# Exercise 1

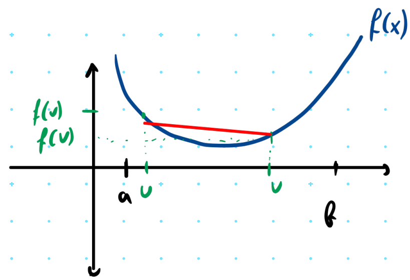
Consider $f \in C^1[a,b]$ monotonically increasing and convex with root $x^* \in [a,b]$

Show that Newtons Method is converging $\forall x_0 \in [x^*, b]$ and

$$x_{n+1} \leq x_n \quad n = 0, 1 \ldots$$

## Hints

- First show $x_{n+1} \leq x_n$
- For konvex functions $\varphi : [a,b] \longrightarrow \mathbb{R}$ we have $\varphi'(x)(z-x) \leq \varphi(z) - \varphi(x) \quad \forall x, z \in [a,b]$
- With the last hint show $x_{n+1} \geq x^*$




$$f(\lambda u + (1-\lambda) v) \leq \lambda f(u) + (1-\lambda) f(v)$$

**Proof:** (1) Show that $x_{n+1} \leq x_n$

Consider Newtons Method

$$x_{n+1} = x - \frac{f(x_n)}{f'(x_n)}$$

- Because $f$ monotonically increasing $f'(x) > 0 \quad \forall x \in [a,b]$

For a convex function we know $f'(x)(y-x) \leq f(y) - f(x) \quad \forall x, y \in [a,b]$

Two conditions
- $f'(x)(y-x) \leq f(y) - f(x)$
- Swap x and y $\quad f'(y)(x-y) \leq f(x) - f(y)$
- Combine $\quad f'(x)(y-x) + f'(y)(x-y) \leq (f(y) - f(x)) + (f(x) - f(y))$

$$\Longleftrightarrow f'(x)y - f'(x)x + f'(y)x - f'(y)y \leq 0$$

$$\Longleftrightarrow (f'(x) - f'(y)) \cdot (y-x) \leq 0$$

$$\Longleftrightarrow (f'(x) - f'(y)) \cdot (x-y) \geq 0$$

$$\left.\begin{array}{l} x > y \Rightarrow x-y > 0 \Rightarrow f'(x) - f'(y) \geq 0 \Rightarrow f'(x) \geq f'(y) \\ x < y \Rightarrow x-y < 0 \Rightarrow f'(x) - f'(y) \geq 0 \Rightarrow f'(x) \geq f'(y) \end{array}\right.$$

In total $f'(x) \geq f'(y)$ when $x > y$

and $f'(x)$ monotonically increasing

Now from Newtons Method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$\left.\begin{array}{l} \bullet\ f \text{ monotonically increasing} \Rightarrow f'(x) > 0 \quad \forall x \in [a,b] \\ \bullet\ f \text{ convex} \Rightarrow f'(x) \text{ monotonically increasing} \\ \bullet\ x_n \in [x^*, b] \Rightarrow f(x_n) \geq 0 \quad \text{because } f(x^*) = 0 \end{array}\right\} \quad \underline{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \leq x_n}$$

② Show that $x_{n+1} \geq x^*$

For convex function $f: [a,b] \to \mathbb{R}$ : $f'(x)(z-x) \leq f(z) - f(x) \quad \forall x, z \in [a,b]$

Let $x = x_n$ and $z = x^*$ $\Rightarrow f'(x_n)(x^* - x_n) \leq f(x^*) - f(x_n)$

Now $f(x^*) = 0 \Rightarrow f'(x_n)(x^* - x_n) \leq -f(x_n)$

$$\Leftrightarrow -f'(x_n)(x^* - x_n) \geq f(x_n)$$

Now divide with $f'(x_n)$ $(>0)$ $\Rightarrow \dfrac{f(x_n)}{f'(x_n)} \leq -(x^* - x_n)$

$$\Leftrightarrow -\frac{f(x_n)}{f'(x_n)} \geq x^* - x_n \Leftrightarrow x_n - \frac{f(x_n)}{f'(x_n)} \geq x^* \Rightarrow x_{n+1} \geq x^*$$

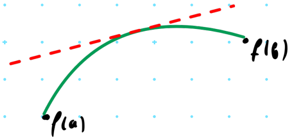
So we know $\{x_n\}$ is monotonically decreasing $x_{n+1} \leq x_n$, and $x^*$ is a lower bound for all elements of the sequence $\Rightarrow$ Monotone Convergence Criterion $x_n \longrightarrow \tilde{x} \in [x^*, b]$ ■

# Exercise 2

Proof the following statements:

Let $f:[a,b] \to \mathbb{R}$ be two times continiously differentialbe and $|f'(x)| \geq m$, $|f''(x)| \leq M$ $\forall x \in [a,b]$

where $m, M > 0$ then:




a) f has in $[a,b]$ one root

Hint: Mean Value Theorem: $f:[a,b]$ continious on $[a,b]$, differentialbe on $(a,b)$ $a<b$
then $\exists c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof Suppose $f$ has two roots $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ $\Rightarrow f(x_1) = f(x_2) = 0$

Because $f$ is continiously differentialbe we use mean value theorem

$$\exists c \in (x_1, x_2) \text{ with } f'(c) = \frac{f(x_2)-f(x_1)}{x_2 - x_1} = 0$$

But we know $|f'(x)| \geq m$ with $m > 0$ $\longrightarrow$ Contradiction $\lightning$ $f$ has one root maximal

b) If $x^*$ is a root in $(a,b)$ then Newtons Method is well defined $\forall x_0 \in U_r(x^*)$
with $r := \min(2mM^{-1}, b-x^*, x^*-a)$

Newtons Method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ is well defined if $f'(x_n) \neq 0$ $\forall n$

Because $|f'(x)| \geq m > 0$ $\forall x \in [a,b]$ always fullfilled

Now consider Taylor Series expansion around $x^*$

$$f(x_n) = f(x^*) + f'(x^*)(x_n - x^*) + \frac{f''(\xi_n)}{2}(x_n - x^*)^2 \quad \text{with } \xi_n \text{ between } x_n \text{ and } x^*$$

$$f(x_n) = f'(x^*)(x_n - x^*) + \frac{f''(\xi_n)}{2}(x_n - x^*)^2$$

From Newton Iteration we know $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{f'(x^*)(x_n - x^*) + \frac{f''(\xi_n)}{2}(x_n - x^*)^2}{f'(x_n)}$$

Subtraction of $x^*$ to estimate error

$$x_{n+1} - x^* = (x_n - x^*) - \frac{f'(x^*)(x_n - x^*) + \frac{f''(\xi_n)}{2}(x_n - x^*)^2}{f'(x_n)}$$

$$\Leftrightarrow \; x_{n-1} - x^* = (x_n - x^*) \cdot \left(1 - \frac{f'(x^*)}{f'(x_n)}\right) + \left(- \frac{f''(\xi_n)}{2f'(x_n)}(x_n - x^*)^2\right)$$

$$|x_{n-1} - x^*| = \left|(x_n - x^*) \cdot \left(1 - \frac{f'(x^*)}{f'(x_n)}\right) + \left(- \frac{f''(\xi_n)}{2f'(x_n)}(x_n - x^*)^2\right)\right|$$

$$\le \left|(x_n - x^*) \cdot \left(1 - \frac{f'(x^*)}{f'(x_n)}\right)\right| + \left|\left(- \frac{f''(\xi_n)}{2f'(x_n)}(x_n - x^*)^2\right)\right|$$

$$\le |x_n - x^*| \cdot \left|1 - \frac{f'(x^*)}{f'(x_n)}\right| + \left|\frac{f''(\xi_n)}{2f'(x_n)}(x_n - x^*)^2\right|$$

- $|f'(x)| \ge m \Leftrightarrow \frac{1}{|f'(x_n)|} \le \frac{1}{m}$
- $|f''(x)| \ge M \Leftrightarrow \left|\frac{f''(\xi_n)}{2}\right| \le \frac{M}{2}$

$$\Rightarrow \; |x_{n+1} - x^*| \le |x_n - x^*| \cdot \left|1 - \frac{f'(x^*)}{f'(x_n)}\right| + \frac{M}{2m}|x_n - x^*|^2$$

Now $|f'(x)| \ge m > 0 \;\; \forall x \in [a,b] \Rightarrow f'(x_n) \neq 0$ well defined

By using $r = \min\left(\frac{2m}{M}, b - x^*, x^* - a\right)$ and $x_0 \in U_r(x^*)$

- Iterations are in $[a,b]$
- Quadratic term $\frac{M}{2m}|x_n - x^*|^2$ is small giving convergence

c) If $q := M(2m)^{-1}|x_0 - x^*| < 1$ then $\forall n \in \mathbb{N}$

i) $|x^* - x_n| \le \frac{M}{2m}|x^* - x_{n-1}|^2$

From part b) we know:

$$|x_{n+1} - x^*| \le |x_n - x^*| \cdot \left|1 - \frac{f'(x^*)}{f'(x_n)}\right| + \frac{M}{2m}|x_n - x^*|^2$$

Plugging in $x_n$

Consider $|x_n - x^*| \le |x_{n-1} - x^*| \cdot \left|1 - \frac{f'(x^*)}{f'(x_n)}\right| + \frac{M}{2m}|x_{n-1} - x^*|^2$

Because $|f''(x)| \leq M \longrightarrow$ Lipschitz continuity of $f'$

$$|f'(x) - f'(y)| \leq M|x-y| \quad \forall x \in [a,b]$$

Consider the term
$$\left|1 - \frac{f'(x^*)}{f'(x_n)}\right| = \left|\frac{f'(x_{n-1}) - f'(x^*)}{f'(x_n)}\right| \leq \frac{|f'(x_{n-1}) - f'(x^*)|}{m} \leq \frac{M}{m}|x_{n-1} - x^*|$$

$$|x_n - x^*| \leq |x_{n-1} - x^*| \cdot \left|1 - \frac{f'(x^*)}{f'(x_n)}\right| + \frac{M}{2m}|x_{n-1} - x^*|^2 \leq |x_{n-1} - x^*| \cdot \frac{M}{m}|x_{n-1} - x^*| + \frac{M}{2m}|x_{n-1} - x^*|^2$$
$$= \frac{3M}{2m}|x_{n-1} - x^*|^2$$

So for all $n \in \mathbb{N}$ $|x_n - x^*| \leq \frac{M}{2m}|x_{n-1} - x^*|^2 \leq \frac{3M}{2m}|x_{n-1} - x^*|^2$ holds

ii) $|x^* - x_n| \leq \frac{2m}{M} q^{2^n}$ with $q := \frac{M}{2m}|x - x^*| < 1$

Induction start $n = 0$ $\quad |x^* - x_0| \leq \frac{2m}{M} q^0 = \frac{2m}{M} \frac{M}{2m}|x - x^0| = |x^* - x_0| \leq |x^* - x_0|$ ✓

Induction hypothesis $n = k$ $\quad |x^* - x_k| \leq \frac{2m}{M} q^{2^k}$ holds

Induction Step $n = k+1$

From c.I we know $|x^* - x_{k+1}| \leq \frac{M}{2m}|x^* - x_k|^2$

use induction hypothesis on $|x^* - x_k|$

$$|x^* - x_{k+1}| \leq \frac{M}{2m}|x^* - x_k|^2 \leq \frac{M}{2m}\left(\frac{2m}{M} q^{2^k}\right)^2$$

$$\Leftrightarrow |x^* - x_{k+1}| \leq \frac{M}{2m} \frac{4m^2}{M^2} q^{2^{k+1}} = \frac{2m}{M} q^{2^{k+1}}$$ exactly what we wanted to show

(iii) $|x^* - x_n| \leq \frac{1}{m}|f(x_n)| \leq \frac{M}{2m}|x_n - x_{n-1}|^2$

Because $|f'(x)| \geq m \quad \forall x \in [a,b]$

Write $f(x_n) = f(x_n) - f(x^*) \quad \exists \xi_n \in (x_n, x^*)$ with $f(x_n) - f(x^*) = f'(\xi_n)(x_n - x^*)$

$$|f(x_n)| = |f'(\xi_n)| \cdot |x_n - x^*| \geq m|x_n - x^*|$$

thus we get $|x_n - x^*| \leq \frac{1}{m}|f(x_n)|$

now show $\frac{1}{m}|f(x_n)| \leq \frac{M}{2m}|x_n - x_{n-1}|$

By Taylor: $f(x_n) = f(x_{n-1}) + f'(x_{n-1})(x_n - x_{n-1}) + \frac{f''(\eta_n)}{2}(x_n - x_{n-1})^2$

And using Newton $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

Substitute into Taylor $f(x_n) = f(x_{n-1}) + f'(x_{n-1})\left(-\frac{f(x_{n-1})}{f'(x_{n-1})}\right) + \frac{f''(\eta_n)}{2}(x_n - x_{n-1})^2$

$$\Leftrightarrow f(x_n) = \frac{f''(\eta_n)}{2}(x_n - x_{n-1})^2$$

$$\Rightarrow |f(x_n)| \leq \frac{M}{2}|x_n - x_{n-1}|^2$$

So we have $|x^* - x_n| \leq \frac{1}{m}|f(x_n)| \leq \frac{M}{2m}|x_n - x_{n-1}|^2$ ■