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9. Interpolation and Sampling

9.1. Preliminaries and Notation

- Interpolation
  - core: interval between samples T_s
  - from digital (discrete) to analog (continuous)

- Sampling
  - continuous time to discrete time

- Notation
  - T_s : sampling period (time between samples)
    - seconds (s)
    - $T_s = 1/F_s$
  - F_s : Sampling frequency
    - Hertz (Hz)
    - $F_s = 1/T_s$
  - Ω_s : Sampling frequency (angular)
    - rad/s
    - $\Omega_s = 2\pi F_s = 2\pi/T_s$
  - Ω_N : Nyquist frequency (angular)
    - rad/s
    - $\Omega_N = \Omega_s/2 = \pi/T_s$

9.2. Continuous-Time Signals

- Modeled by complex functions of a real variable t (usually time in sec)
- the functions can be periodic or aperiodic
- can have finite or infinite support
- common condition on aperiodic signals:
  - modeling function: square integrable (finite energy)

- Inner Product and Convolution
  - Inner product

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f^*(t)\, g(t)\, dt$$

  - convolution

$$(f * g)(t) = \langle f(t-\tau), g(\tau) \rangle = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau$$

↳ linear and time invariant

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- Frequency-Domain Representation of Continuous-Time Signals

  - Fourier transform of $x(t)$:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-\Omega t}\, dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, dt$$

- Convolution Theorem (mirrors theorem for discrete signals)

  - if $h(t) = (f * g)(t)$
    then $H(j\Omega) = F(j\Omega)\, G(j\Omega)$

  - can be used to compute

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\Omega)\, G(j\Omega)\, e^{j\Omega t}\, dt$$

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9.3. Bandlimited signals

- bandlimited : Fourier transform is non-zero only over finite frequency interval

  e.g. $x(t)$ is bandlimited if Ω_N exists such that

  $$X(j\Omega) = 0 \quad \text{for } |\Omega| \geq \Omega_N$$

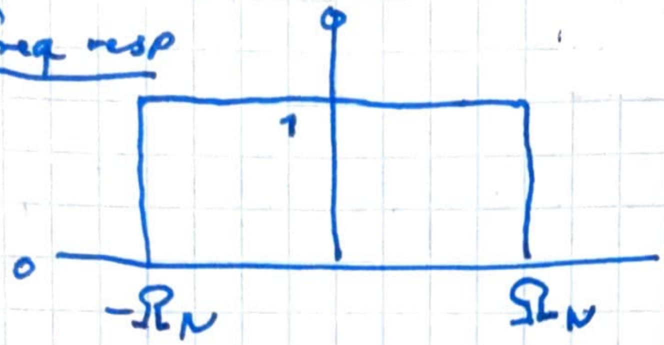
  with Ω_N : Nyquist frequency

- time-limited : non zero over finite time interval (finite support signal)

- a band-limited signal cannot be a time-limited signal and vice versa

- The Sinc Function
  - prototypical Ω_N-band-limited signal

<u>freq resp</u>



$$\mathrm{rect}(x) = \begin{cases} 1 & |x| \leq 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

- Fourier transform:

$$\Phi(j\Omega) = \underbrace{\frac{\pi}{\Omega_N}}_{\text{Normalization term}} \mathrm{rect}\left(\frac{\Omega}{2\Omega_N}\right)$$

- Time domain: inverse Fourier Transf.

$$\varphi(t) = \frac{\sin \Omega_N t}{\Omega_N t} = \mathrm{sinc}\left(\frac{t}{T_s}\right)$$

- with: $T_s = \pi / \Omega_N$
  - $\mathrm{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

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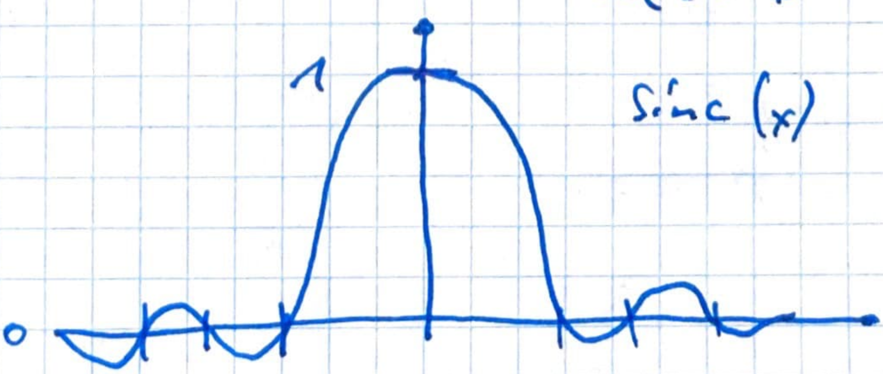
- Sinc Function:
  - symmetric, $\text{sinc}(x) = \text{sinc}(-x)$
  - sinc is 0 for all integer values of argument x except for $x=0$
    ↳ interpolation property
  - square integrable: finite energy
  - not absolutely integrable: discontinuity of its Fourier transform
  - slow decay: asymptotic to $1/x$
  - scaled sinc function: impulse response of ideal continuous-time lowpass filter with cutoff frequency $\omega_c = \Omega_N$



9.4. Interpolation

- convert discrete-time sequence $x[n]$ to a continuous-time function $x(t)$
- interval between samples of $x[n]$:
  - physical time duration T_s
- requirement interpolated function:
  - values at multiples of T_s are equal to values of $x[n]$:

$$x(t)\Big|_{t=T_s \cdot n} = x[n]$$

9.4.1. Local Interpolation

- set $x(t)$ to $x[n]$ for $t = nT_s$ and $x(t)$ linear combination of neighbors when t between interpolation instants

(can be interpreted as mixed-domain convolution)

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] I\left(\frac{t-nT_s}{T_s}\right)$$

I: interpolation function

- Interpolation function must satisfy the fundamental interpolation properties

$$\begin{cases} I(0) = 1 \\ I(k) = 0 \text{ for } k \in \mathbb{Z} \setminus \{0\} \end{cases}$$

- By changing Interpolation function I we can change the type of the interpolation and the resulting signal

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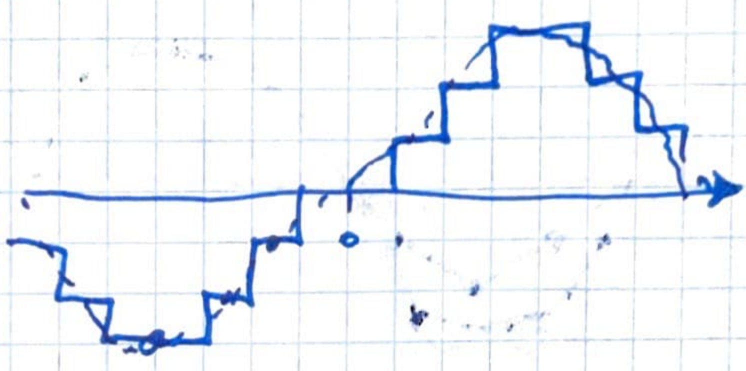
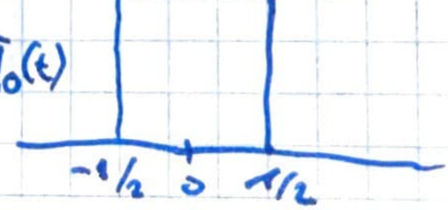
- Zero-Order Hold
  - simplest interpolation function
  - piecewise constant interpolation



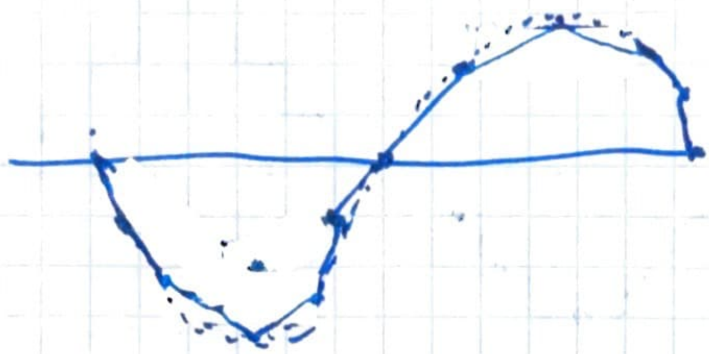
- $x(t) = x[n]$, for $(n-\frac{1}{2})T_s \leq t < (n+\frac{1}{2})T_s$
- not smooth at all, discontinuous
- interpolation function: $I_0(t) = rect(t)$

  0 indicates how many samples, besides the current one use to calculate interpolated values for $x(t)$

- values of $x(t)$ depend only on current discrete-time sample value
- Interpolation function: $I_0(t)$



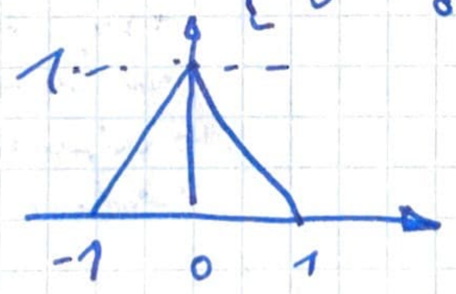
- First-Order Hold (linear interpolator
  - connect two sample values with straight line
  - $x(t)$ depends on two consecutive discrete-time samples



- interpolated function is continuous
  - first derivative is not continuous

- Interpolation Function:

$$I_1(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$



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- Higher-Order Interpolators
  - extend Zero-order- and First-order-Hold to higher dimensions
  - B-spline of higher order
  - $I_N(t)$ is N-th order polynomial int
    - ↳ how many samples (in addition to current one) are used for calculating the interpolation

  - easy to implement in practice
  - disadvantage: N-th derivative is discontinuous
    - ↳ lack of smoothness in interpolated function
    - ↳ high frequency energy

9.4.2. Polynomial Interpolation

- Lagrange interpolation:
  - maximally smooth: continuous including all it's derivatives

↳ see Book or other resources on calculating Lagrange interpolation

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9.4.3. Sinc Interpolation

- local Interpolation
  - Interpolated function is linear combination of shifted versions of the same prototype function (interpolation function $I(t)$)
  - disadvantage: interpolated continuous-time function is not smooth

- Polynomial interpolation
  - perfectly smooth
  - disadvantage: works only for finite-length signals, needs different interpolation function for different signal lengths

↳ combine both approaches:

Lagrange interpolation for infinite-length signals

↳ $L_0^{(N)}(t)$ converges to sinc

$I(t) = \mathrm{sinc}(t)$

- Spectral Properties of the Sinc Interpolation
  - sinc-interpolation of discrete-time sequence → strictly bandlimited continuous-time function
  - Spectrum of interpolated function $X(j\Omega)$

$$X(j\Omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j\Omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \operatorname{sinc}\left(\frac{t-nT_s}{T_s}\right) e^{-j\Omega t} dt$$

- Fourier Transform of scaled and shifted sinc

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\pi}{\Omega_N}\right) \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right) e^{-jnT_s\Omega}$$

- with $T_s = \pi/\Omega_N$

$$X(j\Omega) = \left(\frac{\pi}{\Omega_N}\right) \operatorname{rect}\left(\frac{\Omega}{2\Omega_N}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j\pi\frac{\Omega}{\Omega_N}n}$$

$$= \begin{cases} \frac{\pi}{\Omega_N} X(e^{j\pi\Omega/\Omega_N}) & \text{for } |\Omega| \leq \Omega_N \\ 0 & \text{otherwise} \end{cases}$$

- continuous-time spectrum: scaled and streched version of DTFT of discrete-time sequence between π and $-\pi$

- Spectrum of sinc (cont.)
  - interpolation interval T_s: inversely proportional to bandwidth of interpolated signal
  - slow interpolation (T_s large):
    - spectrum around low frequencies
  - fast interpolation (T_s small):
    - spread-out spectrum, more high frequencies present

9.S. The Sampling Theorem

Any square summable discrete-time signal can be interpolated into a continuous-time signal which is:

- smooth in time
- strictly band-limited in frequency

- Space of Bandlimited Signals
  - class of Ω_N-bandlimited functions of finite energy form a Hilbert space
  - Inner product:

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f^*(t) g(t) dt$$

  - Orthogonal (see book for derivation)
  - space is complete

· The Sampling: Theorem

- if $x(t)$ is a Ω_N bandlimited continuous-time signal then $x(t)$ can sufficiently be represented by discrete-time signal

  $x[n] = x(nT_s)$, with $T_s = \pi/\Omega_N$

- The continuous time signal $x(t)$ can be exactly reconstructed from $x[n]$ as

$$x(t) = \sum_{-\infty}^{\infty} x[n] \,\mathrm{sinc}\left(\frac{t-nT_s}{T_s}\right)$$

- a bandlimited signal $x(t)$ is uniquely represented by all sequences $x[n] = x(nT)$ for which $T \leq T_s = \pi/\Omega_{max}$
  T_s: largest sampling period that guarantees perfect reconstruction
  ↳ we cannot take fewer than $1/T_s$ samples per second to perfectly capture the signal

- The minimum sampling frequency Ω_s for perfect reconstruction is exactly twice the signal's max. frequency: Nyquist frequency
  = higest freq. of bandlimited signal

- sampling Freq Ω:
  $\Omega \geq \Omega_s = 2\,\Omega_{max}$
  in Hz: $F \geq F_s = 2\,F_{max}$

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9.6. Aliasing

- Sampling: measuring values of a continuous signal at uniformly spaced instances in time
- Bandlimited signals: can be perfectly reconstructed from samples $x(nT_s)$

9.6.1. Non-Bandlimited Signals

- Sampling theorem: no loss of information by sampling Ω_N-bandlimited signals with $\Omega_N = \frac{\pi}{T_s}$
- non-bandlimited signals
  - i.e. spectrum is nonzero outside of $[-\Omega_N, \Omega_N]$

  ↳ best approximation (of orthogonal bases)

  ↳ ideal lowpass filter on $x(t)$
    - gain: T_s
    - truncates spectrum outside of $[-\Omega_N, \Omega_N]$

9.6.2. Aliasing: Intuition

- $x[n] = x(nT_s)$
  with $F_s = 1/T_s$: sampling frequency
- if $x(t)$ is
  - not band-limited
  - or sampling frequency is less than twice max. frequency

  → what is the error?
- Sampling of Sinusoids
  - Continuous-time signal:
    $$x(t) = e^{j2\pi f_0 t}$$
  - sampled signal
    $$x[n] = e^{j2\pi(f_0/F_s)n} = e^{j\omega_0 n}$$
    with $\omega_0 = 2\pi \frac{f_0}{F_s}$
  - $x(t)$: bandlimited (as only one sinusoid) for all $\Omega > 2\pi|f_0|$
  - if freq of $x(t)$ satisfies
    $$|f_0| < F_s/2 = F_N$$
    then $\omega_0 \in (-\pi, \pi)$ → original sinusoid can be reconstructed

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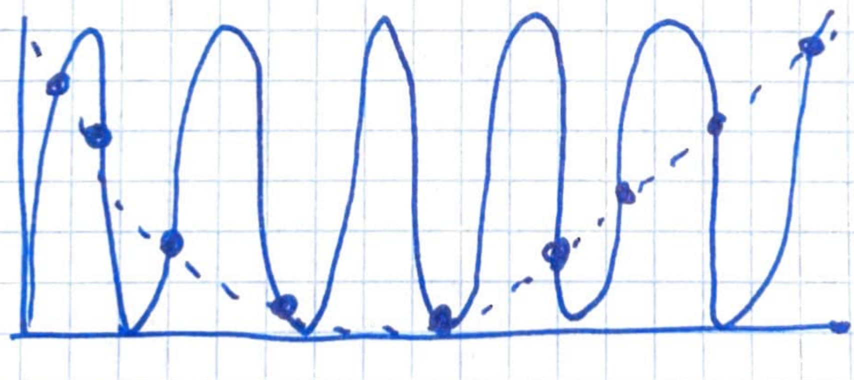
- if $f_0 = F_N = F_S/2$
  - ↳ $x[n] = e^{j\pi n} = e^{-j\pi n}$
  - ambiguity with respect to the direction of rotation of complex exponential
  - cannot determine if original frequency was $f_0 = F_N$ or $f_0 = -F_N$
  - increase frequency: $f_0 = (1+\alpha) \cdot F_N$
    ↳ $x[n] = e^{j(1+\alpha)\pi n} = e^{-j\alpha\pi n}$

    ⇒ ambiguity on direction and frequency value



- two continuous-time frequencies are mapped to same discrete-time frequency
  ⇒ aliasing

- because of 2π-periodicity of discrete-time complex exponential we can always write

$$\omega_0 = (2\pi f_0 T_s) + 2k\pi$$

and choose $k \in \mathbb{Z}$ so that ω_0 falls into $[-\pi, \pi]$

- all continuous-time frequencies of form

$$f = f_0 + k F_s$$

with $f_s < F_N$ are aliased to the same descrete-time frequency ω_0.

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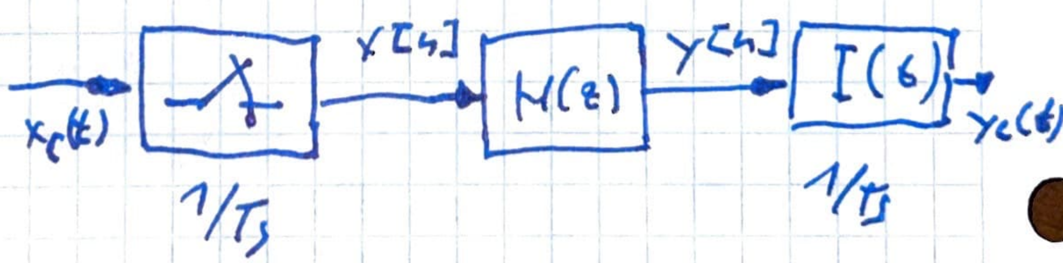
- Energy Folding of the Fourier Transform
  - Fourier Transform:
    - infinitely many complex oscillators initiated with phase and amplitude
    - two frequencies F_s apart are indistinguishable (aliasing)
    - their contribution to DTFT add up

$$\sum_{k=-\infty}^{\infty} X(j2\pi f + j2k\pi F_s)$$

9.7. Discrete-Time Processing of Analog Signals

- Samplers and interpolators (A/D and D/A) are the only interface of the digital to the physical (analog) world.
- digital: timeless, dimensionless
- In most cases the frequencies of the A/D and the D/A are the same taking bandwidth of signals to be processed into account
- band-limited input:

$$\Omega_N = \pi / T_s \quad (\text{or } F_s/2)$$



↳ analog transfer function

$$X(e^{j\omega}) = \frac{1}{T_s} X_c\left(j\frac{\omega}{T_s}\right)$$

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$Y_c(j\Omega) = T_s Y(e^{j\Omega T_s})$$

$$\Rightarrow \quad Y_c(j\Omega) = H(e^{j\Omega T_s}) X_c(j\Omega)$$