14. Nonzero initial velocity. Find the deflection $u(x, t)$ of the string of length $L = \pi$ and $c^2 = 1$ for zero initial displacement and "triangular" initial velocity $u_t(x, 0) = 0.01x$ if $0 \leqq x \leqq \frac{1}{2}\pi$, $u_t(x, 0) = 0.01(\pi - x)$ if $\frac{1}{2}\pi \leqq x \leqq \pi$. (Initial conditions with $u_t(x, 0) \neq 0$ are hard to realize experimentally.)

Vi har her funktionen u_t

$$u_t(x,0) = \begin{cases} 0{,}01x, & 0 \leq x \leq \frac{1}{2}\pi \\ 0{,}01(\pi - x), & \frac{1}{2}\pi \leq x \leq \pi \end{cases}$$

Eftersom at strengen ikke har noget udsving til t=0, men bare en hastighed kan vi finde u(x, t) som

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi}{L}x\right) \cdot \sin\left(\frac{n\pi c}{L}t\right)$$

Her kan vi finde B_n som:

$$B_n = \frac{2}{cn\pi} \cdot \int_0^L g(x) \cdot \sin\left(\frac{n\pi}{L}x\right) dx$$

Her er g(x) start hastigheden på snoren

$$B_n = \frac{2}{cn\pi} \cdot \left(\int_0^{\frac{1}{2}\pi} 0{,}01x \cdot \sin\left(\frac{n\pi}{L}x\right) dx + \int_{\frac{1}{2}\pi}^{\pi} 0{,}01 \cdot (\pi - x) \cdot \sin\left(\frac{n\pi}{L}x\right) dx \right)$$

$$B_n = \frac{2}{100\,cn\pi} \cdot \left(\int_0^{\frac{1}{2}\pi} x \cdot \sin\left(\frac{n\pi}{L}x\right) dx + \int_{\frac{1}{2}\pi}^{\pi} (\pi - x) \cdot \sin\left(\frac{n\pi}{L}x\right) dx \right)$$

Vi løser her integrallet:

$$\int x \cdot \sin(kx)\,dx = \frac{1}{k}\left(-x \cdot \cos(kx) + \int \cos(kx)\,dx\right)$$

$$= \frac{1}{k}\cdot\left(\frac{1}{k}\cdot \sin(kx) - x \cdot \cos(kx)\right)$$

$$= \frac{1}{k^2}\cdot\left(\sin(kx) - k \cdot x \cdot \cos(kx)\right)$$

Dette kan vi så indsætte ind i vores integral

$$\boxed{k = \frac{n\pi}{L}}$$

$$b_n = \frac{1}{50 c n \pi} \cdot \left(\frac{1}{k^2}\cdot\left(\sin(kx) - k \cdot x \cdot \cos(kx)\right)\Big|_0^{\frac{1}{2}\pi} - \frac{\pi}{k}\cdot \cos(kx)\Big|_{\frac{1}{2}\pi}^{\pi} - \frac{1}{k^2}\cdot\left(\sin(kx) - k \cdot x \cdot \cos(kx)\right)\Big|_{\frac{1}{2}\pi}^{\pi}\right)$$

$$b_n = \frac{1}{50 c n \pi k^2} \cdot \left(\sin(kx) - kx \cos(kx)\Big|_0^{\frac{1}{2}\pi} - \pi k \cdot \cos(kx)\Big|_{\frac{1}{2}\pi}^{\pi} - \left(\sin(kx) - kx \cdot \cos(kx)\Big|_{\frac{1}{2}\pi}^{\pi}\right)\right)$$

$$b_n = \frac{1}{50 c n \pi k^2} \cdot \left(\sin\left(\frac{k\pi}{2}\right) - \frac{k\pi}{2} \cdot \cos\left(\frac{k\pi}{2}\right) - \pi k \cdot \cos(\pi k) + \pi k \cdot \cos\left(\frac{k\pi}{2}\right) - \sin(k\pi) + k\pi \cdot \cos(k\pi) + \sin\left(\frac{k\pi}{2}\right) - \frac{k\pi}{2}\cdot \cos\left(\frac{k\pi}{2}\right)\right)$$

$$B_n = \frac{1}{50cn\pi k^2} \cdot \left(2 \cdot \sin\left(\frac{k\pi}{2}\right) - \sin(k\pi)\right)$$

Her kan vi så indsætte k igen $\boxed{k = \frac{n\pi}{L}}$

$$B_n = \frac{1}{50cn\pi \cdot \left(\frac{n\pi}{L}\right)^2} \cdot \left(2 \cdot \sin\left(\frac{n\pi^2}{2L}\right) - \sin\left(\frac{n\pi^2}{L}\right)\right)$$

Vi kan her indsætte værdierne for c og L

$$B_n = \frac{1}{50 n^3 \pi} \cdot \left(2 \cdot \sin\left(\frac{n\pi}{2}\right) - \sin(\pi n)\right)$$

Eftersom at n er et hel tal og sin(pi)=sin(2pi)=0 kan vi skrive

$$B_n = \frac{2}{50 n^3 \pi} \cdot \sin\left(n\frac{\pi}{2}\right)$$

Vi kan nu opskrive vores løsning

$$u(x,t) = \frac{1}{25\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{n^3} \cdot \sin\left(n\frac{\pi}{2}\right) \sin(nx) \cdot \sin(0{,}01\, n \cdot t)$$

$$= \frac{1}{25\pi} \cdot \left(\sin(x) \cdot \sin(0{,}01 \cdot t) - \frac{1}{27} \cdot \sin(3x) \cdot \sin(0{,}03 \cdot t) \ldots\right.$$