Proseminar Numerische Mathematik 2 am 11. März 2025

Exercise 1

Let $(E, \|\cdot\|)$ be a Banach space (a complete normed vector space, with norm denoted by $\|\cdot\|$). Consider a mapping $\phi : E \mapsto E$ that is contractive:

$$\exists L < 1, \text{ such that, } \forall (x, y) \in E^2, \|\phi(x) - \phi(y)\| \leq L\|x - y\|.$$

Starting from a given $x_0 \in E$, we construct the recursive sequence $(x_k)_{k\in\mathbb{N}}$ defined by

$$x_{k+1} = \phi(x_k)\ \forall k \in \mathbb{N}. \tag{1}$$

- Show that the sequence $(x_k)_{k\in\mathbb{N}}$ is well-defined.
- Show that the function ϕ is continuous. Deduce that if the sequence x_k converges, then it converges to a fixed point of ϕ.
- Show that if ϕ admits a fixed point, then this fixed point is unique.
- Show that the sequence $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. Deduce that ϕ admits a unique fixed point, which is given by the limit of the sequence $(x_k)_{k\in\mathbb{N}}$ as k tends to $+\infty$.
- Application: Let $\phi : E \mapsto E$ be a continuous mapping such that $\phi^m = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{m \text{ times}}$ $(m \in \mathbb{N}^*)$ is contractive:
  1. Show that ϕ^m has a unique fixed point $a \in E$.
  2. Show that $\phi(\phi^m(a)) = \phi(a)$. Deduce that ϕ admits a fixed point.
  3. Show that if a is a fixed point of ϕ, then a is a fixed point of ϕ^m.
  4. Using the two previous questions, show that ϕ admits a unique fixed point.

Exercise 2

1. Our objective is to solve the nonlinear equation $2xe^x = 1$.
   (a) Check if the equation above can be rewritten as $x = \frac{1}{2}e^{-x}$.
   (b) Write in Python an algorithm which calculates the 1^{st} three values x_0, x_1, x_2 and x_3 starting with $x_0 = 1$.
   (c) Justify the convergence of your algorithm.
2. Now we want to solve $x^2 - 2 = 0, \quad x > 0$.

(a) Check that the previous equation can be rewritten in the fix point form $x = f(x)$ and determine f.

(b) Write a Python program solving the fix point problem and plot the 1^{st} values x_1, x_2, x_3 choosing $x_0 = 1$ then $x_0 = 2$ as initial guess.

(c) In question 2.a you might have chosen $f(x) = \frac{2}{x}$. Now repeat the question 2.a by considering $x = \frac{x^2+2}{2x}$ and conclude.

(d) For a fixed known iterate x_n, expand (Taylor series) the function $g(x) = x^2 - 2$ between $x^{(n)}$ and $x^{(n+1)}$, replace the equation $g(\bar{x}) = 0$ by $g(x^{n+1}) = 0$ and $g(x^{(n+1)})$ by its Taylor expansion at $x^{(n+1)}$ and deduce the approximation $x^{(n+1)} = x^{(n)} - \frac{g(x^{(n)})}{g'(x^{(n)})}$.

Exercise 3

We consider the nonlinear system

$$x^2 + 2xy = 0, \quad xy + 1 = 0$$

1. Calculate theoretically the solution of the system above.
2. Solve in Python the system using Newton's method and give the conditions under which the Newton sequence is well defined.
3. Calculate the 1^{st} iteration (x_1, y_1) obtained in Newton's method starting at $(x_0, y_0) = (1, -1)$.

Exercise 4

1. 1. We consider the application $f : \mathbb{R} \to \mathbb{R}, \quad x \longmapsto x^2$
   (a) Write a Newton's algorithm to find the zeros of f and prove its convergence $\forall x \in \mathbb{R}$
   (b) Implement this in Python and recover the convergence numerically.
2. We consider now the application $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x, y) = [x^2 - y, y^2]^T$
   (a) Determine the set of the solution to $F(x, y) = (0, 0)$ analytically.
   (b) Show that Newton's algorithm is well defined for all couples (x_0, y_0) such that $x_0 \neq 0$ and $y_0 > 0$.
   (c) We choose $(x_0, y_0) = (1, 1)$ and we denote $(x_k, y_k), \quad k \in \mathbb{N}$ the Newton's iterations
      i. Express y_k in terms of k and y_0 and deduce that the sequence $(y_k)_k$ converge (Note that $y_{k+1} = \frac{y_k}{2}$).
      ii. Show that $x_k \geq 2^{-k/2}, \quad \forall k \in \mathbb{N}$ and deduce that $x_{k+1} \leq x_k, \quad \forall k \in \mathbb{N}$.
      iii. Deduce that Newton's method converge to a solution of $F(x, y) = (0, 0)$.

   Hint: For question 2.b use proof by induction.