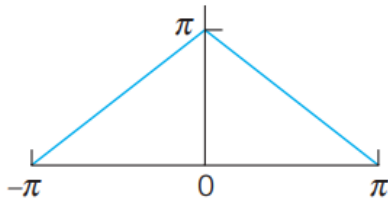


Find the Fourier series of the given function  $f(x)$ , which is assumed to have the period  $2\pi$ . Show the details of your work. Sketch or graph the partial sums up to that including  $\cos 5x$  and  $\sin 5x$ .

17.



Her opskrives først en funktion for grafen

$$f(x) = \begin{cases} x + \pi & -\pi \leq x < 0 \\ \pi - x & 0 \leq x \leq \pi \end{cases}$$

Vi har her en lige funktion hvilket betyder vi kan sætte  $b_n = 0$  og vi finder her  $a_0$

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \cdot \left( \int_{-\pi}^0 x + \pi dx + \int_0^{\pi} \pi - x dx \right) \\ &= \frac{1}{2\pi} \cdot \left( \left( \frac{1}{2}x^2 + \pi x \right) \Big|_{-\pi}^0 + \left( \pi x - \frac{1}{2}x^2 \right) \Big|_0^{\pi} \right) \\ &= \frac{1}{2\pi} \cdot \left( \left( \frac{1}{2}(-\pi)^2 + \pi \cdot (-\pi) \right) + \left( \pi \cdot \pi - \frac{1}{2}\pi^2 \right) \right) \\ &= \frac{1}{2\pi} \cdot \left( -\frac{\pi^2}{2} + \pi^2 + \pi^2 - \frac{1}{2}\pi^2 \right) \\ &= \frac{1}{2\pi} \cdot \left( \pi^2 \right) \\ &= \frac{\pi}{2} \end{aligned}$$

Herefter finder vi så nu  $a_n$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx) dx$$

$$a_n = \frac{1}{\pi} \cdot \left( \int_{-\pi}^0 (x+\pi) \cdot \cos(nx) dx + \int_0^{\pi} (\pi-x) \cdot \cos(nx) dx \right)$$

Her ganges integrallerne ud

$$a_n = \frac{1}{\pi} \cdot \left( \int_{-\pi}^0 x \cdot \cos(nx) dx + \int_{-\pi}^0 \pi \cdot \cos(nx) dx + \int_0^{\pi} \pi \cdot \cos(nx) dx - \int_0^{\pi} x \cdot \cos(nx) dx \right)$$

Her kan der så bruges partiel integration til at integrere  $x \cdot \cos(x)$

$$\int \cos(x) \cdot x dx = \frac{1}{n} \sin(nx) \cdot x - \int \frac{1}{n} \sin(nx) dx$$

$$= \left( \sin(nx) \cdot x + \frac{1}{n} \cos(nx) \right) + C$$

Dette kan så indsættes ind igen

$$a_n = \frac{1}{\pi} \cdot \left( \left( \sin(nx) \cdot x + \frac{1}{n} \cos(nx) \right) \Big|_{-\pi}^0 + \pi \cdot \sin(nx) \Big|_{-\pi}^0 + \frac{\pi}{n} \cdot \sin(nx) \Big|_0^{\pi} + \left( -\sin(nx) \cdot x - \frac{1}{n} \cos(nx) \right) \Big|_0^{\pi} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left( \frac{1}{n} \cdot \left( \sin(\pi n) \cdot \pi + \cos(\pi n) \right) - \left( \sin(-\pi n) \cdot (-\pi) + \cos(-\pi n) \right) + \pi \cdot \sin(\pi n) + \frac{\pi}{n} \cdot \sin(\pi n) - \left( -\sin(0) \cdot 0 - \frac{1}{n} \cos(0) \right) \right)$$

$$a_n = \frac{1}{n^2 \pi} \cdot \left( \cos(0) - \cos(-\pi n) - \cos(\pi n) + \cos(0) \right)$$

$$a_n = \frac{1}{n^2 \pi} \cdot \left( 1 - \cos(\pi n) - \cos(\pi n) + 1 \right)$$

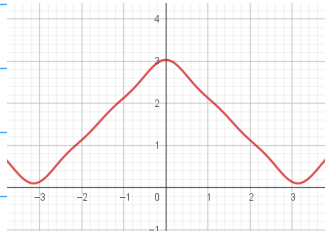
$$a_n = \frac{2 - 2 \cos(\pi n)}{n^2 \pi}$$

$$a_n = \frac{2 - 2 \cdot (-1)^n}{n^2 \pi}$$

Nu kan fourier rækken skrives op

$$S_5 = \frac{\pi}{2} + \frac{4}{\pi} \cos(x) + \frac{4}{9\pi} \cos(3x) + \frac{4}{25\pi} \cos(5x)$$

Den kan ses tegnet her:



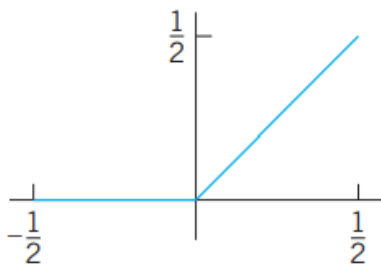
11.2

**8-17**

### FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

13.



Denne funktion er hverken lige eller ulige  
her kan det ses at perioden er 1 da grafen går fra  $-1/2$  til  $1/2$   
funktionen kan også skrives som:

$$f(x) = \begin{cases} 0, & -\frac{1}{2} \leq x < 0 \\ x, & 0 \leq x < \frac{1}{2} \end{cases}$$

Vi kan så her finde fourier rækken

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$a_0 = \frac{1}{2 \cdot \frac{1}{2}} \cdot \left( \int_{-\frac{1}{2}}^0 0 dx + \int_0^{\frac{1}{2}} x dx \right)$$

$$a_0 = \int_0^{\frac{1}{2}} x dx$$

$$a_0 = \frac{1}{2} x^2 \Big|_0^{\frac{1}{2}}$$

$$a_0 = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2$$

$$a_0 = \frac{1}{8}$$

Vi kan nu finde  $a_n$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

$$a_n = \frac{1}{\left(\frac{1}{2}\right)} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho \, dx + \int_0^{\frac{1}{2}} x \cdot \cos\left(\frac{n\pi x}{\frac{1}{2}}\right) dx$$

$$a_n = 2 \cdot \int_0^{\frac{1}{2}} x \cdot \cos(2n\pi x) \, dx$$

Dette integral kan vi løse med partiel integration

$$\begin{aligned} \int x \cdot \cos(kx) \, dx &= \frac{1}{k} \cdot \sin(kx) \, dx - \int \frac{1}{k} \sin(kx) \, dx \\ &= \frac{1}{k} \cdot \sin(kx) \, dx + \frac{1}{k^2} \cdot \cos(kx) + C \\ &= \frac{k \sin(kx) + \cos(kx)}{k^2} + C \end{aligned}$$

Dette kan så indsættes ind i vores originale integral

$$a_n = 2 \cdot \left( \frac{2n\pi \cdot \sin(2n\pi x) + \cos(2n\pi x)}{4n^2\pi^2} \right) \bigg|_{x=0}^{x=\frac{1}{2}}$$

$$a_n = \frac{2}{4n^2\pi^2} \cdot \left( 2n\pi \cdot \sin\left(2n\pi \cdot \frac{1}{2}\right) + \cos\left(2n\pi \cdot \frac{1}{2}\right) - 2n\pi \cdot \sin(0) - \cos(0) \right)$$

$$a_n = \frac{z}{4n^2\pi^2} e \left( 2n\pi \cos(n\pi) + \cos(n\pi) - 1 \right)$$

Vi har her at eftersom at  $n$  kun er hele tal og sinus giver 0 ved  $\pi$  og 0 og har en periode på  $2\pi$ :

$$a_n = \frac{z}{4n^2\pi^2} e \left( \cos(n\pi) - 1 \right)$$

Ligesom med sinus så har vi her cosinus og eftersom vi har  $n\pi$  kan vi skrive det som:

$$a_n = \frac{(-1)^n - 1}{2 \cdot n^2 \cdot \pi^2}$$

Vi prøver nu at finde  $b_n$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{2} \cdot \left( \int_{-\frac{1}{2}}^0 0 dx + \int_0^{\frac{1}{2}} x \cdot \sin\left(\frac{n\pi x}{\frac{1}{2}}\right) dx \right)$$

$$b_n = 2 \cdot \int_0^{\frac{1}{2}} x \cdot \sin(2n\pi x) dx$$

Vi bruger her partiel integration ligesom før

$$\int x \cdot \sin(kx) = -\frac{1}{k} \cos(kx) \cdot x + \frac{1}{k} \int \cos(kx) dx$$

$$= -\frac{1}{k} \cos(kx) \cdot x + \frac{1}{k^2} \sin(kx) + C$$

$$= \frac{\sin(kx) - k \cdot \cos(kx) \cdot x}{k^2} + C$$

Dette kan så indsættes ind i tidligere integral

$$b_n = 2 \cdot \left( \frac{\sin(2n\pi x) - 2n\pi \cdot \cos(2n\pi x)}{4n^2\pi^2} \right) \bigg|_{x=0}^{x=\frac{1}{2}}$$

$$b_n = \frac{1}{2n^2\pi^2} \cdot \left( \sin(n\pi) - 2n\pi \cdot \cos(n\pi) - \sin(0) + 2n\pi \cdot \cos(0) \right)$$

$$b_n = \frac{1}{2n^2\pi^2} \cdot \left( -2n\pi \cdot (-1)^n + 2n\pi \right)$$

$$b_n = \frac{2n\pi - 2n\pi \cdot (-1)^n}{2n^2\pi^2}$$

$$b_n = \frac{1 - (-1)^n}{n\pi}$$

Vi har nu at:

$$a_n = \frac{(-1)^n - 1}{2 \cdot n^2 \pi^2}$$

$$b_n = \frac{1 - (-1)^n}{n\pi}$$

$$a_0 = \frac{1}{8}$$

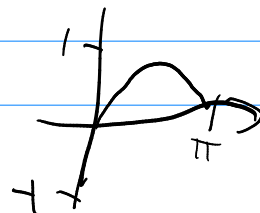
Og vi kan nu skrive rækken op

$$\frac{1}{8} - \frac{1}{\pi^2} \left( \cos(x) + \frac{1}{9} \cos(3x) + \frac{1}{25} \cos(5x) \right) \dots$$

$$+ \frac{2}{\pi} \left( \sin(x) + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) \right) \dots$$

Find **(a)** the Fourier cosine series, **(b)** the Fourier sine series. Sketch  $f(x)$  and its two periodic extensions. Show the details.

**29.**  $f(x) = \sin x$  ( $0 < x < \pi$ )





Finder først  $a_0$  når den er lige

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin(x) dx$$

$$a_0 = \frac{1}{\pi} \cdot (-\cos(\pi) + \cos(0))$$

$$a_0 = \frac{1}{\pi} \cdot (1 + 1)$$

$$a_0 = \frac{2}{\pi}$$

Finder herefter  $a_n$  når den er lige

$$a_n = \frac{2}{L} \int_0^L f(x) \cdot \cos\left(\frac{n x \pi}{L}\right) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cdot \cos(nx) dx$$

Vi kan her bruge additionsformlerne

$$\sin(x) \cdot \cos(nx) = \frac{\sin(x+nx) + \sin(x-nx)}{2}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} \sin(x+nx) + \sin(x-nx) dx$$

$$a_n = \frac{1}{\pi} \cdot \left( -\frac{\cos(x+n\pi)}{1+n} - \frac{\cos(\alpha-n\alpha)}{1-n} \right) \Big|_0^\pi$$

$$a_n = \frac{1}{\pi} \cdot \left( -\frac{\cos(\pi+n\pi)}{1+n} - \frac{\cos(\pi-n\pi)}{1-n} + \frac{\cos(0)}{1+n} + \frac{\cos(0)}{1-n} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left( \frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} + \frac{1}{1+n} + \frac{1}{1-n} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left( \frac{(-1)^n \cdot (1-n) + (-1)^n \cdot (1+n) + (1-n) + (1+n)}{1-n^2} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left( \frac{(-1)^n \cdot ((1-n) + (1+n)) + 2}{1-n^2} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left( \frac{(-1)^n \cdot (2) + 2}{1-n^2} \right)$$

$$a_n = \frac{2}{\pi} \cdot \left( \frac{(-1)^n + 1}{1-n^2} \right)$$

$$a_n = \begin{cases} \frac{4}{\pi} \cdot \frac{1}{1-n^2} & n \text{ er iige} \\ 0 & n \text{ er ulige} \end{cases}$$

$$f_5(x) = \frac{2}{\pi} + \frac{4}{\pi} \cdot \left( -\frac{1}{3} \cos(2x) - \frac{1}{15} \cos(4x) \right)$$

Kigger nu på hvordan den ser ud hvis den er ulige

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n x \pi}{L}\right) dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cdot \sin(nx) dx$$

$$\sin(x) \cdot \sin(nx) = \frac{\cos(x-nx) - \cos(x+nx)}{2}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} \cos(x-nx) - \cos(x+nx) dx$$

$$b_n = \frac{1}{\pi} \cdot \left( \frac{\sin(x-nx)}{1-n} - \frac{\sin(x+nx)}{1+n} \right) \Big|_0^{\pi}$$

$$b_n = \frac{1}{\pi} \cdot \left( \frac{\sin(\pi-n\pi)}{1-n} - \frac{\sin(\pi+n\pi)}{1+n} - \frac{\sin(0)}{1-n} + \frac{\sin(0)}{1+n} \right)$$

$$b_n = \frac{1}{\pi} \left( \frac{\sin(\pi - n\pi)}{1-n} - \frac{\sin(\pi + n\pi)}{1+n} \right)$$

$$b_n = \frac{1}{\pi} \cdot (0)$$

$$b_n = 0$$

Eftersom vi undervejs oplevede at vi dividerede med  $1-n$  hvor  $n$  ikke måtte være 1 test vi nu for det

$$b_1 = \frac{2}{\pi} \cdot \int_0^{\pi} \sin(x) \cdot \sin(x) dx$$

$$b_1 = \frac{2}{\pi} \cdot \left( \frac{1}{2}x - \frac{1}{4} \sin(2x) \right) \Big|_0^{\pi}$$

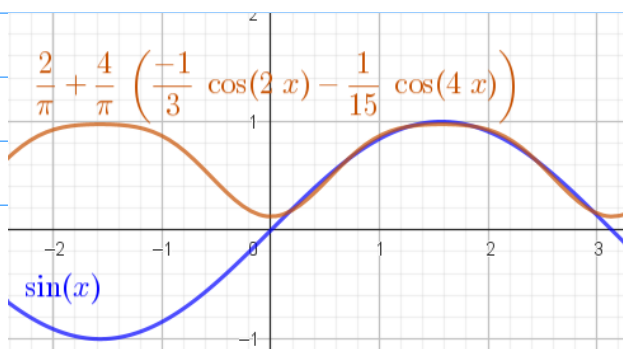
$$b_1 = \frac{2}{\pi} \cdot \left( \frac{1}{2}\pi - \frac{1}{4} \sin(2\pi) - 0 + \frac{1}{4} \sin(0) \right)$$

$$b_1 = \frac{2}{\pi} \cdot \left( \frac{1}{2}\pi - 0 + 0 \right)$$

$$b_1 = 1$$

$$\underline{S_b = \sin(x)}$$

Vi kan nu tegne de 2 tækker vi har fået:



Find the steady-state oscillations of  $y'' + cy' + y = r(t)$  with  $c > 0$  and  $r(t)$  as given. Note that the spring constant is  $k = 1$ . Show the details. In Probs. 14–16 sketch  $r(t)$ .

$$13. r(t) = \sum_{n=1}^N (a_n \cos nt + b_n \sin nt)$$

Vi skriver først vores differentialligning op og sætter en lig med en fourrierrække

$$y'' + cy' + y = r(t)$$

$$y'' + cy' + y = \sum_{n=1}^N (a_n \cos(nt) + b_n \sin(nt))$$

$$y'' + cy' + y = a_n \cos(nt) + b_n \sin(nt)$$

Gætter her på en løsning og differentier den 2 gange for at finde  $y''$  og  $y'$

$$y_n = A_n \cdot \cos(nt) + B_n \cdot \sin(nt)$$

$$y'_n = -n A_n \cdot \sin(nt) + n B_n \cdot \cos(nt)$$

$$y''_n = -n^2 A_n \cdot \cos(nt) - n^2 B_n \cdot \sin(nt)$$

Dette indsættes så ind i vores differentialligning

$$\cos(nt) \cdot (-n^2 A_n + n B_n + A_n) + \sin(nt) \cdot (-n^2 B_n - n A_n + B_n) = a_n \cos(nt) + b_n \sin(nt)$$

Her kigges så på koeficienterne foran cos og sin

$$a_n = -n^2 A_n + n B_n + A_n$$

$$b_n = -n^2 B_n - n A_n + B_n$$

Dette giver dette sæt af lineære ligninger som løses med Cramers rule

$$a_n = A_n \cdot (1 - n^2) + B_n \cdot n \cdot c$$

$$b_n = A_n \cdot (-n) + B_n \cdot (1 - n^2)$$

$$\begin{pmatrix} 1 - n^2 & nc \\ -n & 1 - n^2 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$A_n = \frac{\begin{vmatrix} a_n & nc \\ b_n & 1 - n^2 \end{vmatrix}}{\begin{vmatrix} 1 - n^2 & nc \\ -n & 1 - n^2 \end{vmatrix}} = \frac{a_n(1 - n^2) - b_n \cdot nc}{(1 - n^2)(1 - n^2) + n^2 c^2}$$

$$= \frac{a_n(1 - n^2) - b_n \cdot nc}{1 + n^4 - 2n^2 + n^2 c^2}$$

$$= \frac{a_n(1 - n^2) - b_n \cdot nc}{1 + n^4 + n^2 \cdot (c^2 - 2)}$$

$$B_n = \frac{\begin{vmatrix} 1 - n^2 & a_n \\ -n & b_n \end{vmatrix}}{\begin{vmatrix} 1 - n^2 & n \\ -n & 1 - n^2 \end{vmatrix}}$$

$$= \frac{b_n \cdot (1 - n^2) + a_n n}{1 + n^4 + n^2 \cdot (z^2 - 2)}$$

$$y_h = A_n \cdot \cos(nz) + B_n \cdot \sin(nz)$$

Der er nu fundet en funktion som løser differentialligningen

$$y_h = \frac{a_n \cdot (1 - n^2) - b_n \cdot n}{1 + n^4 + n^2 \cdot (z^2 - 2)} \cdot \cos(nz) + \frac{b_n \cdot (1 - n^2) + a_n \cdot n}{1 + n^4 + n^2 \cdot (z^2 - 2)} \cdot \sin(nz)$$