# Kernel Embedding for Particle Gibbs-Based Optimal Control

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# Motivation

[Xiloyannis, Chiaradia, Frisoli and Masia 2019]

**Challenges**:

- Unknown dynamics
- Latent states
- Safety

# Problem Statement - System

**Given:** Dataset $\mathbb{D} = \{\boldsymbol{u}_t, \boldsymbol{y}_t\}_{t=\text{-}T:\text{-}1}$ from unknown system

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{f}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{v}_t, & \boldsymbol{v}_t &\sim \boldsymbol{\mathcal{V}}, \\ \boldsymbol{y}_t &= \boldsymbol{g}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{w}_t, & \boldsymbol{w}_t &\sim \boldsymbol{\mathcal{W}} \end{aligned}$$

## Assumptions

- Known system structure

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{f}_{\boldsymbol{\theta}}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{v}_t, & \boldsymbol{v}_t &\sim \boldsymbol{\mathcal{V}}_{\boldsymbol{\theta}}, \\ \boldsymbol{y}_t &= \boldsymbol{g}_{\boldsymbol{\theta}}\left(\boldsymbol{x}_t, \boldsymbol{u}_t\right) + \boldsymbol{w}_t, & \boldsymbol{w}_t &\sim \boldsymbol{\mathcal{W}}_{\boldsymbol{\theta}}. \end{aligned}$$

- Known priors $p(\boldsymbol{\theta})$ and $p(\boldsymbol{x}_{\text{-}T})$

# Problem Statement - Optimal Control Problem

**Goal:** Solve optimal control problem (OCP)

**Stochastic OCP**

$$\min_{\boldsymbol{u}_{0:H}} J_H(\boldsymbol{u}_{0:H})$$

subject to:

$$P\left[\max(\boldsymbol{h}(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}, \boldsymbol{y}_{0:H})) \leq 0\right] \geq 1 - \alpha$$

**Problem:** Underlying data distribution $P$ is unknown

# Related Works

Particle Gibbs based optimal control via scenario theory **[Lefringhausen, Srithasan, Lederer and Hirche 2024]**

$\Rightarrow$ Guarantees determined retroactively

Alternative Approaches:

- Wasserstein ambiguity [Hota, Cherukuri and Lygeros 2019]
  - $\Rightarrow$ Constraints limited to affine functions
- Kernel embeddings **[Nemmour, Kremer, Schoelkopf and Zhu 2022]**
  [Thorpe, Lew, Oishi and Zhu 2022]

# Particle Gibbs Scenarios

Particle Gibbs gives us the scenarios $\boldsymbol{\delta}^{[1:N]} = \{\boldsymbol{\theta}, \boldsymbol{x}_0, \boldsymbol{v}_{0:H}, \boldsymbol{w}_{0:H}\}^{[1:N]}$
**Goal:** Reformulate chance-constraint problem with scenarios $\boldsymbol{\delta}^{[1:N]}$

## Chance Constraints

$$P\left[\max(\boldsymbol{h}(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}, \boldsymbol{y}_{0:H}) \leq 0\right] \geq 1 - \alpha$$

$$\Downarrow$$

## Scenario Approach (used in [Lefringhausen+ 2024])

$$\max(\boldsymbol{h}(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}^{[n]}, \boldsymbol{y}_{0:H}^{[n]})) \leq 0, \ \forall n = 1, ..., N$$

$\Rightarrow$ Risk factor $\alpha$ not considered in optimization

# Maximum Mean Discrepancy (MMD) ambiguity sets

**Goal:** Reformulate chance-constraint problem with scenarios $\boldsymbol{\delta}^{[1:N]}$

**Idea:** Replace distribution $P$ with the ambiguity set $\mathcal{P}$

## MMD ambiguity set

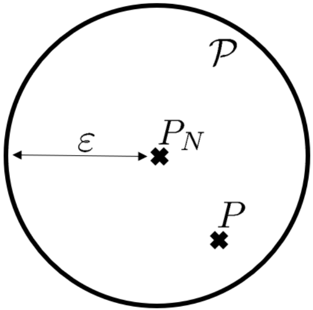
$$\mathcal{P} = \left\{ \tilde{P} : \mathrm{MMD}(\tilde{P}, P_N) \leq \varepsilon \right\}.$$

$$\Downarrow$$

## Expanded Chance-Constraints

$$\inf_{\tilde{P} \in \mathcal{P}} \tilde{P}\left[\tilde{h}(\boldsymbol{u}_{0:H}, \boldsymbol{\delta}) \leq 0\right] \geq 1 - \alpha.$$

with $\tilde{h}(\boldsymbol{u}_{0:H}, \boldsymbol{\delta}) = \max(\boldsymbol{h}(\boldsymbol{u}_{0:H}, \boldsymbol{x}_{0:H}, \boldsymbol{y}_{0:H}))$.



# Constraint Reformulation

**Goal:** Reformulate chance-constraint problem with scenarios $\boldsymbol{\delta}^{[1:N]}$

**Feasible Region of chance constraint**

$$Z := \left\{ \boldsymbol{u}_{0:H} \in \mathcal{U}^{H+1} : \inf_{\tilde{P} \in \mathcal{P}} \tilde{P} \left[ \tilde{h}(\boldsymbol{u}_{0:H}, \boldsymbol{\delta}) \leq 0 \right] \geq 1 - \alpha \right\}$$

$$\Downarrow$$

**Reformulated Feasible Region [Nemmour+ 2022]**

$$\hat{Z} := \left\{ \boldsymbol{u}_{0:H} \in \mathcal{U}^{H+1} : \begin{array}{l} g_0 + \frac{1}{N} \sum_{n=1}^{N} (\boldsymbol{K}\boldsymbol{\gamma})_n + \varepsilon \sqrt{\boldsymbol{\gamma}^\mathsf{T} \boldsymbol{K} \boldsymbol{\gamma}} \leq t' \alpha \\ [\tilde{h}(\boldsymbol{u}_{0:H}, \boldsymbol{\delta}^{[n]}) + t']_+ \leq g_0 + (\boldsymbol{K}\boldsymbol{\gamma})_n, \; n = 1, \ldots, N \\ g_0 \in \mathbb{R}, \boldsymbol{\gamma} \in \mathbb{R}^N, t' \in \mathbb{R} \end{array} \right\}$$

# Problem Formulation

**Goal:** Reformulate chance-constraint problem with scenarios $\boldsymbol{\delta}^{[1:N]}$

$$\min_{\boldsymbol{u}_{0:H}, g_0, \boldsymbol{\gamma}, t'} J_H(\boldsymbol{u}_{0:H})$$

subject to: $\forall n \in \mathbb{N}_{\leq N},\ \forall t \in \mathbb{N}^0_{\leq H},$

$$\left.\begin{aligned} \boldsymbol{x}_{t+1}^{[n]} &= \boldsymbol{f}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_t^{[n]}, \boldsymbol{u}_t\right) + \boldsymbol{v}_t^{[n]} \\ \boldsymbol{y}_t^{[n]} &= \boldsymbol{g}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_t^{[n]}, \boldsymbol{u}_t\right) + \boldsymbol{w}_t^{[n]} \end{aligned}\right\} \text{Dynamic Constraints}$$

$$\boldsymbol{u}_{0:H} \in Z(g_0, \boldsymbol{\gamma}, t') \quad \Big\} \text{ Reformulated Chance Constraints}$$

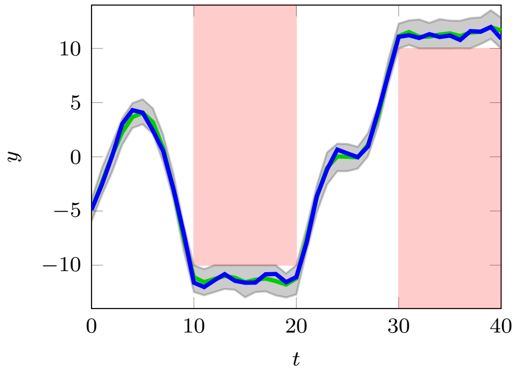
# Simulation Setup

- Unknown system: $$\boldsymbol{f}(\boldsymbol{x}, u) = \begin{bmatrix} 0.8x_1 - 0.5x_2 \\ 0.4x_1 + 0.5x_2 + u \end{bmatrix}$$ $$\boldsymbol{v}_t \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{Q} = \begin{bmatrix} 0.03 & \text{-}0.004 \\ \text{-}0.004 & 0.01 \end{bmatrix}\right).$$
- Known system structure: $\boldsymbol{f}(\boldsymbol{x}, u) = \boldsymbol{A}\,[x_1, x_2, u]^{\mathsf{T}} + \boldsymbol{v}_t,\ \boldsymbol{v}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q})$
- Priors: $$\boldsymbol{Q} \sim \mathcal{IW}(100\boldsymbol{I}_2, 10)$$ $$\boldsymbol{A} \sim \mathcal{MN}(\mathbf{0}, \boldsymbol{Q}, 10\boldsymbol{I}_2)$$ [Andrieu+ 2017] $$\boldsymbol{x}_{\text{-}T} \sim \mathcal{N}([2, 2]^{\mathsf{T}}, \boldsymbol{I}_2)$$
- Known measurement model $g(\boldsymbol{x}, u) = x_1,\ w_t \sim \mathcal{N}(0, 0.1)$
- Cost function $J_H = \sum_{t=0}^{H} u_t^2$
- Input constraints $|u| \leq 10$
- Gaussian kernels with bandwidth $\sigma$ set via the median heuristic [Garreau+ 2018]
- Ambiguity set radius $\varepsilon$ set via bootstrap construction [Nemmour+ 2022].

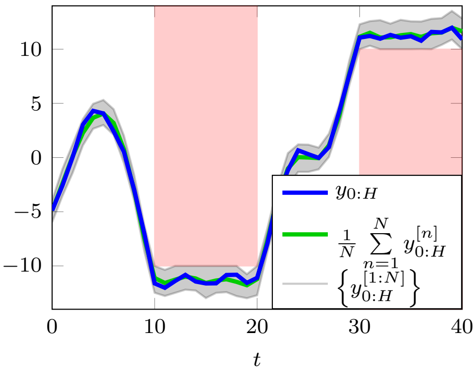
# Optimal Control with Constrained Outputs

Number of scenarios used for optimization: $N = 200$

Scenario Approach



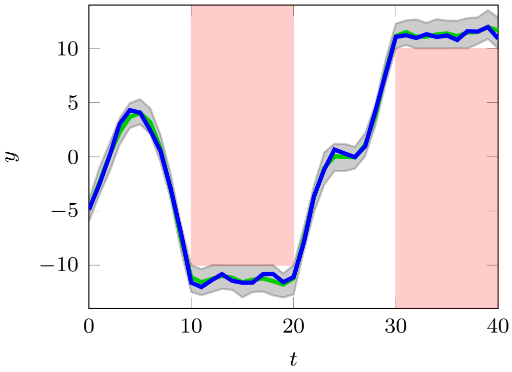

Kernel Approach ($\alpha = 0.01$)



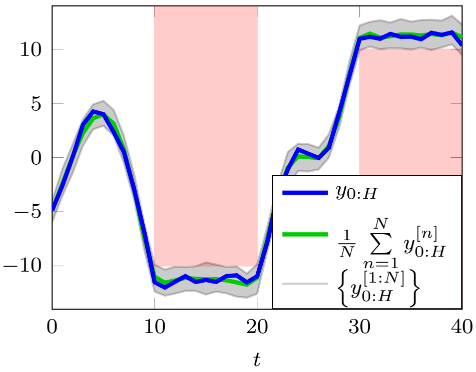
# Optimal Control with Constrained Outputs

Number of scenarios used for optimization: $N = 200$

Scenario Approach




Kernel Approach ($\alpha = 0.2$)

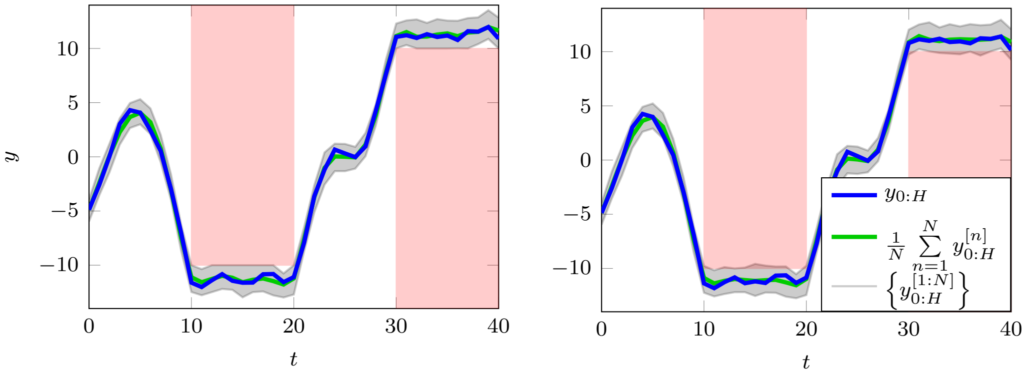


# Optimal Control with Constrained Outputs

Number of scenarios used for optimization: $N = 200$

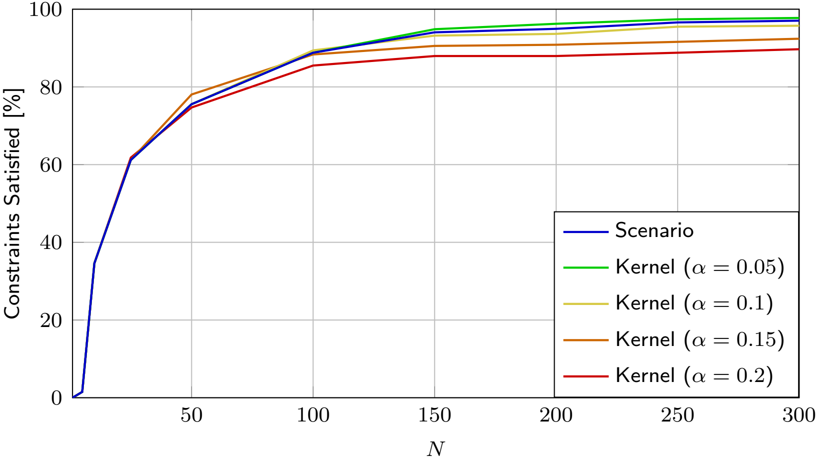
Scenario Approach

Kernel Approach ($\alpha = 0.5$)



# Successrate of Solution

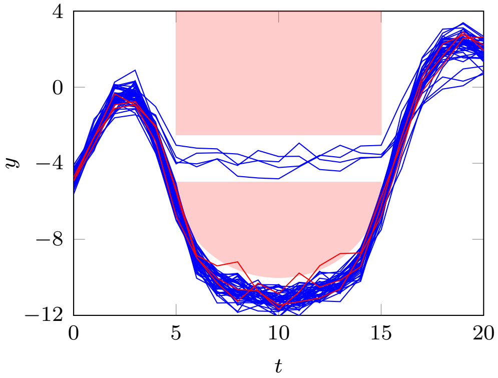
$N' = 2000$ scenarios used to test $u_{0:H}$



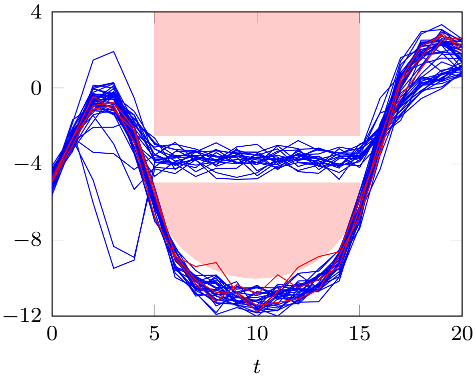
# Corridor Test

Number of scenarios used for optimization: $N = 200$

Scenario Approach




Kernel Approach ($\alpha = 0.1$)

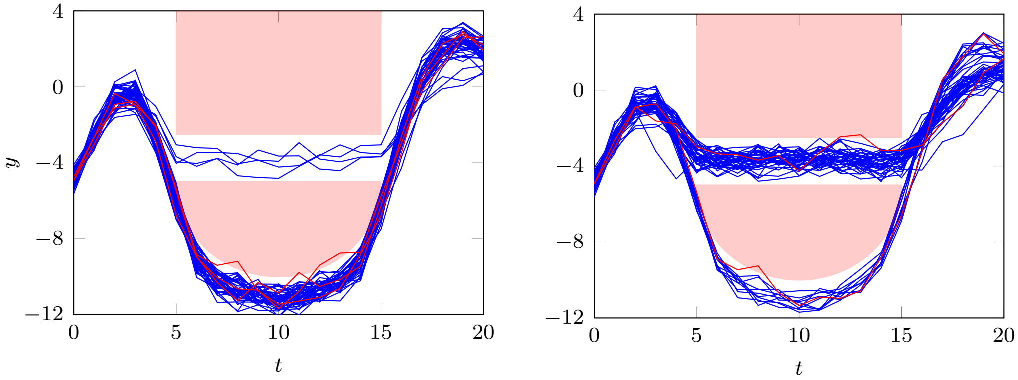


# Corridor Test

Number of scenarios used for optimization: $N = 200$

Scenario Approach

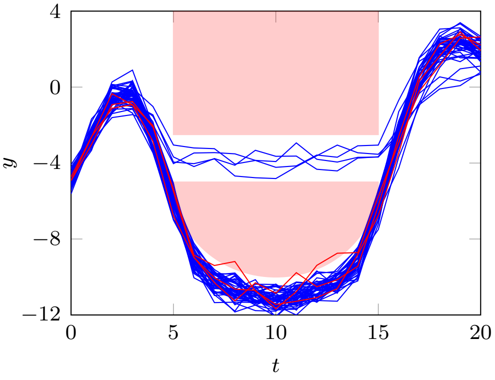
Kernel Approach ($\alpha = 0.2$)



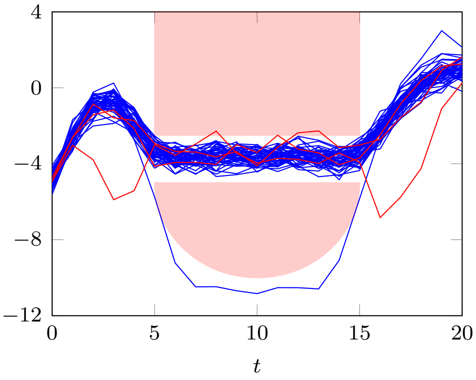
# Corridor Test

Number of scenarios used for optimization: $N = 200$

Scenario Approach




Kernel Approach ($\alpha = 0.3$)



# Nonlinear System

**Linear System (used in previous simulations)**

$$\boldsymbol{f}(\boldsymbol{x}, u) = \begin{bmatrix} 0.8x_1 - 0.5x_2 \\ 0.4x_1 + 0.5x_2 + u \end{bmatrix}$$

$$\Downarrow$$

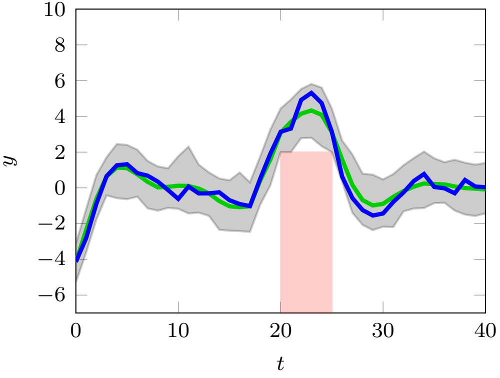
**Nonlinear System**

$$\boldsymbol{f}(\boldsymbol{x}, u) = \begin{bmatrix} 0.8x_1 - 0.5x_2 + 0.1\cos(3x_1)x_2 \\ 0.4x_1 + 0.5x_2 + (1 + 0.3\sin(2x_2))u \end{bmatrix}$$

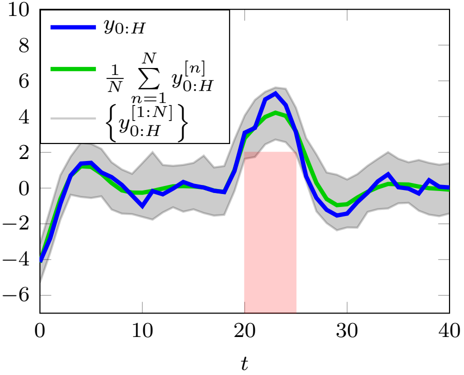
# Optimal Control of Nonlinear Systems

Number of scenarios used for optimization: $N = 200$

Scenario Approach



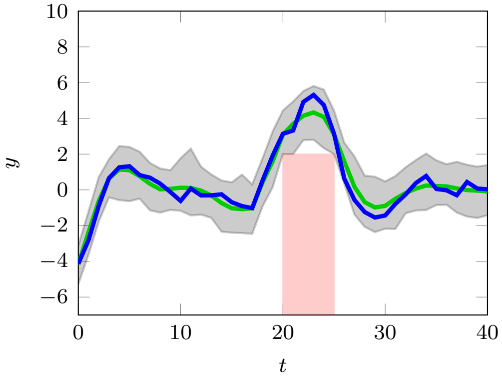

Kernel Approach ($\alpha = 0.2$)



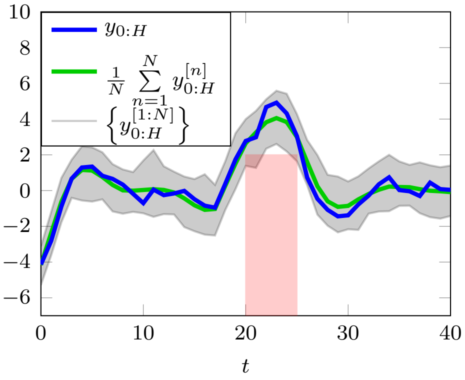
# Optimal Control of Nonlinear Systems

Number of scenarios used for optimization: $N = 200$

Scenario Approach



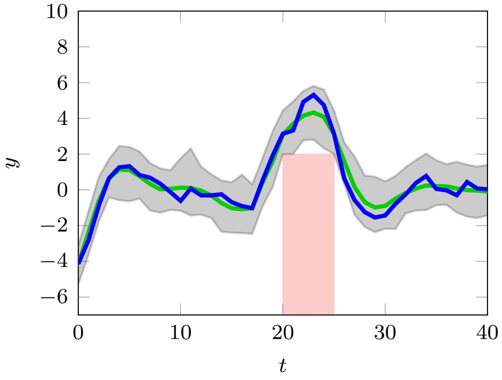

Kernel Approach ($\alpha = 0.4$)



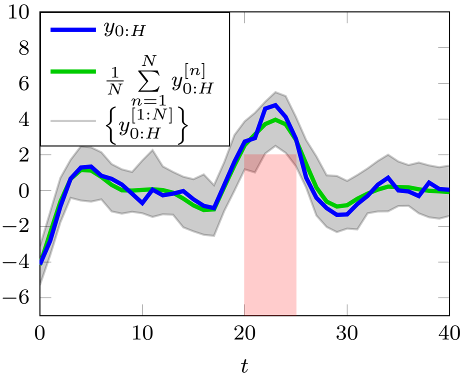
# Optimal Control of Nonlinear Systems

Number of scenarios used for optimization: $N = 200$

Scenario Approach




Kernel Approach ($\alpha = 0.6$)



# Problems

Solver unreliable for kernel approach with non linear systems

Impossible to solve OCPs with ...

- Low number of samples $N$
- Low Risk Factor $\alpha$
- High number of constraints

# Conclusion

**Summary:** Kernel Embeddings allow for ...

- solving chance-constrained OCPs
- Choosing risk factor $\alpha$

**Future Plans:**

- Use kernel embeddings on non-linear systems
- Parameter tuning of $\sigma$
- Alternative approach of reformulating chance constraints

# References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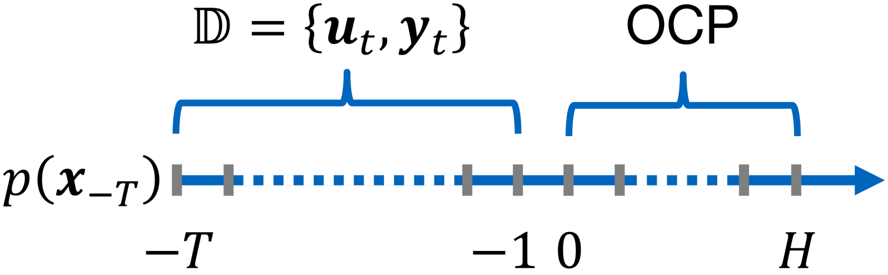
# Timeline


$\mathbb{D} = \{\boldsymbol{u}_t, \boldsymbol{y}_t\}$
OCP
$p(\boldsymbol{x}_{-T})$
$-T$
$-1$
$0$
$H$

# Scenario Generation

**Goal:** Generate scenarios $\boldsymbol{\delta}^{[1:N]}$ using the observations $\mathbb{D}$

**Algorithm: Scenario Generation**

For $n = 1, ..., N$:

1. Sample $\{\boldsymbol{\theta}, \boldsymbol{x}_{\text{-}T:\text{-}1}\}^{[n]}$ from $p\left(\boldsymbol{\theta}, \boldsymbol{x}_{\text{-}T:\text{-}1} \mid \mathbb{D}\right)$ using PMCMC [Lefringhausen+ 2024].
2. Sample $\boldsymbol{v}_t^{[n]}$ from $\boldsymbol{\mathcal{V}}_{\boldsymbol{\theta}^{[n]}}$ and $\boldsymbol{w}_t^{[n]}$ from $\boldsymbol{\mathcal{W}}_{\boldsymbol{\theta}^{[n]}}$ for $t = \text{-}1, ..., H$
3. Set $\boldsymbol{x}_0^{[n]} = \boldsymbol{f}_{\boldsymbol{\theta}^{[n]}}\left(\boldsymbol{x}_{\text{-}1}^{[n]}, \boldsymbol{u}_{\text{-}1}\right) + \boldsymbol{v}_{\text{-}1}^{[n]}$

**Output:** Scenarios $\boldsymbol{\delta}^{[1:N]} = \{\boldsymbol{\theta}, \boldsymbol{x}_0, \boldsymbol{v}_{0:H}, \boldsymbol{w}_{0:H}\}^{[1:N]}$

# Bootstrap Construction

**Algorithm: Bootstrap MMD ambiguity set**

1. $\boldsymbol{K} \leftarrow$ *kernel*$(\boldsymbol{\delta}, \boldsymbol{\delta})$
2. **For** $m = 1, ..., B$
3. $\quad I \leftarrow N$ numbers from $\{1, \ldots N\}$ with replacement
4. $\quad K_x \leftarrow \sum_{i,j=1}^{N} K_{ij},\ K_y \leftarrow \sum_{i,j\in I} K_{ij},\ K_{xy} \leftarrow \sum_{j\in I}\sum_{i=1}^{N} K_{ij}$
5. $\quad \mathrm{MMD}[m] \leftarrow \frac{1}{N^2}\left(K_x + K_y - 2K_{xy}\right)$
6. **End For**
7. MMD $\leftarrow$ *sort*(MMD)
8. $\varepsilon \leftarrow$ MMD[*ceil*$(B\beta)$]

**Output:** Gram matrix $\boldsymbol{K}$, Radius of MMD ambiguity set $\varepsilon$

$B = 1000, \beta = 0.95$

# Maximum Mean Discrepancy (MMD)

## Maximum Mean Discrepancy

$$
\begin{aligned}
\mathsf{MMD}(\tilde{P}, P_N) &= ||\mu_{\tilde{P}} - \mu_{P_N}||_{\mathcal{H}} \\
&= \mathsf{E}_{x,x' \sim \tilde{P}}[k(x, x')] + \mathsf{E}_{y,y' \sim P_N}[k(y, y')] - 2\mathsf{E}_{x \sim \tilde{P}, y \sim P_N}[k(x, y)]
\end{aligned}
$$

## (Biased) MMD estimator

$$
\widehat{\mathsf{MMD}}(P, Q) = \frac{1}{N^2} \sum_{i,j=1}^{N} k(\boldsymbol{\delta}^{[i]}, \boldsymbol{\delta}^{[j]}) + k(\tilde{\boldsymbol{\delta}}^{[i]}, \tilde{\boldsymbol{\delta}}^{[j]}) - 2k(\boldsymbol{\delta}^{[i]}, \tilde{\boldsymbol{\delta}}^{[j]})
$$