

University of Dundee

# Modelling dryland vegetation patterns: Nonlocal dispersal and species coexistence

Mathematical Biology Seminar, St Andrews

17 October 2022

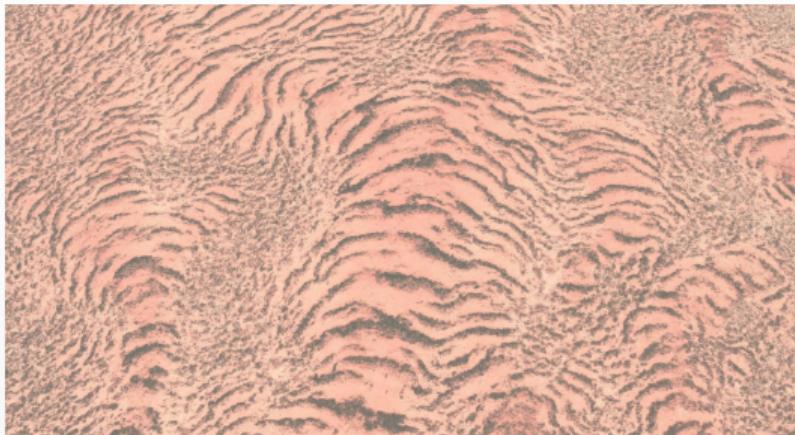
*Lukas Eigentler  
joint work with*

*Jamie JR Bennett (Icahn School of Medicine), Jonathan A Sherratt (Heriot-Watt Univ.)*

# Vegetation patterns

Vegetation patterns are a classic example of a **self-organisation principle** in ecology.

Stripe pattern in Ethiopia<sup>1</sup>.



Gap pattern in Niger<sup>2</sup>.



- Plants increase water infiltration into the soil and thus induce a **positive feedback loop**.

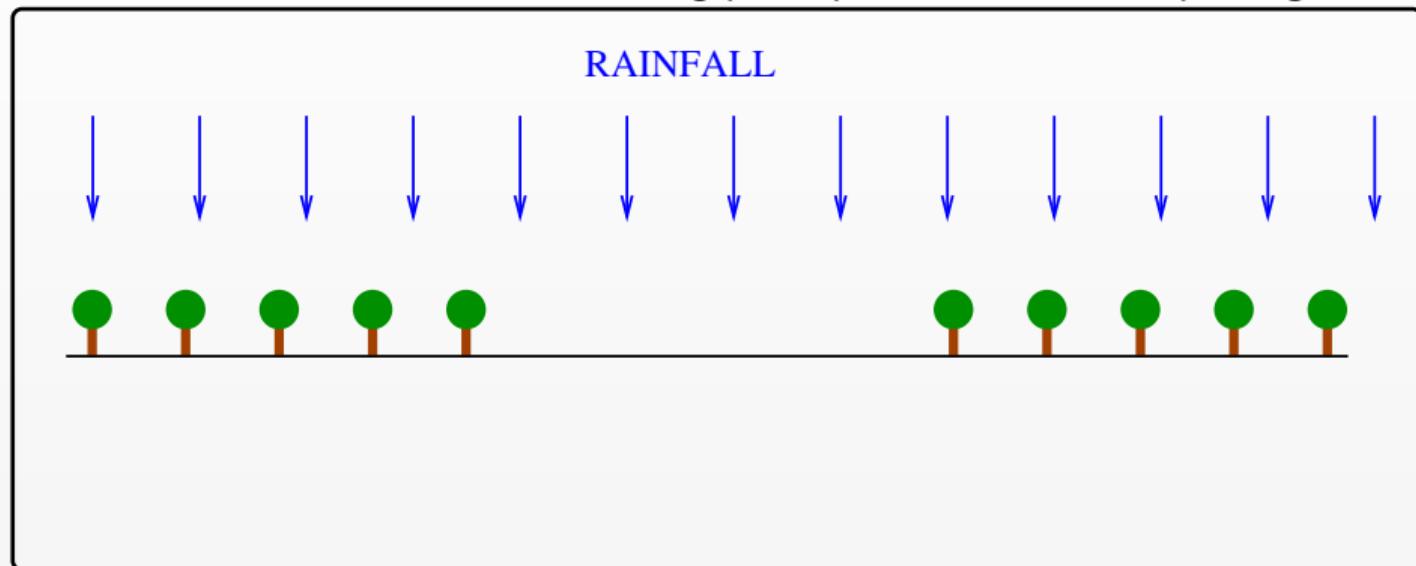
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<sup>1</sup>Source: Google Maps

<sup>2</sup>Source: Wikimedia Commons

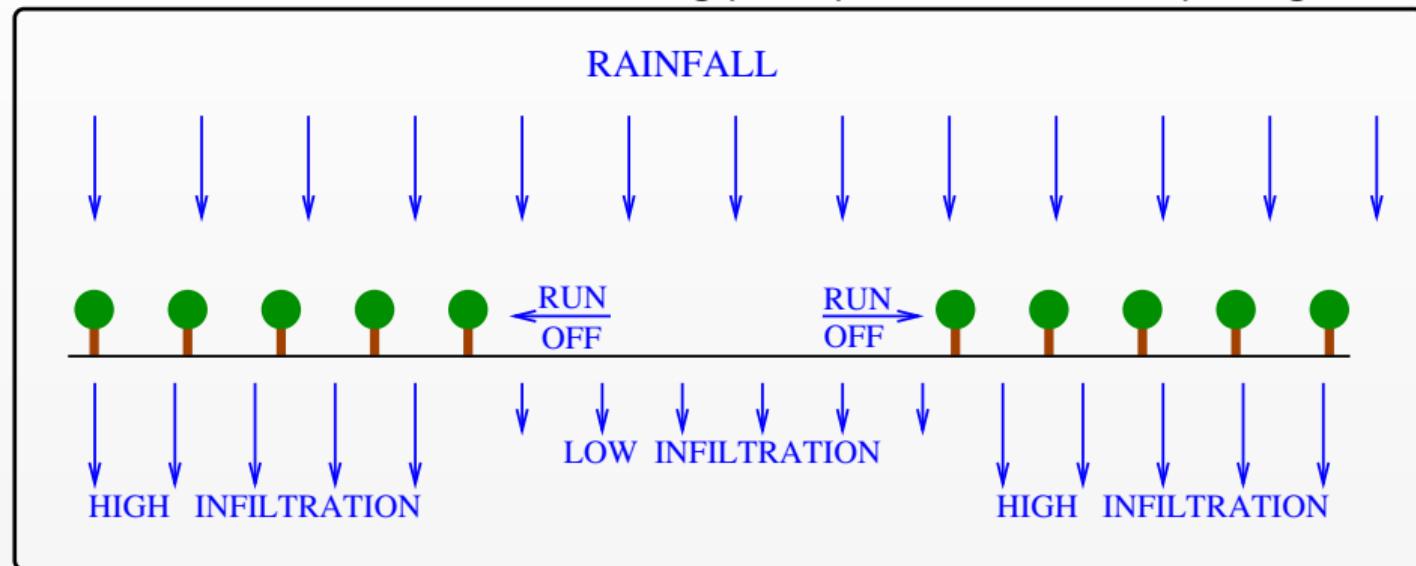
# Local facilitation in vegetation patterns

Positive feedback loop: Water infiltration into the soil depends on local plant density ⇒ redistribution of water towards existing plant patches ⇒ further plant growth.



# Local facilitation in vegetation patterns

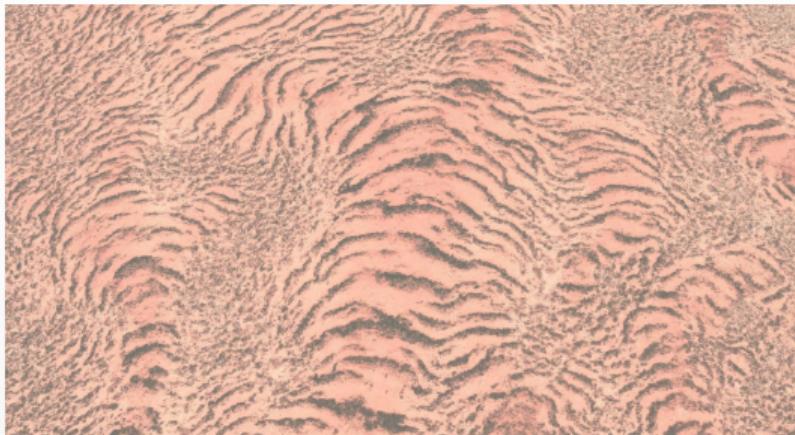
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# Vegetation patterns

Uphill migration due to water gradient.<sup>5</sup>



- On sloped ground, stripes grow **parallel to the contours**.
- Stripes either **move uphill** or are **stationary**.
- **Species coexistence** commonly occurs.

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<sup>5</sup>Dunkerley, D.: *Desert* 23.2 (2018).

# Klausmeier model

One of the most basic phenomenological models is the **extended Klausmeier reaction-advection-diffusion model**.<sup>6</sup>

$$\begin{aligned}\frac{\partial u}{\partial t} &= \underbrace{u^2 w}_{\text{plant growth}} - \underbrace{Bu}_{\text{plant loss}} + \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{plant dispersal}}, \\ \frac{\partial w}{\partial t} &= \underbrace{A}_{\text{rainfall}} - \underbrace{w}_{\text{evaporation}} - \underbrace{u^2 w}_{\text{water uptake by plants}} + \underbrace{\nu \frac{\partial w}{\partial x}}_{\text{water flow downhill}} + \underbrace{d \frac{\partial^2 w}{\partial x^2}}_{\text{water diffusion}}.\end{aligned}$$

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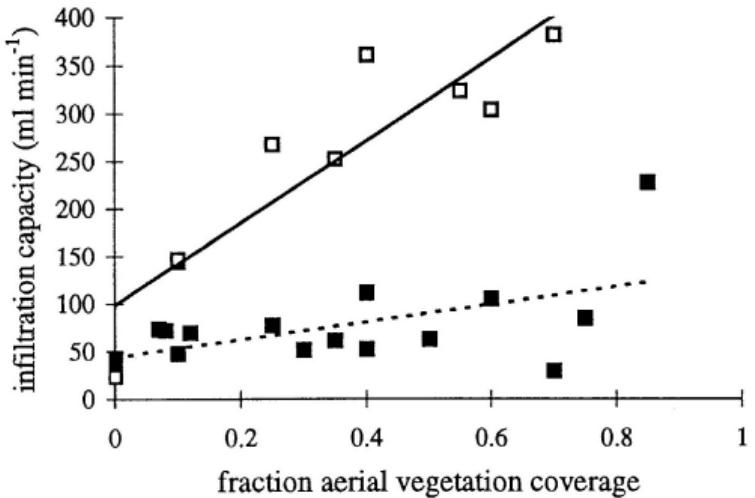
<sup>6</sup>Klausmeier, C. A.: *Science* 284.5421 (1999).

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# Water uptake



Infiltration capacity increases with plant density<sup>7</sup>

The nonlinearity in the water uptake and plant growth terms arises because plants increase the soil's water infiltration capacity.

⇒ Water uptake = Water density × plant density × infiltration rate.

<sup>7</sup>Rietkerk, M. et al.: *Plant Ecol.* 148.2 (2000)

## Research questions

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- How does nonlocal seed dispersal affect onset, existence and stability of patterns?
- How can species coexist despite competition for one limiting resource?

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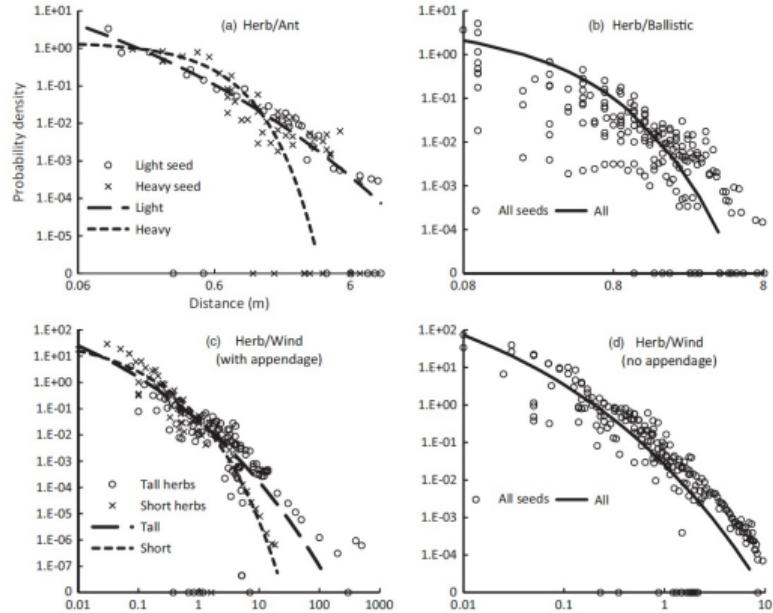
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# Local Model

The Klausmeier model models plant dispersal by a diffusion term, i.e. a local process.

$$\frac{\partial u}{\partial t} = \underbrace{u^2 w}_{\text{plant growth}} - \underbrace{Bu}_{\text{plant loss}} + \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{local plant dispersal}},$$
$$\frac{\partial w}{\partial t} = \underbrace{A}_{\text{rainfall}} - \underbrace{w}_{\text{evaporation}} - \underbrace{u^2 w}_{\text{water uptake by plants}} + \underbrace{\nu \frac{\partial w}{\partial x}}_{\text{water flow downhill}} + \underbrace{d \frac{\partial^2 w}{\partial x^2}}_{\text{water diffusion}}.$$

# Nonlocal seed dispersal



Data of long range seed dispersal<sup>8</sup>

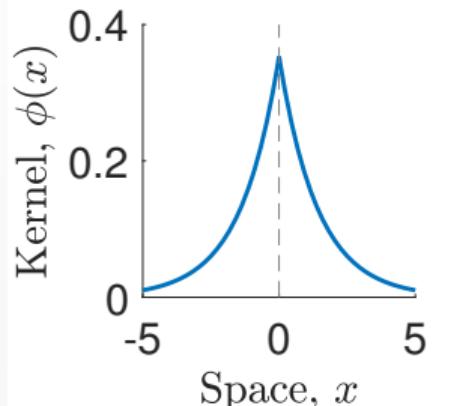
<sup>8</sup>Bullock, J. M. et al.: *J. Ecol.* 105.1 (2017)

More realistic: **Include effects of nonlocal processes**, such as dispersal by wind or large mammals.

# Nonlocal model

Diffusion is replaced by a **convolution of the plant density  $u$  with a dispersal kernel  $\phi$** . The scale parameter  $a$  controls the width of the kernel.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \underbrace{u^2 w}_{\text{plant growth}} - \underbrace{Bu}_{\text{plant loss}} + \overbrace{C(\phi(\cdot; a) * u(\cdot, t) - u)}^{\text{nonlocal plant dispersal}}, \\ \frac{\partial w}{\partial t} &= \underbrace{A}_{\text{rainfall}} - \underbrace{w}_{\text{evaporation}} - \underbrace{u^2 w}_{\text{water uptake by plants}} + \underbrace{\nu \frac{\partial w}{\partial x}}_{\text{water flow downhill}} + \underbrace{d \frac{\partial^2 w}{\partial x^2}}_{\text{water diffusion}}.\end{aligned}$$



## Laplacian kernel

If  $\phi$  decays exponentially as  $|x| \rightarrow \infty$ , and  $C = 2/\sigma(a)^2$ , then the nonlocal model tends to the local model as  $\sigma(a) \rightarrow 0$ .

E.g. Laplace kernel

$$\phi(x) = \frac{a}{2} e^{-a|x|}, \quad a > 0, \quad x \in \mathbb{R}.$$

Useful because

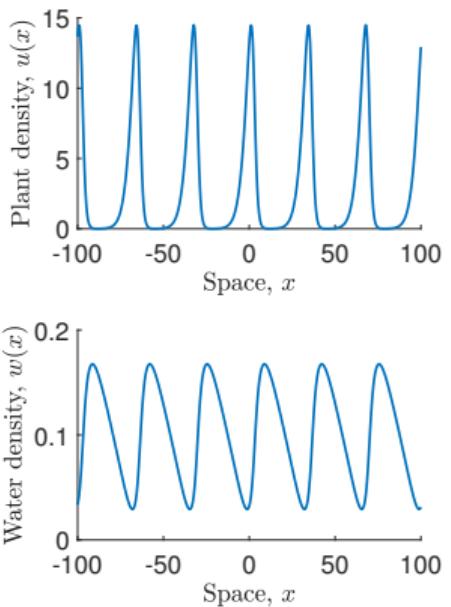
$$\hat{\phi}(k) = \frac{a^2}{a^2 + k^2}, \quad k \in \mathbb{R}.$$

and allows transformation into a local model. If  $v(x, t) = \phi(\cdot; a) * u(\cdot; t)$ , then

$$\frac{\partial^2 v}{\partial x^2}(x, t) = a^2(v(x, t) - u(x, t))$$

# Travelling waves

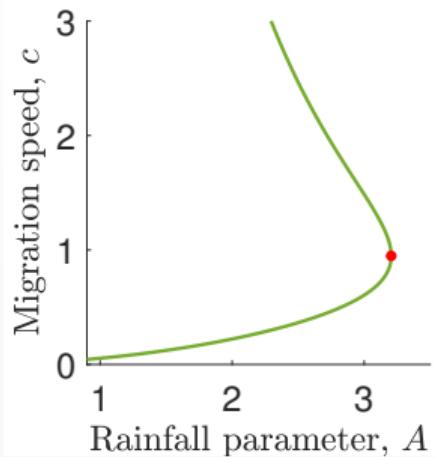
- Numerical simulations of the model on sloped terrain suggest uphill movement  $\Rightarrow$  Periodic travelling waves.
- Patterns correspond to **limit cycles** of the travelling wave integro-ODEs.



Numverical simulation.

# Travelling waves

- Numerical simulations of the model on sloped terrain suggest uphill movement  $\Rightarrow$  Periodic travelling waves.
- Patterns correspond to **limit cycles** of the travelling wave integro-ODEs.
- Numerical continuation shows that **patterns emanate from a Hopf bifurcation** and terminate at a homoclinic orbit.
- In the PDE model, pattern onset occurs at a threshold  $A = A_{\max}$ , the maximum rainfall level of the Hopf bifurcation loci in the travelling wave ODEs.



Location of the Hopf bifurcation in  $A$ - $c$  plane.

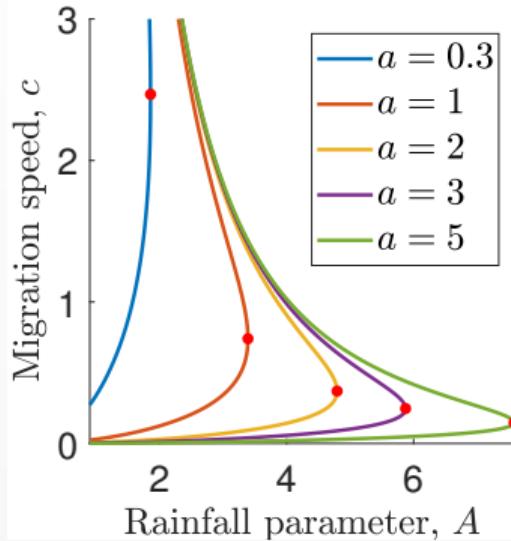
## Pattern onset

Using that  $\nu \gg 1$ ,

$$A_{\max} = \left( \frac{3C - B - 2\sqrt{2C}\sqrt{C - B}}{(B + C)^2} \right)^{\frac{1}{4}} a^{\frac{1}{2}} B^{\frac{5}{4}} \nu^{\frac{1}{2}},$$

to leading order in  $\nu$  as  $\nu \rightarrow \infty$ .

- Note that  $A_{\max} = O(\sqrt{\nu})$ .
- Decrease in  $a$  (i.e. increase in kernel width) causes decrease of  $A_{\max}$ .
- Increase in dispersal rate  $C$  causes decrease of  $A_{\max}$ .

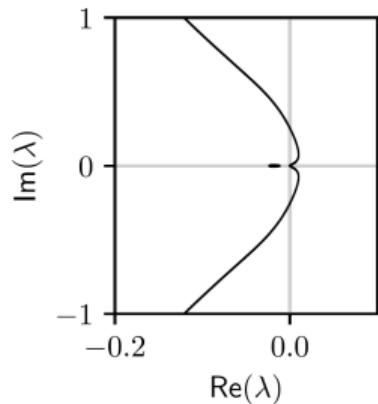


Locus of Hopf bifurcation for fixed  $C$  and varying  $a$ .<sup>9</sup>

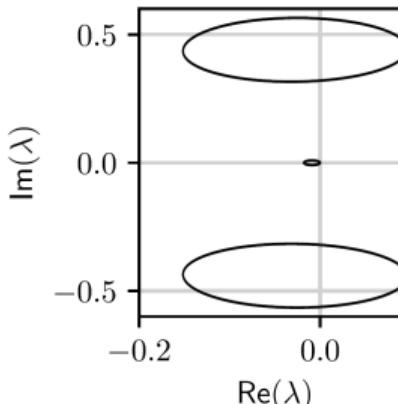
<sup>9</sup>EL and Sherratt, J. A.: *J. Math. Biol.* 77.3 (2018)

# Pattern stability

- The **essential spectrum** of a periodic travelling wave determines the behaviour of small perturbations.  $\Rightarrow$  Tool to determine pattern stability.
- Two different types stability boundaries: **Eckhaus-type** and **Hopf-type**.
- Essential spectra and stability boundaries are calculated using the numerical continuation method by Rademacher et al.<sup>10</sup>



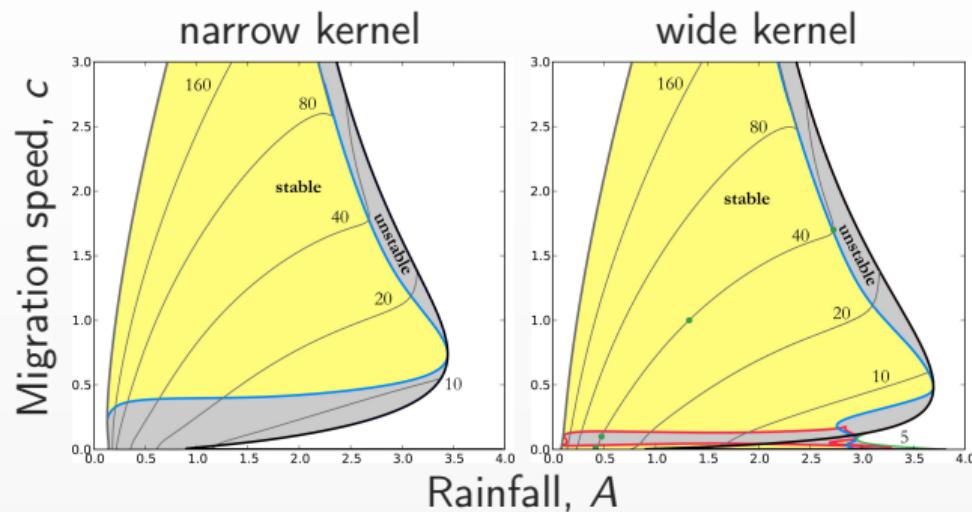
Eckhaus-type



Hopf-type

<sup>10</sup>Rademacher, J. D., Sandstede, B. and Scheel, A.: *Physica D* 229.2 (2007)

# Pattern existence and stability



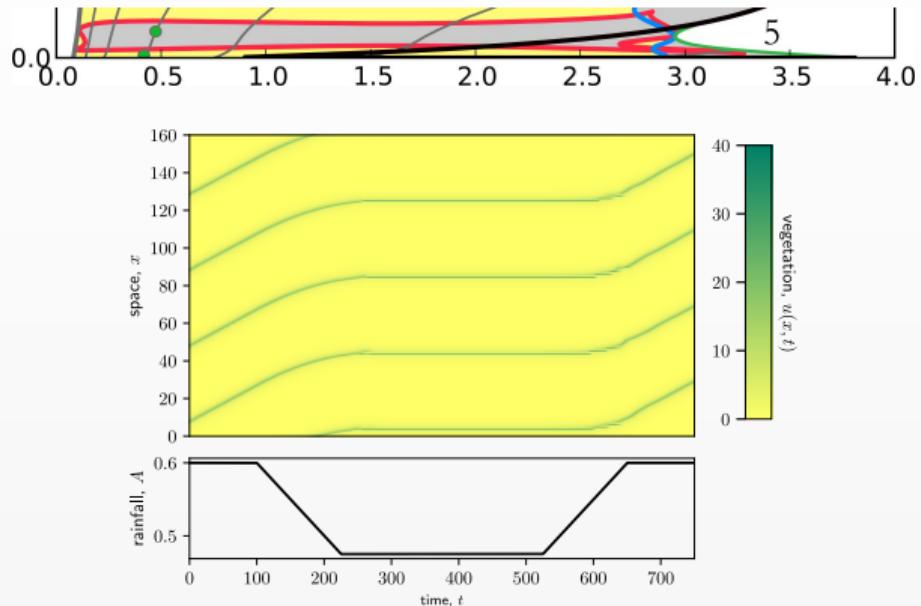
Stability of patterns in the  $A$ - $c$  plane.<sup>11</sup>

For wide kernels, the stability boundary towards the desert state changes from Eckhaus to Hopf-type. This yields

- increased resilience due to oscillating vegetation densities in peaks,

<sup>11</sup> Bennett, J. J. R. and Sherratt, J. A.: *J. Theor. Biol.* 481 (2018)

# Pattern existence and stability



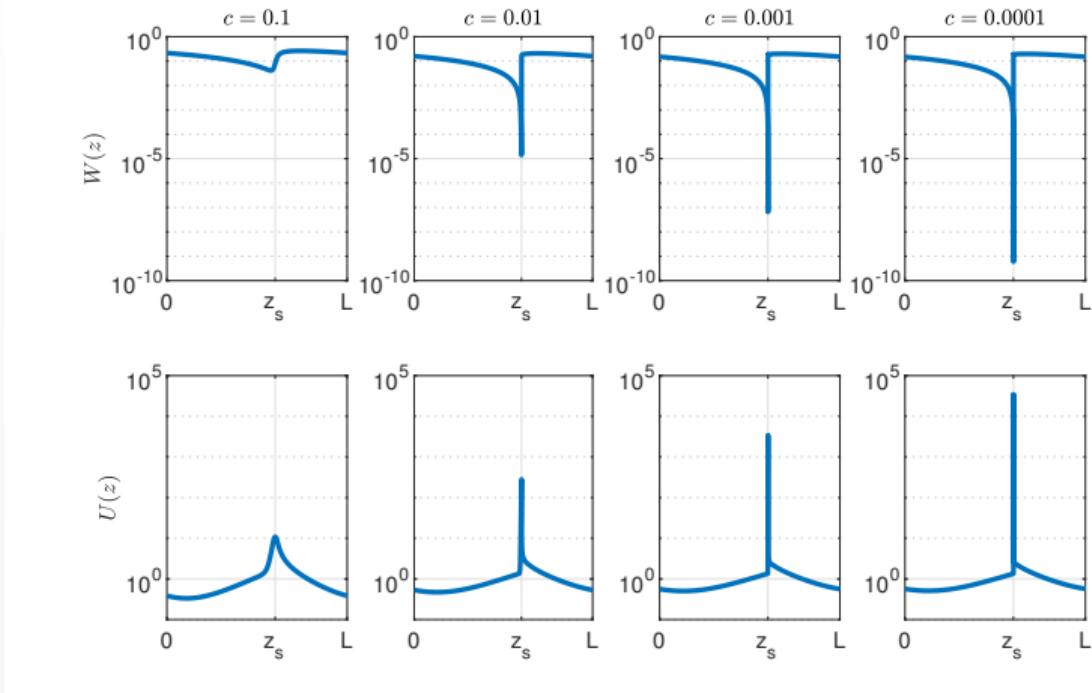
For wide kernels, the stability boundary towards the desert state changes from Eckhaus (sideband) to Hopf-type. This yields

- increased resilience due to oscillating vegetation densities in peaks,
- existence of stable patterns with small migration speed ( $c \ll 1$ ).

Existence of stable (almost) stationary patterns.<sup>12</sup>

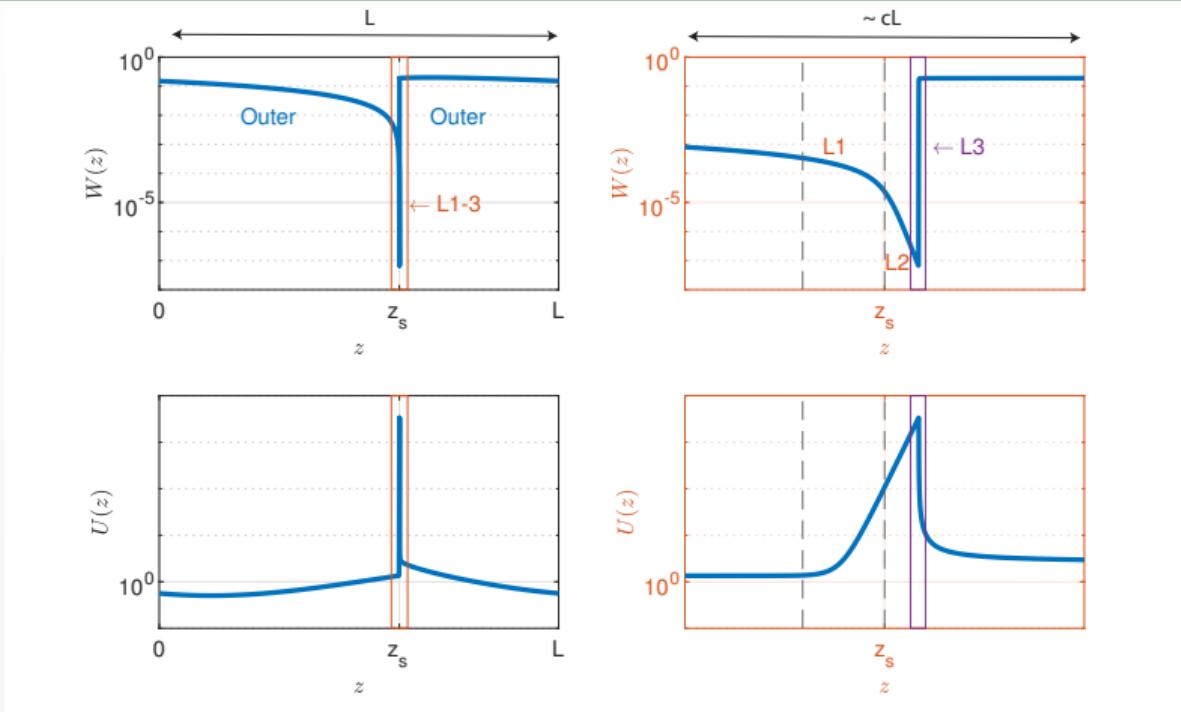
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# Almost stationary spike patterns



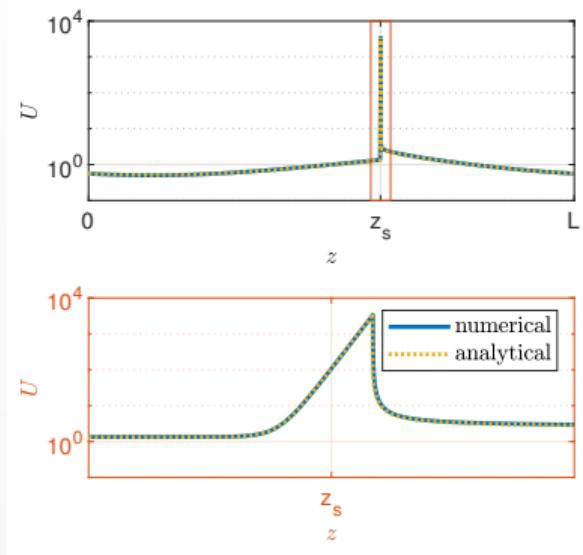
As  $c$  decreases, plant density develops a “spike”.

# Almost stationary spike patterns



Layered structure of spike solution

# Almost stationary spike patterns



Existence of almost stationary patterns is confirmed analytically using a singular perturbation theory approach, exploiting  $c \ll 1$ .

Analytical calculation of (almost) stationary patterns.<sup>13</sup>

<sup>13</sup>EL and Sherratt, J. A.: (submitted)

# Conclusions I

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- Wider kernels and higher dispersal rates inhibit pattern onset.
- Stability analysis of periodic travelling waves provides ecological insights into pattern dynamics: Long-range seed dispersal increases the resilience of a pattern and stabilises (almost) stationary patterns.
- Numerical simulations (pattern onset) and space discretisation to avoid nonlocality (calculation of essential spectra) show no qualitative differences for other kernel functions.

## Research questions

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- How does nonlocal seed dispersal affect onset, existence and stability of patterns?
- How can species coexist despite competition for one limiting resource?

# Klausmeier Model

The **one-species** extended Klausmeier reaction-advection-diffusion model.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \underbrace{u^2 w}_{\text{plant growth}} - \underbrace{B u}_{\text{plant loss}} + \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{plant dispersal}}, \\ \frac{\partial w}{\partial t} &= \underbrace{A}_{\text{rainfall}} - \underbrace{w}_{\text{evaporation}} - \underbrace{u^2 w}_{\text{water uptake by plants}} + \underbrace{\nu \frac{\partial w}{\partial x}}_{\text{water flow downhill}} + \underbrace{d \frac{\partial^2 w}{\partial x^2}}_{\text{water diffusion}}.\end{aligned}$$

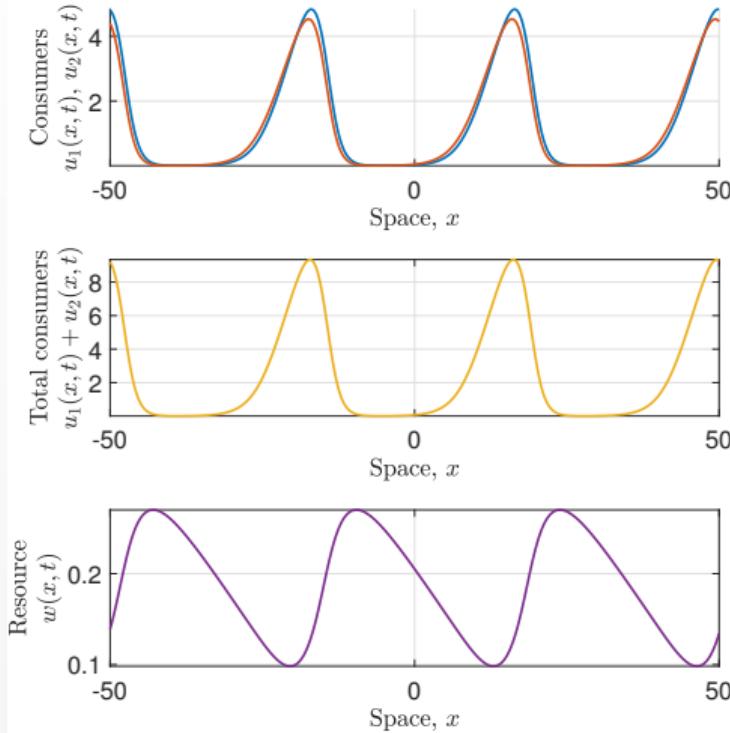
# Multispecies Model

Multispecies model:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \underbrace{wu_1(u_1 + Hu_2)}_{\text{plant growth}} - \underbrace{B_1 u_1}_{\text{plant loss}} \\ &\quad + \underbrace{\frac{\partial^2 u_1}{\partial x^2}}_{\text{plant dispersal}}, \\ \frac{\partial u_2}{\partial t} &= \underbrace{Fwu_2(u_1 + Hu_2)}_{\text{plant growth}} - \underbrace{B_2 u_2}_{\text{plant loss}} \\ &\quad + \underbrace{D \frac{\partial^2 u_2}{\partial x^2}}_{\text{plant dispersal}}, \\ \frac{\partial w}{\partial t} &= \underbrace{A}_{\text{rainfall}} - \underbrace{w}_{\text{evaporation}} - \underbrace{w(u_1 + u_2)(u_1 + Hu_2)}_{\text{water uptake by plants}} + \underbrace{\nu \frac{\partial w}{\partial x}}_{\text{water flow downhill}} + \underbrace{d \frac{\partial^2 w}{\partial x^2}}_{\text{water diffusion}}.\end{aligned}$$

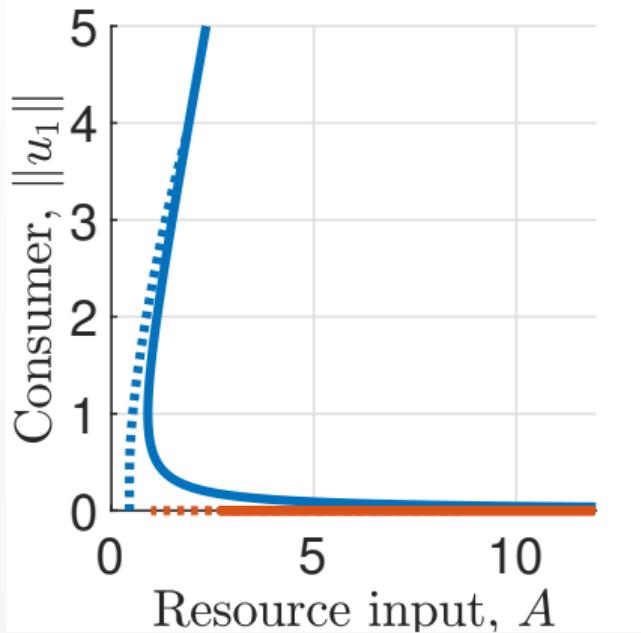
Species only differ quantitatively (i.e. in parameter values) but not qualitatively (i.e. same functional responses). Assume  $u_1$  is superior coloniser;  $u_2$  is locally superior.

# Simulations



- Consumer species coexist in a spatially patterned solution.
- Coexistence requires a balance between species' local average fitness and their colonisation abilities.
- Solutions are periodic travelling waves and move in the direction opposite to the unidirectional resource flux.

# Bifurcation diagram

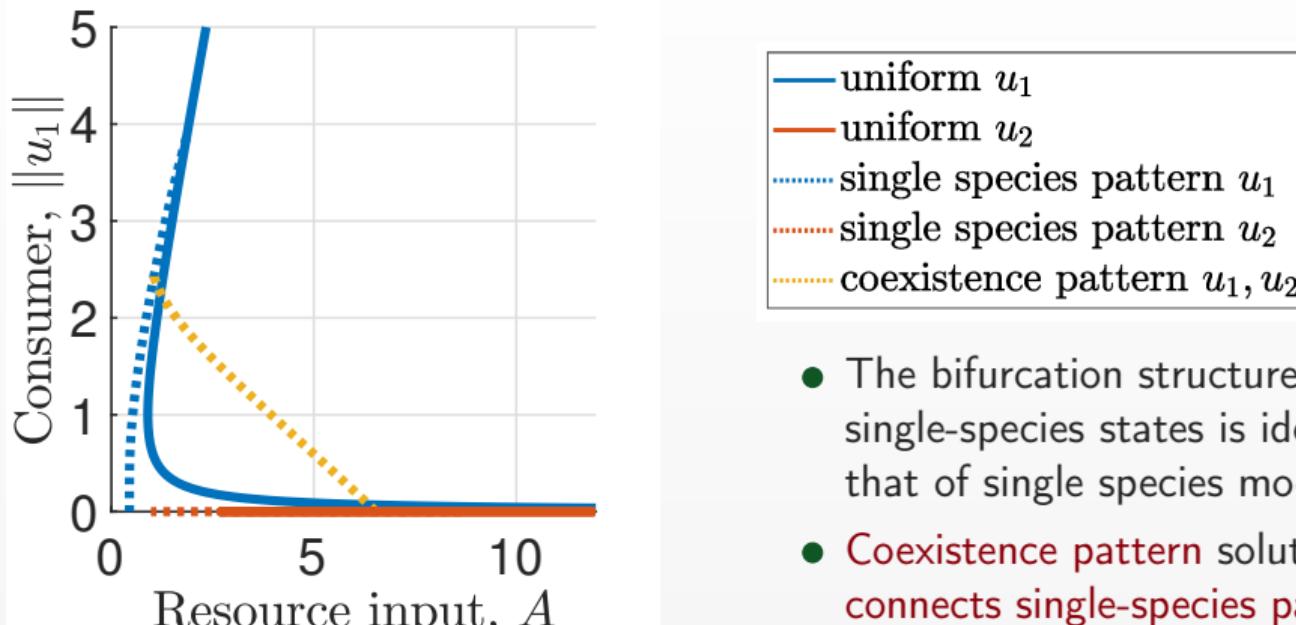


- uniform  $u_1$
- uniform  $u_2$
- single species pattern  $u_1$
- single species pattern  $u_2$
- coexistence pattern  $u_1, u_2$

- The bifurcation structure of single-species states is identical with that of single species model.

Bifurcation diagram: one wavespeed only

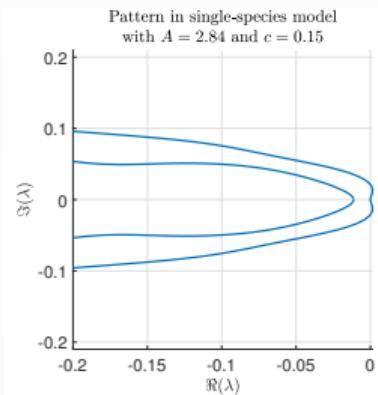
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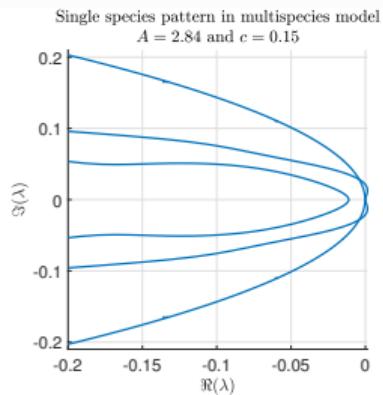
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- **Coexistence pattern** solution branch connects single-species pattern solution branches.

# Pattern onset



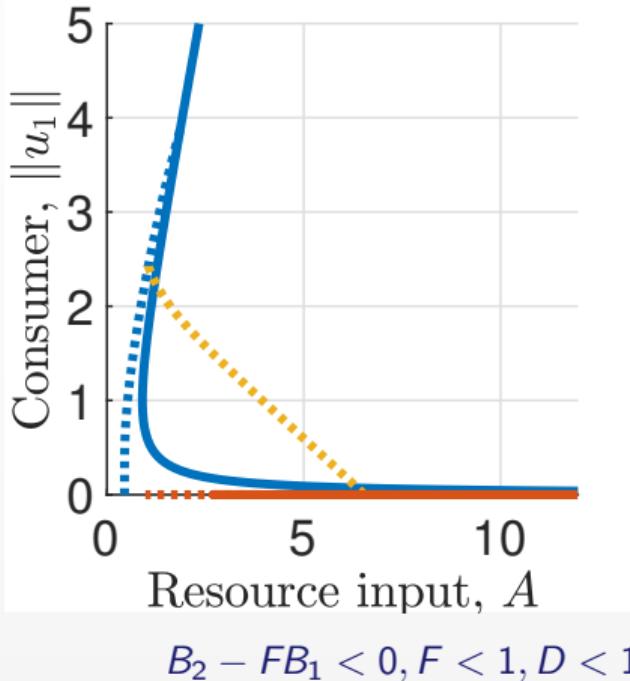
Essential spectrum in single-species model



Essential spectrum in multispecies model

- The key to understand **coexistence pattern onset** is knowledge of single-species pattern's stability.
- Pattern onset occurs as the single-species pattern loses/gains stability to the introduction of a competitor.

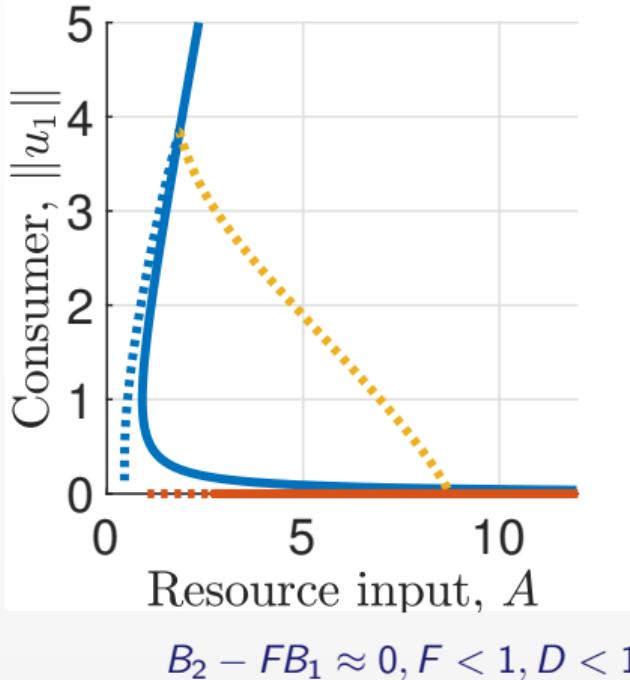
# Pattern existence



- uniform  $u_1$
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- coexistence pattern  $u_1, u_2$

- Key quantity: Local average fitness difference  $B_2 - FB_1$  determines stability of single-species states in spatially uniform setting.
- Condition for pattern existence: Balance between local competitive and colonisation abilities.

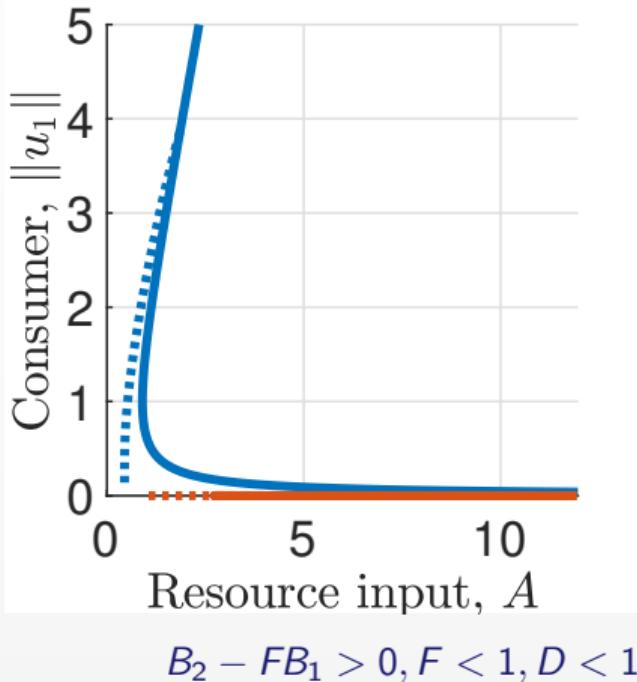
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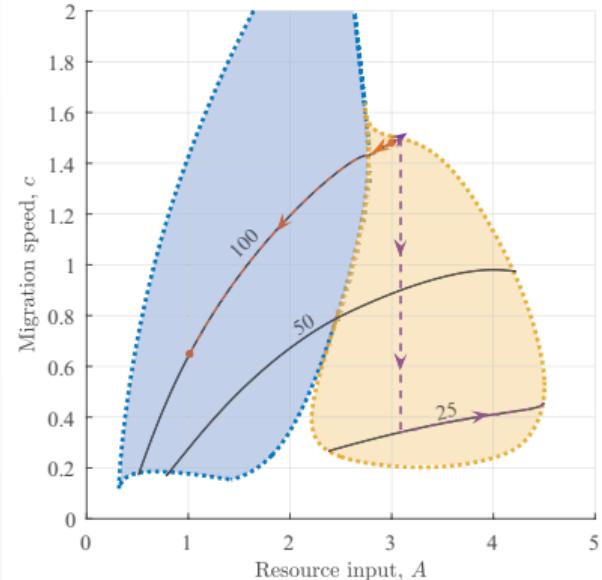
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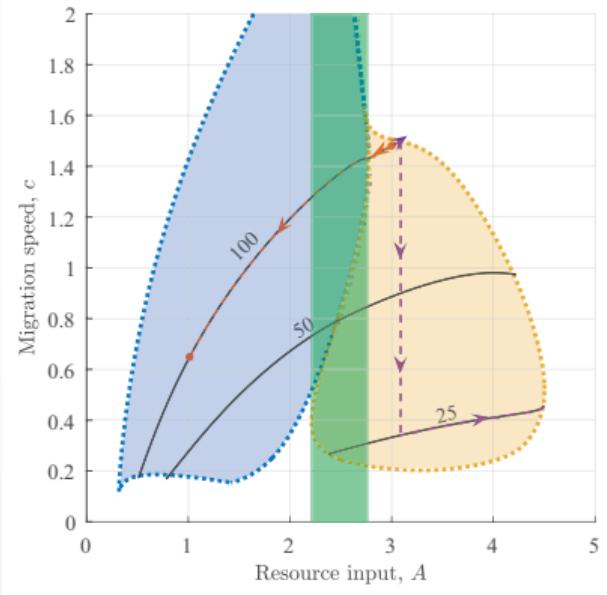
# Pattern stability



- Stability regions of patterned solution can be traced using numerical continuation.
- For decreasing resource input, coexistence state loses stability to single-species pattern of coloniser species.

Stability regions of system states.

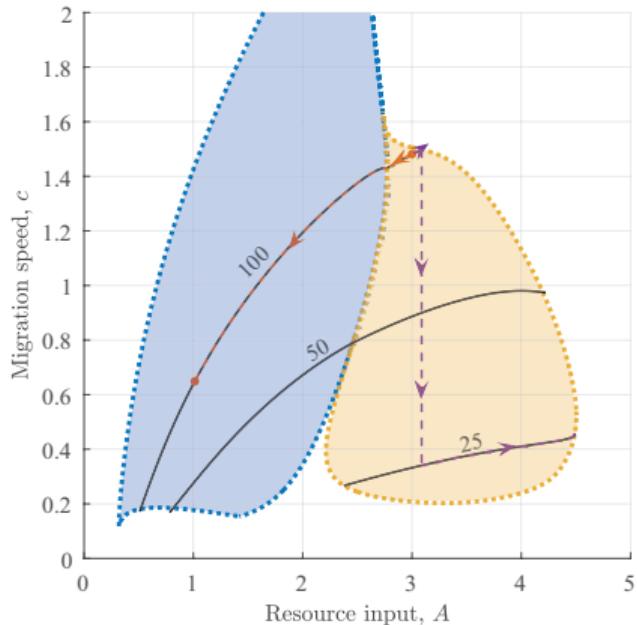
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Stability regions of system states.

- Stability regions of patterned solution can be traced using numerical continuation.
- For decreasing resource input, coexistence state loses stability to single-species pattern of coloniser species.
- **Bistability of single-species coloniser pattern and coexistence pattern occurs.**

# Hysteresis



Wavelength contours of stable patterns

- State transitions are affected by **hysteresis**.
- Extinction can occur despite a coexistence state being stable.
- **Ecosystem resilience depends on both current and past states of the system.**

## Conclusions II

- Spatial self-organisation is a coexistence mechanism<sup>14</sup>.
- Coexistence is enabled by spatial heterogeneities in the resource, caused by the consumers' self-organisation into patterns.
- A balance between species' colonisation abilities and local competitiveness promotes coexistence.
- Coexistence may occur as a metastable state if the average fitness difference between species is small<sup>15</sup>.

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<sup>14</sup>EL and Sherratt, J. A.: *J. Theor. Biol.* 487 (2020), EL: *Oikos* 130.4 (2021), EL: *Ecol. Complexity* 42 (2020).

<sup>15</sup>EL and Sherratt, J. A.: *Bull. Math. Biol.* 81.7 (2019).

## Future Work

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- How does nonlocal consumer dispersal affect species coexistence?<sup>16</sup>
- Do results extend to an arbitrary number of species?
- How do fluctuations in environmental conditions (in particular resource input) affect coexistence?
- In particular, what are the effects of seasonal<sup>17</sup>, intermittent<sup>18</sup> and probabilistic resource input regimes on both single-species and multispecies states?

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<sup>16</sup>EL and Sherratt, J. A.: *J. Math. Biol.* 77.3 (2018).

<sup>17</sup>EL and Sherratt, J. A.: *J. Math. Biol.* 81 (2020).

<sup>18</sup>EL and Sherratt, J. A.: *Physica D* 405 (2020).

## References

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Slides are available on my website.

<http://lukaseigentler.github.io>

- [1] Bennett, J. J. R. and Sherratt, J. A.: *J. Theor. Biol.* 481 (2018), pp. 151–161.
- [2] Eigentler, L.: *Oikos* 130.4 (2021), pp. 609–623.
- [3] Eigentler, L.: *Ecol. Complexity* 42 (2020), p. 100835.
- [4] Eigentler, L. and Sherratt, J. A.: *J. Math. Biol.* 77.3 (2018), pp. 739–763.
- [5] Eigentler, L. and Sherratt, J. A.: (submitted).
- [6] Eigentler, L. and Sherratt, J. A.: *Bull. Math. Biol.* 81.7 (2019), pp. 2290–2322.
- [7] Eigentler, L. and Sherratt, J. A.: *J. Theor. Biol.* 487 (2020), p. 110122.