# Note on Heterogeneous Agent Model with Aggregate Uncertainty: Krusell and Smith (1998)

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April 2007

## 1 Introduction

The model by Aiyagari (1994) is relatively easy to solve because we focus on the steady state equilibrium, where type distribution of agents and thus the prices are constant over time. At the individual level, there is a large degree of uncertainty and ex-post heterogeneity. However, the individual uncertainty disappears at the aggregate level in the model, roughly because of the law of large numbers. That's why we can have constant prices in equilibrium even though individuals face uninsurable idiosyncratic uncertainty.

However, if we want to introduce aggregate uncertainty to the model, we no longer have the luxury. For example, if there is a shock to TFP, all the agents are equally affected by shocks to TFP, so there is no way that the shock disappears at the aggregate level like shocks to the individual productivity.

Since we need prices to solve the optimization problem of the individual agents, and the prices are functions of the aggregate state of the world, we need to add these *aggregate state variables* to the set of state variables. States variables which are associated with a particular agent is called the *individual state variables*. In the model of Aiyagari (1994), we only need to take care of the individual state variables since aggregate state variables turn out to be constant over time.

However, it's a daunting task. In models with ex-post heterogeneity across agents, an aggregate state variable is the type distribution of agents, which is potentially an enormous object. For example, in the model of Aiyagari (1994), the type distribution is represented by a probability measure $x(e, a)$ where $e \in E = \{e_1, e_2, ..., e_{ne}\}$ the current individual productivity and $a \in A \subset \mathbb{R}$ is the current asset holding. How to deal with this huge object? Krusell and Smith (1998) propose that, for the model of Aiyagari (1994) with aggregate uncertainty, we can obtain a very high precision by approximating the type distribution of agents using some moments of the distribution. In particular, they show that, in order to store the distribution of assets, we only need to store the mean asset holding to obtain a reasonable level of precision.

We will start by the general characterization of equilibrium with aggregate uncertainty, then we see an approximated equilibrium proposed by Krusell and Smith (1998). Then we study the solution algorithm of the approximated equilibrium. Good sources of information are Ríos-Rull (1999) and Heer and Maussner (2005). In a separate note, I will cover extensions of the basic model with both aggregate and idiosyncratic uncertainty.

Closely related models are the ones by Castañeda et al. (1998) and Storesletten et al. (2001). Instead of using a model with infinitely-lived agents, both papers use a model with overlapping generations of finitely-lived agents, and with both idiosyncratic and aggregate uncertainty. Basically, their models are Huggett (1996) with aggregate uncertainty. The solution method is essentially the same as the one presented below.

## 2 Model with both Idiosyncratic and Aggregate Uncertainty

Let's start by describing the general environment of the model.

**Model 1 (Krusell and Smith (1998))**

1. *Time is discrete ($t = 0, 1, ...$). There are continuum of infinitely-lived agents. Total number of agents is normalized to one. Each agent has the following preference:*

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

2. *Agents are given asset $a_0$ initially and one unit of time in each period. Agents spend all of their time in working, since leisure is not valued.*

3. *Agents can hold asset $a_t \in A = [\underline{a}, \overline{a}]$ which yields return $r_t$ in period $t$. This is the only way to save. In particular, agents cannot trade Arrow securities to insure against idiosyncratic uncertainty.*

4. *An agent can be either employed $e_t = 1$ or unemployed $e_t = 0$ in each period. The employment status $e_t$ follows a first order Markov process with the transition matrix $p^e_{e,e'|z,z'}$. The transition matrix is assumed to be conditional on $z$ and $z'$. We will discuss more about this assumption later. The labor income of an agent with type $e_t$ can be denoted by $w_t\ e_t$, where $w_t$ is the wage.*

5. *There is a representative firm which has access to the following CRS technology:*

$$Y_t = z_t F(K_t, L_t)$$

   *where $K_t$ is capital input and $L_t$ is labor input. The representative firm rents inputs in competitive markets. capital depreciates at a constant rate $\delta$.*

6. *$z_t \in Z = \{z_1, z_2, ..., z_{n_z}\}$ follows a first order Markov process with the transition matrix $p^z_{z,z'}$. Following Krusell and Smith (1998), we use $n_z = 2$ and call the two states $z_1$ and $z_2$ as expansion and recession, respectively.*

It might seem odd that the transition matrix for $e$ depends not only $z$ but also $z'$. There is a reason to do so. By wisely choosing the transition probabilities $p^e_{e,e'|z,z'}$, we can obtain the constant but different number of the unemployed (and thus the employed) in each state of $z$. For example, let assume $z$ can take one of the two values, expansion ($z_1$) and recession ($z_2$). We can pin down transition probabilities such that the proportion of the unemployed is always 10% in recessions ($z_2$) while the proportion of the unemployed is always 4% in expansions ($z_1$), regardless of the state $z$ in the previous period. If the transition probabilities are not pinned down in this way, we need to keep track of the number of the unemployed (and the employed) to simulate the economy. However, using the trick, the number of the unemployed is perfectly characterized by the current $z$ so we don't need to keep track of the number of unemployed.

Therefore, the aggregate labor supply, which is equal to the proportion of the employed, is only a function of $z$. Let's denote the aggregate labor supply conditional on $z$ as $L(z)$. By construction, $L(z_1) = 0.96$ and $L(z_2) = 0.90$.

We make the following assumptions on the functional forms:

1. The following CRRA utility function is assumed:

$$u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

   where $\sigma$ is the coefficient of relative risk aversion. Intertemporal elasticity of substitution is $\frac{1}{\sigma}$

2. Production function is assumed to be Cobb-Douglas:

$$Y_t = z_t K_t^{\theta} L_t^{1-\theta}$$

As for the parameter values, Krusell and Smith (1998) set one period as a quarter and use the following values:

$\sigma = 1$ (implying *log* utility), $\beta = 0.99$, $\delta = 0.025$, $z_1 = 1.01$, $z_2 = 0.99$, $\theta = 0.36$. $p^z$ are pinned down such that the average duration of both an expansion and a recession is 8 quarters. $p^e$ are pinned down such that *(i)* average duration of unemployment during expansions is 1.5 quarters, *(ii)* average duration of unemployment in recessions is 2.5 quarters, *(iii)* unemployment rate in expansions is 4%, *(iv)* unemployment rate in recessions is 10%, *(v)* $p^e_{0,0|1,2} = 1.25 p^e_{0,0|2,2}$, *(vi)* $p^e_{0,0|2,1} = 0.75 p^e_{0,0|1,1}$. Since the number of free parameters in $p^e$ matrix is 8, and the conditions *(iii)* and *(iv)* are used twice, we can pin down all the transition probabilities using the conditions above.

# 3 Recursive Formulation

As usual, let's formulate the equilibrium recursively. First, let's recursively formulate the problem of an agent.

**Problem 1 (Recursive formulation of agent's problem)**

$$V(z, x, e, a) = \max_{c \in \mathbb{R}^+, a' \in A} \left\{ u(c) + \beta \sum_{z'} \sum_{e'} p^z_{z,z'} p^e_{e,e'|z,z'} V(z', x', e', a') \right\} \tag{1}$$

*subject to*

$$a(1 + r(z, x)) + w(z, x)e = a' + c$$
$$x' = \phi_x(z, x)$$

Notice that $r$ and $w$ are functions of the aggregate state of the world $(z, x)$ in an equilibrium. Remember that the optimal decision of the representative firm implies:

$$r = \theta z K^{\theta-1} L^{1-\theta} - \delta$$

$$w = (1-\theta) z K^{\theta} L^{-\theta}$$

where

$$K = \int_X a \; dx$$

$$L = \int_X e \; dx$$

These imply that $z$ and $x$ are sufficient to compute equilibrium $w$ and $r$.

Denote the optimal decision rules as $c = g_c(z, x, e, a)$ and $a' = g_a(z, x, e, a)$. We can define the recursive competitive equilibrium as follows:

**Definition 1 (Recursive competitive equilibrium)**
*A recursive competitive equilibrium consists of pricing functions $r(z, x)$, $w(z, x)$, forecasting function for $x$, $\phi_x(z, x)$, value function $V(z, x, e, a)$, optimal decision rules $g_c(z, x, e, a)$ and $g_a(z, x, e, a)$, such that:*

1. ***Agent's optimization:*** *Value function $V(z, x, e, a)$ solves the Bellman equation (1). $g_c(z, x, e, a)$ and $g_a(z, x, e, a)$ are the associated optimal decision rules.*
2. ***Firm's optimization:*** *Pricing functions $r(z, x)$ and $w(z, x)$ satisfy the following marginal conditions:*

$$r(z, x) = \theta z K^{\theta-1} L^{1-\theta} - \delta$$

$$w(z, x) = (1 - \theta) z K^{\theta} L^{-\theta}$$

*where*

$$K = \int_X a \; dx$$

$$L = \int_X e \; dx$$

3. ***Consistency:*** *The forecasting function $\phi_x(z, x)$ is consistent with the actual law of motion implied by the optimal decision rule $g_a(z, x, e, a)$ and the transition matrices for $z$ and $e$.*

# 4 Krusell and Smith (1998) Approximation

A big problem in computing the equilibrium defined above is how to deal with the type distribution $x$. $x$ is an argument for many functions if you look at the definition of the equilibrium. If you have a large dimensional object like $x$, a natural way to go is to replace $x$ by simpler objects that represent $x$. How can we choose a set of variables to represent $x$?

Notice first that we don't need to worry about $e$. The distribution of $e$ (the proportion of the employed or the unemployed) is determined by the current $z$, thanks to the way we set up the transition matrix for $e$. In general, however, we need to include the proportion of the employed (or the unemployed) to the set of variables which represent $x$ (for example, see Nakajima (2007)).

Second, how do we represent the distribution of asset holding? A natural choice is to use the moments of the asset distribution. But there is a huge degree of freedom. For example, it is possible to use the fraction of assets held by the certain fraction of people to represent the asset distribution. Or, you can use Gini index. What Krusell and Smith (1998) show is that, for the current model, it is enough to use only the first moment (mean asset holding) to represent the asset distribution. It turns out that adding higher moments to represent $x$ does not change the result substantially.

The intuition of this surprising result is that, because the optimal decision rule with respect to the asset holding is close to linear in the current asset holding for both employed and unemployed agents, type distribution does not matter much. Actually, in case the optimal decision rule is perfectly linear, it can be shown that the distribution doesn't matter at all to know the dynamics of aggregate variables. See Chatterjee (1994).

One interpretation of the approximation method is that agents are assumed to be boundedly rational, or using only the partial information when making their decision. In the *true* equilibrium, agents are supposed to know $(z, x, e, a)$ when solving their optimization problem. However, here we assume that they can only use the first moment of asset distribution instead of $x$. In other words, the quality of approximation depends on to what extent the additional information contained in $x$ but not the first moment of asset holding distribution change the optimal decision of agents. The results obtained by Krusell and Smith (1998) imply that the additional information are not so valuable for agents in solving their optimization problem.

Anyway, for simplicity, let's represent $x$ by the mean asset holding. Having more statistics to represent $x$ is a trivial extension. Because we normalize the population size to be unity, the mean asset holding is equal to the total asset holding $K$, which is:

$$K = \int_X a \; dx$$

Let's use $K$ instead of $x$ to formulate the agent's problem:

**Problem 2 (Recursive formulation of agent's problem with partial information)**

$$V(z, K, e, a) = \max_{c \in \mathbb{R}^+, a' \in A} \left\{ u(c) + \beta \sum_{z'} \sum_{e'} p^z_{z,z'} p^e_{e,e'|z,z'} V(z', K', e', a') \right\} \tag{2}$$

*subject to*

$$a(1 + r(z, K)) + w(z, K)e = a' + c$$
$$K' = \phi_K(z, K)$$

The problem is much easier to handle, since we have $K$, which is a real number, instead of $x$, a probability measure. The recursive equilibrium can be defined as follows:

**Definition 2 (Recursive competitive equilibrium with partial information)**
*A recursive competitive equilibrium consists of pricing functions $r(z,K)$, $w(z,K)$, forecasting function for $K$, $\phi_K(z,K)$, value function $V(z,K,e,a)$, optimal decision rules $g_c(z,K,e,a)$ and $g_a(z,K,e,a)$, such that:*

1. ***Agent's optimization:*** *Value function $V(z,K,e,a)$ solves the Bellman equation (2). $g_c(z,K,e,a)$ and $g_a(z,K,e,a)$ are the associated optimal decision rules.*

2. ***Firm's optimization:*** *Pricing functions $r(z,K)$ and $w(z,K)$ satisfy the following marginal conditions:*

$$r(z,K) = \theta z K^{\theta-1} L(z)^{1-\theta} - \delta$$

$$w(z,K) = (1-\theta) z K^{\theta} L(z)^{-\theta}$$

3. ***Consistency:*** *The forecasting function $\phi_K(z,K)$ is consistent with the actual law of motion implied by the optimal decision rule $g_a(z,K,e,a)$ and the transition matrices for $z$ and $e$.*

# 5 Solution Method

Remember the solution algorithm of Aiyagari (1994) economy. We have an outside loop to find a set of equilibrium prices. Similarly, we need to have an outside loop to find an equilibrium forecasting function (law of motion) $\phi_K(z,K)$. Instead of actually iterating in the space of functions, we will first parameterize the forecasting function and iterate in the space of parameters. Therefore, a key choice that we have to make at this point is how to parameterize the forecasting function. A functional form suggested by Krusell and Smith (1998) and used by many other applications is log-linear form. Using log-linear form, $\phi_K(z,K)$ can be parameterized as follows:

$$logK' = \phi_{K,0,z} + \phi_{K,1,z} logK$$

Since $z$ can take one of the two values, there are 4 $(= 2 \times 2)$ coefficients that characterize the forecasting function.

Now we are ready to summarize the solution method for the model of Krusell and Smith (1998):

**Algorithm 1 (Solution algorithm: model of Krusell and Smith (1998))**

1. *Set $z$ at its unconditional mean ($\overline{z} = 1$), and solve the steady state version of the model. The solution method is the one used for the model by Aiyagari (1994). Denote $\overline{x}$ and $\overline{K}$ as the steady state type distribution and the capital stock, respectively. We will use them later.*

2. *Choose the set of statistics that represent the type distribution. For simplicity, we use the mean asset holding $K$ as the only statistic as an example.*

3. *Parameterize the forecasting function. As an example, we use the following log-linear functional form:*

$$logK' = \phi_{K,0,z} + \phi_{K,1,z} logK$$

4. *Set grid points for the space of $K$. Let's denote them as $\{K_1, K_2, ..., K_{n_K}\}$. Since it is expected that the curvature of the value function or the optimal decision rules with respect to $K$ is not large with respect to $K$, usually not a large $n_K$ is necessary to obtain a good approximation. Krusell and Smith (1998) use $n_K$ as low as 25.*

5. *Set the initial guess for $\phi^0_{K,0,z}$ and $\phi^0_{K,1,z}$. A conservative and reasonable guess is:*

$$\phi^0_{K,0,z} = log\overline{K}$$

$$\phi^0_{K,1,z} = 0$$

6. *Using the guess for $\phi_{K,0,z}$ and $\phi_{K,1,z}$, solve the optimization problem for the agent. Specifically, do the following steps:*

   (a) *Guess $V_0(z, K, e, a)$.*

   (b) *Using $V_0(z, K, e, a)$ as the value in the next period, and using the Bellman equation, update the value function to obtain $V_1(z, K, e, a)$. In order to compute the value in the next period, you need to interpolate the value in the dimension of $K$. You can use either the piecewise-linear approximation, or polynomial approximation.*

   (c) *Compare $V_0(z, K, e, a)$ and $V_1(z, K, e, a)$. If the distance between the two measured by, for example, the sup-norm is less than a predetermined tolerance level, the iteration is done. Otherwise, update $V(z, K, e, a)$ by $V_0(z, K, e, a) = V_1(z, K, e, a)$ and go back to step (b).*

7. *Using the optimal decision rules obtained in the previous step, implement the simulation. Specifically, do the following steps:*

   (a) *Set the number of periods $T$ and the number of periods which will be cut $T_0$. Heer and Maussner (2005) use $T = 3000$ and $T_0 = 500$. Krusell and Smith (1998)use $T = 11000$ and $T_0 = 1000$.*

   (b) *Choose $z_0$ and draw a sequence of $z_t$ for $t = 0, 1, ..., T$, using a random number generator. It's better to keep the draw of $z_t$ and use the same draw again and again every time the economy is simulated. Otherwise, the simulation result depends on the realization of $\{z_t\}$ and thus it is hard to get a convergence.*

   (c) *Set the initial type distribution $x_0$. A reasonable choice is $x_0 = \overline{x}$.*

   (d) *Using $x_0$, you can compute $K_0$, which is the aggregate (mean) capital stock associated with $x_0$.*

   (e) *Using the optimal decision rule $g_a(z, K, e, a)$, and the transition probabilities of $e$, update the distribution $x_0$ and obtain $x_1$. You can also compute $K_1$ from $x_1$.*

   (f) *Keep implementing the previous step until period $t = T$.*

   (g) *Now we have the sequences of $x_t$ and $K_t$.*

8. *Drop the first $T_0$ periods in order to eliminate the influence of the arbitrary (reasonable though) choice of the initial distribution and initial $z$.*

9. *Use* $\{z_t, K_t\}_{t=T_0+1}^{T}$ *and OLS regression to obtain the new set of coefficients for the forecasting function,* $\phi^1_{K,0,z}$ *and* $\phi^1_{K,1,z}$

10. *Compare* $\phi^0_{K,0,z}$ *and* $\phi^1_{K,0,z}$ *and* $\phi^0_{K,1,z}$ *and* $\phi^1_{K,1,z}$*. If the distance of coefficients are all less than a predetermined tolerance level, done. Otherwise, update both* $\phi_{K,0,z}$ *and* $\phi_{K,1,z}$ *and go back to step 6. In updating, it's good to be conservative. Use the following updating formula with a small* $\lambda$*:*

$$\phi^0_{K,0,z} = \lambda\phi^1_{K,0,z} + (1-\lambda)\phi^0_{K,0,z}$$

$$\phi^0_{K,1,z} = \lambda\phi^1_{K,1,z} + (1-\lambda)\phi^0_{K,1,z}$$

11. *If a consistent set of* $\phi_{K,0,z}$ *and* $\phi_{K,1,z}$ *are obtained, check the goodness of fit. You can use* $R^2$ *or other measures for goodness of fit. If the fit is not satisfactory, either try richer functional form for the forecasting function, or increase the set of statistics that represent the distribution (for example, add higher moments of the asset distribution).*

# References

**Aiyagari, S. Rao**, "Uninsured Idiosyncratic Risk, and Aggregate Saving," *Quarterly Journal of Economics*, 1994, *109*, 659–684.

**Castañeda, A., J. Díaz-Giménez, and J.-V. Ríos-Rull**, "Exploring the Income Distribution Business Cycle Dynamics," *Journal of Monetary Economics*, 1998, *42* (1), 93–130.

**Chatterjee, Satyajit**, "Transitional Dynamics and the Distribution of Wealth in a Neoclassical Growth Model," *Journal of Public Economics*, 1994, *54*, 97–119.

**Heer, Burkhard and Alfred Maussner**, *Dynamic General Equilibrium Modelling*, Heidelberg: Springer, 2005.

**Huggett, Mark**, "Wealth Distribution in Life-Cycle Economies," *Journal of Monetary Economics*, 1996, *38*, 469–494.

**Krusell, Per and Anthony Smith**, "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 1998, *106*, 867–896.

**Nakajima, Makoto**, "Business Cycles in the Equilibrium Model of Labor Market Search and Self-Insurance," 2007. Manuscript.

**Ríos-Rull, José-Víctor**, "Computation of Equilibria in Heterogeneous-Agent Models," in Ramon Marimon and Andrew Scott, eds., *Computational Methods for the Study of Economic Dynamics*, Oxford: Oxford University Press, 1999, chapter 11.

**Storesletten, Kjetil, Chris I. Telmer, and Amir Yaron**, "The Welfare Cost of Business Cycles Revisited: Finite Lives and Cyclical Variation in Idiosyncratic Risk," *European Economic Review*, 2001, *45*, 1311–1339.