Computability and Complexity

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All usual aids such as books, notes, exercises, and the like may be used during the exam, but no calculators, computers, cell phones, or similar equipment. The exam must be answered in Danish or English and may be answered in pencil.

This exam has 4 pages and consists of 3 questions, where Question 1 weighs approximately $\frac{1}{2}$ and Questions 2 and 3 each approximately $\frac{1}{4}$.

Question 1

We say $s \in \{0,1\}^*$ is a *binary numeral* if $s = 0$ or the first symbol of s is a 1.
We write $\hat{s}$ for the natural number denoted by binary numeral s, with most significant digit first; for example, if $s = 10111$ then $\hat{s} = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 16 + 0 + 4 + 2 + 1 = 23$.
We let s^R be the *reverse* of s; e.g. $s^R = 11101$ if $s = 10111$.

Define the following languages over $\Sigma = \{0, 1, \#\}$:

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\begin{aligned}
B &= \{s \mid s \text{ is a binary numeral}\} \\
MOD_i &= \{s \mid s \in B \wedge \hat{s} = 0 \pmod{i}\} \\
U_k &= \bigcup_{i=2}^{k} MOD_i \\
REV &= \{s\#t \mid s, t \in B \wedge s = t^R\} \\
SUM &= \{s\#t\#u \mid s, t, u \in B \wedge \hat{u} = \hat{s} + \hat{t}\}
\end{aligned}
$$

MOD_i contains the binary numerals denoting numbers divisible by i; in particular, $MOD_2 = \{0, 10, 100, 110, 1000, 1010, \ldots\}$ and $MOD_3 = \{0, 11, 110, 1001, 1100, 1111, \ldots\}$.
REV consists of pairs of binary numerals separated by a single #-symbol where the second numeral is the reverse of the first; for example, $1011\#1101 \in REV$, but $11\#10 \notin REV$ and $110\#011 \notin REV$ (because 011 is not a binary numeral).
SUM consists of triples of binary numerals separated by single #-symbols such that the third binary numeral is the sum of the first two; for example, $1\#10\#11 \in SUM$, but $1\#11\#0 \notin SUM$ and $1\#11\#011 \notin SUM$ (since 011 is not a binary numeral).

Part 1.1 Show that MOD_2 is regular.

Part 1.2 Show that MOD_3 is regular.

Part 1.3 Show that U_k is regular for any integer $k \geq 2$.

Part 1.4 Show that REV is context-free, but not regular.

Part 1.5 Show that $SUM \in \mathbf{L}$.

Part 1.6 Show that SUM is not context-free.

Question 2

Let *UNBOUNDED* be the language defined by

$$\begin{aligned} \mathit{UNBOUNDED} = \{\langle M\rangle | M &\text{ is a TM and there exists a string } s_M \in \Sigma^* \text{ such that } M \\ &\text{accesses infinitely many distinct tape cells on input } s_M\}. \end{aligned}$$

Here, a TM accesses a tape cell when its tape head is positioned at that cell. Let *ALWAYSHALT* be the language defined by

$$\mathit{ALWAYSHALT} = \{\langle M\rangle | M \text{ is a TM that halts on all strings in } \Sigma^*\}.$$

Part 2.1 In the following, $\subset$ denotes proper set inclusion, i.e., $A \subset B$ means that set A is contained in but not identical to set B.

Which one of the following statements is true?

1. $\overline{\mathit{UNBOUNDED}} \subset \mathit{ALWAYSHALT}$.
2. $\mathit{ALWAYSHALT} = \overline{\mathit{UNBOUNDED}}$,
3. $\mathit{ALWAYSHALT} \subset \overline{\mathit{UNBOUNDED}}$,

Justify your answer.

Part 2.2 It can be shown that $\overline{\mathit{UNBOUNDED}}$ is undecidable (in fact it is not Turing-recognizable). One way to show that a language is undecidable is to apply Rice's theorem (Problem 5.16 in the course book).

Which of the two conditions of Rice's theorem are satisfied for $\overline{\mathit{UNBOUNDED}}$? Explain your answer, i.e., prove for each condition that it holds or give a counter-example showing that the condition fails to hold.

Part 2.3 Show that *UNBOUNDED* is not Turing-recognizable.

Question 3

Given an integer $r \geq 0$, an *r-neighborhood cover* of an undirected unweighted graph $G = (V, E)$ is a subset V' of V such that for all edges $(u, v) \in E$ there exists a vertex $v' \in V'$ such that $d_G(u, v') \leq r$ or $d_G(v, v') \leq r$; here $d_G(x, y)$ denotes the shortest path distance (measured in number of edges) in G between vertices x and y of V. The *r-neighborhood cover problem*, $r-$*NEIGHBORHOOD*, is the problem of determining, for a given graph $G = (V, E)$ and a given $k \in \mathbb{N}$, if G has an r-neighborhood cover of size at most k.

Part 3.1 Fix an integer $r \geq 0$. Formulate $r-$*NEIGHBORHOOD* as a language and show that it belongs to NP.

Part 3.2 Fix an integer $r \geq 0$. Is $r-$*NEIGHBORHOOD* in PSPACE? Justify your answer.

Part 3.3 Show that $2-$*NEIGHBORHOOD* is NP-complete. [Hint: Consider *VERTEX-COVER*, and in the reduction subdivide each edge into a path consisting of three edges.]

Part 3.4 Fix integers $r > 0$ and $k > 0$. Let $(r, k)-$*NEIGHBORHOOD* be the problem of determining, for a given graph $G = (V, E)$, if G has an r-neighborhood cover of size at most k. Formulate $(r, k)-$*NEIGHBORHOOD* as a language and show that it belongs to P.

END OF EXAM SET