Chapter 3: Rigid-Body Motions

Describe mathematically the motion of a rigid body moving in a 3D physical space.

1. Step: 3×3 matrix describing the body's (frame) orientation: rotation matrix

A rotation matrix is parameterized by 3 independent coordinates

The exponential coordinate representation is a natural way to represent a rotation matrix:

Given: R: rotation Matrix
$\hat{\omega} \in \mathbb{R}^3$: unit vector, rotation axis
$\theta \in [0, \pi]$: rotation angle
$\rightsquigarrow$ rotating identity frame I about $\hat{\omega}$ by θ $= R$

The exponential coordinates are defined as
$$\omega = \hat{\omega}\theta \in \mathbb{R}^3 \qquad \text{(3-parameter representation)}$$

Exponential description of rigid-body motions (screw theory)
- twist $\in \mathbb{R}^6$: linear and angular velocities (spacial velocities)
- wrench $\in \mathbb{R}^6$: forces (3d) and moments (3d) (spacial forces)

The configuration of a frame (w.r.t. a fixed frame) is represented as a 4×4 Matrix (Homogenous Transformation Matrix).
This is an implicit representation of the C-Space. 10 constraints are applied to the 16 dimensional space of the 4×4 Matrix.

A rigid body velocity can be represented as a point in $\mathbb{R}^6$: 3 angular velocities and 3 linear velocities.
This point in $\mathbb{R}^6$ (six-vector) is called twist or spacial velocity.

A rigid body moments (torques) and forces can also be represented as a point in $\mathbb{R}^6$.
This point (six-vector) in $\mathbb{R}^6$ is called wrench or spacial force.

Any rigid-body config. can be achieved by starting from the fixed frame and integrating a constant twist for a given time: screw motion along (translation) and around (rotation) a screw-axis.
Screw motion ⇒ six-parameter Representation: exponential coordinates

Exponential coordinates parameters:
- direction of the screw axis
- scalar: how far to follow the screw motion

A twist can also be represented as a combination of angular and linear velocity: screw axis $S = (\omega_z, v_x, v_y)$ where $\omega_z = 1$
and scale it by rotation speed $\dot{\theta}$
The twist is $\underbrace{V = S \cdot \dot{\theta}}_{\text{also a set of exponential coordinates}}$

Alternative way to represent a twist

Exponential Coordinates are one 3-parameter representation of rotation: axis and rotation about that axis.

Other representations of rotation:

- Euler angles
- Roll-pitch-yaw
- Cayley-Rodrigues parameters
- quaternions
- ...

3.2. Rotations and Angular Velocities

Rotation Matrices

The columns of a Rotation Matrix R correspond to the body-frame unit axes $\{\hat{x}_b, \hat{y}_b, \hat{z}_b\}$

$$R = [x_b \; y_b \; z_b] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

of the 9 entries (r_{ij}) in the Rotation Matrix only 3 can be chosen independently. ⇒ 6 constraints

a) $\hat{x}_b, \hat{y}_b, \hat{z}_b$ are unit vectors

$$r_{11}^2 + r_{21}^2 + r_{31}^2 = 1$$
$$r_{12}^2 + r_{22}^2 + r_{32}^2 = 1$$
$$r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$$

$$\hat{=} \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ \underbrace{r_{31}}_{\hat{=}1} & \underbrace{r_{32}}_{\hat{=}1} & \underbrace{r_{33}}_{\hat{=}1} \end{bmatrix}$$

b) $\hat{x}_b \cdot \hat{y}_b = \hat{x}_b \cdot \hat{z}_b = \hat{y}_b \cdot \hat{z}_b = 0$

they are all orthogonal to each other

$$r_{11} \cdot r_{12} + r_{21} r_{22} + r_{31} \cdot r_{32} = 0$$
$$r_{12} \cdot r_{13} + r_{22} \cdot r_{23} + r_{32} \cdot r_{33} = 0$$
$$r_{11} \cdot r_{13} + r_{21} \cdot r_{23} + r_{31} \cdot r_{33} = 0$$

a) and b) can be combined in 1 constraint

$$R^T \cdot R = I$$

it needs to be ~~fo~~ accounted for right-handed frames only:

$$\det R = 1$$

$R^T \cdot R = I$ implies that $\det R = \pm 1$.

$\det R = 1$ means that only right-handed frames are allowed.

Special orthogonal group $SO(n)$: the group of all Rotation Matrices: $R^T R = I$ and $\det R = 1$

not commutative $R_1 R_2 \neq R_2 R_1$

Properties of Rotation Matrices

- Matrix Multiplication $A \cdot B$ — $A \in SO(n)$, $B \in SO(n)$, $C \in SO(n)$
- closure $A \cdot B \in SO(n)$
- associativity: $(AB)C = A(BC)$
- identity element I: $A \cdot I = I \cdot A = A$
- inverse element A^{-1}: $A \cdot A^{-1} = I = A^{-1} \cdot A$
  - ↳ the inverse is equivalent to the transpose: $A^{-1} = A^T$

Uses of Rotation Matrices

a) represent an orientation (R represents a frame)

b) change reference frame (of vector or frame): R is an operator

c) rotate a vector or frame: R is an operator

Representing an orientation (a)

R_c refers to the orientation of frame $\{c\}$ relative to the stationary frame $\{s\}$

$$\boxed{R_c = \underbrace{R_{sc}}_{\{c\}\text{ relative to }\{s\}}}$$

Rules: $R_{de} = R_{ed}^{-1} = R_{ed}^T$

$R_{de} \cdot R_{ed} = I$

Changering the referece frame (b)

$$\boxed{R_{ac} = \underbrace{R_{ab}}_{} \cdot R_{bc} \qquad R_{ab}\text{: chance refeece frame from } \{b\} \text{ to } \{a\} \text{ (operator)}}$$

$R_{a\cancel{b}} \cdot R_{\cancel{b}c} = R_{ac}$: subscript cancellation rule

Rotating a vector or frame (c)

Rotating a frame (or vector) about a unit (vector) axis $\hat{\omega}$ by amount (angle) θ can be represented as a rotation matrix:

$R = R_{sc'}$: rotation that takes $\{s\}$ to $\{c'\}$

For $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$

$$Rot(\hat{\omega}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\omega_3(1-c_\theta) + \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}$$

where $s_\theta = \sin(\theta)$
$c_\theta = \cos(\theta)$

Note: $Rot(\hat{\omega}, \theta) = Rot(-\hat{\omega}, -\theta)$

Rotation axis $\hat{\omega}$ reference frame:

- $\hat{\omega}$ is represented in $\{s\}$ frame: $R_{sb'} = R R_{sb}$ (premultiply R)
- $\hat{\omega}$ is represented in $\{b\}$ frame: $R_{sb''} = R_{sb} \cdot R$ (postmultiply R)

$R = Rot(\hat{\omega}, \theta)$

Rotate a vector (v):

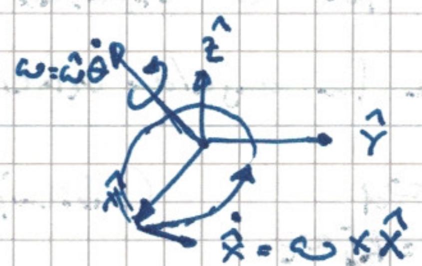
Only one frame involved (the one v is represented in).
Therefore $\hat{\omega}$ is also represented in the same frame as v)

$v_s' = R v_s$

3.2.2 Angular Velocities

$$\omega_s = \hat{\omega}_s \dot{\theta}$$

ω_s: angular velocity
$\hat{\omega}_s$: rotation (unit) axis
$\dot{\theta}$: rate of rotation



with $R(t) = R_{sb}(t) = \begin{array}{ccc} & & \overbrace{\quad}^{r_3(t) \hat{=} \hat{x}} \\ \left[\begin{matrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{matrix}\right. & & \left.\right] \\ \underbrace{\quad}_{r_1(t) \hat{=} \hat{x}} & \underbrace{\quad}_{r_2(t) \hat{=} \hat{x}} & \end{array}$

$$\left.\begin{aligned} \dot{\hat{x}} &= \dot{r}_1 = \omega_s \times r_1 \\ \dot{\hat{y}} &= \dot{r}_2 = \omega_s \times r_2 \\ \dot{\hat{z}} &= \dot{r}_3 = \omega_s \times r_3 \end{aligned}\right\} \Rightarrow \dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3]$$

$$= \omega_s \times R$$

with $R = R_{sb}$

Skew-symmetric Matrix

To eliminate the crossproduct in $\dot{R} = \omega_s \times R$ we define:

Given $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

The set of skew-symmetric matrices is called Lie algebra $so(3)$ of the Lie group $SO(3)$

properties:

$$[x] = -[x]^T$$
$$R[\omega]R^T = [R\omega]$$

Angular velocity

$\dot{R} = \omega_s \times R$ can be written with $so(3)$ (skew-symmetric matrix)

$\hookrightarrow \dot{R} = [\omega_s] \cdot R$

$R = R_{sb}$

$R^{-1} = R^T$

$$\boxed{\begin{aligned} [\omega_s] &= \dot{R} R^{-1} \\ [\omega_b] &= R^{-1} \cdot \dot{R} \end{aligned}}$$

for ω expressed in ~~body~~ fixed-frame

for ω expressed in body-frame

change of reference frame for angular velocity

$\omega_c = R_{cd}\, \omega_d$ (subscript cancellation rule)

$\omega_d = R_{cd}^{-1} \omega_c = R_{cd}^T \omega_c$

Pre- or Postmultiplying R by R^{-1} results in the <u>velocity</u> vector represented as a <u>skew-symmetric</u> matrix.

Exponential Coordinate Representation of Rotation

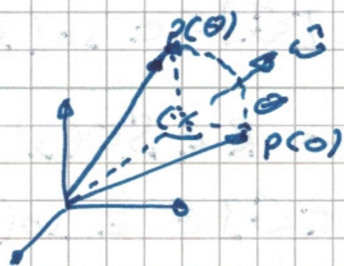
Exponential coordinates parametize a Rotation Matrix in terms of

- Rotation axis $\hat{\omega}$
- Angle of rotation θ (about $\hat{\omega}$)

↳ The vector $\hat{\omega}\theta \in \mathbb{R}^3$ is the 3-parameter exp. coordin. representation of a rotation

Matrix Exponential of Rotations

Axis-angle representation: $\hat{\omega}$ and θ individually



$\dot{p} = \hat{\omega} \times p$

↳ $\dot{p} = [\hat{\omega}]\, p$

↳ $p(t) = e^{[\hat{\omega}]t} p(0)$

$$\Rightarrow \quad \boxed{Rot(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I \cdot \sin(\theta)[\hat{\omega}] + (1-\cos(\theta))[\hat{\omega}]^2 \in SO(3)}$$

this equation is also known as Roriques' formula.

Rotating a vector:

$e^{[\hat{\omega}]\theta} \cdot p$: Rotating $p \in \mathbb{R}^3$ about $\hat{\omega}_s$ by θ
$\hat{=}$ fixed frame axis

Rotating a frame

$R' = e^{[\hat{\omega}]\theta} \cdot R = Rot(\hat{\omega}, \theta) \cdot R$: Rotating R about $\hat{\omega}$ $\hat{\omega}_s$ in fixed frame

$R'' = R e^{[\hat{\omega}]\theta} = R \cdot Rot(\hat{\omega}, \theta)$: Rotating R about $\hat{\omega}$ in body frame $\hat{=} \hat{\omega}_b$

Matrix Logarithm of Rotations

- $\hat{\omega}\theta \in \mathbb{R}^3$ represents exponential coordinates of rot. matrix R
- $[\hat{\omega}\theta] = \underbrace{[\hat{\omega}]}_{\in so(3)}\theta$ is matrix logarithm

Matrix exponential "integrates" an angular velocity (in matrix form: $[\hat{\omega}]\theta \in so(3)$) to give an orientation ($R \in SO(3)$).

Matrix logarithm "differentiates" an $R \in SO(3)$ to find a constant angular velocity $[\hat{\omega}]\theta \in so(3)$.

$$\boxed{\begin{aligned} \exp &: [\hat{\omega}]\theta \in so(3) \to R \in SO(3) \\ \log &: R \in SO(3) \to [\hat{\omega}]\theta \in so(3) \end{aligned}}$$

Algorithm:

Given $R \in SO(3)$ find $\theta \in [0, \pi]$ and $\hat{\omega} \in \mathbb{R}$ ($\|\hat{\omega}\| = 1$) such that $e^{[\hat{\omega}]\theta} = R$.

a) if $R = I$ then $\theta = 0$ $\hat{\omega}$ is undefined

b) if $\operatorname{tr} R = -1$ then $\theta = \pi$

ω is any of the following (choose feasible solution)

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

if $\hat{\omega}$ is a solution then $-\hat{\omega}$ is also a solution

c) otherwise $\theta = \cos^{-1}\left(\frac{1}{2}(\operatorname{tr} R - 1)\right) \in [0, \pi)$

and $[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T)$

Ch. 3.3 Rigid Body Motions and Twists

Rigid body configurations and velocities are analogous to those for rotation and angular velocities:

homogenous transformation matrix	T	$\hat{=}$	rotation matrix	R
Screw Axis	$\mathcal{S}$	$\hat{=}$	rotation axis	$\hat{\omega}$
Twist	$\mathcal{V} = \mathcal{S}\dot{\theta}$	$\hat{=}$	angular velocity	$\omega = \hat{\omega} \cdot \dot{\theta}$
exponential coordinates for rigid body motions	$\mathcal{S}\theta \in \mathbb{R}^6$	$\hat{=}$	exponential coordinates for rotations	$\hat{\omega}\theta \in \mathbb{R}^3$

Homogenous Transformation Matrices

Combine rotation matrix $R \in SO(3)$ to represent orientation of $\{b\}$ in $\{s\}$

and vector $p \in \mathbb{R}^3$ to represent origin of $\{b\}$ in $\{s\}$

$SE(3)$: Special Euclidian group (for 3d)

$$\boxed{T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3)}$$

where $R \in SO(3)$
$p \in \mathbb{R}^3$

SE(2): Special Euclidian group (for 2d)

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & p_1 \\ \sin\Theta & -\cos\Theta & p_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$T \in SE(2)$

$R \in SO(2)$

$p \in \mathbb{R}^2$

$\Theta \in [0, 2\pi)$

Properties of Transformation Matrices

not commutative $T_1 T_2 \neq T_2 T_1$

- Identity I is a trivial transformation matrix

  $I \in SE(3)$

- Inverse:

  $$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

  $T \in SE(3) \quad T^{-1} \in SE(3)$

- Product (closure)

  $T = T_1 \cdot T_2$

  $T \in SE(3) \quad T_1 \in SE(3) \quad T_2 \in SE(3)$

- Associative multiplication (but not commutative)

  $(T_1 T_2) T_3 = T_1 (T_2 \cdot T_3)$

  generally $T_1 T_2 \neq T_2 \cdot T_1$

- Preserving distances and angles

  $$T \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

  $x \in \mathbb{R}^3$

  $(R, p) \hat{=} T \in SE(3)$

  a) preserving distances:

  $\|Tx - Ty\| = \|x - y\|$

  $y \in \mathbb{R}^3$

  $\|\cdot\|$: standard Euclidian Norm ($\|x\| = \sqrt{x^T x}$)

  b) preserves angles:

  $\langle Tx - Tz, Ty - Tz \rangle = \langle x - z, x - y \rangle$

  $\langle \cdot, \cdot \rangle$: standard Euclidian inner product $\langle x, y \rangle = x^T y$

Uses of Transformation Matrices

3 major uses for a transformation matrix T
(like for rotation matrices)

a) represent configuration (position and orientation) of a rigid body

T as operator
$\left\{ \begin{array}{l} \text{b) change reference frame of a vector or a frame} \\ \text{c) displace a vector or a frame} \end{array} \right.$

a) Representing a Configuration:

$$T_{sb} = (R_{sb}, p_{sb})$$

$T \in SE(3)$
$R \in SO(3)$
$p \in \mathbb{R}^3$

T_{sb} is the configuration of frame $\{b\}$ expressed in $\{s\}$ coordinates.

Any frame can be expressed relative to any other frame:
$T_{bc} = (R_{bc}, p_{bc})$ represents $\{c\}$ relative to $\{b\}$

For any two frames $\{d\}$ and $\{e\}$:

$$T_{de} = T_{ed}^{-1}$$

b) Changing reference frame of a vector or a frame

Subscript cancellation rule (analogous to rotations)

$$T_{ab} T_{bc} = T_{a\cancel{b}} T_{\cancel{b}c} = T_{ac}$$

for any frame $\{a\}, \{b\}, \{c\}$

$$T_{ab} v_b = T_{a\cancel{b}} \cdot v_{\cancel{b}} = v_a$$

v_b: v expressed in $\{b\}$
v_a: v expressed in $\{a\}$

c) Displacing (rotating and translating) a vector or a frame

$$T = (R, p) = (\mathrm{Rot}(\hat{\omega}, \theta), p)$$

T can act as operator on a frame T_{sb} by rotating by θ about $\hat{\omega}$ and translating by p.

extend 3x3 rotation $R = \mathrm{Rot}(\hat{\omega}, \theta)$ to 4x4 matrix

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

and similarly

$$\mathrm{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with: $p = \vec{p}\,\|p\|$: translation along unit vector p with distance $\|p\|$

$\hat{\omega}$ axis and p can be inerpeted in fixed- or in body-frame

- ==fixed frame== $\{s\}$ (pre-multiplication by T):

$$T_{sb}' = T T_{sb} = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega}, \theta) \cdot T_{sb}$$
$$= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R R_{sb} & R p_{sb} + p \\ 0 & 1 \end{bmatrix}$$

this can be interpeted as
1. rotating $\{b\}$ frame by θ about $\hat{\omega}$ in $\{s\}$ frame
2. translating by p (in $\{s\}$ frame)

- ==body frame== $\{b\}$ (post-multiplication by T):

$$T_{sb}'' = T_{sb} T = T_{sb} \cdot \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega}, \theta)$$
$$= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb} R & R_{sb} \cdot p + p_{sb} \\ 0 & 1 \end{bmatrix}$$

this can be interpreted as
1. translating $\{b\}$ by p (in the $\{b\}$ frame)
2. rotating about $\hat{\omega}$ in the new body frame $\{b'\}$

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3.3.2 Twists

Linear and angular velocity of a frame.

{s}: fixed (space) frame
{b}: moving body frame

Configuration of {b} as seen from {s}:

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

Angular velocity expressed in body frame ω_b

$$T^{-1}\dot{T} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}$$

$\dot p$ is the linear velocity of the origin of {b} expressed in {s} coordinates. $R^T\dot p = v_b$ is this linear velocity expressed in {b} coordinates

$$\to \; T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

Lie Algebra of the Lie Group SE(3)

where $[\mathcal{V}_b]$ is the skew-symmetric representation of the body twist (or spacial velocity in the body frame)

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6$$

Angular velocity expressed in the fixed frame ω_s

$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{R}R^T & \dot{p} - \dot{R}R^T p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}$$

$v_s = \dot p - \dot R R^T p$ is not the linear velocity of {b} expressed in {s}

$$\to \; \dot{T} \cdot T^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3)$$

where $[\mathcal{V}_s]$ is the skew-symmetric representation of the spacial twist $\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$

Symmetry between $\mathcal{V}_s = (\omega_s, v_s)$ and $\mathcal{V}_b = (\omega_b, v_b)$

a) ω_b : angular velocity expressed in $\{b\}$
ω_s : angular velocity expressed in $\{s\}$

b) v_b : linear velocity of a point at origin of $\{b\}$ expressed in $\{b\}$
v_s : linear velocity of a point at origin of $\{s\}$ expressed in $\{s\}$

Relationships between $\mathcal{V}_b$ and $\mathcal{V}_s$:

$$[\mathcal{V}_b] = T^{-1}\dot{T}$$
$$= T^{-1}[\mathcal{V}_s]T$$

$$[\mathcal{V}_s] = \dot{T}T^{-1}$$
$$= T[\mathcal{V}_b]T^{-1}$$

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Adjoint representation:

Given $T = (R, p) \in SE(3)$

its adjoint representation $[Ad_T]$ is:

$$[Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times6}$$

For any $\mathcal{V} \in \mathbb{R}$: adjoint map associated with T

$$\mathcal{V}' = [Ad_T]\mathcal{V}$$

this can be written

$$\mathcal{V}' = Ad_T(\mathcal{V})$$

Matrix representation $[\mathcal{V}] \in se(3)$ of $\mathcal{V} \in \mathbb{R}^6$

$$[\mathcal{V}'] = T[\mathcal{V}]T^{-1}$$

Properties of adjoint map

Given: · $T_1, T_2 \in SE(3)$

· $\mathcal{V} = (\omega, v)$

Then $Ad_{T_1}(Ad_{T_2}(\mathcal{V})) = Ad_{T_1T_2}(\mathcal{V})$

or $[Ad_{T_1}][Ad_{T_2}]\mathcal{V} = [Ad_{T_1T_2}]\mathcal{V}$

For any $T \in SE(3)$

$$[Ad_T]^{-1} = [Ad_{T^{-1}}]$$

Summary on Twists

Given: . fixed (space) frame $\{s\}$
. body frame $\{b\}$
. $T_{sb} \in SE(3)$ and T_{sb} is differentiable
where $T_{sb}(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$

then: body twists

$$T_{sb}^{-1} \cdot \dot{T}_{sb} = [V_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3)$$

Spacial twist

$$\dot{T}_{sb} T_{sb}^{-1} = [V_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3)$$

Relation between V_s and V_b:

$$V_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [Ad_{T_{sb}}] V_b$$

$$V_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^T & 0 \\ -R^T[p] & R^T \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [Ad_{T_{bs}}] V_s$$

General for any 2 frames $\{c\}$ and $\{d\}$:
. a twist represented as V_c in $\{c\}$ is related to its representation V_d in $\{d\}$

$$V_c = [Ad_{T_{cd}}] V_d$$

$$V_d = [Ad_{T_{dc}}] V_c$$

subscript cancellation rule

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Screw Interpretation of a Twist

Analogous to angular velocity:

angular velocity : $\omega \hat{=} \hat{\omega}\dot{\theta}$

twist : $\mathcal{V} \hat{=} \mathcal{S}\dot{\theta}$

$\hat{\omega}$: unit rot axis
$\dot{\theta}$: rate of rotation (velocity)
$\mathcal{S}$: screw axis
$\dot{\theta}$: rate of rotation (velocity)

Screw axis: · rotating about axis while also
· translating along axis

$\mathcal{S}$ can be represented as : $\mathcal{S} = \{q, \hat{s}, h\}$

where: $q \in \mathbb{R}^3$: any point on the axis
$\hat{s}$: unit vector in direction of the axis
h: screw pitch: ratio of linear velocity to angular velocity ($\dot{\theta}$)

$\mathcal{V} = (\omega, v)$

$$\Rightarrow \quad \mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix} = \mathcal{S}\dot{\theta}$$

Screw axis $\mathcal{S}$ using a normalized version of a twist $\mathcal{V} = (\omega, v)$:

a) if $\omega \neq 0$: $\mathcal{S} = \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$

angular velocity about screw axis:
$\dot{\theta} = \|\omega\|$ such that $\mathcal{S}\dot{\theta} = \mathcal{V}$

b) if $\omega = 0$: $\mathcal{S} = \mathcal{V}/\|v\| = (0, v/\|v\|)$

Screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized to linear velocity vector

linear velocity is along screw axis is
$\dot{\theta} = \|v\|$ such that $\mathcal{S}\dot{\theta} = \mathcal{V}$

Definition of Screw Axis S for a given reference frame

$$S = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

alternative Notation:
$S = (S_v, S_\omega) = \begin{bmatrix} S_v \\ S_\omega \end{bmatrix}$

where either:

i) $\|\omega\| = 1$ then $v = -\omega \times q + h\omega$

$\dot{\theta}$: lin speed

q: a point on the axis

h: pitch is finite

or

ii) $\omega = 0$ and $\|v\| = 1$ then pitch (h) is infinite and the twist is pure translation along the axis

$\dot{\theta}$: rot. speed

The screw axis S is a normalized twist:

$$S = (\omega, v)$$

$$[S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}, \quad [\omega] = \begin{bmatrix} 0 & -\omega_3 & +\omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$[S]$: matrix representation of S

$[\omega] \in so(3)$

Relation of a screw axis represented in different frames $\{a\}$ and $\{b\}$:

$$S_a = [Ad_{T_{ab}}] S_b$$

$$S_b = [Ad_{T_{ba}}] S_a$$

Exponential Coordinates Representation of Rigid-Body Motions 7.9.22

Every rigid-body displacement can be expressed as a displacement along a fixed screw axis S is space.

↳ Chasles - Mozzi Theorem

⇒ 6 dimensional exponential coordinates of a homogeneous transformation T as $S\theta \in \mathbb{R}^6$

- S: Screw axis
- θ: distance to be traveled along S

to take I to T

Matrix exponential: $\exp: [S]\theta \in se(3) \to T \in SE(3)$

Matrix logarithm : $\log: T \in SE(3) \to [S]\theta \in se(3)$

⇒ Given Screw axis $S = (\omega, v)$:

- if $\|\omega\| = 1$ then for any $\theta \in \mathbb{R}$ (distance along axis)

$$e^{[S]\theta} = \begin{bmatrix} \overset{\text{rasw}}{e^{[\omega]\theta}} & (I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v \\ 0 & 1 \end{bmatrix}$$

- if $\omega = 0$ and $\|v\| = 1$ then $= S_v \theta$

$$e^{[S]\theta} \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$$

- Every rotation matrix can be represented as the matrix exponential of some skew-symmetric matrix.
- The matrix exponential map transforms skew-symmetric matrices into orthogonal matrices

Matrix Logarithm of Rigid-Body Motions

Given a $T = (R, p) \in SE(3)$
find a screw axis $S = (\omega, v)$
and a scalar θ such that

$$e^{[S]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

i.e. the matrix $[S]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix} \in se(3)$
is the matrix logarithm of $T = (R, p)$

Algorithm:

Given $T = (R, p) \in SE(3)$
find a $\theta \in [0, \pi)$ and
screw axis $S = (\omega, v) \in \mathbb{R}^6$ (where at least one of $\|\omega\|$ and $\|v\|$ is 1)
such that $e^{[S]\theta} = T$

- $S\theta \in \mathbb{R}^6$: exponential coordinates of T
- $[S]\theta \in se(3)$: matrix logarithm of T

a) if $R = I$ then:
- $\omega = 0$
- $v = p / \|p\|$
- $\theta = \|p\|$

b) otherwise :
- matrix logarithm on $SO(3)$ (rotation matrix) to determine ω ($[\omega]$) and θ for R
- then :

$$v = G^{-1}(\theta)\, p$$

where

$$G^{-1}(\theta) = \frac{1}{\theta} I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2$$

Wrenches

10.9.22

A linear force f acting on a rigid body at point r creates a torque or moment:

$$m_a = r_a \times f_a$$

The linear force and the twist can be combined to a spacial force or wrench:

$$F_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \mathbb{R}^6$$

Transform a wrench between coordinate frames:

$$F_b = Ad^T_{T_{ab}}(F_a) = \left[Ad_{T_{ab}}\right]^T F_a$$

$$F_a = Ad^T_{T_{ba}}(F_b) = \left[Ad_{T_{ba}}\right]^T F_b$$