Kernel Embedding for Particle Gibbs-Based Optimal Control

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Intermediate Report Master's Thesis
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Motivation

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Related Works

Particle Markov Chain [Lefringhausen+ 2024], [Andrieu+ 2010]
Kernel Embeddings: [Nemmour+ 2022]

Problem Statement - System

Given: Dataset $\mathbb{D} = \{\boldsymbol{u}_t, \boldsymbol{y}_t\}_{t=\text{-}T:\text{-}1}$ from unknown system

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{f}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{v}_t, && \boldsymbol{v}_t \sim \boldsymbol{\mathcal{V}}, \\ \boldsymbol{y}_t &= \boldsymbol{g}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{w}_t, && \boldsymbol{w}_t \sim \boldsymbol{\mathcal{W}} \end{aligned}$$

Assumptions

- Known system structure

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{f}_{\boldsymbol{\theta}}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{v}_t, && \boldsymbol{v}_t \sim \boldsymbol{\mathcal{V}}_{\boldsymbol{\theta}}, \\ \boldsymbol{y}_t &= \boldsymbol{g}_{\boldsymbol{\theta}}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{w}_t, && \boldsymbol{w}_t \sim \boldsymbol{\mathcal{W}}_{\boldsymbol{\theta}}. \end{aligned}$$

- Known priors $p(\boldsymbol{\theta})$ and $p(\boldsymbol{x}_{\text{-}T})$

Problem Statement - Optimal Control Problem

Goal: Solve optimal control problem (OCP)

Stochastic OCP

$$\min_{\boldsymbol{u}_{0:H}} \overline{J_H}$$

subject to:

$$P_0\left[J_H \leq \overline{J_H}\right] \geq 1 - \alpha,$$

$$P_0\left[h_i(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}, \boldsymbol{y}_{0:H}) \leq 0\right] \geq 1 - \alpha, \; \forall i = 1, ..., n_c$$

Idea: Reformulate stochastic OCP to deterministic OCP

Scenario Generation

Goal: Generate scenarios $\boldsymbol{\delta}^{[1:N]}$ using the observations $\mathbb{D}$

Algorithm: Scenario Generation

For $n = 1, \ldots, N$:

1. Sample $\{\boldsymbol{\theta}, \boldsymbol{x}_{\text{-}T:\text{-}1}\}^{[n]}$ from $p\left(\boldsymbol{\theta}, \boldsymbol{x}_{\text{-}T:\text{-}1} \mid \mathbb{D}\right)$ using PMCMC [Lefringhausen+ 2024].
2. Sample $\boldsymbol{v}_t^{[n]}$ from $\boldsymbol{\mathcal{V}}_{\boldsymbol{\theta}^{[n]}}$ and $\boldsymbol{w}_t^{[n]}$ from $\boldsymbol{\mathcal{W}}_{\boldsymbol{\theta}^{[n]}}$ for $t = \text{-}1, \ldots, H$
3. Set $\boldsymbol{x}_0^{[n]} = \boldsymbol{f}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_{\text{-}1}^{[n]}, \boldsymbol{u}_{\text{-}1}\right) + \boldsymbol{v}_{\text{-}1}^{[n]}$

Output: Scenarios $\boldsymbol{\delta}^{[1:N]} = \{\boldsymbol{\theta}, \boldsymbol{x}_0, \boldsymbol{v}_{0:H}, \boldsymbol{w}_{0:H}\}^{[1:N]}$

Maximum Mean Discrepancy (MMD) ambiguity sets

Goal: Reformulate chance-constraint problem with scenarios $\boldsymbol{\delta}^{[1:N]}$

Expanded Chance-Constraints

$$\inf_{P \in \mathcal{P}} P\left[h(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}, \boldsymbol{y}_{0:H}) \leq 0\right] \geq 1 - \alpha.$$

$\mathcal{P}$ is constructed with the samples $\boldsymbol{\delta}^{[1:N]}$ as

MMD ambiguity set

$$\mathcal{P} = \left\{P : \mathsf{MMD}(P, P_N) \leq \varepsilon\right\}.$$

Constraint Reformulation

Goal: Reformulate chance-constraint problem with scenarios $\boldsymbol{\delta}^{[1:N]}$

Feasible Region of chance constraint

$$Z_i := \left\{ \boldsymbol{u}_{0:H} \in \mathcal{U}^{H+1} : \inf_{P \in \mathcal{P}} P\left[\tilde{h}_i(\boldsymbol{u}_{0:H}, \boldsymbol{\delta}) \leq 0\right] \geq 1 - \alpha \right\}$$

$$\Downarrow$$

Reformulated Feasible Region [Nemmour+ 2022]

$$Z_i := \left\{ \boldsymbol{u}_{0:H} \in \mathcal{U}^{H+1} : \begin{array}{l} g_0 + \frac{1}{N} \sum_{n=1}^{N} (\boldsymbol{K}\boldsymbol{\gamma})_n + \varepsilon \sqrt{\boldsymbol{\gamma}^\top \boldsymbol{K} \boldsymbol{\gamma}} \leq t\alpha \\ [\tilde{h}_i(\boldsymbol{u}_{0:H}, \boldsymbol{\delta}^{[n]}) + t]_+ \leq g_0 + (\boldsymbol{K}\boldsymbol{\gamma})_n, \ n = 0, \ldots, N \\ g_0 \in \mathbb{R}, \boldsymbol{\gamma} \in \mathbb{R}^N, t \in \mathbb{R} \end{array} \right\}$$

Problem Formulation

Goal: Reformulate chance-constraint problem with $\boldsymbol{\delta}^{[1:N]}$

$$\min_{\boldsymbol{u}_{0:H}, \overline{J_H}} \overline{J_H}$$

subject to: $\forall n \in \mathbb{N}_{\leq N},\ \forall t \in \mathbb{N}^0_{\leq H}$

$$\left.\begin{aligned} \boldsymbol{x}_{t+1}^{[n]} &= \boldsymbol{f}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_t^{[n]}, \boldsymbol{u}_t\right) + \boldsymbol{v}_t^{[n]} \\ \boldsymbol{y}_t &= \boldsymbol{g}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_t^{[n]}, \boldsymbol{u}_t\right) + \boldsymbol{w}_t^{[n]} \end{aligned}\right\} \text{Dynamic Constraints}$$

$$J_H(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}^{[n]}, \boldsymbol{y}_{0:H}^{[n]}) \leq \overline{J_H}$$

$$\left.\boldsymbol{u}_{0:H} \in Z_i, \forall i = 1, \ldots, n_c \right\} \text{Reformulated Chance Constraints}$$

Simulation Setup (1/2)

- Unknown system: $$\boldsymbol{f}(\boldsymbol{x}, u) = \begin{bmatrix} 0.8x_1 - 0.5x_2 \\ 0.4x_1 + 0.5x_2 + u \end{bmatrix}$$ $$\boldsymbol{v}_t \sim \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{Q} = \begin{bmatrix} 0.03 & \text{-}0.004 \\ \text{-}0.004 & 0.01 \end{bmatrix}\right).$$
- Known measurement model $g(\boldsymbol{x}, u) = x_1,\ w_t \sim \mathcal{N}(0, 0.1)$
- Known initial state for measurements $\boldsymbol{x}_{\text{-}T} \sim \mathcal{N}([2, 2]^\mathsf{T}, \boldsymbol{I}_2)$
- Cost function $J_H = \sum_{t=0}^{H} u_t^2$
- Input constraints $|u| \leq 10$
- Output constraints $y_{10:20} \leq (\text{-}10)$ and $10 \leq y_{30:40}$

Simulation Setup (2/2)

Particle Marcov chain Monte Carlo [Andrieu+ 2017]:

- Known system structure: $\boldsymbol{f}(\boldsymbol{x}, u) = \boldsymbol{A}\left[x_1, x_2, u\right]^{\mathsf{T}} + \boldsymbol{v}_t, \ \boldsymbol{v}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q})$
- Model parameters: $\boldsymbol{\theta} = \{\boldsymbol{A}, \boldsymbol{Q}\}$
- Priors: $\boldsymbol{Q} \sim \mathcal{IW}(100\boldsymbol{I}_2, 10)$
  $\boldsymbol{A} \sim \mathcal{MN}(\mathbf{0}, \boldsymbol{Q}, 10\boldsymbol{I}_2)$

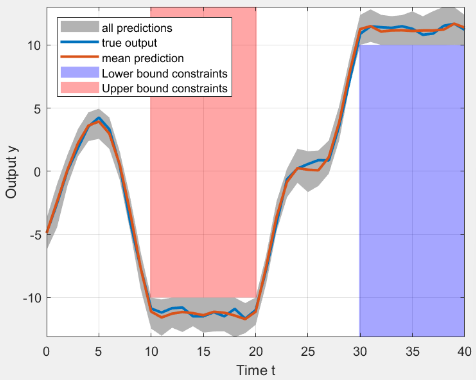
Kernel Embedding:

- Gaussian kernels $k(x, x') = \exp\left(-\frac{1}{2\sigma^2}||x - x'||_2^2\right)$ for all 4 random varibles
- Bandwidth σ set via the median heuristic [Garreau+ 2018] and scaled with factors $[1.5, 5, 5, 1]^{\mathsf{T}}$
- Ambiguity set radius ε set via bootstrap construction $(B = 1000, \beta = 0.95)$ [Nemmour+ 2022].

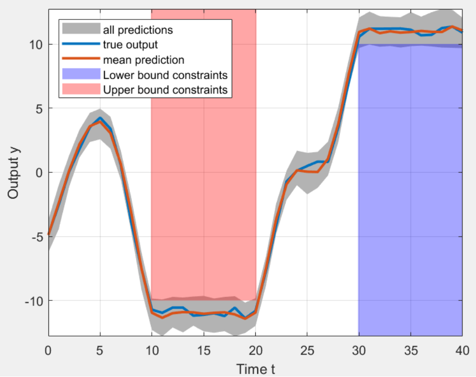
Optimal Control with Constrained Outputs (1/2)

Number of scenarios used for optimization: $N = 200$

Scenario Approach



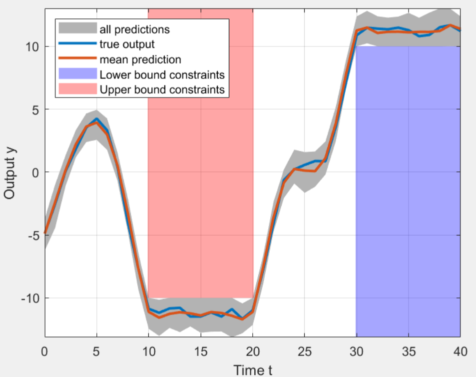
Kernel Approach ($\alpha = 0.1$)



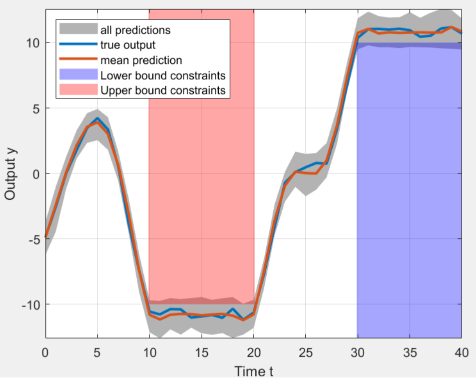
Optimal Control with Constrained Outputs (2/2)

Number of scenarios used for optimization: $N = 200$

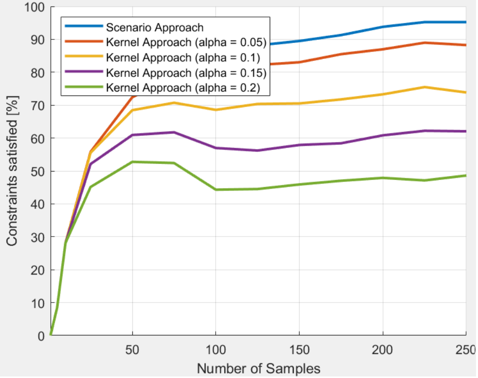
Kernel Approach ($\alpha = 0.01$)



Kernel Approach ($\alpha = 0.3$)



Successrate of Solution

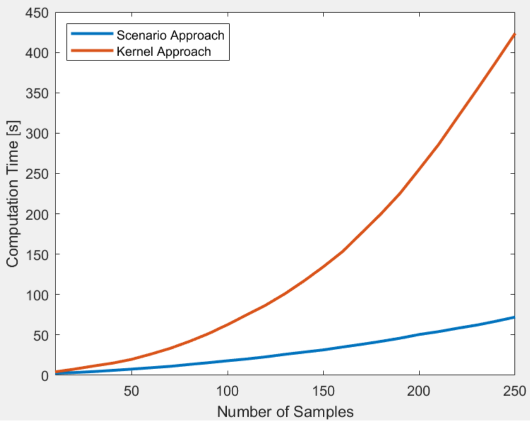


$N = 2000$: Number of Scenarios used to test $u_{0:H}$

Successrate does **not** converge to $(1-\alpha)$
Potential explaination:
Guarantee constraint is applied to each output constraint seperately

Computation Time

Computation time increases faster for the kernel approach
But: Kernel approach comes with adjustable risk factor α



Summary

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Future Plans

- Use Kernel Embeddings on non-linear systems
- Parameter tuning of σ
- Risk factor for all constraints at once instead of individually

References

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