DSP Coursera II 6.5.24

6. The z-Transform

Mapping between complex sequences and analytical functions (on the complex plane).

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}, z \in \mathbb{C}$$

z-Transform is not an analysis tool per se.
↳ no new physical insight.

2 key features:

- easily solve CCDE's as algebraic functions (equations)
- close association to the DTFT:
  - stability criteria for design of filters
  - z-Transform calculated at unit circle $z = e^{j\omega}$
    ↳ DTFT of the sequence

6.1. Filter Analysis

- Linearity:

  Given sequences $x[n]$ and $y[n]$ and their z-transforms $X(z)$ and $Y(z)$

  $$\mathcal{Z}\{\alpha x[n] + \beta y[n]\} =$$
  $$\alpha X(z) + \beta Y(z)$$

- Time-shift:

  Given a sequence $x[n]$ and its z-transform $X(z)$

  $$\mathcal{Z}\{x[n-N]\}$$
  $$= z^{-N} X(z)$$

6.1.1. Solving CCDEs

Generic CCDE:

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] - \sum_{k=1}^{N-1} a_k y[n-k]$$

↳ apply z-transform to both sides:

$$Y(z) = \sum_{k=0}^{M-1} b_k z^{-k} X(z) - \sum_{k=1}^{N-1} a_k z^{-k} Y(z)$$

$$= \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N-1} a_k z^{-k}} X(z)$$

$$= H(z) X(z)$$

$H(z)$: transfer function of the LTI system

Properties of transfer function ($H(z)$)

- transfer function of a realizable filter is a rational transfer function
  - ↳ ratio of finite-degree polynomials in z
- transfer function evaluated on the unit circle:
  - ↳ frequency response of filter
  - ↳ obtaining frequency response directly from the CCDE
- transfer function is the z-transform of the impulse response
- z-transform version of convolution theorem

$$y[n] = x[n] * h[n]$$

$x[n], h[n]$: square-summable

$$\mathcal{Z}\{y[n]\} = Y(z) = X(z)\,H(z)$$

6.1.3. Region of Convergence (ROC)

Given: sequence $x[n]$

↳ set of points on complex plane for which $\sum x[n]z^{-n}$ exists and is finite

↳ is called: Region of convergence (ROC)

Absolute convergence:

$$z \in ROC\{X(z)\} \Longleftrightarrow \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty$$

Properties of ROC:

- ROC has circular symmetry
  ↳ if $z_0 \in RO$ then the set $\{z \mid |z| = |z_0|\}$ is also in ROC
- see book for more properties

6.1.4. ROC and System Stability

z-transform: quick and easy way to test stability of a linear system

stability:

- BIBO stable
- absolute summability of impulse responce

↳ equivalent to:

z-transform of impulse response is absolutely convergent in $|z| = 1$

→ ROC of transfer function includes unit circle

6.1.5 ROC of Rational transfer Functions and Filter Stability

Poles: denominator is zero

↳ must lie outside of ROC

Determine stability of realizable filter:

Causal filter:

- Find poles → p_0: pole with largest magnitude
- ROC: area outside of circle with radius $|p_0|$ (on complex plane)
- Sability: all poles inside unit circle

Anticausal: all poles outside unit circle

6.2. The Pole-Zero Plot

Rational transfer function can be written explicitley (with CCDE coefficients)

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_{M-1} z^{-(M-1)}}{1 + a_1 z^{-1} + \dots a_{N-1} z^{-(N-1)}}$$

degree of numerator polynomial : $M-1$
" " denominator " : $N-1$

→ in refactored form:

$$H(z) = b_0 \frac{\prod_{n=1}^{M-1} (1 - z_n z^{-1})}{\prod_{n=1}^{N-1} (1 - p_n z^{-1})}$$

where • z_n are the $M-1$ complex roots of the numerator polynome
• p_n are the $N-1$ complex roots of the denominator polynome

$\left.\begin{array}{l} z_n : \text{zeros} \\ p_n : \text{poles} \end{array}\right\}$ of the LTI system graphically:

- zeroes: o
- poles: x

Stability (visually):

- causal system:
  all poles (crosses) inside of unit circle
- acausal system:
  all poles (crosses) outside of unit circle

6.2.1. Pole-Zero Patterns

Real valued filters

- coefficients are real valued (common case, only this case is handled in the book)
- roots: - either real
  - or complex conjugate pairs

↳ pole-zero plot symmetric around real axis

Linear-Phase FIR Filters

- FIR filters have no poles
- Symmetry:
  - if z_0 is a (complex) zero of syst. then $1/z_0$ is a zero as well
  - for real valued FIR filters the complex conjugates are zeros, too

6.2.2. Pole-Zero Cancellation

By cascading filters poles and zeros of different (cascaded) filter can cancel each other out.
This can be used to stabilise a filter.
In practical realization: consider numerical instability!