

14. **Nonzero initial velocity.** Find the deflection $u(x, t)$ of the string of length $L = \pi$ and $c^2 = 1$ for zero initial displacement and "triangular" initial velocity $u_t(x, 0) = 0.01x$ if $0 \leq x \leq \frac{1}{2}\pi$, $u_t(x, 0) = 0.01(\pi - x)$ if $\frac{1}{2}\pi \leq x \leq \pi$. (Initial conditions with $u_t(x, 0) \neq 0$ are hard to realize experimentally.)

Vi har her funktionen u_t

$$u_t(x, 0) = \begin{cases} 0.01x, & 0 \leq x \leq \frac{1}{2}\pi \\ 0.01(\pi - x), & \frac{1}{2}\pi \leq x \leq \pi \end{cases}$$

Eftersom at strengen ikke har noget udsving til $t=0$, men bare en hastighed kan vi finde $u(x, t)$ som

$$u(x, t) = \sum_{n=1}^{\infty} B_n \cdot \sin\left(\frac{n\pi}{L}x\right) \cdot \sin\left(\frac{n\pi c}{L}t\right)$$

Her kan vi finde B_n som:

$$B_n = \frac{2}{cn\pi} \cdot \int_0^L g(x) \cdot \sin\left(\frac{n\pi}{L}x\right) dx$$

Her er $g(x)$ start hastigheden på snoren

$$B_n = \frac{2}{cn\pi} \cdot \left(\int_0^{\frac{1}{2}\pi} 0.01x \cdot \sin\left(\frac{n\pi}{L}x\right) dx + \int_{\frac{1}{2}\pi}^{\pi} 0.01(\pi - x) \cdot \sin\left(\frac{n\pi}{L}x\right) dx \right)$$

$$B_n = \frac{2}{100cn\pi} \cdot \left(\int_0^{\frac{1}{2}\pi} x \cdot \sin\left(\frac{n\pi}{L}x\right) dx + \int_{\frac{1}{2}\pi}^{\pi} (\pi - x) \cdot \sin\left(\frac{n\pi}{L}x\right) dx \right)$$

Vi løser her integrallet:

$$\int x \cdot \sin(kx) dx = \frac{1}{k} \left(-x \cdot \cos(kx) + \int \cos(kx) dx \right)$$

$$= \frac{1}{k} \cdot \left(\frac{1}{k} \cdot \sin(kx) - x \cdot \cos(kx) \right)$$

$$= \frac{1}{k^2} \cdot (\sin(kx) - k \cdot x \cdot \cos(kx))$$

Dette kan vi så indsætte ind i vores integral

$$k = \frac{n\pi}{L}$$

$$B_n = \frac{1}{50 \text{ cm } \pi} \cdot \left(\frac{1}{k^2} \cdot (\sin(kx) - k \cdot x \cdot \cos(kx)) \right) \Big|_0^{\frac{1}{2}\pi} - \frac{\pi}{k} \cdot \cos(kx) \Big|_{\frac{1}{2}\pi}^{\pi} - \frac{1}{k^2} \cdot (\sin(kx) - k \cdot x \cdot \cos(kx)) \Big|_{\frac{1}{2}\pi}^{\pi}$$

$$B_n = \frac{1}{50 \text{ cm } \pi k^2} \cdot \left(\sin(kx) - kx \cos(kx) \right) \Big|_0^{\frac{1}{2}\pi} - \pi k \cdot \cos(kx) \Big|_{\frac{1}{2}\pi}^{\pi} - \left(\sin(kx) - kx \cdot \cos(kx) \right) \Big|_{\frac{1}{2}\pi}^{\pi}$$

$$B_n = \frac{1}{50 \text{ cm } \pi k^2} \cdot \left(\sin\left(\frac{k\pi}{2}\right) - \frac{k\pi}{2} \cdot \cos\left(\frac{k\pi}{2}\right) \right)$$

$$- \pi k \cdot \cos(\pi k) + \pi k \cdot \cos\left(\frac{k\pi}{2}\right) - \sin(k\pi) + k\pi \cdot \cos(k\pi)$$

$$+ \sin\left(\frac{k\pi}{2}\right) - \frac{k\pi}{2} \cdot \cos\left(\frac{k\pi}{2}\right)$$

$$B_n = \frac{1}{50cn\pi k^2} \cdot \left(2 \cdot \sin\left(\frac{k\pi}{2}\right) - \sin(k\pi) \right)$$

Her kan vi så indsætte k igen

$$k = \frac{n\pi}{L}$$

$$B_n = \frac{1}{50cn\pi \cdot \left(\frac{n\pi}{L}\right)^2} \cdot \left(2 \cdot \sin\left(\frac{n\pi^2}{2L}\right) - \sin\left(\frac{n\pi^2}{L}\right) \right)$$

Vi kan her indsætte værdierne for c og L

$$B_n = \frac{1}{50n^3\pi} \cdot \left(2 \cdot \sin\left(\frac{n\pi}{2}\right) - \sin(\pi n) \right)$$

Eftersom at n er et helt tal og $\sin(\pi) = \sin(2\pi) = 0$ kan vi skrive

$$B_n = \frac{2}{50n^3\pi} \cdot \sin\left(n\frac{\pi}{2}\right)$$

Vi kan nu opskrive vores løsning

$$\begin{aligned} u(x,t) &= \frac{1}{25\pi} \sum_{n=1}^{\infty} \frac{1}{n^3} \cdot \sin\left(n\frac{\pi}{2}\right) \sin(nx) \cdot \sin(0,01n \cdot t) \\ &= \frac{1}{25\pi} \cdot \left(\sin(x) \cdot \sin(0,01t) - \frac{1}{27} \cdot \sin(3x) \cdot \sin(0,03t) \cdot \dots \right) \end{aligned}$$

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GRAPHING SOLUTIONS

Using (13) sketch or graph a figure (similar to Fig. 291 in Sec. 12.3) of the deflection $u(x, t)$ of a vibrating string (length $L = 1$, ends fixed, $c = 1$) starting with initial velocity 0 and initial deflection (k small, say, $k = 0.01$).

5. $f(x) = k \sin \pi x$

Vi bruger her D'Alemberts metode til at finde $u(x, t)$ der står i opgaven at start hastigheden er 0 hvilket betyder vi kan finde $u(x, t)$ som:

$$u(x, t) = \frac{1}{2} \cdot (f(x+t) + f(x-t))$$

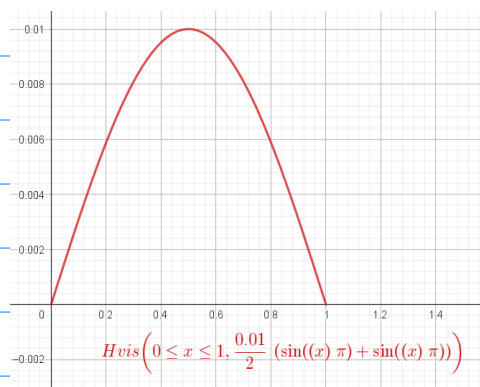
Vi kan her indsætte dette ind i vores funktion, og $c=1$

$$u(x, t) = \frac{1}{2} \cdot (k \cdot \sin((x+t)\pi) + k \cdot \sin((x-t)\pi))$$

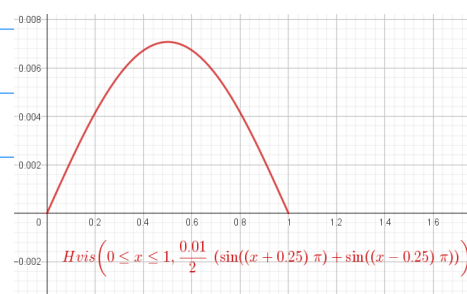
$$u(x, t) = \frac{k}{2} \cdot (\sin((x+t)\pi) + \sin((x-t)\pi))$$

Vi kan her tegne dette:

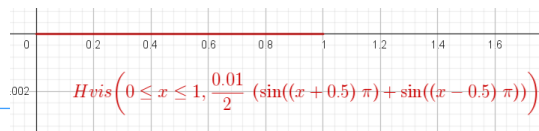
$$t = 0$$



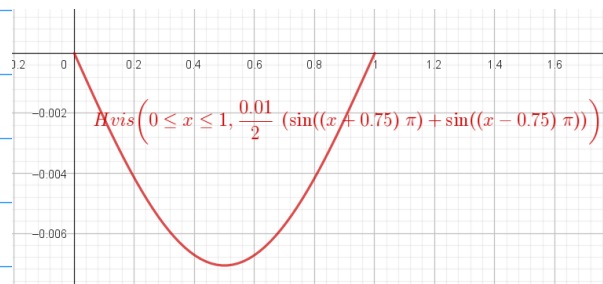
$$t = \frac{1}{4}$$



$$t = \frac{1}{2}$$



$$t = \frac{3}{4}$$



$$t = 1$$

