Proseminar Numerische Mathematik 2 am 17 Juni 2025

Exercise 1

1. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric positive definite matrix, $b \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Our objective is to solve the system $Ax = b$ iteratively. To this end we consider the scheme

   - Initialization: $x^{(0)} \in \mathbb{R}^n$.
   - Iterations: $x^{(k+1)} = x^{(k)} + \alpha(b - Ax^{(k)})$.

   (a) For which values of α (in terms of the eigenvalues of A) the scheme above converges?

   (b) Calculate α_0 (in terms of the eigenvalues of A) such that $\rho(Id - \alpha_0 A) = \min\{\rho(Id - \alpha A),\ \alpha \in \mathbb{R}\}$.

   Note: This method is called **Richardson method**: It can be seen as a method of gradient with fix step where the objective is to minimize the function $f : \mathbb{R}^n \to \mathbb{R}, \quad x \longmapsto f(x) = \frac{1}{2}Ax \cdot x - b \cdot x$. Since the matrix A is symmetric positive definite, the function f has only one minimum (obtained by making the gradient of f vanishing). Or $\nabla f(x) = Ax - b$ and making $\nabla f(x) = 0$ is equivalent to solve the system $Ax = b$.

2. Let $a \in \mathbb{R}$ and the matrix
$$A = \begin{pmatrix} 1 & \alpha & \alpha \\ a & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}$$
   Show that A is symmetric positive definite iff $-1/2 < \alpha < 1$ and thus the Jacobi method converges iff $-1/2 < \alpha < 1/2$.

Exercise 2

Our objective in this exercise is to compare Gauss-Seidel and Jacobi methods. Given $x^{(0)} \in \mathbb{R}^3$ and consider the matrix A and the vector b given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

- Jacobi

  1. Write the Jacobi method to solve the system $Ax = b$ into the form $x^{(k+1)} = B_J x^{(k)} + C_J$ where you determine B_J and C_J.
  2. Determine the kernel of B_J and give a basis.

3. Calculate the spectral radius of B_J. Does Jacobi method converge?
4. Calculate the 1^{st} two iterations $x^{(1)}$ and $x^{(2)}$ starting with

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad x^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

5. Implement the method in Python and validate your code based on your results from the previous questions.

- Gauss-Seidel

1. Write the Gauss-Seidel method to solve the system $Ax = b$ into the form $x^{(k+1)} = B_{GS}x^{(k)} + C_{GS}$ where you determine B_{GS} and C_{GS}.
2. Determine the kernel of B_{GS}.
3. Calculate the spectral radius of B_{GS}. Does the Gauss-Seidel method converge?
4. Calculate the 1^{st} two iterations $x^{(1)}$ and $x^{(2)}$ starting with

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad x^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

5. Implement the method in Python and validate your code based on your results from the previous questions.

Compare the spectral radius of B_J and B_{GS}. What is your conclusion? Check if your code is consistent with the conclusion.

Exercise 3

Our objective is to analyze Jacobi, Gauss-Seidel and relaxation method in the case of a tridiagonal matrix obtained by discretizing the diffusion operator. The matrix $A \in \mathcal{M}_n(\mathbb{R})$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}$$

1. Write a Python code which factorize A using Jacobi, Gauss-Seidel and SOR methods.
2. Application: Using as Right-hand-side the vector $b = (1, 0, \ldots, 0, 1)^T$ and $n = 10$ solve the system $Ax = b$ using $Niter = 100$ (maximum number of iterations). In the case of SOR method, use $\omega = 3/2$.

3. Now consider $n = 20$ and determine the optimal parameter for the SOR method. Plot the spectral radius of the matrix of iteration obtained in term of ω.
4. For different n compare the necessary number of iterations required to get an approximation of 10^{-12} between the exact and approximated solutions. What is the relationship between the iteration number and the spectral radius?

Exercise 4

Analyze the convergence of the sequence $(x^{(k)})_{k\in\mathbb{N}} \subset \mathbb{R}^n$ given by $x^{(0)}$ is given, $x^{(k+1)} = Bx^{(k)} + c$ in the following cases

$$B = \begin{pmatrix} 2/3 & 1 \\ 0 & 2/3 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad\qquad B = \begin{pmatrix} 2/3 & 1 \\ 0 & 2 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$