DSP3 – Practice Homework

Exercise 1. Interpolation.

Consider a finite-energy discrete-time sequence $x[n]$ with DTFT $X(e^{j\omega})$ and the continuous-time interpolated signal

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{rect}(t-n)$$

i.e. a signal obtained from the discrete-time sequence using a zero-centered zero-order hold with interpolation period $T_s = 1$s. Let $X_0(f)$ be the Fourier transform of $x_0(t)$.

(a) Express $X_0(f)$ in terms of $X(e^{j\omega})$.

(b) Compare $X_0(f)$ to $X(f)$, where $X(f)$ is the spectrum of the continuous-time signal obtained using an ideal sinc interpolator with $T_s = 1$:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{sinc}(t-n)$$

Comment on the result: you should point out two major problems.

(c) The signal $x(t)$ can be obtained back from the zero-order hold interpolation via a continuous-time filtering operation:

$$x(t) = x_0(t) * g(t).$$

Sketch the frequency response of the filter $g(t)$.

(d) Propose two solutions (one in the continuous-time domain, and another in the discrete-time domain) to eliminate or attenuate the distortion due to the zero-order hold. Discuss the advantages and disadvantages of each.

Exercise 2. A bizarre interpolator

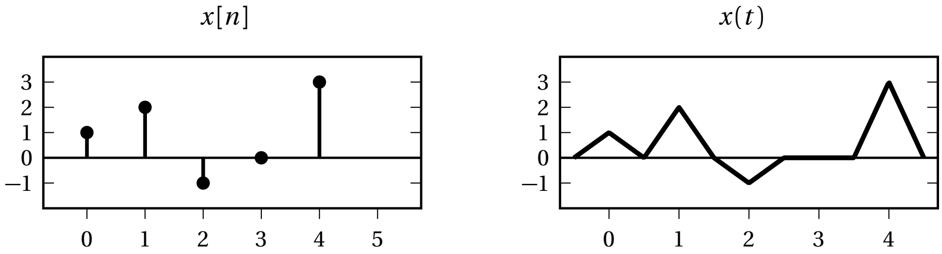
Consider a local interpolation scheme as in the previous exercise but now the characteristic of the interpolator is:

$$i(t) = \begin{cases} 1-2|t| & |t| \le 1/2 \\ 0 & \text{otherwise} \end{cases}$$

This is a triangular characteristic but the same unit support as the zero-order hold. If we pick an interpolation interval T_s and interpolate a given discrete-time signal $x[n]$ with $I(t)$, we obtain the continuous-time signal

$$x(t) = \sum_n x[n]\, i\left(\frac{t-nT_s}{T_s}\right)$$

an example of which is shown here



Assume that the spectrum of $x[n]$ between $-\pi$ and π is

$$X(e^{j\omega}) = \begin{cases} 1 & |\omega| \le 2\pi/3 \\ 0 & \text{otherwise} \end{cases}$$

(with the obvious 2π-periodicity over the entire frequency axis).

(a) Compute and sketch the Fourier transform $I(f)$ of the interpolating function $i(t)$. Recall that the triangular function can be expresses as the convolution of a suitably scaled rect with itself.

(b) Sketch the Fourier transform $X(f)$ of the interpolated signal $x(t)$; in particular, clearly mark the Nyquist frequency $F_s/2$

(c) The use of $i(t)$ instead of a sinc interpolator introduces two types of errors: briefly describe them.

(d) To eliminate the error in the baseband $[-F_s/2, F_s/2]$ we can pre-filter the signal $x[n]$ before interpolating with $i(t)$. Write the frequency response of the required discrete-time pre-filter $H(e^{j\omega})$.

Exercise 3. Another view of Sampling

An alternative way of describing the sampling operation relies on the concept of *modulation by a pulse train.* Given a sampling interval T_s, a continuous-time pulse train $p(t)$ is an infinite collection of equally spaced Dirac deltas:

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s).$$

The pulse train is the used to modulate a continuous-time signal:

$$x_s(t) = p(t)\,x(t).$$

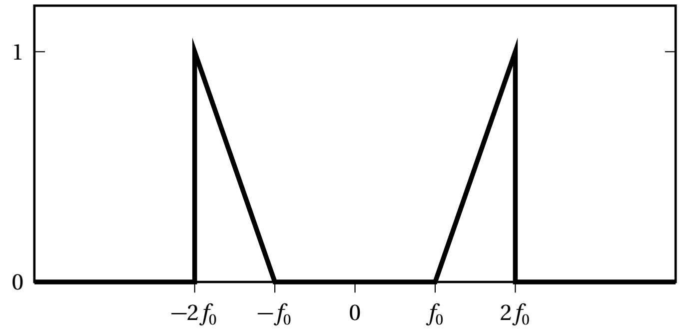
Intuitively, $x_s(t)$ represents a “hybrid” signal where the nonzero values are only those of the discrete time samples that would be obtained by raw-sampling $x(t)$ with period T_s; however, instead of representing the samples a countable sequence (i.e. with a different mathematical object) we are still using a continuous-time signal that is nonzero only over infinitesimally short instants centered on the sampling times. Using Dirac deltas allows us to embed the instantaneous sampling values in the signal.
Note that the Fourier Transform of the pulse train is

$$P(f) = F_s \sum_{k=-\infty}^{\infty} \delta\big(f - kF_s\big)$$

(where, as per usual, $F_s = 1/T_s$). This result is a bit tricky to show, but the intuition is that a periodic set of pulses in time produces a periodic set of pulses in frequency and that the spacing between pulses frequency is inversely proportional to the spacing between pulses in time.
Derive the Fourier transform of $x_s(t)$ and show that if $x(t)$ is bandlimited to $F_s/2$, where $F_s = 1/T_s$, then we can reconstruct $x(t)$ from $x_s(t)$.

Exercise 4. Bandpass sampling

Consider a real, continuous-time signal $x_c(t)$ with the following spectrum $X_c(f)$:



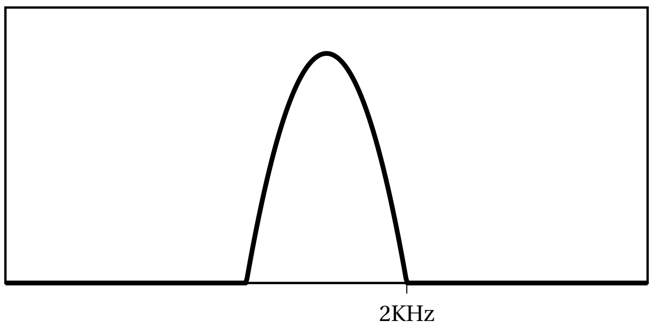
(a) What is the bandwidth of the signal? What is the minimum sampling frequency that satisfies the sampling theorem?

(b) If we sample the signal with a sampling frequency $F_a = 2f_0$, clearly there will be aliasing. Plot the DTFT of the resulting discrete-time signal $x_a[n] = x_c(n/F_a)$.

(c) Suggest a way to perfectly reconstruct $x_c(t)$ from $x_a[n]$.

(d) From the previous example it would appear that we can exploit "gaps" in the original spectrum to reduce the sampling frequency necessary to losslessly sample a bandpass signal. In general, what is the minimum sampling frequency that we can use to sample with no loss a real-valued signal whose frequency support on the positive axis is $[f_0, f_1]$ (with the usual symmetry around zero, of course)?

Exercise 5. Aliasing or not.
Consider a bandlimited continuous-time signal $x(t)$ with the following spectrum $X(f)$:



Sketch the DTFT of the discrete-time signal $x[n] = x(n/F_s)$ for the cases $F_s = 4\text{KHz}$ and $F_s = 2\text{KHz}$.

Exercise 6. Multirate identities
Prove the following two identities:

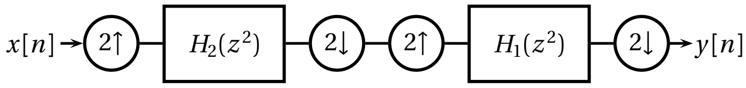
(a) Downsampling by 2 followed by filtering by $H(z)$ is equivalent to filtering by $H(z^2)$ followed by downsampling by 2.

(b) Filtering by $H(z)$ followed by upsampling by 2 is equivalent to upsampling by 2 followed by filtering by $H(z^2)$.

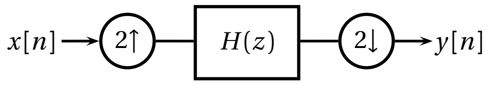
Exercise 7. Multirate systems.
Consider the input-output characteristic of the following multirate systems. Remember that, technically, one cannot talk of transfer functions in the case of multirate systems since sampling rate changes are not

time-invariant. It may happen, though, that by carefully designing the processing chain, the input-output characteristic does indeed implement a time-invariant transfer function.

(a) Find the overall transformation operated by the following system:



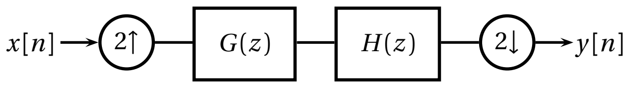
(b) Assume $H(z) = A(z^2) + z^{-1}B(z^2)$ for arbitrary $A(z)$ and $B(z)$. Show that the transfer function of the following system is equal to $A(z)$.

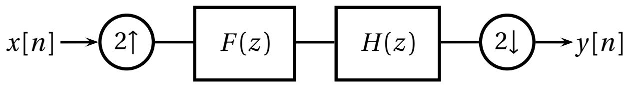


(c) Let $H(z)$, $F(z)$ and $G(z)$ be filters satisfying

$$H(z)G(z) + H(-z)G(-z) = 2$$
$$H(z)F(z) + H(-z)F(-z) = 0$$

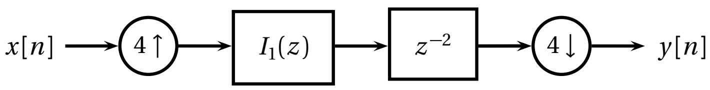
Prove that one of the following systems is unity and the other zero:





Exercise 8. Fractional resampling with multirate.

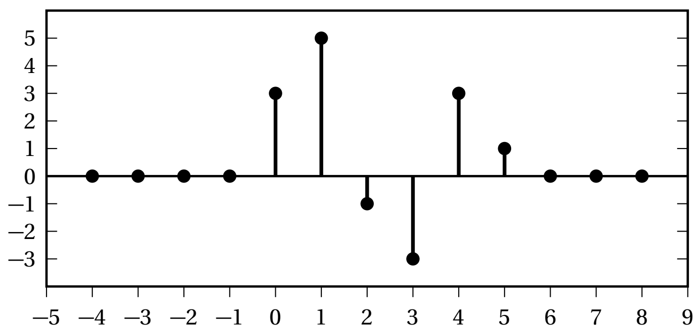
Consider the following multirate processing system:



where $I_1(z)$ is the first-order discrete-time interpolator with impulse response

$$i_1[n] = \begin{cases} 1 - |n|/4 & \text{for } |n| < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Assume $x[n]$ is the finite-support signal shown here:



Compute the values of $y[n]$ for $0 \le n \le 6$, showing your calculation method.

Exercise 9. Quantization.

Consider a stationary i.i.d. random process $x[n]$ whose samples are uniformly distributed over the $[-1,1]$ interval. Consider a quantizer $\mathcal{Q}\{\cdot\}$ with the following characteristic:

$$\mathcal{Q}\{x\} = \begin{cases} -1 & \text{if } -1 \le x < -0.5 \\ 0 & \text{if } -0.5 \le x \le 0.5 \\ 1 & \text{if } 0.5 < x \le 1 \end{cases}$$

Compute the power of the quantization error.

Exercise 10. Quantization.

Consider a stationary i.i.d. random process $x[n]$ whose samples are uniformly distributed over the $[-1,2]$ interval. The process is uniformly quantized with a 1-bit quantizer with the following characteristic:

$$\mathcal{Q}\{x\} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$$

Compute the signal to noise ratio at the output of the quantizer

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Exercise 11. Deadzone quantizers.

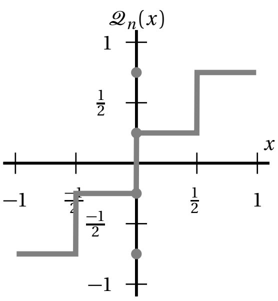
A *deadzone* quantizer is a quantizer that has a quantization interval centered around zero. To see the effects of the deadzone quantizer on SNR consider an i.i.d. discrete-time process $x[n]$ whose values are in the $[-1,1]$ interval. Consider the following uniform 2-bit quantizers for the interval:

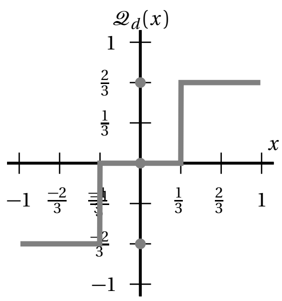
normal quantizer

$$\mathcal{Q}_n(x) = \begin{cases} 3/4 & \text{if } 1/2 \le x \le 1 \\ 1/4 & \text{if } 0 \le x < 1/2 \\ -1/4 & \text{if } -1/2 \le x < 0 \\ -3/4 & \text{if } -1 \le x < -1/2 \end{cases}$$

deadzone quantizer

$$\mathcal{Q}_n(x) = \begin{cases} 2/3 & \text{if } 1/3 \le x \le 1 \\ 0 & \text{if } |x| < 1/3 \\ -2/3 & \text{if } -1 \le x \le -1/3 \end{cases}$$





Both quantizers operate at two bits per sample but the deadzone quantizer "wastes" a fraction of a bit since it has only 3 quantization intervals instead of 4; for a uniformly distributed input, therefore, the SNR of the deadzone quantizer is smaller than the SNR of the standard quantizer. Assume now that the probability distribution for each input sample is the following:

$$P[x[n] = \alpha] = \begin{cases} 0 & \text{if } |\alpha| > 1 \\ p & \text{if } |\alpha| = 0 \\ (1-p)/2 & \text{otherwise} \end{cases}$$

In other words, each sample is either zero with probability p or drawn from a uniform distribution over the $[-1,1]$ interval; we can express this distribution as a pdf like so:

$$f(x) = \frac{1-p}{2} + p\delta(x)$$

Determine the minimum value of p for which it is better to use the deadzone quantizer, i.e. the value of p for which the SNR of the deadzone quantizer is larger that the SNR of the uniform quantizer.