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Homework 4 for
Mathematics I (winter term 25/26)

Submit your solutions to **Problems 1 and 2** until Sunday, **November 23**, 11:59 pm at the latest, using **abGabi**. Only one of these problems will be chosen by us to be corrected and graded. We strongly suggest that you **submit in pairs**. Please state the names and matriculation numbers of both persons on your sumbissions and only submit once per group (the other person will still receive credit). Submission in larger groups is not permitted.

Problem 1 *Sequences*

Check the sequences $(x_n)_{n\in\mathbb{N}}$ for convergence and determine the limit. For convergent sequences determine the limit. In case of divergence, check whether $x_n \to +\infty$ or $x_n \to -\infty$.

(a) $x_n = \sqrt{9n^4+1} - (3n^2+1)$

(b) $x_n = (-1)^n \sqrt[n]{n^2 2^n}$

(c) $x_n = \dfrac{\left(p^{\frac{n-2}{n}} \cdot p^{\frac{2}{n}}\right)^{n+1} \cdot q^{-n+1}}{q^{\frac{1}{n}} \cdot (q^{n-1})^{\frac{1}{n}} \cdot p^2}$ with $p, q \in \mathbb{R}$ and $0 < p < q$

Problem 2 *Sequences*

Let $x_1 := 1$ and $x_{n+1} := \sqrt{6 + x_n}$ for $n \in \mathbb{N}$. Investigate whether (x_n) is monotone, bounded, or convergent, and—provided it is convergent—determine the limit $\lim_{n\to\infty} x_n$.

Problem 3 *Cross product and inner product*

Let $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the cross product for $\mathbb{R}^3$ and let $\langle\cdot,\cdot\rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the standard inner product of $\mathbb{R}^3$. Show that for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ one has

(a) $\langle \mathbf{a}, \mathbf{a} \times \mathbf{b} \rangle = 0$,

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\langle \mathbf{a}, \mathbf{c} \rangle - \mathbf{c}\langle \mathbf{a}, \mathbf{b} \rangle$.

Problem 4 *Vectors*

(a) Find a vector $\mathbf{v}_1 \in \mathbb{R}^2$ which is orthogonal to

$$\mathbf{u}_1 := \begin{pmatrix} -5 \\ 12 \end{pmatrix} \in \mathbb{R}^2$$

and which satisfies $\|\mathbf{v}_1\|_2 = 1$.

(b) Consider the vector

$$\mathbf{u}_2 := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \in \mathbb{R}^3.$$

Find two vectors $\mathbf{v}_2, \mathbf{w}_2 \in \mathbb{R}^3 \setminus \{\mathbf{o}\}$ such that any two vectors from $\mathbf{u}_2, \mathbf{v}_2, \mathbf{w}_2$ are orthogonal.

(c) Consider a triangle whose edges are given by the vectors

$$\mathbf{u}_3 := \begin{pmatrix} -3 \\ 6 \\ 6 \end{pmatrix}, \quad \mathbf{v}_3 := \begin{pmatrix} 8 \\ -4 \\ 8 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 - \mathbf{u}_3.$$

Determine the circumference and the area of the triangle.

Problem 2 *Sequences*

Let $x_1 := 1$ and $x_{n+1} := \sqrt{6 + x_n}$ for $n \in \mathbb{N}$. Investigate whether (x_n) is monotone, bounded, or convergent, and—provided it is convergent—determine the limit $\lim_{n\to\infty} x_n$.

Monotonicity:
At least monotonically increasing ($\sqrt{6+x_n}$ won't drop below x_n).
Not analyzing strictness.

Boundedness:
Has lower bound 1 ($\sqrt{6+x_n}$ won't drop below x_1).
From convergence: $L = (1, 3)$

Convergence:
Assuming limit exists:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$$

let's call these limits l

$$l = \sqrt{6+l} \quad |\,()^2$$

$$l^2 = 6 + l \quad |-(6+l)$$

$$l^2 - l - 6 = 0$$

ABC-Formula: $l_{1/2} = \dfrac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot (-6)}}{2}$

$$= \frac{1 \pm 5}{2} \to l_1 = \frac{6}{2} = 3$$

$$\left(l_2 = -\frac{4}{2} = -2\right)$$

↳ we've determined a lower limit 1, so this can be disregarded

Sequence converges to $\lim_{n\to\infty} x_n = 3$