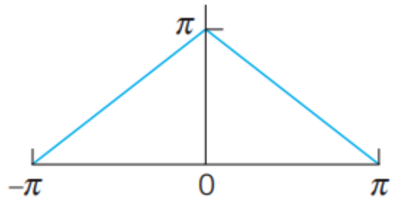
Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π. Show the details of your work. Sketch or graph the partial sums up to that including $\cos 5x$ and $\sin 5x$.

17.



Her opskrives først en funktion for grafen

$$f(x) = \begin{cases} x+\pi & -\pi \leq x < 0 \\ \pi - x & 0 \leq x \leq \pi \end{cases}$$

Vi har her en lige funktion hvilket betyder vi kan sætte b_n=0 og vi finder her a_0

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi} \cdot \left(\int_{-\pi}^{0} x+\pi \, dx + \int_{0}^{\pi} \pi - x \, dx \right)$$

$$= \frac{1}{2\pi} \cdot \left(\left(\frac{1}{2}x^2 + \pi x\right)\Big|_{-\pi}^{0} + \left(\pi x - \frac{1}{2}x^2\right)\Big|_{0}^{\pi} \right)$$

$$= \frac{1}{2\pi} \cdot \left(-\left(\frac{1}{2}(-\pi)^2 + \pi \cdot (-\pi)\right) + \pi \cdot \pi - \frac{1}{2}\pi^2 \right)$$

$$= \frac{1}{2\pi} \cdot \left(-\frac{\pi^2}{2} + \pi^2 + \pi^2 - \frac{1}{2}\pi^2 \right)$$

$$= \frac{1}{2\pi} \cdot \left(2\pi^2 - \pi^2 \right)$$

$$= \frac{1}{2\pi} \cdot \left(\pi^2 \right)$$

$$a_0 = \frac{\pi}{2}$$

Herefter finder vi så nu a_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(xn)\, dx$$

$$a_n = \frac{1}{\pi} \cdot \left(\int_{-\pi}^{0} (x+\pi) \cdot \cos(xn)\, dx + \int_{0}^{\pi} (\pi - x) \cdot \cos(xn)\, dx \right)$$

Her ganges integrallerne ud

$$a_n = \frac{1}{\pi} \cdot \left(\int_{-\pi}^{0} x \cdot \cos(xn)\, dx + \int_{-\pi}^{0} \pi \cdot \cos(xn)\, dx + \int_{0}^{\pi} \pi \cdot \cos(xn)\, dx - \int_{0}^{\pi} x \cdot \cos(xn)\, dx \right)$$

Her kan der så bruges partiel integration til at integrerer x*cos(x)

$$\int \cos(xn) \cdot x\, dx = \frac{1}{n} \cdot \sin(xn) \cdot x - \int \frac{1}{n} \sin(xn)\, dx$$

$$= \underbrace{\left(\sin(xn) \cdot x + \frac{1}{n} \cdot \cos(xn) \right)}_{n} + C$$

Dette kan så indsættes ind igen

$$a_n = \frac{1}{\pi} \cdot \left(\underbrace{\left(\sin(xn) \cdot x + \frac{1}{n} \cdot \cos(xn) \right)}_{n} \Big|_{-\pi}^{0} + \frac{\pi}{n} \cdot \sin(xn) \Big|_{-\pi}^{0} + \frac{\pi}{n} \cdot \sin(xn) \Big|_{0}^{\pi} + \underbrace{\left(-\sin(xn) \cdot x - \frac{1}{n} \cdot \cos(xn) \right)}_{n} \Big|_{0}^{\pi} \right)$$

$$a_n = \frac{1}{\pi} \left(\frac{1}{n^2} \cdot \left(\left(\sin(xn) \cdot xn + \cos(xn) \right) \Big|_{-\pi}^{0} \right) + \frac{1}{n^2} \left(\left(-\sin(xn) \cdot xn - \cos(xn) \right) \Big|_{0}^{\pi} \right) \right)$$

$$a_n = \frac{1}{n^2 \pi} \left(\cos(0) - \cos(-\pi n) - \cos(\pi n) + \cos(0) \right)$$

$$a_n = \frac{1}{n^2 \pi} \left(1 - \cos(-\pi n) - \cos(\pi n) + 1 \right)$$

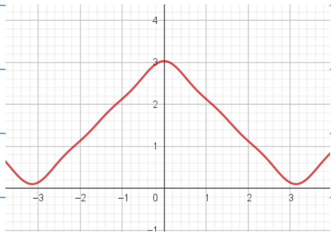
$$a_n = \frac{2 - 2 \cdot \cos(\pi n)}{n^2 \cdot \pi}$$

$$a_n = \frac{2 - 2 \cdot (-1)^n}{n^2 \cdot \pi}$$

Nu kan fourrier rækken skrives op

$$S_5 = \frac{\pi}{2} + \frac{4}{\pi}\cos(x) + \frac{4}{9\pi}\cos(3x) + \frac{4}{25\pi}\cos(5x)$$

Den kan ses tegnet her:

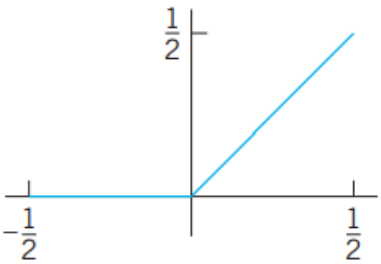


11.2

8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

13.



Denne funktion er hverken lige eller ulige
her kan det ses at perioden er 1 da grafen går fra −1/2 til 1/2
funktionen kan også skrives som:

$$f(x) = \begin{cases} 0, & -\frac{1}{2} \leq x < 0 \\ x, & 0 \leq x < \frac{1}{2} \end{cases}$$

Vi kan så her finde fourier rækken

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_0 = \frac{1}{2\cdot\frac{1}{2}}\cdot\left(\int_{-\frac{1}{2}}^{0} 0\,dx + \int_{0}^{\frac{1}{2}} x\,dx\right)$$

$$a_0 = \int_{0}^{\frac{1}{2}} x\,dx$$

$$a_0 = \frac{1}{2}x^2\Big|_{0}^{\frac{1}{2}}$$

$$a_0 = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2$$

$$a_0 = \frac{1}{8}$$

Vi kan nu finde a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$a_n = \frac{1}{\left(\frac{1}{2}\right)}\left(\int_{-\frac{1}{2}}^{0} 0 \, dx + \int_{0}^{\frac{1}{2}} x \cdot \cos\left(\frac{n\pi x}{\frac{1}{2}}\right) dx\right)$$

$$a_n = 2 \cdot \int_{0}^{\frac{1}{2}} x \cdot \cos(2n\pi x) \, dx$$

Dette integral kan vi løse med partiel integration

$$\int x \cdot \cos(kx) \, dx = \frac{1}{k} \cdot \sin(kx) \, x - \int \frac{1}{k} \sin(kx) \, dx$$

$$= \frac{1}{k} \cdot \sin(kx) \cdot x + \frac{1}{k^2} \cdot \cos(kx) + C$$

$$= \frac{kx\sin(kx) + \cos(kx)}{k^2} + C$$

Dette kan så indsættes ind i vores originale integral

$$a_n = 2 \cdot \left(\left[\frac{2n\pi \cdot x\sin(2n\pi x) + \cos(2n\pi x)}{4n^2\pi^2}\right]_{x=0}^{x=\frac{1}{2}}\right)$$

$$a_n = \frac{2}{4n^2\pi^2} \cdot \left(2n\pi \cdot \frac{1}{2}\sin\left(2n\pi \cdot \frac{1}{2}\right) + \cos\left(2n\pi \cdot \frac{1}{2}\right) - 2n\pi \cdot 0 \cdot \sin(0) - \cos(0)\right)$$

$$a_n = \frac{2}{4n^2\pi^2} \cdot \left(2n\pi \cdot \sin(n\pi) + \cos(n\pi) - 1\right)$$

Vi har her at eftersom at n kun er hele tal og sinus giver 0 ved pi og 0 og har en periode på 2 pi:

$$a_n = \frac{2}{4n^2\pi^2} \cdot \left(\cos(n\pi) - 1\right)$$

Ligesom med sinus så har vi her cosinus og eftersom vi har n*pi kan vi skrive det som:

$$a_n = \frac{(-1)^n - 1}{2 \cdot n^2 \cdot \pi^2}$$

Vi prøver nu at finde b_n

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{\frac{1}{2}} \cdot \left(\int_{-\frac{1}{2}}^{0} 0 \, dx + \int_{0}^{\frac{1}{2}} x \cdot \sin\left(\frac{n\pi x}{\frac{1}{2}}\right) dx \right)$$

$$b_n = 2 \cdot \int_{0}^{\frac{1}{2}} x \cdot \sin(2n\pi x) \, dx$$

Vi bruger her partiel integration ligesom før

$$\int x \cdot \sin(kx) = -\frac{1}{k}\cos(kx) \cdot x + \frac{1}{k}\int \cos(kx)\,dx$$

$$= -\frac{1}{k}\cos(kx) \cdot x + \frac{1}{k^2} \cdot \sin(kx) + C$$

$$= \frac{\sin(kx) - k \cdot \cos(kx)}{k^2} + C$$

Dette kan så indsættes ind i tidligere integral

$$b_n = 2 \cdot \left(\frac{\sin(2n\pi x) - 2n\pi \cdot \cos(2n\pi x)}{4n^2\pi^2} \Bigg|_{x=0}^{x=\frac{1}{2}} \right)$$

$$b_n = \frac{1}{2n^2\pi^2} \cdot \left(\sin(n\pi) - 2n\pi \cdot \cos(n\pi) - \sin(0) + 2n\pi \cdot \cos(0) \right)$$

$$b_n = \frac{1}{2n^2\pi^2} \cdot \left(-2n\pi \cdot (-1)^n + 2n\pi \right)$$

$$b_n = \frac{2n\pi - 2n\pi \cdot (-1)^n}{2n^2\pi^2}$$

$$b_n = \frac{1 - (-1)^n}{n\pi}$$

Vi har nu at:

$$a_n = \frac{(-1)^n - 1}{2 \cdot n^2 \cdot \pi^2} \qquad b_n = \frac{1 - (-1)^n}{n\pi}$$

$$a_0 = \frac{1}{8}$$

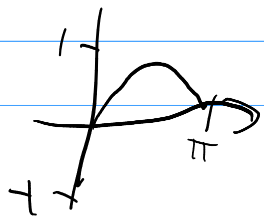
Og vi kan nu skrive rækken op

$$\frac{1}{8} - \frac{1}{\pi^2} \cdot \left(\cos(x) + \frac{1}{9} \cos(3x) + \frac{1}{25} \cos(5x) \right) \cdots$$

$$+ \frac{2}{\pi} \cdot \left(\sin(x) + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) \right) \cdots$$

Find **(a)** the Fourier cosine series, **(b)** the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. Show the details.

29. $f(x) = \sin x \ (0 < x < \pi)$



Finder først a_0 når den er lige

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_0^\pi \sin(x)\,dx$$

$$a_0 = \frac{1}{\pi}\cdot\left(-\cos(\pi) + \cos(0)\right)$$

$$a_0 = \frac{1}{\pi}\cdot(1+1)$$

$$\underline{\underline{a_0 = \frac{2}{\pi}}}$$

Finder herefter a_n når den er lige

$$a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{n x \pi}{L}\right)dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin(x)\cdot\cos(nx)\,dx$$

Vi kan her bruge additionsformlerne

$$\sin(x)\cdot\cos(nx) = \frac{\sin(x+nx) + \sin(x-nx)}{2}$$

$$a_n = \frac{1}{\pi}\int_0^\pi \sin(x+nx) + \sin(x-nx)\,dx$$

$$a_n = \frac{1}{\pi} \cdot \left(-\frac{\cos(x+nx)}{1+n} - \frac{\cos(x-nx)}{1-n} \right) \Bigg|_0^{\pi}$$

$$a_n = \frac{1}{\pi} \cdot \left(-\frac{\cos(\pi+\pi n)}{1+n} - \frac{\cos(\pi-n\pi)}{1-n} + \frac{\cos(0)}{1+n} + \frac{\cos(0)}{1-n} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left(\frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} + \frac{1}{1+n} + \frac{1}{1-n} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left(\frac{(-1)^n \cdot (1-n) + (-1)^n \cdot (1+n) + (1-n) + (1+n)}{1-n^2} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left(\frac{(-1)^n \cdot \big((1-n) + (1+n) \big) + 2}{1-n^2} \right)$$

$$a_n = \frac{1}{\pi} \cdot \left(\frac{(-1)^n \cdot (2) + 2}{1-n^2} \right)$$

$$a_n = \frac{2}{\pi} \cdot \left(\frac{(-1)^n + 1}{1-n^2} \right)$$

$$a_n = \begin{cases} \frac{4}{\pi} \cdot \frac{1}{1-n^2} & n \text{ er lige} \\ 0 & n \text{ er ulige} \end{cases}$$

$$S_5(x) = \frac{2}{\pi} + \frac{4}{\pi} \cdot \left(-\frac{1}{3} \cos(2x) - \frac{1}{15} \cdot \cos(4x) \right)$$

Kigger nu på hvordan den ser ud hvis den er ulige

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n x \pi}{L}\right) dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cdot \sin(nx) \, dx$$

$$\sin(x) \cdot \sin(nx) = \frac{\cos(x - nx) - \cos(x + nx)}{2}$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} \cos(x - nx) - \cos(x + nx) \, dx$$

$$b_n = \frac{1}{\pi} \cdot \left(\frac{\sin(x - nx)}{1-n} - \frac{\sin(x + nx)}{1+n} \right) \Bigg|_0^{\pi}$$

$$b_n = \frac{1}{\pi} \cdot \left(\frac{\sin(\pi - n\pi)}{1-n} - \frac{\sin(\pi + n\pi)}{1+n} \right.$$

$$\left. - \frac{\sin(0)}{1-n} + \frac{\sin(0)}{1+n} \right)$$

$$b_n = \frac{1}{\pi}\left(\frac{\sin(\pi - n\pi)}{1-n} - \frac{\sin(\pi + n\pi)}{1+n}\right)$$

$$b_n = \frac{1}{\pi}\cdot(0)$$

$$b_n = 0$$

Eftersom vi undervejs oplevede at vi dividerede med 1-n hvor n ikke måtte være 1 test vi nu for det

$$b_1 = \frac{2}{\pi}\cdot\int_0^{\pi} \sin(x)\cdot\sin(x)\,dx$$

$$b_1 = \frac{2}{\pi}\cdot\left(\frac{1}{2}x - \frac{1}{4}\cdot\sin(2x)\right)\Big|_0^{\pi}$$

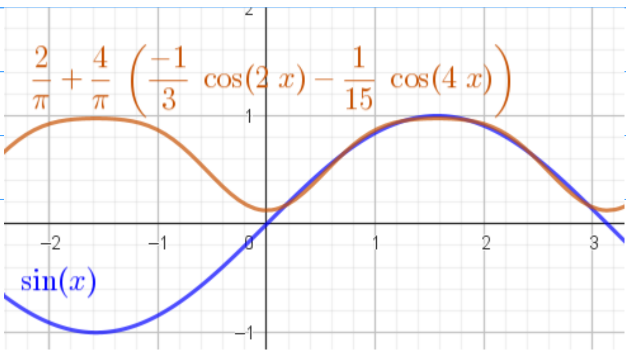
$$b_1 = \frac{2}{\pi}\cdot\left(\frac{1}{2}\pi - \frac{1}{4}\cdot\sin(2\pi) - 0 + \frac{1}{4}\cdot\sin(0)\right)$$

$$b_1 = \frac{2}{\pi}\cdot\left(\frac{1}{2}\pi - 0 + 0\right)$$

$$b_1 = 1$$

$$S_6(x) = \sin(x)$$

Vi kan nu tegne de 2 tækker vi har fået:



Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. Note that the spring constant is $k = 1$. Show the details. In Probs. 14–16 sketch $r(t)$.

13. $r(t) = \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$

Vi skriver først vores differentialligning op og sætter en lig med en fourrierrække

$$y'' + cy' + y = r(t)$$

$$y'' + cy' + y = \sum_{n=1}^{N} \left(a_n \cos(nt) + b_n \sin(nt)\right)$$

$$y'' + cy' + y = a_n \cos(nt) + b_n \sin(nt)$$

Gætter her på en løsning og differentier den 2 gange for at finde y'' og y'

$$y_n = A_n \cdot \cos(nt) + B_n \cdot \sin(nt)$$

$$y'_n = -cn\, A_n \cdot \sin(nt) + cn B_n \cdot \cos(nt)$$

$$y''_n = -n^2 \cdot A_n \cdot \cos(nt) - n^2 B_n \cdot \sin(nt)$$

Dette indsættes så ind i vores differentialligning

$$\cos(nt) \cdot \left(-n^2 A_n + cn B_n + A_n\right) + \sin(nt) \cdot \left(-n^2 B_n - ncA_n + B_n\right) = a_n \cdot \cos(nt) + b_n \cdot \sin(nt)$$

Her kigges så på koeficienterne foran cos og sin

$$a_n = -n^2 A_n + ncB_n + A_n$$

$$b_n = -n^2 B_n - ncA_{n} + B_n$$

Dette giver dette sæt af linære ligninger som løses med crampers rule

$$a_n = A_n \cdot (1-n^2) + B_n \cdot n \cdot c$$

$$b_n = A_n \cdot (-cn) + B_n \cdot (1-n^2)$$

$$\begin{pmatrix} 1-n^2 & nc \\ -nc & 1-n^2 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

$$A_n = \frac{\begin{vmatrix} a_n & nc \\ b_n & 1-n^2 \end{vmatrix}}{\begin{vmatrix} 1-n^2 & nc \\ -nc & 1-n^2 \end{vmatrix}} = \frac{a_n \cdot (1-n^2) - b_n \cdot nc}{(1-n^2) \cdot (1-n^2) + n^2c^2}$$

$$= \frac{a_n \cdot (1-n^2) - b_n \cdot nc}{1 + n^4 - 2n^2 + n^2c^2}$$

$$= \frac{a_n \cdot (1-n^2) - b_n \cdot nc}{1 + n^4 + n^2 \cdot (c^2-2)}$$

$$B_n = \frac{\begin{vmatrix} 1-n^2 & a_n \\ -n & b_n \end{vmatrix}}{\begin{vmatrix} 1-n^2 & n \\ -n & 1-n^2 \end{vmatrix}}$$

$$= \frac{b_n \cdot (1-n^2) + a_n n}{1+n^4+n^2 \cdot (c^2-2)}$$

$$y_n = A_n \cdot \cos(nt) + B_n \cdot \sin(nt)$$

Der er nu fundet en funktion som løser differentialligningen

$$y_n = \frac{a_n \cdot (1-n^2) - b_n \cdot n c}{1+n^4+n^2 \cdot (c^2-2)} \cdot \cos(nt) + \frac{b_n \cdot (1-n^2) + a_n n c}{1+n^4+n^2 \cdot (c^2-2)} \cdot \sin(nt)$$