Finite Elements - IOC5107
Final Report Homework 4

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1. Introduction

This report extends the finite-element study of Poisson's equation in two dimensions by building a complete FEM solver that reads an unstructured triangular mesh, assembles and solves the variational problem using both Constant Strain Triangle (CST) and Linear Serendipity Triangle (LST) elements, and rigorously verifies its accuracy. The solver begins by parsing node coordinates, element connectivities and physical tags from the mesh file, thereby capturing the spatially varying element size h directly as defined by the mesh generator. Using inner-product and Sobolev-space theory, the Poisson problem with Dirichlet boundary conditions is recast into its weak form and discretized via the Galerkin method. Element-by-element stiffness matrices are computed by numerical integration of the gradients of the shape functions, then assembled into the global system, to which boundary values are applied and the resulting linear system is solved.

To verify correctness and quantify convergence, the Method of Manufactured Solutions (MMS) is employed. An analytic u_{MMS} is prescribed, its Laplacian yields a source term $f = -\Delta u_{\mathrm{MMS}}$, and Dirichlet boundary data $g = u_{\mathrm{MMS}}|_{\partial\Omega}$ are enforced. The solver is run on a sequence of uniformly refined and non-uniformly refined meshes; the discrete solution u_h is compared to u_{MMS} in the L^2 and H^1 norms, and log–log plots of error versus h are used to extract observed convergence rates, which should match the theoretical orders ($O(h)$ and $O(h^2)$ in H^1, and one order higher in L^2 for CST and LST respectively).

This assignment thus integrates mesh handling, variational theory, element assembly, boundary condition enforcement, solver implementation, and systematic code verification, providing a thorough hands-on experience in modern finite-element analysis.

2. Theoretical Background

This section gathers the mathematical foundations and numerical techniques used to formulate, discretize, and verify the solution of the Poisson problem via the Finite Element Method (FEM).

2.1. Inner Product Spaces and Norms

Inner product spaces provide the abstract setting for variational formulations and error analysis. An inner product space is a real vector space V endowed with a bilinear form

$$(u,v) = \langle u,v \rangle, \tag{1}$$

which is symmetric, linear in its first argument, and positive-definite. It induces the norm

$$\|v\| = \sqrt{\langle v,v \rangle}, \tag{2}$$

and satisfies the Cauchy–Schwarz inequality $|\langle u,v \rangle| \leq \|u\|\|v\|$. Completeness under this norm yields a Hilbert space structure, essential for Lax–Milgram arguments **?**.

2.2. Sobolev Spaces H^1 and H_0^1

Sobolev spaces extend inner product concepts to functions with weak derivatives. The space

$$H^1(\Omega) = \{ v \in L^2(\Omega) \mid \nabla v \in [L^2(\Omega)]^d \} \tag{3}$$

carries the norm

$$\|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2. \tag{4}$$

Imposing homogeneous Dirichlet data leads to

$$H_0^1(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1}}, \tag{5}$$

the natural trial space for PDEs with zero boundary conditions **?**.

2.3. Hilbert Spaces

A Hilbert space is an inner product space that is complete with respect to the norm induced by its inner product. Completeness means that every Cauchy sequence $\{v_n\} \subset V$ satisfies

$$\lim_{m,n\to\infty} \|v_n - v_m\| = 0 \quad \Longrightarrow \quad \exists v \in V : \lim_{n\to\infty} \|v_n - v\| = 0. \tag{6}$$

Key examples include $L^2(\Omega)$ with inner product

$$(u,v)_{L^2} = \int_\Omega u\, v\, dx, \tag{7}$$

and the Sobolev space $H^1(\Omega)$ itself. The Riesz representation theorem in a Hilbert space H states that every continuous linear functional $F : H \to \mathbb{R}$ can be written uniquely as

$$F(v) = (u_F, v)_H \quad \text{for some } u_F \in H. \tag{8}$$

This result underpins the variational theory of PDEs, since the mapping $v \mapsto \ell(v)$ in the weak formulation can be identified with an element of $H_0^1(\Omega)$, guaranteeing existence and uniqueness of solutions. **?**

2.4. Poisson Problem in 2D

The Poisson equation models steady-state diffusion or potential fields. In two dimensions:

$$-\Delta u = f \quad \text{in } \Omega, \tag{9}$$

subject to boundary conditions on $\partial\Omega$. Here, $\Delta u = \partial_{xx}u + \partial_{yy}u$ and f is a source term. Dirichlet conditions $u = g$ or Neumann conditions $\partial_n u = h$ prescribe values on $\partial\Omega$ **???**.

2.5. Weak (Variational) Formulation

Rewriting the boundary-value problem in Sobolev spaces allows FEM discretization. For $u - g \in H_0^1(\Omega)$, multiply by test $v \in H_0^1(\Omega)$ and integrate by parts:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx - \int_{\partial\Omega} h v \, ds. \tag{10}$$

Defining

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \ell(v) = \int_\Omega f v \, dx - \int_{\partial\Omega} h v \, ds, \tag{11}$$

the problem becomes: find $u \in H^1(\Omega)$ such that

$$a(u,v) = \ell(v) \quad \forall v \in H_0^1(\Omega). \tag{12}$$

Continuity and coercivity of $a(\cdot,\cdot)$ guarantee a unique solution via Lax–Milgram **?**.

2.6. Galerkin Method

The Galerkin method projects the infinite-dimensional weak problem onto a finite subspace $V_h \subset H_0^1(\Omega)$. One seeks $u_h \in V_h$ such that

$$a(u_h, v_h) = \ell(v_h) \quad \forall v_h \in V_h. \tag{13}$$

This ensures *Galerkin orthogonality*

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h, \tag{14}$$

which is fundamental to derive error estimates **?**.

2.7. Finite Element Spaces: CST and LST

Finite element spaces consist of piecewise-defined basis functions over a mesh.

2.7.1. CST Elements (Constant Strain Triangle)

CST uses three linear shape functions per triangle T, each associated with a vertex and satisfying $\varphi_i(x_j) = \delta_{ij}$. The stiffness matrix entry is

$$K_{ij}^e = \int_T \nabla\varphi_i \cdot \nabla\varphi_j \, dx, \tag{15}$$

with constant gradients on T, yielding a simple, first-order accurate scheme **?**.

2.7.2. LST Elements (Linear Serendipity Triangle)

LST augments CST by adding three mid-edge nodes, producing six quadratic shape functions per triangle. The same formula

$$K_{ij}^e = \int_T \nabla \varphi_i \cdot \nabla \varphi_j \, dx \tag{16}$$

applies, but gradients vary within T, giving second-order convergence at increased computational cost **?**.

2.8. Manufactured Solution Method

The MMS provides a systematic code verification test. One selects an analytic u_{MMS}, then computes

$$f = -\Delta u_{\text{MMS}}, \quad g = u_{\text{MMS}}\big|_{\partial\Omega}. \tag{17}$$

Solving the FEM system with these data and comparing u_h to u_{MMS} in various norms reveals implementation errors **?**.

2.9. Convergence Study

Error analysis predicts for CST

$$\|u - u_h\|_{H^1} = O(h), \quad \|u - u_h\|_{L^2} = O(h^2), \tag{18}$$

and for LST

$$\|u - u_h\|_{H^1} = O(h^2), \quad \|u - u_h\|_{L^2} = O(h^3). \tag{19}$$

Numerical experiments on successive mesh refinements, plotted as $\log(\|e\|)$ vs. $\log(h)$, confirm these rates **?**.

3. Results