Exercises for Algebra 2

Due Monday, 19/5/2025

Exercise 1

Prove Proposition 7.2.3 in the lecture notes: Let p prime, $e, m, n \in \mathbb{N}$ with n, m relatively prime, and $\varphi\colon \mathbb{N} \to \mathbb{N}; n \mapsto \#(\mathbb{Z}/n\mathbb{Z})^\times$. Show that

(*i*) $\varphi(p^e) = p^{e-1}(p-1)$

(*ii*) $\varphi(mn) = \varphi(m)\varphi(n)$.

Then, when using the prime factorisation $n = p_1^{e_1} \cdots p_r^{e_r}$, we have

$$\varphi(n) = p_1^{e_1-1}(p_1-1) \cdots p_r^{e_r-1}(p_r-1).$$

Exercise 2

Show that the following data defines functors. For each of them, choose appropriate source and target categories, and also define how the functor acts on morphisms. In each case, decide if the functor is covariant or contravariant.

(*i*) $\mathrm{Hom}(X, \cdot)\colon M \mapsto \mathrm{Hom}(X, M)$.

(*ii*) $\mathrm{Hom}(\cdot, X)\colon M \mapsto \mathrm{Hom}(M, X)$.

(*iii*) $\cdot \otimes X\colon M \mapsto M \otimes X$.

(*iv*) $G \mapsto \mathcal{K}(G)$, for G a group and $\mathcal{K}(G)$ the set of its conjugacy classes.

(*v*) $M \mapsto M^{\mathrm{tor}}$.

Exercise 3

Let $\mathcal{C}$ be a category, $X, Y \in \mathrm{Obj}(\mathcal{C})$ and $f \in \mathcal{C}(X, Y), g \in \mathcal{C}(Y, X)$. If $g \circ f = \mathrm{id}_X$, we call g a *retraction* and f a *section*.

(*i*) Show that covariant functors map retractions to retractions and sections to sections. What happens for contravariant functors?

(*ii*) Show that in the categories of sets, groups, rings, k-vector spaces, R-modules, and topological spaces, retractions are surjective and sections are injective.

(*iii*) Find a covariant functor between two categories in (*ii*) which maps at least one injective morphism to a non-injective morphism. Find another covariant functor between two categories in (*ii*) which maps at least one surjective morphism to a non-surjective morphism.

Exercise 4

Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of rings on the topological space X. Show that

$$\mathcal{H}(U) := \mathcal{F}(U) \times \mathcal{G}(U)$$

together with the obvious restriction morphisms defines a sheaf of rings $\mathcal{H}$ on X.