Chapter 11 Dynamic Macro II: The stochastic OLG model

11.1 General structure and long-run equilibrium

11.1.1 Demographics, Behaviour, and Markets

- The economy is populated by J overlapping generations indexed by $j = 1, \ldots, J$.
- Cohort sizes grow over time at the constant rate n_p.
- Let N_t denote the size of the cohort that enters the labor market at time t.

$$N_t = (1 + n_p) \cdot N_{t-1}$$

- On a balanced growth path (BGP), all aggregate variables grow at rate n_p. Normalize aggregate variables by the size of the youngest cohort living in this period, then they are constant.

$$m_j = \frac{N_{t-j+1}}{N_t} = \frac{N_{t-j+1}}{(1+n_p)^{j-1} N_{t-j+1}} = \frac{1}{(1+n_p)^{j-1}} = (1+n_p)^{1-j}$$

- Preferences and labor-productivity risk

$$U_0 = E\left[\sum_{j=1}^{J} \beta^{j-1} * u(c_{jt}, 1 - l_{jt})\right] \qquad (1)$$

$$u(c_{jt}, 1 - l_{jt}) = \frac{[c_{jt}^{v}(1 - l_{jt})^{1-v}]^{1-1/\gamma}}{1 - 1/\gamma}$$

Individuals differ with respect to their labour productivity h_{jt}, which depends on a (deterministic) age profile of earnings e_j, a fixed productivity effect θ that is drawn at the beginning of the life cycle, and an autoregr essive component η_{jt} that evolves over time. At the mandatory retirement age j_r, labour productivity falls to zero and households receive a flat pension benefit pen_{jt} computed as a fraction κ of average labour income in period t.

- Households maximize the expected utility function (1) subject to the (periodical) budget constraints

$$s.t.\ a_{j+1,t} = (1 + r_t^n) a_{jt} + w_t^n h_{jt} l_{jt} + pen_{jt} - p_t c_{jt} \qquad (2)$$

$$h_{jt} = \begin{cases} e_j * \exp(\theta + \eta_{jt}), & j < j_r \\ 0, & j \geq j_r \end{cases}$$

where net wage rate $w_t^n = w_t(1-\tau_t^w-\tau_t^p)$, net interest rate $r_t^n = r_t(1-\tau_t^r)$, consumer price $p_t = 1+\tau_t^c$. And τ_t^c, τ_t^w, τ_t^p and τ_t^r are consumption, labor-income, payroll (pension) and capital-income tax rates. Finally, the household has to respect a non-negativity constraint on savings $a_{j+1,t} \geq 0$ at all ages in every period.

The dynamic programming problem

- The optimization problem of households reads

$$V_t(z) = \max_{c,l,a^+} \; u(c, 1-l) + \beta E[V_{t+1}(z^+)|\eta] \tag{3}$$

$$s.t. \;\; a^+ + p_t c = (1+r_t^n)a + w_t^n hl + pen, \qquad a^+ \geq 0, l \geq 0$$

$$\eta^+ = \rho\eta + \varepsilon^+ \; with \;\; \varepsilon^+ \sim N(0, \sigma_\varepsilon^2)$$

where $z = (j, a, \theta, \eta)$ is the vector of individual state variables. Note that we put a time index on the value function and on prices. This will be necessary as soon as we compute transitional dynamics of the model.

- The terminal condition for the value function is

$$V_t(z) = 0 \; for \; z = (J+1, a, \theta, \eta)$$

which means we assume that the household doesn't value what is happening after death.

Solve the DP problem

$$\mathcal{L} = \frac{[c^v(1-l)^{1-v}]^{1-1/\gamma}}{1-1/\gamma} + \beta E[V(z^+)|\eta] + \lambda[(1+r_t^n)a + w_t^n hl + pen - a^+ - p_t c]$$

$$\frac{\partial \mathcal{L}}{\partial c} = [c^v(1-l)^{1-v}]^{-1/\gamma} * vc^{v-1} * (1-l)^{1-v} - \lambda p_t = 0$$

$$\Rightarrow \frac{v}{c}[c^v(1-l)^{1-v}]^{1-1/\gamma} = \lambda p_t$$

$$\frac{\partial \mathcal{L}}{\partial l} = [c^v(1-l)^{1-v}]^{-1/\gamma} * (1-v)(1-l)^{-v}(-1)c^v + \lambda w_t^n h = 0$$

$$\Rightarrow \frac{1-v}{1-l}[c^v(1-l)^{1-v}]^{1-1/\gamma} = \lambda w_t^n h$$

Combining the above two F.O.C. equations, we have

$$\frac{\frac{v}{c}[c^v(1-l)^{1-v}]^{1-1/\gamma}}{\frac{1-v}{1-l}[c^v(1-l)^{1-v}]^{1-1/\gamma}} = \frac{\lambda p_t}{\lambda w_t^n h}$$

$$\frac{v}{1-v}\frac{1-l}{c} = \frac{p_t}{w_t^n h}$$

$$\Rightarrow c = \frac{v}{1-v}\frac{w_t^n h(1-l)}{p_t}$$

$$\frac{\partial \mathcal{L}}{\partial a^+} = \beta E[V_{a^+}(z^+)|\eta] - \lambda = 0$$

Recall $V(z^+) = u(z^+) + \beta E[V(z^{++})|\eta^+] = u\big(c(z^+), 1 - l(z^+)\big) + \beta E[V(z^{++})|\eta^+]$, and

$$c(z^+) = [(1 + r_{t+1}^n)a^+ + w_{t+1}^n h^+ l^+ + pen^+ - a^{++}]/p_{t+1}$$

Given the Envelope theory, $\frac{\partial V}{\partial a} = V_a + V_c \frac{\partial c}{\partial a} = V_a + 0 * \frac{\partial c}{\partial a} = V_a$, then

$$V_a(z^+) = \frac{\partial u\big(c(z^+), 1 - l(z^+)\big)}{\partial a^+}$$

$$= \frac{\partial u\big(c(z^+), 1 - l(z^+)\big)}{\partial c(z^+)} * \frac{\partial c(z^+)}{\partial a^+}$$

$$= \frac{v}{c(z^+)}[c(z^+)^v(1 - l(z^+))^{1-v}]^{1-1/\gamma} * \frac{(1 + r_{t+1}^n)}{p_{t+1}}$$

$$= \frac{(1 + r_{t+1}^n)}{p_{t+1}}\frac{v}{c(z^+)}[c(z^+)^v(1 - l(z^+))^{1-v}]^{1-1/\gamma}$$

Given $\frac{\partial \mathcal{L}}{\partial c} = 0$, then $\beta E[V_{a^+}(z^+)|\eta] - \lambda = 0$ becomes

$$\beta E\left[\frac{(1 + r_{t+1}^n)}{p_{t+1}}\frac{v[c_{t+1}(z^+)^v(1 - l_{t+1}(z^+))^{1-v}]^{1-1/\gamma}}{c_{t+1}(z^+)}\middle|\eta\right] = \lambda = \frac{v[c(a^+)^v(1 - l(a^+))^{1-v}]^{1-1/\gamma}}{c(a^+) * p_t}$$

Given the budget constraint and $c = \frac{v}{1-v}\frac{w_t^n h(1-l)}{p_t}$, we have

$$l = l(a^+) = [a^+ + p_t c - (1 + r_t^n)a - pen]/(w_t^n h)$$

$$= \left[a^+ + \frac{v}{1-v}w_t^n h(1 - l(a^+)) - (1 + r_t^n)a - pen\right]/(w_t^n h)$$

$$w_t^n h l(a^+) = a^+ + \frac{v}{1-v}w_t^n h(1 - l(a^+)) - (1 + r_t^n)a - pen$$

$$(1 - v)w_t^n h l(a^+) = (1 - v)[a^+ - (1 + r_t^n)a - pen] + v w_t^n h(1 - l(a^+))$$

$$w_t^n h l(a^+) = (1 - v)[a^+ - (1 + r_t^n)a - pen] + v w_t^n h$$

$$l(a^+) = v + \frac{1-v}{w_t^n h}[a^+ - (1+r_t^n)a - pen]$$

$$l \underset{=}{\geq} 0 \max\left\{v + \frac{1-v}{w_t^n h}[a^+ - (1+r_t^n)a - pen], 0\right\}$$

$$l \underset{=}{\leq} 1 \min\left\{\max\left\{v + \frac{1-v}{w_t^n h}[a^+ - (1+r_t^n)a - pen], 0\right\}, 1\right\}$$

Therefore,

$$l(a^+) = \min\left\{\max\left\{v + \tfrac{1-v}{w_t^n h}[a^+ - (1+r_t^n)a - pen], 0\right\}, 1\right\} \tag{4}$$

$$c = c(a^+) = \tfrac{1}{p_t}[(1+r_t^n)a + w_t^n h l(a^+) + pen - a^+] \tag{5}$$

$$\tfrac{v[c(a^+)^v(1-l(a^+))^{1-v}]^{1-1/\gamma}}{p_t c(a^+)} = \beta(1+r_{t+1}^n)E\left[\tfrac{v[c_{t+1}(z^+)^v(1-l_{t+1}(z^+))^{1-v}]^{1-1/\gamma}}{p_{t+1}c_{t+1}(z^+)}\middle|\eta\right] \tag{6}$$

where a^+ is the unknown.

Aggregation

In order to aggregate individual decisions at each element of the state space to economy-wide quantities, we need to determine the distribution of households $\phi_t(z)$ across the state space. For the sake of simplicity, we assume that we have already discretized the state space.

Specifically, we know that at age $j = 1$ households hold zero assets, experience a permanent productivity shock $\hat{\theta}_i$; with probability $\hat{\pi}_i$, as well as a transitory productivity shock of $\eta_1 = 0$. Hence, we have

$$\phi_t(z) = \phi_t(j, a, \theta, \eta) = \phi_t(1, 0, \hat{\theta}_i, \hat{\eta}_g) = \begin{cases} \pi_i^\theta & if \;\; g = \dfrac{m+1}{2} \\ 0 & otherwise \end{cases}$$

Knowing the distribution of households over the state space at age 1, we can compute the distribution at any successive age-year combination using the policy function $a_t^+(z)$. Specifically, for each element of the state space $z = (j, a, \theta, \eta)$ at age j and time t, we compute the left and right interpolation nodes $\hat{a}_l$ and $\hat{a}_r$ as well as the corresponding interpolation weight φ. The nodes and the weight satisfy

$$a_t^+(z) = \varphi * \hat{a}_l + (1-\varphi) * \hat{a}_r$$

Taking into account the transition probabilities for the transitory productivity shock η_{gg^+}, we then distribute the mass of individuals at state z to the state space at the next age $j+1$ and year $t+1$ according to

$$\phi_{t+1}(z^+) = \phi_{t+1}\big(j+1, \hat{a}_v, \hat{\theta}_i, \hat{\eta}_{g^+}\big) = \begin{cases} \phi_{t+1}(z^+) + \varphi * \pi_{gg^+} * \phi_t(z) & if \;\; v = l, \\ \phi_{t+1}(z^+) + (1-\varphi) * \pi_{gg^+} * \phi_t(z) & if \;\; v = r. \end{cases}$$

Note that for any age j and time t, the distributional measure $\phi_t(z)$ satisfies

$$\sum_{v=0}^{n} \sum_{i=1}^{2} \sum_{g=1}^{m} \phi_t(z) = 1$$

We can hence use it to calculate cohort-specific aggregates

$$\bar{c}_{jt} = \sum_{v=0}^{n} \sum_{i=1}^{2} \sum_{g=1}^{m} \phi_t(z) * c_t(z)$$

$$\bar{l}_{jt} = \sum_{v=0}^{n} \sum_{i=1}^{2} \sum_{g=1}^{m} \phi_t(z) * h_t(z) l_t(z)$$

$$\bar{a}_{jt} = \sum_{v=0}^{n} \sum_{i=1}^{2} \sum_{g=1}^{m} \phi_t(z) * \hat{a}_v$$

From these cohort values, we can in turn generate economy-wide quantities. We have only to weight the cohort variables with by the respective relative cohort sizes m_j. Consequently, we get

$$C_t = \textstyle\sum_{j=1}^{J} m_j * \bar{c}_{jt}, \;\; L_t^s = \sum_{j=1}^{J} m_j * \bar{l}_{jt} \;\text{ and }\; A_t = \sum_{j=1}^{J} m_j * \bar{a}_{jt} \qquad (7)$$

Firms

The Cobb-Douglas production technology

$$Y_t = \Omega K_t^{\alpha} L_t^{1-\alpha}$$

Capital depreciates at rate δ, so that the capital stock evolves as

$$\big(1+n_p\big)K_{t+1} = (1-\delta)K_t + I_t$$

Under the assumption of perfect competition, the (inverse) demand functions of the firm for capital and labor are

$$r_t = \alpha\Omega \left[\frac{L_t}{K_t}\right]^{1-\alpha} - \delta \;\; and \;\; w_t = (1-\alpha)\Omega \left[\frac{K_t}{L_t}\right]^{\alpha} \qquad (8)$$

The government

The government runs two separate systems: the tax system and the pension system.

- At any point in time the budget of the tax system is balanced if

$$\tau_t^c C_t + \tau_t^w w_t L_t^s + \tau_t^r r_t A_t + (1+n_p)B_{t+1} = G_t + (1+r_t)B_t \tag{9}$$

In addition to revenue from taxation, the government finances expenditure from issuing new debt $(1+n_p)B_{t+1}$. However, it has to repay current debt including interest payments $(1+r_t)B_t$. Therefore, in steady state, expenditure $(r-n_p)B$ reflects the cost needed to keep the debt level constant.

- The pension system operates on a pay-as-you-go basis. The budget-balance equation of the pension system then reads

$$\tau_t^p w_t L_t^s = \overline{pen}_t * N^R = \overline{pen}_t * \sum_{j=j_r}^{J} m_j \tag{10}$$

- For the evolution of pension payments over time we assume that they are linked to the average labor earnings of the previous period, i.e.,

$$\overline{pen}_t = \kappa_t * \frac{w_{t-1}L_{t-1}^s}{N^L} = \kappa_t * \frac{w_{t-1}L_{t-1}^s}{\sum_{j=1}^{j_r-1} m_j}$$

where κ_t is the replacement rate of the pension system and N^L is the (fixed) size of working-age cohorts. We assume that the replacement rate κ_t is given exogenously while the contribution rate τ_t^p adjusts in order to balance the budget.

Markets

There are three markets in our economy: The factor markets for capital and labor and the goods market.

- On the factor markets, prices for capital r_t and labor w_t adjust so that markets clear, i.e.,

$$K_t + B_t = A_t \quad and \quad L_t = L_t^s \tag{11}$$

- The goods market equilibrium reads

$$Y_t = C_t + G_t + I_t \tag{12}$$

Using Walras' law, we know that if the factor markets clear, then the goods market will also be in equilibrium.

Equilibrium

Define an equilibrium path of the OLG economy.

- **Definition (Equilibrium path)**

Given a path for government expenditure and debt $\{G_t, B_t\}_{t=0}^{\infty}$, a path for tax rates $\{\tau_t^c, \tau_t^w, \tau_t^r\}_{t=0}^{\infty}$, and a characterization of the pension system $\{\tau_t^p, \kappa_t\}_{t=0}^{\infty}$, and a recursive competitive equilibrium ofthe economy is a set of policy functions

$$\{c_t(z), l_t(z), a_t(z)\}_{t=0}^{\infty}$$

for the households, a set of input choices $\{K_t, L_t\}_{t=0}^{\infty}$ for the firms, prices $\{r_t, w_t\}_{t=0}^{\infty}$ and a measure of households $\{\phi_t\}_{t=0}^{\infty}$ that are such that

1.[Household maximization]:

Given prices $\{r_t, w_t\}_{t=0}^{\infty}$, the policy functions solve the household optimization problem as stated in (3).

2.[Firms maximization]:

Given prices, the firms' factor inputs satisfy their demand equations in (8)

3.[Government budget constraints]:

The government budget constraints (9) and (10) are satisfied.

4.[Market clearing]:

The market-clearing equations (11) and (12) hold with aggregate quantities being computed from (7).

5.[Consistency of the measure of households]:

The measure of households is consistent with the assumptions about stochastic processes and individual decisions.

The above definition applies to any general equilibrium path of the economy. We can furthermore specify a long-run equilibrium of the model or a steady state.

- **Definition (Long-run equilibrium)**

A long-run equilibrium of the economy is an equilibrium path on which prices, tax rates, and all individual variables are constant over time and aggregate quantities all grow at the rate of the population n_p.

Fortran Code (https://github.com/fabiankindermann/ce-fortran/tree/main/code-book/prog11/prog11_01)

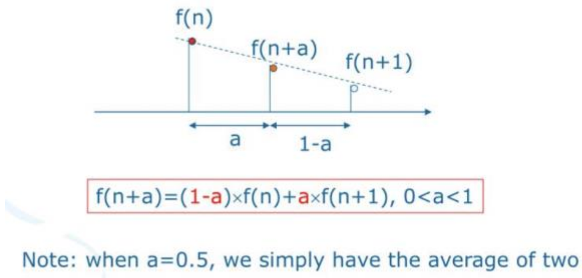
1. Basics (https://github.com/fabiankindermann/ce-fortran/wiki/toolbox-documentation)

Interpolation: a useful Statistical tool that is used to estimate values between any two given points.

1. subroutine linint_Equi(x, left, right, n, il, ir, phi)

This subroutine calculates nodes and weights for the linear interpolation of data $\{y_i\}_{i=0}^{n}$ that is located on an equidistant grid of nodes $\{x_i\}_{i=0}^{n}$, see grid_Cons_Equi. It therefore calculates the indices of a left and right gridpoints i_l and i_r as well as an interpolation weight φ, so that we can calculate the linear interpolant as

$$y = \varphi * y_{i_l} + (1 - \varphi) * y_{i_r}$$



Input arguments:

- real*8 :: x or real*8 :: x(:)

  The point(s) where to evaluate the interpolating polynomial. This can be either a scalar or a one-dimensional array of arbitrary size. In the latter case, the polynomial is evaluated at each point supplied in the array x.

- real*8 :: left

  The left interval endpoint of the equidistant grid $\{x_i\}_{i=0}^{n}$.

- real*8 :: right

  The right interval endpoint of the equidistant grid $\{x_i\}_{i=0}^{n}$.

- integer :: n

  The number of gridpoints the equidistant grid is made up of.

Output arguments:

- integer :: il or integer :: il(:)

  The left interpolation node(s) i_l. If the input value x is a scalar, this needs to be a scalar. If x is a one-dimensional array, this also needs to be a one-dimensional array with exactly the same length.

- integer :: ir or integer :: ir(:)

  The right interpolation node(s) i_r. If the input value x is a scalar, this needs to be a scalar. If x is a one-dimensional array, this also needs to be a one-dimensional array with exactly the same length.

- real*8 :: phi or real*8 :: phi(:)

  The interpolation weight(s) φ. If the input value x is a scalar, this needs to be a scalar. If x is a one-dimensional array, this also needs to be a one-dimensional array with exactly the same length.