LU decomposition algorithm

LU decomposition of a matrix is the factorization of a given square matrix into two triangular matrices, one upper triangular matrix and one lower triangular matrix, such that the product of these two matrices gives the original matrix. (where ‘LU’ stands for ‘lower upper’)

A square matrix A can be decomposed into two square matrices L and U such that A = L U where U is an upper triangular matrix formed as a result of applying the Gauss Elimination Method on A, and L is a lower triangular matrix with diagonal elements being equal to 1. Doolittle’s method provides an alternative way to factor A into an LU decomposition without going through the hassle of Gaussian Elimination.

$$\begin{bmatrix} A00 & A01 & A02 \\ A10 & A11 & A12 \\ A20 & A21 & A22 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ L10 & 1 & 0 \\ L20 & L21 & 1 \end{bmatrix}}^{\text{Lower Triangular}} \underbrace{\begin{bmatrix} U00 & U01 & U02 \\ 0 & U11 & U12 \\ 0 & 0 & U22 \end{bmatrix}}_{\text{Upper Triangular}}$$

It is always possible to factor a square matrix into a lower triangular matrix and an upper triangular matrix. That is, [A] = [L][U]

For a general n×n matrix A, we assume that an LU decomposition exists, and write the form of L and U explicitly. We then systematically solve for the entries in L and U from the equations that result from the multiplications necessary for A=LU.

Terms of U matrix are given by:

$$\forall j$$
$$i = 0 \rightarrow U_{ij} = A_{ij}$$
$$i > 0 \rightarrow U_{ij} = A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj}$$

And the terms for L matrix:

$$\forall i$$
$$j = 0 \rightarrow L_{ij} = \frac{A_{ij}}{U_{jj}}$$
$$j > 0 \rightarrow L_{ij} = \frac{A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj}}{U_{jj}}$$

Example:

$$\begin{bmatrix} 2 & -1 & -2 \\ -4 & 6 & 3 \\ -4 & -2 & 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}}_{\text{Lower Triangular}} \underbrace{\begin{bmatrix} 2 & -1 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}}_{\text{Upper Triangular}}$$

det(A) = det(L) * det(U)

The determinant of a triangle matrix is the product of its diagonal

Solve systems of linear equations:

We know from last semester that we can write system linear equations as a matrix equation like

A*x = b

Definition 2.17. Given a linear system $Ax = b$ with coefficient matrix A and RHS b, then we denote the **set of solutions** by

$$L(A, b) := \{x \in \mathbb{R}^n : Ax = b\}.$$

Therefore we can solve the system using LU Decomposition:

Let A = L*U and substitute into A*x = b.

Solve L*U*x = b for x to solve the system.

Let U*x = y

L*y = b and U*x = y

First solve L*y = b for y and then solve U*x for x

Sources:
https://www.youtube.com/watch?v=m3EojSAgIao
https://www.geeksforgeeks.org/doolittle-algorithm-lu-decomposition/