Least-Squares Finite-Element Evaluation of Flow Nets

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Abstract: A novel least-squares implementation of the finite-element method is presented to evaluate stream functions in the solution of field problems. The method is programmatically similar to the solution of the Laplace equation, and is based on the development of a stream field that is orthogonal to an already calculated potential field. The main advantage of the method comes from the fact that it eliminates the need of identification of boundary conditions for the stream functions. Implementation of this method requires that the Laplace equation be solved first to calculate the nodal potentials. The Laplace equation, with an identity conductivity matrix is then solved again to calculate the nodal values of the stream functions. One arbitrary boundary condition is sufficient for the second solution. Examples of cofferdam and curtained dam flow with isotropic as well as orthotropic soil conductivity are presented to demonstrate the method.

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Introduction

Flow nets are important in the interpretation process of field problems in engineering. Civil engineers find flow nets an invaluable tool in understanding fluid flow through porous media. Whereas calculations of the hydraulic potential using the finite-element method are the result of simple solutions of the Laplace equation, evaluating and drawing the stream functions is often a harder problem. This is because in many cases the identification of boundary conditions for the stream functions can be quite complex. The wide and multidisciplinary interest in this problem has produced numerous related publications (Christian 1980; Aalto 1984; Tracy and Radhakrishnan 1989; Fan et al. 1992). The approaches that have been suggested can be divided into two general categories: (a) those that solve the Laplace equation of the field (or stream) lines ψ, with the appropriate boundary conditions (Christian 1980; Aalto 1984; Tracy and Radhakrishnan 1989); and (b) those that develop the field lines using graphical approaches (Fan et al. 1992). The methods of the first group are easier to program, as they are governed by identical mathematics to the basic problem of solving for the potential (Zienkiewicz and Taylor 1991). Tracy and Radhakrishnan (1989) in a very interesting study, present a case-by-case analysis to demonstrate how to evaluate boundary conditions for the stream functions. Similarly, Griffiths (1994) examines seepage underneath unsymmetric cofferdams. This is a problem that introduces stream function boundary conditions that are not trivial to evaluate. Griffiths presents a process by which the boundary stream functions and the flow-dividing stream line are evaluated. Christian (1987) presents a process by which once the head solution is found, the boundary can be traversed to sum the flow values. The graphical approach (Fan et al. 1992), on the other hand, is free of the burden of identifying boundary conditions for the field functions, but requires the development of complex graphical algorithms and extensive programing.

The purpose of this paper is to present an alternative approach, which is programmatically similar to solving the Laplace equation, while at the same time excluding the difficulties of having to identify stream function boundary conditions.

Basic Equations

The potential ϕ satisfies the following equation:

$$K_x \frac{\partial^2 \phi}{\partial X^2} + K_y \frac{\partial^2 \phi}{\partial Y^2} = 0 \tag{1}$$

Finite-element solution in terms of potential ϕ using Eq. (1) requires the introduction of interpolation functions for the potential and its gradient:

$$\phi = \mathbf{N}\hat{\phi} \quad \nabla\phi = \mathbf{B}\hat{\phi} \tag{2}$$

where $\hat{\phi}$ = array of nodal values of the interpolated potential and $\mathbf{N}$ = typical interpolation array

$$\nabla\phi = \left[\frac{\partial\phi}{\partial x} \quad \frac{\partial\phi}{\partial y}\right]^T$$

where $\nabla\phi$ = gradient of ϕ, and $\mathbf{B}$ is defined as

$$\left[\frac{\partial N}{\partial x} \quad \frac{\partial N}{\partial y}\right]^T$$

All these are standard processes of the finite-element procedures and are not discussed further.

In the following it is assumed that the potential ϕ has been solved over the entire domain of the problem using the finite-element method.

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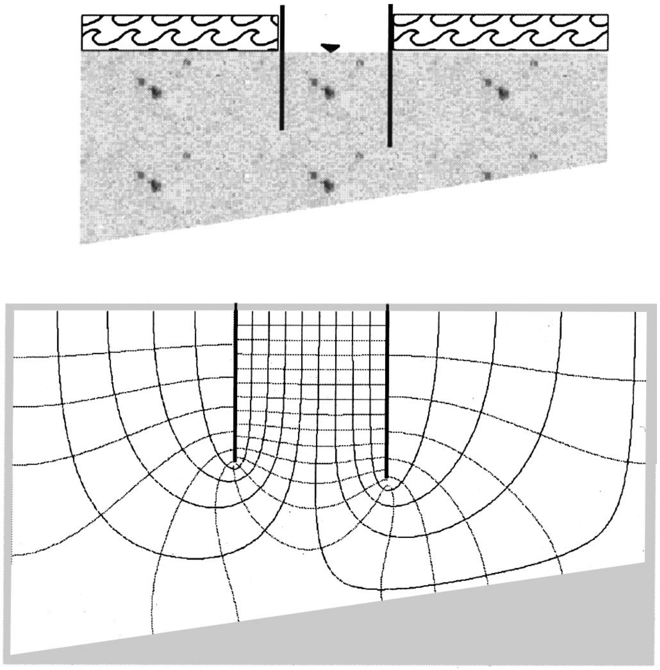

Fig. 1. Flow net of asymmetric cofferdam

The definition of the stream or field lines ψ is as follows:

$$\frac{\partial\psi}{\partial x}=K_y\frac{\partial\phi}{\partial y}\quad\frac{\partial\psi}{\partial y}=-K_x\frac{\partial\phi}{\partial x}\tag{3}$$

Note from the above that in the case of isotropic media ($K_x=K_y$) the orthogonality of ϕ and ψ is automatically satisfied:

$$\nabla\phi^T\nabla\psi=0\tag{4}$$

Eq. (4) can be rewritten due to Eq. (3) as follows:

$$\begin{gathered}\mathbf{B}_1\hat{\psi}=K_y\mathbf{B}_2\hat{\phi}\\ \mathbf{B}_2\hat{\psi}=-K_x\mathbf{B}_1\hat{\phi}\end{gathered}\tag{5}$$

where $\mathbf{B}_1$ = first line of matrix $\mathbf{B}$, leave; in $\mathbf{B}_2$ = second line of matrix $\mathbf{B}$; and $\hat{\phi}$ and $\hat{\psi}$ = arrays of nodal values of ϕ and ψ in the usual finite-element interpolation process. Here, it is assumed that the stream functions ψ are interpolated within an element using the interpolations functions for ϕ. *This assumption is not essential nor is it necessary although it was found convenient for the presentation of this paper*.

The expressions in Eq. (5) are not satisfied exactly due to the errors introduced by the interpolation of the values of ϕ and ψ. This is an inherent problem in all but the simplest cases of finite-element solutions. Eq. (5) may be rewritten in a more precise and concise manner as follows:

$$\mathbf{B}\hat{\psi}-\mathbf{K}\overline{\mathbf{B}}\hat{\phi}=E\tag{6}$$

where

$$\overline{\mathbf{B}}=\begin{bmatrix}\mathbf{B}_2\\-\mathbf{B}_1\end{bmatrix}\quad\mathbf{K}=\begin{bmatrix}K_y&0\\0&K_x\end{bmatrix}$$

= conductivity matrix, and E = error that has been generated due to the interpolation process.

Evaluation of the error E over a region requires that a positive measure of E be used to eliminate counteracting effects. Thus, the expression $E^2=E^{\mathrm{T}}E$ is preferred

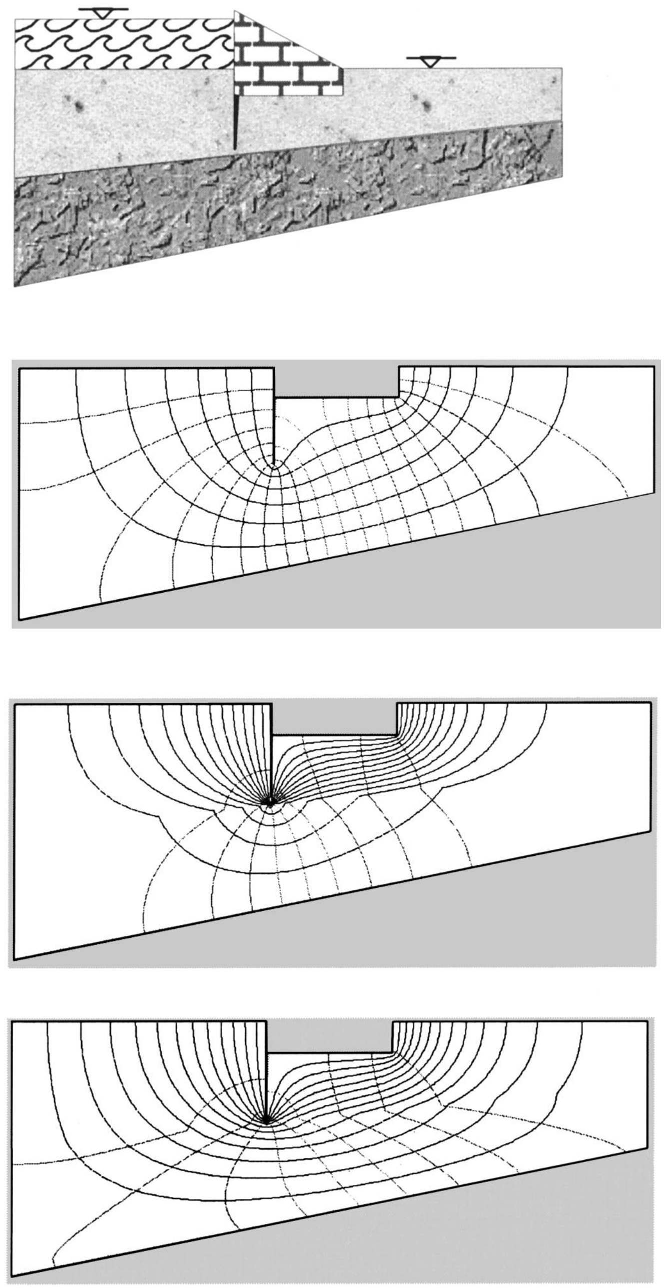

Fig. 2. Seepage around a curtained dam

$$\begin{aligned}E^2&=(\hat{\psi}^T\mathbf{B}^T-\hat{\phi}^T\overline{\mathbf{B}}^T\mathbf{K})(\mathbf{B}\hat{\psi}-\mathbf{K}\overline{B}\hat{\phi})=\hat{\psi}^T\mathbf{B}^T\mathbf{B}\hat{\psi}-2\hat{\psi}^T\mathbf{B}^T\mathbf{K}\overline{\mathbf{B}}\hat{\phi}\\&\quad+\hat{\phi}^T\overline{\mathbf{B}}^T\mathbf{K}\mathbf{K}\overline{B}\hat{\phi}\end{aligned}\tag{7}$$

The above expression of error is point-wise. It can be integrated over the entire domain of the problem to give

$$\begin{aligned}\overline{E}&=\hat{\psi}^T\sum_N\int\mathbf{B}^T\mathbf{B}\,dV\,\hat{\psi}-2\hat{\psi}^T\sum_N\int\mathbf{B}^T\mathbf{K}\overline{\mathbf{B}}\,dV\,\hat{\phi}\\&\quad+\hat{\phi}^T\sum_N\int\overline{\mathbf{B}}^T\mathbf{K}\mathbf{K}\overline{\mathbf{B}}\,dV\,\hat{\phi}\end{aligned}\tag{8}$$

where the integrals = element-wise, and Σ_N = usual ''assembly'' operator that adds the element matrices and vectors into the proper global positions.

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The nodal values $\hat{\phi}$ are known, having been calculated using a standard finite-element solution as stated earlier. Thus, the error $\overline{E^2}$ is only a function of the nodal values $\hat{\psi}$. Minimization of the error thus requires that the derivative of $\overline{E^2}$ with respect to every nodal value of $\hat{\psi}$ be equal to zero

$$\frac{\partial \overline{E^2}}{\partial \hat{\psi}} = 0 \quad \therefore \quad \sum_N \int \mathbf{B}^T\mathbf{B}\, dV\, \hat{\psi} - \sum_N \int \mathbf{B}^T\mathbf{K}\overline{\mathbf{B}}\, dV\, \hat{\phi} = 0 \quad (9)$$

Eq. (9) is the finite-element equation of the system that produces the nodal values for ψ. The first part of Eq. (9) is identical to the stiffness matrix of the regular Laplace equation using the identity permeability matrix. The second part of Eq. (9) provides a simple calculation for the right-hand side of the system of equations to be solved. The most important feature of Eq. (9), apart from its programing simplicity, is the fact that it does not require knowledge of the boundary conditions for ψ. One arbitrary boundary condition, however, is required to produce a unique solution. The author commonly uses $\hat{\psi}_1 = 1$. Any one value on any one node is acceptable as it produces a constant shift to the nodal values of $\hat{\psi}$ with no consequence to the flow value contours.

Application of Method

Two examples are presented to demonstrate the effectiveness of this method. Both solutions have been produced by finite-element meshes that are very refined (using approximately 4,000 bilinear quadrilateral elements). The longest computer run using a Pentium III 1 GHz W2K laptop PC was approximately 3 s. Thus, whereas reasonable results may also be obtained with less-refined meshes, the speed of solution renders coarser meshes irrelevant. The meshes are not presented here for economy of space.

In the first example, consider the asymmetric cofferdam of Fig. 1(a). In this problem there are three boundary stream lines: Two associated to the sheet piles, and a third associated to the impervious bottom boundary. The flow net that results from the use of the present method is shown in Fig. 1(b), illustrating a successful solution that does not require the knowledge of the boundary stream function values.

The second example is related to the underground flow around the base of a dam with a curtain (Fig. 2). The problem is solved for three different soil configurations. In the first case, the entire soil is homogeneous and isotropic. In the second case the soil consists of two layers with isotropic properties. The top layer is ten times more permeable than the bottom layer. Finally, the third case represents a two-layered soil where the top layer is isotropic while the bottom layer is orthotropic. The coefficient of permeability of the top layer is 10^{-5} cm/s, while the bottom layer has horizontal permeability equal to 0.5×10^{-5} cm/s and vertical permeability equal to 10^{-6} cm/s.

Conclusions

A novel least-squares implementation of the finite-element method is presented to develop complete flow nets for the general field problem. The significance of the method is that it provides the stream function solution without requiring knowledge of the related field boundary conditions. The method is capable of handling orthotropic as well as inhomogeneous materials, and can be applied to an existing code with small modifications as this is dictated by Eq. (9).

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