LQR Tracking Example

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This example uses a linear system to show how to implement LQR based tracking and some of the tradeoffs between feedfoward and feedback. Integral action is also implemented.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import control as ct
```

System definition

We use a simple linear system to illustrate the concepts. This system corresponds to the linearized lateral dynamics of a vehicle driving down a road at 10 m/s.

In [2]:
```
# Define a simple linear system that we want to control
sys = ct.ss([[0, 10], [-1, 0]], [[0], [1]], np.eye(2), 0, name='sys')
print(sys)
```

```
<LinearIOSystem>: sys
Inputs (1): ['u[0]']
Outputs (2): ['y[0]', 'y[1]']
States (2): ['x[0]', 'x[1]']

A = [[ 0. 10.]
     [-1.  0.]]

B = [[0.]
     [1.]]

C = [[1. 0.]
     [0. 1.]]

D = [[0.]
     [0.]]
```

Controller design

We start by defining the equilibrium point that we plan to stabilize.

In [3]:
```
# Define the desired equilibrium point for the system
x0 = np.array([2, 0])
u0 = np.array([2])
Tf = 4
```

Then construct a simple LQR controller (gain matrix) and create the controller + closed loop system models:

In [4]:
```
# Construct an LQR controller for the system
K, _, _ = ct.lqr(sys, np.eye(sys.nstates), np.eye(sys.ninputs))
ctrl, clsys = ct.create_statefbk_iosystem(sys, K)
print(ctrl)
print(clsys)
```

```
<LinearIOSystem>: sys[2]
Inputs (5): ['xd[0]', 'xd[1]', 'ud[0]', 'y[0]', 'y[1]']
Outputs (1): ['u[0]']
States (0): []

A = []

B = []

C = []

D = [[ 0.41421356  3.04701021  1.         -0.41421356 -3.04701021]]

<LinearICSystem>: u[0]
Inputs (3): ['xd[0]', 'xd[1]', 'ud[0]']
Outputs (3): ['y[0]', 'y[1]', 'u[0]']
States (2): ['sys_x[0]', 'sys_x[1]']

A = [[ 0.         10.        ]
     [-1.41421356 -3.04701021]]

B = [[0.         0.         0.        ]
     [0.41421356 3.04701021 1.        ]]

C = [[ 1.          0.        ]
     [ 0.          1.        ]
     [-0.41421356 -3.04701021]]

D = [[0.         0.         0.        ]
     [0.         0.         0.        ]
     [0.41421356 3.04701021 1.        ]]
```

Note that the name of the second system is `u[0]`. This is a bug in control-0.9.3 that will be fixed in a future release.

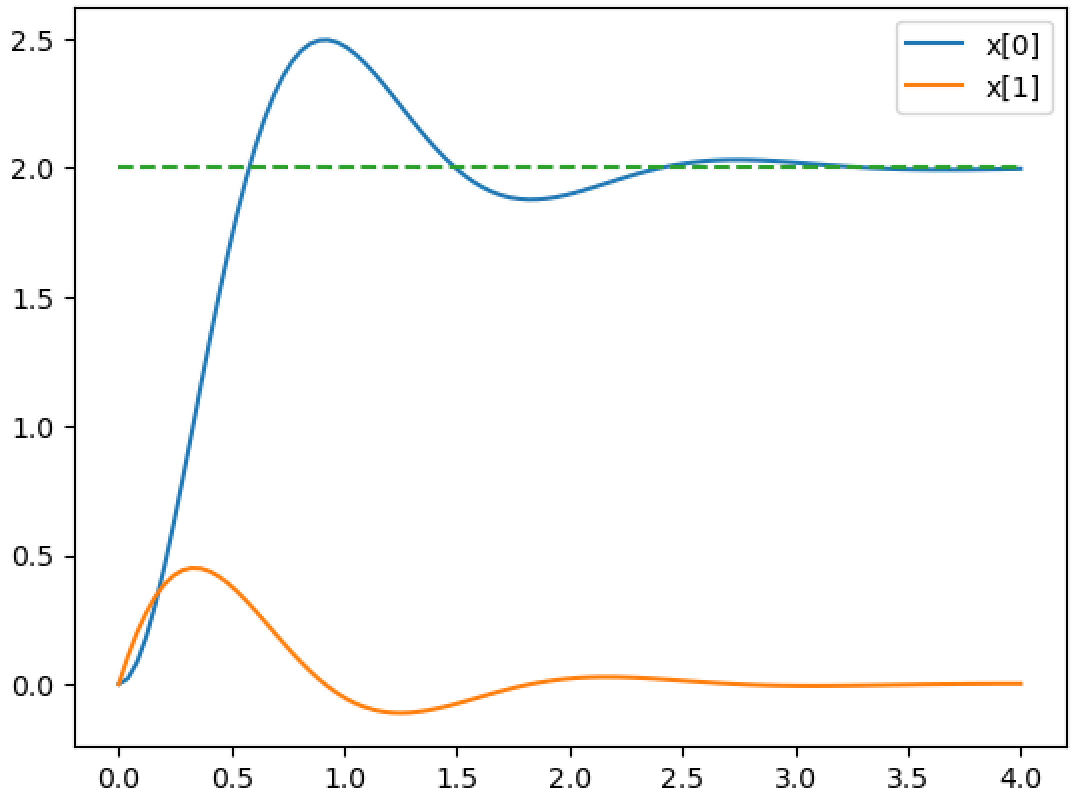
System simulations

Baseline controller

To see how the baseline controller performs, we ask it to track a step change in (xd, ud):

In [5]:
```
# Plot the step response with respect to the reference input
tvec = np.linspace(0, Tf, 100)
```

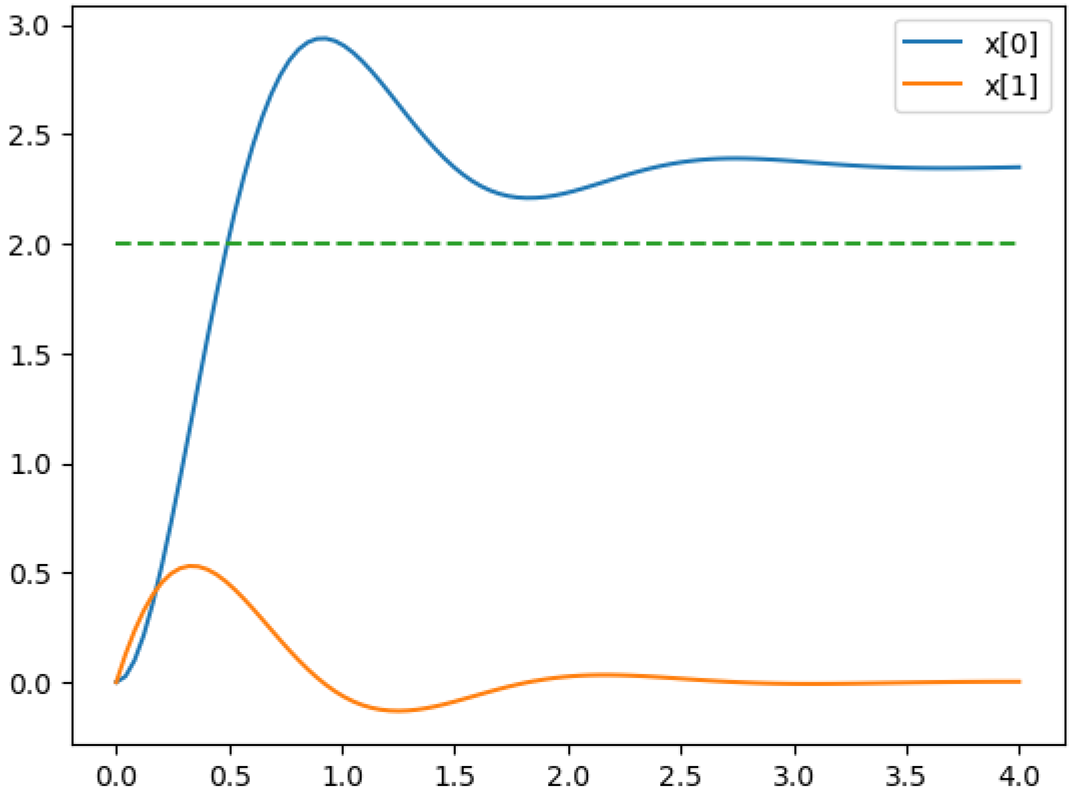
```
xd = x0
ud = u0

# U = np.hstack([xd, ud])
U = np.outer(np.hstack([xd, ud]), np.ones_like(tvec))
time, output = ct.input_output_response(clsys, tvec, U)
plt.plot(time, output[0], time, output[1])
plt.plot([time[0], time[-1]], [xd[0], xd[0]], '--');
plt.legend(['x[0]', 'x[1]']);
```



Disturbance rejection

We add a disturbance to the system by modifying ud (since this enters directly at the system input u).

In [6]:
```
# Resimulate with a disturbance input
delta = 0.5
U = np.outer(np.hstack([xd, ud + delta]), np.ones_like(tvec))
time, output = ct.input_output_response(clsys, tvec, U)
plt.plot(time, output[0], time, output[1])
plt.plot([time[0], time[-1]], [xd[0], xd[0]], '--')
plt.legend(['x[0]', 'x[1]']);
```



We see that this leads to steady state error, since some amount of system error is required to generate the force to offset the disturbance.

Integral feedback

A standard approach to compensate for constant disturbances is to use integral feedback. To do this, we have to decide what output we want to track and create a new controller with integral feedback.

We do this by creating an "augmented" system that includes the dynamics of the process along with the dynamics of the controller (= integrators for the errors that we choose):

In [7]:
```
# Create a controller with integral feedback
C = np.array([[1, 0]])

# Define an augmented state space for use with LQR
A_aug = np.block([
    [sys.A, np.zeros((sys.nstates, 1))],
    [C, 0]
])
B_aug = np.vstack([sys.B, 0])
print("A =", A_aug, "\nB =", B_aug)
```

```
A = [[ 0. 10.  0.]
 [-1.  0.  0.]
 [ 1.  0.  0.]]
B = [[0.]
 [1.]
 [0.]]
```

Now generate an LQR controller for the the augmented system:

In [8]:
```
# Create an LQR controller for the augmented system
K_aug, _, _ = ct.lqr(
    A_aug, B_aug, np.diag([1, 1, 1]), np.eye(sys.ninputs))
print(K_aug)
```

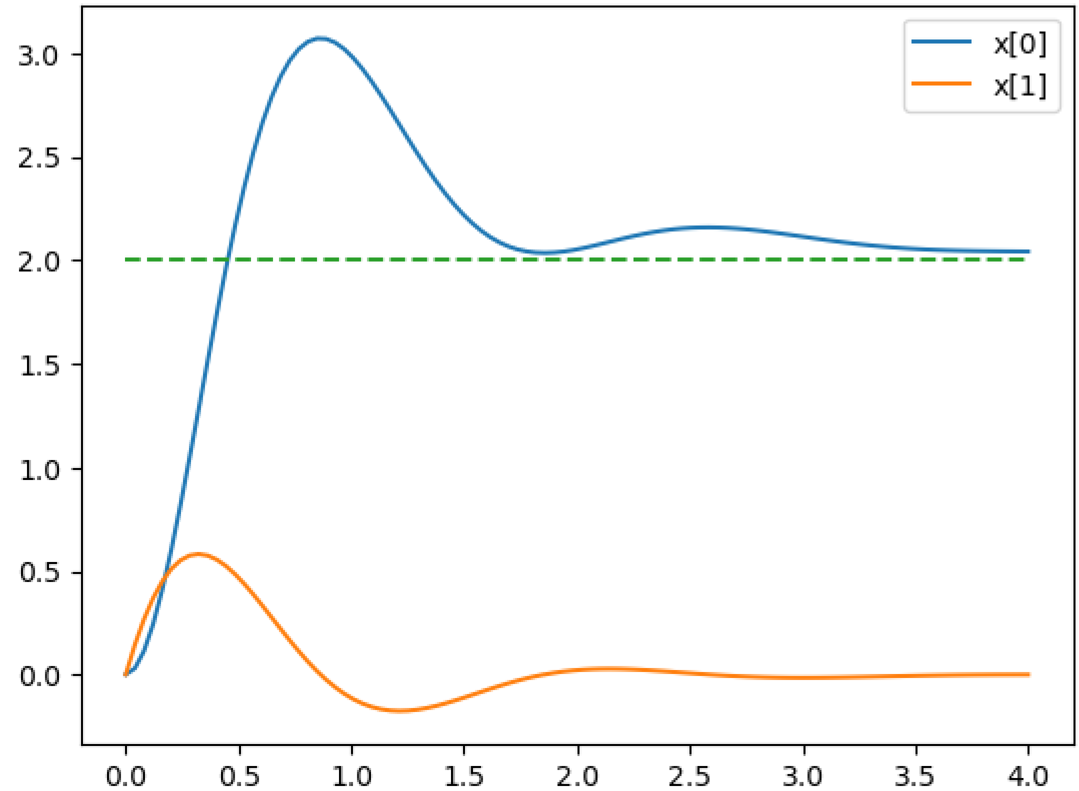
```
[[0.65930346 3.76643986 1.        ]]
```

We can think about this gain as `K_aug = [K, ki]` and the resulting contoller becomes

$$u = u_{\mathrm{d}} - K(x - x_{\mathrm{d}}) - k_{\mathrm{i}} \int_0^t (y - y_{\mathrm{d}})\, d\tau.$$

In [9]:
```
# Construct an LQR controller for the system
integral_ctrl, sys_integral = ct.create_statefbk_iosystem(sys, K_aug, integr
print(integral_ctrl)
print(sys_integral)

# Resimulate with a disturbance input
delta = 0.5
U = np.outer(np.hstack([xd, ud + delta]), np.ones_like(tvec))
time, output = ct.input_output_response(sys_integral, tvec, U)
plt.plot(time, output[0], time, output[1])
plt.plot([time[0], time[-1]], [xd[0], xd[0]], '--')
plt.legend(['x[0]', 'x[1]']);
```

```
<LinearIOSystem>: sys[4]
Inputs (5): ['xd[0]', 'xd[1]', 'ud[0]', 'y[0]', 'y[1]']
Outputs (1): ['u[0]']
States (1): ['x[0]']

A = [[0.]]

B = [[-1.  0.  0.  1.  0.]]

C = [[-1.]]

D = [[ 0.65930346  3.76643986  1.         -0.65930346 -3.76643986]]

<LinearICSystem>: u[0]
Inputs (3): ['xd[0]', 'xd[1]', 'ud[0]']
Outputs (3): ['y[0]', 'y[1]', 'u[0]']
States (3): ['sys_x[0]', 'sys_x[1]', 'sys[4]_x[0]']

A = [[ 0.         10.          0.        ]
     [-1.65930346 -3.76643986 -1.        ]
     [ 1.          0.          0.        ]]

B = [[ 0.          0.          0.        ]
     [ 0.65930346  3.76643986  1.        ]
     [-1.          0.          0.        ]]

C = [[ 1.          0.          0.        ]
     [ 0.          1.          0.        ]
     [-0.65930346 -3.76643986 -1.        ]]

D = [[0.         0.         0.        ]
     [0.         0.         0.        ]
     [0.65930346 3.76643986 1.        ]]
```



Things to try

- Play around with the gains and see whether you can reduce the overshoot (50%!)
- Try following more complicated trajectories (hint: linear systems are differentially flat...)

In [ ]: