Discrete Algebraic Structures

WiSe 2025/2026

Prof. Dr. Antoine Wiehe
Research Group for Theoretical Computer Science

Exam: 10.03.

- What a relation is
- How to draw a relation $R \subseteq A \times B$
- How to compute the composition and transpose of binary relations
- The basic properties
  (anti)reflexivity, (anti)symmetry, transitivity
- How to compute the closures

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Things can share a common property:

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- x, y work in the same company

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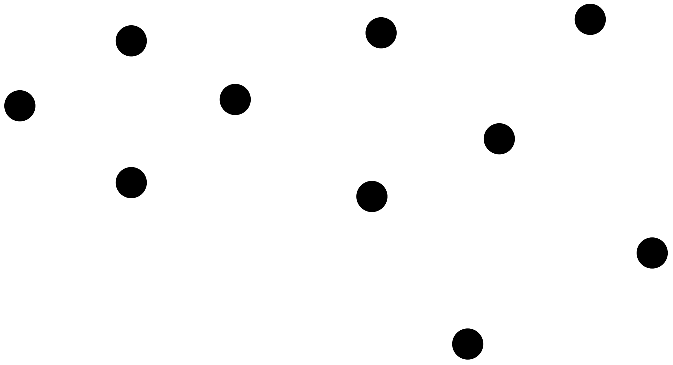

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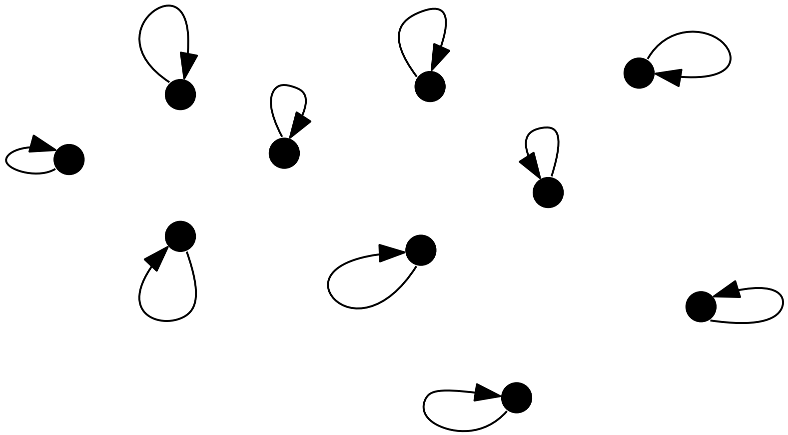

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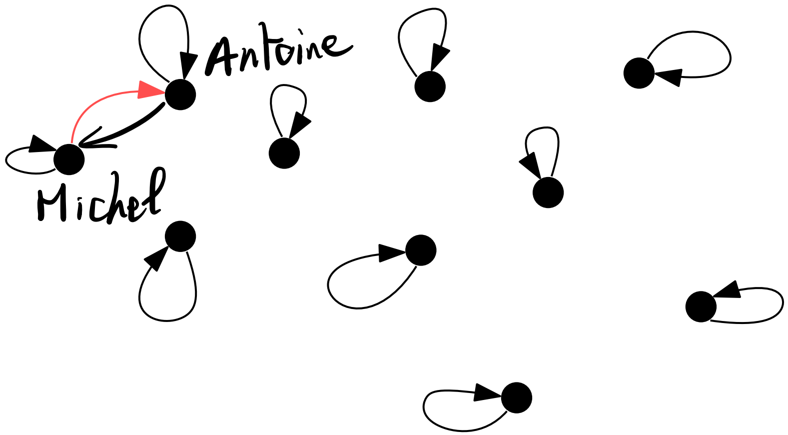

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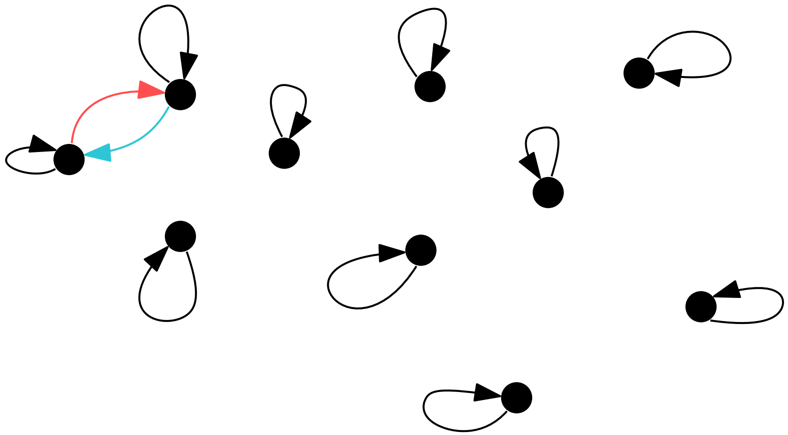

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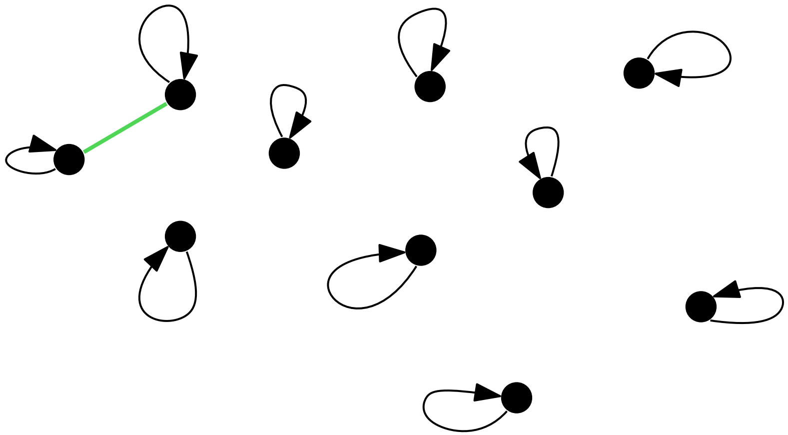

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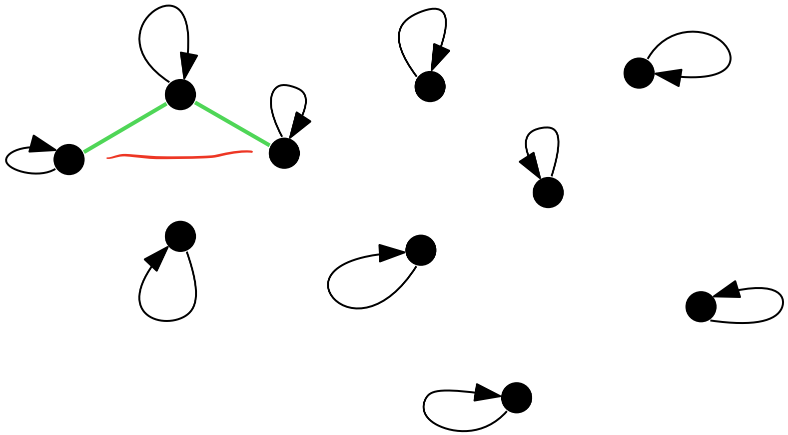

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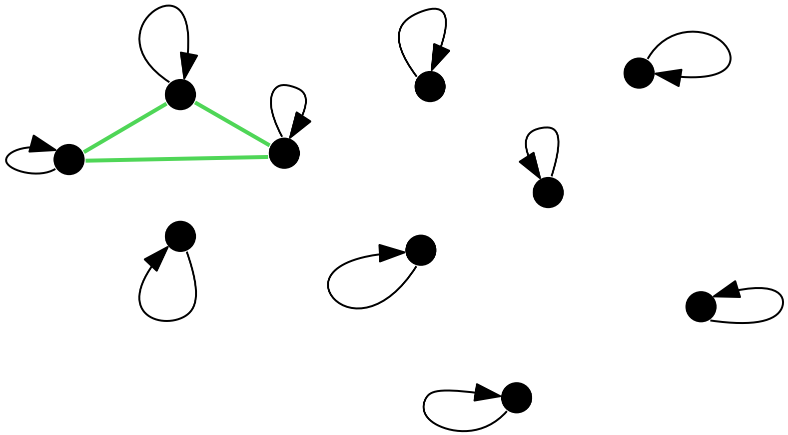

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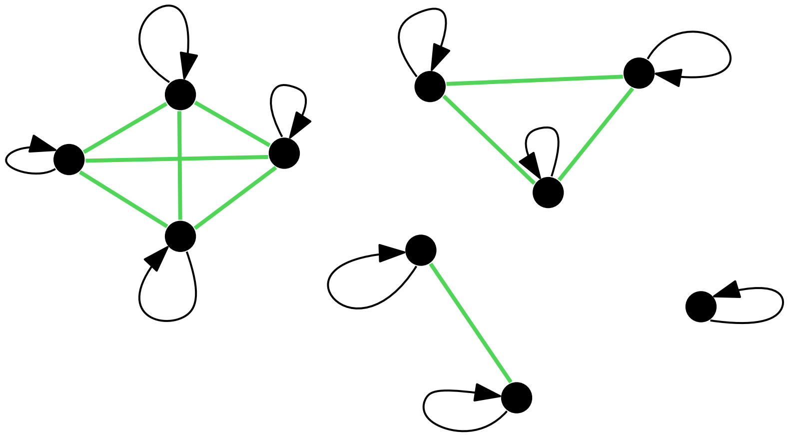

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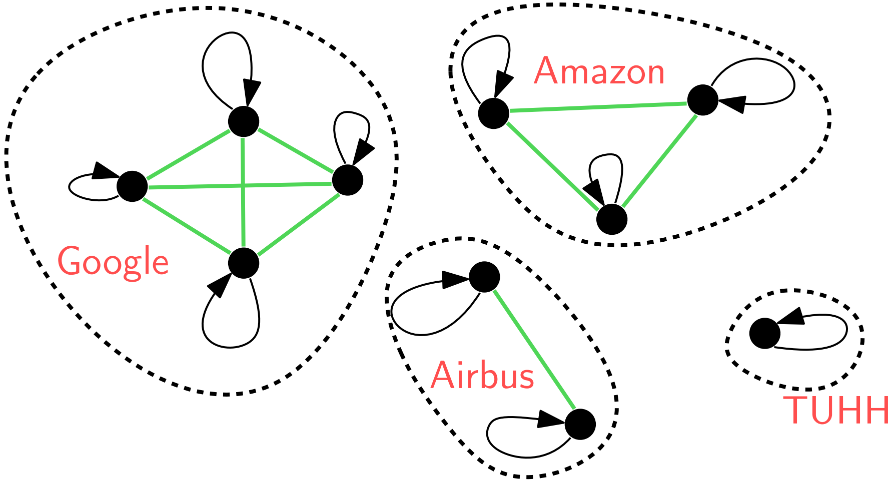

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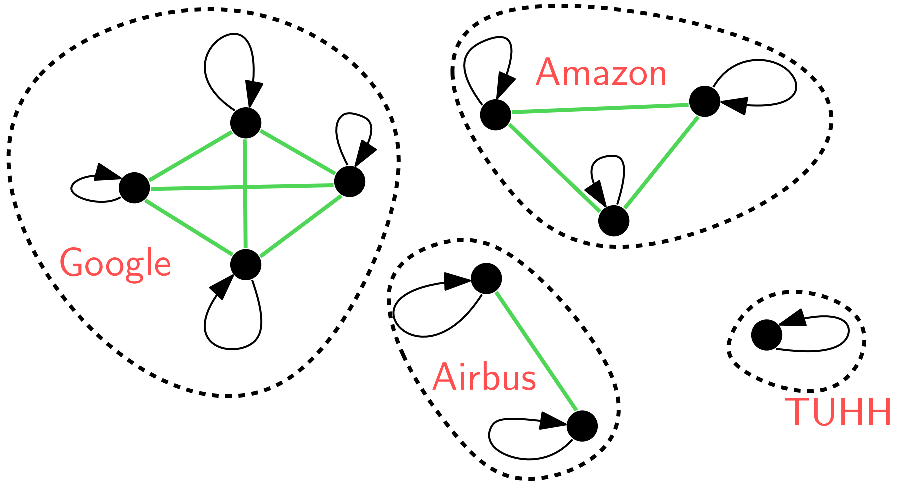

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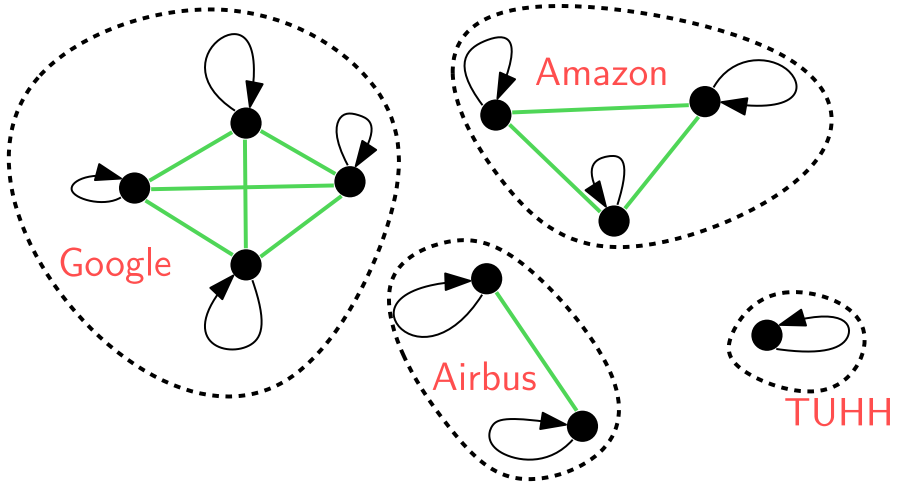

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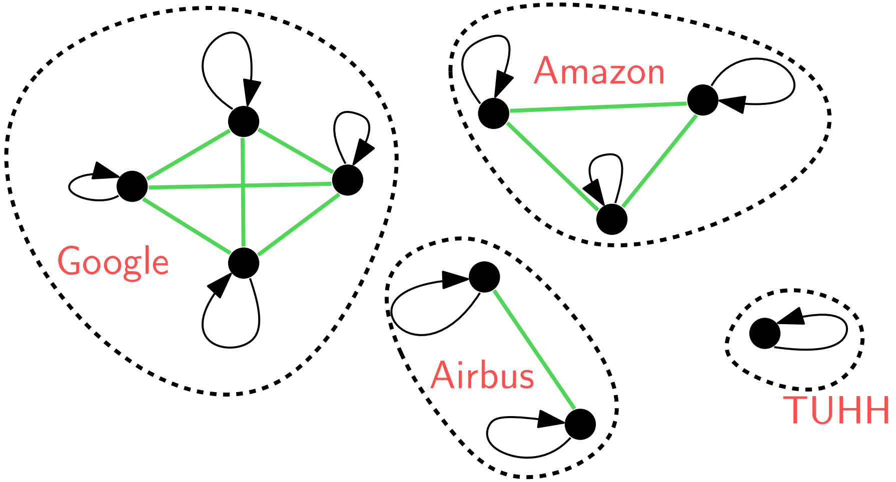

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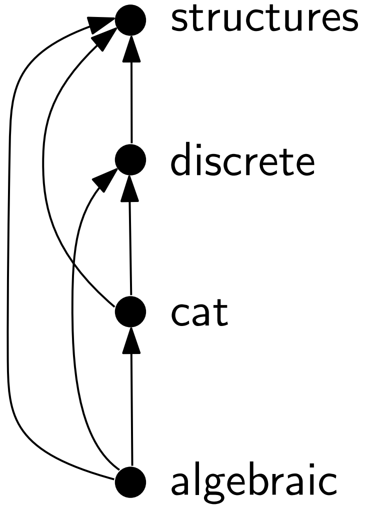

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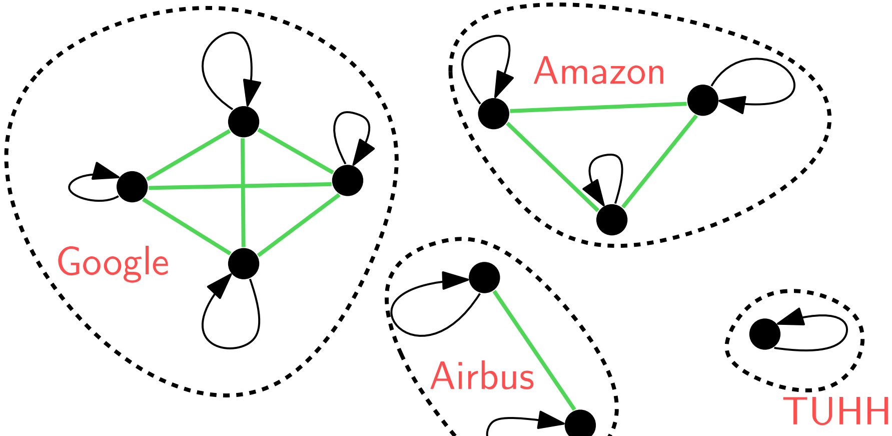

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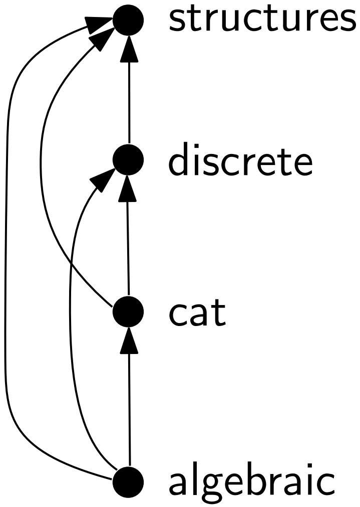
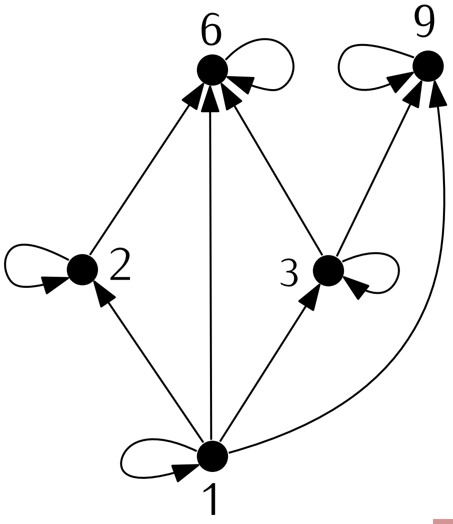
$\{1, 2, 3, 6, 9\}$





Definition. A relation $R \subseteq A \times A$ is:

- reflexive: if for all $a \in A$, $(a, a) \in R$
- antireflexive: if for all $a \in A$, $(a, a) \notin R$
- symmetric: if for all $a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$
- antisymmetric: if for all $a, b \in A$, if $(a, b) \in R$ and $a \neq b$ then $(b, a) \notin R$
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Equivalence relation	✓		✓		✓
Order	✓			✓	✓
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Quasi-order	✓				✓

Equivalence relations

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Examples. All of these are equivalence relations:

- $\{(x, y) \in \mathbb{Z}^2 \mid x, y \text{ have the same sign}\}$ $\{0, +, -\}$
- $\{(x, y) \in (\mathbb{N}_0)^2 \mid x, y \text{ have the same parity}\}$ $\{\text{even}, \text{odd}\}$
- $\{(x, y) \in \text{People}^2 \mid x, y \text{ have the same height}\}$ $\mathbb{N}_0$
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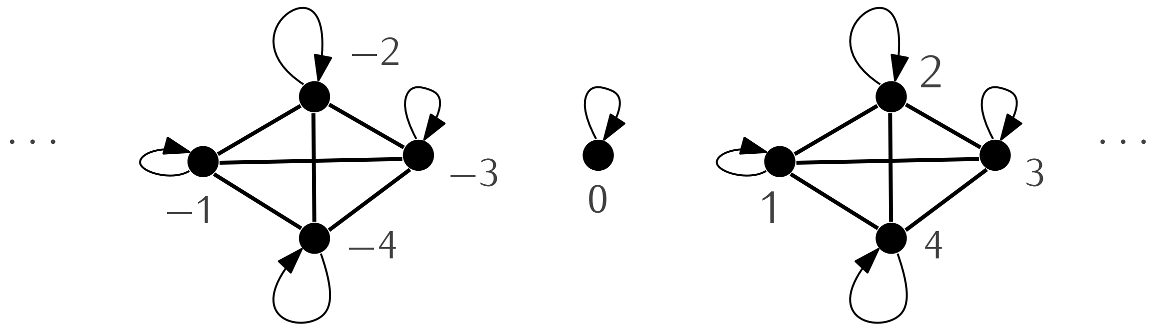

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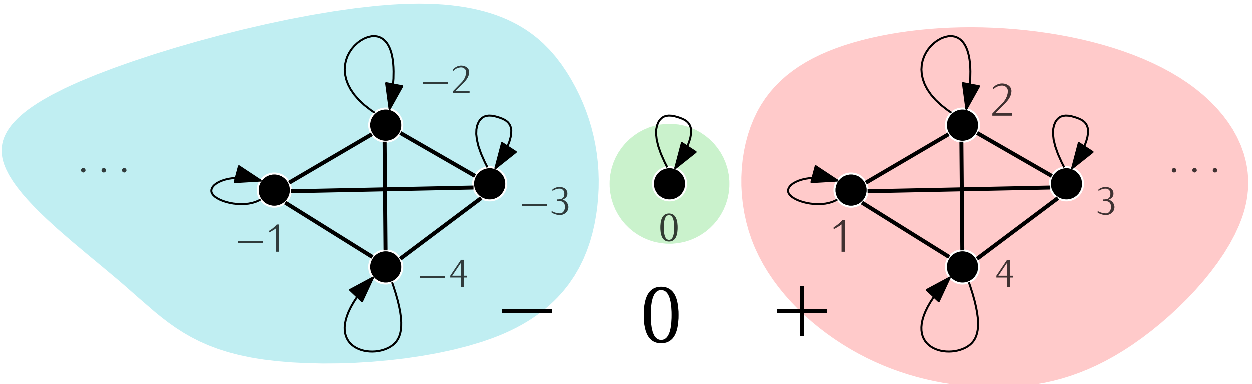

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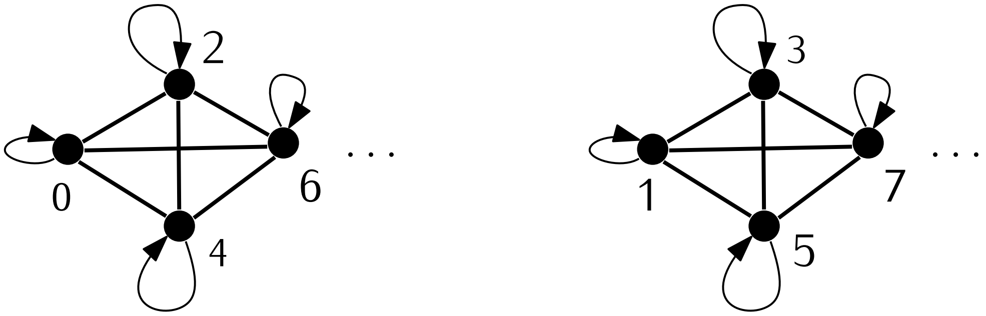

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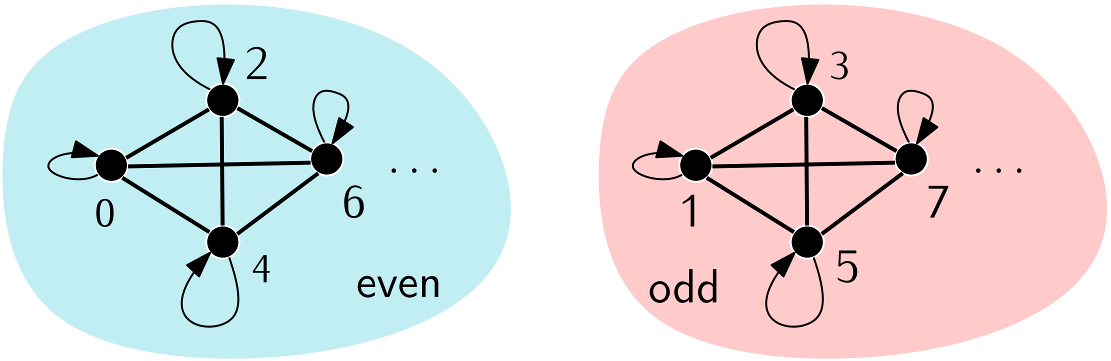

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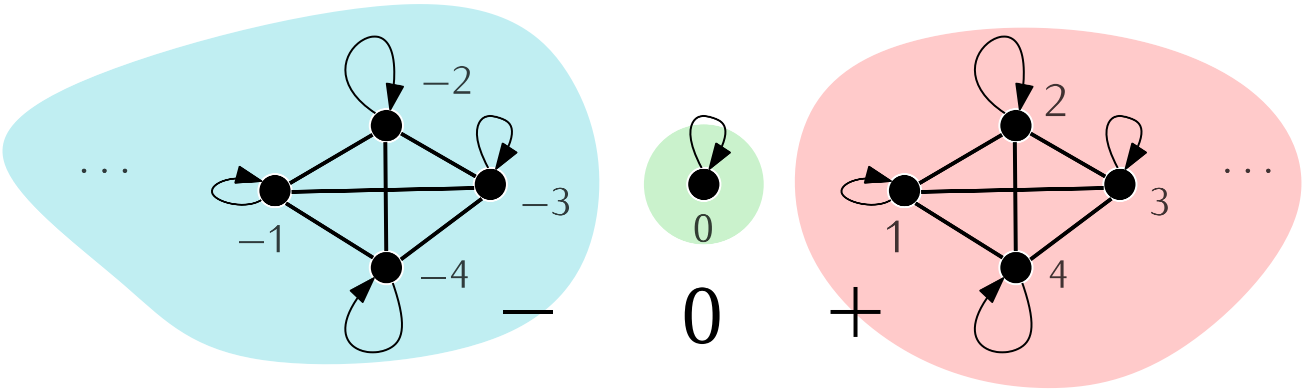

$x \sim y$ if x, y have same sign

$[-1]_\sim = \{-2, -3, -4, -1, \dots\}$

$[0]_\sim = \{0\}$

$[1]_\sim = \{1, 2, 3, 4, \dots\}$

$[-2]_\sim = \{-1, -2, -3, -4, \dots\}$

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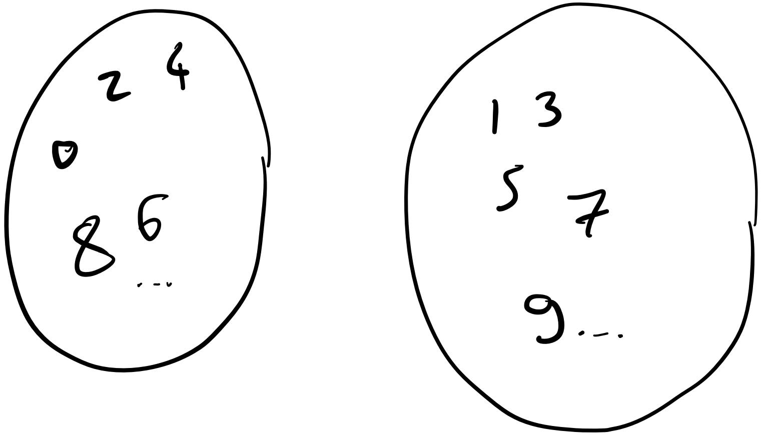

Define $a \sim b$ on $\mathbb{N}_0$ as "a and b have the same parity".
What is $[0]_\sim$?

$\cancel{\emptyset}$ $\cancel{\{0\}}$ $\cancel{\{1\}}$ $\{0, 1, 2, 4, \ldots\}$ $\{0, 2, 4, \ldots\}$ ✓

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Theorem. Let $\sim$ be an equivalence relation on A, and let $a, b \in A$. The following are equivalent:

1. $a \sim b$
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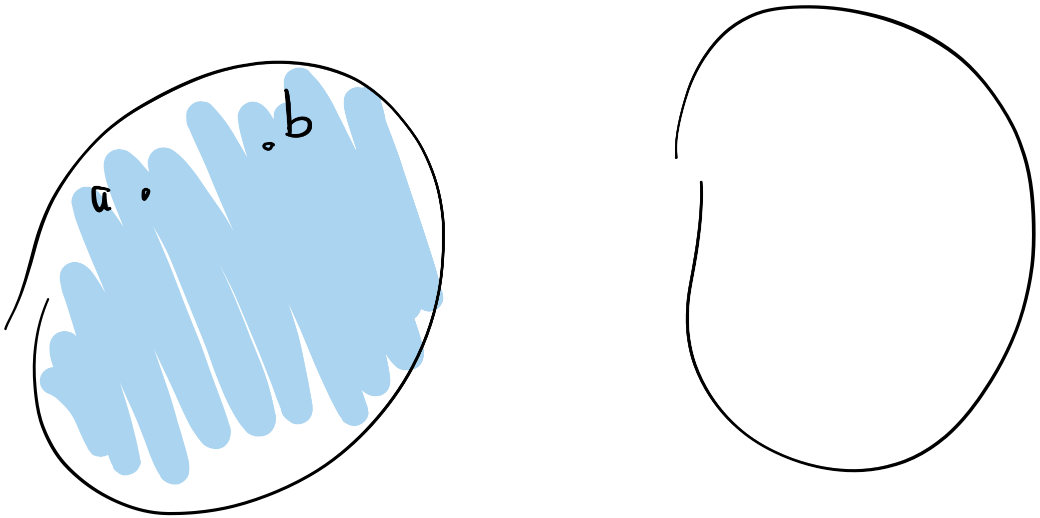

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Theoretically, one needs to prove 6 implications:

$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_6$

$\equiv$

$\varphi_1 \wedge \varphi_2 \wedge \varphi_3$

$$\begin{tikzcd}
 & a \sim b \arrow[dl, Rightarrow, shift right] \arrow[dr, Rightarrow, shift left] & \\
{[a]_\sim \cap [b]_\sim \neq \emptyset} \arrow[ur, Rightarrow, shift right] \arrow[rr, Rightarrow, shift right] & & {[a]_\sim = [b]_\sim} \arrow[ul, Rightarrow, shift left] \arrow[ll, Rightarrow, shift right]
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Hyp: · $a \sim b$
· $\sim$ is an eq. relation
· $x \in [a]_\sim$ so in fact $x \sim a$
· $x \sim b$ by transitivity of $\sim$

$$a \sim b$$

$$\Downarrow$$

$$[a]_\sim = [b]_\sim$$

Goal: $x \in [b]_\sim$

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$$a \in [a]_\sim \neq \emptyset$$

$$[a]_\sim \cap [b]_\sim \neq \emptyset \quad \Longleftarrow \quad [a]_\sim = [b]_\sim$$

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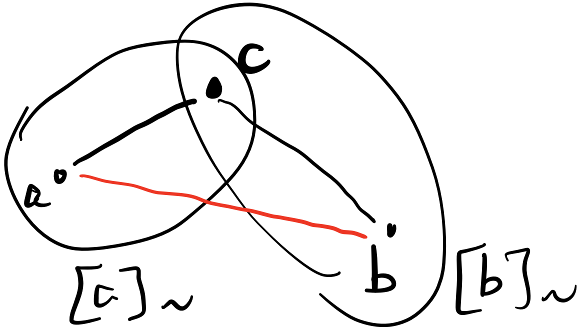
$$a \sim b$$



$$[a]_\sim \cap [b]_\sim \neq \emptyset$$

By logical equivalence, it suffices to prove 3 implications!

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- the parts are not empty $\forall S \in P\ S \neq \emptyset$
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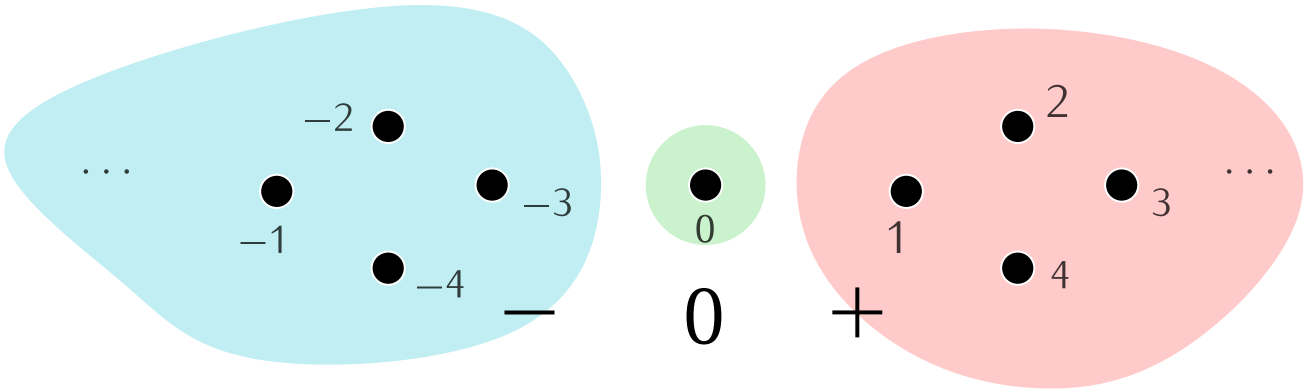

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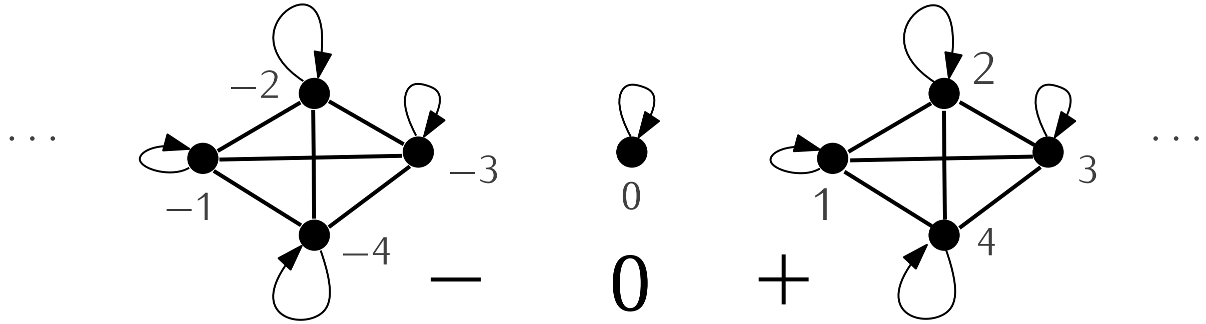

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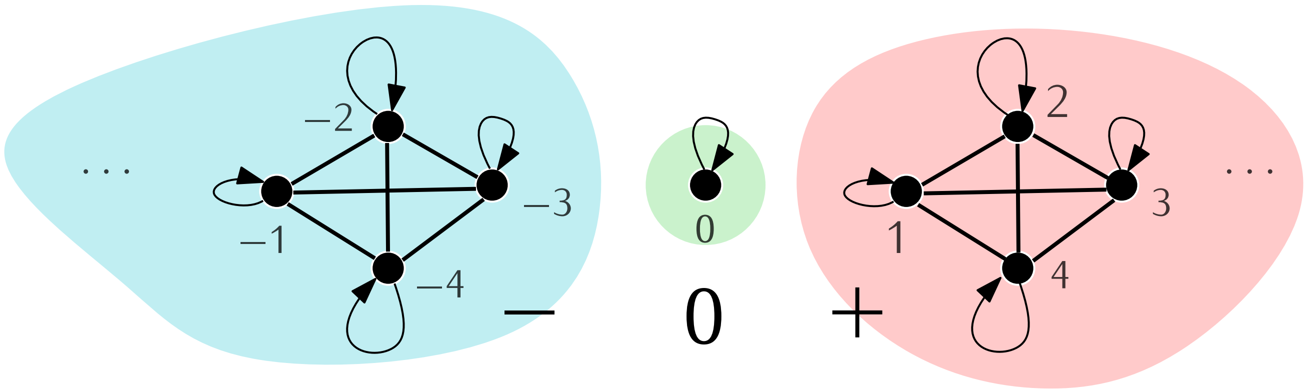

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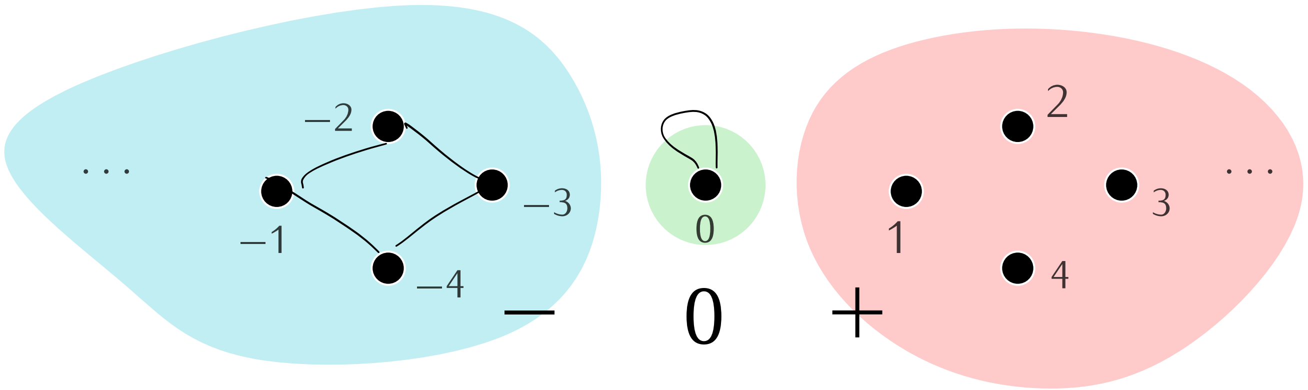

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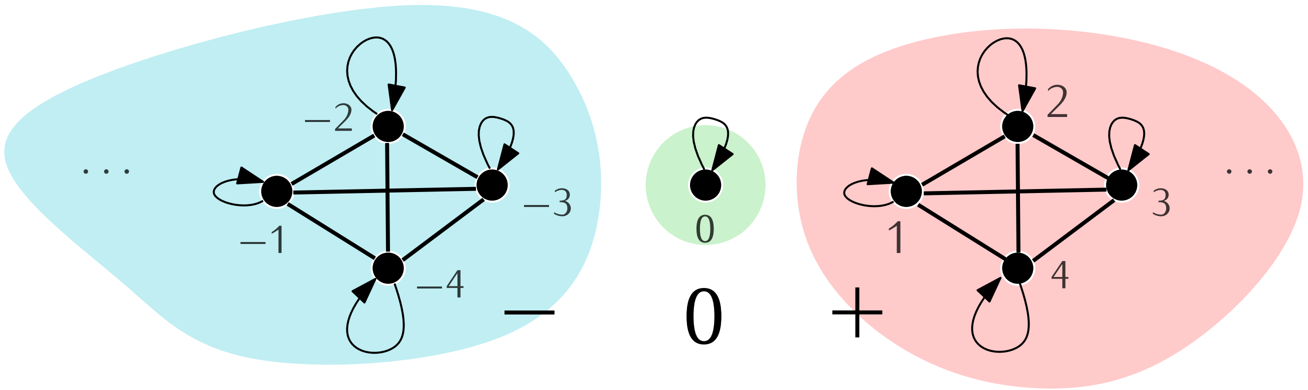

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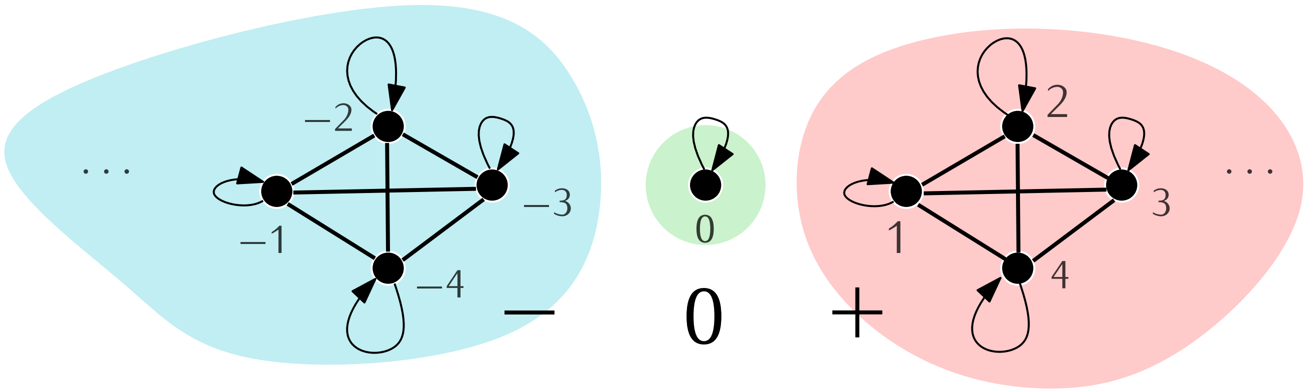

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these two functions are inverse of each other!

$$(g \circ f)(\sim) = \sim$$
$$(f \circ g)(P) = P$$

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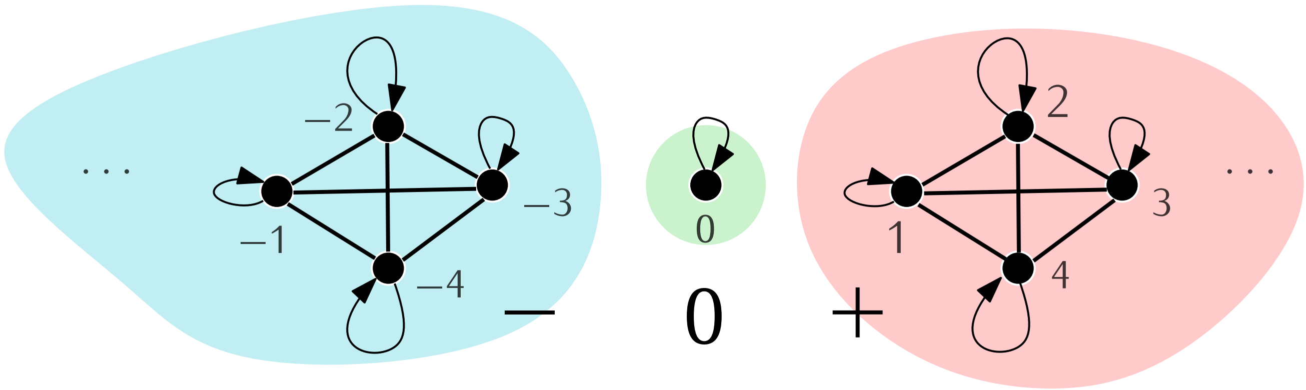

$P = \{\mathbb{Z}_-, \{0\}, \mathbb{Z}_+\} = \mathbb{Z}/\sim$
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We have proved:

Theorem. There exists a bijection between the set of all equivalence relations on A, and the set of all partitions of A.

Partitions

Examples. All of these are equivalence relations:

- $\{(x, y) \in \mathbb{Z}^2 \mid x, y \text{ have the same sign}\}$ $\{0, +, -\}$
- $\{(x, y) \in (\mathbb{N}_0)^2 \mid x, y \text{ have the same parity}\}$ $\{\text{even}, \text{odd}\}$
- $\{(x, y) \in \text{People}^2 \mid x, y \text{ have the same height}\}$ $\mathbb{N}_0$
- $\{(\varphi, \psi) \in \text{PropositionalFormulas}^2 \mid \varphi \equiv \psi\}$
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Notation. We write $A/{\sim}$ for the partition of A given by the equivalence classes of $\sim$.

$$A/{\sim} = \{[a]_\sim \mid a \in A\}$$

Partitions and quotients will appear many many times in your studies, for example:

- modular arithmetic, Lagrange's theorem (later this semester): RSA cryptosystem
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Intuition: forget some differences between elements,
only remember some specific aspects

Orders

- Generalize properties of $(\mathbb{N}, <)$ and $(\mathbb{Q}, <)$ to other sets
- Having an order can help to design good algorithms (binary search)
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Question. What properties does the relation $\leq$ have (on $\mathbb{N}, \mathbb{Z}, \ldots$)?

- Reflexive ✓
- Antireflexive ✗
- Symmetric ✗
- Antisymmetric ✓
- Transitive ✓

all the loops

- For all x, $x \leq x$
- For all x, y: $x \leq y \Rightarrow y \leq x$
- For all x, y: $(x \leq y \wedge y \leq x) \Rightarrow x = y$

Definition. A relation $R \subseteq A \times A$ is an order if it is reflexive, antisymmetric, and transitive.

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Examples. The following relations are all orders:

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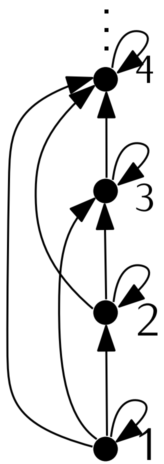

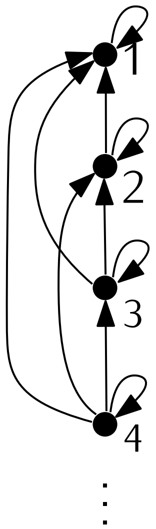

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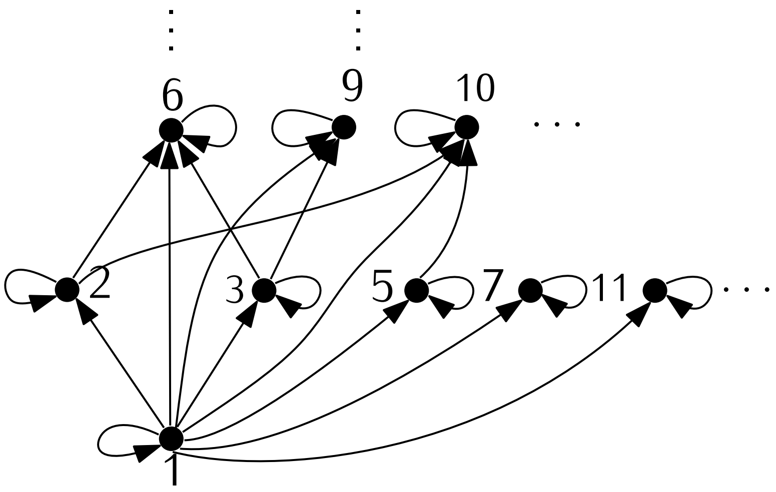

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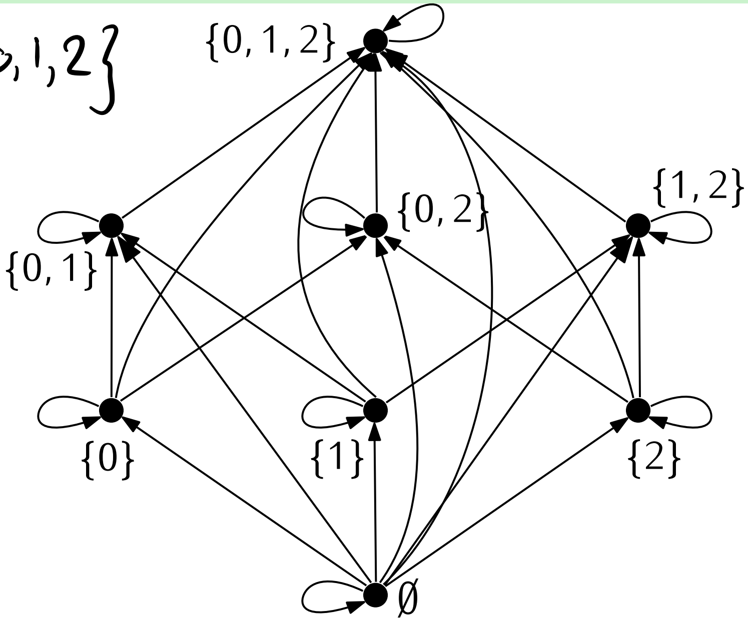
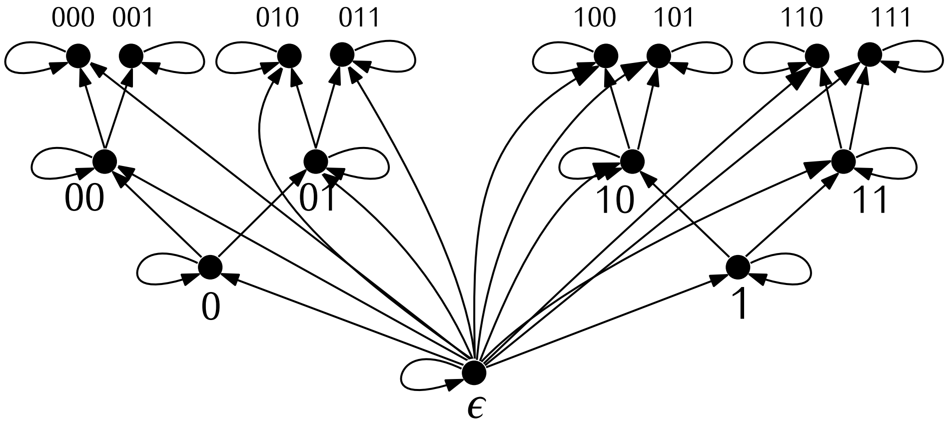
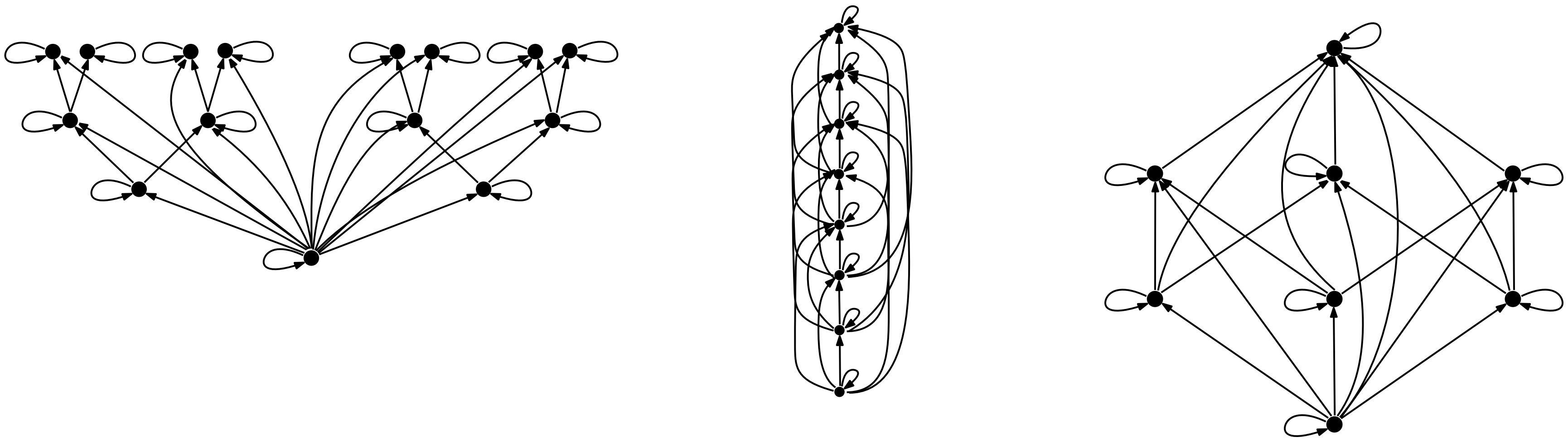
$A = \{0, 1, 2\}$







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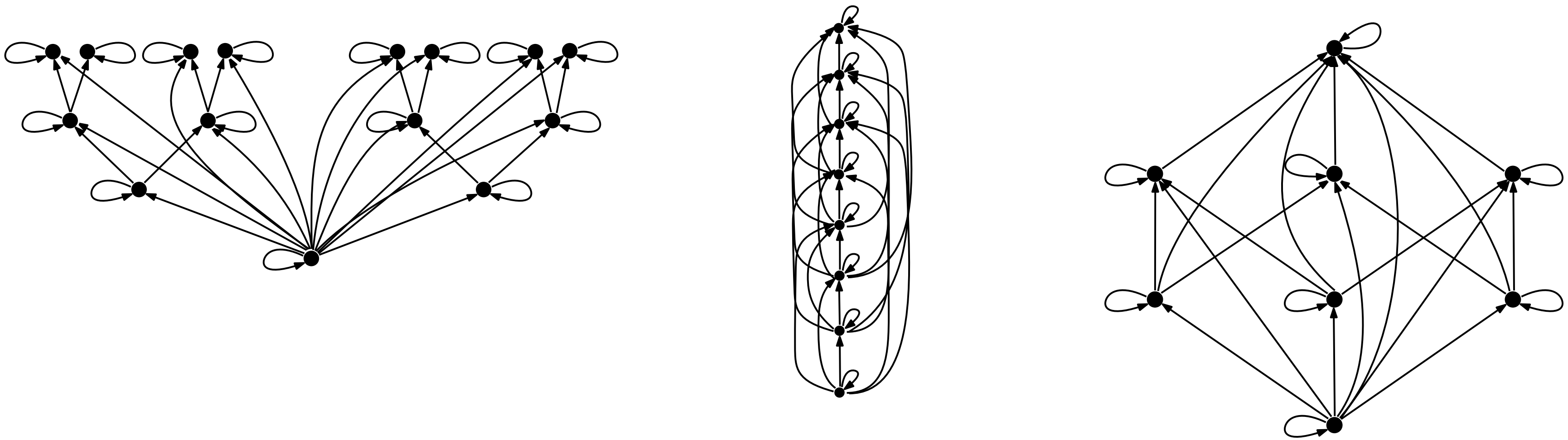

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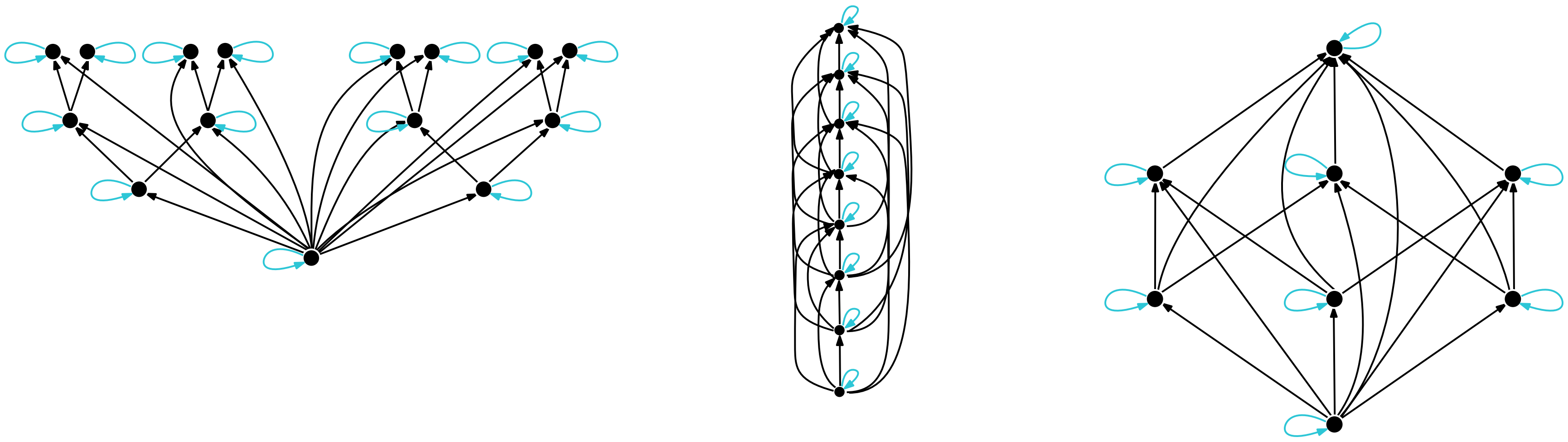

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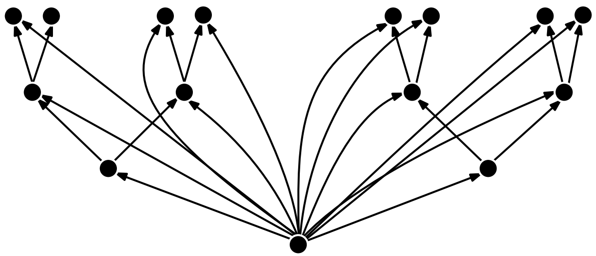

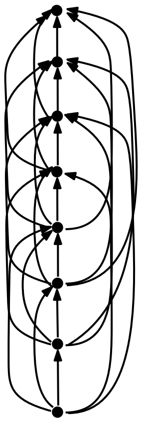

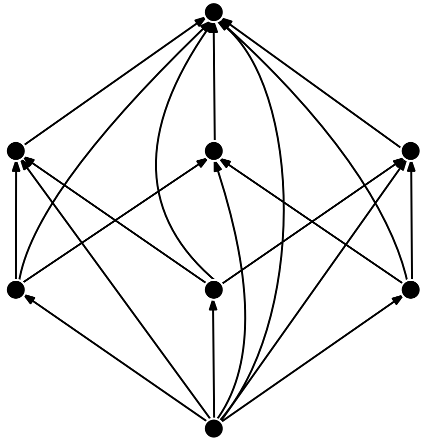

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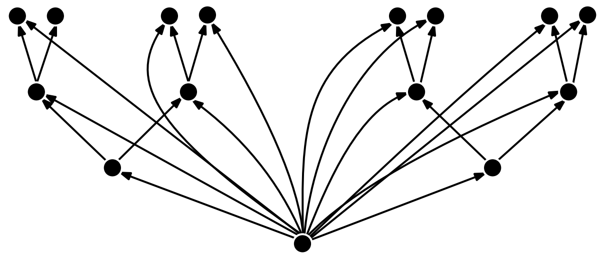
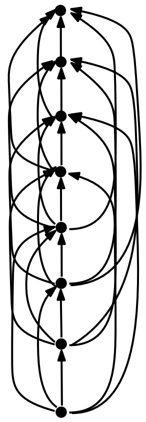
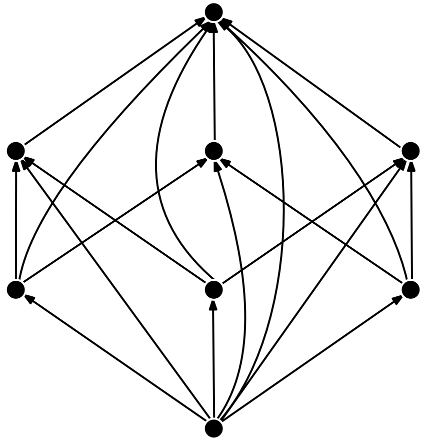

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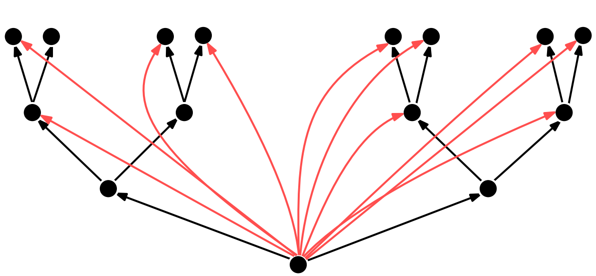

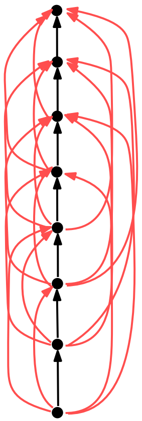

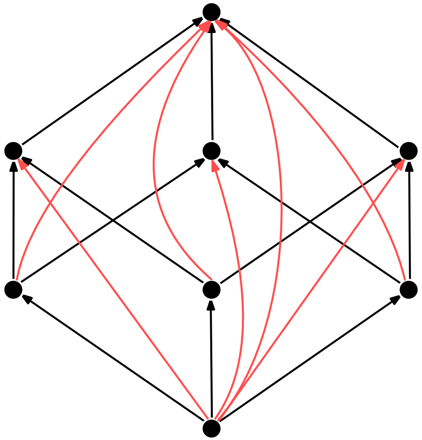

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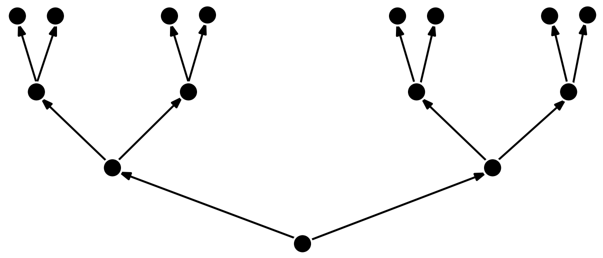

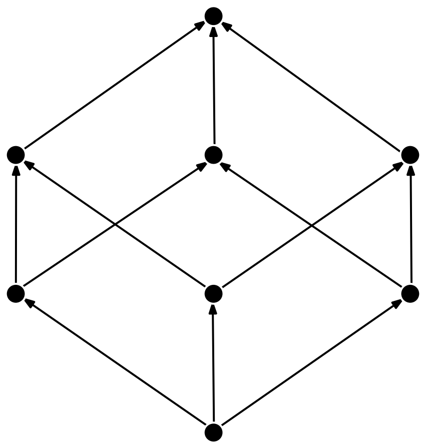

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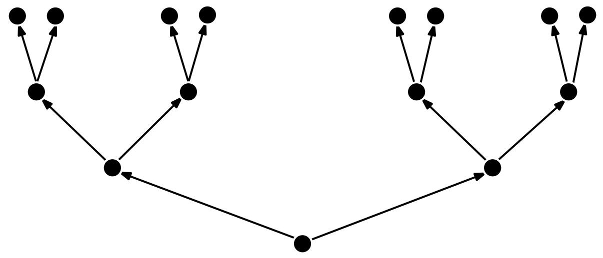

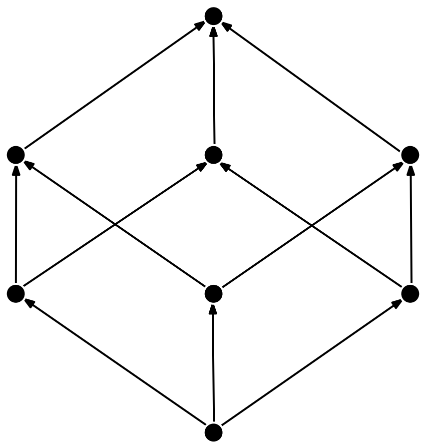

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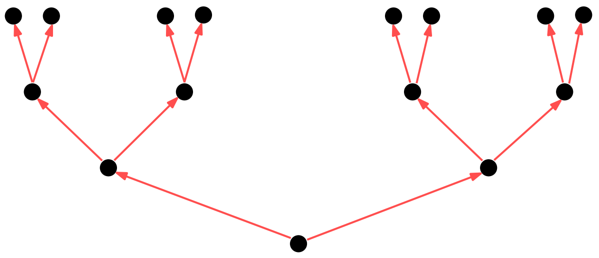

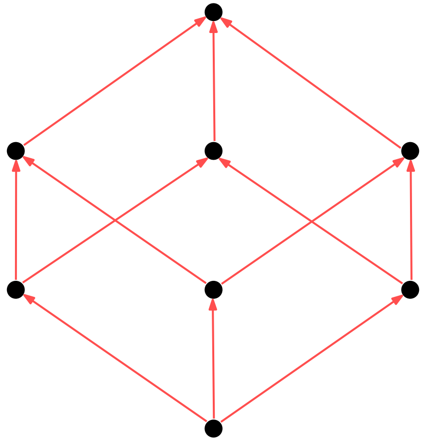

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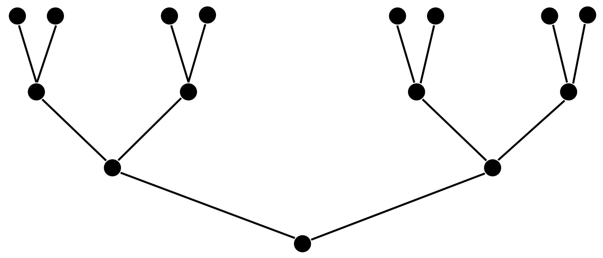
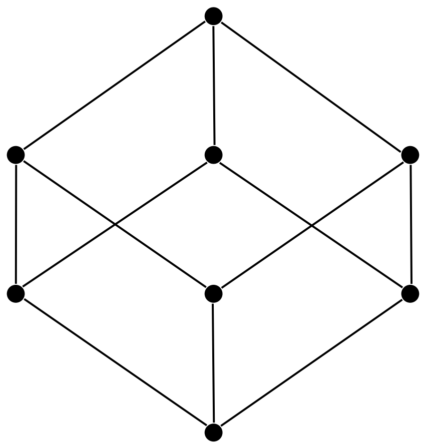
Drawing orders: Hasse diagrams





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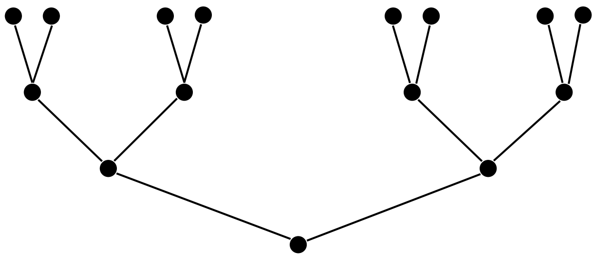

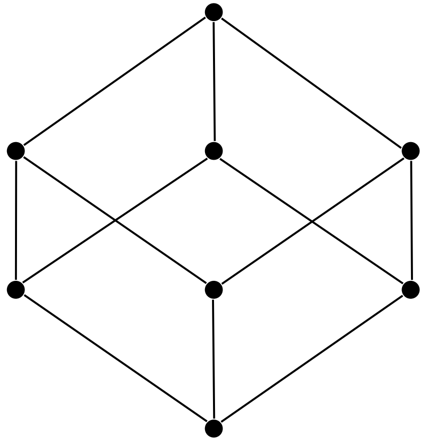

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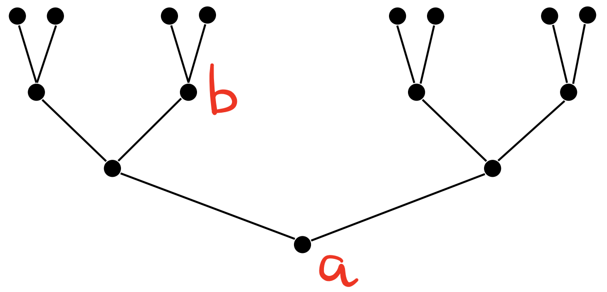

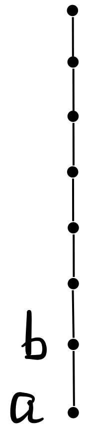

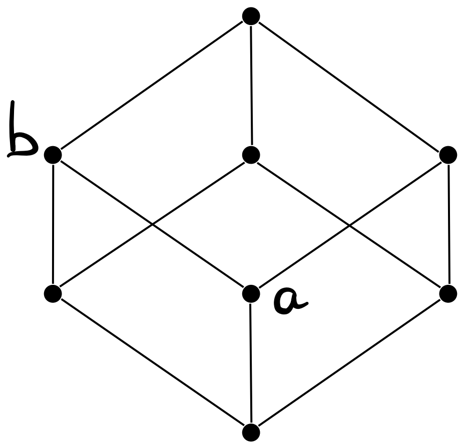

Question. How much information is actually necessary to "remember" the order we are drawing?

- It's reflexive, so we know the loops are there
- It's transitive, so some edges are implied by others
- The increasing direction is already shown by going up, so arrows are not necessary

These are the Hasse diagrams of the orders.

In general: consider an order $\leq$ on A and $a, b \in A$ distinct with $a \leq b$

We say b covers a if there is nothing between a and b $\quad \forall c \in A(a \leq c \leq b \Rightarrow c \in \{a, b\})$

Definition. If A is finite, then a Hasse diagram of $\leq$ is a drawing of $\leq_{\text{cover}} = \{(a, b) \in A^2 \mid b \text{ covers } a\}$, where if $a \leq_{\text{cover}} b$, then b is above a.

Drawing orders: Hasse diagrams

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For infinite orders, sometimes it is not possible to draw a Hasse diagram:
in $\mathbb{Q}$ with the usual ordering, does 1 cover 0?

1

1/2

1/4

0

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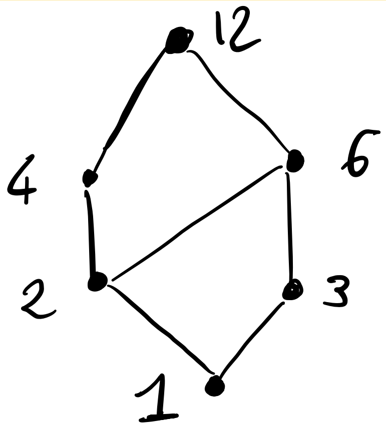
Let $A = \{n \in \mathbb{N} \mid n \text{ divides } 12\}$, and let $R = \{(n, m) \in A^2 \mid n \text{ divides } m\}$.
Draw the Hasse diagram of R.

$A = \{1, 2, 3, 4, 6, 12\}$



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Definition. A relation $R \subseteq A \times A$ is an order if it is reflexive, antisymmetric, and transitive.

Consider the relation R defined by $\{(\varphi, \psi) \mid \text{if } \varphi \text{ is true, then } \psi \text{ is true}\}$

- $(p, p \vee q) \in R$
- $((p \Rightarrow q) \wedge p, q) \in R$
- $(p \wedge q, q \wedge p) \in R$
- $(p, p \wedge q) \notin R$

Small variations: strict orders, quasi orders, linear orders

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$$p \wedge q \rightleftarrows q \wedge p$$

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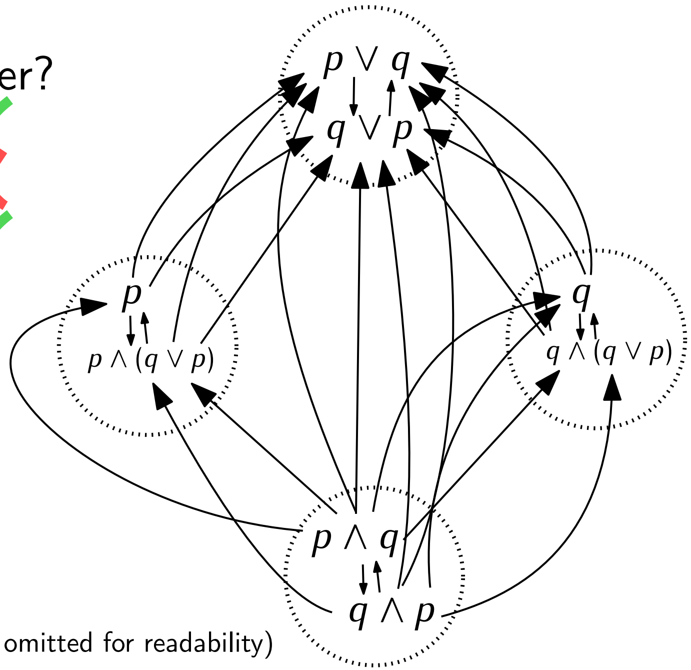

(loops omitted for readability)

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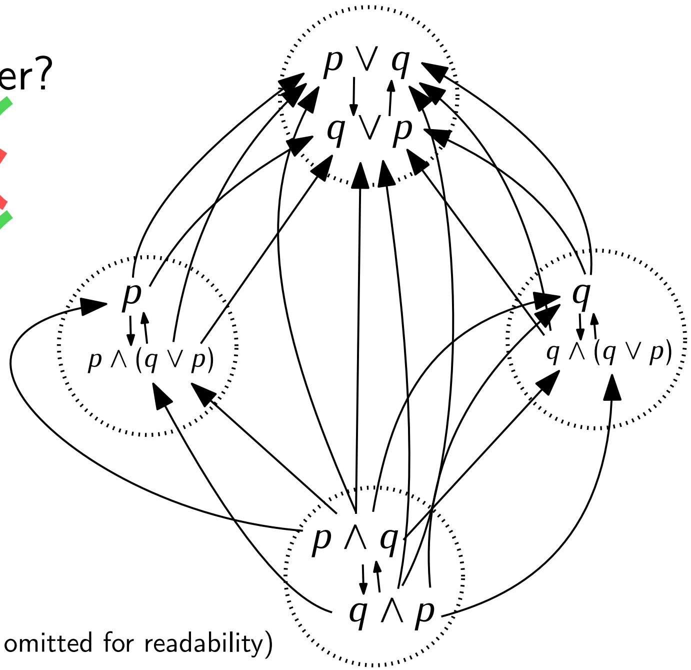

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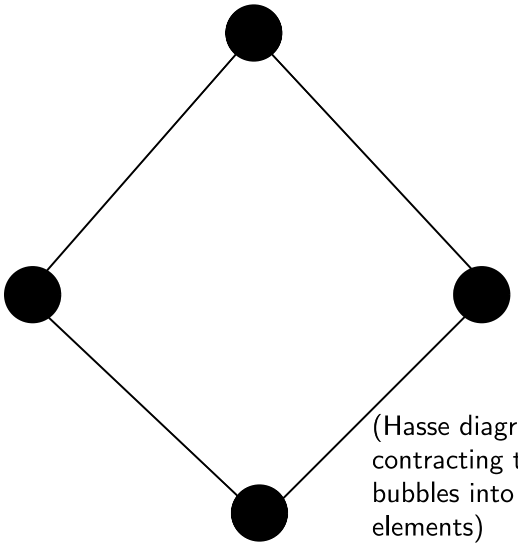
$\rightsquigarrow$



(Hasse diagram after contracting the dotted bubbles into single elements)

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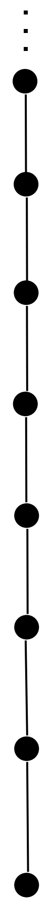

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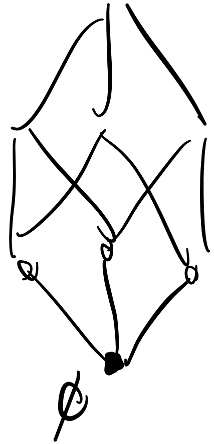

Which of the following orders are linear?

- $\leq$ on $\mathbb{Q}$
- $\preceq_{\text{prefix}}$ on binary strings
- $\leq$ on $\mathbb{R}$
- $\subseteq$ on $\mathcal{P}(A)$

Definition. Let $\leq$ be an order on A and $a \in A$.
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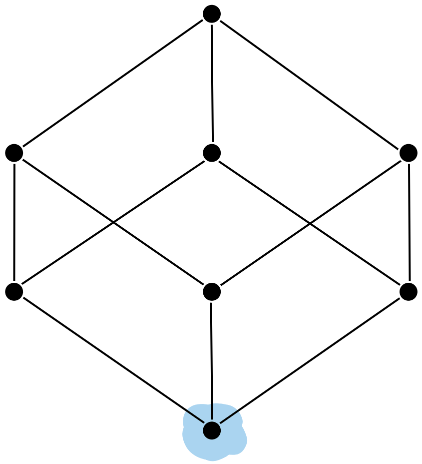

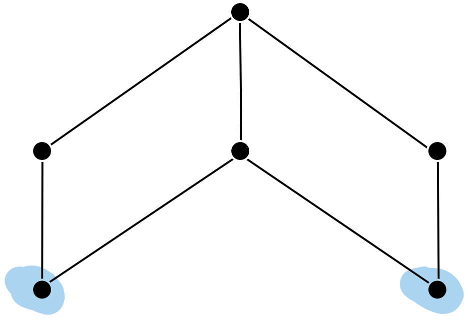

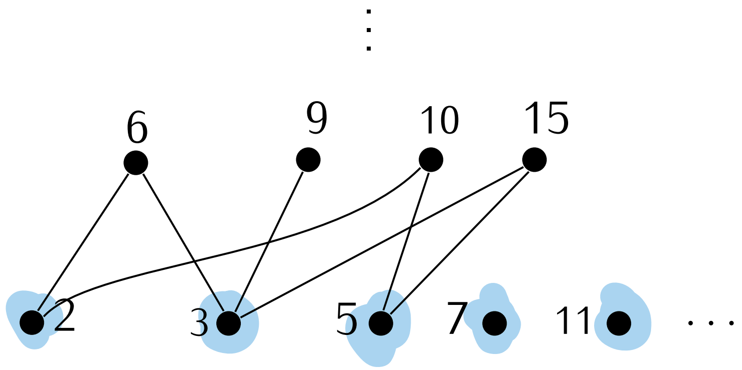

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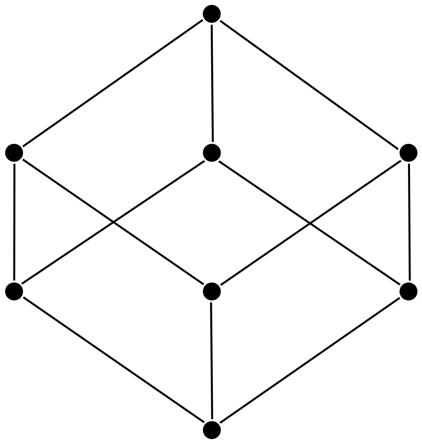

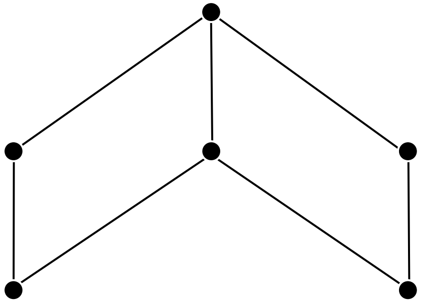

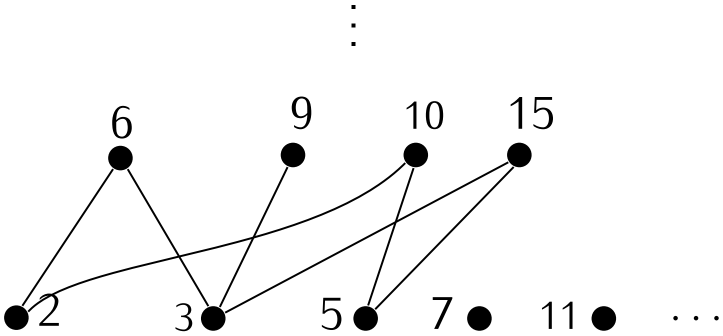

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Being able to formalize the order you care about allows you to use known algorithms/methods.

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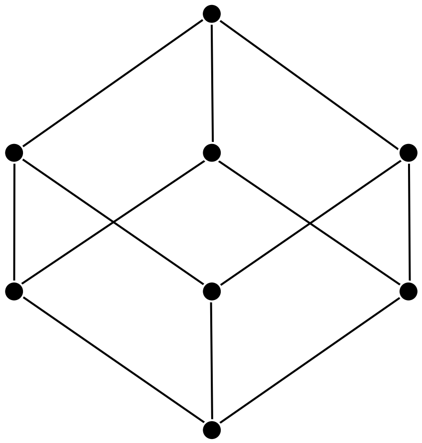

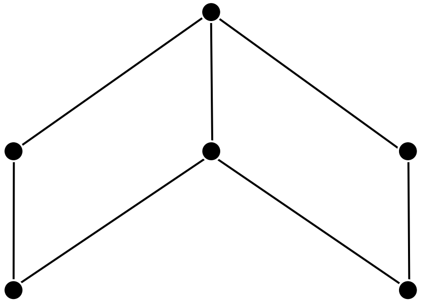

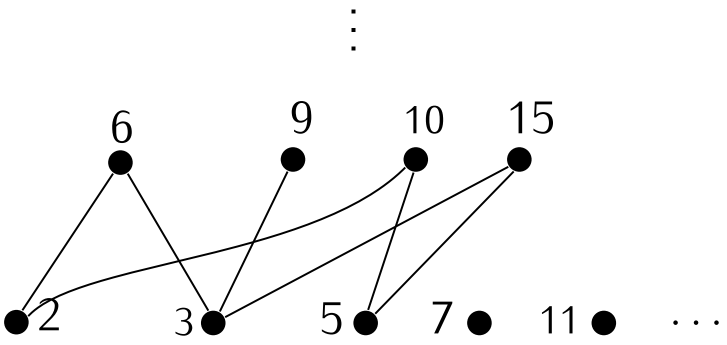

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Theorem. Every non-empty finite order has a minimal element.

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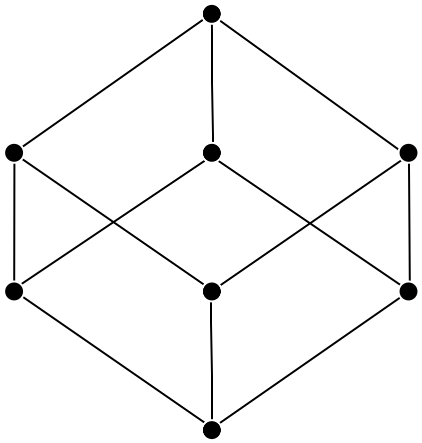
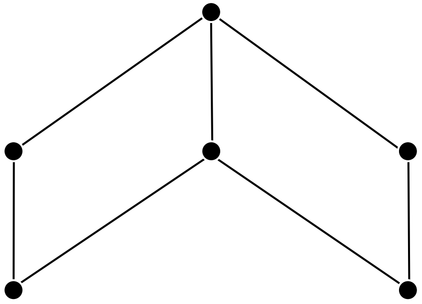
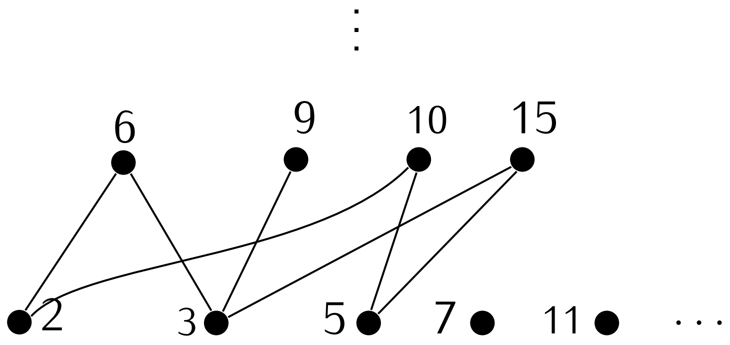

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Theorem. Every non-empty finite order has a minimal element.

Proof by contradiction!

Definition. Let $\leq$ be an order on A, $S \subseteq A$, and $a \in A$. We say:

- a is a lower bound of S if for all $s \in S$, we have $a \leq s$
- a is a greatest lower bound of S if it is a lower bound and for every lower bound b of S, we have $b \leq a$.

We write $a \wedge b$ for the greatest lower bound of $\{a, b\}$, if it exists.

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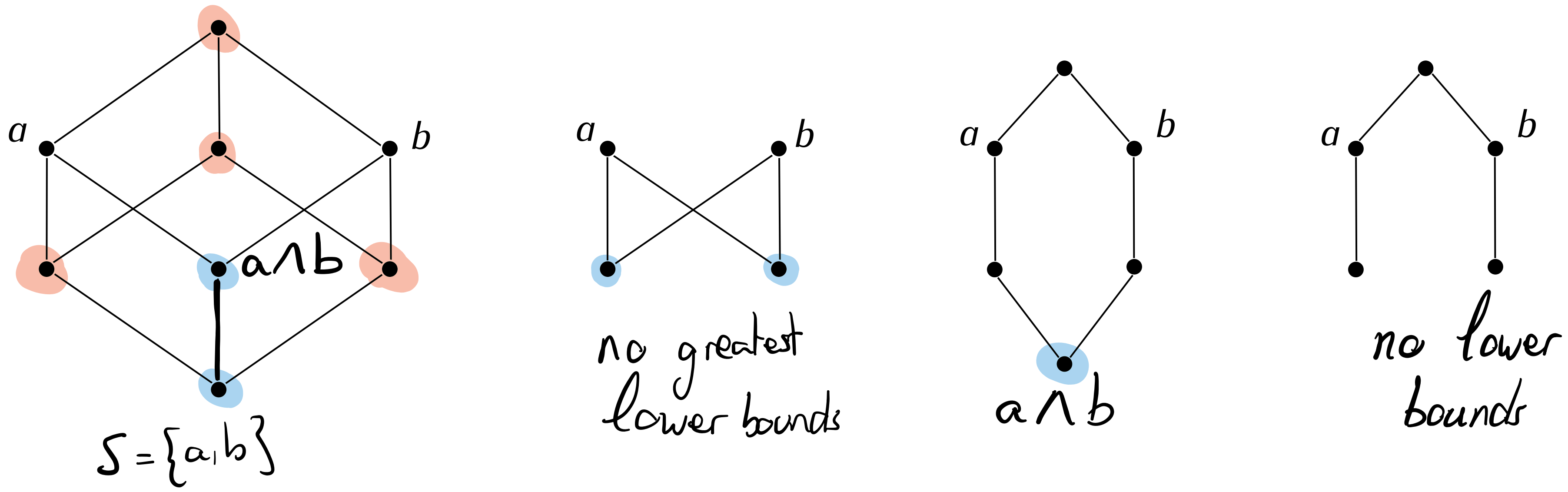

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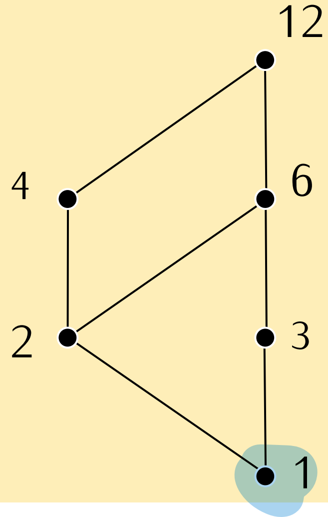
Let $A = \{n \in \mathbb{N} \mid n \text{ divides } 12\}$, and let $R = \{(n, m) \in A^2 \mid n \text{ divides } m\}$.
What is $4 \wedge 3$?



Lower/upper bounds

Definition. Let $\leq$ be an order on A, $S \subseteq A$, and $a \in A$. We say:

- a is an upper bound of S if for all $s \in S$, we have $s \leq a$
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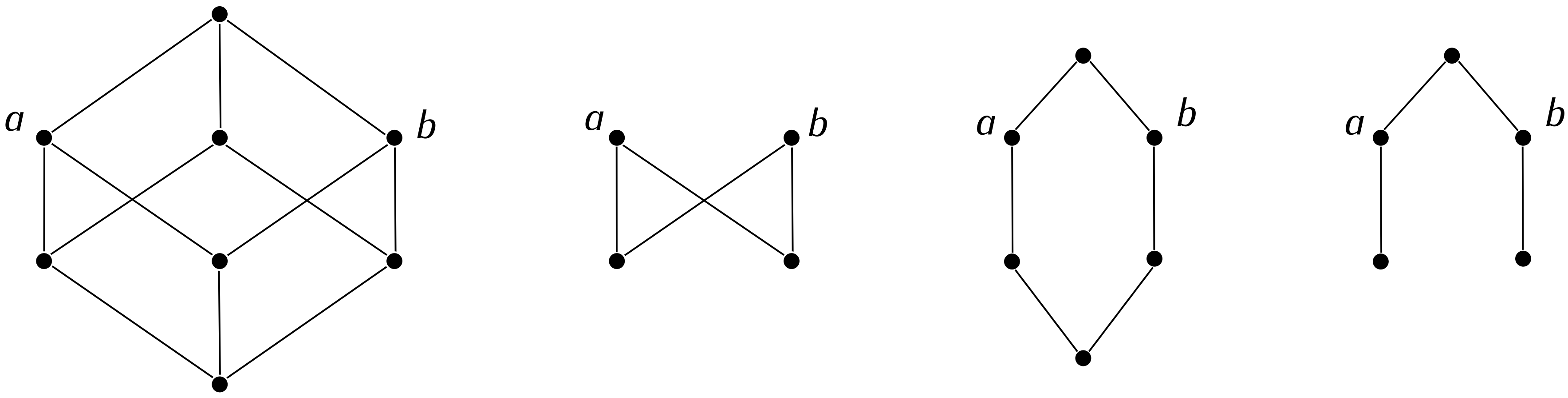

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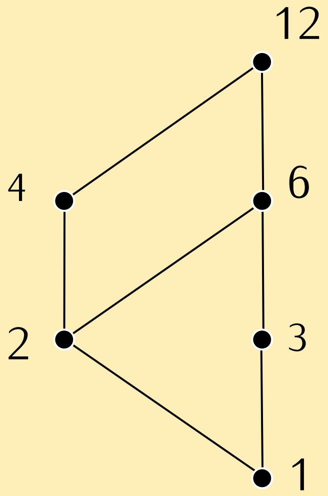
What is $2 \vee 3$?



Relations (part 2): what to know

- How to recognize an equivalence relation, an order
- Compute equivalence classes
- Compute $A/\sim$
- Draw a Hasse diagram
- Recognize minimal/maximal elements
- Recognize lower/upper bounds

	Reflexive	Antireflexive	Symmetric	Antisymmetric	Transitive
Equivalence relation	✓		✓		✓
Order	✓			✓	✓
Strict order		✓		✓	✓
Quasi-order	✓				✓