Computability and Complexity

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Written exam for 4 hours, January 23rd, 2013

All usual aids such as books, notes, exercises, and the like may be used during the exam, but no calculators, computers, cell phones, or similar equipment. The exam must be answered in Danish or English, and pencil may be used.

This set has 2 pages and consists of 5 questions that have an equal weight.

Question 1

For $k = 1, 2, 3, \ldots$ define the languages J_k, M_k and N_k over the alphabet $\{a, b\}$ as follows:

$$\begin{aligned} J_k &= \{ub^k \mid u \in \{a,b\}^*, |u| \leq k\} \\ M_k &= \{a^k u b^k \mid u \in \{a,b\}^*, |u| \leq k\} \\ N_k &= \{a^k u b^k \mid u \in \{a,b\}^*, |u| \geq k\} \end{aligned}$$

Part 1.1 Show that $M_1 \setminus N_1 \neq \emptyset$ and that $M_k \subseteq N_1$ for $k = 2, 3, \ldots$.

Now consider the languages J, M and N defined by:

$$J = \bigcup_{k=1}^{\infty} J_k \qquad M = \bigcup_{k=1}^{\infty} M_k \qquad N = \bigcup_{k=1}^{\infty} N_k$$

Part 1.2 Show that J is context-free.

Part 1.3 Show that J is not regular.

Part 1.4 Show that M is in the class L of languages decidable in logarithmic space.

Part 1.5 Show that M is not context-free.

Part 1.6 Show that N is regular.

Question 2

For a string w and an integer $i \geq 1$, w^i denotes the concatenation of w with itself i times. As an example, $(001)^3 = 001001001$. Define the language A by

$$A = \{\langle M \rangle \mid M \text{ is a TM that accepts a string } w \text{ if and only if it accepts } w^i \text{ for all } i > 1\}.$$

Define another language B by

$$B = \{\langle M \rangle \mid M \text{ is an LBA that accepts at least one string, rejects at least one string, and loops on at least one string}\}.$$

Part 2.1 Show that A is undecidable by reduction from A_{TM}.

Part 2.2 Is B decidable? Is B Turing-recognizable? Is $\overline{B}$ Turing-recognizable?

Question 3

The problem *PARTITION* is that of deciding whether a given multiset S of integers admits a partition into two subsets of equal sum:

$$PARTITION = \{\langle S\rangle \mid S \text{ is a multiset of integers and for some } X \subseteq S, \sum_{x\in X} x = \sum_{x\in S\setminus X} x\}.$$

Consider the related problem *PSPARTITION* which has the additional requirement that the sum of elements in S is a square:

$$\begin{aligned} PSPARTITION = \{\langle S\rangle \mid\ & S \text{ is a multiset of integers with } \textstyle\sum_{x\in S} x = n^2 \text{ for an integer } n, \\ & \text{and for some } X \subseteq S, \textstyle\sum_{x\in X} x = \sum_{x\in S\setminus X} x\}. \end{aligned}$$

Part 3.1 Show that *PARTITION* is NP-complete.

Part 3.2 Show that *PSPARTITION* is NP-complete by reduction from *PARTITION*.

Question 4

Consider two strings s and t. We say that s *yields* t if t can be obtained from s by adding or removing some symbol from s. As examples, *abb* yields *abab* and *bbaab* yields *bbab*.

Let Y be the following language:

$$\begin{aligned} Y = \{\langle M, s, t\rangle \mid\ & M \text{ is a DFA and there are strings } s_1, \ldots, s_k \in L(M) \text{ where } s_1 = s,\ s_k = t, \\ & \text{and for } i = 1, \ldots, k-1, s_i \text{ yields } s_{i+1} \text{ and } |s_i| \leq |s| + |t|\}. \end{aligned}$$

Part 4.1 Show that $Y \in PSPACE$.
[Hint: Recall the proof of Savitch's theorem.]

Question 5

A directed graph is *strongly connected* if for every two vertices u and v there is a directed path from u to v and a directed path from v to u. Define the language

$$STRONGLY\text{-}CONNECTED = \{\langle G\rangle \mid G \text{ is a strongly connected directed graph}\}$$

Part 5.1 Prove that *STRONGLY-CONNECTED* is NL-complete.
[Hint: To prove NL-hardness, reduce from *PATH*.]

END OF THE EXERCISES