(1) Filter on a finite set: Consider the set $X = \{0, 1, 2\}$.

  (a) Determine all filters on X.

  (b) Which of these filters are ultrafilters?

  (c) Which of these filters converge, if X is equipped with the discrete topology

(2) Convergence of filters and initial topology: Consider a family of topological spaces $(X_i)_{i\in I}$, a set X and maps $f_i \colon X \to X_i$. We define on X the initial topology with respect to these maps and consider a filter $\mathcal{F}$ on X. Prove that $\mathcal{F}$ converges to some $x \in X$ if and only if for every $i \in I$ the filter $f_i(\mathcal{F})$ converges to $f_i(x)$.

(3) Continuity and sequential continuity: Consider $\mathbb{R}$ equipped with the cocountable topology $\mathcal{T}_{\text{cco}} = \{A \subset \mathbb{R} \colon \mathbb{R} \setminus A \text{ countable}\} \cup \{\varnothing\}$.

  (a) Show that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in \mathbb{R}$ if and only if there is an $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

  (b) Show that for every nonempty subset $A \subset \mathbb{R}$ and every convergent sequence $x_n \to x$ with $x_n \to x$ it follows that $x \in A$. Give an example of a set $A \subset \mathbb{R}$ for which $\overline{A}$ does not only consist of limits of sequences in A.

  (c) Show that $(\mathbb{R}, \mathcal{T}_{\text{cco}})$ is not a Hausdorff space, but limits of convergent sequences are unique.

  (d) Let $\mathcal{T}_{\mathbb{R}}$ denote the natural topology on $\mathbb{R}$. Show that the identity map $(\mathbb{R}, \mathcal{T}_{\text{cco}}) \to (\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ is sequentially continuous but not continuous.

(4) Compact intervals are compact: Let $a, b \in \mathbb{R}$, $a < b$. Show that the interval $[a, b] \subset \mathbb{R}$ is compact, i.e., that every open cover $(O_i)_{i\in I}$ admits a finite subcover.
Hint: Consider the set $A := \{x \in [a, b] : [a, x] \text{ is covered by finitely many } O_i\}$ and show that $b = \sup A$.

(5) Uniqueness of the product topology: Let $(X_i)_{i\in I}$ be compact Hausdorff spaces. Show that the product topology on $\prod_{i\in I} X_i$ is the unique topology with the following properties:

  (a) The projections $\pi_j \colon \prod_{i\in I} X_i \to X_j$ are continuous for every $j \in I$.

  (b) $\prod_{i\in I} X_i$ is compact.