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ch11. Multirate Signal Processing

- Changing sampling rate in the discrete-time domain

11.1. Downsampling

- downsampling by N
  - lower-rate sequence
  - keep only one out of N samples
- also called subsampling or decimation
- downsampling operator $\mathcal{D}_N$:

$$x_{ND}[n] = \mathcal{D}_N\{x[n]\} = x[nN]$$

- loss of information: discards $N-1$ out of N samples

- $x[n] \longrightarrow N\downarrow \longrightarrow x_{ND}[n]$

11.1.1. Properties of the Downsampling Operator

Example: · downsampling by 2
· operator $\mathcal{D}_2$
· if $x_{2D}[n] = \mathcal{D}_2\{x[n]\}$

$\rightsquigarrow$ $x[n] = \dots, x[-2], x[-1], x[0], x[1], x[2], \dots$

$\Rightarrow$ $x_{2D}[n] = \dots, x[-4], x[-2], x[0], x[2], x[4] \dots$

· time origin is important

$\mathcal{D}_2\{x[n+1]\} = \dots, x[-5], x[-3], x[1], \dots$

$\Rightarrow$ downsampling operator is not time-invariant

· periodically time-varying

· downsampling operator is linear

· not LTI due to no time-invariance

· complex sinusoids are no longer 'eigensequences'

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11.1.2 Frequency-Domain Representation

- Downsampling by N
- z-tranform of downsampled signal

$$X_{ND}(z) = \sum_{n=-\infty}^{\infty} x[nN] z^{-n}$$

- "auxiliary" z-transform

$$X_a(z) = \sum_{n=-\infty}^{\infty} x[nN] z^{-nN}$$

$\hookrightarrow \quad X_{ND}(z) = X_a(z^{1/N})$

- $X_a(z)$: derive from $X(z)$ (original signals z-transform) by "killing off" terms in z-transform whose index are not multiple of N

$$\hookrightarrow \quad X_a(z) = \sum_{n=-\infty}^{\infty} \xi_N[n] x[n] z^{-n}$$

where $\xi_N[n]$ is selector

$$\xi_N = \begin{cases} 1 & \text{for } n \text{ multiple of } N \\ 0 & \text{otherwise} \end{cases}$$

- Fourier transform of downsampled signal:

$$X_{ND}(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X(W_N^k z^{1/N})$$

- Aliasing can occur due to the loss of information in the down-sampled signal

- non-aliasing condition
  - max. positive freq. of original spectrum (ω_M)

$$\omega_M \overset{!}{<} \pi/N$$

- fulfillment of non-aliasing condition indicates that original signal is intrinically redundant
  - ↳ downsampled signal can be used to completely reconstruct original signal

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Upsampling (Video)

Increase number of samples of discrete-time sequence

$$x[n] \longrightarrow N\uparrow \longrightarrow x_{NU}[n]$$

- Interpolate between samples
- Perfect upsampling: interpolate and low-pass filter

Downsampling (Video)

Discard samples from sequence

$$x[n] \longrightarrow N\downarrow \longrightarrow x_{ND}[n]$$

- loss of information
- aliasing occurs
- first low-pass with $\omega_0 = \pi/N$

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11.1.4. Downsampling and Filtering

- Filter signal prior to downsampling because of aliasing
- eliminate aliasing by removing the high frequency components
- high frequencies would fold back onto lower frequencies
- for downsampling by N:

  lowpass filter with cutoff freq:

  $$\omega_c = \pi / N$$

$$x[n] \longrightarrow \boxed{LP\{\pi/N\}} \longrightarrow (N\downarrow) \longrightarrow y[n]$$

- some information is lost by filtering but the distortion is controlled and less disruptive than foldover aliasing

11.2. Upsampling

- create higher-rate sequence by creating N samples out of every sample of original signal
  ↳ upsampling by N
- simply inserting $N-1$ zeroes between every two input samples

- $x_{Nu}[n]\ \mathcal{U}_N\{x[n]\} = \begin{cases} x[k] & \text{for } n = kN,\ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$

$\mathcal{U}_N$: upsampling operator

- upsampling is „nicer" than downsampling:
  no information is lost
- original signal can be recovered by downsampling
  ↳ $\mathcal{D}_N\{\mathcal{U}_N\{x[n]\}\} = x[n]$
- spectral description of upsampling is simpler in z-transform domain:

$$X_{Nu}(z) = \sum_{n=-\infty}^{\infty} x_{Nu}[n]\, z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} x[k]\, z^{-k\cdot N} = X(z^N)$$

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↳ $X_{N\uparrow}(e^{j\omega}) = X(e^{j\omega N})$

- Upsampling is only a contraction of the frequency axis by factor N

- 2π periodicity of spectrum must be taken into account

- Upsampling operator:

$$x[n] \longrightarrow N\uparrow \longrightarrow x_{N\uparrow}[n]$$

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11.2.1. Upsampling and Interpolation

- Upsampled signal: 2 drawbacks
  - time domain: signal not "natural" since $N-1$ zeros between every sample
    ↪ not "smooth"
  - frequency domain:
    - repetition of base spectrum drawn in by upsampling do not look as if they belong to the $[-\pi, \pi]$ interval

↪ these two problems are one and the same

⇒ solve by appropriate filter

$$x[n] \longrightarrow \boxed{N\uparrow} \longrightarrow \boxed{LP\{\pi/N\}} \longrightarrow y[n]$$

- filling the gaps between nonzero samples in upsampled sequence is similar to discrete- to continuous-time interpolation, except now we operate entirely in discrete time

- Zero-Order Hold
  - piecewise-constant interpolation
  - after upsampling by N, use filter with impulse response

$$h_0[n] = \begin{cases} 1 & n = 0, 1, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

↳ simply repeats the original sample N times (staircase approximation)

- First-Order Hold
  - piecewise linear interpolation after upsampling by N

$$h_1[n] = \begin{cases} 1 - \frac{|n|}{N}, & |n| < N \\ 0 & \text{otherwise} \end{cases}$$

- impulse response: triangular function
  ↳ triangular fn: convolution of two rects $h_1[n] = \frac{1}{N}(h_0[n] * h_0[n])$

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- Sinc Interpolation
  - smoothest interpolation
  - interpolation filter: discrete time sinc

$$h[n] = \mathrm{sinc}\left(\frac{n}{N}\right)$$

11.3. Rational Sampling Rate Changes

- Arbitrary rational sampling rate changes
  - combining upsampling and downsampling

- Typically: rate change by N/M cascading
  - upsample by N
  - lowpass filter
  - downsample by M

- Filters cutoff freq: $\min(\{\pi/N, \pi/M\})$
  - both (up- and downsampling) require a lowpass filter). The one with the minimum cutoff freq. dominates the cascade

$$x[n] \to \boxed{N\uparrow} \to \boxed{LP\{\min\{\pi/N, \pi/M\}\}} \to \boxed{M\downarrow} \to y[n]$$

- upsampling and downsampling operators are generally not commutative
  olny if N and M are coprime they do commute

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11.4. Oversampling

- use higher sampling rate than needed by sampling theorem
- improve performance of A/D and D/A converters

11.4.1. Oversampled A/D conversion

- Sampling theorem: if
  - continous-time signal $x(t)$
  - bandlimited
  - choose sampling period T_s

  ↳ no error introduced by sampling

⇒ Only source of error in A/D conversion
↳ quantization error

⇒ oversampling reduces this error

- quantization error: additive noise
- if $x(t)$ is Ω_N-bandlimited signal and $T_s = \pi / \Omega_N$

$$\hat{x}[n] = x(nT_s) + e[n]$$

- with $e[n]$: white process
- variance

$$P_e = \frac{\Delta^2}{12}$$

where Δ: quantization intervall

- signal to noise ratio of oversampled signal is still the same
- It's easy to implement a digital filter that removes the noise outside of the support
  ↳ this improves SNR
- after out-of-band noise is removed
  ↳ use downsampler to obtain critically sampled signal