Computability and Complexity

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All usual aids such as books, notes, exercises, and the like may be used during the exam, but no calculators, computers, cell phones, or similar equipment. The exam must be answered in Danish or English, and pencil may be used.

This set has 3 pages and consists of 4 questions of roughly equal weight.

Question 1

Let

$$\begin{aligned} L_0 &= \{a^k b^l c^m d^n \mid k, l, m, n \geq 0\} \\ L_1 &= \{a^k b^l c^m d^n \mid k, l, m, n \geq 0 \text{ and at least 2 of } k, l, m, n \text{ are equal}\} \\ L_2 &= \{a^k b^l c^m d^n \mid k, l, m, n \geq 0 \text{ and at least 3 of } k, l, m, n \text{ are equal}\} \end{aligned}$$

Show:

Part 1.1 L_0 is regular.

Part 1.2 L_1 is context-free, but not regular.
[Hint: Give context-free grammar for L_1 and use pumping lemma for regular languages.]

Part 1.3 L_2 is in L.

Question 2

In this question M and N denote nondeterministic finite automata.
Let $E_{\text{NFA}} = \{\langle M \rangle \mid L(M) = \emptyset\}$.
Let $EQ_{\text{NFA}} = \{\langle M, N \rangle \mid L(M) = L(N)\}$.

Part 2.1 Show that E_{NFA} is in NL.
[Hint: Show that E_{NFA} is in coNL.]

Part 2.2 Show that EQ_{NFA} is in PSPACE.
[Hint: Describe a nondeterministic Turing Machine that accepts $\overline{EQ_{\text{NFA}}}$, the complement of EQ_{NFA}, in polynomial space. Observe that a language A is in PSPACE if $\overline{A}$ is in PSPACE.]

Question 3

Let $ONE\text{-}HALT_{TM}$ be the language consisting of descriptions of pairs of Turing machines where for every string, exactly one of the two Turing machines halts on that string, i.e.,

$$ONE\text{-}HALT_{TM} = \{\langle A, B\rangle | A, B \text{ TMs and for all } w \in \Sigma^*, \text{ exactly one of } A, B \text{ halts on } w\}.$$

For this question, we define an enumerator for a language L to be a TM that outputs every string in L on its tape (using some special symbol to separate each string) and never halts even if L is finite.

Part 3.1 Show that $ONE\text{-}HALT_{TM}$ is not Turing-recognizable. You may use, without proof, that $\overline{HALT_{TM}}$ is not Turing-recognizable.

Part 3.2 Show that every finite language is decidable.

Part 3.3 Show that a language L is decidable if and only if there exists an enumerator that enumerates L in lexicographic order.

[Hint: The result you obtained in the previous part may help. Please note the definition of enumerator given above.]

Question 4

The problem *SUBGRAPH-ISOMORPHISM* takes two undirected graphs G_1 and G_2 and asks whether G_1 is a subgraph of G_2. In other words, the problem asks whether there is a one-to-one function f from the vertices of G_1 to vertices of G_2 such that there is an edge (u, v) in G_1 if and only if $(f(u), f(v))$ is an edge in G_2. Written as a language:

$$SUBGRAPH\text{-}ISOMORPHISM = \{\langle G_1, G_2\rangle | G_1 \text{ is a subgraph of } G_2\}.$$

It is known that *SUBGRAPH-ISOMORPHISM* is NP-complete by reduction from CLIQUE. Let *SUBGRAPH*-100-*ISOMORPHISM* be the problem consisting of those pairs $\langle G_1, G_2\rangle$ of *SUBGRAPH-ISOMORPHISM* where G_1 has at most 100 vertices:

$$SUBGRAPH\text{-}100\text{-}ISOMORPHISM = \{\langle G_1, G_2\rangle | G_1 \text{ is a subgraph of } G_2 \text{ and } |V(G_1)| \le 100\}.$$

Part 4.1 Show that *SUBGRAPH*-100-*ISOMORPHISM* belongs to P.

Recall that *CIRCUIT-SAT* is the set of descriptions of satisfiable circuits. A circuit is represented as a directed graph where each vertex has a label denoting if it is an input gate, an output gate (which is unique), or an AND, OR, or NOT gate. It is shown in Sipser that *CIRCUIT-SAT* is NP-complete. We consider a variant of this problem, called *PLANAR-CIRCUIT-SAT*.

A graph is *planar* if it can be drawn in the plane such that no two edges intersect, except possibly in their endpoints. We say that a Boolean circuit is planar if its representation as a graph is planar. In a planar Boolean circuit, we allow XOR gates in addition to the types of gates above and wires can have branch points (see, e.g., the endpoint of the wire starting in x_1 in Figure 1). Like gates, branch points are vertices in the graph representation of the circuit. Problem *PLANAR-CIRCUIT-SAT* is the set of descriptions of satisfiable planar circuits.

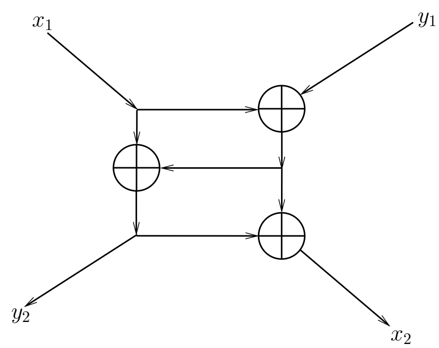

Figure 1: A planar circuit with three XOR gates.

Let `PlanarityTest` be a deterministic TM which, given the description $\langle G \rangle$ of an undirected graph G, outputs whether G is planar or not. Let `Draw2D` be a deterministic TM which on input $\langle G \rangle$ outputs a representation of a drawing of G in the plane such that

- the coordinates of vertices are integers,
- each edge is drawn as a line segment,
- no vertices are contained in such a segment other than its endpoints, and
- at most two segments cross in the same point.

The representation of the drawing that `Draw2D` outputs specifies for each vertex of G its integer coordinates in standard binary format.

Part 4.2 Show that *PLANAR-CIRCUIT-SAT* is NP-complete. You may assume, without proof, that TMs `PlanarityTest` and `Draw2D` exist and run in polynomial time.

[Hint: For a suitable reduction, consider a drawing of a circuit in the plane and consider gadgets of the form shown in Figure 1. What does such a gadget compute? Remember to explain your answer.]

END OF EXAM SET