Computability and Complexity

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All usual aids such as books, notes, exercises and the like may be used during the exam, but no calculators, computers, cell phones or similar equipment. The exam may be answered in pencil.

This set has 4 pages and consists of 7 exercises that have an equal weight.

Exercise 1

Consider the following languages over the two-letter alphabet $\{0,1\}$:

$$\begin{array}{rcl} L_1 & = & \{w \in \{0,1\}^* \mid w \text{ contains an even number of occurrences of the substring } 01\} \\ & = & \{\varepsilon, 0, 1, 00, 10, 11, 000, 100, 110, 111, 0000, 0101, 1000, 1100, 1110, 1111, \ldots\} \\ L_2 & = & \{0^i 1^j 0^{i+j} \mid i, j \in \mathcal{N}\} \\ & = & \{\varepsilon, 00, 10, 0100, 001000, 011000, \ldots\} \\ L_3 & = & \{0^i 1^j 0^{i \cdot j} \mid i, j \in \mathcal{N}\} \\ & = & \{\varepsilon, 0, 1, 010, 00100, 01100, \ldots\} \end{array}$$

Question 1.1 Give a formal description of L_1 using either a DFA, an NFA, a GNFA or a regular expression.

Question 1.2 Prove that L_2 is not regular.

Question 1.3 Prove that L_2 is context-free.

Question 1.4 Prove that L_3 is not context-free.

Question 1.5 Prove that L_3 belongs to the class L.

Exercise 2

For languages A and B over the same alphabet Σ, let the ***shuffle*** of A and B be the language

$$\{a_1 b_1 \cdots a_k b_k \mid k \in \mathcal{N}, a_1, \ldots, a_k, b_1, \ldots, b_k \in \Sigma^*, a_1 \cdots a_k \in A, b_1 \cdots b_k \in B\}$$

Question 2.1 Show that the class of decidable languages is closed under shuffle.

Question 2.2 Show that the class NP is closed under shuffle.

Exercise 3

Let

$$\begin{array}{rcl} \mathit{EXHAUST}_{\mathrm{DFA}} & = & \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are DFAs, and } L(M_1) \cup L(M_2) = \Sigma^*\} \\ \mathit{EXHAUST}_{\mathrm{CFG}} & = & \{\langle G_1, G_2 \rangle \mid G_1 \text{ and } G_2 \text{ are CFGs, and } L(G_1) \cup L(G_2) = \Sigma^*\} \end{array}$$

Question 3.1 Show that $\mathit{EXHAUST}_{\mathrm{DFA}}$ is decidable.

Question 3.2 Show that $\mathit{EXHAUST}_{\mathrm{CFG}}$ is undecidable.

Exercise 4

Let A_{ENUM} denote the following Turing-undecidable language

$$A_{\text{ENUM}} = \{\langle E, w\rangle \mid w \text{ is one of the strings that is enumerated by the enumerator } E\}$$

Question 4.1 Show that a language A may be enumerated by an enumerator if and only if $A \leq_m A_{\text{ENUM}}$.

Exercise 5

The *absolute difference* of two natural numbers a and b, denoted $|a-b|$, is defined by

$$|a-b| \quad = \quad \begin{cases} a-b & \text{if } a \geq b \\ b-a & \text{if } a \leq b \end{cases}$$

In this exercise we consider well-formed formulas of the predicate calculus with one 3-ary relation symbol R. Examples of such formulas are (1), (2) and (3) below:

$$\begin{array}{ll} (1) & \forall x \forall y \exists z\, R(x,y,z) \\ (2) & \exists z \forall x\, R(x,x,z) \\ (3) & \exists x \forall y (R(x,x,y) \wedge R(y,y,x)) \end{array}$$

Let $(\mathcal{N}, |-|)$ be the model whose universe is the natural numbers $\mathcal{N}$ and which assigns the relation $|a-b| = c$ to the relation symbol $R(a,b,c)$. In this model, sentences (1) and (2) above are true, while (3) is false.

Question 5.1 Show a well-formed formula with x as its only free variable that is true in the model for $x = 0$ only.

Question 5.2 Show a well-formed formula with x as its only free variable that is true in the model for $x = 1$ only.

Question 5.3 The theory $\text{Th}(\mathcal{N}, |-|)$ is the collection of true sentences in the model. Show that $\text{Th}(\mathcal{N}, |-|)$ is decidable.

Exercise 6

As usual, for a language A we let $\overline{A} = \complement A = \Sigma^* \setminus A$ denote the complementary language.

Question 6.1 Show that $A \leq_T \overline{A}$ for all languages A.

Question 6.2 Show that $A \leq_m \overline{A}$ is not the case for all languages A.

Exercise 7

For a finite set U, in this exercise define an *exhausting subset family over* U to be a finite set $\mathbf{S} = \{S_1, S_2, \ldots, S_n\}$ of subsets of U exhausting U: $S_i \subseteq U$ for $i = 1, 2, \ldots, n$, $U = \bigcup_{i=1}^{n} S_i$.

For an integer k, $1 \leq k \leq n$, a *k-setcover of* $\mathbf{S}$ is a size k subset $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\} \subseteq \mathbf{S}$ that covers U: $\bigcup_{j=1}^{k} S_{i_j} = U$

Question 7.1 Let $\mathbf{S}$ be the exhausting subset family $\{\{1, 2, 3\}, \{2, 4, 8\}, \{3, 5, 7, 8\}, \{1, 5, 6\}\}$ over $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Show that $\mathbf{S}$ has a 3-setcover but no 2-setcover.

Question 7.2 Define the language

$$SETCOVER = \{\langle \mathbf{S}, k\rangle \mid \text{The exhausting subset family } \mathbf{S} \text{ over } \bigcup_{S \in \mathbf{S}} S \text{ has a } k\text{-setcover}\}$$

Show that *SETCOVER* is NP-complete.

END OF THE EXERCISES