Proseminar Numerische Mathematik 2 am 27.May 2025

Exercise 1 (Application of Gerschgörin's theorem)

Let λ be eigenvalue of an $n \times n$ matrix A and u its associated eigenvector. Let k an index such that $|u_k| = \max_{1\leq j\leq n} |u_j|$.

1. (in the script!) Show that $|\lambda - a_{kk}| \leq \sum_{j=1,j\neq k}^{n} |a_{kj}|$.
2. Consider the matrix $A = \begin{pmatrix} 1+i & i & 2 \\ -3 & 2+i & 1 \\ 1 & i & 6 \end{pmatrix}$ Draw the corresponding Gerschgörin disks and localize the eigenvalues of A.
3. Deduce the plot of Gerschgörin disks of A^T the transpose matrix of A.
4. Using Gerschgörin's theorem give an upper bound of the eigenvalues of A.
5. Using Gerschgörin's theorem show that if A is diagonally strictly dominant, then A is invertible. (A is diagonally strictly dominant if $|a_{kk}| > \sum_{j=1,j\neq k}^{n} |a_{kj}|, \forall k, 1 \leq k \leq n$.)
6. Based on the previous question, is the matrix $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & 0 & 1 \end{pmatrix}$ invertible ?

Exercise 2

Let A be a symmetric positive definite matrix with n eigenvalues such that $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. The matrix A can then be decomposed as $A = P\boldsymbol{\lambda}P^T$ with $\boldsymbol{\lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and P is an orthogonal matrix. We suppose further that P admits an LU decomposition where the diagonal coefficients of U are strictly positive. Our aim is to prove $A_k \to \boldsymbol{\lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where the sequence A_k is constructed as

$$A_1 = A = Q_1R_1, \quad R_1Q_1 = A_2 = Q_2R_2, \ldots, R_kQ_k = A_k = Q_{k+1}R_{k+1}. \qquad (*)$$

1. (a) Let Q_i and R_i orthogonal matrices and upper triangular as defined in $(*)$. Show that $A^2 = \tilde{Q}_2\tilde{R}_2$ with $\tilde{Q}_2 = Q_1Q_2$ and $\tilde{R}_2 = R_2R_1$.
   (b) Show by induction over k that $A^k = \tilde{Q}_k\tilde{R}_k$, where $\tilde{Q}_k = Q_1Q_2\ldots Q_{k-1}Q_k$ and $\tilde{R}_k = R_kR_{k-1}\ldots R_2R_1$.
   (c) Justify (briefly) why $\tilde{Q}_k$ is an orthogonal matrix and $\tilde{R}_k$ is a matrix triangular which the diagonal elements are positive.
2. Let $M_k = \boldsymbol{\lambda}^kL\boldsymbol{\lambda}^{-k}$

(a) Prove that $PM_k = \tilde{Q}_k T_k$ where $T_k = \tilde{R}_k U^{-1} \lambda^{-k}$ is an upper triangular matrix which the diagonal coefficients are positive.

(b) Calculate the coefficients of M_k in terms of those of L and the eigenvalues of A.

(c) Deduce that M_k converges to the identity matrix Id and that $\tilde{Q}_k T_k$ converges to P when $k \to \infty$.

3. Let $(B_k)_{k\in\mathbb{N}}$ and $(C_k)_{k\in\mathbb{N}}$ two sequence matrices such that B_k are orthogonal and the matrices C_k upper triangular with positive diagonal elements. We want to show that if $B_k C_k \to B$ then $B_k \to B$ and $C_k \to Id$ when $k \to \infty$. Therefore, suppose that $B_k C_k \to B$. We denote $b_1, b_2, \ldots, b_n$ the columns of the matrix B and $b_1^{(k)}, b_2^{(k)}, \ldots, b_n^{(k)}$ the columns of the matrix B_k. We denote also $c_{i,j}^{(k)}$ the coefficients of C_k.
   (a) Show that the 1^{st} column of $B_k C_k$ is equal to $c_{1,1}^{(k)} b_1^{(k)}$. Deduce that $c_{1,1}^{(k)} \to 1$ and that $b_1^{(k)} \to b_1$.
   (b) Show that the 2^{nd} column of $B_k C_k$ is equal to $c_{1,2}^{(k)} b_1^{(k)} + c_{2,2}^{(k)} b_2^{(k)}$. Deduce that $c_{1,2}^{(k)} \to 0$ and then $c_{2,2}^{(k)} \to 1$ and that $b_2^{(k)} \to b_2$.
   (c) Show that when $k \to \infty$, we have $c_{i,j}^{(k)} \to 0$ if $i \neq j$, then $c_{i,i}^{(k)} \to 1$ and $b_i^{(k)} \to b_i$.
4. Deduce from questions 3 and 4 that $\tilde{Q}_k \to P$ and $T_k \to Id$ when $k \to \infty$.
5. Show that $\tilde{R}_k(\tilde{R}_{k-1})^{-1} = T_k \lambda T_{k-1}$. Deduce that R_k and A_k converge to λ.

Exercise 3

Consider the matrix $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & 0 \end{pmatrix}$ with $\theta \in [0, \pi]$. Our objective is to apply the QR factorization to A and analyze the effect of θ on the convergence t the eigenvalues of A.

1. Compute analytically the eigenvalues of A.
2. Perform theoretically the QR factorization of the matrix A and verify that $A = QR$, where Q is orthogonal and R is upper triangular.
3. Implement the previous question in Python (You can also use the Numpy's qr function). Iteratively compute $A_k = R_{k-1} Q_{k-1}$ starting from $A_0 = A$.
4. Observe the convergence of the sequence A_k to a diagonal matrix containing the eigenvalues of A. This should validate your algorithm if you recover the results from question 1.
5. Investigate how different values of $\theta = 0, \pi/4, \pi/2$ affect the convergence of the QR algorithm.
6. For which vales of θ the QR decomposition is unique?