Exercise 1

Let $(E, \|\cdot\|)$ be a Banach Space. Consider a mapping $\phi : E \longrightarrow E$ that is contractive

$\exists L < 1$ such that $\forall (x,y) \in E^2, \|\phi(x) - \phi(y)\| \leq L\|x-y\|$

Start from a given $x_0 \in E$ and construct the recursive sequence $x_{k+1} = \phi(x_k) \ \forall k \in \mathbb{N}$

• Show that the sequence is well defined

• Initial point $x_0 \in E$ well defined

• $\phi : E \to E$ so $\forall x_k \in E \Rightarrow \phi(x_k) \in E$

• Just by induction if x_k is well defined $x_{k+1} = \phi(x_k) \in E$ is also well defined

• Show that function ϕ is continuous, deduce that if sequence x_k converges it converges to a fixed point of ϕ.

Def Continuity: If φ is continuous $\forall$ sequences (x_n) in $(E, \|\cdot\|)$: if $x_n \xrightarrow{n \to \infty} x \quad \phi(x_n) \xrightarrow{n \to \infty} \phi(x)$

e.g. $\lim_{n \to \infty} x_n = x \Rightarrow \lim_{n \to \infty} \phi(x_n) = \phi(x)$

We now ϕ is contractive $\|\phi(x) - \phi(y)\| \leq L\|x-y\|$

Let x_n be arbitrary sequence with $\lim_{n \to \infty} x_n = x \iff \|x_n - x\| = 0$ (for $n \to \infty$)

$\|\phi(x_n) - \phi(x)\| \leq L\|x_n - x\|$ because $L < 1 \Rightarrow \|\phi(x_n) - \phi(x)\| \longrightarrow 0$ for $n \longrightarrow \infty$

this shows $\lim_{n \to \infty} \varphi(x_n) = \varphi(x)$

Part II: If (x_n) converges, it converges to a fixed point of ϕ

(x_k) is defined by $x_{k+1} = \phi(x_k)$ assume $x_k \longrightarrow x^* \in E$

$$\lim_{k \to \infty} x_{k+1} = \lim_{k \to \infty} \phi(x_k) \xRightarrow{\phi \text{ continuous}} x^* = \varphi(x^*)$$

- Show if ϕ admits a fixed point, then the fixed point is unique

Suppose ϕ admits two fixed points x^* and y^* $\quad \phi(x^*) = x^* \wedge \phi(y^*) = y^*$

we want to show $x^* = y^*$

$$\| x^* - y^* \| = \| \phi(x^*) - \phi(y^*) \| \leq L \| x^* - y^* \|$$

Inequality $\| x^* - y^* \| \leq L \| x^* - y^* \|$ can only hold if $\| x^* - y^* \| = 0$ as $L < 1 \Rightarrow x^* = y^*$

- Show that the sequence $(x_k)_{k \in \mathbb{N}}$ is a Cauchy sequence. Deduce that ϕ admits one fixed point which is given by the limit on the sequence $(x_k)_{k \in \mathbb{N}}$ $k \longrightarrow \infty$

Def. Cauchy Sequence $\quad \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall m, n \geq N : \| x_n - x_m \| < \varepsilon$

Because ϕ is contractive $\| \varphi(x) - \varphi(y) \| \leq L \| x - y \|$

Consider two terms in the sequence x_n, x_m with $n > m$ then the recursive relationship gives us

$$x_n = \phi(x_{n-1}), \ x_{n-1} = \phi(x_{n-2}) \ \dots \ x_{m+1} = \phi(x_m)$$

$$\| x_n - x_m \| = \| \phi(x_{n-1}) - \phi(x_{m-1}) \| \leq L \| x_{n-1} - x_{m-1} \| = L \| \phi(x_{n-2}) - \phi(x_{m-2}) \| \leq L^2 \| x_{n-2} - x_{m-2} \|$$

if we further iterate $\Longrightarrow \| x_n - x_m \| \leq L^n \| x_1 - x_0 \|$ with $L < 1$

$$\| x_n - x_m \| = \| x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} \dots - x_{m+1} + x_{m+1} - x_m \| \leq \| x_n - x_{n-1} \| + \| x_{n-1} - x_{n-2} \| + \dots \| x_{m+1} - x_m \|$$

$$= \sum_{k=m}^{n-1} \| x_{k+1} - x_k \| \leq \| x_1 - x_0 \| \sum_{k=m}^{n-1} L^k$$

Because $L < 1$ this is geometric series

Def. Convergent Geometric Series

$$S_n = \sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}$$

$$\sum_{k=m}^{n-1} L^k = L^m \sum_{k=0}^{n-m} L^k = \frac{L^m (1 - L^{n-m})}{1 - L} \leq \frac{L^m}{1 - L}$$

Thus $\| x_n - x_m \| \leq \frac{L^m}{1 - L} \| x_1 - x_0 \|$ $\quad m \to \infty$ right side goes to 0

and we find $\varepsilon > 0$ such that $\| x_n - x_m \| < \varepsilon$

$\longrightarrow (x_k)_{k\in\mathbb{N}}$ is a Cauchy Sequence, $(E, \|\cdot\|)$ a Banach Space where every Cauchy seq is convergent. Let $x^* \in E$ be the limit $x_k \longrightarrow x^*$ as $k \longrightarrow +\infty$

Since function is continuous $x_{k+1} = \phi(x_k) \xRightarrow[\text{Limit}]{\text{Apply}} \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} \phi(x_k) \Rightarrow x^* = \varphi(x^*)$

Application: Let $\phi : E \to E$ be a continuous mapping such that $\phi^m = \underbrace{\phi \circ \phi \circ \phi \dots \phi}_{m\text{-times}}$ $(m \in \mathbb{N})$ is contractive

1. Show that ϕ^m has a unique fixed point in $a \in E$

This can be deduced from Banach Fixed-Point Theorem (Satz 1.4)

- $\phi : E \to E$ is a mapping into itself
- ϕ^m is contractive $\exists\, L < 1 \;\forall x, y \in E : \; \|\phi^m(x) - \phi^m(y)\| \le L\|x - y\|$

$\Longrightarrow \exists$ unique $a \in E$ such that $\phi^m(a) = a$

2. Show that $\phi(\phi^m(a)) = \phi(a)$, deduce that ϕ admits a fixed point

Since $a \in E$ is fixed point of $\phi^m(a) = a \Rightarrow \phi(\phi^m(a)) = \phi(a)$

3. Show that if a is a fixed point of ϕ, a is a fixed point of ϕ^m

Assume a is fixed point of $\phi \Rightarrow \phi(a) = a$

now $\phi^m(a) = \phi(\phi(\dots(\phi(a)\dots)) = a$

4. Using the two previous questions show that ϕ admits one unique fixed point

- From Part 1 ϕ^n has unique fixed point $a \in E$
- From Part 3 if a is fixed point of $\phi \Rightarrow a$ is fixed point of ϕ^m
- Since fixed point ϕ^n is unique and ϕ admits a fixed point it must be unique

Exercise 2

1. Our objective is to solve the nonlinear equation $2xe^x = 1$

a) Check if the equation can be written as $x = \frac{1}{2}e^{-x}$

Just algebraic manipulation ? $2xe^x = 1 \iff 2x = e^{-x}$ as $\exp: \mathbb{R} \to \mathbb{R}_{>0}$ this is certainly possible

$\iff x = \frac{1}{2}e^{-x}$

b) Python Algorithm can be found in Jupyter File

c) Justify the convergence of the algorithm

Banach Fixed point theorem

1. $g(x)$ map interval to itself
2. $g(x)$ is contractive mapping

For this we analyze if $g(x) = \frac{1}{2}e^{-x}$ has a fixed point

① Show that $g(x)$ maps arbitrary interval to itself take arbitrary interval

Derivative $g'(x) = -\frac{1}{2}e^{-x} \Rightarrow |g'(x)| = \frac{1}{2}e^{-x}$ Since $e^{-x} > 0 \quad \forall x \in X \quad |g(x)| \le \frac{1}{2}$ for $x \ge 0$

if now $x < 0$, e^{-x} grows exponentially $|g'(x)| >> 1$

Choose arbitrary interval $[a,b]$ with $a \ge 0$ if $a = 0 \Rightarrow g(0) = \frac{1}{2}e^0 = \frac{1}{2}$

↳ minimal condition $a \le \frac{1}{2}e^{-b} \wedge \frac{1}{2}e^{-a} \le b$ then it always works

② Show that the mapping is contractive

Since $g'(x) = -\frac{1}{2}e^{-x}$ $\quad |g'(x)| = \frac{1}{2}e^{-x} \le \frac{1}{2} \Rightarrow$ Contractive Mapping with $L = \frac{1}{2}$ (Analysis II Mean Value Theorem)

$\Rightarrow$ the means $g(x)$ has a unique fixed point and the iteration $x_{n+1} = g(x_n) = \frac{1}{2}e^{-x_n}$ converges

2. Now we want to solve $x^2 - 2 = 0 \quad x > 0$

(a) Check that the previous equation can be written in fixed point form $x = f(x)$ and determine f

$x^2 - 2 = 0 \iff x^2 = 2$ Taking the root would give us $x = \sqrt{2}$
this is not a valid fixed point form as no
iteration can be performed

↳ $x = \frac{2}{x}$ so $f(x) = \frac{2}{x}$ if we set $x = f(x) \Rightarrow x = \frac{2}{x} \xrightarrow{x \neq 0} x^2 - 2 = 0$ ✓

b) Write a python code solving the fixed point problem and plot the 1st values x_1, x_2, x_3 by choosing $x_0 = 1$ then $x_0 = 2$ as an initial guess

c) In 2.a you might have chosen $f(x) = \frac{2}{x}$. Now repeat the question 2.a by considering $x = \frac{x^2+2}{2x}$ and conclude

We consider fixed point form: $x = f(x)$ where $f(x) = \frac{x^2+2}{2x}$

Set $x = \frac{x^2+2}{2x} \xrightarrow{x \neq 0} 2x^2 = x^2 + 2 \iff x^2 - 2 = 0$ which is our function

Check if mapping is contractive $\quad |f(x) - f(y)| \leq L|x-y|$

Take derivative $f(x) = \frac{x}{2} + \frac{1}{x} \longrightarrow f'(x) = \frac{1}{2} - \frac{1}{x^2}$

$|f'(x)| = \left|\frac{1}{2} - \frac{1}{x^2}\right| < \frac{1}{2}$ if $x > \sqrt{2}$ if $x < \sqrt{2}$ $|f'(x)|$ can be greater than 1

which means the function is contraction if $x > \sqrt{2}$ and also $\sqrt{2}$ is the fixed point

Python algorithm for iteration

d) For a fixed point x_n, expand (Taylor Series) the function $g(x) = x^2 - 2$ between x^n and $x^{(n+1)}$ replace the equation $g(\bar{x}) = 0$ by $g(x^{n+1}) = 0$ and $g(x^{(n+1)})$ by its Taylor expansion at $x^{(n+1)}$ and deduce the approximation $x^{(n+1)} = x^n - \frac{g(x^n)}{g'(x^n)}$

Taylor Series expansion $\quad \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Expanding $g(x)$ around x_n by Taylor series: $g(x) = g(x_n) + g'(x)(x - x_n) + \frac{g''(x_n)}{2}(x - x_n)^2$

Evaluate $g(x_{n+1}) \approx g(x_n) + g'(x_{n+1} - x_n) + \frac{g''(x_n)}{2}(x_{n+1} - x_n)^2$

Now replace $g(\bar{x}) = 0$ by $g(x^{n+1}) = 0 \implies 0 = g_n(x_n) + g'(x_n)\,(x_{n+1} - x_n) + \dots$ (Truncate derivatives)

So now we see: $g'(x_n)(x_{n+1} - x_n) = -g_n(x_n)$

$$\iff x_{n+1} - x_n = \frac{-g(x_n)}{g'(x_n)} \iff x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

Exercise 3

Consider the non-linear system $x^2 + 2xy = 0 \qquad xy + 1 = 0$

Calculate the theoretical solutions

$$xy + 1 = 0 \iff xy = -1 \overset{x \neq 0}{\Longrightarrow} y = \frac{-1}{x}$$

Substitute: $x^2 + 2x \cdot \left(\frac{-1}{x}\right) = 0 \iff x^2 - 2 = 0 \iff x = \pm\sqrt{2}$

By this we can solve for $y = \frac{-1}{x}$:

$x = +\sqrt{2} \quad y = \frac{-1}{\sqrt{2}}$

$x = -\sqrt{2} \quad y = \frac{+1}{\sqrt{2}}$

Final Solutions $\quad (x,y) = \left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right) \qquad (x,y) = \left(-\sqrt{2}, \frac{1}{\sqrt{2}}\right)$

Calculate the first iteration (x_1, y_1) obtained by Newtons method starting at $(x_0, y_0) = (1,-1)$

1. Initialize the System

$$F(x) = \left[x^2 + 2xy,\ xy + 1\right]^T$$

Jacobian Matrix
$$J(x) = \begin{bmatrix} \frac{\delta f_1}{\delta x} & \frac{\delta f_1}{\delta y} \\ \frac{\delta f_2}{\delta x} & \frac{\delta f_2}{\delta y} \end{bmatrix} = \begin{bmatrix} 2x + 2y & 2x \\ y & x \end{bmatrix}$$

Solve for $\Delta x^1 \Rightarrow$
$$\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 2 & -1 \end{array}\right] \xrightarrow[\mathrm{I}\cdot(-1)]{\mathrm{II}/2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ & 1 & 1/2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{array}\right]$$

$$x^1 = x^0 + \Delta x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$$

If we plug this back into our equation

$$F(x) = \left[x^2 + 2xy\ ,\ xy + 1\right]^T = \begin{bmatrix} 2{,}25 - 1{,}5 \\ -0{,}75 + 1 \end{bmatrix} = \begin{bmatrix} 0{,}75 \\ 0{,}25 \end{bmatrix}$$

Exercise 4

2. We consider the application $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y) = \left[x^2 - y, y^2\right]^T$

a) Determine the set of solution to $F(x,y) = (0,0)$ analytically

Consider the System $\begin{pmatrix} x^2 - y \\ y^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Step ① $y^2 = 0 \Rightarrow y = 0$

Step ② $x^2 - y = 0 \Rightarrow x^2 - 0 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

Therefore our final solution has the form $F(x,y) = (0,0)$ $(x,y) = (0,0)$

b) Show that Newton's algorithm is well defined for all couples (x_0, y_0) such that $x_0 \neq 0$ $y_0 > 0$

Newtons algorithm in two dimensions is given by $z_{n+1} = z_n - J_F^{-1}(z_n)\, F(z_n)$ $z_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

Consider the Jacobian-Matrix $J_F = \begin{pmatrix} \frac{\delta}{\delta x}(x^2 - y) & \frac{\delta}{\delta y}(x^2 - y) \\ \frac{\delta}{\delta x} y^2 & \frac{\delta}{\delta y}(y^2) \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ 0 & 2y \end{pmatrix}$

If Newton method is well defined $\Rightarrow z_n$ is defineable $\Rightarrow$ Inverse $J_F(x,y)$ exists

$\det(J_F) = 4xy \Rightarrow 4xy \neq 0 \Leftrightarrow x \neq 0 \wedge y \neq 0$

Now use induction let (x_0, y_0) be initial guess, $x_0 \neq 0$, $y_0 > 0$

we must show $\forall n$ $x_n \neq 0$ $y_n > 0$

Base Case $n=0$ $\quad (x_0, y_0)$ satisfies $x_0 \neq 0,\ y_0 > 0$

Inverse $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Inductive Step Given the statement holds for $z_n = (x_n, y_n)$

By using Newtons Algorithm $\quad z_{n+1} = z_n - J_F^{-1}(z_n) F(z_n)$

Inverse of $J_F(x_n, y_n) = \begin{pmatrix} 2x_n & -1 \\ 0 & 2y_n \end{pmatrix} \Rightarrow J_F(x_n, y_n)^{-1} = \frac{1}{4x_n y_n} \begin{pmatrix} 2y_n & +1 \\ 0 & 2x_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2x_n} & \frac{1}{4x_n y_n} \\ 0 & \frac{1}{2y_n} \end{pmatrix}$

Determine Iteration $\quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \underline{\begin{pmatrix} \frac{1}{2x_n} & \frac{1}{4x_n y_n} \\ 0 & \frac{1}{2y_n} \end{pmatrix} \begin{pmatrix} x_n^2 - y_n \\ y_n^2 \end{pmatrix}}$

$$* \quad \begin{pmatrix} \frac{1}{2x_n} & \frac{1}{4x_n y_n} \\ 0 & \frac{1}{2y_n} \end{pmatrix} \begin{pmatrix} x_n^2 - y_n \\ y_n^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2x_n}(x_n^2 - y_n) + \frac{1}{4x_n} y_n \\ \frac{y_n}{2} \end{pmatrix}$$

$$-\frac{2y_n}{4x_n} + \frac{y_n}{4x_n} = -\frac{y_n}{4x_n}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \frac{x_n}{2} - \frac{y_n}{2x_n} + \frac{y_n}{4x_n} \\ \frac{y_n}{2} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} \frac{x_n}{2} - \frac{y_n}{4x_n} \\ \frac{y_n}{2} \end{pmatrix}$$

Then we have equations $\quad x_{n+1} = \frac{x_n}{2} - \frac{y_n}{4x_n}$

$\underline{y_{n+1} = \frac{y_n}{2}} \Rightarrow y_{n+1} > 0$ because induction hypothesis gives us $y_n > 0$

For $\quad x_{n+1} = \frac{x_n}{2} - \frac{y_n}{4x_n} \overset{!}{=} 0 \Rightarrow \frac{x_n}{2} = \frac{y_n}{4x_n} \quad / \cdot 4x_n$ as $x_n \neq 0$ by hypothesis

$$\Rightarrow 2x_n^2 = y_n \Rightarrow x_n = \sqrt{\frac{y_n}{2}}$$

Since $y_n > 0$ this is well defined and $x_n \neq 0$

c) We chose $(x_0, y_0) = (1,1)$ and denote (x_k, y_k) $k \in \mathbb{N}$ the Newton Iterations

i) Express y_k in terms of k and y_0 and deduce that $(y_k)_k$ converge $\left(\text{Note } y_{k+1} = \frac{y_k}{2}\right)$

We are given $\quad y_{k+1} = \frac{y_k}{2}$

Recursive expansion $\quad y_1 = \frac{y_0}{2} \quad y_2 = \frac{y_1}{2} = \frac{y_0}{2^2} \quad y_3 = \frac{y_2}{2} = \frac{y_0}{2^3} \ \dots \Rightarrow \underline{y_k = \frac{y_0}{2^k}}$

Convergence of sequence $\quad y_k = \frac{y_0}{2^k} \xrightarrow{k \to \infty} 0$ and y_k converges to 0

ii) Show that $x_k \geq 2^{-k/2}$ $\forall k \in \mathbb{N}$ then deduce that $x_{k+1} \leq x_k$ $\forall k \in \mathbb{N}$

We now from proof above: $x_{n+1} = \frac{x_n}{2} - \frac{y_n}{4x_n} \Rightarrow x_{k+1} = \frac{x_k}{2} - \frac{y_k}{4x_k}$

Now we insert $y_k = \frac{1}{2^k}$ into the iteration $x_{k+1} = \frac{x_k}{2} - \frac{y_0}{4x_k 2^k} \overset{y_0=1}{=} \frac{x_k}{2} - \frac{1}{4x_k 2^k}$

Show $x_k \geq 2^{-k/2}$ by induction

$k = 0 = 1 \geq 2^{-0/2} = 1$ ✓

Assume that for $k \in \mathbb{N}$ $x_k \geq 2^{-k/2}$ and consider the inductive step (show) $x_{k+1} \geq 2^{-(k+1)/2}$

$x_{k+1} = \frac{x_k}{2} - \frac{y_k}{4x_k}$ For y_k we now $y_k = \frac{1}{2^k}$

$x_{k+1} = \frac{x_k}{2} - \frac{1}{2^k 4x_k} \xRightarrow[\text{Hypothesis}]{\text{Induction}} x_{k+1} \geq \frac{2^{-k/2}}{2} - \frac{1}{2^k 4 \cdot 2^{-k/2}}$

$x_{k+1} \geq 2^{-(k/2+1)} - \frac{1}{4 \cdot 2^{k/2}} \Rightarrow x_{k+1} \geq \frac{1}{4 \cdot 2^{k/2}} = 2^{-(k/2+2)} \geq 2^{-(k+1)/2}$

Now $2^{-(k/2+1)} = \frac{1}{2 \cdot 2^{k/2}}$

so the inductive step holds

Since we now now $x_{k+1} = \frac{x_k}{2} - \frac{y_k}{4x_k}$

And $y_k = \frac{1}{2^k} \geq 0$ and $x_k > 2^{-k/2} > 0$ we now $x_{k+1} \leq \frac{x_k}{2} \Rightarrow x_{k+1} \leq x_k$ $\forall k \in \mathbb{N}$

iii) Deduce that Newton Method converges to a solution of $F(x,y) = (0,0)$

From part (i) we now $y_k \to 0$ as $k \to \infty$

From part (ii) we now x_k is decreasing and bounded below by 0

So x_k also converges to a limit

Taking the limit in the formula $x_{k+1} = \frac{x_k}{2} - \frac{y_k}{4x_k} \xrightarrow{k\to\infty} x^* = \frac{x^*}{2} - 0$

$\Rightarrow x^* = \frac{x^*}{2} \Rightarrow x^* = 0$

and the solution converges to $(0,0)$