = (x_0 - R, x_0 + R)$,
  - where they are also continous as a function of x,

  - they may or may not converge at $x_0 \pm R$,
  - they diverge at any other point.
  - We have formulas for the convergence radius R.

- The exponential function $\exp : \mathbb{R} \to (0, +\infty)$
  - is given by the power series, $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$, that converges on all of $\mathbb{R}$,
  - turns $+$ into $\cdot$,
  - is continuous, strictly increasing and bijective,

- The corresponding inverse function $\ln : (0, +\infty) \to \mathbb{R}$
  - is called the natural logarithm,
  - is continuous, strictly increasing and bijective,
  - turns $\cdot$ into $+$,

- Now we can also define $a^b := \exp(b \cdot \ln a)$ for $a > 0$ and $b \in \mathbb{R}$.
- The well-known potency laws hold.
- Also for the logarithm, one can pass to an arbitrary basis $a > 0$ ($a \neq 1$), leading to nothing but a scaled version of the ln function with scaling factor $\frac{1}{\ln a}$.
- sin and cos
  - are also given by a power series with $R = +\infty$ and are hence continuous,
  - are equal to themselves when shifted by 2π
  - and equal to one another when shifted by $\frac{\pi}{2}$,
  - where π is defined as two times the smallest positive zero of the cos function.
  - Their quotients are tan and cot.
  - On appropriate pairs of intervals, all four functions are invertible.

Overview: powers, roots and logarithms

We have $2^5 = 32$. *Often two of the values are given and the third is to be determined:*

operation	*problem*	*solution*	*functions*	
powers	$2^5 = ?$	$? = 2^5 = 32$	$x \mapsto x^5$	$x \mapsto 2^x$
roots	$?^5 = 32$	$? = \sqrt[5]{32} = 2$	$x \mapsto \sqrt[5]{x}$	$x \mapsto \sqrt[x]{32}$
logarithms	$2^? = 32$	$? = \log_2 32 = 5$	$x \mapsto \log_2 x$	$x \mapsto \log_x 32$

Roots are special cases of powers: $\sqrt[5]{x} = x^{\frac{1}{5}}$.
Since $\log_x 32 = \frac{1}{\log_{32} x}$, *the latter is not really a new function either.*
Thus, the three function classes remaining are

$x \mapsto x^a$ *(power function),*

$x \mapsto b^x$ *(exponential function) and*

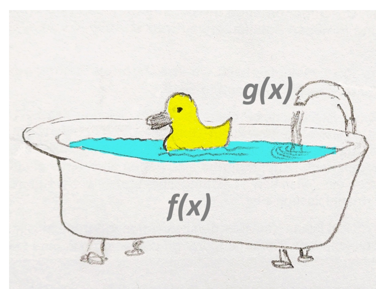
$x \mapsto \log_c x$ *(logarithmic function).*

Chapter 7

Differentiation in one variable

GER vs. ENG – this has a long tradition; not only in football or battles over sun loungers by the hotel pool. Around 1670, Leibniz and Newton independently discovered differential and integral calculus and disputed till death (of the english man) over glory and fame. What is it about? A brief example:

We fill a bathtub: Plug in, turn the tap on, water flows. The water level in the tub (in liters) rises with time x and is denoted by $f(x)$. The flow rate at the tap (in liters per second) is $g(x)$. We might open the tap more or less at times, changing the flow rate $g(x)$ with time x. How are the functions f and g related? As follows:



- The filling level $f(x)$ is the accumulated inflow $g(x)$ over time;
- The flow $g(x)$ is the current rate of change of the fill level $f(x)$.

Such pairs f and g are found everywhere: for example, $f(x)$ could be the distance traveled so far and $g(x)$ the current speed. Or $f(x)$ could be the account balance and $g(x)$ the current inflow and outflow.

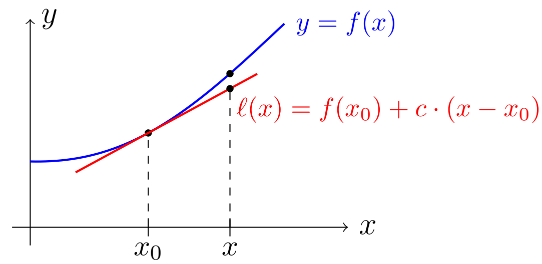
f is always a kind of accumulator for g, and g is the current rate of change of f. We will soon write $g = f'$, say "derivative" instead of "rate of change", and "integral" instead of "accumulator". And the fact that the two processes are essentially inverses of each other will be called the "fundamental theorem of calculus". But let's start from the beginning:

7.1 Differentiation and derivative

Let $D \subset \mathbb{R}$ be an open set and $f : D \to \mathbb{R}$. We fix a point $x_0 \in D$.

Fundamental idea of differentiation:
We approximate f locally (nearby x_0) by an (affine-)linear function:

$$\ell(x) \;=\; f(x_0) + c \cdot (x - x_0)$$



Thus, $\ell(x_0) = f(x_0)$, i.e., at the point x_0 the approximation is exact.
The optimal choice for c (slope of ℓ) is the slope/derivative of f at x_0:

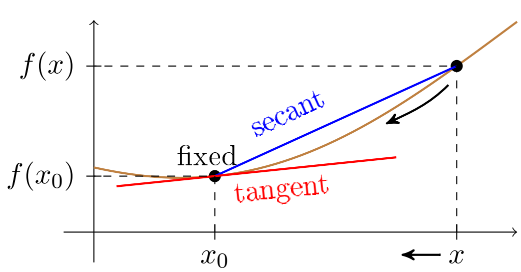
Definition 7.1. Differentiability, derivative

The function $f : D \to \mathbb{R}$ is called differentiable at $x_0 \in D$ if the following limit exists:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad \textit{often shortened as:} \qquad \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}(x_0). \tag{7.1}$$

The limit is then called the derivative of f at x_0. Notations: $f'(x_0)$, $\frac{df}{dx}(x_0)$, $\frac{d}{dx}f(x_0)$.
The function f is called differentiable if it is differentiable at every $x_0 \in D$.

The quotient in (7.1) is called the difference quotient, and it indicates the change in output $f(x)$ relative to the change in input x when $x \approx x_0$. Geometrically, it describes the slope of the so-called secant through the points $(x_0, f(x_0))$ and $(x, f(x))$:



„tangent(x_0) $= \lim_{x \to x_0}$ secant(x_0, x)“

Proposition 7.2. Differentiability reformulated

For $f : D \to \mathbb{R}$, $x_0 \in D$ and $c \in \mathbb{R}$, the following statements are equivalent:

(i) f is differentiable at x_0 and $f'(x_0) = c$.

(ii) The function $\varphi : x \mapsto \frac{f(x) - f(x_0)}{x - x_0}$ is continuously extended to x_0 via $\varphi(x_0) := c$.

(iii) It holds that $f(x) = f(x_0) + c \cdot (x - x_0) + r(x)$, where $\frac{r(x)}{x - x_0} \to 0$ as $x \to x_0$.

Proof: Simple algebra. ∎

Example 7.3. Some elementary functions and their derivatives

a) *Let* $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = ax + b$ *for fixed,* $b \in \mathbb{R}$*. Then for every* $x_0 \in \mathbb{R}$

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{ax + b - ax_0 - b}{x - x_0} = \frac{ax - ax_0}{x - x_0} = a,$$

so that the limit exists for $x \to x_0$ *and is equal to* a*, i.e.,* $f'(x_0) = a$*.*

b) *Let* $f : \mathbb{R} \to \mathbb{R}$ *with* $f(x) = x^2$*. For every* $x_0 \in \mathbb{R}$*, it holds that* $f'(x_0) = 2x_0$*:*

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{(x - x_0)(x + x_0)}{x - x_0} = x + x_0 \to 2x_0 \quad \textit{as} \quad x \to x_0.$$

c) *A power series* $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ *is differentiable at the expansion point* x_0 *and* $f'(x_0) = a_1$ *holds. Initially holds* $f(x) = a_0 + a_1(x - x_0) + r(x)$ *with*

$$r(x) = \sum_{n=2}^{\infty} a_n(x - x_0)^n = (x - x_0)^2 \sum_{n=2}^{\infty} a_n(x - x_0)^{n-2},$$

so that

$$|r(x)| = |x - x_0|^2 \left| \sum_{n=2}^{\infty} a_n(x - x_0)^{n-2} \right| \le |x - x_0|^2 \sum_{n=2}^{\infty} |a_n||x - x_0|^{n-2}$$

and hence

$$\left| \frac{r(x)}{x - x_0} \right| = \frac{|r(x)|}{|x - x_0|} \le |x - x_0|^1 \underbrace{\sum_{n=2}^{\infty} |a_n||x - x_0|^{n-2}}_{<\infty} \to 0 \quad \textit{for} \quad x \to x_0,$$

since the radius of convergence of the power series $\sum_n |a_n|(x - x_0)^n$ *is the same according to Cauchy-Hadamard. It remains to apply (i)* $\Leftrightarrow$ *(iii) from Proposition 7.2 .*

d) *A power series is differentiable at every interior point of its region of convergence* D*, because by moving the center, the new center can be anywhere in* $\operatorname{int} D$ *and then part* **c)** *can be applied.*

e) *Let* $f = \exp$*, i.e.,* $f(x) = e^x$ *and let* $x_0 \in D = \mathbb{R}$ *be arbitrary as well as* $x = x_0 + h$*.*

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{e^{x_0 + h} - e^{x_0}}{h} = e^{x_0} \cdot \underbrace{\frac{e^h - 1}{h}}_{\to 1 \ (\nearrow \text{HÜ})} \stackrel{h \to 0}{\longrightarrow} e^{x_0} =: f'(x_0).$$

f) *Let* $f = \sin$ *and* $x_0 \in D = \mathbb{R}$*. By (6.3) it holds* $f'(x_0) = \cos x_0$*, because*

$$\frac{\sin x - \sin x_0}{x - x_0} = \frac{2 \cos \frac{x + x_0}{2} \sin \frac{x - x_0}{2}}{x - x_0} = \underbrace{\cos}_{\text{continuous}} \underbrace{\frac{x + x_0}{2}}_{\to x_0} \cdot \underbrace{\frac{\sin \frac{x - x_0}{2}}{\frac{x - x_0}{2}}}_{\to 1 \ (\nearrow \text{HÜ})} \stackrel{x \to x_0}{\longrightarrow} \cos x_0.$$

Proposition 7.4. differentiable $\Longrightarrow$ continuous

If $f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$, then f is also continuous at x_0.

Proof: Let f be differentiable at x_0. Then(i) from Proposition **??** holds and thus (iii):

$$f(x) \;=\; f(x_0) + c \cdot (x - x_0) + \frac{r(x)}{x - x_0}(x - x_0) \;\to\; f(x_0) \quad \text{for} \quad x \to x_0. \qquad \blacksquare$$

The converse of Proposition 7.4 does not hold. An example is $f : x \mapsto |x|$ at $x_0 = 0$.

7.2 Rules for differentiation

Proposition 7.5. The derivative commutes with + and − but not with · and :

Let f and g be differentiable in $x_0 \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then $\alpha f, f + g, f - g, f \cdot g$ and in case $g(x_0) \neq 0$ also $\frac{f}{g}$ are differentiable at x_0 and:

a) $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

b) $(\alpha f)'(x_0) = \alpha \cdot f'(x_0)$

c) $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ *(product rule)*

d) $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$ *(quotient rule)*

Proof: Taking the limit $h \to 0$ in each case leads to the assertion:

a)

$$\frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} \;=\; \frac{f(x_0+h) - f(x_0)}{h} \;+\; \frac{g(x_0+h) - g(x_0)}{h}$$

b)

$$\frac{(\alpha f)(x_0+h) - (\alpha f)(x_0)}{h} \;=\; \alpha \cdot \frac{f(x_0+h) - f(x_0)}{h}$$

c)

$$\frac{(f \cdot g)(x_0+h) - (f \cdot g)(x_0)}{h} \;=\; \frac{f(x_0+h) - f(x_0)}{h} g(x_0+h) \;+\; \frac{g(x_0+h) - g(x_0)}{h} f(x_0)$$

d) Let $g(x_0) \neq 0$. Due to the continuity of g (Proposition 7.4), $g(x_0 + h) \neq 0$ holds for sufficiently small $|h|$. For such h it holds

$$\frac{\left(\frac{f}{g}\right)(x_0+h) - \left(\frac{f}{g}\right)(x_0)}{h} \;=\; \frac{\frac{f(x_0+h)-f(x_0)}{h} g(x_0) \;-\; f(x_0)\frac{g(x_0+h)-g(x_0)}{h}}{g(x_0+h)g(x_0)}. \qquad \blacksquare$$

Example 7.6. Derivative of tangent

As shown in Example 7.3 f), analog to $\sin' = \cos$*, we have* $\cos' = -\sin$*. From the quotient rule it follows:*

$$\tan' x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\sin' x \cos x - \sin x \cos' x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

The next rule concerns the derivative of the composition $f \circ g$:

Proposition 7.7. Chain rule

In the situation $D \xrightarrow{g} D' \xrightarrow{f} \mathbb{R}$ *let* g *in* $x_0 \in D$ *and* f *in* $g(x_0) \in D'$ *be differentiable. Then* $f \circ g$ *is differentiable in* x_0 *as well, and the following holds*

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0). \qquad (\underline{\textit{chain rule}})$$

Proof: Roughly speaking, to prove the chain rule, you have to use

$$\frac{f(g(x)) - f(g(x_0))}{x - x_0} = \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \cdot \frac{g(x) - g(x_0)}{x - x_0}, \quad \text{i.e.} \quad \frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x},$$

and to pass to the continuous extension of the quotients, see Proposition 7.2 (ii). ■

Example 7.8. ... on the chain rule

a) *Let* $f(x) = \exp(2x) = \exp(g(x))$ *with* $g(x) = 2x$*.*

$$\Rightarrow \quad f'(x) = \exp'(g(x)) \cdot g'(x) = \exp(2x) \cdot 2 = 2 \cdot f(x).$$

Further $f(x) = \exp(2x) = (\exp x)^2 = h(\exp x)$ *with* $h(y) = y^2$*.*

$$\Rightarrow \quad f'(x) = h'(\exp x) \cdot \exp' x = 2 \cdot \exp x \cdot \exp x = 2 \cdot f(x)$$

b) *Consider* $\ln$ *on* $(0, +\infty)$*. Let* $x_0, x \in (0, +\infty)$ *and* $y_0 := \ln x_0$*,* $y := \ln x$*.*

- *1. Path: Using the definition and a few transformations:*
$$\ln' x_0 = \lim_{x \to x_0} \frac{\ln x - \ln x_0}{x - x_0} = \lim_{y \to y_0} \frac{y - y_0}{e^y - e^{y_0}} = \frac{1}{\lim\limits_{y \to y_0} \frac{e^y - e^{y_0}}{y - y_0}} = \frac{1}{e^{y_0}} = \frac{1}{x_0}$$
- *2. Path:* $\mathrm{id} = \ln \circ \exp$*, i.e., for all* $x \in \mathbb{R}$ $x = \ln(\exp x)$ *counts. Derivative:* $\Longrightarrow \forall x \in \mathbb{R}: \; 1 = \ln'(\exp x) \cdot \exp' x = \ln'(\underbrace{\exp x}_{y}) \cdot \underbrace{\exp x}_{y} \;\Rightarrow\; \ln' y = \frac{1}{y}$
- *3. Path:* $\mathrm{id} = \exp \circ \ln \overset{\textit{Diff.}}{\Rightarrow} 1 = \exp'(\ln x) \cdot \ln' x = \underbrace{\exp(\ln x)}_{x} \cdot \ln' x \;\Rightarrow\; \ln' x = \frac{1}{x}$

Nerd box 7.9. Our first differential equation

$f(x) = e^{2x}$ from example 7.8 a) solves the so-called differential equation $f' = 2f$.

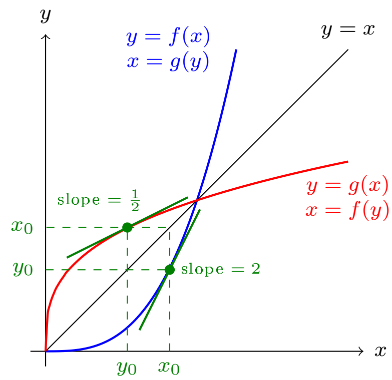
Result and method from Example 7.8 b) are not limited to ln and exp :

Proposition 7.10. Derivative of the inverse function

Let $f : D \to f(D)$ be continuous and strictly monotonic, and $g : f(D) \to D$ be the inverse function of f.

If f is differentiable in $x_0 \in D$ with $f'(x_0) \neq 0$, then g is differentiable in $y_0 = f(x_0) \in f(D)$ and the following holds

$$g'(y_0) \;=\; \frac{1}{f'(x_0)} \;=\; \frac{1}{f'\big(g(y_0)\big)}.$$



Proof: f is injective by Proposition 5.25 and surjective by construction. So the inverse function exists, $g : f(D) \to D$. The differentiability of g in $y_0 = f(x_0)$ can be seen by continuous extension of the quotients

$$\frac{g(y) - g(y_0)}{y - y_0} \;=\; \frac{x - x_0}{f(x) - f(x_0)} \;=\; \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}, \qquad \text{i.e.} \qquad \frac{\Delta g}{\Delta y} = \frac{\Delta x}{\Delta f} = \frac{1}{\frac{\Delta f}{\Delta x}},$$

and the formula for $g'(y_0)$ follows from the above formula or as in Example 7.8 b). ■

Example 7.11. ...on the derivative of the inverse function

a) $g = \arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$, *as inverse function of* $f = \tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.
Example 7.6 $\Rightarrow$ tan *is differentiable on* $(-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $\tan' x = 1 + \tan^2 x$
Proposition 7.10 $\Rightarrow \forall y \in \mathbb{R} :$ arctan *is differentiable in* y *and*

$$\arctan' y = \frac{1}{\tan' x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}$$

b) $g = \arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$, *as inverse function of* $f = \sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$.
$\sin' x = \cos x = \sqrt{1 - \sin^2 x}$ *(↗ Example 7.3f))*
Proposition 7.10 $\Rightarrow$ arcsin *is differentiable with*

$$\arcsin' y \;=\; \frac{1}{\sin' x} \;=\; \frac{1}{\cos x} \;=\; \frac{1}{\sqrt{1 - \sin^2 x}} \;=\; \frac{1}{\sqrt{1 - y^2}}$$

With the help of $\exp'$ and $\ln'$ we are able to derive the derivative of $x \mapsto x^a$ and $x \mapsto a^x$:

Example 7.12. Derivative of power functions and exponential function

a) *Let $a \in \mathbb{R}$ and $f(x) = x^a = \exp(a \cdot \ln x)$ for $x \in D = (0, +\infty)$.*

$$\Rightarrow \; f'(x) = \exp'(a \cdot \ln x) \cdot \frac{d}{dx}(a \cdot \ln x) = \exp(a \cdot \ln x) \cdot a \cdot \frac{1}{x} = x^a \cdot a \cdot \frac{1}{x} = a \cdot x^{a-1}$$

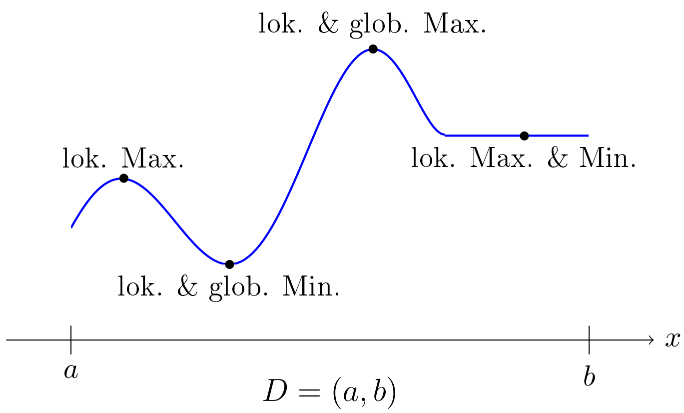
Of course, we have known this formula for a long time. Now finally with proof.

b) *Let $a > 0$ and $f(x) = a^x = \exp(x \cdot \ln a)$ for $x \in D = \mathbb{R}$.*

$$\Rightarrow \; f'(x) \; = \; \exp'(x \cdot \ln a) \cdot \frac{d}{dx}(x \cdot \ln a) \; = \; \exp(x \cdot \ln a) \cdot \ln a \; = \; a^x \cdot \ln a$$

7.3 Local extrema

One of the main applications of differential calculus is to find local extrema (i.e. peaks and valleys in the graph) of f.



Let $D \subset \mathbb{R}$ be an open set and $f : D \to \mathbb{R}$ a function.

Definition 7.13. local extrema

f has a <u>*local minimum*</u> *in $x_0 \in D$ if a neighborhood $U_\varepsilon(x_0) \subset D$ exists with*

$$\forall x \in U_\varepsilon(x_0) : \; f(x_0) \leq f(x).$$

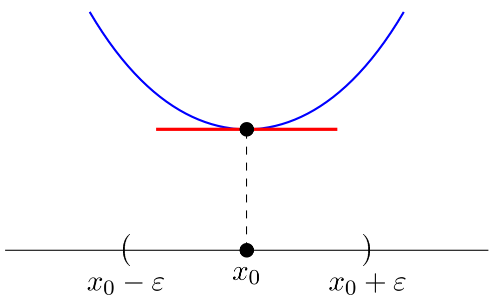
If $f(x_0) \geq f(x)$ in the last term then f is said to have a <u>*local maximum*</u> *at x_0.*
In both cases we say that f has a <u>*local extremum*</u> *in x_0.*

Proposition 7.14. Fermats proposition **(Pierre de Fermat 1607 - 1665)**

If f is differentiable and has a local extremum in $x_0 \in D$, then

$$f'(x_0) = 0.$$

Proof: W.l.o.g. let it be a local minimum (maximum works analogously).
Let $\varepsilon > 0$ be as in Definition 7.13. Then, for all



$$x \in (x_0 - \varepsilon, x_0 + \varepsilon): \;\; f(x) \geq f(x_0).$$

$$\Rightarrow \;\; \forall x \in (x_0 - \varepsilon, x_0): \;\; \frac{\overbrace{f(x) - f(x_0)}^{\geq 0}}{\underbrace{x - x_0}_{<0}} \leq 0$$

$$\text{and } \forall x \in (x_0, x_0 + \varepsilon): \;\; \frac{\overbrace{f(x) - f(x_0)}^{\geq 0}}{\underbrace{x - x_0}_{>0}} \geq 0.$$

Because

$$0 \;\geq\; \lim_{x \nearrow x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\leq 0} \;=\; f'(x_0) \;=\; \lim_{x \searrow x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\geq 0} \;\geq\; 0$$

we are left with $f'(x_0) = 0$. ■

In short:

Necessary criterion for local extrema: For differentiable functions f,

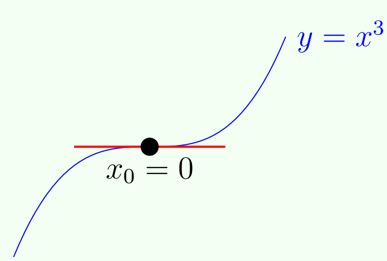
$$f \text{ has a local extremum in } x_0 \quad \Longrightarrow \quad f'(x_0) = 0.$$

Example 7.15. The opposite direction, „$\Leftarrow$“ does not hold.

The function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) \;=\; x^3$$

has the derivative $f'(x_0) = 3x_0^2 = 0$ at $x_0 = 0$,
but it has no local extremum there.



A point $x_0 \in D$ with $f'(x_0) = 0$ is called a <u>*stationary point*</u> or an <u>*extreme value suspect*</u>.

Example 7.16. another third grade polynomial

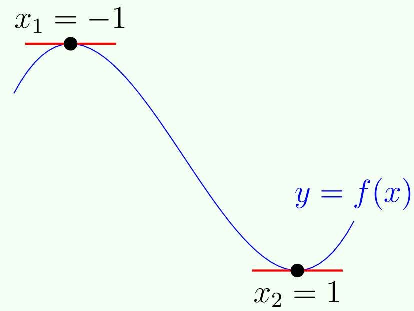
The function $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) \;=\; x^3 - 3x$$

has the following derivative

$$f'(x) \;=\; 3x^2 - 3 \;=\; 3(x-1)(x+1)$$

with roots at $x_1 = -1$ and $x_2 = 1$.
This time we have two local extrema.



Whether there is actually a local extremum at a stationary point (as in Example 7.16) or not (as in Example 7.15), can be found out after further investigations:

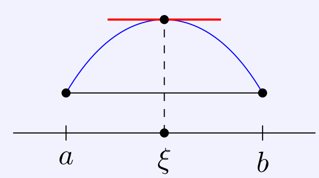
7.4 The mean value theorem and the monotonicity rule

We start with a small auxiliary result:

Proposition 7.17. Rolle's proposition **(Michel Rolle 1652 - 1719)**

Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable in (a,b) . If $f(a) = f(b)$, then there is a $\xi \in (a,b)$ with

$$f'(\xi) \;=\; 0.$$



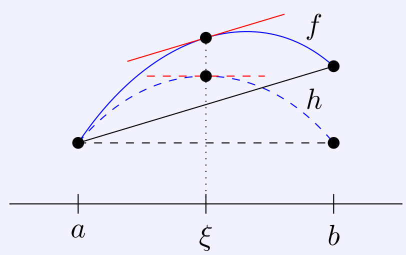
Proof: If f is constant then, for each $\xi \in (a,b) : f'(\xi) = 0$. If not, then f (being continuous) assumes its minimum and maximum on $[a,b]$ (which is compact) by Proposition 5.19. At least one of the two values is different from $f(a) = f(b)$ and therefore lies at a point $\xi \in (a,b)$. So f has a local extremum in ξ, from which, by Proposition 7.14, $f'(\xi) = 0$ follows. ∎

Here is a straightforward consequence:

Proposition 7.18. Mean value theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable in (a, b) . Then there is a $\xi \in (a, b)$ with

$$f'(\xi) \;=\; \frac{f(b) - f(a)}{b - a}.$$



Proof: We „straighten“ the picture: Let

$$h(x) \;:=\; f(x) \;-\; \underbrace{(x-a) \cdot \frac{f(b) - f(a)}{b - a}}_{\text{„slope“}}.$$

Then $h(a) = f(a) = h(b)$, h is continuous on $[a, b]$ and differentiable on (a, b). It remains to apply Proposition 7.17 (Rolle) to h :

$$\Longrightarrow \quad \exists \xi \in (a, b) : \qquad 0 = h'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a}$$

∎

Example 7.19. The mean value theorem and quadratic polynomials

For $f : [a, b] \to \mathbb{R}$ with $f : x \mapsto x^2$, the mean value ξ from the mean value theorem lies exactly at the middle of the interval:

$$\frac{f(b) - f(a)}{b - a} \;=\; \frac{b^2 - a^2}{b - a} \;=\; b + a \;=\; 2 \cdot \frac{a + b}{2} \;=\; f'\left(\frac{a + b}{2}\right)$$

This is therefore the case for all quadratic polynomials. (: And only for those!)

Between us: speed control!

If your vehicle was flashed in town at a speed of $60\,\frac{km}{h}$, then approximately the following has happened: You only took $b - a = 0.6$ seconds for a distance of $f(b) - f(a) = 10$ metres. The difference quotient

$$\frac{f(b) - f(a)}{b - a} \;=\; \frac{10\,m}{0.6\,s} \;\approx\; 16.66\,\frac{m}{s} \;\approx\; 60\,\frac{km}{h} \quad \textit{(minus } 3\,\frac{km}{h} \textit{ tolerance)}$$

is what you receive by post. Strictly speaking, this was not an instantaneous speed. (An $0.6\,s$ long moment?) No one actually caught you with over $50\,\frac{km}{h}$!
However, if the letter quotes the mean value theorem correctly, you have to pay...

A simple conclusion of the mean value theorem is:

Corollary 7.20. monotonicity rule

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Then the following holds

a) $f' > 0$ *on* (a, b) $\Longrightarrow$ f *strictly increasing on* (a, b)

b) $f' < 0$ *on* (a, b) $\Longrightarrow$ f *strictly decreasing on* (a, b)

c) $f' \geq 0$ *on* (a, b) $\Longleftrightarrow$ f *monotonically increasing on* (a, b)

d) $f' \leq 0$ *on* (a, b) $\Longleftrightarrow$ f *monotonically decreasing on* (a, b)

e) $f' = 0$ *on* (a, b) $\Longleftrightarrow$ f *is constant on* (a, b)

If f is continous on $[a, b]$, all the statements on the right even apply to $[a, b]$.

Proof: All „$\Leftarrow$“ implications follow from Definition 7.1.

All „$\Rightarrow$“ implications follow from the mean value theorem, i.e. from

$$f(x_2) - f(x_1) = f'(\xi) \cdot (x_2 - x_1)$$

with $x_1, x_2 \in (a, b)$ resp. $[a, b]$ and with corresponding $\xi \in (x_1, x_2) \subset (a, b)$. ■

Note that in a) and b) „$\Leftarrow$“ does not apply (example $f : x \mapsto \pm x^3$).

The monotonicity criterion now provides an initial decision-making aid for stationary points:

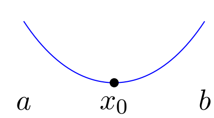
Corollary 7.21. first sufficient criterion for local extrema

Let $f : (a, b) \to \mathbb{R}$ be differentiable and $x_0 \in (a, b)$ a stationary point, i.e., $f'(x_0) = 0$.

a) *If f' changes its sign at x_0 from $-$ to $+$,*
i.e., $f' \leq 0$ in (a, x_0) and $f' \geq 0$ on (x_0, b), then f has a local minimum at x_0.

b) *If f' changes its sign at x_0 from $+$ to $-$,*
i.e., $f' \geq 0$ on (a, x_0) and $f' \leq 0$ on (x_0, b), then f has a local maximum at x_0.

Proof: **a)** decreasing on (a, x_0), increasing on (x_0, b) $\Rightarrow$ local minimum



b) analogously ■

Example 7.22. back to our two polynomials of degree three

a) *Looking at $f : x \mapsto x^3$ from Example 7.15, $x_0 = 0$ is a stationary point, since $f'(x) = 3x^2 = 0$ at $x_0 = 0$. However, since f' has no sign change at x_0, Corollary 7.21 cannot make any statement here.*

b) *Looking at $f : x \mapsto x^3 - 3x$ from Example 7.16 on the other side, $f'(x) = 3x^2 - 3$ changes its sign at $x_1 = -1$ from $+$ to $-$ and at $x_2 = 1$ from $-$ to $+$. Corollary 7.21 shows: local maximum at x_1, local minimum at x_2.*

7.5 De l'Hospitals rule

First, we need a new and improved version of the mean value theorem:

Proposition 7.23. generalized mean value theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous and on (a, b) differentiable. Let furthermore $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists a $\xi \in (a, b)$ with

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Attention!

a) *To prove this, it is not sufficient to apply the mean value theorem to f and g and divide the formulae. The result on the right-hand side would be $\frac{f'(\xi_1)}{g'(\xi_2)}$, where ξ_1 and ξ_2 do not need to be equal.*

b) *The case $g(b) - g(a) = 0$ (denominator of the left-hand side) is impossible, because then there would be (Rolle's proposition) a $\xi \in (a, b)$ with $g'(\xi) = 0$, which has been excluded.*

Proof: We refer back to Rolle's proposition with a similar construction as in the proof of the mean value theorem: Be this time

$$h(x) := f(x) - \big(g(x) - g(a)\big) \cdot \frac{f(b) - f(a)}{g(b) - g(a)}, \qquad x \in (a, b).$$

Then h satisfies the conditions of Rolle's proposition, and thus

$$\exists \xi \in (a, b) : \quad 0 = h'(\xi) = f'(\xi) - g'(\xi) \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \blacksquare$$

Proposition 7.24. de l'Hospitals rule **(Marquis de l'Hospital 1661 - 1704)**

Let $f, g : (a, b) \to \mathbb{R}$ *be differentiable with* $g'(x) \neq 0$ *for all* $x \in (a, b)$.
If $\lim_{x \searrow a} f(x) = 0$ *and* $\lim_{x \searrow a} g(x) = 0$ *as well as* $\lim_{x \searrow a} \frac{f'(x)}{g'(x)}$ *exists, then* $\lim_{x \searrow a} \frac{f(x)}{g(x)}$ *exists and*

$$\lim_{x \searrow a} \frac{f(x)}{g(x)} = \lim_{x \searrow a} \frac{f'(x)}{g'(x)}.$$

Proof: We extend f and g continuously (by 0) to a. Now let $x \in (a, b)$ be arbitrary. Then f and g are continuous on $[a, x]$ and differentiable on (a, x), so that, according to the generalised mean value theorem,

$$\exists \xi \in (a, x) : \quad \frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x) - 0}{g(x) - 0} = \frac{f(x)}{g(x)}.$$

For $x \searrow a$ it follows $\xi \searrow a$ and our claim follows. ■

All versions of de l'Hospital's rule summarised:

The following rule

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

holds in the setting of

- *one-sided limits* $x \searrow a$ *of the form* $\frac{0}{0}$ *(Proposition 7.24) and*
- *one-sided limits* $x \nearrow b$ *of the form* $\frac{0}{0}$ *(proof analogously).*

With a little trick, the rule can be transferred to

- *one-sided limits* $x \searrow a$ *and* $x \nearrow b$ *of the form* $\frac{\infty}{\infty}$,
- *but also to limits* $x \to +\infty$ *and* $x \to -\infty$ *of the form* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$.

We accept all these versions as proven and can now start to apply de l'Hospital's rule in different situations:

Example 7.25. ...on the rule of de l'Hospital

a) $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$ *(of the form* $\frac{0}{0}$*)*

b) $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$ *(of the form* $\frac{0}{0}$*)*

c) $\lim_{x \to +\infty} \frac{\ln x}{x^a} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{a x^{a-1}} = \lim_{x \to +\infty} \frac{1}{a x^a} = 0$ *for* $a > 0$ *(of the form* $\frac{\infty}{\infty}$*)*

d) $\lim\limits_{x\to+\infty} \dfrac{x^n}{e^x} \;=\; \lim\limits_{x\to+\infty} \dfrac{nx^{n-1}}{e^x} \;=\; \ldots \;=\; \lim\limits_{x\to+\infty} \dfrac{n(n-1)\cdots 2\cdot 1}{e^x} \;=\; 0$ *for* $n \in \mathbb{N}$

e) $\lim\limits_{x\to 0}(x\cdot\cot x) \;=\; \lim\limits_{x\to 0} \dfrac{x}{\tan x} \;=\; \lim\limits_{x\to 0} \dfrac{1}{1+\tan^2 x} \;=\; 1$ *(of the form* $0\cdot\infty \sim \frac{0}{0}$*)*

f) $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right) \;=\; \lim\limits_{x\to 0} \dfrac{x-\sin x}{x\sin x} \;=\; \ldots$ *(of the form* $\frac{1}{0} - \frac{1}{0} \sim \frac{0-0}{0\cdot 0}$*)*

7.6 The second derivative

Let $D \subset \mathbb{R}$ be open and $f : D \to \mathbb{R}$ differentiable. Then by $f' : D \to \mathbb{R}$ with $x \mapsto f'(x)$ again a function is given. If this function is as well differentiable, you can consider its derivative $(f')'$ and denote it by f''. This is then again a function $f'' : D \to \mathbb{R}$ and is called the *second derivative* of f.

As long as differentiability holds on, you can continue this process and obtain ever higher derivatives: $f''', f'''', f''''', \ldots$ At the latest from $n \geq 4$ dashes you write $f^{(n)}$.

Definition 7.26. multiple (continuous) differentiability

$f : D \to \mathbb{R}$ is called n times differentiable for some $n \in \mathbb{N}$, if $f^{(n)}$ exists. If $f^{(n)}$ is still continuous, we call f n times continuously differentiable; we write $f \in C^n(D)$.

Example 7.27. an exactly n times differentiable function

For $n \in \mathbb{N}_0$, let $f_n : \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(x) \;=\; x^n \cdot |x| \;=\; \begin{cases} x^{n+1}, & x \geq 0, \\ -x^{n+1}, & x < 0. \end{cases}$$

Then f_n is n times differentiable but not $n+1$ times. This can be seen inductively: f_0 is continuous but not differentiable and $f'_{n+1} = (n+1)f_n$ holds.

For the rest of this section, we restrict ourselves to $n = 2$, i.e., to the 2nd derivative, f''.

Proposition 7.28. second sufficient criterion for local extremas

Let $f : (a,b) \to \mathbb{R}$ be 2-times differentiable and $x_0 \in (a,b)$ be a stationary point.

a) *If $f''(x_0) > 0$, then f has a local minimum in x_0.*

b) *If $f''(x_0) < 0$, then f has a local maximum in x_0.*

Proof: **a)** With $f''(x_0) > 0$, also the difference quotient of f' is positive nearby x_0, let's say in $U_\varepsilon(x_0)$:

$$0 < \frac{f'(x) - f'(x_0)}{x - x_0} = \frac{f'(x) - 0}{x - x_0} = \frac{f'(x)}{x - x_0}, \qquad x \in U_\varepsilon(x_0)$$

Since $x - x_0$ changes its sign at x_0 from $-$ to $+$, $f'(x)$ does the same. It remains to apply Corollary 7.21 a). Part **b)** is proven in the same way. ■

Example 7.29. and again: back to our two polynomials of degree three

a) *Looking back to $f : x \mapsto x^3$ from Example 7.15, $x_0 = 0$ is a stationary point, because $f'(x) = 3x^2 = 0$ at $x_0 = 0$. By $f''(x) = 6x = 0$ in $x_0 = 0$, there is nothing more that Proposition 7.28 could say.*

b) *However, looking back to $f : x \mapsto x^3 - 3x$ from Example 7.16 on the other side, $f''(x) = 6x$ is negative at $x_1 = -1$ and positive at $x_2 = 1$, so that Proposition 7.28 comes to the following conclusion: local maximum at x_1, local minimum at x_2.*

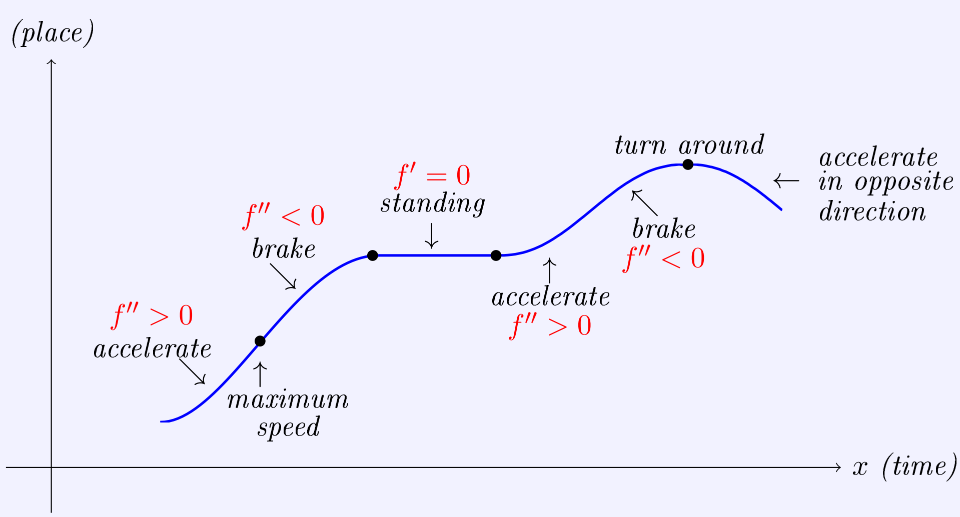
If $f'(x_0) = 0$ and $f''(x_0) = 0$, higher derivatives may have to be considered.

Interaction of f, f' and f'': An object moves along a line

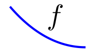
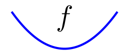
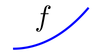
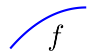
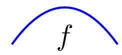
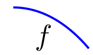
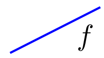
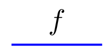
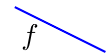
$f(x)$ ... *location at which the object is located at the time x,*

$f'(x)$ ... *speed at time x,*

$f''(x)$ ... *acceleration at time x*



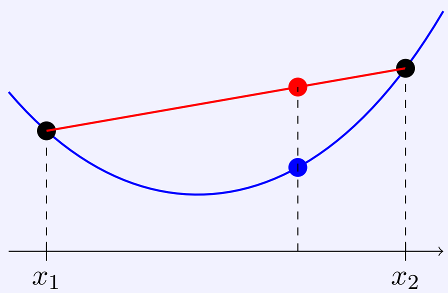
The sign of f'' determines the direction in which the graph of f curves, and the greater the magnitude of f'', the more the graph of f is curved:

- $f'' > 0 \quad \Longrightarrow \quad f'$ increases, which means  or  or 
- $f'' < 0 \quad \Longrightarrow \quad f'$ decreases, which means  or  or 
- $f'' = 0 \quad \Longrightarrow \quad f'$ is constant, which means  or  or 

The two directions of curvature are called convex and concave:

Definition 7.30. convex, concave

Let $D \subset \mathbb{R}$ be an interval. A function $f : D \to \mathbb{R}$ is called convex if for all $x_1, x_2 \in D$ with $x_1 < x_2$ the graph of f on (x_1, x_2) lies below the secant to f through x_1 and x_2 :



In formulas: For all $x_1, x_2 \in D$ with $x_1 < x_2$ the following applies

$$\forall \lambda \in (0,1): \quad f\Big((1-\lambda)x_1 + \lambda x_2\Big) \leq (1-\lambda)f(x_1) + \lambda f(x_2). \tag{7.2}$$

f is called concave if $-f$ is convex, which means if in (7.2) the relation „$\geq$“ applies. f is called strictly convex (strictly concave), if in (7.2) „$<$“ (or „$>$“) holds.

Example 7.31. convex functions

a) *$f : x \mapsto x^2$ is strictly convex on $\mathbb{R}$.*

b) *Every affine-linear function $f : x \mapsto ax + b$, with $a, b \in \mathbb{R}$ fixed, is convex and concave at the same time, but in both not strictly.*

c) *The absolute value function $x \mapsto |x|$ is convex (but not strictly convex) on $\mathbb{R}$.*

d) *The function $f : [0,1] \to \mathbb{R}$ with* $f(x) = \begin{cases} 3\,, & x = 0, \\ 3\,, & x = 1, \\ 2\,, & x \in (0,1), \end{cases}$

is convex (but not strictly convex) – and discontinuous at the boundary points.

As already indicated, convexity can be characterised by the monotonicity of the first or the sign of the second derivative:

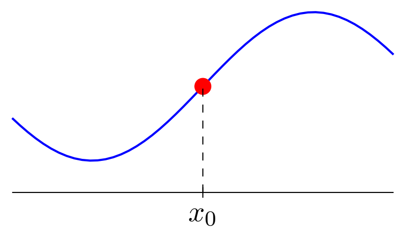
Proposition 7.32. convexity vs. f' and f''

Let $f : [a,b] \to \mathbb{R}$ be continuous. If f is one or two times differentiable on (a,b) then the corresponding parts of the following statements apply (analogously for concave):

$$\begin{array}{rcccl} f \text{ is convex} & \Longleftrightarrow & f' \text{ is monotonically increasing} & \Longleftrightarrow & f'' \geq 0, \\ f \text{ is strictly convex} & \Longleftrightarrow & f' \text{ is strictly increasing} & \Longleftarrow & f'' > 0. \end{array}$$

A point $x_0 \in (a,b)$, in which the behaviour of f changes from convex to concave, or vice versa, is called *inflection point* of f.

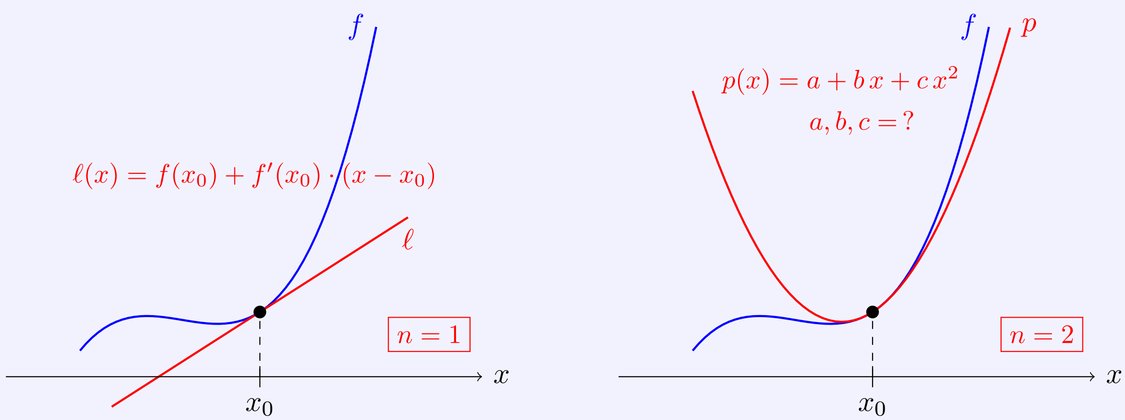
For 2-times continuously differentiable f, this is equivalent to f'' having a change of sign in x_0. It is then necessary that $f''(x_0) = 0$.



7.7 Taylor expansion

Taylor Polynomial. Since the beginning of the previous section, we now have higher derivatives $f^{(n)}$ available, and we ask ourselves the following question:

If one can use $f(x_0)$ and $f'(x_0)$ to approximate the function f in $U_\varepsilon(x_0)$ beautifully in an affine-linear manner, can one perhaps use $f(x_0), f'(x_0), \ldots, f^{(n)}(x_0)$ to approximate f even better by a polynomial of degree n?



Answer: Yes.

Taylor polynomial: problem description

given: $f(x_0),\ f'(x_0), \ldots,\ f^{(n)}(x_0)$

desired: *polynomial of degree n, p, that behaves in x_0 exactly (value, slope, curvature, ...) as f, i.e., it should hold that:*

$$\left.\begin{array}{rcl} p(x_0) & = & f(x_0), \\ p'(x_0) & = & f'(x_0), \\ & \vdots & \\ p^{(n-1)}(x_0) & = & f^{(n-1)}(x_0), \\ p^{(n)}(x_0) & = & f^{(n)}(x_0). \end{array}\right\} \tag{7.3}$$

Approach: Put

$$p(x) \ := \ a_0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n$$

with initially unknown coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

We now compute the left hand sides of (7.3) and then construct $a_0, a_1, \ldots, a_n$ so, that the equations (7.3) fit. ($n+1$ equations for $n+1$ unknowns: sounds fair.)

$$\begin{array}{rcl} p(x) & = & a_0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \ldots + a_n(x-x_0)^n \\ \Rightarrow\ p(x_0) = a_0 & & \\ p'(x) & = & 1\cdot a_1 + 2\cdot a_2(x-x_0)^1 + 3\cdot a_3(x-x_0)^2 + \ldots + n\cdot a_n(x-x_0)^{n-1} \\ \Rightarrow\ p'(x_0) = 1\cdot a_1 & & \\ p''(x) & = & 1\cdot 2\cdot a_2 + 2\cdot 3\cdot a_3(x-x_0)^1 + \ldots + (n-1)\cdot n\cdot a_n(x-x_0)^{n-2} \\ \Rightarrow\ p''(x_0) = 1\cdot 2\cdot a_2 & & \\ & \vdots & \\ p^{(n)}(x) & = & 1\cdot 2\cdot \ldots \cdot n\cdot a_n. \end{array}$$

Solving for a_k, we get the formula for the so-called <u>*Taylor coefficients*</u>:

$$a_k \ = \ \frac{p^{(k)}(x_0)}{k!} \ \overset{(7.3)}{=} \ \frac{f^{(k)}(x_0)}{k!}, \qquad k = 0, \ldots, n \quad (\text{mit } p^{(0)} = p \text{ und } 0! = 1).$$

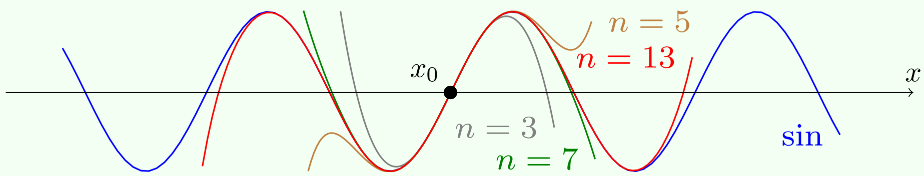
Plugging a_k into the polynomial p above, we get:

Taylor polynomial of order n **(Brook Taylor, 1685 – 1731)**

$$p(x) \ = \ \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\,(x-x_0)^k \tag{7.4}$$

Example 7.33. Taylor polynomial of $f = \sin$ at $x_0 = 0$

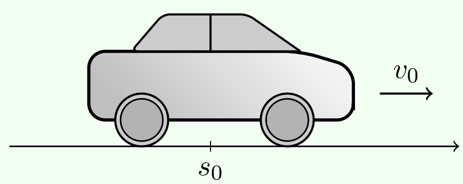
$$p(x) \;=\; \sum_{k=0}^{n} \frac{\sin^{(k)}(0)}{k!} x^k = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - + \ldots (\pm) \frac{x^n}{n!}, \qquad n \in \mathbb{N} \text{ odd}$$



Example 7.34. Second order Taylor polynomial in driving

$f(x)$ … position at time x,
$f'(x)$ … speed at time x,
$f''(x)$ … acceleration at time x.

Let $x_0 = 0$ (for simplicity).



<u>$n = 0$:</u> All I know: Lately, my car was at point $s_0 \in \mathbb{R}$.

$$f(0) = s_0 \qquad \text{(last known location)}$$

$\Longrightarrow$ *Best possible prediction:*

$$p(x) = s_0 \quad \forall x$$

<u>$n = 1$:</u> All I know:

$$\begin{aligned} f(0) &= s_0 &&\text{(last known location)} \\ f'(0) &= v_0 &&\text{(last known speed)} \end{aligned}$$

$\Longrightarrow$ *Prediction:*

$$p(x) = s_0 + v_0 x \quad \Longrightarrow \quad \begin{cases} p(0) = s_0 = f(0) \\ p'(0) = v_0 = f'(0) \end{cases}$$

<u>$n = 2$:</u> I know:

$$\begin{aligned} f(0) &= s_0 &&\text{(last known location)} \\ f'(0) &= v_0 &&\text{(last known speed)} \\ f''(0) &= a_0 &&\text{(last known acceleration)} \end{aligned}$$

$\Longrightarrow$ *Prediction:*

$$p(x) = s_0 + v_0 x + \frac{a_0}{2} x^2 \quad \Longrightarrow \quad \begin{cases} p(0) = s_0 = f(0) \\ p'(0) = v_0 = f'(0) \\ p''(0) = a_0 = f''(0) \end{cases}$$

Next, we address the following questions:

1.) How well does the Taylor polynomial p approximate the original function f?

2.) What happens to p as we let $n \to \infty$? Convergence? To what limit? f?

1.) Error estimation for the approximation $p \approx f$. First, we estimate the approximation error of the Taylor polynomial p to the original function f:

Proposition 7.35. Taylor with Lagrange remainder term (J.-L. Lagrange 1736 - 1813)

Let $D \subset \mathbb{R}$ be an open interval, $n \in \mathbb{N}$, and $f \in C^{n+1}(D)$.
For all $x_0, x \in D$, there exists a ξ between x_0 and x such that

$$f(x) \;=\; \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{p(x)} \;+\; \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrange remainder}}.$$

Proof: Let $x_0, x \in D$ and let M be such that $f(x) = p(x) + \frac{M}{(n+1)!}(x-x_0)^{n+1}$. Through $(n+1)$ applications of Rolle's theorem, one shows: $\exists \xi : \; f^{(n+1)}(\xi) = M$. ∎

Example 7.36. Error estimate for the exponential function when $n = 5$

Let $f = \exp$ and $x_0 = 0$. Then $f^{(n)}(x_0) = \exp(0) = 1$ for all n.

$$e^x \;=\; \underbrace{1 + x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n}_{p(x)} \;+\; \underbrace{\frac{e^\xi}{(n+1)!}x^{n+1}}_{=:\,R(x)} \quad \text{with } \xi \in [0, x].$$

For $n = 5$, we conclude at $x = 1$ that $\xi \leq 1$ and $|R(1)| \;\leq\; \frac{e^1}{6!}1^6 \;=\; \frac{e}{720} \;<\; 0.0038$.

Taylor polynomial with Lagrange remainder (alternative notation)

With $x = x_0 + h$ and ξ between x_0 and x from Proposition 7.35, we get:

$$f(x_0 + h) \;=\; f(x_0) + \frac{f'(x_0)}{1!}h + \frac{f''(x_0)}{2!}h^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}h^n \;+\; \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

Based on upper bounds for $|f^{(n+1)}(\xi)|$ and $|x - x_0|$, the Lagrange remainder and thus the worst-case approximation error can be estimated:

Corollary 7.37. Taylor polynomial: error estimate

Let $D = (a, b)$, $n \in \mathbb{N}$, *and let* $f \in C^{n+1}(D)$ *and* $M \geq \sup_{\xi \in D} \left| f^{(n+1)}(\xi) \right|$.
Then for all $x, x_0 \in D$, *it holds that*

$$|f(x) - p(x)| \;\leq\; \frac{M}{(n+1)!} |x - x_0|^{n+1}$$

and thus for all $x_0 \in D$

$$\sup_{x \in D} |f(x) - p(x)| \;\leq\; \frac{M}{(n+1)!} \sup_{x \in D} |x - x_0|^{n+1} \;=\; \frac{M}{(n+1)!} d^{n+1},$$

where $d := \max(x_0 - a, b - x_0)$ *is the distance from* x_0 *to the further end point.*

2.) Now the Taylor series. Now let us send $n \to \infty$ in the Taylor polynomial.

To ensure that all Taylor coefficients $a_n = \frac{f^{(n)}(x_0)}{n!}$ are defined, let $f \in C^\infty(D) := \bigcap_{n \in \mathbb{N}} C^n(D)$.

Definition 7.38. Taylor series

Let $D \subset \mathbb{R}$ *be an open interval. For* $f \in C^\infty(D)$ *and* $x_0 \in D$, *let* Tf *be the following power series,*

$$(Tf)(x) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \, (x - x_0)^n \, ,$$

which we call the <u>*Taylor series*</u> *of* f *expanded / centered at* x_0.

Let $f \in C^\infty(D)$ already be given by a power series with radius of convergence $R > 0$,

$$f(x) \;=\; \sum_{n=0}^{\infty} a_n \, (x - x_0)^n, \qquad x \in D = U_R(x_0). \tag{7.5}$$

In this case, f is also called <u>*analytic at* x_0</u>.
Then f is differentiable in $D = U_R(x_0)$ (Example 7.3 d), and one has

$$f'(x) \;=\; \sum_{n=1}^{\infty} n a_n \, (x - x_0)^{n-1}, \qquad x \in D.$$

In particular, the power series can be differentiated term by term and the radius of convergence remains the same, $R > 0$, because

$$\sqrt[n]{|n a_n|} \;=\; \sqrt[n]{n} \;\cdot\; \sqrt[n]{|a_n|} \qquad \text{and} \qquad \sqrt[n]{n} \to 1.$$

Just like in the derivation of (7.4), we get $f(x_0) = a_0$, $f'(x_0) = 1 \cdot a_1$ as well as $f''(x_0) = 1 \cdot 2 \cdot a_2$, etc.: The a_n in (7.5) are automatically the Taylor coefficients of f.

$$f \text{ analytic at } x_0 \quad \Longrightarrow \quad f = Tf \text{ in } U_R(x_0). \tag{7.6}$$

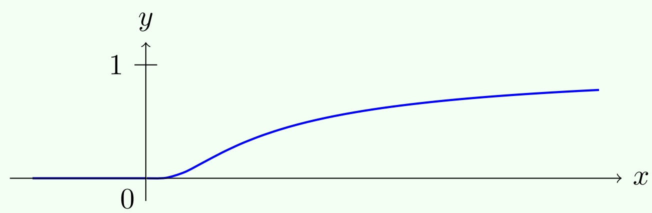
Thus, the known power series of $\exp, \sin, \cos, \ldots$ are indeed their Taylor series.

The radius of convergence of Tf can also be $R = 0$. And even if $R > 0$, Tf does not necessarily converge to f! (In which case f is of course not analytic, see (7.6).)

Example 7.39. Taylor series Tf of $f \in C^\infty(\mathbb{R})$ converges, but $Tf \neq f$

Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0, \\ 0 & x \leq 0. \end{cases}$$



It can be shown that $f \in C^\infty(\mathbb{R})$ with $f^{(n)}(0) = 0,\ n = 0, 1, 2, \ldots,$ so that $Tf \equiv 0 \neq f$ holds with Taylor expansion at $x_0 = 0$.

7.8 Curve discussion

Let a function $f : D \to \mathbb{R}$ be given.

Typical steps of a curve discussion

1.) *Find the domain $D \subset \mathbb{R}$ (if not given)*
2.) *Symmetries (for example $f(x) = f(-x)$ or $f(x) = -f(-x)$ for all $x \in D$)*
3.) *Continuity*
4.) *Poles and behavior at gaps in the domain*
5.) *Asymptotic behavior for $x \to \pm\infty$*
6.) *Zeros*
7.) *Monotonicity intervals and local extrema*
8.) *Convexity and concavity as well as inflection points*

A quick example:

Example 7.40. Curve discussion

Let the rule $f : x \mapsto \frac{2x^2-12x+10}{x^2}$ *be given. We simplify it a bit:*

$$f(x) \;=\; \frac{2x^2-12x+10}{x^2} \;=\; 2-\frac{12}{x}+\frac{10}{x^2}.$$

1.) *Due to the division by* x^2, $x=0$ *must be excluded, whence* $D=\mathbb{R}\setminus\{0\}$.

2.) *Obviously,* $f(-x)=\;2+\frac{12}{x}+\frac{10}{x^2}$ *is neither equal to* $f(x)$ *nor to* $-f(x)$.

3.) *According to Example 5.17d),* f *is continuous on the entire domain* $D=\mathbb{R}\setminus\{0\}$.

4.) *The only gap in the domain is at* $x_0=0$. *By* $2x^2-12x+10\to 10\neq 0$ *and* $x^2\to 0$ *as* $x\to 0$, *it is an expression of the form "*$\frac{10}{0}$*" and thus a pole.*
By $10>0$ *and* $x^2>0$ *for* $x\neq 0$, *it follows that* $f(x)\to+\infty$ *as* $x\to 0$.

5.) *For* $x\to+\infty$ *and for* $x\to-\infty$, $\frac{12}{x}\to 0$ *and* $\frac{10}{x^2}\to 0$, *so that* $f(x)\to 2$.

6.) *It holds that* $f(x)=0 \iff 2x^2-12x+10=0 \iff x=1$ *or* $x=5$.

7.) *Here we investigate when the derivative* f' *has which sign. It holds that*

$$f'(x) \;=\; \frac{12}{x^2}-\frac{20}{x^3} \;=\; \frac{12x-20}{x^3}.$$

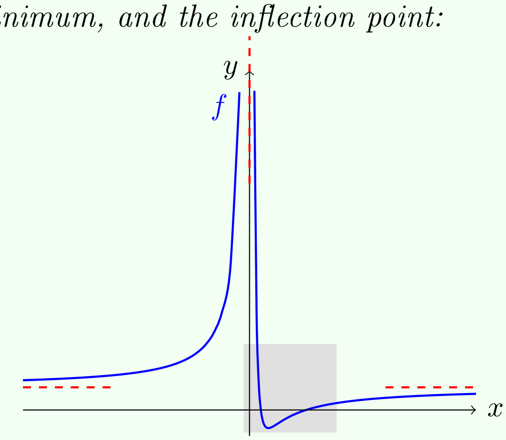
$12x-20$ *changes its sign at* $x=\frac{20}{12}=\frac{5}{3}$ *from* $-$ *to* $+$ *and* x^3 *changes its sign at* $x=0$ *from* $-$ *to* $+$. *So* $f'(x)$ *is positive (*f *is increasing) for* $x<0$, *negative (*f *is decreasing) for* $0<x<\frac{5}{3}$ *and positive again (*f *is increasing) for* $\frac{5}{3}<x$. *Thus, at* $x=\frac{5}{3}$, *there is a local (and even global) minimum.*

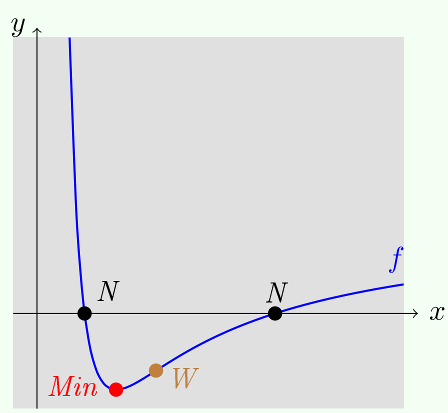
8.) *Finally, let's examine when the second derivative* f'' *has which sign. It holds that*

$$f''(x) \;=\; -\frac{24}{x^3}+\frac{60}{x^4} \;=\; \frac{-24x+60}{x^4}.$$

$-24x+60$ *changes its sign at* $x=\frac{60}{24}=\frac{5}{2}$ *from* $+$ *to* $-$ *and* x^4 *is always* ≥ 0. *So* $f''(x)$ *is positive (*f *is convex) for* $x<\frac{5}{2}$ *and negative (*f *is concave) for* $\frac{5}{2}<x$. *Thus, at* $x=\frac{5}{2}$, *there is an inflection point.*

We conclude the example with a graph sketch, showing the asymptotics at the pole and as $x\to\pm\infty$, *and a zoom into the gray area showing the two zeros, the minimum, and the inflection point:*





Summary

- $f : D \to \mathbb{R}$ is differentiable at $x_0 \in \operatorname{int} D$ if $\frac{f(x)-f(x_0)}{x-x_0}$ has a limit as $x \to x_0$.
- The limit value is then the derivative $f'(x_0)$.
- Continuity follows from differentiability – but not vice versa.
- With f and g, also $f+g, f-g, f \cdot g, f : g, f \circ g$ and f^{-1} are all differentiable, and we know a formula for the derivative in each case.
- For f to have a local extremum in x_0, it is necessary that $f'(x_0) = 0$ applies. (assuming $x_0 \in \operatorname{int} D$ and differentiability of f in x_0)
- However, $f'(x_0) = 0$ alone is not sufficient to conclude a local extremum.
  - This is then only referred to as an extreme value suspect or stationary point.
  - However, if f' also changes its sign at x_0, we can conclude a local extremum,
  - If $f''(x_0) \neq 0$, then also.
  - Minimum or maximum? Depends on the sign change of f' or the sign of $f''(x_0)$.
- Monotonicity of f can be characterized equivalently via $f' \geq 0$ or $f' \leq 0$.
- Strict monotonicity follows from $f' > 0$ or $f' < 0$ – but not vice versa.
- If $\frac{f(x)}{g(x)}$ for $x \to x_0$ or $x \to \pm\infty$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the limit value of $\frac{f(x)}{g(x)}$ can be calculated by the limit value of $\frac{f'(x)}{g'(x)}$, if the latter exists.
- The second derivative f'' has something to do with the curvature of the graph of f:
  - For $f'' = 0$ on the whole of D, the graph is a straight line,
  - for $f'' > 0$ the graph is curved like a "U"; f is then strictly convex,
  - for $f'' < 0$ the curve goes the other way round and f is called strictly concave,
  - the magnitude $|f''|$ is related to the intensity of the curvature.
- Just as $f(x_0)$ and $f'(x_0)$ are used to determine the tangent to f in x_0, all of $f(x_0)$, $f'(x_0), \ldots, f^{(n)}(x_0)$ can be used to determine a polynomial of nth degree that optimally matches f at x_0, the so-called Taylor polynomial.
  - The polynomial coefficient in front of x^k is calculated via $a_k = \frac{f^{(k)}(x_0)}{k!}$,
  - the approximation error is estimated using the Lagrange remainder,
  - if we send $n \to \infty$, we get the Taylor series.
- In a curve discussion, you determine the domain of definition, symmetries, any discontinuities, in particular poles, the asymptotics at $\pm\infty$, zeros, intervals with monotonicity, local extrema, intervals with convexity, concavity and inflection points.

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