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Homework 2 for
Mathematics I (winter term 25/26)

Submit your solutions to **Problems 1 and 4** until Sunday, **November 9**, 11:59 pm at the latest, using **abGabi**. Only one of these problems will be chosen by us to be corrected and graded. We strongly suggest that you **submit in pairs**. Please state the names and matriculation numbers of both persons on your sumbissions and only submit once per group (the other person will still receive credit). Submission in larger groups is not permitted.

Problem 1 *Functions*

(a) Consider the function $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ given by

$$f(x) := \begin{cases} x^2 & \text{if } x \leq 2, \\ 8 - x & \text{if } x \geq 3. \end{cases}$$

Decide (with justification) whether f is (i) injective or (ii) surjective.

(b) Consider the sets

$$A := \{2|x - 1| : \ x \in \mathbb{Z} \cap [-2, 2]\} \ \text{ and}$$
$$B := \{y^2 : \ y \in \mathbb{N} \text{ und } y \leq 5\}.$$

If possible, give functions satisfying the following properties (e.g. by using a table of values) and otherwise justify why such functions do not exist:

(i) $g_1 : A \to B$ such that g_1 is surjective and not injective.

(ii) $g_2 : A \to B$ such that g_2 is injective and not surjective.

Problem 2 *Function*

(a) Give the compositions $f \circ g$ and $g \circ f$ of the functions $f : \mathbb{R} \to \mathbb{R}$ with $f : x \mapsto 3x$ and $g : \mathbb{R} \to \mathbb{R}$ with $g : x \mapsto x^2$.

(b) Describe the function $h : \mathbb{R} \to \mathbb{R}$ with $h : x \mapsto (x - 3)^2$ as the composition of two functions f and g.

(c) For which sets $X, Y \subset \mathbb{R}$ is the function $h : X \to Y$ with $h : x \mapsto (x-3)^2$ from part (b) bijective? Give the inverse function in this case.

Problem 3 *Induction*

Show by induction that all numbers of the form $a_n = 5^n - 1$ with $n \in \mathbb{N}$ are divisible by 4.

Problem 4 *Induction*

With an induction, prove that

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

holds for every positive integer n.

1. a)

function $f: \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$

$$f(x) := \begin{cases} x^2 & \text{if } x \leq 2 \\ 8-x & \text{if } x \geq 3 \end{cases}$$

$f(1) = 1^2 = 1$
$f(2) = 2^2 = 4$
$f(3) = 8-3 = 5$
$f(4) = 8-4 = 4$
$f(5) = 8-5 = 3$
$f(6) = 8-6 = 2$

Functional values: $W = \{1,2,3,4,5\}$

function f is not injective, due to f(2) and f(4) both having the produkt 4, as injective functions cannot have the same product twice.

function f is also not surjective due to the definition set of f and the functional values w not being the same.

1b)

$A := \{2|x-1| : x \in \mathbb{Z} \cap [-2,2]\}$
$B := \{y^2 : y \in \mathbb{N} \text{ und } y \leq 5\}$

interval of A $[-2,2]$

$x = -2 \to 2|-2-1| = 2|-3| = 2 \cdot 3 = 6$
$x = -1 \to 2|-1-1| = 2|-2| = 2 \cdot 2 = 4$
$x = 0 \to 2|0-1| = 2|-1| = 2 \cdot 1 = 2$
$x = 1 \to 2|1-1| = 2|0| = 2 \cdot 0 = 0$
$x = 2 \to 2|2-1| = 2|1| = 2 \cdot 1 = 2$

$A = \{0,2,4,6\}$

interval of B $[y \in \mathbb{N}\ y \leq 5] = \{1,2,3,4,5\}$

$y = 1 \to 1^2 = 1$
$y = 2 \to 2^2 = 4$
$y = 3 \to 3^2 = 9$
$y = 4 \to 4^2 = 16$
$y = 5 \to 5^2 = 25$

$B = \{1,4,9,16,25\}$

$g_1\ A \to B$: is surjective, but not injective ?

g1 is not possible due to the cardinality of A and B not being the same (A has a cardinality of 4 and B of 5).

example:

$g_2(0) = 1$ $g_2(4) = 9$
$g_2(2) = 4$ $g_2(6) = 16$

$g_2(A) = \{1,4,9,16\}$

It is not surjective due to g2 being a part of B where 25 E B not part of it is. It is unjective due to every element of A being one unique element of B.

P4

Thesis: $\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$

1. BC for $n=1$:

1.1. $\frac{1}{2} = 2 - \frac{3}{2} \Leftrightarrow \frac{1}{2} = \frac{1}{2}$ ✓ BC holds for $n=1$

1.2. Induction statement: $\sum_{k=1}^{n+1} \frac{k}{2^k} = 2 - \frac{n+3}{2^{n+1}}$

1.3. $LS = \sum_{k=1}^{n} \frac{k}{2^k} + \frac{n+3}{2^{n+1}} \overset{IH}{=} 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}}$

$= 2 - \frac{2n+4}{2^{n+1}} + \frac{n+1}{2^{n+1}}$

$= 2 - \frac{n+3}{2^{n+1}}$

Q.E.D.