# Kernel Embedding for Particle Gibbs-Based Optimal Control

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# Motivation

[Xiloyannis, Chiaradia, Frisoli and Masia 2019]

# Problem Statement - System

**Given:** Dataset $\mathbb{D} = \{\boldsymbol{u}_t, \boldsymbol{y}_t\}_{t=\text{-}T:\text{-}1}$ from unknown system

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{f}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{v}_t, & \boldsymbol{v}_t &\sim \boldsymbol{\mathcal{V}}, \\ \boldsymbol{y}_t &= \boldsymbol{g}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{w}_t, & \boldsymbol{w}_t &\sim \boldsymbol{\mathcal{W}} \end{aligned}$$

## Assumptions

- Known system structure

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{f}_{\boldsymbol{\theta}}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{v}_t, & \boldsymbol{v}_t &\sim \boldsymbol{\mathcal{V}}_{\boldsymbol{\theta}}, \\ \boldsymbol{y}_t &= \boldsymbol{g}_{\boldsymbol{\theta}}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{w}_t, & \boldsymbol{w}_t &\sim \boldsymbol{\mathcal{W}}_{\boldsymbol{\theta}}. \end{aligned}$$

- Known priors $p(\boldsymbol{\theta})$ and $p(\boldsymbol{x}_{\text{-}T})$

# Problem Statement - Optimal Control Problem

**Goal:** Solve optimal control problem (OCP)

**Stochastic OCP**

$$\min_{\boldsymbol{u}_{0:H}} J_H(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}, \boldsymbol{y}_{0:H})$$

subject to:

$$P\left[h_i(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}, \boldsymbol{y}_{0:H}) \leq 0\right] \geq 1 - \alpha, \ \forall i = 1, \ldots, n_c$$

**Problem:** Underlying data distribution $P$ is unknown

# Related Works

Particle Markov chain Monte Carlo [Andrieu, Doucet and Holenstein 2010]

Chance Constraints:

- Scenario Approach [Lefringhausen, Srithasan, Lederer and Hirche 2024]
- Wasserstein Ambiguity [Hota, Cherukuri and Lygeros 2019]
- Kernel Embeddings [Thorpe, Lew, Oishi and Zhu 2022]
  [Nemmour, Kremer, Schoelkopf and Zhu 2022]

# Scenario Generation

**Goal:** Generate scenarios $\boldsymbol{\delta}^{[1:N]}$ using the observations $\mathbb{D}$

**Algorithm: Scenario Generation**

For $n = 1, ..., N$:

1. Sample $\{\boldsymbol{\theta}, \boldsymbol{x}_{\text{-}T:\text{-}1}\}^{[n]}$ from $p\left(\boldsymbol{\theta}, \boldsymbol{x}_{\text{-}T:\text{-}1} \mid \mathbb{D}\right)$ using PMCMC [Lefringhausen+ 2024].
2. Sample $\boldsymbol{v}_t^{[n]}$ from $\boldsymbol{\mathcal{V}}_{\boldsymbol{\theta}^{[n]}}$ and $\boldsymbol{w}_t^{[n]}$ from $\boldsymbol{\mathcal{W}}_{\boldsymbol{\theta}^{[n]}}$ for $t = \text{-}1, ..., H$
3. Set $\boldsymbol{x}_0^{[n]} = \boldsymbol{f}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_{\text{-}1}^{[n]}, \boldsymbol{u}_{\text{-}1}\right) + \boldsymbol{v}_{\text{-}1}^{[n]}$

**Output:** Scenarios $\boldsymbol{\delta}^{[1:N]} = \{\boldsymbol{\theta}, \boldsymbol{x}_0, \boldsymbol{v}_{0:H}, \boldsymbol{w}_{0:H}\}^{[1:N]}$

# Maximum Mean Discrepancy (MMD) ambiguity sets

**Goal:** Reformulate chance-constraint problem with scenarios $\boldsymbol{\delta}^{[1:N]}$

**Expanded Chance-Constraints**

$$\inf_{\tilde{P} \in \mathcal{P}} \tilde{P}\left[h(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}, \boldsymbol{y}_{0:H}) \leq 0\right] \geq 1 - \alpha.$$

$\mathcal{P}$ is constructed with the samples $\boldsymbol{\delta}^{[1:N]}$ as

**MMD ambiguity set**

$$\mathcal{P} = \left\{ \tilde{P} : \mathsf{MMD}(\tilde{P}, P_N) \leq \varepsilon \right\}.$$

With large enough $N \Rightarrow P$ is an element of $\mathcal{P}$

# Constraint Reformulation

**Goal:** Reformulate chance-constraint problem with scenarios $\boldsymbol{\delta}^{[1:N]}$

## Feasible Region of chance constraint

$$Z_i := \left\{ \boldsymbol{u}_{0:H} \in \mathcal{U}^{H+1} : \inf_{\tilde{P} \in \mathcal{P}} \tilde{P} \left[ \tilde{h}_i(\boldsymbol{u}_{0:H}, \boldsymbol{\delta}) \leq 0 \right] \geq 1 - \alpha \right\}$$

$$\Downarrow$$

## Reformulated Feasible Region [Nemmour+ 2022]

$$Z_i := \left\{ \boldsymbol{u}_{0:H} \in \mathcal{U}^{H+1} : \begin{array}{l} g_0 + \frac{1}{N} \sum_{n=1}^{N} (\boldsymbol{K}\boldsymbol{\gamma})_n + \varepsilon \sqrt{\boldsymbol{\gamma}^\mathsf{T} \boldsymbol{K} \boldsymbol{\gamma}} \leq t\alpha \\ [\tilde{h}_i(\boldsymbol{u}_{0:H}, \boldsymbol{\delta}^{[n]}) + t]_+ \leq g_0 + (\boldsymbol{K}\boldsymbol{\gamma})_n, \ n = 0, \ldots, N \\ g_0 \in \mathbb{R}, \boldsymbol{\gamma} \in \mathbb{R}^N, t \in \mathbb{R} \end{array} \right\}$$

# Problem Formulation

**Goal:** Reformulate chance-constraint problem with $\boldsymbol{\delta}^{[1:N]}$

$$\min_{\boldsymbol{u}_{0:H}, J_H, \{g_0, \boldsymbol{\gamma}, t\}^{[1:n_c]}} J_H(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}^{[n]}, \boldsymbol{y}_{0:H}^{[n]})$$

subject to: $\forall n \in \mathbb{N}_{\leq N},\ \forall t \in \mathbb{N}_{\leq H}^{0},\ \forall i \in \mathbb{N}_{\leq n_c}$

$$\left.\begin{aligned} \boldsymbol{x}_{t+1}^{[n]} &= \boldsymbol{f}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_t^{[n]}, \boldsymbol{u}_t\right) + \boldsymbol{v}_t^{[n]} \\ \boldsymbol{y}_t^{[n]} &= \boldsymbol{g}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_t^{[n]}, \boldsymbol{u}_t\right) + \boldsymbol{w}_t^{[n]} \end{aligned}\right\} \text{Dynamic Constraints}$$

$$\left.\boldsymbol{u}_{0:H} \in Z_i(g_0^{[i]}, \boldsymbol{\gamma}^{[i]}, t^{[i]})\right\} \text{Reformulated Chance Constraints}$$

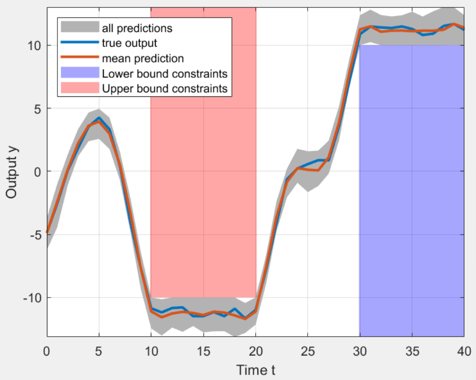
# Simulation Setup (1/2)

- Unknown system: $$\boldsymbol{f}(\boldsymbol{x},u) = \begin{bmatrix} 0.8x_1 - 0.5x_2 \\ 0.4x_1 + 0.5x_2 + u \end{bmatrix}$$ $$\boldsymbol{v}_t \sim \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{Q} = \begin{bmatrix} 0.03 & \text{-}0.004 \\ \text{-}0.004 & 0.01 \end{bmatrix}\right).$$
- Known system structure: $\boldsymbol{f}(\boldsymbol{x},u) = \boldsymbol{A}\,[x_1, x_2, u]^\mathsf{T} + \boldsymbol{v}_t,\ \boldsymbol{v}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q})$
- Priors: $\boldsymbol{Q} \sim \mathcal{IW}(100\boldsymbol{I}_2, 10)$
  $\boldsymbol{A} \sim \mathcal{MN}(\boldsymbol{0}, \boldsymbol{Q}, 10\boldsymbol{I}_2)$ [Andrieu+ 2017]
  $\boldsymbol{x}_{\text{-}T} \sim \mathcal{N}([2,2]^\mathsf{T}, \boldsymbol{I}_2)$
- Known measurement model $g(\boldsymbol{x},u) = x_1,\ w_t \sim \mathcal{N}(0, 0.1)$
- Cost function $J_H = \sum_{t=0}^{H} u_t^2$
- Input constraints $|u| \leq 10$
- Gaussian kernels with bandwidth $\sigma$ set via the median heuristic [Garreau+ 2018]
- Ambiguity set radius $\varepsilon$ set via bootstrap construction [Nemmour+ 2022].

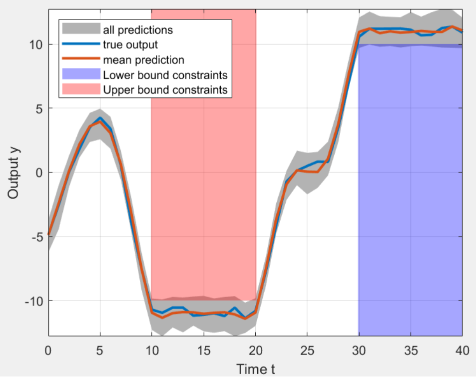
# Optimal Control with Constrained Outputs (1/2)

Number of scenarios used for optimization: $N = 200$

Scenario Approach



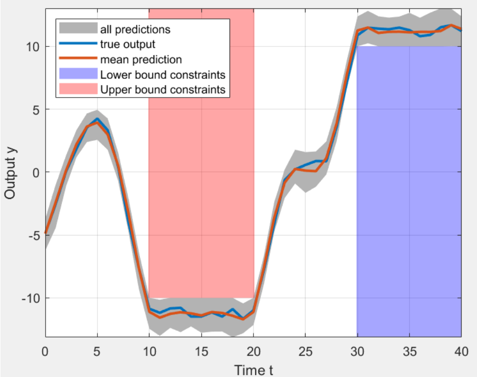

Kernel Approach ($\alpha = 0.1$)



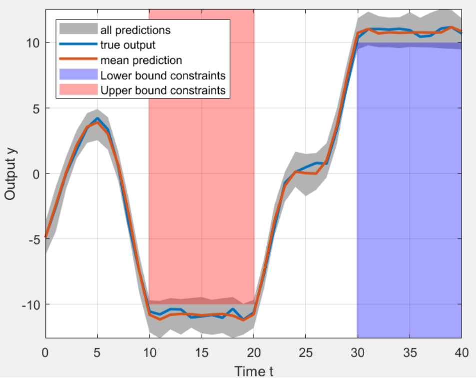
# Optimal Control with Constrained Outputs (2/2)

Number of scenarios used for optimization: $N = 200$

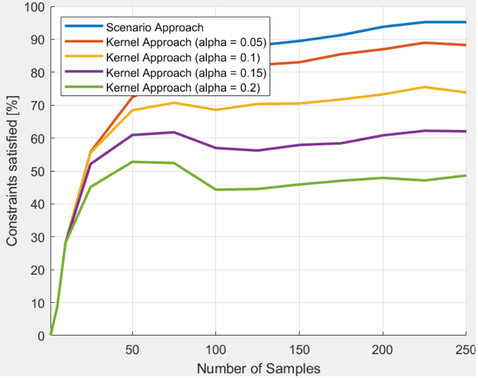
Kernel Approach ($\alpha = 0.01$)




Kernel Approach ($\alpha = 0.3$)



# Successrate of Solution




$N = 2000$: Number of Scenarios used to test $u_{0:H}$

Successrate does **not** converge to $(1 - \alpha)$
Potential explaination:
Guarantee constraint is applied to each output constraint seperately

# Conclusion

**Summary:** Kernel Embeddings allow for ...

- Solving of chance-constrained OCPs
- Controlling of risk factor $\alpha$

**Future Plans:**

- Use Kernel Embeddings on non-linear systems
- Parameter tuning of $\sigma$
- Alternative approach of reformulating chance constraints

# References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# Timeline

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# Scenario Generation

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3. Set $\boldsymbol{x}_0^{[n]} = \boldsymbol{f}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_{\text{-}1}^{[n]}, \boldsymbol{u}_{\text{-}1}\right) + \boldsymbol{v}_{\text{-}1}^{[n]}$

**Output:** Scenarios $\boldsymbol{\delta}^{[1:N]} = \{\boldsymbol{\theta}, \boldsymbol{x}_0, \boldsymbol{v}_{0:H}, \boldsymbol{w}_{0:H}\}^{[1:N]}$

# Bootstrap Construction

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# Computation Time

Computation time increases faster for the kernel approach
**But:** Kernel approach comes with adjustable risk factor $\alpha$