Differential Flatness

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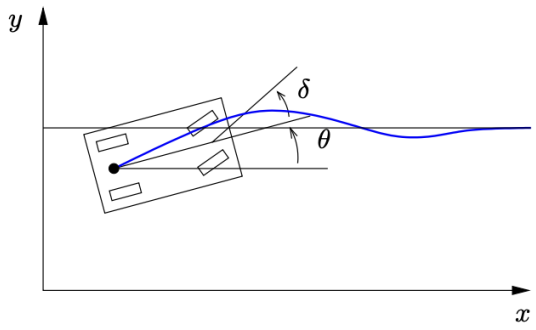
This notebook contains an example of using differential flatness as a mechanism for trajectory generation for a nonlinear control system. A differentially flat system is defined by creating an object using the `FlatSystem` class, which has member functions for mapping the system state and input into and out of flat coordinates. The `point_to_point()` function can be used to create a trajectory between two endpoints, written in terms of a set of basis functions defined using the `BasisFamily` class. The resulting trajectory is return as a `SystemTrajectory` object and can be evaluated using the `eval()` member function.

In [1]:
```
# Import the packages needed for the examples included in this notebook
import numpy as np
import matplotlib.pyplot as plt
import control as ct
import control.flatsys as fs
import control.optimal as opt
import time
```

Example: bicycle model

To illustrate the methods of generating trajectories using differential flatness, we make use of a simple model for a vehicle navigating in the plane, known as the "bicycle model". The kinematics of this vehicle can be written in terms of the contact point (x, y) and the angle θ of the vehicle with respect to the horizontal axis:



$$\dot{x} = \cos\theta\, v$$
$$\dot{y} = \sin\theta\, v$$
$$\dot{\theta} = \frac{v}{l}\tan\delta$$

The input v represents the velocity of the vehicle and the input δ represents the turning rate. The parameter l is the wheelbase.

We will generate trajectories for this system that correspond to a "lane change", in which we travel longitudinally at a fixed speed for approximately 40 meters, while moving from the right to the left by a distance of 4 meters.

It will be convenient to define a function that we will use to plot the results in a uniform way. In addition to the subplot, we also change the size of the figure to make the figure wider.

In [2]:
```
# Plot the trajectory in xy coordinates
def plot_motion(t, x, ud):
    # Set the size of the figure
    # plt.figure(figsize=(10, 6))

    # Top plot: xy trajectory
    plt.subplot(2, 1, 1)
    plt.plot(x[0], x[1])
    plt.xlabel('x [m]')
    plt.ylabel('y [m]')
    plt.axis([x0[0], xf[0], x0[1]-1, xf[1]+1])

    # Time traces of the state and input
    plt.subplot(2, 4, 5)
    plt.plot(t, x[1])
    plt.ylabel('y [m]')

    plt.subplot(2, 4, 6)
    plt.plot(t, x[2])
    plt.ylabel('theta [rad]')

    plt.subplot(2, 4, 7)
    plt.plot(t, ud[0])
    plt.xlabel('Time t [sec]')
    plt.ylabel('v [m/s]')
    plt.axis([0, Tf, u0[0] - 1, uf[0] + 1])

    plt.subplot(2, 4, 8)
    plt.plot(t, ud[1])
    plt.xlabel('Ttime t [sec]')
    plt.ylabel('$\delta$ [rad]')
    plt.tight_layout()
```

Flat system mappings

To define a flat system, we have to define the functions that take the state and compute the flat "flag" (flat outputs and their derivatives) and that take the flat flag and return the state and input.

The `forward()` method computes the flat flag given a state and input:

```
zflag = sys.forward(x, u)
```

The `reverse()` method computes the state and input given the flat flag:

```
x, u = sys.reverse(zflag)
```

The flag $\bar{z}$ is implemented as a list of flat outputs z_i and their derivatives up to order q_i:

`zflag[i][j]` = $z_i^{(j)}$

The number of flat outputs must match the number of system inputs.

In addition, a flat system is an input/output system and so we define and update function ($f(x, u)$) and output (use `None` to get the full state).

In [3]:
```
# Function to take states, inputs and return the flat flag
def bicycle_flat_forward(x, u, params={}):
    # Get the parameter values
    b = params.get('wheelbase', 3.)

    # Create a list of arrays to store the flat output and its derivatives
    zflag = [np.zeros(3), np.zeros(3)]

    # Flat output is the x, y position of the rear wheels
    zflag[0][0] = x[0]
    zflag[1][0] = x[1]

    # First derivatives of the flat output
    zflag[0][1] = u[0] * np.cos(x[2])  # dx/dt
    zflag[1][1] = u[0] * np.sin(x[2])  # dy/dt

    # First derivative of the angle
    thdot = (u[0]/b) * np.tan(u[1])

    # Second derivatives of the flat output (setting vdot = 0)
    zflag[0][2] = -u[0] * thdot * np.sin(x[2])
    zflag[1][2] =  u[0] * thdot * np.cos(x[2])

    return zflag

# Function to take the flat flag and return states, inputs
def bicycle_flat_reverse(zflag, params={}):
    # Get the parameter values
    b = params.get('wheelbase', 3.)

    # Create a vector to store the state and inputs
    x = np.zeros(3)
    u = np.zeros(2)

    # Given the flat variables, solve for the state
    x[0] = zflag[0][0]  # x position
    x[1] = zflag[1][0]  # y position
    x[2] = np.arctan2(zflag[1][1], zflag[0][1])  # tan(theta) = ydot/xdot

    # And next solve for the inputs
    u[0] = zflag[0][1] * np.cos(x[2]) + zflag[1][1] * np.sin(x[2])
    thdot_v = zflag[1][2] * np.cos(x[2]) - zflag[0][2] * np.sin(x[2])
    u[1] = np.arctan2(thdot_v, u[0]**2 / b)

    return x, u
```

```
# Function to compute the RHS of the system dynamics
def bicycle_update(t, x, u, params):
    b = params.get('wheelbase', 3.)             # get parameter values
    dx = np.array([
        np.cos(x[2]) * u[0],
        np.sin(x[2]) * u[0],
        (u[0]/b) * np.tan(u[1])
    ])
    return dx

# Return the entire state as output (instead of default flat outputs)
def bicycle_output(t, x, u, params):
    return x

# Create differentially flat input/output system
bicycle_flat = fs.FlatSystem(
    bicycle_flat_forward, bicycle_flat_reverse,
    bicycle_update, bicycle_output,
    inputs=('v', 'delta'), outputs=('x', 'y', 'theta'),
    states=('x', 'y', 'theta'), name='bicycle_model')

print(bicycle_flat)
```

```
<FlatSystem>: bicycle_model
Inputs (2): ['v', 'delta']
Outputs (3): ['x', 'y', 'theta']
States (3): ['x', 'y', 'theta']

Update: <function bicycle_update at 0x7fb77111add0>
Output: <function bicycle_output at 0x7fb77111ae60>

Forward: <function bicycle_flat_forward at 0x7fb77111acb0>
Reverse: <function bicycle_flat_reverse at 0x7fb77111ad40>
```

Point to point trajectory generation

In addition to the flat system description, a set of basis functions $\phi_i(t)$ must be chosen. The `BasisFamily` class is used to represent the basis functions. A polynomial basis function of the form $1, t, t^2, \ldots$ can be computed using the `PolyFamily` class, which is initialized by passing the desired order of the polynomial basis set:

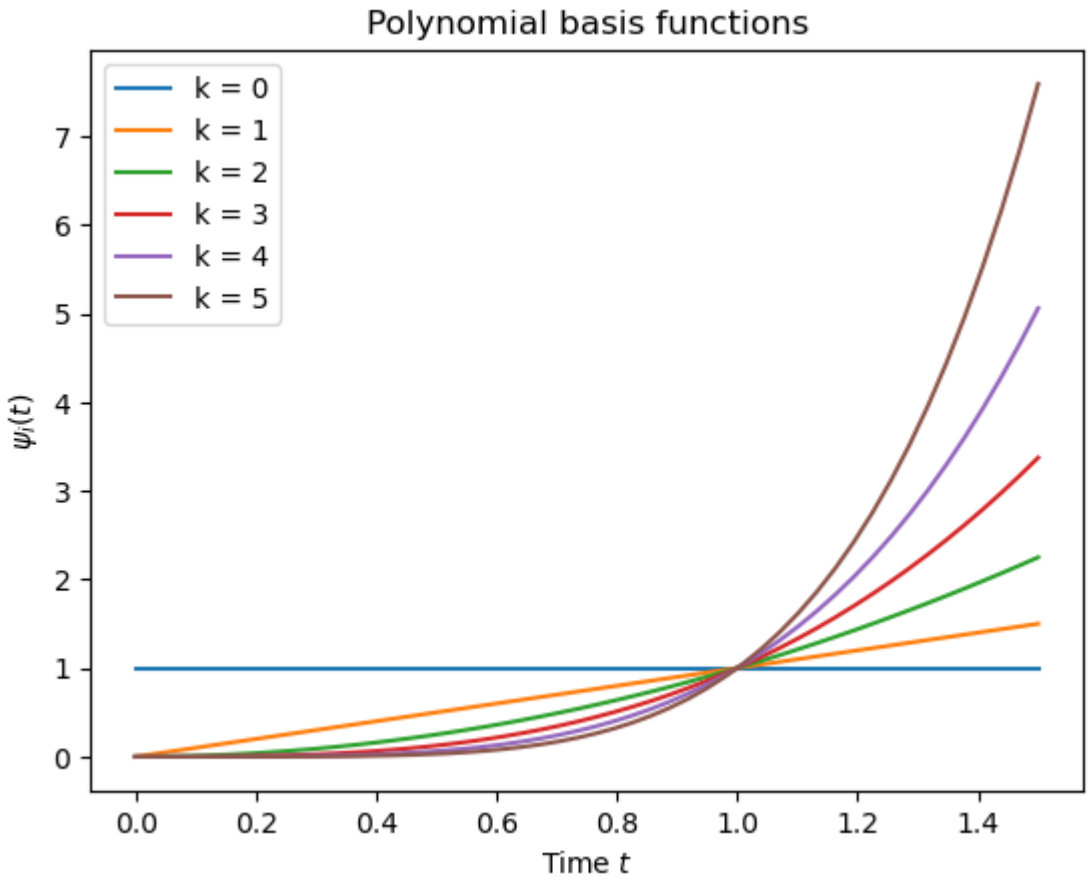
```
polybasis = control.flatsys.PolyFamily(N)
```

In [4]:

```
print(fs.BasisFamily.__doc__)
print(fs.PolyFamily.__doc__)

# Define a set of basis functions to use for the trajectories
poly = fs.PolyFamily(6)

# Plot out the basis functions
t = np.linspace(0, 1.5)
for k in range(poly.N):
    plt.plot(t, poly(k, t), label=f'k = {k}')
```

```
plt.legend()
plt.title("Polynomial basis functions")
plt.xlabel("Time $t$")
plt.ylabel("$\psi_i(t)$");
```

```
Base class for implementing basis functions for flat systems.

    A BasisFamily object is used to construct trajectories for a flat syste
m.
    The class must implement a single function that computes the jth
    derivative of the ith basis function at a time t:

      :math:`z_i^{(q)}(t)` = basis.eval_deriv(self, i, j, t)

    A basis set can either consist of a single variable that is used for
    each flat output (nvars = None) or a different variable for different
    flat outputs (nvars > 0).

    Attributes
    ----------
    N : int
        Order of the basis set.


Polynomial basis functions.

    This class represents the family of polynomials of the form

    .. math::
         \phi_i(t) = \left( \frac{t}{T} \right)^i

    Parameters
    ----------
    N : int
        Degree of the Bezier curve.

    T : float
        Final time (used for rescaling).
```



Approach 1: point to point solution, no cost or constraints

Once the system and basis function have been defined, the `point_to_point()` function can be used to compute a trajectory between initial and final states and inputs:

```
traj = control.flatsys.point_to_point(sys, Tf, x0, u0, xf,
uf, basis=polybasis)
```

The returned object has class `SystemTrajectory` and can be used to compute the state and input trajectory between the initial and final condition:

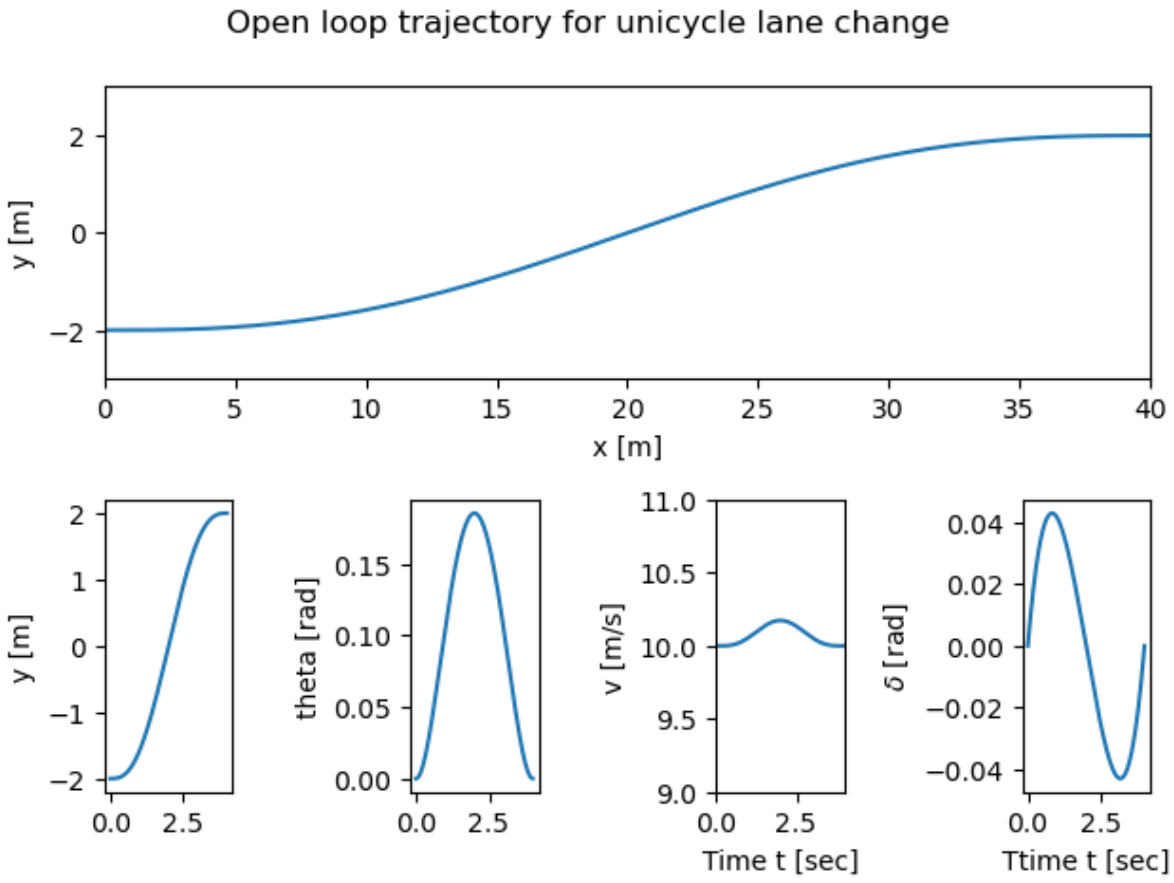
```
xd, ud = traj.eval(timepts)
```

where `timepts` is a list of times on which the trajectory should be evaluated (e.g., `timepts = numpy.linspace(0, Tf, M)`).

In [5]:
```
# Define the endpoints of the trajectory
x0 = np.array([0., -2., 0.]); u0 = np.array([10., 0.])
xf = np.array([40., 2., 0.]); uf = np.array([10., 0.])
Tf = 4

# Generate a normalized set of basis functions
poly = fs.PolyFamily(6, Tf)
```

```
# Find a trajectory between the initial condition and the final condition
traj = fs.point_to_point(bicycle_flat, Tf, x0, u0, xf, uf, basis=poly)

# Create the desired trajectory between the initial and final condition
timepts = np.linspace(0, Tf, 500)
xd, ud = traj.eval(timepts)

# Simulation the open system dynamics with the full input
t, y, x = ct.input_output_response(
    bicycle_flat, timepts, ud, x0, return_x=True)

# Plot the open loop system dynamics
plt.figure(1)
plt.suptitle("Open loop trajectory for unicycle lane change")
plot_motion(t, x, ud)

# Make sure the initial and final points are correct
print("x[0] = ", xd[:, 0])
print("x[T] = ", xd[:, -1])
```

```
x[0] =  [-5.90673952e-15 -2.00000000e+00 -4.06619183e-16]
x[T] =  [4.00000000e+01 2.00000000e+00 7.10542736e-16]
```



A look inside the code

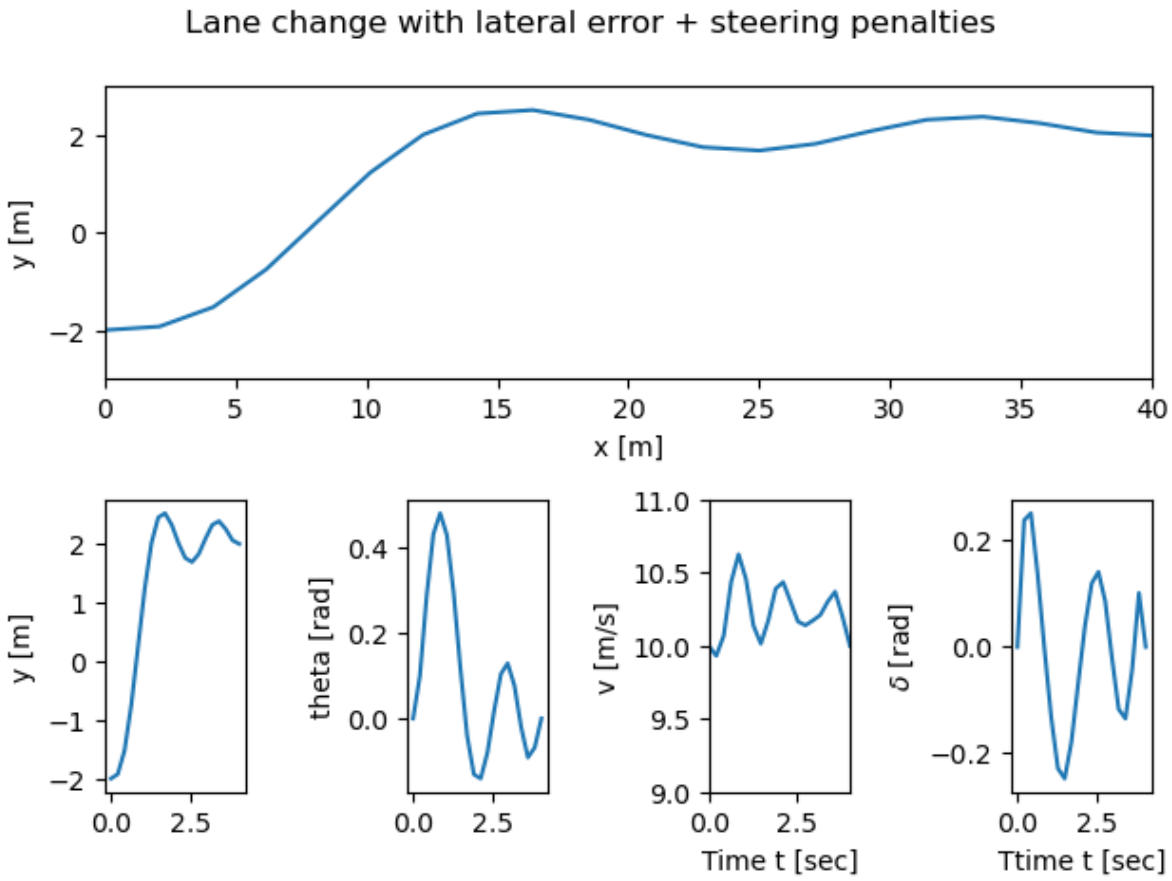
The code to solve this problem is inside the file flatsys.py in the python-control package. Here is what operative code inside the `point_to_point()` looks like:

```
#
# Map the initial and final conditions to flat output
conditions
#
# We need to compute the output "flag": [z(t), z'(t), z''(t),
...]
# and then evaluate this at the initial and final condition.
#

zflag_T0 = sys.forward(x0, u0)
zflag_Tf = sys.forward(xf, uf)

#
# Compute the matrix constraints for initial and final
conditions
#
# This computation depends on the basis function we are
using.  It
# essentially amounts to evaluating the basis functions and
their
# derivatives at the initial and final conditions.

# Compute the flags for the initial and final states
M_T0 = _basis_flag_matrix(sys, basis, zflag_T0, T0)
M_Tf = _basis_flag_matrix(sys, basis, zflag_Tf, Tf)

# Stack the initial and final matrix/flag for the point to
point problem
M = np.vstack([M_T0, M_Tf])
Z = np.hstack([np.hstack(zflag_T0), np.hstack(zflag_Tf)])

#
# Solve for the coefficients of the flat outputs
#
# At this point, we need to solve the equation M alpha =
zflag, where M
# is the matrix constrains for initial and final conditions
and zflag =
# [zflag_T0; zflag_tf].
#
# If there are no constraints, then we just need to solve a
linear
# system of equations => use least squares.  Otherwise, we
have a
# nonlinear optimal control problem with equality constraints
=> use
# scipy.optimize.minimize().
#

# Start by solving the least squares problem
alpha, residuals, rank, s = np.linalg.lstsq(M, Z, rcond=None)
```

Approach #2: add cost function to make lane change quicker

In [6]:
```
# Define timepoints for evaluation plus basis function to use
timepts = np.linspace(0, Tf, 20)
basis = fs.PolyFamily(12, Tf)

# Define the cost function (penalize lateral error and steering)
traj_cost = opt.quadratic_cost(
    bicycle_flat, np.diag([0, 0.1, 0]), np.diag([0.1, 1]), x0=xf, u0=uf)

# Solve for an optimal solution
start_time = time.process_time()
traj = fs.point_to_point(
    bicycle_flat, timepts, x0, u0, xf, uf, cost=traj_cost, basis=basis,
)
print("* Total time = %5g seconds\n" % (time.process_time() - start_time))

xd, ud = traj.eval(timepts)

plt.figure(2)
plt.suptitle("Lane change with lateral error + steering penalties")
plot_motion(timepts, xd, ud);
```

```
* Total time = 0.449223 seconds
```

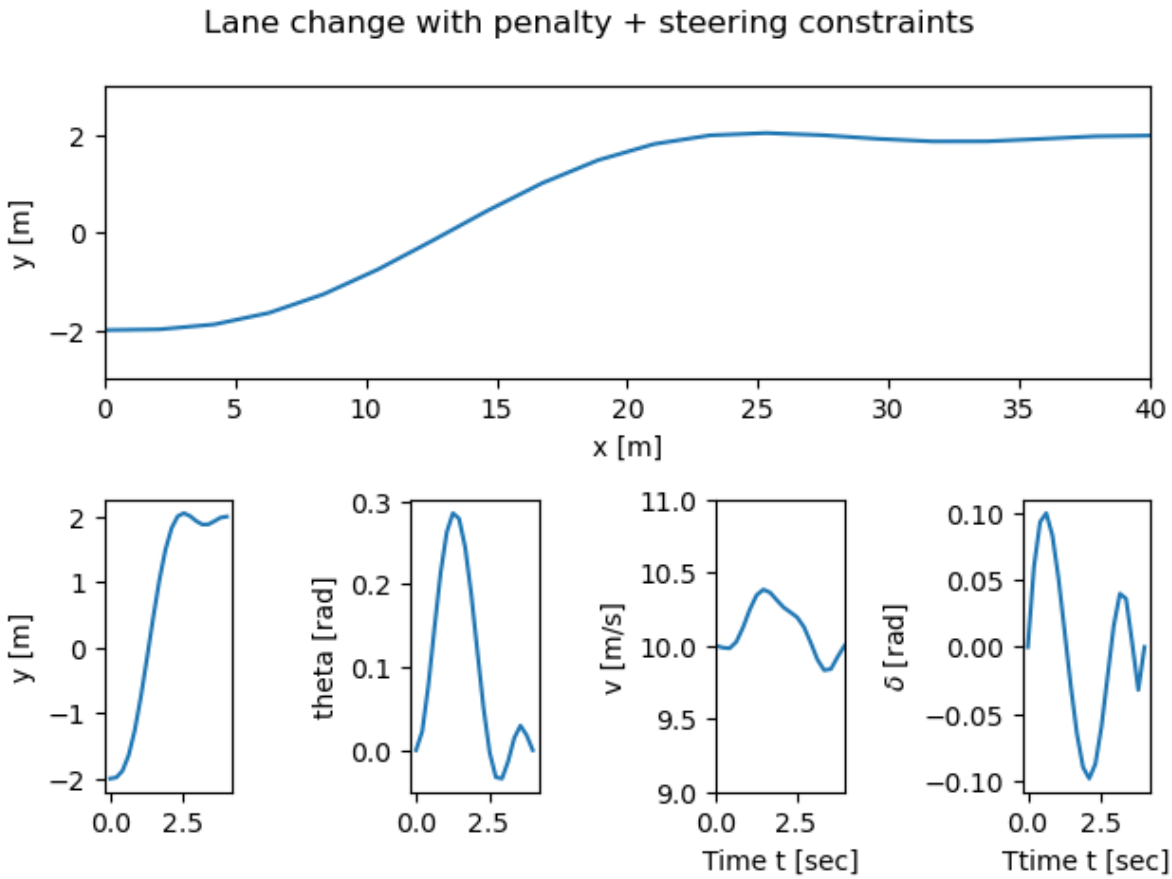


Note that the solution has a very large steering angle (0.2 rad = ~12 degrees).

Approach #3: optimal cost with trajectory constraints

To get a smaller steering angle, we add constraints on the inputs.

In [7]:
```
constraints = [
    opt.input_range_constraint(bicycle_flat, [8, -0.1], [12, 0.1]) ]

# Solve for an optimal solution
traj = fs.point_to_point(
    bicycle_flat, timepts, x0, u0, xf, uf, cost=traj_cost,
    trajectory_constraints=constraints, basis=basis,
)
xd, ud = traj.eval(timepts)

plt.figure(3)
plt.suptitle("Lane change with penalty + steering constraints")
plot_motion(timepts, xd, ud)
```

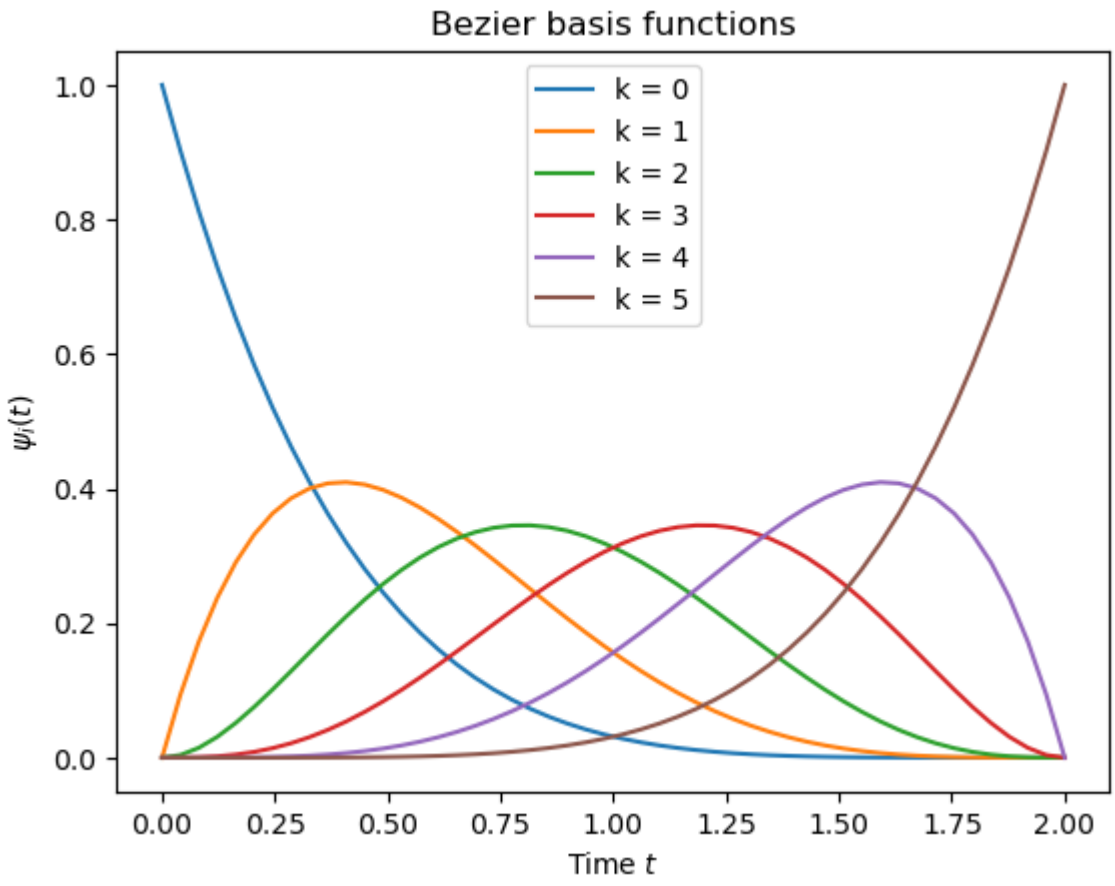


Ideas to explore

- Change the number of basis functions
- Change the number of time points
- Change the type of basis functions: BezierFamily

In [8]:
```
# Define a set of basis functions to use for the trajectories
poly = fs.BezierFamily(6, 2)
```

```
# Plot out the basis functions
t = np.linspace(0, 2)
for k in range(poly.N):
    plt.plot(t, poly(k, t), label=f'k = {k}')

plt.legend()
plt.title("Bezier basis functions")
plt.xlabel("Time $t$")
plt.ylabel("$\psi_i(t)$");
```



In [ ]: