Proseminar Numerische Mathematik 2 am 08.April 2025

Exercise 1

Let $a \geqslant 0$, We define the function $\phi_a : \mathbb{R}^+ \to \mathbb{R}^+, \quad x \longmapsto x^a$. For which values of a the function ϕ_a is locally lipschitz? (We say that the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is locally lipschitz if $\forall A > 0, \exists M_A > 0$ such that $|\phi(x)| \leq M_A|x|$.)
Hint: Consider first $a = 1/2$. Then observe what happens for $a = 0$, $a \geq 1$ and $0 < a < 1$.
We consider the Chauchy problem

$$y'(t) = \phi_a(y(t)), \quad t \in [0, \infty[, \qquad y(0) = 0.$$

Show that if ϕ_a is locally lipschitz, the problem above has a unique solution and if ϕ_a is not locally lipschitz, the problem has at least 2 solutions.

Exercise 2

Given a function $f \in C(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^n)$ locally Lipschitz and let x be the solution to the differential problem

$$x'(t) = f(x(t), t), \quad t \in]0, T], \qquad x(0) = x0.$$

We aim to find an approximation to the solution x. Therefore we descritize the interval $[0, T]$ into $0 = t_0 < t_1 < \cdots < t_n = T$ and we take $\Delta t = t_{k+1} - t_k, \quad \forall k = 0, 1, \ldots, N-1$. Our objective is to prove the convergence of some schemes presented in the lecture (Pages 44 and 52).

1. Suppose that $f \in C^1(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^n)$. Prove that forward Euler scheme is consistent convergent of order 1.
2. Suppose that $f \in C^1(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^n)$. prove that the modified Euler scheme and Heun scheme are consistent and convergent of order 2.
3. Suppose that $f \in C^4(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^n)$. prove that Runge-Kutta 4 scheme is consistent convergent of order 4.
4. Suppose that $f \in C^1(\mathbb{R}^n \times \mathbb{R}^+, \mathbb{R}^n)$. prove that the backward Euler scheme is consistent convergent of order 1.

Exercise 3

We consider the scalar nonlinear differential equation

$$x'(t) = x^2(t), \quad t \in]0, 0.9], \qquad x(0) = 1.$$

1. Solve the problem above in Python using explicit Euler scheme.
2. Now consider Newton's method and solve the problem using implicit Euler scheme and return the number of iterations until convergence.
3. Compare the results with the analytical solution.

Exercise 4

Consider the 2^{nd}-order differential equation which describes the motion of a pendulum

$$x"(t) + \sin(x(t)) = 0, \quad t > 0, \qquad x(0) = \xi, \quad x'(0) = 0,$$

where $\xi \in [0, 2\pi[$ is the initial position of the pendulum.

1. Rewrite the problem above into a differential system of 1^{st}-order.
2. Write the implicit Euler scheme which allows solving the system yielding x^{n+1} and y^{n+1}, the approximations of x and y at time t^{n+1}. Conclude that at each time step Δt, we need to solve a nonlinear system of the form

$$x - ay = \alpha, \qquad a\sin(x) + y = \beta, \qquad (*).$$

3. Reformulate the system $(*)$ into the form $F(X) = 0$ where you determine the function F and write Newton's method solving $F(X) = 0$. For which values of a Newton's method is well defined? (For any choice of the initial condition).
4. Let $(\bar{x}, \bar{y})$ be solution to $(*)$. Suppose that a is such that Newton's method is well defined. Show that $\exists\epsilon > 0$ such that if (x_0, y_0) is in $B_\epsilon(\bar{x}, \bar{y})$ (the ball centered in $(\bar{x}, \bar{y})$ and of radius ϵ), the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ constructed by Newton's method converges to $(\bar{x}, \bar{y})$ when $n \to \infty$.
5. Now reformulate the system $(*)$ into

$$x - ay = \alpha, \qquad f(x) = 0,$$

where you determine f and write Newton's method solving $f(x) = 0$ giving again the values of a for which the sequence constructed by Newton's method always converge independently of the initial guess.
6. Implement Newton's method for question 3 and question 5 and compare the number of iterations required for the convergence.
7. (Challenge yourself!) Based on the previous questions, study Newton's method for the implicit Euler scheme for the following problem which models the pendulum with damping:

$$x''(t) + \mu x'(t) + \sin x(t) = 0, \quad t > 0,$$

$$x(0) = \xi,$$

$$x'(0) = 0,$$

where $\xi \in [0, 2\pi[$ is the initial position of the pendulum and $\mu \in \mathbb{R}^+$ is the damping coefficient.