

ch11. Multirate Signal Processing

- Changing sampling rate in the discrete-time domain

11.1. Downsampling

- downsampling by N
 - lower-rate sequence
 - keep only one out of N samples
- also called subsampling or decimation
- downsampling operator \mathcal{D}_N :

$$x_{ND}[n] = \mathcal{D}_N\{x[n]\} = x[nN]$$

- loss of information: discards $N-1$ out of N samples



11.1.1. Properties of the Downsampling Operator

Example : : downsampling by 2

• operator D_2

• if $x_{2D}[n] = D_2\{x[n]\}$

$\Rightarrow x[n] = \dots, x[-2], x[-1], x[0], x[1], x[2], \dots$

$\Rightarrow x_{2D}[n] = \dots, x[-4], x[-2], x[0], x[2], x[4], \dots$

• time origin is important

$D_2\{x[n+1]\} = \dots, x[-3], x[-1], x[1], \dots$

\Rightarrow downsampling operator is
not time-invariant

• periodically time-varying

• downsampling operator is linear

• not LTI due to no time-invariance

• complex sinusoids are no longer
'eigensequences'

11.1.2 Frequency-Domain Representation

- Downsampling by N
- z -transform of downsampled signal

$$X_{ND}(z) = \sum_{n=-\infty}^{\infty} x[nN] z^{-n}$$

- "auxiliary" z -transform

$$X_a(z) = \sum_{n=-\infty}^{\infty} x[nN] z^{-nN}$$

$$\hookrightarrow X_{ND}(z) = X_a(z^{1/N})$$

- $X_a(z)$: derive from $X(z)$ (original signal's z -transform) by "killing off" terms in z -transform whose index are not multiple of N

$$\hookrightarrow X_a(z) = \sum_{n=-\infty}^{\infty} \sum_N[n] x[n] z^{-n}$$

where $\sum_N[n]$ is selector

$$\sum_N = \begin{cases} 1 & \text{for } n \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

• Fourier transform of downsampled signal:

$$X_{ND}(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X(W_N^k z^{1/N})$$

• Aliasing can occur due to the loss of information in the downsampled signal

• non-aliasing condition

• max. positive freq. of original spectrum (ω_M)

$$\omega_M < \pi/N$$

• fulfillment of non-aliasing condition indicates that original signal is intrinsically redundant
↳ downsampled signal can ^{used} be completely reconstruct original signal

Upsampling (Video)

Increase number of samples of discrete-time sequence



- Interpolate between samples
- Perfect upsampling: interpolate and low-pass filter

Downsampling (Video)

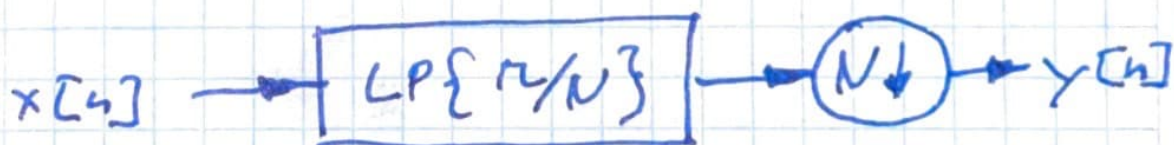
Discard samples from sequence



- loss of information
- aliasing occurs
- first low-pass with $\omega_0 = \pi/N$

11.1.4. Downsampling and Filtering

- Filter signal prior to downsampling because of aliasing
- eliminate aliasing by removing the high frequency components
- high frequencies would fold back onto lower frequencies
- for downsampling by N :
lowpass filter with cutoff freq:
 $\omega_c = \pi/N$



- some information is lost by filtering but the distortion is controlled and less disruptive than foldover aliasing

11.2. Upsampling

- create higher-rate sequence by creating N samples out of every sample of original signal
 - ↳ upsampling by N
- simply inserting $N-1$ zeroes between every two input samples
- $x_{Nu}[n] \mathcal{U}_N \{x[n]\} = \begin{cases} x[k] & \text{for } n = kN, \\ 0 & \text{otherwise} \end{cases} \quad \begin{matrix} k \in \mathbb{Z} \\ k \in \mathbb{Z} \end{matrix}$

\mathcal{U}_N : upsampling operator

- upsampling is "nicer" than downsampling
 - ↳ no information is lost
- original signal can be recovered by downsampling

$$\text{↳ } \mathcal{D}_N \{ \mathcal{U}_N \{ x[n] \} \} = x[n]$$

- spectral description of upsampling is simpler in z -transform domain:

$$\begin{aligned} X_{Nu}(z) &= \sum_{n=-\infty}^{\infty} x_{Nu}[n] z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x[k] z^{-kN} = X(z^N) \end{aligned}$$

$$\hookrightarrow X_{N4}(e^{j\omega}) = X(e^{j\omega N})$$

- Upsampling is only a contraction of the frequency axis by factor N
- 2π periodicity of spectrum must be taken into account
- Upsampling operator:



11.2.1. Upsampling and Interpolation

- Upsampled signal: 2 drawbacks
 - time domain: signal not "natural" since $N-1$ zeros between every sample \rightarrow not "smooth"
 - frequency domain:
 - repetition of base spectrum drawn in by upsampling do not look as if they belong to the $[-\pi, \pi]$ interval
- \rightarrow these two problems are one and the same
- \Rightarrow solve by appropriate filter



- filling the gaps between nonzero samples in upsampled sequence is similar to discrete - to continuous-time interpolation, except now we operate entirely in discrete time

• Zero-Order Hold

- piecewise-constant interpolation
- after upsampling by N , use filter with impulse response

$$h_0[n] = \begin{cases} 1 & n=0, 1, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

↳ simply repeats the original sample N times (staircase approximation)

• First-Order Hold

- piecewise linear interpolation after upsampling by N

$$h_1[n] = \begin{cases} 1 - \frac{|n|}{N}, & |n| < N \\ 0 & \text{otherwise} \end{cases}$$

- impulse response: triangular function

↳ triangular fn: convolution of two rects
 $h_1[n] = \frac{1}{N} (h_0[n] * h_0[n])$

- Sinc Interpolation

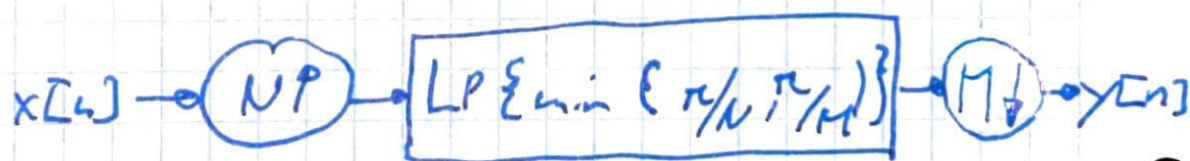
- smoothest interpolation

- interpolation filter: discrete time sinc

$$h[n] = \text{sinc}\left(\frac{n}{N}\right)$$

11.3. Rational Sampling Rate Changes

- Arbitrary rational sampling rate changes
 - combining upsampling and downsampling
- Typically : rate change by N/M cascading
 - upsample by N
 - lowpass filter
 - downsample by M
- Filter's cutoff freq: $\min(\{\pi/N, \pi/M\})$
 - both (up- and downsampling) require a lowpass filter. The one with the minimum cutoff freq. dominates the cascade



- Upsampling and downsampling operators are generally not commutative
only if N and M are coprime they do commute

11.4. Oversampling

- use higher sampling rate than needed by sampling theorem
- improve performance of A/D and D/A converters

11.4.1. Oversampled A/D conversion

- Sampling theorem: if
 - continuous-time signal $x(t)$
 - bandlimited
 - choose sampling period T_s
- ↳ no error introduced by sampling

⇒ Only source of error in A/D conversion
↳ quantization error

⇒ oversampling reduces this error

- quantization error: additive noise
 - if $x(t)$ is Ω_N -bandlimited signal
and $T_s = \pi/\Omega_N$
- $$\hat{x}[n] = x(nT_s) + e[n]$$

- with $e[n]$: white process
- variance

$$P_e = \frac{\Delta^2}{12}$$

where Δ : quantization interval

- signal to noise ratio of oversampled signal is still the same
- It's easy to implement a digital filter that removes the noise outside of the support
 - ↳ this improves SNR
- after out-of-band noise is removed
 - ↳ use downsampler to obtain critically sampled signal