Random Walks on Cayley Graphs

Lukas Nießen

Geboren am 07. Juli 2001 in Bad Honnef

December 17, 2024

Bachelorarbeit Mathematik

Betreuer: Prof. Dr. Eveliina Peltola

Zweitgutachter: Dr. Alexis Langlois-Rémillard

Institute for Applied Mathematics

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Deutsche Zusammenfassung

Diese Arbeit befasst sich mit Irrfahrten auf Cayley-Graphen, insbesondere mit der Frage, wann eine solche Irrfahrt transient und wann rekurrent ist. Sie basiert auf dem Buch [LP16], welches Wahrscheinlichkeitstheorie auf Bäumen und Netzwerken untersucht. Der Hauptsatz dieser Arbeit besagt, dass es für Random Walks auf Cayley-Graphen, die einem natürlichen "Abnehmephänomen" genügen, einen sogenannten *kritischen Wert* gibt. Wenn die "Abnahme" schneller erfolgt als durch diesen Wert vorgegeben, liegt Rekurrenz vor, ist die „Abnahme" größer, ergibt sich Transienz.

Der Beweis dieses Theorems stützt sich hauptsächlich auf Methoden der elektrischen Netzwerktheorie, aber auch auf Grundlagen der diskreten Mathematik, der Stochastik und der Gruppentheorie. Im ersten Kapitel werden grundlegende Definitionen und Begriffe eingeführt, während Kapitel 2 einen kurzen Überblick über harmonische Funktionen gibt. Kapitel 3 behandelt die elektrische Netzwerktheorie, stellt wichtige Theoreme vor und enthält eine kleine Anwendung: den Beweis von Pólyas Theorem. Das vierte Kapitel widmet sich dem Beweis des Hauptsatzes und betrachtet anschließend, als Kontrast, Irrfahrten mit einem weniger natürlichen "Zunahmephänomen".

Contents

Introduction **3**

Notes **4**

Acknowledgments **5**

1 Definitions and Terminology **6**
1.1 Graphs 6
1.2 Random Walks and Markov Chains 7

2 Harmonic Functions **8**

3 Electrical Networks **10**
3.1 Bijection between Markov Chains and Random Walks 11
3.2 Hilbert Spaces and Operators 12
3.3 Basic Concepts 14
3.4 Hitting Function 15
3.5 Flows 17
3.6 Ohm's and Kirchhoff's Laws 19
3.7 Effective Conductance 21
3.8 Energy 22
3.9 Infinite Networks 24
3.10 Thomson's Principle and Monotonicity 25
3.11 Green Function 30
3.12 Recurrence and Transience 30

4 Random Walks on Cayley Graphs **36**
4.1 Admissible Flows 37
4.2 Decaying Conductances and Branching Number 37
4.3 Cayley Graphs 40
4.4 Subperiodic Subtree of a Cayley Graph 40
4.5 Growth Rate and Random Walk 43

References **47**

Introduction

Random walks are processes that describe objects randomly moving away from where they started. They have applications in diverse areas, such as particle diffusion in physics and modeling stock price fluctuations in finance. Mathematically, a random walk can be modeled as a sequence of steps on a predefined structure, such as the real line, a lattice, or more generally, a graph. In the context of graph theory, this corresponds to randomly moving from one node to another, following the edges of the graph. A common research question is whether a random walk is *recurrent*, meaning it returns to the starting point infinitely often, or *transient*, meaning it eventually escapes and does not return to the starting point.

One famous result in this field is Pólya's theorem, which states that the simple random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ and transient otherwise. In his paper from 1921 [Pól21], he proved the theorem using elementary methods of probability theory and analysis. We will present a very short and elegant partial proof in Chapter 3, primarily using electrical network theory, which is a branch of mathematics inspired by physical principles; see [DS84]. This thesis is based on the book [LP16] and focuses on a certain type of random walk where the probability of moving farther from the starting point decreases exponentially with the distance. A fundamental result in this context is the existence of a *critical value*: if the probability of stepping farther decays faster than the critical value "allows", the walk is recurrent; otherwise, it is transient.

Our interest lies in determining this critical value for random walks on Cayley graphs, which are graphs associated with groups and their generators. It turns out that this critical value is deeply connected to "how fast the graph grows". The main theorem of this thesis establishes that, for Cayley graphs with finite generating sets, the critical value is exactly the exponential growth rate of the graph. Here is a simplified version of the main theorem.

Theorem 0.1. *Let M_n be the number of nodes with distance n to the root. Then the "exponential-decay random walk" described above on an infinite Cayley graph has the following critical value:*

$$\lim_{n\to\infty} M_n^{1/n}.$$

In other words, the critical value is precisely the exponential growth rate of the graph.

To prove this, we will analyze a subtree of the graph. Our theory primarily involves electrical network theory, focusing on concepts such as finite energy flows and admissible flows. However, we also use results from probability theory, group theory, and discrete mathematics. The main theorem is particularly interesting because it builds a bridge between group structure and probability theory.

We begin with definitions and terminology in Chapter 1. Then, in Chapter 2, we provide an overview of discrete harmonic functions, which play a key role in this thesis because many of the functions we will use, such as the *voltage function*, are harmonic. Chapter 3 is devoted to electrical network theory, presenting crucial results like Thomson's Principle, Rayleigh's Monotonicity Principle, the Nash–Williams Inequality, and the Nash–Williams Criterion. In that chapter, we also establish the connection between recurrence/transience and finite energy flows which is due to T. Lyons (1983) in [Lyo83]. Finally, in Chapter 4, we prove the main theorem and also look at the "opposite case" of random walks where the probability of moving farther from the starting point grows exponentially with the distance.

Notes

Contribution

No content in this thesis has been copied one-to-one from references. All material has been reworded to reflect my own style. This work is based on the book [LP16], which means that we follow the basic structure of the book leading to the main theorem and its proof. However, the book contains 16 chapters, of which we only use the first three. As a result, I have adapted many definitions to our field of application, I have restructured many things, e.g. instead of defining the term "flow" three times in different contexts in different places, I have put all the basic definitions at the beginning and adapted them. Another example is the chapter Harmonic Functions, which does not exist as a separate chapter in the book, but is presented in the middle of other theory. I have also simplified or even omitted many things that were not needed. Many statements in the book were labelled as exercises and did not contain proofs or solutions. In other cases, I chose to use my own proof instead of a given one. For this reason, many of the proofs in this work are entirely my own. I have also added comments, illustrations or other additions from other sources, while most illustrations were created by myself. At the end we consider the "opposite case" of "decaying-random walks". This part is not taken from or inspired by any book or paper, but is entirely my own idea and contribution.

For clarity, some mathematical elements such as lemmas, corollaries, theorems, remarks, proofs or figures are annotated as follows:

- $\circledast$: indicates that it is entirely my own creation and has not been taken from a book or paper in any way. For example, a proof labeled with $\circledast$ is a proof that I came up with myself, a figure with this label was created by me, and a lemma with this label is a lemma whose statement is my own creation.

- $\ominus$: indicates that the material has been taken from a reference but contains significant changes, such as a different proof technique, additional details or alternative examples.

Code

I have created a GitHub repository at www.github.com/LukasNiessen/random-walks-on-cayley-graphs ($\circledast$) which includes Python scripts that execute different random walks and visualise them — from simple random walk on the integer lattices to random walks on Cayley graphs.

Acknowledgments

I owe special thanks to several people who made this thesis possible. Professor Eveliina Peltola and Doctor Alexis Langlois-Rémillard have shown great support and have given amazing feedback throughout the past half-year of writing this thesis. I want to thank both of you very much for everything.

To my parents, dear Loretta and Wolfgang, your relentless drive and iron will have been the foundation on which I have built my life, and for that, I am and will remain forever grateful.

1 Definitions and Terminology

In this chapter we will give a series of definitions and terminology, largely related to graphs, random walks and Markov chains.

1.1 Graphs

A *directed graph* is a pair $G = (V, E)$, where V is a set of vertices and E is an irreflexive subset of $V \times V$, the edge set. Irreflexive means $(x, x) \notin E$ for all vertices x. Since we will only look at directed graphs we will just say *graph*. If we look at more than one graph, we distinguish the vertex and edge sets by using the following notation: $V(G)$ and $E(G)$. A *subgraph* of G is a graph whose vertex set is a subset of $V(G)$ and whose edge set is a subset of $E(G)$. For an edge $e = (x, y)$ we call x and y the *endpoints*. We call $e^- := x$ the *tail* and $e^+ := y$ the *head*. For an edge $e = (x, y)$ we write $-e$ for the "inversely directed" edge (y, x).

An edge e such that $e^- = e^+$ is called a *loop*. Edges e_1, e_2 with the same tails and heads, that is $e_1^- = e_2^-$ and $e_1^+ = e_2^+$, are called *parallel*. Vertices x and y are *adjacent* or *neighbors* if $(x, y) \in E$, denoted $x \sim y$. Two edges are adjacent if they share an endpoint. A vertex x is *incident* to an edge e if it is one of the edge's endpoints, denoted $x \sim e$. The *degree* of a vertex v is the number of its neighbors:

$$d(v) := \#\{e \in E :\ e^- = v\}.$$

A graph is *locally finite* if each vertex has finite degree. A sequence of vertices where each consecutive pair is connected by an edge is called a *path*. We say *it joins its first and last vertices*. A path is *vertex simple* if no vertex is repeated. Similarly, *edge simple* means no edge is repeated. If we say just *simple*, we mean simple vertex, which implies simple edge. A graph is *connected* if for any two vertices $x \neq y$, there exists a path between them. The minimum number of edges in a path between two vertices is *distance* and we it denote it by $d(x, y)$. Similarly, we define the *distance* between of an edge e and a vertex x as the minimum number of edges in a path that starts at x and whose last edge is e.

A *cycle* is a path that starts and ends at the same vertex. A cycle is called *simple* if the only repeated vertices are the first and last ones. A *forest* is a graph without cycles. A *tree* is a connected forest. In a tree, the *parent* of a vertex x is the neighbor y of x that has smallest distance to the root. A *network* is a graph where edges are assigned weights $c(e)$ where c is a function $c : E \to \mathbb{R}$. Given a network $G = (V, E)$ with edge weights $c(\cdot)$ and a subset K of its vertices, the *induced subnetwork* $G_{|K}$ consists of the vertex set K, the edge set $(K \times K) \cap E$, and weights c restricted to the edges in $(K \times K) \cap E$. We call weights *symmetric* if $c(x, y) = c(y, x)$ for all adjacent vertices x, y.

A *multigraph* is a generalization of a graph that allows loops and multiple edges between vertices. Given a subset $K \subseteq V$, the multigraph G/K, formed by *identifying* K into a single vertex $z \notin V$, has vertex set $(V \setminus K) \cup \{z\}$ and edge set modified from E by adjusting the endpoints that were previously in K so that they now have the value z.

The *contraction* of an edge e is obtained by first deleting e and then merging its endpoints. We denote the contraction by G/e.

We say a graph, network, or tree is finite if both V and E are, otherwise it is infinite. We say it is countable if both E and V are countable.

1.2 Random Walks and Markov Chains

Definition 1.1. A *Markov chain* is a stochastic process $\{X_n\}_{n\geq 0}$ taking values in a state space S such that the probability of transitioning from a current state to the next state depends only on the present state and not on the past history. Formally, for all $n \geq 0$ and for any states $x_0, x_1, \ldots, x_{n+1} \in S$,

$$P(X_{n+1} = x_{n+1} \mid X_n = x_n, \ldots, X_0 = x_0) = P(X_{n+1} = x_{n+1} \mid X_n = x_n).$$

Unless defined differently, we will always write S for the state space of a given Markov chain throughout this thesis.

Definition 1.2. Given a graph $G = (V, E)$, a *random walk* is a time-homogeneous Markov chain $\{X_n\}_{n\geq 0}$ on V. At each time step n, the process moves from the current vertex to a neighboring vertex chosen according to a specified probability distribution. If all neighboring vertices are chosen with equal probability, the random walk is said to be *simple.*

Example 1.1. One example of a simple random walk is *the simple random walk on* $\mathbb{Z}^d$. This is a random walk $\{X_n\}_{n\geq 0}$ on the integer lattice $\mathbb{Z}^d$, where the state space is the set of all points in $\mathbb{Z}^d$, and the process starts at some initial point $X_0 = x \in \mathbb{Z}^d$. At each step, the walk moves to one of the $2d$ nearest neighbors of the current position, each with probability $1/(2d)$:

$$P(X_{n+1} = y \mid X_n = x) = \begin{cases} \frac{1}{2d} & \text{if } y \text{ is a nearest neighbor of } x, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.3. *Given an irreducible Markov chain, we have either that all states are* recurrent*, meaning that every state is visited infinitely often with probability* 1*, or all states are* transient*, meaning that each state is visited only finitely many times with probability* 1*.*

A proof and more about the topic of Markov chains can be found in [Nor98].

Definition 1.4. A Markov chain is called *reversible* if there exists a function $\pi : S \to \mathbb{R}_{\geq 0}$, called *stationary distribution*, on the state space S such that:

$$\pi(x)p(x,y) = \pi(y)p(y,x) \quad \text{for all } x, y \in S.$$

Remark 1.5. In this thesis, when we use sums or products over potentially infinite sets, such as V or E, we mean

$$\lim_{n\to\infty} \sum_{i=1}^{n} (\text{n-th term}) \quad \text{or} \quad \lim_{n\to\infty} \prod_{i=1}^{n} (\text{n-th term})$$

respectively. We can say "n-th term" because in these cases we will assume always countability. This does not depend on the term order because we will always implicitly assume absolute convergence.

2 Harmonic Functions

Some of the most important functions in this thesis are discrete harmonic functions. One example is the voltage function which is harmonic on all states except the sources and sinks. For this reason, we study this type of functions in this chapter, which is primarily based on chapter 2 of [LP16]. Throughout the chapter, S denotes the state space of a Markov chain and we assume that S is countable.

Definition 2.1. For $x \in S$ we call a function f *harmonic* at x if

$$f(x) = \sum_{y \in S,\, x \sim y} p(x,y) f(y).$$

If f is harmonic at each point of some subset $W \subset S$, we say that f is harmonic on W.

Remark 2.2. ($\circleddash$) One might call the defining equality in the above definition an "averaging property". So to say, through the lens of f, the state equals the weighted sum of all adjacent states, where the weights are based on a probability measure; in particular, the weights always sum to 1.

Lemma 2.3 (Linearity of Harmonic Functions)**.** *($\circledast$) Let f, g be harmonic functions and $a \in \mathbb{R}$. Then $a \cdot f$ and $f + g$ are harmonic too.*

Proof. We have $f(x) = \sum_{y \in S} p(x,y) f(y)$ for all $x \in W$. Thus

$$(a \cdot f)(x) = a \cdot \sum_{y \in S} p(x,y) f(y) = \sum_{y \in S} p(x,y)(a \cdot f(y)),$$

showing that $a \cdot f$ is harmonic. For harmonic functions f and g, we have

$$(f+g)(x) = \sum_{y \in S} p(x,y) f(y) + \sum_{y \in S} p(x,y) g(y) = \sum_{y \in S} p(x,y)(f(y) + g(y)),$$

showing that their sum is harmonic. $\square$

Definition 2.4. Let $W \subset S$ be a subset. The *boundary* of a function that is harmonic on W is defined as $V \setminus W$. We write W^+ for the set that contains every state $w \in W$, and every state $s \in S \setminus W$ where there exists some reachable state $w' \in W$, that is, we have $p(s, w') > 0$.

Proposition 2.5 (Maximum Principle). *Let $W \subset S$ be a subset and let $f : S \to \mathbb{R}$ be a function that is harmonic on W. If f achieves its supremum at some $x_0 \in W$, then f is constant on all states in W^+ that can be reached from x_0.*

Proof. ($\ominus$) Let $y \in W^+$ be reachable from x_0, that is, $p(x_0, y) > 0$. We want to show that $f(y) = f(x_0)$. Since f is harmonic at x_0, we have

$$f(x_0) = \sum_{x_0 \sim z} p(x_0, z) f(z).$$

By assumption we have $f(z) \leq f(x_0)$ for every vertex $z \sim x$. But then, as long as $p(x_0, z) > 0$, we must have $f(z) = f(x_0)$ because otherwise the sum on the right-hand side would be strictly smaller than $f(x_0)$. Because y is among these z and $p(x_0, y) > 0$, the conclusion follows. □

Proposition 2.6. *($\circledast$) Let $W \subset S$ be a subset and let f be a function that is harmonic on W. Suppose that $S \setminus W$ is accessible from every state in W, that is, for every state $x \in W$ there exists a state $y \in S \setminus W$ such that $p(x, y) > 0$. Then the supremum of f is achieved on the boundary.*

Proof. Let the supremum be achieved at some $x_0 \in S$. If $x_0 \notin W$, we have nothing to show. So assume that $x_0 \in W$. By the Maximum Principle (Lemma 2.5), we know that f is constant on all states that are reachable from x_0. By assumption, there exists a state $y \in S \setminus V$ that is reachable from x_0. Thus we have $f(y) = f(x_0) = \sup f$, so the supremum is achieved at the boundary, concretely on y. □

Proposition 2.7 (Uniqueness Principle). *Let S be a finite set, let W be a proper subset, and let $S \setminus W$ be accessible from every state in W. If we have two functions $f, g : S \to \mathbb{R}$ that are harmonic on W and agree on $S \setminus W$, then $f = g$.*

Proof. Set $h \coloneqq f - g$. We have h is harmonic on W by Lemma 2.3, and $h_{|S \setminus W} = 0$ since f and g agree on $S \setminus W$. Let $x_0 \in S$ be a state where h achieves its supremum.

Suppose $x_0 \in W$. By assumption, we have that $S \setminus W$ is reachable from x_0. By Proposition 2.6, we obtain that $h(x_0) \leq h_{|S \setminus W}$. Therefore we have

$$h_{|W} \leq h(x_0) \leq h_{|S \setminus W} = 0, \quad \text{in other words,} \quad h \leq 0.$$

If, on the other hand, we have that $x_0 \notin W$, so $x_0 \in S \setminus W$, we have $h \leq \sup h = h(x_0) = 0$ since $h_{|S \setminus W} = 0$. So we also get that $h \leq 0$.

By symmetry we obtain $h \geq 0$, and thus $h = 0$. □

Proposition 2.8 (Superposition Principle)**.** *Let $W \subsetneq S$ be a finite subset and let $a, b \in \mathbb{R}$. Suppose $S \setminus W$ is accessible from every state in W. If f, g and h are harmonic on W and $f = a \cdot g + b \cdot h$ on $S \setminus W$, then $f = a \cdot g + b \cdot h$ everywhere.*

Proof. We apply Linearity of Harmonic Functions (Lemma 2.3) and the Uniqueness Principle (Proposition 2.7) and get the claim. □

Proposition 2.9 (Existence Principle)**.** *Let $W \subsetneq S$ be a proper subset. If $f_0 : S \setminus W \to \mathbb{R}$ is bounded, then there exists a function $f : S \to \mathbb{R}$ that is harmonic on W and agrees with f_0 on $S \setminus W$, that is, $f_{|S \setminus W} = f_0$.*

Proof. (⊝) For any starting point x of the Markov chain, let a be the first vertex in $S \setminus W$ visited by the Markov chain if $S \setminus W$ is visited. Let $b \coloneqq f_0(a)$ if $S \setminus W$ is visited and $b \coloneqq 0$ otherwise. Now set $f(x) \coloneqq \mathbb{E}_x[b]$. We see that this works using *first step analysis*:

$$\begin{aligned} f(x) &= \sum_{y \in S} P_x[\text{first step is to } y] \cdot \mathbb{E}_x[b \mid \text{first step is to } y] \\ &= \sum_{y \in S} p(x, y) \mathbb{E}_y[b] = \sum_{y \in S} p(x, y) f(y). \end{aligned}$$

□

Remark 2.10. In the Existence Principle (Proposition 2.9) we can have $f \geq 0$ by choosing an $f_0 \geq 0$, indeed, then we have $Y \geq 0$, so $f(x) = \mathbb{E}_x[Y] \geq 0$.

Corollary 2.11. *(⊝) Let S be a finite set and let $f : S \to \mathbb{R}$ be a function that is harmonic on S. If every state is accessible from every other state, then we have that f is constant on every connected component of S.*

Proof. We pretend that S is connected and show that f is constant on S, which implies the general statement. Pick an arbitrary $s \in S$ and set $W \coloneqq S \setminus \{s\}$. We have that $S \setminus W = \{s\}$ is reachable from W by assumption. Recall that f is harmonic on S. Now define $g(x) \coloneqq f(s)$ for all $x \in S$, making g a constant function. We have that f and g agree off W and both are harmonic on W, thus, applying the Uniqueness Principle (Proposition 2.7) gives us $f = g$ everywhere, meaning that f is constant. □

3 Electrical Networks

This chapter examines electrical network theory and is largely based on chapter 2 of [LP16]. We begin by establishing a bijective correspondence between certain Markov chains and random walks, then introduce key definitions and fundamental concepts. While initially we focus on finite networks only, we later extend the theory to infinite networks by *exhausting them* through finite networks. We then prove key results like Thomson's Principle, Rayleigh's Monotonicity Principle, and the Nash–Williams Criterion, and conclude with a partial proof of Pólya's Theorem.

3.1 Bijection between Markov Chains and Random Walks

There is a natural bijection between reversible Markov chains on a countable state set and random walks on countable connected networks that have non-negative symmetric weights and where the sum of the weights incident to every vertex is finite. We use it constantly throughout this thesis.

Definition 3.1. ($\ominus$) Given a reversible Markov chain on a countable state set, one can construct a network G, which might have loops, using the following steps:

1. Take the states of the Markov chain as the vertices of G.

2. Join two vertices x and y by an edge if $p(x, y) > 0$, where $p(x, y)$ is the transition probability from x to y.

3. Assign a weight to each edge as follows:

$$c(x, y) := \pi(x)p(x, y),$$

   where $\pi(x)$ is the stationary distribution of the Markov chain. Note that the stationary distribution exists and is unique, see [LPW09, Chapter 1]. Since the Markov chain is reversible, this definition gives us symmetry:

$$c(x, y) = c(y, x).$$

Note that the edge weights are non-negative and the sum of the weights incident to each vertex is finite. The former is because $\pi \geq 0$ and $p \geq 0$, the latter is because for any fixed vertex x we have

$$\sum_{y\sim x} c(x, y) = \sum_{y\sim x} \pi(x)p(x, y) = \pi(x) \sum_{y\sim x} p(x, y) = \pi(x) < \infty.$$

With this weighted graph, the Markov chain can be interpreted as a random walk on G. At a vertex x, the random walk transitions to a neighboring vertex y with probability proportional to the weight of the edge between x and y. Concretely, the probability to go from x to y is

$$\frac{c(x, y)}{\sum_{x\sim z} c(x, z)} = \frac{\pi(x)p(x, y)}{\sum_{x\sim z} \pi(x)p(x, z)} = \frac{p(x, y)}{\sum_{x\sim z} p(x, z)} = \frac{p(x, y)}{1} = p(x, y).$$

Note that we do not divide by 0 here because $c(x, y)$ or $p(x, y)$ is in the denominator sum and is positive.

Conversely, suppose that we are given a countable connected network with non-negative symmetric edge weights such that the sum of the weights incident to each vertex is finite. This, like above, induces a random walk with transition probabilities proportional to the edge weights. This random walk corresponds to a Markov chain, and it is reversible by setting $\pi(x) := \sum_{x \sim y} c(x, y)$, as we easily check:

$$\pi(x)p(x, y) = \sum_{x \sim z} c(x, z) \cdot \frac{c(x, y)}{\sum_{x \sim z} c(x, z)} = c(x, y),$$

$$\pi(y)p(y, x) = \sum_{y \sim z} c(y, z) \cdot \frac{c(y, x)}{\sum_{y \sim z} c(y, z)} = c(y, x) = c(x, y).$$

Here again we did not divide by 0. Note that $\pi(x)$ is well defined as we assumed that the sum of the weights incident to each vertex is finite.

Remark 3.2. If we have an unweighted graph, we set the weights to be 1 on every edge. If all edge weights are equal, for example to 1, the resulting random walk is simple.

Remark 3.3. Unless defined differently, we will write V for the vertex set of a network that corresponds to a random walk.

Remark 3.4. If we have $x \sim y$ for two vertices x, y, then $p(x, y) > 0$ and thus $c(x, y) > 0$. Also note that $\pi(x) > 0$ always because we assume connectedness of the graph or, respectively, reversibilty of the Markov chain, so that there exists a positive summand $c(x, y)$ in $\pi(x) = \sum_{x \sim y} c(x, y)$.

3.2 Hilbert Spaces and Operators

In this chapter, we introduce Hilbert spaces and operators that are essential for definitions and concepts developed later in this thesis. For this chapter, we assume or define $G = (V, E)$ to be a network and, unless defined differently, we assume or define G to be countable, meaning that both V and E are countable.

By $\ell^2(V)$ we denote the space of square-summable functions on the vertices:

$$\ell^2(V) := \left\{ f : V \to \mathbb{R} : \sum_{x \in V} f(x)^2 < \infty \right\}.$$

Note that the square-summability condition is always satisfied if V is finite. We define an inner product $(f, g)_V := \sum_{x \in V} f(x)g(x)$ on $\ell^2(V)$.

Remark 3.5. (⊛) The inner product is well defined because

$$|f(x)g(x)| \leq \frac{1}{2}(f(x)^2 + g(x)^2).$$

By $\ell^2_-(E)$ we denote the space of antisymmetric functions on the edges:

$$\ell^2_-(E) := \left\{ \theta : E \to \mathbb{R} : \ \theta(-e) = -\theta(e) \text{ for all } e \in E, \text{ and } \sum_{e \in E} \theta(e)^2 < \infty \right\}.$$

Note that the summability condition is always satisfied if E is finite. We equip this space with the inner product $(\theta, \theta')_E := \sum_{e \in E_{1/2}} \theta(e)\theta'(e)$, where $E_{1/2} \subseteq E$ is a set of edges containing exactly one of each pair e and $-e$ for an edge e.

Remark 3.6. (⊛) Note that the above definition does not depend on the choice of $E_{1/2}$ because we assume antisymmetry for θ and θ'. Concretely, choosing $-e$ instead of e for a certain edge would change only one summand as $\theta(-e)\theta'(-e) = (-1)\theta(e) \cdot (-1)\theta'(e) = \theta(e)\theta'(e)$. So the inner product is well defined.

Remark 3.7. (⊛) The inner product is well defined by the same argument described in Remark 3.5.

Moreover, given a function $h : E \to \mathbb{R}$, we define the inner product $(\theta, \theta')_h := (\theta h, \theta')_E = (\theta, \theta' h)_E$ on $\ell^2_-(E)$, for $\theta, \theta' \in \ell^2$. We define the norm $\|f\|_h := \sqrt{(f, f)_h}$.

When we use the inner product $(\cdot, \cdot)_h$ we write $\ell^2_-(E, h)$ and adapt the definition as follows:

$$\ell^2_-(E, h) := \left\{ \theta : E \to \mathbb{R} : \ \theta(-e) = -\theta(e) \, \forall e \in E \text{ and } \sum_{e \in E} \theta(e)^2 h(e) < \infty \right\}.$$

We define the *coboundary operator* $d : \ell^2(V) \to \ell^2_-(E)$ as

$$(df)(e) := f(e^-) - f(e^+)$$

for $f \in \ell^2(V)$ and $e \in E$. If $\sum_{e^- = x} |\theta(e)| < \infty$, we define the *boundary operator* $d^* : \ell^2_-(E) \to \ell^2(V)$ as follows:

$$(d^*\theta)(x) := \sum_{e^- = x} \theta(e),$$

for $\theta \in \ell^2_-(E)$. In the case where G is finite, the condition is always satisfied and d^* is defined.

Lemma 3.8. *Assume that d^* is always defined, which, for example, is satisfied when G is finite. Then d and d^* are both linear.*

Proof. (⊛) Let $f, g \in \ell^2(V)$ and $a, b \in \mathbb{R}$. Then:

$$\begin{aligned} d(af + bg)(e) &= (af + bg)(e^-) - (af + bg)(e^+) \\ &= af(e^-) + bg(e^-) - \big(af(e^+) + bg(e^+)\big) \\ &= a\big(f(e^-) - f(e^+)\big) + b\big(g(e^-) - g(e^+)\big) \\ &= a(df)(e) + b(dg)(e). \end{aligned}$$

Let $\theta, \eta \in \ell^2_-(E)$ and $a, b \in \mathbb{R}$. Then:

$$\begin{aligned} d^*(a\theta + b\eta)(x) &= \sum_{e^-=x} (a\theta(e) + b\eta(e)) \\ &= a \sum_{e^-=x} \theta(e) + b \sum_{e^-=x} \eta(e) \\ &= a(d^*\theta)(x) + b(d^*\eta)(x). \end{aligned}$$

□

Lemma 3.9. *Assume that d^* is always defined, which, for example, is satisfied when G is finite. Then for all $f \in \ell^2(V)$ and $\theta \in \ell^2_-(E)$, the following holds:*

$$(\theta, df)_E = (d^*\theta, f)_V.$$

In other words, d and d^* are adjoints of one another, which is the reason we use the superscript $*$ notation.

Proof. (⊛) We have

$$\begin{aligned} (\theta, df)_E &= \sum_{e \in E_{1/2}} \theta(e)(f(e^-) - f(e^+)) \\ &= \sum_{e \in E_{1/2}} \theta(e)f(e^-) - \sum_{e \in E_{1/2}} \theta(e)f(e^+) \\ &= \sum_{e \in E_{1/2}} \theta(e)f(e^-) + \sum_{e \in E_{1/2}} \theta(-e)f(e^+). \end{aligned}$$

This summands of this sum are every edge multiplied by f applied to its tail, and every such summand occurs precisely once. Next we see that

$$(d^*\theta, f)_V = \sum_{v \in V} (\sum_{e^-=x} \theta(e))f(x) = \sum_{x \in V} \sum_{e^-=x} \theta(e)f(x).$$

In this sum, we also have every edge multiplied by f applied to its tail and every such summand occurs precisely once. This is because every edge has a tail x, so it occurs in $\sum_{e^-=x} \theta(e)f(x)$, and it occurs only in that one $\sum_{e^-=x} \theta(e)f(x)$ because every edge has precisely one tail.

We conclude that both sums have the same summands and are therefore equal. □

3.3 Basic Concepts

We first present the basic notions needed for this work. Many of them are inspired from electrical networks in physics. However, we do not pay much attention to the physical inspiration besides mentioning it and refer to [LP16] or [DS84] for details.

An *electrical network* is a countable locally finite connected network with positive, symmetric edge weights, that is $c(e) = c(x, y) = c(y, x) = c(-e)$ and $c(e) > 0$ for each edge $e = (x, y)$. We allow the underlying graph to be a multigraph. We call $c(e)$ the *conductance* of the edge e. The *resistance* $r(e)$ of e is defined as $c(e)^{-1}$, or as ∞ if $c(e) = 0$. Given subsets $A, Z \subset S$, a function $v : V \to \mathbb{R}_{\geq 0}$ is called a *voltage function* if v is harmonic at $S \setminus (A \cup Z)$. Often we set $v_{|A} = 1$ and $v_{|Z} = 0$. The associated *current function* $i : E \to \mathbb{R}$ is defined by

$$i(x, y) = c(x, y)(v(x) - v(y)). \tag{3.1}$$

Remark 3.10. Due to the Existence Principle (Proposition 2.9) and Remark 2.10, a voltage function always exists. To see that, set $W = S \setminus (A \cup Z)$ and pick an arbitrary $f_0 \geq 0$ that is bounded on $S \setminus W = A \cup Z$.

Remark 3.11. ($\ominus$) Current flows in the direction of decreasing voltage because $c(x, y) \geq 0$. Concretely, if $v(x) > v(y)$, the current is positive and we say "it flows from x to y"; if $v(x) < v(y)$, it is negative and we say "it flows from y to x". Similar to how water flows downhill. Also note that $i(x, y) = -i(y, x)$.

Remark 3.12. By definition of current, if the voltage is scaled by multiplying all values by a constant, then the current is scaled by the same factor.

Definition 3.13. For convenience, we often scale the voltage in a way that

$$\sum_{e \text{ incident to some } a \in A} i(e) = 1.$$

We then refer to the current as a *unit flow*.

Remark 3.14. ($\circledast$) We can therefore always scale a voltage in a way that the resulting flow is a unit flow. The converse is true also. Due to Remark 3.10, there always exists a voltage that satisfies the wished values on A and Z. So we can always impose a voltage that will give us a flow of any desired total current flow, in particular, a total flow of 1.

3.4 Hitting Function

The *hitting fuction* is important because we need it for defining key concepts like the *hitting probability* or *effective conductance*. This chapter defines the hitting function and examines its basic properties.

Definition 3.15. By τ_A we denote the first time the chain *hits* A, that is, it visits a vertex in A. So if we start in A, we have $\tau_A = 0$. We will write τ_A^+ to denote the first time after 0 that we hit A, which is only different from τ_A if we start in A.

Definition 3.16. Let $A, Z \subset S$ be disjoint and let x be the starting point of the Markov chain. We write $F(x)$ for the probability that the Markov chain visits A before it visits Z. This is called the *hitting function*:

$$F(x) := P_x[\tau_A < \tau_Z].$$

By P_x we denote that we start at state x.

Lemma 3.17. *The hitting function F is harmonic on $S \setminus (A \cup Z)$.*

Proof. We have $F(x) = 1$ for all $x \in A$, since the chain has already "hit" A by definition. Similarly, $F(x) = 0$ for all $x \in Z$. Now, for $x \in S \setminus (A \cup Z)$, we have:

$$\begin{aligned} F(x) &= \sum_y P_x[\text{first step is to } y] \cdot P_x[\tau_A < \tau_Z \colon \text{first step is to } y] \\ &= \sum_{x \sim y} p(x,y) F(y), \end{aligned}$$

where the sum is over all states y adjacent to x. The first equality above is because $\sum_y p(x,y) = 1$. So F is harmonic on $S \setminus (A \cup Z)$. □

Definition 3.18. We denote the *hitting probability* that a random walk starting at a will hit a set Z before returning to a as

$$P[a \to Z] := P_a[\tau_Z < \tau_a^+].$$

Lemma 3.19. *Assume V is finite and impose a voltage $v(a)$ at vertex a and 0 at Z. We have*

$$P_x[\tau_a < \tau_Z] = \frac{v(x)}{v(a)}$$

and

$$P[a \to Z] = \frac{1}{v(a)\pi(a)} \sum_{x \in V} i(a,x).$$

Proof. As seen in Lemma 3.17, the hitting function $F(x) = P_x[\tau_a < \tau_Z]$ is harmonic on $S \setminus (\{a\} \cup Z)$ and so is v. Clearly, F and v agree on $\{a\} \cup Z$. By the Superposition Principle (Proposition 2.8) we conclude that

$$P_x[\tau_a < \tau_Z] = \frac{v(x)}{v(a)}$$

everywhere. Finally we have

$$
\begin{aligned}
P[a \to Z] &= \sum_{x \in V} p(a,x)\,(1 - P_x[\tau_a < \tau_Z]) \\
&= \sum_{x \in V} \frac{c(a,x)}{\pi(a)} \Big(1 - \frac{v(x)}{v(a)}\Big) \\
&= \frac{1}{v(a)\pi(a)} \sum_{x \in V} c(a,x)\big(v(a) - v(x)\big) \\
&= \frac{1}{v(a)\pi(a)} \sum_{x \in V} i(a,x).
\end{aligned}
$$

□

3.5 Flows

In this chapter, we introduce *flows*, one of the most important concepts in this thesis. Initially, it might not seem like flows relate to whether a network is transient or recurrent, but we will later see that they are closely linked. Royden's Criterion, for example, states that an infinite electrical network is transient if and only if there exists a certain flow from some vertex to infinity.

Definition 3.20. We call $\theta \in \ell^2_-(E)$ a *flow* if for every vertex x we have

$$\sum_{e^- = x} |\theta(e)| < \infty.$$

Note that we take the absolute value of each summand, so the above is not the same as $(d^*\theta)(x)$. But we have, due to the finiteness condition above, that $(d^*\theta)(x)$ is always defined for a flow θ and a vertex x. We call it a flow between A and Z if $(d^*\theta)(x) = 0$ for any $x \in V \setminus (A \cup Z)$. If $d^*\theta \geq 0$ on A and $d^*\theta \leq 0$ on Z, we say θ is a flow from A to Z. If $(d^*\theta)(x) = 0$ for any $x \notin A$, we say θ is a flow from A to infinity. The *amount of flow into the network* at a vertex a is given by $(d^*\theta)(a)$. The *strength* or *total amount flowing into the network* is $\mathrm{Strength}(\theta) := \sum_{a \in A} (d^*\theta)(a)$. A flow of strength 1 is called a *unit flow*.

Figure 1 shows an illustrated example flow of strength 16.

Remark 3.21. (⊛) Scaling a flow from A to Z by a bounded factor $a \in \mathbb{R}$ still results in a flow from A to Z because

$$\sum_{e^- = x} |a \cdot \theta(e)| = |a| \cdot \sum_{e^- = x} |\theta(e)| < \infty,$$

and $d^*(a \cdot \theta) = a \cdot d^*\theta = 0$ on $V \setminus A$ or on $V \setminus (A \cup Z)$ respectively.

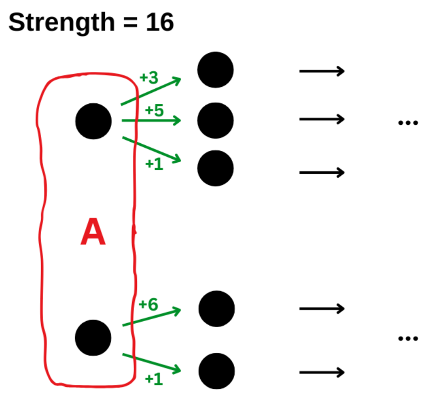

Figure 1: (⊛) Flow strength example

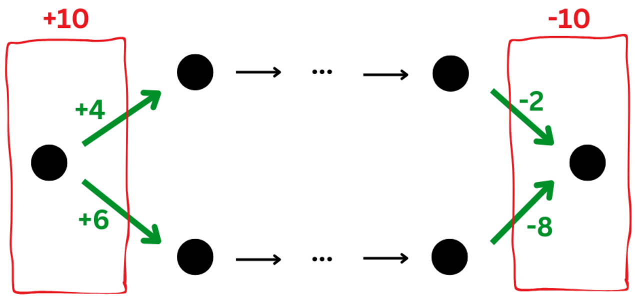

Figure 2: (⊛) Flow Conservation illustration

The following lemma states a very important, almost characterizing, property about flows, which we will often use.

Lemma 3.22 (Flow Conservation)**.** *Let G be a finite graph, and let A and Z be two disjoint subsets of vertices. If θ is a flow between A and Z, then*

$$\sum_{a\in A}(d^*\theta)(a) = -\sum_{z\in Z}(d^*\theta)(z).$$

Proof.

$$\sum_{x\in A}(d^*\theta)(x) + \sum_{x\in Z}(d^*\theta)(x) = \sum_{x\in A\cup Z}(d^*\theta)(x) = (d^*\theta, 1) = (\theta, d1) = (\theta, 0) = 0,$$

since $(d^*\theta)(x) = 0$ for $x \notin A\cup Z$. □

Figure 2 shows an illustration of Flow Conservation.

Remark 3.23. (⊝) One can compare a flow to water flowing through pipes. It starts at A, called the sources, and ends in Z, called the sinks. In between the sources and sinks "no water is lost" and "no new water is created" so to say, which is the second condition in above definition.

Lemma 3.24. *Let G be a finite graph, and let A and Z be disjoint vertex subsets. If θ is a flow from A to Z and $f_{|A} = \alpha$ and $f_{|Z} = \zeta$ for $\alpha, \zeta \in \mathbb{R}$, respectively, then*

$$(\theta, df) = (\alpha - \zeta) \cdot \mathrm{Strength}(\theta).$$

Proof. We have

$$\begin{aligned}(\theta, df) = (d^*\theta, f) &= \sum_{a \in A} (d^*\theta)(a)\alpha + \sum_{z \in Z} (d^*\theta)(z)\zeta \\ &= \alpha \cdot \sum_{a \in A} (d^*\theta)(a) + \zeta \cdot \sum_{z \in Z} (d^*\theta)(z).\end{aligned}$$

By applying Flow Conservation (Lemma 3.22) we can write this as

$$\begin{aligned}\alpha \cdot \sum_{a \in A} (d^*\theta)(a) - \zeta \cdot \sum_{a \in A} (d^*\theta)(a) &= (\alpha - \zeta) \cdot \sum_{a \in A} (d^*\theta)(a) \\ &= (\alpha - \zeta) \cdot \mathrm{Strength}(\theta).\end{aligned}$$

□

3.6 Ohm's and Kirchhoff's Laws

Ohm's and Kirchhoff's Laws are key concepts in electrical network theory and take their names from physics. While Ohm's law is simply something we already know, written in a different way, Kirchhoff's Laws require proofs. All three laws are used frequently in this thesis.

Definition 3.25. For $e \in E$ we define $\chi^e : E \to \mathbb{R}$ via

$$\chi^e(e') := 1_{\{e\}}(e') - 1_{\{-e\}}(e') \quad \text{for } e' \in E.$$

Proposition 3.26 (Ohm's Law)**.** *(⊝) For every edge $e \in E$ we have*

$$dv = ir, \quad \textit{that is,} \quad (dv)(e) = i(e)r(e).$$

In other words, for adjacent vertices x, y, the current $i(x, y)$ from x to y satisfies

$$v(x) - v(y) = i(x, y)r(x, y).$$

Proposition 3.27 (Kirchhoff's Node Law)**.** *(⊝) For every vertex that is not in A or Z we have*

$$(d^*i)(x) = 0,$$

in other words, every current function is a flow between A and Z. We can also state this using the inner product:

$$\left(\sum_{e^- = x} c(e)\chi^e, i \right)_r = 0.$$

Recall that $(\cdot, \cdot)_r$ is the inner product defined in Chapter 3.2, and r is the resistance function.

Proof. By the definition of current function, $i(x, y) = -i(y, x)$. For any $x \in S \setminus (A \cup Z)$, since v is harmonic, we have

$$\begin{aligned} v(x) &= \sum_{x \sim y} p(x, y)v(y) \\ &= \sum_{x \sim y} \frac{c(x, y)}{\sum_{x \sim z} c(x, z)} v(y) \\ &= \frac{1}{\sum_{x \sim z} c(x, z)} \sum_{x \sim y} c(x, y)v(y). \end{aligned}$$

Thus

$$v(x) \sum_{x \sim y} c(x, y) = \sum_{x \sim y} c(x, y)v(y),$$

and therefore

$$0 = v(x) \sum_{x \sim y} c(x, y) - \sum_{x \sim y} c(x, y)v(y) = \sum_{x \sim y} i(x, y).$$

□

Proposition 3.28 (Kirchhoff's Cycle Law). *(⊝) For any cycle $x_1 \sim x_2 \sim \cdots \sim x_n \sim x_{n+1} = x_1$ we have:*

$$\sum_{k=1}^{n} i(x_k, x_{k+1})r(x_k, x_{k+1}) = 0.$$

We can also state it in the following way. If $e_1, e_2, \ldots, e_n$ form an oriented cycle in G, then

$$\left(\sum_{k=1}^{n} \chi_{e_k}, i \right)_r = 0.$$

Proof. We sum up the voltage differences around the cycle:

$$0 = \sum_{k=1}^{n} (v(x_k) - v(x_{k+1})) = \sum_{k=1}^{n} i(x_k, x_{k+1})r(x_k, x_{k+1}),$$

where the last equality follows from Ohm's Law (Proposition 3.26). □

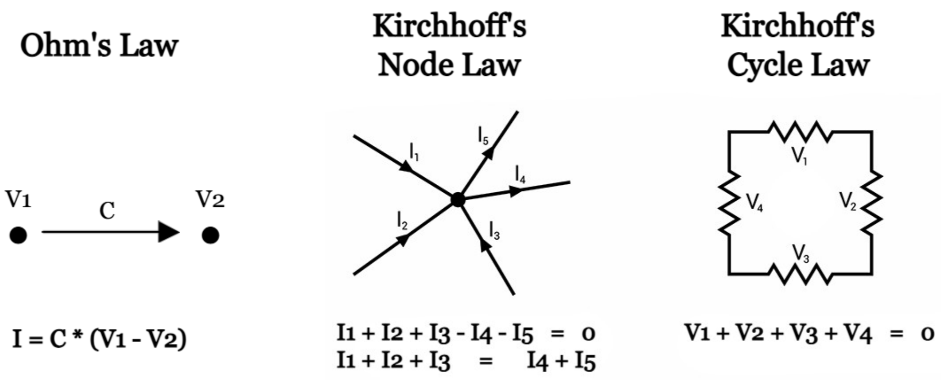

Figure 3: (⊛) Illustration of Ohm's and Kirchhoff's laws

Lemma 3.29. *(⊛) Let v be a function on an electrical network with conductances c, for any vertex x the following are equivalent*

1. $\sum_{y\sim x}(v(x)-v(y))c(x,y)=0$ *(Kirchhoff's Node Law),*
2. $v(x)=\sum_{y\sim x}p(x,y)v(y)$ *(Harmonicity).*

Proof. The first condition $\sum_{y\sim x}\big(v(x)-v(y)\big)c(x,y)=0$ is equivalent to

$$\begin{aligned} v(x)\sum_{y\sim x}c(x,y) &= \sum_{y\sim x}v(y)c(x,y) \\ \Leftrightarrow v(x)\cdot\pi(x) &= \sum_{y\sim x}v(y)c(x,y) \\ \Leftrightarrow v(x) &= \sum_{y\sim x}\frac{c(x,y)}{\pi(x)}v(y) \\ \Leftrightarrow v(x) &= \sum_{y\sim x}p(x,y)v(y), \end{aligned}$$

which is (2). □

3.7 Effective Conductance

Let G be an electrical network and let a be a vertex and $Z\subset V$. In this chapter, we introduce a key concept with the idea to look at the entire circuit between a and Z as a single conductor and assign it an *effective conductance*.

Definition 3.30. The *effective conductance* between a and Z is defined as

$$C(a\leftrightarrow Z):=\pi(a)P[a\to Z]. \tag{3.2}$$

We write $C(a \leftrightarrow Z; G)$ if we want to indicate the dependence on G. The *effective resistance* between a and Z is the reciprocal of the effective conductance:

$$R(a \leftrightarrow Z) = \frac{1}{C(a \leftrightarrow Z)}.$$

If $a \in Z$, we define $R(a \leftrightarrow Z) := 0$.

For a vertex subset A we define $C(A \leftrightarrow Z)$ as the effective conductance $C(a \leftrightarrow Z)$ that would result if all vertices in A were identified into a single vertex $a \in A$.

Remark 3.31. ($\circledast$) The reason we picked the value $\pi(a)P[a \to Z]$ can be motivated as follows. Think of "$\pi(a) \cdot P[a \to Z]$" as

$$\text{(total conductance coming out of } a) \cdot \text{(scale-down-constant)}.$$

Recall that $\pi(a) = \sum_{x \sim a} c(a, x)$. Here, the "scale-down-constant" is needed because the total conductance coming out of a would "account" for parts that are not the target as well. So we scale it down by the probability to hit the target, which is precisely the hitting probability $P[a \to Z]$.

Remark 3.32. The hitting probability $P[a \to Z]$ can be now be written as:

$$P[a \to Z] = \frac{C(a \leftrightarrow Z)}{\pi(a)}.$$

Remark 3.33. ($\ominus$) The number of visits to a before hitting Z is a geometric random variable. That is because we set the success to be "hitting Z" and set p to be its probability. Note that this indeed creates a geometric random variable as the event "success" is independent from previous events by definition of a Markov chain. We see that $p = P[a \to Z]$ and thus the mean of the random variable is

$$\frac{1}{p} = \frac{1}{P[a \to Z]} = \pi(a)R(a \leftrightarrow Z).$$

Suppose there is unit current flowing from a to Z. Then, by Lemma 3.19, this equals $\pi(a)v(a)$, so $R(a \leftrightarrow Z) = v(a)$.

3.8 Energy

Energy is another key concept in electrical network theory. One example of its application is Royden's Criterion, which tells us that an infinite network is transient if and only if there is a flow from some vertex to infinity with finite energy.

Definition 3.34. For an antisymmetric function θ, define its *energy* as

$$E(\theta) := \|\theta\|_r^2,$$

where r denotes the resistances.

Lemma 3.35. *($\circleddash$) Suppose G is a finite electrical network with vertex subsets A and Z. Impose a voltage v such that $v|_A$ is constant and $v|_Z = 0$. Then*

$$v|_A = I_{AZ} R(A \leftrightarrow Z),$$

where

$$I_{AZ} := \sum_{a \in A} \sum_{x \in V} i(a, x)$$

is the total current flowing from A to Z.

Proof. ($\circledast$) We will prove the statement in the form $v|_A \cdot C(A \leftrightarrow Z) = I_{AZ}$. Identify A into a single vertex a and denote the resulting network as H. On H we have $v(a) = v|_A$ clearly. In the context of G the term $v|_A \cdot C(A \leftrightarrow Z)$ is the same as $v(a) \cdot C(a \leftrightarrow Z)$ in the context of H.

Now we will stay in the context of H. By Lemma 3.19 we have

$$P(a \to Z) = \frac{1}{v(a)\pi(a)} \sum_{x \in V} i(a, x).$$

Thus

$$\begin{aligned} v(a) \cdot C(a \leftrightarrow Z) &= v(a)\pi(a) P(a \to Z) \\ &= v(a)\pi(a) \cdot \frac{1}{v(a)\pi(a)} \sum_{x \in V} i(a, x) \\ &= \sum_{x \in V} i(a, x). \end{aligned}$$

But $\sum_x i(a, x)$ is, read in the context of G, the same as

$$\sum_{a \in A} \sum_{x \in V} i(a, x) = I_{AZ}.$$

In the context of G, we conclude $v|_A \cdot C(A \leftrightarrow Z) = I_{AZ}$. $\square$

Remark 3.36. Note that given a finite network we can write

$$E(i) = (i, i)_r = (i, dv)$$

for a flow i. If i is a unit current flow from A to Z with voltages v_A constant on A and voltage 0 on Z, then by Lemma 3.24 and Lemma 3.35,

$$E(i) = v_A - v_Z = v_A = R(A \leftrightarrow Z). \tag{3.3}$$

This will be useful for estimating effective resistances.

3.9 Infinite Networks

This chapter shows how we can apply the theory of the previous chapters to infinite networks. We do this by *exhausting* infinite networks with finite networks.

For an infinite locally finite electrical network G, let $\{G_n\}$ be a sequence of finite subgraphs that *exhausts* G. This means $G_n \subseteq G_{n+1}$ and $G = \bigcup G_n$. Such a sequence always exists because by definition $V(G)$ and $E(G)$ are countable. Each edge in G_n is also an edge in G, and we assign it the same conductance as in G. We assume that G_n is the subgraph induced by the vertex set $V(G_n)$. Let Z_n be the set of vertices in $G \setminus G_n$. Define G_n^W as the graph obtained by identifying all vertices in Z_n into a single vertex z_n, and then removing loops but keeping multiple edges. Clearly $V(G_n^W)$ is finite, but $E(G_n^W)$ is finite too because we assumed that G is locally finite.

Remark 3.37. For the rest of the thesis, given an infinite electrical network G we will write G_n, Z_n, z_n and G_n^W without defining it again and refer to the above definitions.

Remark 3.38. Note that Remark 3.14 holds true also for flows from A to ∞ as we can just apply it to G_n^W and let $n \to \infty$.

Remark 3.39. Similarly, under the same definitions as in Remark 3.33, we get $v(a) = R(a \leftrightarrow \infty)$.

Definition 3.40. Given a random walk on an infinite electrical network G, if we stop the walk the first time it reaches Z_n, this induces a random walk on G_n^W that continues until it reaches z_n. For any $a \in G$, the events $\{a \to Z_n\}$ are decreasing in n, so the limit $\lim_{n\to\infty} P[a \to Z_n]$ gives the probability that the random walk never returns to a in G, which we call the *escape probability from* a.

Definition 3.41. Given an infinite electrical network G, the limit

$$C(a \leftrightarrow \infty) \coloneqq \lim_{n\to\infty} C(a \leftrightarrow z_n; G_n^W)$$

is called the effective conductance from a to infinity in G. The reciprocal of this, the effective resistance, is denoted $R(a \leftrightarrow \infty)$.

The following theorem is one reason why we are interested in the concept of effective conductance.

Theorem 3.42 (Transience and Effective Conductance)**.** *A random walk on an infinite electrical network is transient if and only if the effective conductance from any vertex to infinity is positive.*

Proof. For any $a \in V$ we see that

$$\begin{aligned}
&C(a \leftrightarrow \infty) > 0 \\
&\Leftrightarrow \lim_{n\to\infty} C(a \leftrightarrow z_n) > 0 \\
&\Leftrightarrow \lim_{n\to\infty} \pi(a)P(a \to z_n) > 0 \\
&\Leftrightarrow \pi(a) \lim_{n\to\infty} P(a \to z_n) > 0 \\
&\Leftrightarrow \lim_{n\to\infty} P(a \to z_n) > 0,
\end{aligned}$$

because $\pi(a) \geq 0$. The escape probability is positive if and only if the random walk is transient, as we know from Lemma 1.3. □

3.10 Thomson's Principle and Monotonicity

Here, we prove two very important theorems, namely Thomson's Principle and Rayleigh's Monotonicity Principle. In particular the latter one will be of key importance for the rest of this thesis. Before that, however, we need to introduce new notation and prove helping lemmas.

Definition 3.43. Define the *star space* $\star \subset \ell^2_-(E)$ as the subspace spanned by the stars at each vertex x, that is the functions $\sum_{e^-=x} c(e)\chi^e$. Similarly, define the *cycle space* $\diamond \subset \ell^2_-(E)$ as the subspace spanned by oriented cycles in G, represented as sums $\sum_{k=1}^n \chi^{e_k}$ for edges $e_1, e_2, \ldots, e_n$ forming a cycle. Note that since we defined $\ell^2_-(E)$ on a countable network, we have defined $\star$ and $\diamond$ on countable networks only.

Remark 3.44. It is clear that the spanning functions in both parts of the above definition are indeed antisymmetric since χ^e is antisymmetric by definition.

We need the following helping lemma before we continue.

Lemma 3.45. *Let W be a subset of vertices of a finite connected network. For an antisymmetric function $j : E \to \mathbb{R}$, that is $j(x,y) = -j(y,x)$, we assume that j satisfies Kirchhoff's Cycle Caw (Proposition 3.28) and Kirchhoff's Node Law (Proposition 3.27) in the form $\sum_{y\sim x} j(x,y) = 0$ for all $x \in W$. Then j is the current corresponding to some voltage function v, and v is unique up to an additive constant.*

Proof. ($\circledast$) Fix an arbitrary reference vertex x_0 and set $v(x_0) = 0$. For any other vertex x, we define $v(x)$ by choosing any path $\gamma = (x_0, x_1, \ldots, x_n = x)$ from x_0 to x and setting

$$v(x) = \sum_{i=0}^{n-1} r(x_{i+1}, x_i) j(x_{i+1}, x_i).$$

To see that this is well defined, let

$$\gamma_1 = (x_0, x_1, \dots, x_n = x) \quad \text{and} \quad \gamma_2 = (x_0, y_1, \dots, y_m = x)$$

be two different paths from x_0 to x. Consider the path γ_2 in the reverse direction: $(y_m, y_{m-1}, \dots, y_1, x_0)$. Then γ_1 followed by γ_2 reversed forms a cycle C. Since j satisfies Kirchhoff's Cycle Law (Proposition 3.28), we obtain

$$\sum_{(a,b)\in C} r(a,b)j(a,b) = 0.$$

In other words, we have

$$\sum_{i=0}^{n-1} r(x_{i+1}, x_i)j(x_{i+1}, x_i) + \sum_{i=0}^{m-1} r(y_i, y_{i+1})j(y_i, y_{i+1}) = 0.$$

Since j is antisymmetric and $r(a,b) = r(b,a)$, we obtain

$$\sum_{i=0}^{n-1} r(x_{i+1}, x_i)j(x_{i+1}, x_i) = \sum_{i=0}^{m-1} r(y_{i+1}, y_i)j(y_{i+1}, y_i).$$

Thus the sum is independent of the chosen path.

For adjacent vertices $x \sim y$, consider some path from x_0 to x followed by edge (x, y), and some path from x_0 to y. By the above argument, these give the same sum. Thus we have

$$v(y) = v(x) + r(y,x)j(y,x),$$

and therefore

$$v(x) - v(y) = -r(y,x)j(y,x) = r(x,y)j(x,y),$$

as required for a voltage function generating current j.

We see that v is harmonic due to Lemma 3.29, as Kirchhoff's Node Law (Proposition 3.27) holds true by assumption. Therefore, v is indeed a voltage function generating j.

For uniqueness, let v_1, v_2 be two voltage functions generating j. Then for any edge (x, y) we get

$$v_1(y) - v_1(x) = r(x,y)j(x,y) = v_2(y) - v_2(x).$$

Therefore $(v_1 - v_2)(y) = (v_1 - v_2)(x)$ for all adjacent vertices x and y. Since the network is connected, we conclude that $v_1 - v_2$ is constant. $\square$

Lemma 3.46. *Assume that the network is finite. Then the star space $\star$ and cycle space $\diamond$ are orthogonal with respect to the inner product $(\cdot,\cdot)_r$. Furthermore, their sum covers all of $\ell^2_-(E,r)$. In other words, every $\theta \in \ell^2_-(E,r)$ can be decomposed uniquely as a sum of elements from $\star$ and $\diamond$, that is, $\ell^2_-(E,r) = \star \oplus \diamond$.*

Proof. For orthogonality, note that for any star $\sum_{e^-=x} c(e)\chi^e \in \star$ and any cycle $\sum_{k=1}^n \chi^{e_k} \in \diamond$, the inner product $(\cdot,\cdot)_r$ results in zero. To see that, we count "overlaps" of the star and the circle, since any other edges will vanish anyway because one of the factors is 0. For any edge e in the star, that is $e^- = x$ or $e^+ = x$, and $e = e_k$ for some k, we can without loss of generality assume $e^- = x$. But then $e_{k-1}^+ = x$ and thus in the inner product we are adding $+1$ and -1, meaning, it vanishes.

To show that the sum covers all of $\ell^2_-(E,r)$, we prove that only the zero vector in $\ell^2_-(E,r)$ is orthogonal to both $\star$ and $\diamond$. Suppose $\theta \in \ell^2_-(E,r)$ is orthogonal to both $\star$ and $\diamond$. Since θ is antisymmetric by definition and because it is orthogonal to $\diamond$ we know that it satisfies Kirchhoff's Cycle Law (Proposition 3.28) and thus we can apply Lemma 3.45, and we apply it for $W := \varnothing$. This gives us that there is a voltage function $F : V \to \mathbb{R}_{\geq 0}$ such that $\theta = cdF$. Since θ is also orthogonal to $\star$, we have $(cdF, \star) = 0$. In other words, for any vertex x we have

$$\begin{aligned}
&\sum_{e^-=x} (cdF)(e) = 0 \\
&\Leftrightarrow \sum_{e^-=x} c(e)(F(e^-) - F(e^+)) = 0 \\
&\Leftrightarrow \sum_{e^-=x} c(e)F(e^-) - \sum_{e^-=x} c(e)F(e^+)) = 0 \\
&\Leftrightarrow \sum_{e^-=x} c(e)F(x) = \sum_{e^-=x} c(e)F(e^+)) \\
&\Leftrightarrow F(x) = \sum_{e^-=x} c(e)F(e^+)),
\end{aligned}$$

which means that F is harmonic on V. Since G is finite and every state can be reached from any other state, we can apply Corollary 2.11, which gives us that F is constant. Thus we have $\theta = 0$. $\square$

Lemma 3.47. *($\ominus$) Assuming the network is finite, the following statements hold.*

1. *Any current i is in $\star$.*
2. *Any $i \in \star$ is a current.*
3. *Any current i and $\theta \in \ell^2_-(E,r)$ that satisfies $d^*i = d^*\theta$, meaning i and θ have the same sources and sinks, induce a flow $\theta - i \in \diamond$.*
4. *$\theta = i + (\theta - i)$ is the orthogonal decomposition of θ.*
5. *$\|\theta\|_r^2 = \|i\|_r^2 + \|\theta - i\|_r^2$.*

Proof. Kirchhoff's Cycle Law (Proposition 3.28) gives that a current i is orthogonal to $\diamond$ and thus is in $\star$, proving (1). We obtain (2) by applying Lemma 3.45 with $W \coloneqq \{x\colon (d^*i)(x) = 0\}$ to any $i \in \star$.

To prove (3) we will show that for any vertex x we have

$$(\sum_{e^-=x} c(e)\chi^e, \theta - i)_r = 0.$$

This means that $\theta - i$ is orthogonal to $\star$ and therefore an element of $\diamond$. To do that we first note that for any flow η we have

$$\begin{aligned}\Big(\sum_{e^-=x} c(e)\chi^e, \eta\Big)_r &= \sum_{e\in E_{1/2}} \Big(r(e)\eta(e) \cdot \sum_{e_1^-=x} c(e_1)\chi^{e_1}(e)\Big)\\ &= \sum_{e^-=x} c(e)r(e)\eta(e) = (d^*\eta)(x).\end{aligned}$$

Here, the second last equality holds because all summands corresponding to an edge e that is not incident to x will vanish due to the "$\chi^{e_1}(e)$", and, as we know, the result is independent of the choice of $E_{1/2}$, so among those edges incident to x we pick those edges e with $e^- = x$. Now, to prove (3), we fix a vertex x and write

$$\Big(\sum_{e^-=x} c(e)\chi^e, \theta - i\Big)_r = \big(d^*(\theta - i)\big)(x) = (d^*\theta)(x) - (d^*i)(x) = 0.$$

The last equality holds true because we assumed $d^*i = d^*\theta$.

From here, (4) directly follows. Lastly, we obtain (5) as follows:

$$\begin{aligned}\|\theta\|_r^2 &= (\theta, \theta)_r\\ &= (i + (\theta - i), i + (\theta - i))_r\\ &= (i, i)_r + (i, \theta - i)_r + (\theta - i, i)_r + (\theta - i, \theta - i)_r\\ &= \|i\|_r^2 + (i, \theta - i)_r + (\theta - i, i)_r + \|\theta - i\|_r^2.\end{aligned}$$

Since $i \in \star$ and $(\theta - i) \in \diamond$, they are orthogonal to each other, meaning $(i, \theta - i)_r = (\theta - i, i)_r = 0$. □

Theorem 3.48 (Thomson's Principle)**.** *Let G be a finite network, and let A and Z be two disjoint subsets of its vertices. Let θ be a flow from A to Z, and let i be the current flow from A to Z such that $d^*i = d^*\theta$. We then have $E(\theta) > E(i)$ unless $\theta = i$.*

Proof. The claim follows immediately from item (5) of Lemma 3.47. □

Theorem 3.49 (Rayleigh's Monotonicity Principle)**.** *Let G be a countable connected graph with two assignments c and c' of conductances on G such that $c \leq c'$ everywhere.*

1. *If G is finite and A and Z are two disjoint subsets of its vertices, then*

$$C_c(A \leftrightarrow Z) \leq C_{c'}(A \leftrightarrow Z).$$

2. *If G is infinite and a is one of its vertices, then*

$$C_c(a \leftrightarrow \infty) \leq C_{c'}(a \leftrightarrow \infty).$$

In particular, if (G, c) is transient, then by Theorem 3.42 so is (G, c').

Proof. ($\ominus$) For any unit current flow i from A to Z, by (3.3) we have

$$C(A \leftrightarrow Z) = \frac{1}{E_c(i)}.$$

Let i_c be the unit current flow induced by c and $i_{c'}$ be the one induced by c'. Then

$$E_c(i_c) \geq E_{c'}(i_c) \geq E_{c'}(i_{c'}).$$

The first inequality is immediate when looking at the definition of energy and knowing that, for $c \leq c'$, we have $r \geq r'$. The second inequality follows from Thomson's Principle (Theorem 3.48). Note that we can apply Thomson's Principle since $d{*}i_c = d{*}i_{c'}$ holds true as they are unit current flows. Taking the reciprocals of these inequalities gives the desired result.

Item (2) now follows immediately. We have

$$C_c(a \leftrightarrow z_n; G_n^W) \leq C_{c'}(a \leftrightarrow z_n; G_n^W) \quad \text{for all } n \in \mathbb{N},$$

for all n, and thus

$$\begin{aligned} C_c(a \leftrightarrow \infty) &= \lim_{n\to\infty} C_c(a \leftrightarrow z_n; G_n^W) \\ &\leq \lim_{n\to\infty} C_{c'}(a \leftrightarrow z_n; G_n^W) \\ &= C_{c'}(a \leftrightarrow \infty). \end{aligned}$$

□

Remark 3.50. ($\ominus$) Suppose G is an infinite electrical network and the associated random walk is transient. From Theorem 3.42 we know that the effective conductance from any vertex to ∞ is positive. Now if the conductances increase, meaning they are still positive, we have, by Rayleigh's Monotonicity Principle (Theorem 3.49), that the effective conductances increase too. So Theorem 3.42 gives us that the new electrical network remains transient.

3.11 Green Function

In this chapter, we define the *Green Function* and show its connection to voltage functions. The Green Function is a very central concept in potential theory, and one can learn more about this topic in [LP16]. In this chapter, we define or assume G to be an electrical network, a to be a vertex, and Z to be a subset of V.

Definition 3.51. Let $G_Z(a,x)$ be the expected number of visits to x strictly before hitting Z by a random walk starting at a. We call this the *Green Function.*

Proposition 3.52 (Green Function as Voltage)**.** *Suppose G is finite and we have a voltage imposed on $\{a\} \cup Z$ so that a unit current flows from a to Z and the voltage is* 0 *on Z. Then, for any vertex x, we have*

$$v(x) = \frac{G_Z(a,x)}{\pi(x)}.$$

Proof. We will prove the claim in the form of $\pi(x)v(x) = G_Z(a,x)$. In the definition of the Green Function, we saw that

$$G_Z(a,a) = \pi(a)R(a \leftrightarrow Z),$$

and in Remark 3.33 we saw that $R(a \leftrightarrow Z) = v(a)$. Together, this proves the wanted equation for $x = a$. For $x \in Z$, we have

$$v(x) = 0 = \frac{G_Z(a,x)}{\pi(x)} = \frac{0}{\pi(x)} = 0.$$

Next we observe that for $x \in V \setminus (\{a\} \cup Z)$ we have

$$G_Z(a,x) = \sum_{x \sim y} p(y,x) G_Z(a,y).$$

This means that $\frac{G_Z(a,x)}{\pi(x)}$ is harmonic on $V \setminus (\{a\} \cup Z)$ and the result follows by the Uniqueness Principle (Proposition 2.7). □

3.12 Recurrence and Transience

In this chapter, we focus on recurrence and transience and prove key results like the Nash–Williams Inequality, Nash–Williams Criterion, and Royden's Criterion. We also present an interesting application of the Nash–Williams Criterion, namely a partial proof of Pólya's Theorem.

Definition 3.53. Given disjoint subsets A and Z of vertices, we call a set Π of edges a *cutset* if it *separates* a and Z. By this we mean that every path with one endpoint in A and one in Z must include an edge in Π.

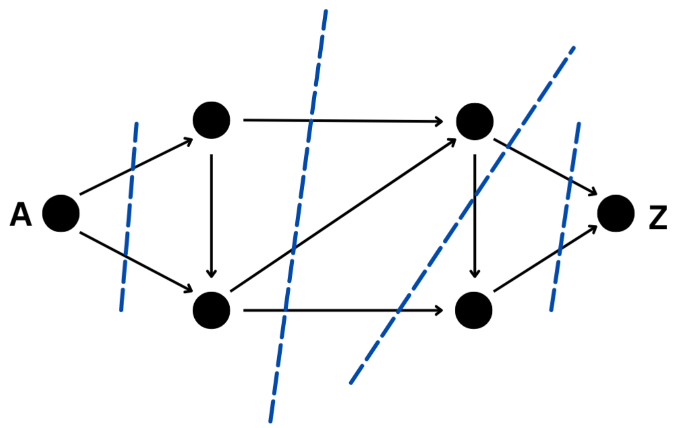

Figure 4: (⊛) Examples of cutsets

We say that a *cuset* Π *separates* a and ∞ if every infinite simple path from a includes an edge from Π. Figure 4 shows an example.

We call a cutset *minimal* if removing any element of it results in a set that is not a cutset.

Lemma 3.54 (Nash–Williams Inequality)**.** *Let a and z be distinct vertices in a finite electrical network that are separated by pairwise disjoint cutsets $\Pi_1, \dots, \Pi_n$. Then*

$$R(a \leftrightarrow z) \geq \sum_{k=1}^{n} \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}. \tag{3.4}$$

Proof. Let i be the unit current flow from a to z. By (3.3) it suffices to show that $E(i)$ is greater or equal than the right-hand side of (3.4). Given a finite cutset Π that separates a from z, let Z denote the endpoints of Π separated by Π from a, and let K denote vertices not separated from a by Π. We could say Z is on the "z-side" and K is on the "a-side". Now let H be the subnetwork induced by $K \cup Z$. Then i induces a unit flow i_H from a to Z, so by Flow Conservation (Lemma 3.22) applied to H we get

$$1 = -\sum_{x \in Z} d^* i_H(x) = -\sum_{e^- \in Z,\, e \in H} i(e).$$

If both endpoints of some e lie in Z, both $i(e)$ and $i(-e)$ occur in the sum and cancel. Furthermore, all edges in H with only one endpoint in Z must lie in Π. Thus we get

$$\sum_{e \in \Pi} |i(e)| \geq \sum_{e^- \in Z,\, e \in H} |i(e)| \geq \left| -\sum_{e^- \in Z,\, e \in H} i(e) \right| = 1.$$

Using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\sum_{e\in\Pi}\left(i(e)^2r(e)\right)\sum_{e\in\Pi}c(e) &\geq \left(\sum_{e\in\Pi}\left(|i(e)|\frac{1}{\sqrt{c(e)}}\right)\cdot\sqrt{c(e)}\right)^2\\ &\geq \left(\sum_{e\in\Pi}|i(e)|\right)^2\\ &\geq 1,\end{aligned}$$

and therefore

$$\sum_{e\in\Pi} i(e)^2 r(e) \geq \Big(\sum_{e\in\Pi} c(e)\Big)^{-1}.$$

Now, we substitute Π for Π_k and sum over $k = 1, \ldots, n$ to get the inequality to be proven. □

Theorem 3.55 (Nash–Williams Criterion)**.** *If $\{\Pi_n\}$ is a sequence of pairwise disjoint finite cutsets in an infinite electrical network G, each separating a from ∞, then*

$$R(a \leftrightarrow \infty) \geq \sum_{k=1}^{\infty}\left(\sum_{e\in\Pi_k} c(e)\right)^{-1}.$$

In particular, note that if the right-hand side is infinite, then by Theorem 3.42 we have that G is recurrent.

Proof. (⊝) Define G_n as usual, but in a way that $\bigcup_{k=1}^{n}\Pi_k \subset G_n$. Then we get the desired inequality we applying the Nash–Williams Inequality (Lemma 3.54) to the following

$$R(a \leftrightarrow \infty) = \lim_{n\to\infty} R(a \leftrightarrow z_n).$$

□

We now apply the theory developed above and provide a quick and elegant proof for the following Theorem. This famous theorem was first published by George Pólya in 1921 in the paper [Pól21].

Theorem 3.56 (Pólya's Theorem – First part)**.** *Simple random walk on the nearest-neighbor graph of $\mathbb{Z}^d$ is recurrent for $d = 1, 2$.*

Proof. (⊝) First, note that $\mathbb{Z}^d$ is countable and locally finite. For $d = 1, 2$, we define the cutsets

$$\Pi_n := \{e :\ d(0, e^-) = n-1,\ d(0, e^+) = n\},$$

where 0 is the origin and $d(\cdot,\cdot)$ denotes graph distance. Figure 5 shows Π_3. The cutsets are clearly pairwise disjoint and separate 0 from ∞. Also note that for all $e \in E$ we have $c(e) = 1$.

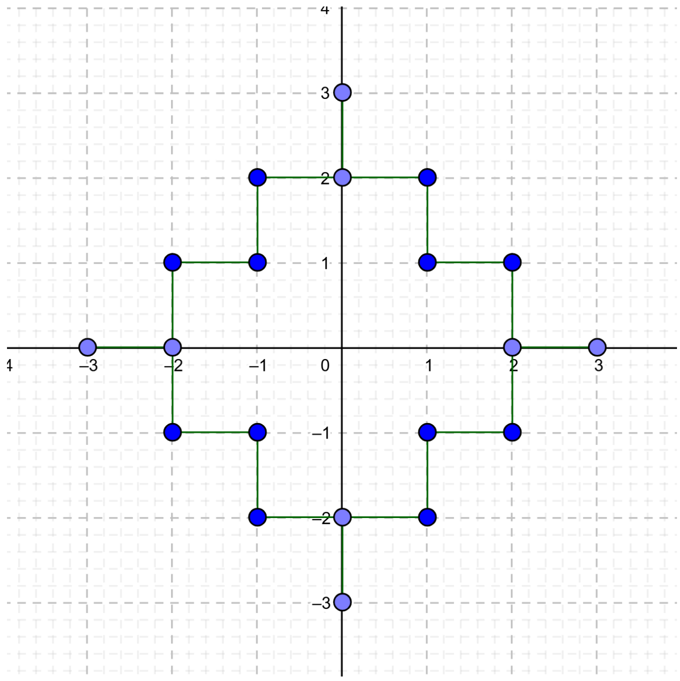

Figure 5: (⊛) The cutset in Theorem 3.56's proof for $n = 3$, created via GeoGebra

In the case of $d = 1$ we have $\#\Pi_n = 2$. Substituting into the right-hand side of the Nash–Williams Criterion (Theorem 3.55) we get:

$$\sum_n \left(\sum_{e \in \Pi_n} c(e) \right)^{-1} = \sum_n \frac{1}{\#\Pi_n} = \sum_n \frac{1}{2}.$$

This is clearly not a finite sum and thus, by Nash–Williams Criterion (Theorem 3.55), the random walk is recurrent.

In the case of $d = 2$ we have $\#\Pi_n = 8(n-1)+4$ by simple combinatorics. The right-hand side of the Nash–Williams Criterion becomes:

$$\sum_n \left(\sum_{e \in \Pi_n} c(e) \right)^{-1} = \sum_n \frac{1}{\#\Pi_n} = \sum_n \frac{1}{8(n-1)+4}.$$

This is not a finite sum either. Therefore, by Nash–Williams Criterion the random walk is recurrent. □

Remark 3.57. (⊛) The second part of Pólya's Theorem shows that for $d \geq 3$ the above random walk is transient. The approach in the proof above fails for $d \geq 3$ as $\#\Pi_n$ would be too large and the sums would be finite. Although we will not give a rigorous proof of the second part we will later briefly present the idea of the proof.

Lemma 3.58. *Let $G = (V, E)$ be an electrical network, and let $\theta_n \in \ell^2_-(E, r)$. Suppose $\mathcal{E}(\theta_n) \leq M < \infty$ and that for some $\theta \in \ell^2_-(E, r)$ we have*

$$\theta_n(e) \to \theta(e) \quad \text{for all } e \in E.$$

Then θ is antisymmetric and

$$\mathcal{E}(\theta) \leq \liminf_{n\to\infty} \mathcal{E}(\theta_n) \leq M. \tag{3.5}$$

Proof. (⊛) The antisymmetry of θ is a simple calculation. For any $e \in E$ we have

$$\begin{aligned}\theta(-e) &= \lim_{n\to\infty} \theta_n(-e) = \lim_{n\to\infty} -\theta_n(e)\\ &= -\lim_{n\to\infty} \theta_n(e) = -\theta(e).\end{aligned}$$

For proving (3.5), we first note that $\lim_{n\to\infty} \theta_n(e)^2 = \theta(e)^2$ for every $e \in E$ since $\theta_n(e) \to \theta(e)$. By definition we have

$$\mathcal{E}(\theta_n) = \sum_{e\in E_{1/2}} r(e)\theta_n(e)^2.$$

By Fatou's Lemma we have that

$$\begin{aligned}\liminf_{n\to\infty} \sum_{e\in E_{1/2}} r(e)\theta_n(e)^2 &\geq \sum_{e\in E_{1/2}} r(e) \liminf_{n\to\infty} \theta_n(e)^2\\ &= \sum_{e\in E_{1/2}} r(e)\theta(e)^2\\ &= \mathcal{E}(\theta).\end{aligned}$$

Since $\mathcal{E}(\theta_n) \leq M$, we can conclude

$$\mathcal{E}(\theta) \leq \liminf_{n\to\infty} \mathcal{E}(\theta_n) \leq M.$$

□

The following theorem provides an important criterion for transience of a random walk. It is due to T. Lyons (1983) in [Lyo83] and was adapted from a Theorem by Royden (1952) in [Roy52].

Theorem 3.59 (Royden's Criterion)**.** *Let G be an infinite electrical network. The induced random walk on G is transient if and only if there exists a unit flow on G of finite energy from some (or any) vertex to infinity.*

Proof. (⊝) Fix any vertex $a \in G$, which, without loss of generality, belongs to each G_n. Let i_n be the unit current flow in G_n^W from a to z_n and let v_n be the corresponding voltage. Note that each edge of G_n^W corresponds to an edge in G, although one endpoint may differ.

For "$\Leftarrow$" suppose θ is a unit flow on G from a to ∞ with finite energy. Then the restriction $\theta|_{G_n^W}$ is a unit flow from a to z_n, so by Thomson's Principle (Theorem 3.48) we have

$$E(i_n) \leq E(\theta|_{G_n^W}) \leq E(\theta) < \infty.$$

Thus we get $\lim E(i_n) < \infty$, implying that the random walk is transient.

For "$\Rightarrow$" suppose that G is transient. Due to Theorem 3.42 we have $R(a \leftrightarrow \infty) < \infty$. By (3.3), we can write this as

$$R(a \leftrightarrow \infty) = \lim_{n \to \infty} R(a \leftrightarrow z_n) = \lim_{n \to \infty} E(i_n) < \infty.$$

Thus an $M \in \mathbb{R}$ exists such that $E(i_n) < M$ for all $n \in \mathbb{N}$. Let $Y_n(x)$ denote the number of visits to x before hitting Z_n and let $Y(x)$ denote the total number of visits to x. Then

$$E(Y_1(x)) \leq E(Y_2(x)) \leq \ldots \leq E(Y(x)) < \infty$$

because of transience. By the Monotone Convergence Theorem, which can be found in [Rud87], and Proposition 3.52, we have

$$\begin{aligned} \mathbb{E}[Y(x)] &= \lim_{n \to \infty} \mathbb{E}[Y_n(x)] = \lim_{n \to \infty} \pi(x) v_n(x) \\ &= \pi(x) \lim_{n \to \infty} v_n(x) = \pi(x) v(x), \end{aligned}$$

where $v(x) \coloneqq \lim_{n \to \infty} v_n(x)$ for $x \in V$.

We have $v(x) < \infty$ because $\mathbb{E}[Y(x)] < \infty$. Now we set $i := c \cdot dv$ and obtain

$$\begin{aligned} i &= c \cdot dv = c \cdot \lim_{n \to \infty} dv_n \\ &= \lim_{n \to \infty} c \cdot dv_n = \lim_{n \to \infty} i_n. \end{aligned}$$

So $\lim_{n \to \infty} i_n$ exists and thus i is a unit current flow from a to infinity with energy at most M by Lemma 3.58. □

Remark 3.60. Another important concept is the "capacity" of a set. One can proof that transience is also equivalent to the existence of a vertex x such that capacity($\{x\}$) > 0. There are many other applications of this concept, see Chapter 2 of [Woe00].

Corollary 3.61. *The type of a random walk, that is, transient or recurrent, remains unchanged when conductances are adjusted by bounded factors.*

Proof. (⊛) Suppose we have a transient random walk. Let

$$a < \infty,\ c' \coloneqq c \cdot a$$

and let θ be a unit current flow from some vertex v to ∞ such that

$$\sum_{e \in E_{1/2}} \theta(e)^2 r(e) < \infty.$$

Such a flow exists due to Theorem 3.59. We have $\sum_{e \in E_{1/2}} \theta(e)^2 \cdot \frac{r(e)}{a} < \infty$, so θ is a flow of finite energy for c' as well. Scaling it to a unit current flow will leave the energy finite and thus the random walk induced by c' is transient as well.

Notice that we can apply the same argument to go from the transience of the c'-induced random walk to the transience of the c-induced random walk. Therefore, scaling the conductances by a in case of a non-transient random walk must result in a non-transient random walk. □

Theorem 3.62 (Pólya's Theorem – Second part). *Simple random walk on the nearest-neighbor graph of $\mathbb{Z}^d$ is transient for all $d \geq 3$.*

> A drunk man will find his way home, but a drunk bird may get lost forever.
>
> —Shizuo Kakutani

The idea of the proof is to use a certain technique that creates flows from random paths. This gives us that there are unit flows with finite energy, which means we have transience. This technique is the same as used in the proof of Pólya's theorem in [Lyo83], and it is also presented in [Mor54, page 173]. Since the additional theory needed for the proof is not relevant for this thesis, we cite [LP16] for a rigorous proof.

In the next chapter we will look at applications of the theory developed above. There are many other interesting questions, such as how to calculate the expected time it takes for a random walk to hit a certain set of vertices, but we will not deal with them in this thesis. Answers to them can be found in [LP16] or [CT98].

4 Random Walks on Cayley Graphs

In this chapter, we prove the main theorem of this thesis: the critical value, which distinguishes transience from recurrence, of a "decaying random walk" on a Cayley graph is identical to its *growth rate*. To prove that, we explore the concept of *admissible flows* and relate it to current flows and cutsets. Using this, we prove that the critical value of a tree equals its *branching* *number*. Rather than directly analyzing Cayley graphs, we look at certain subtrees and show that their branching number equals their growth rate. However, Rayleigh's Monotonicity Principle (Theorem 3.49) allows us then to "extend" the derived statements to the Cayley graph, which gives us the main theorem. Finally, we look at the "opposite case" of "increasing random walks" and analyze their behavior in terms of transience and recurrence. This chapter is primarily based on chapter 3 of [LP16], however, the "opposite case" part is entirely my own idea and contribution.

4.1 Admissible Flows

In this chapter, we introduce a very simple type of flows, called *admissible flows*. They are relevant because they help us relate the growth of networks or trees to transience and recurrence. Specifically, we later introduce the *branching number* of a tree, which essentially marks the point where "admissible flows stop existing". This branching number is closely tied to whether the network is transient or recurrent.

Definition 4.1. Given an electrical network, a flow θ is called *admissible* if it is bounded by the conductances c, that is, we have $\theta(e) \leq c(e)$ for every edge e.

Remark 4.2. The idea is that admissible flows are like water in pipes, where each pipe e allows only "$c(e)$-much" water to flow.

Lemma 4.3. *Given a flow from a to ∞ with finite energy on an electrical network, there exists a nonzero admissible flow (from a on ∞).*

Proof. Let i be the unit current flow from a to infinity (recall that i exists due to Remark 3.38) and let v be the corresponding voltage. Due to Proposition 2.6 we have $v(x) \leq v(a)$ for any vertex x since the flow starts at a. Thus, for all vertices x and y we have

$$|v(x) - v(y)| \leq v(a) \quad \text{since } v \geq 0.$$

By Remark 3.39 we have

$$|i(e)| = |c(e) \cdot (dv)(e)| \leq v(a)c(e) = R(a \leftrightarrow \infty)c(e) \quad \text{for all } e \in E.$$

So $i/v(a)$ is an admissible flow. □

4.2 Decaying Conductances and Branching Number

We now focus on particular networks where the conductances of an edge decrease exponentially with distance from the root. This decay mirrors a natural pattern, seen for example in resource flows, weakening connection strength, signal attenuation in networks, or water pressure in branching pipes.

Definition 4.4. Impose the following conductances on the edges of a tree:

$$c(e) := \lambda^{-|e|} \quad \text{for some } \lambda > 1.$$

By RW_λ we denote the random walk on T induced by c.

Definition 4.5. We define the *branching number* of a tree T as

$$\operatorname{br} T \coloneqq \sup\{\lambda \in \mathbb{R}_{>1} : \Theta_\lambda \neq \emptyset\},$$

where Θ_λ is the set of nonzero flows θ from the root to ∞ on T such that for every edge $e \in T$ we have $0 \leq \theta(e) \leq \lambda^{-|e|}$. In other words, the branching number is the largest $\lambda > 1$ for which there exists a nonzero admissible flow from the root to ∞ on the network T with conductances $c(e) = \lambda^{-|e|}$.

Definition 4.6. We say that RW_λ has *critical value* λ_c if $\lambda < \lambda_c$ implies that RW_λ is transient, and $\lambda > \lambda_c$ implies that RW_λ is recurrent. This value exists due to Rayleigh's Monotonicity Principle (Theorem 3.49).

Before we can demonstrate that for locally finite infinite trees, the critical value equals the branching number, we first need to introduce some definitions and establish two lemmas.

Definition 4.7. For a tree T with fixed root ℓ and any edge $e \in T$, we denote by $|e|$ the distance from e to ℓ, defined as the number of edges on the shortest path between e and ℓ.

Lemma 4.8. *Let T be a locally finite tree with root ℓ and let θ be a flow from ℓ to ∞. For the cutset $\Pi_n = \{e \in E : |e| = n\}$ we have*

$$\mathrm{Strength}(\theta) = \sum_{e \in \Pi_n} \theta(e).$$

Proof. ($\circledast$) By $e(x)$ denote the edge connecting a vertex x to its parent in T_{n-1}. By definition we have

$$\mathrm{Strength}(\theta) = \sum_{e \in \Pi_1} \theta(e).$$

Now consider the subtree T_x induced by $V \setminus \{\ell\}$ that is rooted at x for some vertex x with distance 1 from ℓ. Denote θ restricted to T_x as θ_x. By the same logic we obtain

$$\mathrm{Strength}(\theta_x) = \sum_{e \in \Pi_2 \text{ incident to } x} \theta(e).$$

Summing over these x we get

$$\sum_{\substack{x \in T \\ d(x,\ell)=1}} \mathrm{Strength}(\theta_x) = \sum_{\substack{x \in T \\ d(x,\ell)=1}} \Big(\sum_{\substack{e \in \Pi_2 \\ x \sim e}} \theta(e) \Big) = \sum_{e \in \Pi_2} \theta(e).$$

By Flow Conservation (Lemma 3.22) applied to the finite subnetwork induced by $\{v \in V : d(\ell, v) \leq 2\}$, we have that $\mathrm{Strength}(\theta_x) = \theta(e(x))$. Putting this together, we conclude

$$\sum_{e \in \Pi_2} \theta(e) = \sum_{\substack{x \in T \\ d(x,\ell)=1}} \mathrm{Strength}(\theta_x) = \sum_{e \in \Pi_1} \theta(e) = \mathrm{Strength}(\theta).$$

Repeating this argument n times gives the result. □

Proposition 4.9. *Let c be conductances on a locally finite infinite tree T and w_n be positive numbers with $\sum_{n \geq 1} w_n < \infty$. Every flow θ from 0 to ∞ on T satisfying $0 \leq \theta(e) \leq w_{|e|} c(e)$ for all edges e has finite energy.*

Proof. (⊝) We apply Lemma 4.8 to the cutset $\{e \in E : |e| = n\}$ and get

$$\begin{aligned} E(\theta) &= \sum_{e \in T} \theta(e)^2 r(e) = \sum_{n \geq 1} \sum_{|e|=n} \theta(e)^2 r(e) \\ &\leq \sum_{n \geq 1} w_n \cdot \sum_{|e|=n} \theta(e) = \Big(\sum_{n \geq 1} w_n \Big) \cdot \mathrm{Strength}(\theta) < \infty. \end{aligned}$$

□

We can now present a key theorem which is due to Lyons (1990) in [Lyo90].

Theorem 4.10. *If T is a locally finite infinite tree, then RW_λ has critical value $\mathrm{br}\, T$.*

Proof. (⊝) We will show that transience is equivalent to $\lambda < \mathrm{br}\, T$, which finishes the proof.

For "⇒" suppose transience. So there is a current flow from some vertex to infinity with finite energy and therefore, by Lemma 4.3, there exists a nonzero admissible flow. By definition of $\mathrm{br}\, T$ this gives us that $\lambda \leq \mathrm{br}\, T$.

For "⇐" suppose $\lambda < \mathrm{br}\, T$. Choose $\lambda' \in (\lambda, \mathrm{br}\, T)$ and set

$$w_n := \left(\frac{\lambda}{\lambda'} \right)^n.$$

By definition of $\mathrm{br}\, T$, there is a nonzero flow θ from 0 to ∞ satisfying

$$0 \leq \theta(e) \leq \lambda^{-|e|} \leq (\lambda')^{-|e|} = w_{|e|} \lambda^{-|e|}.$$

Note that $\sum_n w_n < \infty$ since $\lambda/\lambda' < 1$. Due to Proposition 4.9 we get that θ has finite energy, so we have transience. □

Remark 4.11. When $\lambda = \mathrm{br}\, T$, the random walk RW_λ can be either transient or recurrent. Both scenarios are possible, as demonstrated in [Lyo83].

4.3 Cayley Graphs

In this final chapter, we introduce *Cayley graphs* and construct particular subgraphs that are *subperiodic* trees. We continue by showing that this tree's growth rate exists and prove that it is identical to the tree's critical value. This result extends to the original Cayley graph. Lastly, we consider the "opposite case", where conductances increase with distance to the root, and prove that the induced random walk is transient.

Definition 4.12. Given a group Γ and a generating finite subset S, we form the associated *Cayley graph* G with vertices Γ and edges

$$(x, y) \in \Gamma^2 \quad \text{such that } x^{-1}y \in S \cup S^{-1}.$$

In other words:

$$E = \{(g, gs) : \; g \in \Gamma, \, s \in S\}.$$

Remark 4.13. In this chapter we will assume or define Γ as a group, S as a finite subset of Γ, and G as the associated Cayley graph.

Remark 4.14. ($\ominus$) Because S generates Γ, the graph is connected. Some authors do not require that S is a generating set. Note that left multiplication by yx^{-1} is an automorphism on Γ that carries x to y. In other words, Cayley graphs look the same from every vertex. We call this important fact *vertex-transitivity* and we will use it later. Here we used the group property that there is always an inverse element, this is one reason we cannot replace "group" by an algebraic structure like "monoid". Moreover, since we assume the generating set to be finite, we have that Cayley graphs are locally finite.

Figure 6 shows a simple example of a Cayley graph, concretely of the dihedral group D_4 on two generators a and b.

4.4 Subperiodic Subtree of a Cayley Graph

To prove the main theorem of this thesis, we focus on a specific subtree of the Cayley graph rather than the full graph. This subtree is chosen in a way that it is *subperiodic*, giving us that its branching number equals its *growth rate*. As established in Theorem 4.10, the critical value is equal to the branching number, hence it is identical to the growth rate in this context. This is almost the main theorem. Considering a subgraph is equivalent to setting certain edge conductances to zero. Therefore Rayleigh's Monotonicity Principle (Theorem 3.49) allows us to extend this result back to the full Cayley graph, which gives us the main theorem.

We start by constructing the subtree and giving more definitions.

Definition 4.15. We assume that the inverse of each generator is also in the generating set S. Let $x \in G$ be a node. Fix an order of the generating set $S = \{s_1, \ldots, s_n\}$. There are words $(s_{i_1}, \ldots, s_{i_m})$ such that $\prod_{j=1}^{m} s_{i_j} = x$. We

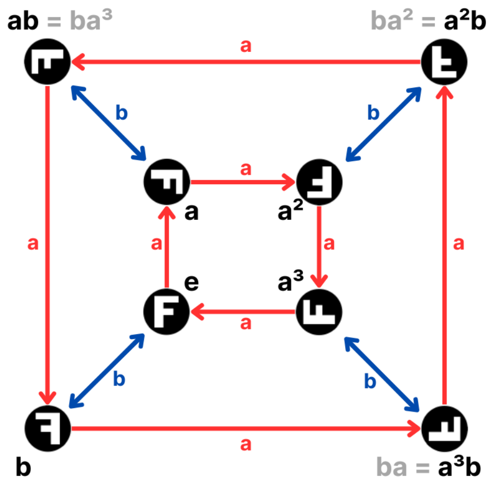

Figure 6: (⊛) Cayley graph of the dihedral group $D_4 = \langle a, b \mid a^4 = b^2 = 1, ab = ba^{-1} \rangle$ with generating set $S = \{a, b\}$. Here a is a rotation by 90 degrees and b is a reflection.

define the *minimal word* $w_x = (s_{j_1}, \ldots, s_{j_p})$ as the lexicographically minimal word among these words, that is, if $(s_{k_1}, \ldots, s_{k_q})$ is another such word and r is the first index i such that $s_{j_i} \neq s_{k_i}$, then $s_{j_r} < s_{k_r}$. Note that the minimal word of the identity e is the empty word.

Definition 4.16. We denote by T_G the subgraph of G with vertex set Γ where two vertices x and y are adjacent if $|x| + 1 = |y|$ and w_x is an initial segment of w_y or $|y| + 1 = |x|$ and w_y is an initial segment of w_x.

Definition 4.17. In a tree T with root ℓ, we write $x \leq y$ if x is on the shortest path from ℓ to y and $x < y$ if $x \leq y$ and $x \neq y$. We write $x \to y$ if $x \leq y$ and $|y| = |x| + 1$. We write T^x for the subtree of T containing the vertices $y \geq x$.

Definition 4.18. Let $N \geq 0$. An infinite tree T is called N*-subperiodic* if for all vertices x there exists an adjacency-preserving injection $f : T^x \to T^{f(x)}$ with $|f(x)| \leq N$. A tree is *subperiodic* if there is some N for which it is N-subperiodic.

Next we will prove that this tree is countable and subperiodic.

Lemma 4.19. *(⊛) The subgraph T_G is countable.*

Proof. Since Γ is generated by a finite subset S, every element of Γ can be written as a finite product of elements from S, hence Γ is countable. By definition, T_G contains all vertices of G, and its edges connect vertices $x, y \in G$ if $|x| + 1 = |y|$ and w_x is an initial segment of w_y, or vice versa. The edge set of T_G is thus a subset of the Cartesian product $\Gamma \times \Gamma$, which is also countable since Γ is countable. □

Lemma 4.20. *(⊛) Let $g \in \Gamma$ be a vertex. If $w_g = (s_{j_1}, \ldots, s_{j_m})$ is the minimal word of g, then*

$$(s_{j_1}), (s_{j_1}, s_{j_2}), \ldots, (s_{j_1}, \ldots, s_{j_{m-1}})$$

are minimal words too.

Proof. Fix an arbitrary index k such that $1 \leq k \leq m-1$ and set $x := \prod_{i=1}^{k} s_{j_k}$. We will show that $w_x = (s_{j_1}, \ldots, s_{j_k})$.

For contradiction, suppose we have $w_x = (s_{i_1}, \ldots, s_{i_n})$ for some indices $i_1, \ldots, i_n$, where for the first p, such that $s_{i_p} \neq s_{j_p}$, we have $s_{i_p} < s_{j_p}$. Then:

$$\begin{aligned} g = \prod_{i=1}^{m} s_{j_i} &= \prod_{i=1}^{k} s_{j_i} \cdot \prod_{i=k+1}^{m} s_{j_i} \\ &= x \cdot \prod_{i=k+1}^{m} s_{j_i} = \prod_{z=1}^{n} s_{i_z} \cdot \prod_{i=k+1}^{m} s_{j_i}. \end{aligned}$$

But $(s_{i_1}, \ldots, s_{i_n}, s_{j_{k+1}}, \ldots, s_{j_m})$ is smaller than $(s_{j_1}, \ldots, s_{j_m})$ which contradicts the assumption that w_g is minimal. □

Proposition 4.21. *The subgraph T_G is a subperiodic tree rooted at identity.*

Proof. (⊛) First, we prove that T_G is a tree. It is clear that the root of T_G is the identity element ℓ, which has the minimal word $w_\ell = \emptyset$. We note that for two vertices $x \sim y$ we have $|x| + 1 = |y|$. So for a path of edges the endpoint distance to e strictly increases for every step, thus a cycle cannot exist.

Moreover, suppose $w_x = (s_{j_1}, \ldots, s_{j_m})$ is the minimal word of an arbitrary vertex x. Then clearly $\emptyset, (s_{j_1}), (s_{j_1}, s_{j_2}), \ldots, (s_{j_1}, \ldots, s_{j_m})$ is a path connecting x to ℓ. It is a valid path because the words in it are indeed minimal due to Lemma 4.20. We can do the same for an arbitrary vertex y, so $x \sim \ell$ and $y \sim \ell$, and therefore $x \sim y$. In other words, T_G is connected and thereby a tree rooted a ℓ.

Next, we show that T_G is subperiodic, concretely, 0-subperiodic. Fix a vertex $g \in T_G$ and let $w_g = (t_1, \ldots, t_m)$ for some $m \in \mathbb{N}$. Consider the subtree T^g rooted at g. We will show that $f : T^g \to T$ with $f(x) := g^{-1}x$ is an adjacency-preserving injection with $|f(g)| \leq 0$. It is clear that this is an injection because it is a bijection. It is also clear that $|f(g)| \leq 0$ since $f(g) = \ell$. We only need to show that it preserves adjacency.

Let $x \sim y$ for $x, y \in T^g$. There are $s_1, \ldots, s_{n+1}$ such that

$$\Big(\prod_{i=1}^{m} t_i\Big) \cdot \Big(\prod_{i=1}^{n} s_i\Big) = x \text{ and } \Big(\prod_{i=1}^{m} t_i\Big) \cdot \Big(\prod_{i=1}^{n+1} s_i\Big) = y.$$

We claim that their minimal words in T are of exactly that form:

$$w_x = (t_1, \ldots, t_m, s_1, \ldots, s_n) \text{ and } w_y = (t_1, \ldots, t_m, s_1, \ldots, s_{n+1}).$$

In other words, we claim that the minimal words "start with" $(t_1, \ldots, t_m)$. Suppose not, so let $w_x = (u_1, \ldots, u_k)$ for some index k with

$$(u_1, \ldots, u_k) < (t_1, \ldots, t_m, s_1, \ldots, s_n).$$

We get that

$$g = x \cdot (s_1 \cdot \ldots \cdot s_n)^{-1} = \Big(\prod_{i=1}^{k} u_i\Big) \cdot (s_1 \cdot \ldots \cdot s_n)^{-1} = \Big(\prod_{i=1}^{k} u_i\Big) \cdot s_n^{-1} \cdot \ldots \cdot s_1^{-1}.$$

But $(u_1, \ldots, u_k, s_n^{-1}, \ldots, s_1^{-1}) < (t_1, \ldots, t_m)$, which contradicts the assumption. Therefore we have

$$w_x = (t_1, \ldots, t_m, s_1, \ldots, s_n) \text{ and } w_y = (t_1, \ldots, t_m, s_1, \ldots, s_{n+1})$$

and claim that after applying f to both terms we get minimal words in T; so we need to show that $(s_1, \ldots, s_n)$ and $(s_1, \ldots, s_{n+1})$ are minimal. It suffices, by Lemma 4.20, to show that $(s_1, \ldots, s_{n+1})$ is minimal. Suppose not, so let

$$(v_1, \ldots, v_p) < (s_1, \ldots, s_{n+1}) \text{ and } \prod_{i=1}^{p} v_i = \prod_{i=1}^{n+1} s_i.$$

Then we get

$$\prod_{1=1}^{m} t_i \cdot \prod_{i=1}^{p} v_i = \prod_{1=1}^{m} t_i \cdot \prod_{i=1}^{n+1} s_i = x,$$

but

$$(t_1, \ldots, t_m, v_1, \ldots, v_p) < (t_1, \ldots, t_m, s_1, \ldots, s_{n+1}),$$

which contradicts the assumption. This finishes the proof because $(s_1, \ldots, s_n)$ and $(s_1, \ldots, s_{n+1})$ have a length difference of 1 and one is the initial segment of the other. □

Figures 7 and 8 show two examples of the subtree of a Cayley graph.

4.5 Growth Rate and Random Walk

Definition 4.22. Let $x \in T$ be a vertex. The *level* n of T is defined as $T_n := \{x \in T : |x| = n\}$ and set $M_n := \#T_n$. The *growth rate* of T, if it exists, is defined as

$$\operatorname{gr} T := \lim_{n \to \infty} M_n^{1/n}.$$

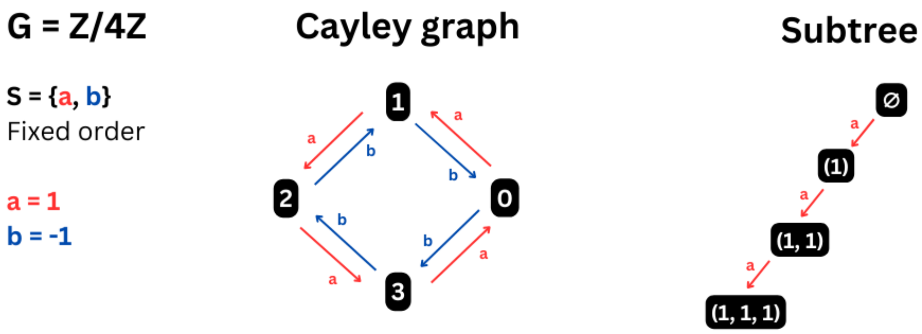

Figure 7: (⊛) Cayley graph and subtree of $\mathbb{Z}_4$ on $S = \{\pm 1\}$

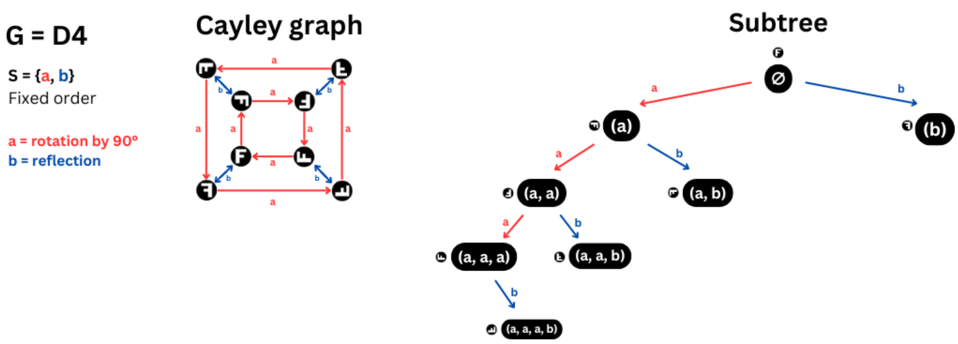

Figure 8: (⊛) Cayley graph and subtree of $\mathbb{D}_4$ on $S = \{a, b\}$

Theorem 4.23 (Subperiodicity and Branching Number)**.** *Let T be a subperiodic infinite tree. Then* $\operatorname{gr} T$ *exists and* $\operatorname{br} T = \operatorname{gr} T$.

This theorem is due to Furstenberg [Fur67] and a proof can be found in Theorem 3.8 of [LP16]. Before we can state the main theorem, we need two more lemmas.

Lemma 4.24 (Fekete's Lemma)**.** *For every subadditive sequence $(a_n)_{n\in\mathbb{N}}$ we have*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}.$$

A proof can be found in [Gos24].

Lemma 4.25. *For the subtree T_G of a Cayley graph, we have:*

$$M_{m+n} \leq M_m M_n \quad \text{and} \quad \lim_{n\to\infty} M_n^{1/n} \text{ exists.}$$

Proof. (⊖) For the first part, consider a vertex of distance $m+n$ to the identity. The shortest path to it can be divided into two paths, one of length m and one of length n. So we get $M_{m+n} \leq M_m \cdot d$, where d is an upper bound of how many vertices can reached from a vertex x that is of distance m to the identity. But due to vertex-transitivity such a bound is given by M_n. This shows the first part of the lemma. Next we conclude

$$\log M_{m+n} \leq \log M_m + \log M_n.$$

Therefore we can apply Fekete's Lemma (Lemma 4.24) to $(\log M_n)_n$, giving us that $\lim M_n^{1/n}$ exists. Note that we can indeed make use of Fekete's Lemma here as all M_n are finite since G is a locally finite graph. □

Finally we can prove the main theorem of this thesis.

Theorem 4.26 (Group Growth and Random Walk)**.** *The random walk RW_λ on an infinite Cayley graph G has critical value λ_c equal to the growth rate of the graph:*

$$\lambda_c = \lim_{n\to\infty} M_n^{1/n}.$$

Proof. (⊖) Note that T_G is a spanning tree, that is, it includes every vertex $x \in \Gamma$ and the distances to the identity are the same as in G. Therefore we have $\operatorname{gr} T_G = \lim_{n\to\infty} M_n^{1/n}$. But since T_G is subperiodic, we know by Theorem 4.23 that

$$\operatorname{br} T_G = \operatorname{gr} T_G = \lim_{n\to\infty} M_n^{1/n}.$$

By Theorem 4.10 we have that RW_λ is transient on T if $\lambda < \lim_{n\to\infty} M_n^{1/n}$, and then on G as well. If, however, $\lambda > \lim_{n\to\infty} M_n^{1/n}$, we conclude recurrence by the Nash–Williams Criterion (Theorem 3.55) as follows. Define the cutsets $\Pi_n = \{e \in E : |e| = n\}$. We have

$$\begin{aligned}\sum_{n\in\mathbb{N}}(\sum_{e\in\Pi_n} c(e))^{-1} &= \sum_{n\in\mathbb{N}}(\frac{1}{M_n \cdot \lambda^{-n}}) \\ &= \sum_{n\in\mathbb{N}} \frac{\lambda^n}{M_n} \\ &= \sum_{n\in\mathbb{N}}(\frac{\lambda}{M_n^{1/n}}).\end{aligned}$$

The right-hand side is infinite since $\lambda > \lim_{n\to\infty} M_n^{1/n}$. □

Lastly we want to look at the much simpler "opposite case" where the conductances grow exponentially with distance to the root.

Definition 4.27. (⊛) For $c(e) := \lambda^{|e|}$, $\lambda > 1$, we denote the induced random walk by RW_λ^g.

Theorem 4.28. *(⊛) The random walk RW_λ^g on a locally finite infinite tree is transient.*

Proof. We know that there exists a unit flow θ from the root to infinity; see Remark 3.38. The resistance on edge e is given by $r(e) = \lambda^{-|e|}$. Furthermore, the absolute value of the current $|i(e)|$ on any edge is bounded by 1. Thus we get

$$E(\theta) = \sum_{e\in E_{1/2}} \theta(e)^2 r(e) \leq \sum_{e\in E} r(e) = \sum_{e\in E} \lambda^{-|e|}.$$

We know that λ^{-k} decays exponentially as $\lambda > 1$. Also recall that the tree is locally finite. Thus:

$$\sum_{e\in E} \lambda^{-|e|} = \sum_{k=0}^{\infty} \#\{e : |e| = k\} \cdot \lambda^{-k} < \infty.$$

So the flow has finite energy and by Theorem 3.59 the random walk is transient. □

Corollary 4.29. *(⊛) The random walk RW_λ^g on an infinite Cayley graph is transient.*

Proof. We apply Theorem 4.28 to T_G of the Cayley graph and get that G is transient too. □

References

[CT98] F. R. K. Chung and Prasad Tetali. "Isoperimetric inequalities for Cartesian products of graphs". In: *Combin. Probab. Comput.* 7.2 (1998), pp. 141–148. ISSN: 0963-5483,1469-2163. DOI: 10.1017/S0963548397003350.

[DS84] Peter G. Doyle and J. Laurie Snell. *Random walks and electric networks*. Vol. 22. Carus Mathematical Monographs. Mathematical Association of America, Washington, DC, 1984, pp. xiv+159. ISBN: 0-88385-024-9.

[Fur67] Harry Furstenberg. "Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation". In: *Math. Systems Theory* 1 (1967), pp. 1–49. ISSN: 0025-5661. DOI: 10.1007/BF01692494. URL: https://doi.org/10.1007/BF01692494.

[Gos24] Angshuman R. Goswami. *Revisiting Fekete's Lemma, Subadditive and Periodic Sequences*. 2024. arXiv: 2410.21437 [math.CA]. URL: https://arxiv.org/abs/2410.21437.

[LP16] Russell Lyons and Yuval Peres. *Probability on Trees and Networks*. Vol. 42. Cambridge Series in Statistical and Probabilistic Mathematics. Available at https://rdlyons.pages.iu.edu/. Cambridge University Press, New York, 2016, pp. xv+699. ISBN: 978-1-107-16015-6. DOI: 10.1017/9781316672815. URL: http://dx.doi.org/10.1017/9781316672815.

[LPW09] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times*. With a chapter by James G. Propp and David B. Wilson. American Mathematical Society, Providence, RI, 2009, pp. xviii+371. ISBN: 978-0-8218-4739-8. DOI: 10.1090/mbk/058. URL: https://doi.org/10.1090/mbk/058.

[Lyo83] Terry Lyons. "A simple criterion for transience of a reversible Markov chain". In: *Ann. Probab.* 11.2 (1983), pp. 393–402. ISSN: 0091-1798,2168-894X.

[Lyo90] Russell Lyons. "Random walks and percolation on trees". In: *Ann. Probab.* 18.3 (1990), pp. 931–958. ISSN: 0091-1798,2168-894X.

[Mor54] Akira Mori. "A note on unramified abelian covering surfaces of a closed Riemann surface". In: *J. Math. Soc. Japan* 6 (1954), pp. 162–176. ISSN: 0025-5645,1881-1167. DOI: 10.2969/jmsj/00620162. URL: https://doi.org/10.2969/jmsj/00620162.

[Nor98] J. R. Norris. *Markov chains*. Vol. 2. Cambridge Series in Statistical and Probabilistic Mathematics. Reprint of 1997 original. Cambridge University Press, Cambridge, 1998, pp. xvi+237. ISBN: 0-521-48181-3.

[Pól21] Georg Pólya. “Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz”. In: *Math. Ann.* 84.1-2 (1921), pp. 149–160. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01458701. URL: https://doi.org/10.1007/BF01458701.

[Roy52] H. L. Royden. “Harmonic functions on open Riemann surfaces”. In: *Trans. Amer. Math. Soc.* 73 (1952), pp. 40–94. ISSN: 0002-9947,1088-6850. DOI: 10.2307/1990823. URL: https://doi.org/10.2307/1990823.

[Rud87] Walter Rudin. *Real and complex analysis*. Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416. ISBN: 0-07-054234-1.

[Woe00] Wolfgang Woess. *Random walks on infinite graphs and groups*. Vol. 138. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2000, pp. xii+334. ISBN: 0-521-55292-3. DOI: 10.1017/CBO9780511470967. URL: https://doi.org/10.1017/CBO9780511470967.