Linear quadratic optimal control example

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This example works through the linear quadratic finite time optimal control problem. We assume that we have a linear system of the form

$$\dot{x} = Ax + Bu$$

and that we want to minimize a cost function of the form

$$\int_0^T (x^T Q_x x + u^T Q_u u) dt + x^T P_1 x.$$

We show how to compute the solution the the Riccati ODE and use this to obtain an optimal (time-varying) linear controller.

In [1]:
```
import numpy as np
import scipy as sp
import matplotlib.pyplot as plt
import control as ct
import control.optimal as opt
import time
```

System dynamics

We use the linearized dynamics of the vehicle steering problem as our linear system. This is mainly for convenient (since we have some intuition about it).

In [2]:
```
# Use the linearized dynamics of the vehicle control problem
# (you can find kincar.py on the course website)
from kincar import kincar, plot_lanechange

# Initial conditions
x0 = np.array([-40, -2., 0.])
u0 = np.array([10, 0])                # only used for linearization
Tf = 4

# Linearized dynamics
sys = kincar.linearize(x0, u0)
print(sys)
```

```
<LinearIOSystem>: sys[2]
Inputs (2): ['u[0]', 'u[1]']
Outputs (3): ['y[0]', 'y[1]', 'y[2]']
States (3): ['x[0]', 'x[1]', 'x[2]']

A = [[ 0.0000000e+00  0.0000000e+00 -5.0004445e-06]
     [ 0.0000000e+00  0.0000000e+00  1.0000000e+01]
     [ 0.0000000e+00  0.0000000e+00  0.0000000e+00]]

B = [[1.         0.        ]
     [0.         0.        ]
     [0.         3.33333333]]

C = [[1. 0. 0.]
     [0. 1. 0.]
     [0. 0. 1.]]

D = [[0. 0.]
     [0. 0.]
     [0. 0.]]
```

Optimal trajectory generation

We generate an trajectory for the system that minimizes the cost function above. Namely, starting from some initial function $x(0) = x_0$, we wish to bring the system toward the origin without using too much control effort.

In [3]:
```
# Define the cost function and the terminal cost
# (try changing these later to see what happens)
Qx = np.diag([1, 1, 1])       # state costs
Qu = np.diag([1, 1])          # input costs
Pf = np.diag([1, 1, 1])       # terminal costs
```

Finite time, linear quadratic optimization

The optimal solution satisfies the following equations, which follow from the maximum principle:

$$
\begin{aligned}
\dot{x} &= \left(\frac{\partial H}{\partial \lambda}\right)^T = Ax + Bu, &\qquad x(0) &= x_0, \\
-\dot{\lambda} &= \left(\frac{\partial H}{\partial x}\right)^T = Q_x x + A^T \lambda, &\qquad \lambda(T) &= P_1 x(T), \\
0 &= \left(\frac{\partial H}{\partial u}\right)^T = Q_u u + B^T \lambda.
\end{aligned}
$$

The last condition can be solved to obtain the optimal controller

$$u = -Q_u^{-1} B^T \lambda,$$

which can be substituted into the equations for the optimal solution.

Given the linear nature of the dynamics, we attempt to find a solution by setting $\lambda(t) = P(t)x(t)$ where $P(t) \in \mathbb{R}^{n\times n}$. Substituting this into the necessary condition, we obtain

$$\begin{aligned}\dot\lambda = \dot P x + P \dot x &= \dot P x + P(Ax - BQ_u^{-1}B^T P)x, \\ \implies \qquad -\dot P x - PAx + PBQ_u^{-1}BPx &= Q_x x + A^T P x.\end{aligned}$$

This equation is satisfied if we can find $P(t)$ such that

$$-\dot P = PA + A^T P - PBQ_u^{-1}B^T P + Q_x, \qquad P(T) = P_1.$$

To solve a final value problem with $P(T) = P_1$, we set the "initial" condition to P_1 and then invert time, so that we solve

$$\frac{dP}{d(-t)} = -\frac{dP}{dt} = -F(P), \qquad P(0) = P_1$$

Solving this equation from time $t = 0$ to time $t = T$ will give us an solution that goes from $P(T)$ to $P(0)$.

In [4]:
```
# Set up the Riccatti ODE
def Pdot_reverse(t, x):
    # Get the P matrix from the state by resizing
    P = np.reshape(x, (sys.nstates, sys.nstates))

    # Compute the right hand side of Riccati ODE
    Prhs = P @ sys.A + sys.A.T @ P + Qx - \
        P @ sys.B @ np.linalg.inv(Qu) @ sys.B.T @ P

    # Return P as a vector, *backwards* in time (no minus sign)
    return Prhs.reshape((-1))

# Solve the Riccati ODE (converting from matrix to vector and back)
P0 = np.reshape(Pf, (-1))
Psol = sp.integrate.solve_ivp(Pdot_reverse, (0, Tf), P0)
Pfwd = np.reshape(Psol.y, (sys.nstates, sys.nstates, -1))

# Reorder the solution in time
Prev = Pfwd[:, :, ::-1]
trev = Tf - Psol.t[::-1]

print("Trange = ", trev[0], "to", trev[-1])
print("P[Tf] =", Prev[:,:,-1])
print("P[0] =", Prev[:,:,0])

# Internal comparison: show that initial value is close to algebraic solutic
_, P_lqr, _ = ct.lqr(sys.A, sys.B, Qx, Qu)
print("P_lqr =", P_lqr)
```

```
Trange =  0.0 to 4.0
P[Tf] = [[1. 0. 0.]
 [0. 1. 0.]
 [0. 0. 1.]]
P[0] = [[ 1.00000000e+00  3.86208813e-07 -1.15917383e-07]
 [ 3.86208813e-07  2.64685426e-01  3.00060130e-01]
 [-1.15917383e-07  3.00060130e-01  7.93554820e-01]]
P_lqr = [[ 1.00000000e+00  3.86261437e-07 -1.15878431e-07]
 [ 3.86261437e-07  2.64575131e-01  3.00000000e-01]
 [-1.15878431e-07  3.00000000e-01  7.93725393e-01]]
```

For solving the x dynamics, we need a function to evaluate $P(t)$ at an arbitrary time (used by the integrator). We can do this with the SciPy `interp1d` function:

In [5]:
```
# Define an interpolation function for P
P = sp.interpolate.interp1d(trev, Prev)

print("P(0) =", P(0))
print("P(3.5) =", P(3.5))
print("P(4) =", P(4))
```

```
P(0) = [[ 1.00000000e+00  3.86208813e-07 -1.15917383e-07]
 [ 3.86208813e-07  2.64685426e-01  3.00060130e-01]
 [-1.15917383e-07  3.00060130e-01  7.93554820e-01]]
P(3.5) = [[ 1.00000000e+00  3.85128042e-07 -1.19928513e-07]
 [ 3.85128042e-07  2.73611300e-01  3.19711737e-01]
 [-1.19928513e-07  3.19711737e-01  8.38939788e-01]]
P(4) = [[1. 0. 0.]
 [0. 1. 0.]
 [0. 0. 1.]]
```

We now solve the $\dot{x}$ equations *forward* in time, using $P(t)$:

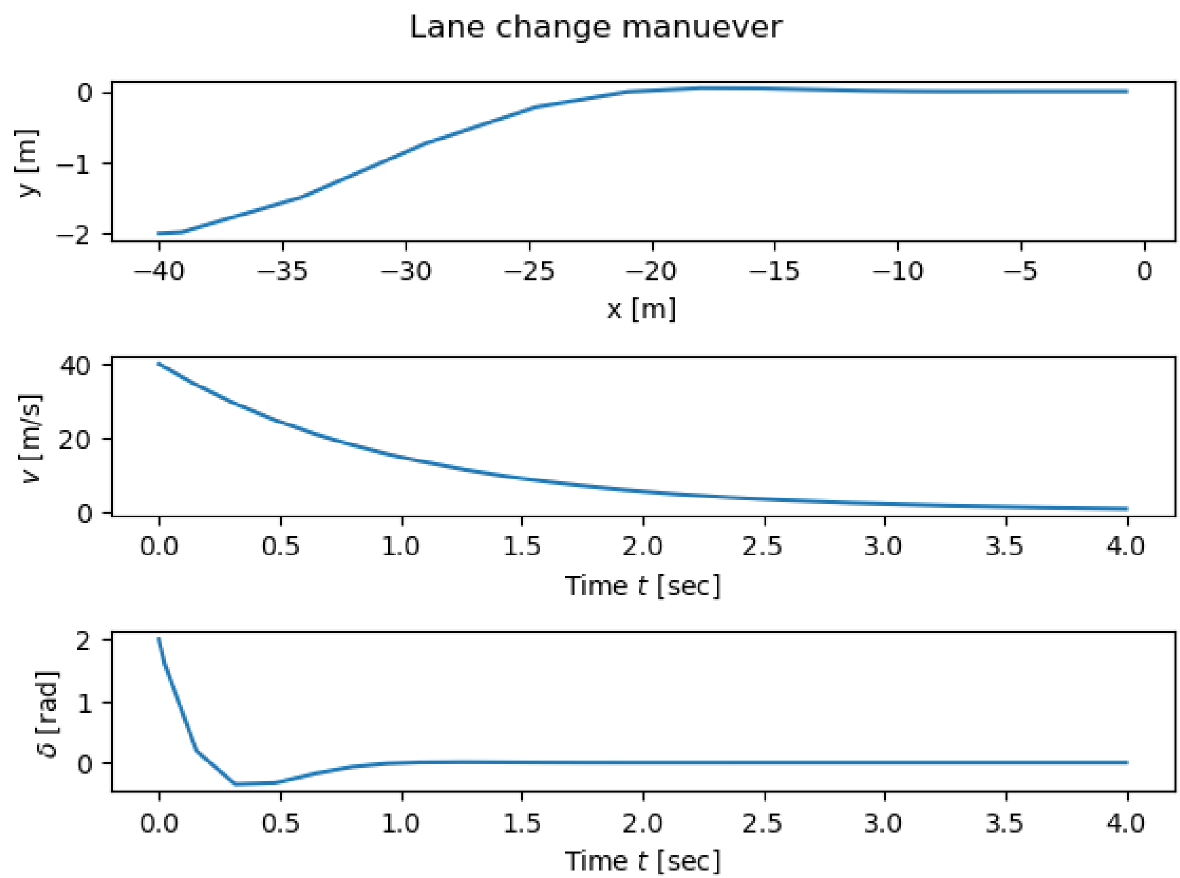
In [6]:
```
# Now solve the state forward in time
def xdot_forward(t, x):
    u = -np.linalg.inv(Qu) @ sys.B.T @ P(t) @ x
    return sys.A @ x + sys.B @ u

# Now simulate from a shifted initial condition
xsol = sp.integrate.solve_ivp(xdot_forward, (0, Tf), x0)
tvec = xsol.t
x = xsol.y
print("x[0] =", x[:, 0])
print("x[Tf] =", x[:, -1])
```

```
x[0] = [-40.  -2.   0.]
x[Tf] = [-7.32629521e-01 -2.56435711e-07 -3.40703665e-07]
```

In [7]:
```
# Finally compute the "desired" state and input values
xd = x
ud = np.zeros((sys.ninputs, tvec.size))
for i, t in enumerate(tvec):
  ud[:, i] = -np.linalg.inv(Qu) @ sys.B.T @ P(t) @ x[:, i]

plot_lanechange(tvec, xd, ud)
```



Note here that we are stabilizing the system to the origin (compared to some of other examples where we change langes and so the final y position is $y_{\mathrm{f}} = 2$.

In [ ]: