Computability and Complexity

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All usual aids such as books, notes, exercises, and the like may be used during the exam, but no calculators, computers, cell phones, or similar equipment. The exam must be answered in Danish or English, and pencil may be used.

This set has 2 pages and consists of 5 questions that have an equal weight.

Question 1

For two integers a and b ($b > 0$), the principal remainder of the division of a with b is denoted, as usual, $a \bmod b$:

$$a \bmod b = a - \left\lfloor \frac{a}{b} \right\rfloor \cdot b$$

The formal languages C_3 and C over the two-letter alphabet $\{0, 1\}$ are defined by:

$$\begin{aligned} C_3 &= \{0^a 1110^r \mid a \text{ and } r \text{ are non-negative integers}, r = a \bmod 3\}. \\ C &= \{0^a 1^b 0^r \mid a \text{ and } r \text{ are non-negative integers}, b \text{ is a positive integer}, r = a \bmod b\}. \end{aligned}$$

For example, $00001110 \in C_3$ and $000111 \in C$, whereas $011100 \notin C_3$ and $0001100 \notin C$.

Part 1.1 Show that C_3 is regular.

Part 1.2 Show that C is *not* regular.

Part 1.3 Show that C is *not* context-free. [**Hint:** Assume that C is context-free, and consider the string $0^{p-1}1^p0^{p-1}$ where p is the pumping length of C.]

Part 1.4 For each of the languages C_3 and C explain whether or not it belongs to the class L of languages decidable in logarithmic space.

Question 2

Consider first-order predicate calculus as defined in Sipser (Section 6.2) without constants or function symbols and with one quaternary (4-ary) relation symbol R. Some examples of sentences in this language are:

$$\exists x(\neg R(x,x,x,x) \wedge \forall y \forall z(\neg R(z,z,z,z) \vee R(x,y,z,y))) \tag{1}$$

$$\forall w \forall x \forall y \forall z(\neg R(z,z,z,z) \vee \neg R(x,y,z,w) \vee R(y,x,z,w)) \tag{2}$$

Formulas of the language are interpreted in the model $(\mathcal{N}, \times{+}{=})$ whose universe is $\mathcal{N}$, the non-negative integers, and where $R(a,b,c,d)$ denotes the relation $a \times b + c = d$ among the integers a, b, c, and d. As an example, under this interpretation both of the above sentences are true. The sentence (1) expresses the existence of a multiplicative unit, and the sentence (2) expresses commutativity of multiplication.

Th($\mathcal{N}, \times +=$) is the collection of true sentences in this model.

Part 2.1 Show that Th($\mathcal{N}, \times +=$) is undecidable.

Question 3

An instance of the *RESTRICTED-SUBSET-SUM* problem consists (as for the ordinary *SUBSET-SUM* problem) of a multiset S of numbers and a target number t, with the additional requirement that the sum of the elements of S have the value $2t$.

We want to determine whether S has a subcollection with sum t:

$$\textit{RESTRICTED-SUBSET-SUM} = \{ \langle S, t \rangle \mid \begin{array}{l} S \text{ is a multiset of numbers and } t \text{ is a number,} \\ \sum_{x \in S} x = 2t, \text{ and} \\ \text{for a subcollection } S' \subseteq S \text{ we have } \sum_{y \in S'} y = t \ \}. \end{array}$$

Part 3.1 Show that *RESTRICTED-SUBSET-SUM* is NP-complete. [**Hint:** Reduce from *SUBSET-SUM* by adding an extra element to the *SUBSET-SUM* instance.]

Question 4

Let A_{REX} denote the acceptance problem for regular expressions:

$$A_{\mathsf{REX}} = \{\langle R, w \rangle \mid R \text{ is a regular expression that generates the string } w\}.$$

Part 4.1 Show that A_{REX} is in PSPACE.

Question 5

Let EQ_{DFA} denote the problem whether two deterministic finite automata (over the same alphabet) recognize the same language:

$$EQ_{\mathsf{DFA}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are DFAs with } L(M_1) = L(M_2)\}.$$

Part 5.1 Show that EQ_{DFA} is NL-complete. [**Hint:** For the reduction part reduce from $\overline{PATH}$, using the automaton alphabet $\{1, \ldots, n\}$, where n is the highest degree of any node in the graph G of the $\overline{PATH}$ instance. You may use without proof that $\overline{PATH}$ is NL-complete.]

END OF THE EXERCISES