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Institute of Mathematics

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Homework 1 for
Mathematics I (winter term 25/26)

Submit your solutions to **Problems 3 and 4** until Sunday, **November 02**, 11:59 pm at the latest, using **abGabi**. Only one of these problems will be chosen by us to be corrected and graded. We strongly suggest that you **submit in pairs**. Please state the names and matriculation numbers of both persons on your sumbissions and only submit once per group (the other person will still receive credit). Submission in larger groups is not permitted.

Problem 1 *Sets*

Consider the sets

$$\begin{aligned} A &:= \big\{ -4a + 3 : \ a \in \mathbb{N} \text{ and } a \leq 4 \big\}, \\ B &:= \big\{ z \in \mathbb{Z} : \ |z| \leq 1 \text{ or } z^2 = 16 \big\}, \\ C &:= \big\{ |a + b| : \ a \in \{1, 2\} \text{ and } b \in \{-2, -3, -4\} \big\}. \end{aligned}$$

How many elements do the following sets contain? Explain your solution.

(a) A (b) B (c) C (d) $A \cup B$ (e) $B \setminus A$ (f) $B \cap C$

Problem 2 *Set operations*

For two sets A and B, the *Cartesian product*

$$A \times B := \{(a, b) : \ a \in A \text{ and } b \in B\}$$

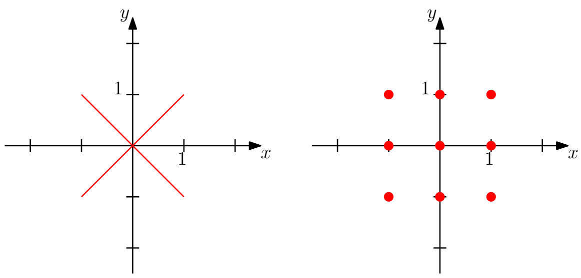
is defined as the set of all (ordered) pairs with the first component belonging to A and the second component belonging to B.

(a) Consider the sets

$$\begin{aligned} R &:= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ 0.5 \leq x \leq 1 \ \text{ and } \ 0.5 \leq |y| \leq 1\} \\ S &:= \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ x + y \leq 2 \ \text{ and } \ x, y \geq 0\} \ . \end{aligned}$$

Sketch the sets R, S, $R \cap S$ and $S \setminus R$ in an appropriate coordinate system.

(b) For the sets which are sketched in the following picture, find a describing representation.



Problem 3 *Set operations*

Decide which of the following equations are true for every choice of sets X, Y, Z, and explain your decision shortly, e.g. by using a Venn diagram.

(a) $(Z \setminus Y) \cup (Y \setminus X) = (Z \cup Y) \setminus (X \cap Y)$

(b) $X \cup (Z \setminus Y) = (X \cup Z) \setminus Y$

(c) $(Z \setminus Y) \setminus X = Z \setminus (X \cup Y)$

Problem 4 *Sums*

Calculate the following sums. Give your way of calculation and simplify fractions!

(a) $\displaystyle\sum_{k=0}^{3}(2k^3 - 2k + 1)$ (b) $\displaystyle\sum_{k=1}^{3} \frac{(-1)^k}{(k+1)^2}$ (c) $\displaystyle\prod_{n=1}^{3}\left(\sum_{k=-n}^{n} 2k\right)$ (d) $\displaystyle\prod_{m=1}^{999} \frac{m}{m+1}$

Hint: Please do not solve (d) by writing down all the 999 factors of the product. Instead of this, find out which numbers cancel.

Problem 5 *Proofs*

Prove the following statements:

(a) If $n \in \mathbb{Z}$ is even and $m \in \mathbb{Z}$ is arbitrary, then $n \cdot m$ is even.

(b) Let $n \in \mathbb{Z}$. Then $n + 1$ is even if and only if n is odd.

P1:

a) Perform the formula on $[1,4] \cap \mathbb{N}$:

$A := \emptyset$ (all insertions produce numbers $\notin \mathbb{N}$)

$\Rightarrow |A| = 0$

b) $B := \{-1, 0, 1, -4, 4\}$ (apply the condition from -1 to 1 and $[-\sqrt{16}, \sqrt{16}]$ in your head)

$\Rightarrow |B| = 5$

c) Performing $|a+b|$ for all combinations of a, b will yield all members of C:

$C := \{1, 2, 3, 0,\}$

$\Rightarrow |C| = 4$

d) $A \cup B = \emptyset \cup B = \{-1, 0, 1, -4, 4\}$ (combine the sets)

e) find all elements exclusive to B between A, B:

$B \setminus A = B = \{-1, 0, 1, -4, 4\}$

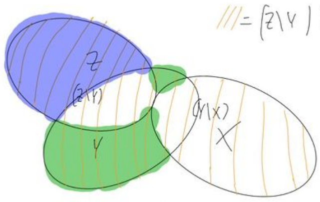
f) $B \cap C = \{1, 0\}$ ($\forall x \in B \wedge \in C$)

P2

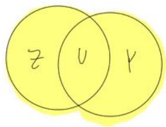
3. a)

$(Z \setminus Y) \cup (Y \setminus X) = (Z \cup Y) \setminus (X \cap Y)$

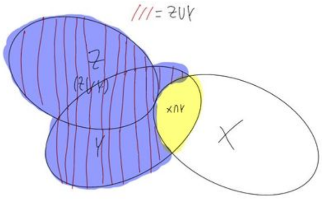
$/// = (Z \setminus Y) \cup (Y \setminus X)$



For the first part in 3.a) we have to get the difference from (Z\Y) and (Y\X) due to the U (union), which later combines both products. As shown in blue, the difference in (Z\Y) is Z due to it being the Elements off Z and not in Y. We do the same thing to (Y\X), where Y has the elements of Y and X not having the elements of Y. So we got in the first set of the equation the Elements Z and Y. And due to it being a set of a U (union), we get the final set (Z,Y).



$/// = Z \cup Y$



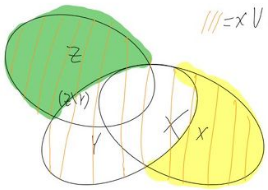
For the second equation, we only need to figure the union between (ZUY), because we will later need to find the „\" (difference) between (ZUY) and (X⋂Y) where we can figure out beforehand that the difference will be (ZUY) due to it being the elements of (ZUY). Now to figure out the union from the set (ZUY), which is the set containing the elements of Z ans Y. So we get the final set (Z,Y).

Finally we compare both final sets, and as shown, we got (Z,Y)=(Z,Y). With that, we can conclude that 3.a) is true.

3b)

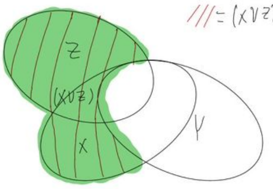
$X \cup (Z \setminus Y) = (X \cup Z) \setminus Y$

$/// = X \cup (Z \setminus Y)$



For the first part in 3b we need to figure out what the difference in (Z\Y) is and then put it in the union from XU(…). So first, we have the difference of (Z\Y) and that is Z due to Z being the elements of Z and not the elements of Y. Now we put it in a union XUZ, so we get for the first part of the equation (X,Z).

$/// = (X \cup Z) \setminus Y$

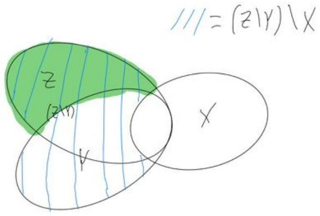


For the second part of the equation we can drop out Y due to it not being the element of whatever will be in (XUZ) due to it not being the elements of X nor Z nor (XUZ). So we only need to get the union off (XUZ) and that is (X,Z).

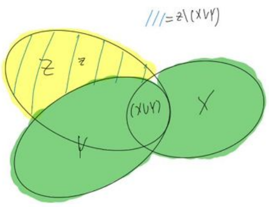
Finally we compare both final sets in the equation; (X,Z)=(X,Z). With that we can conclude that the equation of 3b) is true.

3c)

$(Z \setminus Y) \setminus X = Z \setminus (X \cup Y)$



In the first part of the equation of 3c) we are getting rid of X, due to the difference X not being an element of (Z\Y), Z or Y. Now we need to figure out what the difference in (Z\Y) is. And that is Z due to Y not being an element of Z. So our first produkt is Z.



The second part of 3c) equation consists of the difference Z\(XUY). Here we know that only Z is the answer due to only Z being the element of Z and nothing from (XUY). So the answer for the second part of the equation of 3c is Z.

The final equation is Z=Z. with that we can conclude that the equation of 3c) is true.

Problem 4 *Sums*

Calculate the following sums. Give your way of calculation and simplify fractions!

(a) $\sum_{k=0}^{3}(2k^3 - 2k + 1)$ (b) $\sum_{k=1}^{3}\frac{(-1)^k}{(k+1)^2}$ (c) $\prod_{n=1}^{3}\left(\sum_{k=-n}^{n} 2k\right)$ (d) $\prod_{m=1}^{999}\frac{m}{m+1}$

Hint: Please do not solve (d) by writing down all the 999 factors of the product. Instead of this, find out which numbers cancel.

P4

(a) $\sum_{k=0}^{3}(2k^3-2k+1) = 1+1+(16-4+1)+(54-6+1)$

$= 2+13+49 = 64$

(b) $\sum_{k=1}^{3}\frac{(-1)^k}{(k+1)^2} = \frac{-1}{4}+\frac{1}{9}+\frac{-1}{16}$

$= \frac{-5}{16}+\frac{1}{9} = \frac{-45}{144}+\frac{16}{144} = \frac{-29}{144}$

(c) $\prod_{n=1}^{3}\left(\sum_{k=-n}^{n} 2k\right) = 0\cdot 0\cdot 0 = 0$ ($2\cdot 0$ is always 0, everything else cancels out because it's symmetric around 0)

(d) $\prod_{m=1}^{999}\frac{m}{m+1} = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot \ldots \cdot\frac{998}{999}\cdot\frac{999}{1000}$

each numerator cancels with preceeding denominator, except for the first numerator and last denominator

$\Rightarrow = \frac{1}{1000} = 0{,}01$