Computability and Complexity

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All usual aids such as books, notes, exercises, and the like may be used during the exam, but no calculators, computers, cell phones, or similar equipment. The exam must be answered in Danish or English, and pencil may be used.

This set has 2 pages and consists of 4 questions that have an equal weight.

Question 1

Let $\Sigma = \{0, 1, 2\}$ and let $D = \{w \mid w$ contains an equal number of occurrences of the substrings 01 and 10$\}$. Show that:

Part 1.1 $D \in \mathbf{L}$.

Part 1.2 D is not regular.

Part 1.3 D is context-free.

Question 2

Consider the languages $EVEN_{TM}$ and $INFINITE_{TM}$, defined by:

$$EVEN_{TM} = \{\langle M \rangle \mid M \text{ is a TM that does not accept any string of odd length}\},$$
$$INFINITE_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } |L(M)| \text{ is infinite}\}.$$

Recall that $HALT_{TM} = \{\langle M, w \rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$.

Part 2.1 Use Rice's theorem to show that $EVEN_{TM}$ is undecidable.

Part 2.2 Show that $INFINITE_{TM}$ is not Turing-recognizable. You may use, without proof, that $\overline{HALT_{TM}}$ is not Turing-recognizable.

[Hint: Give a reduction. Consider simulating a TM M on a string w for $|x|$ steps where x is the input string.]

Question 3

Consider the language *SET-SPLITTING*, defined by:

$$\textit{SET-SPLITTING} = \{\langle S, C\rangle \mid S \text{ is a finite set and } C = \{C_1, \ldots, C_k\} \text{ is a collection of subsets of } S,\ \text{for some } k > 0,\ \text{such that elements of } S \text{ can be colored } \textit{red} \text{ or } \textit{blue} \text{ so that no } C_i \text{ has all its elements colored with the same color}\}.$$

Part 3.1 Show that *SET-SPLITTING* belongs to NP.

Part 3.2 Show that *SET-SPLITTING* is NP-complete.

[Hint: reduce from 3SAT. In the reduction, it is useful to introduce an additional Boolean variable z which is fixed to 'false' and to expand each clause with z; for instance, the clause $(x_1 \vee x_2 \vee \neg x_3)$ is expanded to $(x_1 \vee x_2 \vee \neg x_3 \vee z)$. This ensures that any satisfied clause contains at least one false literal. Let truth values correspond to colors red and blue and let the color associated with z represent 'false'.]

Part 3.3 If it can be shown that *SET-SPLITTING* can be decided in polynomial time by a deterministic TM, does this imply that HAMPATH belongs to P?

Question 4

Given alphabets Σ and Γ, a string homomorphism is a function $h : \Sigma^* \to \Gamma^*$ such that $h(xy) = h(x)\,h(y)$ for all $x, y \in \Sigma^*$. Note that h is determined by its application to strings of length 1.

Let $L \subseteq \Sigma^*$ be a language. Write $h(L)$ for the language $\{h(x) \mid x \in L\} \subseteq \Gamma^*$.

Part 4.1 Show that if L is regular, then $h(L)$ is regular.

Part 4.2 Show that if $h(L)$ is regular, then L is not necessarily regular.

END OF THE EXERCISES