Prep Course Mathematics

Integration

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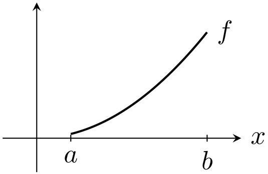
INSTITUTE OF MATHEMATICS

Inhalt

Integration

Motivation and integrability

For $f\colon [a,b] \to \mathbb{R}$: Determine area under f in $[a,b]$:



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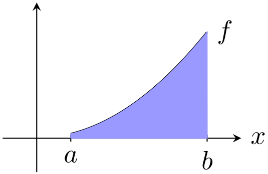

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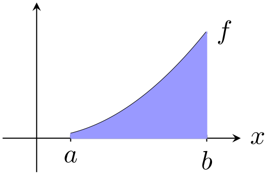
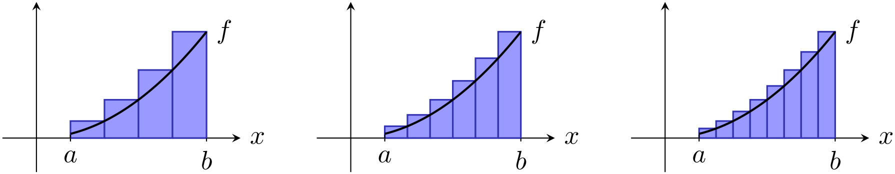


Approximate the area with vertical stripes:



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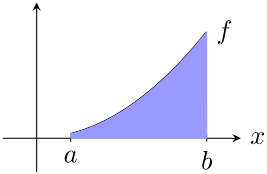

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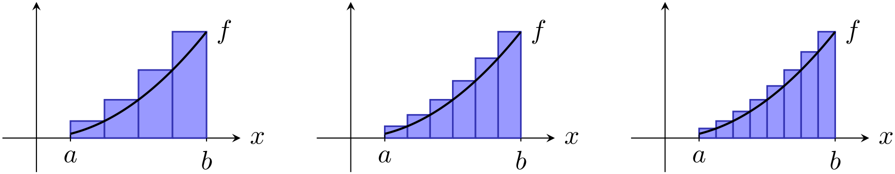


The narrower the stripes, the better the approximation should be.

⚠ There exist functions, where no area can be determined in this way.

- f integrable, if approximation of area possible.

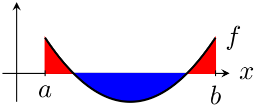
The definite integral

For $f\colon [a,b] \to \mathbb{R}$ integrable: We write the definite integral

$$\int\limits_a^b f(x)\,\mathrm{d}x$$

for signed area between f and x-axis in $[a,b]$, i.e.:

$$\int_a^b f(x)\,\mathrm{d}x = \text{area where } f \geqslant 0 - \text{area where } f < 0.$$



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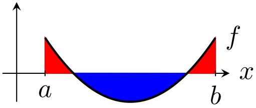

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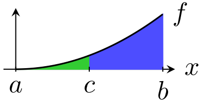
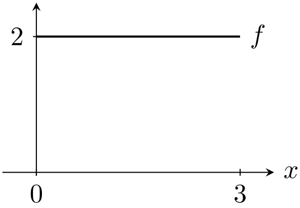
Calculation rules: for integrable f, g:

- constant factor rule: for $c \in \mathbb{R}$: $\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$
- sum rule: $\int_a^b f(x) \pm g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$
- partition: for $a < c < b$: $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$
- $\int_b^a f(x)\,\mathrm{d}x := -\int_a^b f(x)\,\mathrm{d}x$, and $\int_a^a f(x)\,\mathrm{d}x = 0$



Example

For function f given by $f(x) := 2$ for $x \in [0, 3]$:



Area between f and x-axis: rectangle of area $2 \cdot 3 = 6$.

Exercise

Compute $\int_0^4 (x-1)\,\mathrm{d}x$.

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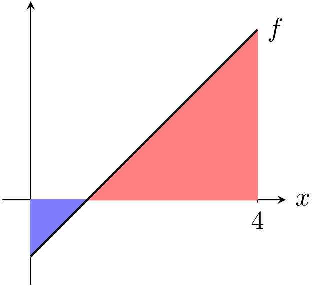

Signed area between f given by $f(x) := x - 1$ for $x \in [0,4]$ and x-axis:

- triangular area for $x \in [0,1]$: $\frac{1}{2}$,
- triangular area for $x \in [1,4]$: $\frac{9}{2}$.

Therefore,

$$\int_0^4 (x-1)\,\mathrm{d}x = \frac{9}{2} - \frac{1}{2} = 4.$$

Antiderivatives and indefinite integrals

For interval $D \subset \mathbb{R}$, function $f\colon D \to \mathbb{R}$:
$F\colon D \to \mathbb{R}$ antiderivative of f, if F differentiable and $F' = f$.

Antidifferentiation is the opposite operation to differentiation.

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- For f given by $f(x) := 2x$ for $x \in [0, 1]$:
  antiderivative: F given by $F(x) := x^2$ for $x \in [0, 1]$.

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Properties:

- If F antiderivative of f, then so is $F + c$ for all $c \in \mathbb{R}$.
- Antiderivatives are unique up to a constant.

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Properties:

- If F antiderivative of f, then so is $F + c$ for all $c \in \mathbb{R}$.
- Antiderivatives are unique up to a constant.

We write the indefinite integral

$$\int f(x)\,\mathrm{d}x$$

for all antiderivatives $F + c$ with $c \in \mathbb{R}$ of f.

Determination of antiderivatives

"Differentiation is a skill, integration is art."

$f(x)$	$f'(x)$
$c \quad (c \in \mathbb{R})$	0
$x^\alpha \quad (\alpha \neq 0)$	$\alpha x^{\alpha-1}$
$\log\lvert x\rvert$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
e^x	e^x
$\int f(x)\,\mathrm{d}x + c$	$f(x)$

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For integrable f, g:

Constant factor rule: for $c \in \mathbb{R}$

$$\int c f(x)\,\mathrm{d}x = c \int f(x)\,\mathrm{d}x$$

Sum rule:

$$\int f(x) \pm g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$$

Products: integration by parts

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

(Special) compositions: substitution

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) + c$$

Example

Indefinite integral for function f given by:

- $f(x) := 4x^3 - 6x^2 + x - 1$:

$$\int f(x)\,\mathrm{d}x = x^4 - 2x^3 + \frac{1}{2}x^2 - x + c \quad \text{with } c \in \mathbb{R}.$$

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- $f(x) := \frac{1}{x} + 3\mathrm{e}^x - 2\sin(x)$:

$$\int f(x)\,\mathrm{d}x = \log|x| + 3\mathrm{e}^x + 2\cos(x) + c \quad \text{with } c \in \mathbb{R}.$$

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- $f(x) := x\cos(x)$:
  We set $u(x) := x$, $v'(x) := \cos(x)$. Then $u'(x) = 1$, $v(x) = \sin(x)$, and

$$\int x\cos(x)\,\mathrm{d}x = x\sin(x) - \int \sin(x)\,\mathrm{d}x = x\sin(x) + \cos(x) + c \quad \text{with } c \in \mathbb{R}.$$

Exercise

Determine the indefinite integral for the function f given by:

- $f(x) := \sqrt{x} + \frac{1}{x}$ for $x > 0$:

- $f(x) := 3\cos(x) - 2\mathrm{e}^x$ for $x \in \mathbb{R}$:

Exercise

Determine the indefinite integral for the function f given by:

- $f(x) := \sqrt{x} + \frac{1}{x}$ for $x > 0$:
  We compute
  $$\int f(x)\,\mathrm{d}x = \frac{2}{3}\sqrt{x^3} + \log|x| + c \quad \text{with } c \in \mathbb{R}.$$

- $f(x) := 3\cos(x) - 2\mathrm{e}^x$ for $x \in \mathbb{R}$:
  We compute
  $$\int f(x)\,\mathrm{d}x = 3\sin(x) - 2\mathrm{e}^x + c \quad \text{with } c \in \mathbb{R}.$$

Relation between indefinite and definite integral

Fundamental theorem of calculus

For $f\colon [a,b] \to \mathbb{R}$ integrable:

…from indefinite to definite integrals:
For antiderivative F of f:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) =: F(x)\Big|_a^b.$$

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. . . from definite to indefinite integrals:
For f continuous: F given by

$$F(x) := \int_a^x f(t)\,\mathrm{d}t$$

is antiderivative of f.

Example

We calculate $\int_{-\pi}^{3\pi} \big(2\cos(x) - \sin(x) + 2\big)\, \mathrm{d}x$:
With $f(x) := 2\cos(x) - \sin(x) + 2$ we obtain that

$$F(x) := 2\sin(x) + \cos(x) + 2x$$

provides an antiderivative F of f. Hence

$$\begin{aligned}
\int_{-\pi}^{3\pi} \big(2\cos(x) - \sin(x) + 2\big)\, \mathrm{d}x &= \big(2\sin(x) + \cos(x) + 2x\big)\Big|_{-\pi}^{3\pi} \\
&= 2\sin(3\pi) + \cos(3\pi) + 6\pi - \big(2\sin(-\pi) + \cos(-\pi) - 2\pi\big) \\
&= 0 - 1 + 6\pi - \big(0 - 1 - 2\pi\big) \\
&= 8\pi.
\end{aligned}$$

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Calculate $\int_0^1 \left(\frac{1}{\sqrt{x}} - x \right) \mathrm{d}x$.

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With $f(x) := \frac{1}{\sqrt{x}} - x$ we obtain that

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provides an antiderivative F of f. Hence

$$\begin{aligned} \int_0^1 \left(\frac{1}{\sqrt{x}} - x \right) \mathrm{d}x &= \left(2\sqrt{x} - \frac{1}{2}x^2\right)\Big|_0^1 \\ &= 2 - \frac{1}{2} - 0 = \frac{3}{2}. \end{aligned}$$

Methods for integration

Integration by parts

For interval $D \subset \mathbb{R}$, $f, g\colon D \to \mathbb{R}$ differentiable:
Reminder: product rule of differentiation: $(fg)' = f'g + fg'$

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and for $D = [a, b]$:

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Typical applications:

- polynomial $\cdot$ $(\sin, \cos, \exp)$
  $g =$ polynomial, multiple application possible
- polynomial $\cdot$ $\log$
  $g = \log$, vanishes after application
- $(\sin, \cos, \exp) \cdot (\sin, \cos, \exp)$
  here also rearranging terms (trigonometric identities, Pythagoras) required

Example

We calculate $\int x^2 \mathrm{e}^x \, \mathrm{d}x$:

Setting $u(x) := x^2$ and $v'(x) := \mathrm{e}^x$ we obtain $u'(x) = 2x$ and $v(x) = \mathrm{e}^x$. Hence

$$\int x^2 \mathrm{e}^x \, \mathrm{d}x = x^2 \mathrm{e}^x - \int 2x \mathrm{e}^x \, \mathrm{d}x.$$

Now another integration by parts: setting $u(x) := 2x$ and $v'(x) := \mathrm{e}^x$ we obtain $u'(x) = 2$ and $v(x) = \mathrm{e}^x$, hence

$$\begin{aligned} \int 2x \mathrm{e}^x \, \mathrm{d}x &= 2x \mathrm{e}^x - \int 2 \mathrm{e}^x \, \mathrm{d}x \\ &= 2x \mathrm{e}^x - 2\mathrm{e}^x + c \quad \text{with } c \in \mathbb{R}. \end{aligned}$$

Thus,

$$\begin{aligned} \int x^2 \mathrm{e}^x \, \mathrm{d}x &= x^2 \mathrm{e}^x - (2x \mathrm{e}^x - 2\mathrm{e}^x + c) \\ &= x^2 \mathrm{e}^x - 2x \mathrm{e}^x + 2\mathrm{e}^x + \tilde{c} \\ &= (x^2 - 2x + 2)\mathrm{e}^x + \tilde{c} \quad \text{with } \tilde{c} \in \mathbb{R}. \end{aligned}$$

Exercise

Calculate $\int_0^\pi x\cos(x)\,\mathrm{d}x$.

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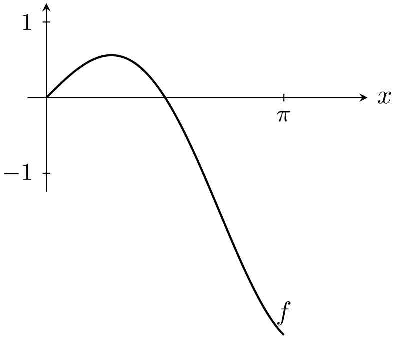
Setting $u(x) := x$ and $v'(x) := \cos(x)$ we obtain $u'(x) = 1$ and $v(x) = \sin(x)$. Hence

$$\begin{aligned}\int_0^\pi x\cos(x)\,\mathrm{d}x &= x\sin(x)|_0^\pi - \int_0^\pi \sin(x)\,\mathrm{d}x \\ &= 0 + \cos(x)|_0^\pi \\ &= -2.\end{aligned}$$



Substitution

For intervals D_f, D_g and $f\colon D_f \to \mathbb{R}$, $g\colon D_g \to D_f$ differentiable:
Reminder: chain rule for differentiation: $f(g)' = f'(g) \cdot g'$

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and for $D_g = [a, b]$

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Typical applications:

- $\int f(\alpha x + \beta)\,\mathrm{d}x$
  Set $g(x) := \alpha x + \beta$
- $\int f(\sin(x))\cos(x)\,\mathrm{d}x$
  Set $g(x) := \sin(x)$. Analogously: $\sin$ and $\cos$ interchanged
- $\int \frac{h'(x)}{h(x)}\,\mathrm{d}x$
  Set $f(x) := \frac{1}{x}$, $g(x) := h(x)$

Example

We calculate $\int_2^5 3\pi \sin(2\pi x - \pi)\,\mathrm{d}x$:

By substituting $t := 2\pi x - \pi$ and $\frac{\mathrm{d}t}{\mathrm{d}x} = 2\pi$ we have

$$\begin{aligned}
\int_2^5 3\pi \sin(2\pi x - \pi)\,\mathrm{d}x &= \int_{3\pi}^{9\pi} \frac{3}{2}\sin(t)\,\mathrm{d}t \\
&= -\frac{3}{2}\cos(t)\Big|_{3\pi}^{9\pi} \\
&= -\frac{3}{2}\big(\cos(9\pi) - \cos(3\pi)\big) \\
&= -\frac{3}{2}\big(-1 - (-1)\big) = 0.
\end{aligned}$$

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Calculate $\int_0^3 \frac{2x}{x^2+1}\,\mathrm{d}x$.

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By substituting $t := x^2 + 1$ and $\frac{\mathrm{d}t}{\mathrm{d}x} = 2x$ we have

$$\begin{aligned}\int_0^3 \frac{2x}{x^2+1}\,\mathrm{d}x &= \int_1^{10} \frac{1}{t}\,\mathrm{d}t \\ &= \log|t|\big|_1^{10} \\ &= \log 10.\end{aligned}$$