Proof that the language $L = \{a^j b^j c a^j b^j \mid j \in \mathbb{N}\}$ is not context-free:

- Let $n \in \mathbb{N}$ be an arbitrary fixed natural number.
- Select $z = a^n b^n c a^n b^n$ from the language L so that $|z| = 4n + 1 \geq n$.
- Consider any division of the word z on 5 substrings $u, v, w, x, y \in \Sigma^*$, for which $z = uvwxy$, $|vwx| \leq n$ a $vx \neq \varepsilon$. For any such division, we consider the following cases based on which substring contains c:

  Letter c is in substring y (so $vx = a^k b^l$, where $k + l \geq 1$).

  Let $i = 0$, then $uv^iwx^iy = a^{n-k}b^{n-l}ca^nb^n$ and since the pumped part is non-empty, we shortened the part of the word before letter c, and so $uv^iwx^iy \notin L$.

  Letter c is in substring v or x. Let $i = 0$, then uv^iwx^iy does not include c. Then, it is not of a form $a^jb^jca^jb^j$, and so $uv^iwx^iy \notin L$.

  Letter c is in substring w (then $vx = b^k a^l$, where $k + l \geq 1$).

  Let $i = 0$, then $uv^iwx^iy =$

  Letter c is in substring u (then $vx = a^k b^l$, where $k + l \geq 1$).

To conclude, for every natural number n we found a word z from the language L longer than n, so that for any its division on five substrings u, v, w, x, y that fulfill the conditions of Pumping lemma there exists a non-negative integer i, such that uv^iwx^iy is not in L. Therefore, based on the Pumping lemma for context-free languages, L is not context-free.

Proof that the language $L = \{a^j b^j c a^j b^j \mid j \in \mathbb{N}\}$ is not context-free:

- Let $n \in \mathbb{N}$ be an arbitrary fixed natural number.
- Select a word $z = a^{\lceil \frac{n}{2} \rceil} b^{\lceil \frac{n}{2} \rceil} c a^{\lceil \frac{n}{2} \rceil} b^{\lceil \frac{n}{2} \rceil}$ from the language L, so that z is longer than n.
- Consider any division of the word z on 5 substrings $u, v, w, x, y \in \Sigma^*$, for which $z = uvwxy$, $|vwx| \leq n$ a $vx \neq \varepsilon$:

To conclude, for every natural number n we found a word z from the language L longer than n, so that for any its division on five substrings u, v, w, x, y that fulfill the conditions of Pumping lemma there exists a non-negative integer i, such that uv^iwx^iy is not in L. Therefore, based on the Pumping lemma for context-free languages, L is not context-free.

Proof that the language $L = \{ucv \mid u, v \in \{a,b\}^*, \#_a(u) = \#_b(v) \text{ a } \#_b(u) = \#_a(v)\}$ is not context-free:

- Let $n \in \mathbb{N}$ be an arbitrary fixed natural number.
- Select a word $z =$
- Consider any division of the word z on 5 substrings $u, v, w, x, y \in \Sigma^*$, for which $z = uvwxy$, $|vwx| \leq n$ a $vx \neq \varepsilon$:

To conclude, for every natural number n we found a word z from the language L longer than n, so that for any its division on five substrings u, v, w, x, y that fulfill the conditions of Pumping lemma there exists a non-negative integer i, such that uv^iwx^iy is not in L. Therefore, based on the Pumping lemma for context-free languages, L is not context-free.

Proof that the language $L = \{a^j b^k c^l \mid j < k < l\}$ is not context-free: