Randomized Algorithms, Lecture 5

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Good Afternoon.

Today's Lecture

Chernoff Applications

- Chernoff Bound Recap
- Routing in a parallel computer
- A Wiring Problem

Last time I proved some Chernoff bounds, which for many problems allows us to get *exponentially small* bounds on the probability of deviating far from expectation.

Today we will show two algorithms, and analyze them via Chernoff bounds.

Chernoff Bound

Let $X_1, \ldots, X_n$ be independent Poisson trials such that, for $1 \leq i \leq n$, $\Pr[X_i = 1] = p_i$, where $0 < p_i < 1$. Then, for $X = \sum_{i=1}^n X_i$, $\mu = \mathbb{E}[X] = \sum_{i=1}^n p_i$, and any $\delta > 0$.

$$\Pr[X > (1+\delta)\mu] < F^+(\mu, \delta)$$

Where

$$F^+(\mu,\delta) := \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu < \begin{cases} 2^{-(1+\delta)\mu} & \text{if } \delta > 2e-1 \\ e^{-\frac{\delta^2 \mu}{4}} & \text{if } \delta \leq 2e-1 \end{cases}$$

For $0 < \epsilon < 1$ define

$$\Delta^+(\mu,\epsilon) := \min\{\delta > 0 \mid F^+(\mu,\delta) \leq \epsilon\}$$
$$\leq \begin{cases} \frac{\log_2 \frac{1}{\epsilon}}{\mu} - 1 & \text{if } F^+(\mu, 2e-1) > \epsilon \\ \sqrt{\frac{4 \ln \frac{1}{\epsilon}}{\mu}} & \text{if } F^+(\mu, 2e-1) \leq \epsilon \end{cases}$$

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These bounds come from Excercise 4.1 and Theorem 4.3 in the book (page 72).

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The book phrases this differently, as the unique value of δ that satisfies $F^+(\mu, \delta) = \epsilon$, but that is equivalent since for fixed μ and $\delta > 0$, $F^+(\mu, \delta)$ is strictly decreasing.

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This follows from the bounds above. If $F^{+}(\mu, 2e-1) > \epsilon$ then the smallest $\delta > 0$ such that $F^{+}(\mu, \delta) \leq \epsilon$ must be $> 2e-1$, and we can use the bound for $\delta > 2e-1$.
Conversely, if $F^{+}(\mu, 2e-1) \leq \epsilon$ then there is a $\delta \leq 2e-1$ satisfying $F^{+}(\mu, \delta) \leq \epsilon$, and so the smallest such delta is $\leq 2e-1$, and we can use the bound for $\delta \leq 2e-1$.

Note however the comment in the middle of page 73. As a rule of thumb, these bounds are only good when $\epsilon \in \mathcal{O}(n^{-c})$ and $\mu \in \Omega(\log n)$.

Permutation Routing Problem

Given a directed graph on N nodes, where each node i initially contains one *packet* destined for some node $d(i)$, s.t. $d(\cdot)$ is a permutation.

In each *step*, every edge can carry a single packet. A node that may send a packet on each outgoing edge (if it has the packets).

How many *steps* are necessary and sufficient?

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Permutation Routing Problem

A *route* for a packet is a list of edges it can follow from its source to its destination.

If two packets want to use the same edge, one may have to wait. The *queueing dicipline* for an algorithm is how it decides which packet goes first.

A routing algorithm is *oblivious*, if the route followed by the packet starting at v_i depends only on $d(i)$, not on $d(j)$ for any $j \neq i$.

Any algorithm must (implicitly) specify routes for all packets.

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Oblivious routing is attractive, because it is simple to implement in hardware. No comparison of different packets needed to decide route.

Routing, Lower Bound

Theorem

For any deterministic oblivious permutation routing algorithm on a network of N nodes each of out-degree d, there is an instance of permutation routing requiring $\Omega\left(\sqrt{N/d}\right)$ steps.

Excercise 4.2 shows that this holds even if the graph is the d-dimensional hypercube.

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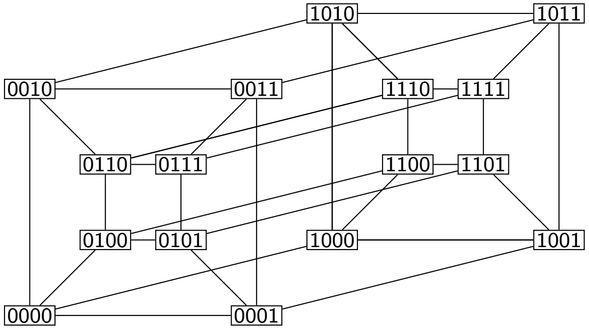
Routing, Bit-Fixing Strategy

Suppose the routing graph is the n-dimensional hypercube (having $N = 2^n$ vertices and Nn edges).



To send packet from e.g. 0110 to 1001, fix one bit at a time starting from the left, until done.

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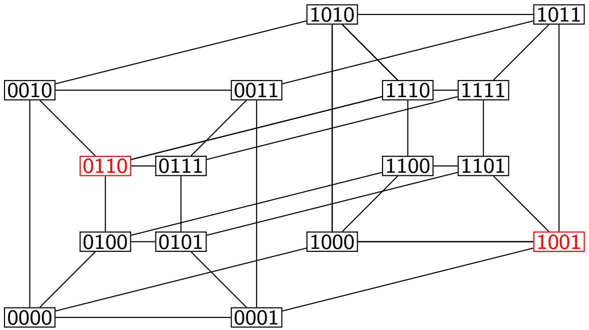

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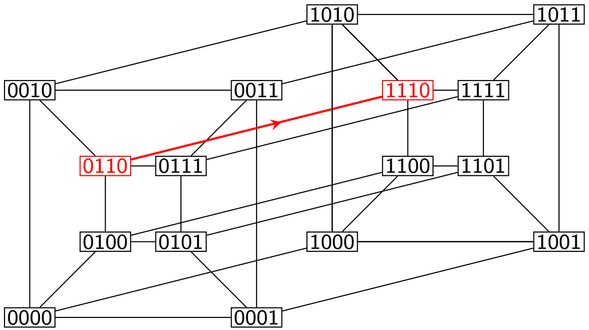

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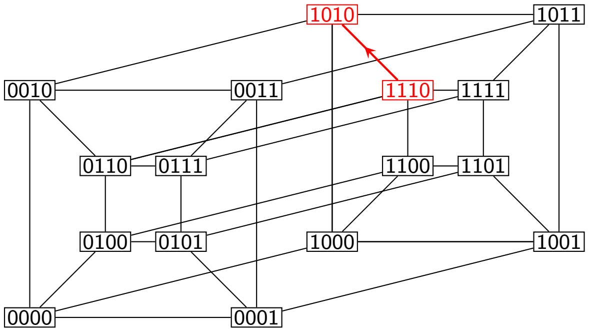

To send packet from e.g. 1110 to 1001, fix one bit at a time starting from the left, until done.

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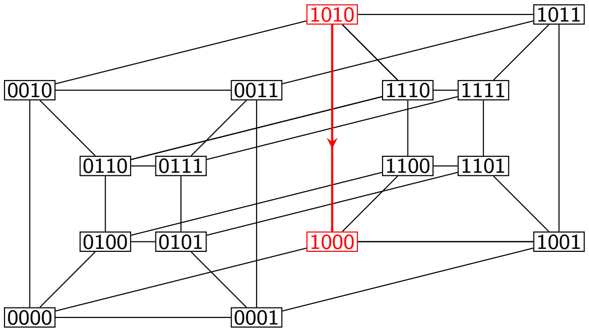

To send packet from e.g. 1010 to 1001, fix one bit at a time starting from the left, until done.

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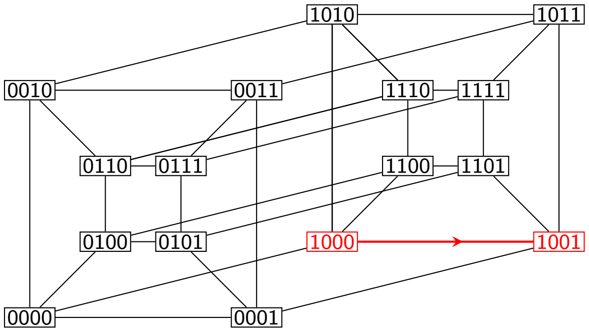

To send packet from e.g. 1000 to 1001, fix one bit at a time starting from the left, until done.

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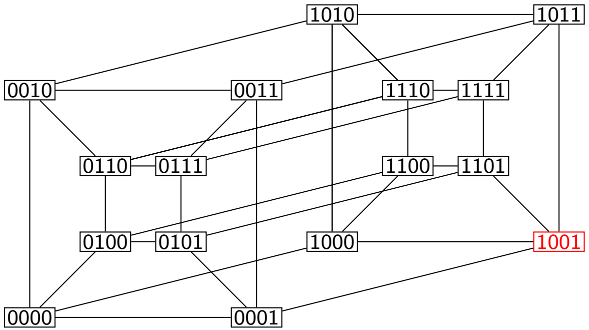

To send packet from e.g. 1001 to 1001, fix one bit at a time starting from the left, until done.

Randomized Routing (Valiant)

For each packet v_i, independently, define its route as follows:

Phase I Pick random $\sigma(i) \in \{1, \ldots, N\}$. Packet v_i travels to $\sigma(i)$ using bit-fixing strategy.

Phase II Packet v_i travels from $\sigma(i)$ to $d(i)$, using bit-fixing strategy.

Queueing dicipline: Arbitrary (FIFO).

We will show that this algorithm is significantly better than $\sqrt{N/d}$.
For simplicity, we will analyze the algorithm as if all packets finish Phase I before starting Phase II.

Routing Delay Phase I

Let delay(v_i) denote the number of steps v_i spends in queues waiting for other packets to move during Phase I. Total #steps for v_i in Phase I is at most $n + \text{delay}(v_i)$.

Lemma

Let $p_i = (e_1, \ldots, e_k)$ be the route for v_i, and let S_i be the set of other paths intersecting p_i. Then $\text{delay}(v_i) \leq |S_i|$.

Before we start considering any probabilities we need a bit of analysis.

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Routing Delay, Proof

Define the *lag* (wrt. p_i) of a packet $v \in S_i \cup \{v_i\}$ that is ready to move along edge $e_j \in p_i$ at time t to be $t - j$.

delay(v_i) is then the lag of v_i when it finally gets to traverse e_k.

We say packet $v \in S_i$ *leaves* p_i in the last time step where it traverses an edge in p_i.

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$\Longrightarrow$ At some time t, the lag of v_i increase to $\ell + 1$.

$\Longrightarrow$ At time t, some $v \in S_i$ follows e_j where $\ell = t - j$. At time t this v has lag ℓ wrt p_i.

$\Longrightarrow$ There is a last time t' where some $v \in S_i$ has lag ℓ wrt p_i. In step t', some $v \in S_i$ is ready to follow $e_{j'} \in p_i$, where $t' - j' = \ell$.

$\Longrightarrow$ In step t', some $\omega \in S_i$ follows $e_{j'} \in p_i$, where $t' - j' = \ell$. By choice of t', ω leaves p_i with lag ℓ wrt p_i (otherwise ω would be ready to traverse $e_{j'+1}$ in step $t' + 1$ and thus have lag $(t' + 1) - (j' + 1) = \ell$ in step $t' + 1$).

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Otherwise, v_i would be following e_j.

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Note that every $\omega \in S_i$ can leave p_i only once, so every packet in S_i contributes at most once to the set of distinct lag values wrt p_i that packets leaving p_i had.

Routing Delay, Expectation

Let H_{ij} indicate that p_i and p_j share at least one edge. Then for any fixed i, $\mathrm{delay}(v_i) \leq |S_i| = \sum_{j=1}^{N} H_{ij}$.

Since the $\sigma(\cdot)$ are all independent, the H_{ij} for $j \neq i$ are independent Poisson trials.

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For each edge e in the hypercube, let $T(e)$ count the number of routes using e. Fix the route $p_i = (e_1, \ldots, p_k)$ with $k \leq n$. Then

$$\sum_{j=1}^{N} H_{ij} \leq \sum_{\ell=1}^{k} T(e_\ell)$$

$$\mathbb{E}\left[\sum_{j=1}^{N} H_{ij}\right] \leq \mathbb{E}\left[\sum_{\ell=1}^{k} T(e_\ell)\right] = \sum_{\ell=1}^{k} \mathbb{E}\left[T(e_\ell)\right]$$

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Routing Delay, Expectation

The expected length of each route is $\frac{n}{2}$, so the expected total length of all routes is

$$\sum_{e \in E} \mathbb{E}\left[T(e)\right] = \sum_{j=1}^{N} \mathbb{E}\left[|p_j|\right] = \sum_{j=1}^{N} \frac{n}{2} = \frac{Nn}{2}$$

By symmetry, $\mathbb{E}\left[T(e)\right] = \mathbb{E}\left[T(e')\right]$ for all $e, e' \in E$, so for any $e \in E$

$$\mathbb{E}\left[T(e)\right] = \frac{1}{|E|} \sum_{e \in E} \mathbb{E}\left[T(e)\right] = \frac{1}{Nn} \frac{Nn}{2} = \frac{1}{2}$$

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Routing Delay, Chernoff Bound

Thus, for $p_i = (e_1, \ldots, p_k)$

$$\mathbb{E}\Big[\sum_{j=1}^{N} H_{ij}\Big] \leq \sum_{\ell=1}^{k} \mathbb{E}\big[T(e_\ell)\big] = \frac{k}{2} \leq \frac{n}{2}$$

Now we can apply our first Chernoff bound to get

$$\Pr\Big[\sum_{j=1}^{N} H_{ij} > 6n\Big] < F^{+}\Big(\frac{n}{2}, 11\Big) < 2^{-(1+11)\frac{n}{2}} = 2^{-6n}$$

Since $\text{delay}(v_i) \leq \sum_{j=1}^{N} H_{ij}$, this gives

$$\Pr\big[\text{delay}(v_i) > 6n\big] < 2^{-6n}$$

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Here we are using that $\mu \leq \frac{n}{2}$, so $6n \geq 12\mu = (1+\delta)\mu$, and we can choose $\delta = 12 - 1 = 11$ to get this bound.

Routing Delay, Chernoff Bound

Thus, for $p_i = (e_1, \ldots, p_k)$

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Since $\delta = 12 > 2e - 1 \approx 4.43656$, by Excercise 4.1 we have $F^+(\mu, \delta) < 2^{-(1+\delta)\mu}$.

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Routing Steps, Phase I Total

Since the probability of a union of events is at most the sum of probabilities, the probability that *any* delay exceeds $6n$ is at most

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Thus, since each path has length at most n

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Routing Steps, Phase II Total

So what about Phase II?

Theorem

With probability at least $1 - 2^{1-5n} \geq 1 - N^{-4}$, *every packet reaches its destination in* $14n$ *or fewer steps.*

Contrast with the deterministic lower bound of $\Omega\left(\sqrt{N/n}\right)$.

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Randomized Routing, Summary

We have seen that for every deterministic oblivious permutation routing algorithm there is an instance requiring $\Omega(\sqrt{N/n})$.

In contrast, Valiant's randomized routing scheme uses only $14n$ steps with high probability. This is *exponentially* better.

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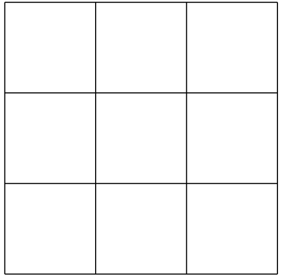
A Wiring Problem


$n = 9$

Given a *gate-array* of $\sqrt{n} \times \sqrt{n}$ gates, and a sequence of *nets* (pairs of gates), find a *global wiring* with at most one bend per net, minimizing the maximal number of wires crossing each boundary segment.

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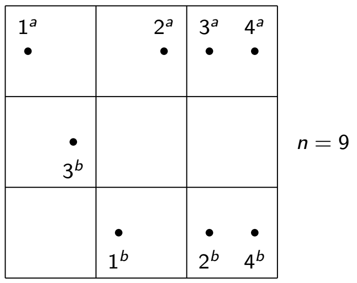

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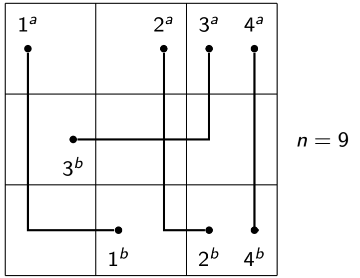

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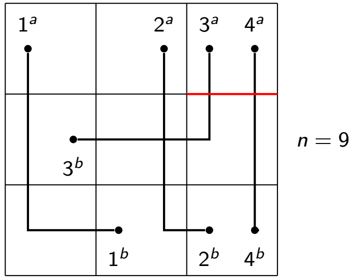

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A Wiring Problem, ILP Formulation

For each net our only choice is between going horizontal or vertical first.

For net $i = (i^a, i^b)$ let $x_{i0} = 1$ if the segment connected to i^a is horizontal, and $x_{i0} = 0$ otherwise.

Conversely, let $x_{i1} = 1$ if the segment connected to i^a is vertical, and $x_{i1} = 0$ otherwise.

Note that this way, we always have $x_{i1} = 1 - x_{i0}$, even for lines without a bend.

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For each net our only choice is between going horizontal or vertical first.

A Wiring Problem, ILP Formulation

For each boundary b define

$$T_{b0} := \{i \mid \text{net } i \text{ passes through } b \text{ if } x_{i0} = 1\}$$
$$T_{b1} := \{i \mid \text{net } i \text{ passes through } b \text{ if } x_{i1} = 1\}$$

Then an *Integer Linear Program* (ILP) for the problem is

$$\begin{array}{rll} \text{minimize} & w & \\ \text{where} & x_{i0}, x_{i1} \in \{0,1\} & (\forall i) \\ \text{subject to} & x_{i0} + x_{i1} = 1 & (\forall i) \\ & \displaystyle\sum_{i \in T_{b0}} x_{i0} + \sum_{i \in T_{b1}} x_{i1} \leq w & (\forall b) \end{array}$$

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A Wiring Problem, LP Relaxation

Solving ILPs is NP-hard, also for this special case.

Instead, we can approximate by solving the following *LP-relaxation*, and use *randomized rounding*.

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A Wiring Problem, Randomized Rounding

Let $(\hat{x}, \hat{w})$ be the solution to the LP-relaxation.

Construct a feasible solution $(\bar{x}, \bar{w})$ to the ILP as follows: Independently for each i, set $\bar{x}_{i0} = 1$ and $\bar{x}_{i1} = 0$ with probability $\hat{x}_{i0}$; otherwise set $\bar{x}_{i0} = 0$ and $\bar{x}_{i1} = 1$.

A Wiring Problem, Randomized Rounding

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A Wiring Problem, Analysis

Theorem

Let $0 < \epsilon < 1$. *With probability at least* $1 - \epsilon$

$$\begin{aligned}\bar{w} &\leq (1 + \Delta^+(\hat{w}, \tfrac{\epsilon}{2n}))\hat{w} \\ &\leq (1 + \Delta^+(w_O, \tfrac{\epsilon}{2n}))w_O\end{aligned}$$

Where w_O *is the optimum for the original ILP.*

Since every feasible solution to the ILP is also feasible in the LP-relaxation, we have $\hat{w} \leq w_O$ so the second inequality is trivial.

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In other words, we have an approximation algorithm, with approximation factor $1 + \Delta^+(W_O, \frac{\epsilon}{2n})$.

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A Wiring Problem, Proof

Let $\bar{w}_b$ count the wires crossing b in $\bar{x}$. Then

$$\bar{w}_b = \sum_{i \in T_{b0}} \bar{x}_{i0} + \sum_{i \in T_{b1}} \bar{x}_{i1}$$

$$\mathbb{E}[\bar{w}_b] = \sum_{i \in T_{b0}} \mathbb{E}[\bar{x}_{i0}] + \sum_{i \in T_{b1}} \mathbb{E}[\bar{x}_{i1}]$$

$$= \sum_{i \in T_{b0}} \hat{x}_{i0} + \sum_{i \in T_{b1}} \hat{x}_{i1} \leq \hat{w}$$

$\bar{w}_b$ is the sum of independent Poisson trials, so $\mu_b = \mathbb{E}[\bar{w}_b]$ and $\delta_b = \Delta^+(\mu, \frac{\epsilon}{2n})$ gives Chernoff bound

$$\Pr[\bar{w}_b > (1 + \delta_b)\mu_b] < \frac{\epsilon}{2n}$$

By definition

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By taking $\mathbb{E}$ on both sides

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$$\Pr[\bar{w}_b > (1 + \delta_b)\mu_b] < \frac{\epsilon}{2n}$$

By definition of our randomized rounding,

$$\mathbb{E}[\bar{x}_{i0}] = \Pr[\bar{x}_{i0} = 1] = \hat{x}_{i0}$$
$$\mathbb{E}[\bar{x}_{i1}] = \Pr[\bar{x}_{i1} = 1] = \hat{x}_{i1}$$

A Wiring Problem, Proof

By definition of our LP-relaxation.

Let $\bar{w}_b$ count the wires crossing b in $\bar{x}$. Then

$$\bar{w}_b = \sum_{i \in T_{b0}} \bar{x}_{i0} + \sum_{i \in T_{b1}} \bar{x}_{i1}$$

$$\mathbb{E}[\bar{w}_b] = \sum_{i \in T_{b0}} \mathbb{E}[\bar{x}_{i0}] + \sum_{i \in T_{b1}} \mathbb{E}[\bar{x}_{i1}]$$

$$= \sum_{i \in T_{b0}} \hat{x}_{i0} + \sum_{i \in T_{b1}} \hat{x}_{i1} \leq \hat{w}$$

$\bar{w}_b$ is the sum of independent Poisson trials, so $\mu_b = \mathbb{E}[\bar{w}_b]$ and $\delta_b = \Delta^+(\mu, \frac{\epsilon}{2n})$ gives Chernoff bound

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$$\Pr[\bar{w}_b > (1 + \delta_b)\mu_b] < \frac{\epsilon}{2n}$$

Because T_{b0} and T_{b1} are disjoint, so in the sum for $\bar{w}_b$ each i occurs at most once as an index.

For each i, the one of $\bar{x}_{i0}$ and $\bar{x}_{i1}$ that is included is chosen independently of all the others.

A Wiring Problem, Proof

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$$\begin{aligned}
\bar{w}_b &= \sum_{i \in T_{b0}} \bar{x}_{i0} + \sum_{i \in T_{b1}} \bar{x}_{i1} \\
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&= \sum_{i \in T_{b0}} \hat{x}_{i0} + \sum_{i \in T_{b1}} \hat{x}_{i1} \leq \hat{w}
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A Wiring Problem, Proof

Let $\delta = \Delta^+(\hat{w}, \frac{\epsilon}{2n}) \geq \delta_b$. Then

$$(1+\delta)\hat{w} \geq (1+\delta_b)\mu_b$$

So

$$\Pr[\bar{w}_b > (1+\delta)\hat{w}] \leq \Pr[\bar{w}_b > (1+\delta_b)\mu_b] < \tfrac{\epsilon}{2n}$$

And finally

$$\begin{aligned}\Pr[\bar{w} > (1+\delta)\hat{w}] &= \Pr[\max_b \bar{w}_b > (1+\delta)\hat{w}] \\ &\leq \sum_b \Pr[\bar{w}_b > (1+\delta)\hat{w}] \\ &< (2n - 2\sqrt{n}) \cdot \tfrac{\epsilon}{2n} < \epsilon \qquad \square\end{aligned}$$

Because $\hat{w} \geq \bar{w}_b$ and $\delta_b = \Delta^+(\bar{w}_b, \frac{\epsilon}{2n})$.

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Since the probability of a union of events it at most the sum of probabilities.

A Wiring Problem, Proof

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Since there are exactly $2n - 2\sqrt{n}$ boundaries, and each term in the sum is $< \frac{\epsilon}{2n}$

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A Wiring Problem, Final

How good is the bound $\bar{w} \leq (1 + \Delta^+(w_O, \frac{\epsilon}{2n}))w_O$?

Depends on w_O. For $w_O = n^\gamma$ we can show

$$\bar{w} \leq \left(1 + \sqrt{\frac{4 \ln \frac{2n}{\epsilon}}{n^\gamma}}\right) n^\gamma = (1 + o(1))w_O$$

On the other hand, for "small" w_O, e.g. $w_O = 20$ we get

$$\bar{w} \leq \left(1 + \Theta\left(\frac{\ln n}{\ln \ln n}\right)\right) w_O$$

In this case, a much better rounding method exists, and gives $\bar{w} \leq 2w_O$ (Exercise 4.7).

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This uses Theorem 4.3.

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This is part of Assignment #3.

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- We have seen two very different algorithms where Chernoff Bounds were essential for the analysis.
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