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Ch. 3 Signals and Hilbert Spaces

3.1. Euclidian Geometry

- Inner product

$$\langle x, y \rangle = \sum_{n=0}^{N-1} x_n y_n \qquad x, y \in \mathbb{R}^N$$

orthogonality : x and y are orthogonal if inner product is zero: $x \perp y \Leftrightarrow \langle x, y \rangle = 0$

- Norm ('length')

$$\|x\|_2 = \sqrt{\sum_{n=0}^{N-1} x_n^2} = \langle x, x \rangle^{1/2}$$

- Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \|x\|_2 \|y\|_2 \qquad x, y \in \mathbb{R}^N$$

- Distance (euclidian distance)

$$d(x, y) = \|x - y\|_2 = \sqrt{\sum_{n=0}^{N-1} (x_n - y_n)^2}$$

↳ Pythagorean theorem for N dimensions
if vectors are orthogonal $x \perp y$ and sum $z = x + y$

$$\|z\|_2^2 = \|x\|_2^2 + \|y\|_2^2$$

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- Bases

Bases span $\mathbb{R}^N$ and can be used to express any vector in $\mathbb{R}^n$ as linear compinations from the vectors in the base.

a base must contain at least N linearly independent vectors:

$$\sum_{k=0}^{N-1} \beta_k y^{(k)} = 0$$

for N vectors $\{y^{(k)}\}_{k=0 \ldots N-1}$

orthonormal bases:

$$\langle y^{(k)}, y^{(h)} \rangle = \begin{cases} 1 \text{ if } k=h \\ 0 \text{ otherwise} \end{cases}$$

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3.2. From Vector Spaces to Hilbert Spaces

Hilbert space: vector space that
generalize vectors $\mathbb{C}^N$ for $N \to \infty$

constraints on elements in Hilbert space
↳ signals need to have finite energy

3.2.1 The Recipe for Hilbert Space

Vectors in Hilbert space are not just
N-tuples (as in Euclidian Space). Can be anything, also a function

Hilbert spaces:
- vector space with 2 additional features
  1. existence of inner product
  2. property of completeness

- Vector Space
  - Set of vectors V
  - Set of scalars S $S \in \mathbb{C}^N$ (or $S \in \mathbb{R}^N$)

Vector space $H(V, S)$ completely defined

- vector addition
- Scalar multiplication

satisfying following properties for
vectors $x, y, z \in V$
scalars $\alpha, \beta \in S$

- Addition is commutative: $x + y = y + x$
- Addition is associative: $(x+y)+z = x+(y+z)$
- Scalar multiplication is distributive:

$$\alpha(x+y) = \alpha x + \alpha y$$
$$(\alpha+\beta)x = \alpha x + \beta x$$
$$\alpha(\beta x) = (\alpha\beta)x$$

- Null vector exists (0):

$$x + 0 = 0 + x = x, \forall x \in V$$

- Additive inverse vector exists $(-x)$:

$$x + (-x) = (-x) + x = 0, \forall x \in V$$

- identy element for scalar multiplication exists (1)

$$1 \cdot x = x \cdot 1 = x \quad \forall x \in V$$

- Inner Product Space

  Inner product is essential to build a notion of distance between elements

  Inner product for $H(V, S)$ is a function from $V \times V \to S$ with following properties:

  - distribution with respect to vector addition

    $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

  - scaling property with respect to scalar multiplication

    $\langle x, \alpha y\rangle = \alpha \langle x, y\rangle$

    $\langle \alpha x, y\rangle = \alpha^* \langle x, y\rangle$! Note: complex-conjugation!

  - commutative within complex conjugation

    $\langle x, y\rangle = \langle y, x\rangle^*$

- self product is positive

$$\langle x, x \rangle \geq 0$$

$$\langle x, x \rangle = 0 \Leftrightarrow x = 0$$

↳ $x \perp y$ iff $\langle x, y \rangle = 0$

⇒ norm

$$\|x\| = \langle x, x \rangle^{1/2}$$

the norm satisfies the Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

triangle inequality

$$\|x+y\| \leq \|x\| \cdot \|y\| \text{ iff } x = \alpha \cdot y$$

• Hilbert Spaces

Another requirement for Hilbert Spaces is completeness:

⇒ Convergent sequences of vectors i V have a limit that is also in V

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3.2.2 Examples of Hilbert Spaces

- Finite Euclidian Spaces

  Vector space $\mathbb{C}^N$

  ↳ Sum: $z = x + y$
  element wise summation
  $z_n = x_n + y_n$

  ↳ inner product:
  $$\langle x, y \rangle = \sum_{n=0}^{N-1} x_n^* y_n$$

- Polynomial Functions

  Vector space $\mathbb{P}_N([0,1])$
  - polynomial functions on interval $[0,1]$
  - max. degree N

  ! $\mathbb{P}_\infty([0,1])$ is not complete

  ↳ converges to:

  $$\lim_{n\to\infty} p_n(x) \to e^x \in \mathbb{P}_\infty([0,1])$$

· Square Summable Functions

(square integrable functions)

e.g : $L_2([-\pi,\pi])$: space of real or complex functions on interval $[-\pi,\pi]$

· Inner product:

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f^*(t)\, g(t)\, dt$$

· Norm of $f(t)$

$$\|f\| = \sqrt{\int_{-\pi}^{\pi} |f(t)|^2\, dt}$$

for $f(t)$ to belong to $L_2([-\pi,\pi])$ it must be $\|f\| < \infty$

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3.2.3 Inner Product and Distances

- Inner product allows definition of distance
- inner product $\langle x, y \rangle$: orthogonal projection of y over x

  in $\mathbb{R}^2$: $\langle x, y \rangle = \|x\| \|y\| \cos\theta$
- inner product: qualitative measure of similarity between vectors
- quantitative measure of similarity: distance: $d(x,y) = \langle x-y, x-y \rangle^{1/2}$

  $= \|x-y\|$
- Distance for $\mathbb{C}^N$:

$$d(x,y) = \sqrt{\sum_{n=0}^{N-1} |x_n - y_n|^2}$$

- Distance for $L_2([-\pi, \pi])$

$$d(x,y) = \sqrt{\int_{-\pi}^{\pi} |x(t) - y(t)|^2 \, dt}$$

- Euclidian distance:

  root mean square error

often the square of RMS:
mean square error MSE

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3.3. Subspaces, Bases, Projections

- Basis: inner structure of Hilbert Space
- Change of basis: change structure of space to access particular information
  ↳ linear operation
- Subspace: specific subset of a given vector space, restricted region of the global space
  ↳ "closed" under usual vector operations

3.3.1 Definitions

- Hilbert space $H(V, S)$
- Vector space V
- Scalars S (i.e $\mathbb{C}$)

Subspace of V:

subset $P \subseteq V$ with properties

- closure under addition:
  $x \in P$ and $y \in P \Rightarrow x + y \in P$
- closure under scalar multiplication
  $x \in P$ and $\alpha \in S \Rightarrow \alpha x \in P$

Span:

Given arbitrary set of M vectors:

$$W = \{x^{(m)}\}_{m=0,1,\dots M-1}$$

Span:

$$\text{span}(W) = \sum_{m=0}^{N-1} a_m x^{(m)}, \quad a_m \in S$$

↳ span of W is the set of all possible linear combinations of the vectors in W

W: linearly independent, if:

$$\sum_{m=0}^{N-1} a_m x^{(m)} = 0 \Leftrightarrow a_m = 0 \quad \text{for } m=0,\dots,N-1$$

Basis:

A set of K vectors $W = \{x^{(k)}\}_{k=0,\dots,K-1}$ from subspace P is a basis if:

- set W is linearly independent
- Span(W) covers P: Span(W) = P
  - ↳ any $y \in P$ can be written as linear combinations of $\{x^{(k)}\}_{k=0,1,\dots,N-1}$

⇒ set W is complete in P

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Orthonormal Basis

- for subspace P:

  a set of K basis vectors $W = \{x^{(k)}\}_{k=0,\dots,K-1}$

  such that

  $\langle x^{(i)}, y^{(j)} \rangle = \delta[i-j] \quad 0 \leq i,j < K$

  ↳ orthogonality across vectors and unit norm

- the most "beautiful" bases
- important property:

  A set of N orthogonal vectors in an N-dimensional subspace is a basis for that subspace

  ↳ = in finite dimensions: if we find a full set of orthogonal vectors, we know they span the set

Orthogonal Basis:

- if set of vectors orthogonal but not unit norm

  ↳ use normalization factor

  or Gramm-Schmidt procedure

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3.3.2 Properties of Orthonormal Bases

$W = \{x^{(k)}\}_{k=0,\dots,K-1}$: orthonormal basis for a (sub)space

- Synthesis Formula:

$$y = \sum_{k=0}^{K-1} a_k x^{(k)}$$

- Analysis Formula

$$a_k = \langle x^{(k)}, y \rangle$$

a_k : Fourier coefficients

- Parseval's Identity

Norm conservation property for an orthonormal basis.

$$\|y\|^2 = \sum_{k=0}^{K-1} |\langle x^{(k)}, y \rangle|^2$$

- norm is equivalent to energy in pysical systems

$\Rightarrow$ energy conservation formula

- Bessel's Inequality
  Generalization of Parseval's equality
  - Subspace P
  - L orthogonal vectors $G \subset P$

  $G = \{g^{(l)}\}_{l=0,\dots,L-1}$ (not necessarily a basis since ~~to~~ maybe $L < K$)

  norm of any vector $y \in P$ is lower bounded:

  $$\|y\|^2 \geq \sum_{l=0}^{L-1} |\langle g^{(l)}, y \rangle|^2$$

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- Best Approximations
  - Approximations that minimize $\|y-\hat{y}\|$
  - Orthonormal basis:

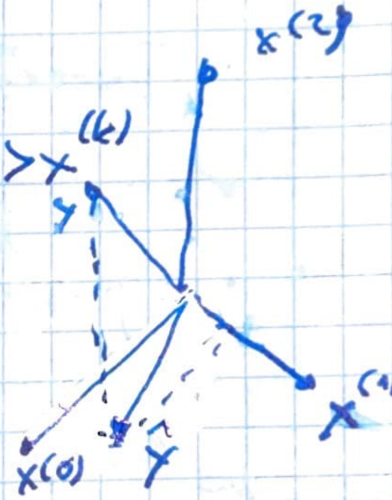
$$P \subset V \qquad \text{(subspace } P\text{)}$$

approximate vector $y \in V$ by linear combination of basis vectors from P

  - W: orthonormal basis for P

Projection:

$$\hat{y} = \sum_{k=0}^{K-1} \langle x^{(k)}, y \rangle x^{(k)}$$



  - Approximation error:

$$d = y - \hat{y}$$

→ error is orthogonal to approximation

$$d \perp \hat{y}$$

→ approximation minimizes error square norm

$$\arg\min_{\hat{y} \in P} \|y-\hat{y}\|_2 = \sum_{k=0}^{K-1} \langle x^{(k)}, y \rangle x^{(k)}$$

→ approximation can be successively refined

3.3.3 Examples of Bases

- Finite Euclidian spaces
  - $\mathbb{C}^N$ orthonormal bases
    - most intuitive
    - N mutually orthogonal vectors of unit length
  - Cannonical basis $\{\delta^{(k)}\}_{k=0,\dots,N-1}$

    $$\delta_n^{(k)} = \delta[n-k]$$

  - Fourier basis $\{w^{(k)}\}$

    $$w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$$

  - Change of basis matrix

$$M_{mn} = \left(x_n^{(m)}\right)^*$$

Analysis formula ↳ $a = My$ — $a \in \mathbb{C}^N$: expansion coefficients $\{a_k\}_{k=0,\dots,N-1}$

Synthesis formula ↳ $y = M^H x$

M^H: Hermitian transpose of M
↳ transposition and conjugation

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- Polynomial Functions
  basis for $\mathbb{P}_N([0,1])$ is $\{x^k\}_{0 \le k < N}$
  - not orthonormal
  - can be transformed to orthonormal basis with Gramm-Schmidt procedure
  - basis vector obtained: Legendre polynomials

- Square Summable Functions
  - for $L_2([-\pi,\pi])$ an orthonormal basis set is $\{(1/\sqrt{2\pi})e^{jnt}\}_{n \in \mathbb{Z}}$
  - Classic fourier basis for functions on an intervall
  - number of basis vectors is infinite

3.4 Signal Spaces Revisited

3.4.1 Finite-Length Signals

Cannonical basis for $\mathbb{C}^N$
(k-th cannonical basis):

$\delta[n-k]$ $\quad n = 0, \dots, N-1$
$\quad k = 0, \dots, N-1$

3.4.2. Periodic Signals

- equivalent to length-N signals
- isomorphic to $\mathbb{C}^N$
  - summation is elementwise

    $z[n] = x[n] + y[n] \quad n \in \mathbb{Z}$

  - inner product over a period only

    $$\langle x[n], y[n] \rangle = \sum_{n=0}^{N-1} x^*[n]\, y[n]$$

- explicitely periodized basis

$$\delta^{(k)} = \sum_{i=-\infty}^{\infty} \delta[n-k-iN]$$

↳ pulse train

2.4.3 Infinite Sequences

· inner Product:

$$\langle x[n], y[n] \rangle = \sum_{n=-\infty}^{\infty} x^*[n]\, y[n]$$

can diverge!

↳ restrict to $l_2(\mathbb{Z})$: square summable sequences for which:

$$\|x\|^2 = \sum_{n \in \mathbb{Z}} |x[n]|^2 < \infty$$

↳ "finite energy"

Cannonical basis for $l_2(\mathbb{Z})$:

$$\delta^{(k)} = \delta[n-k], \quad n,k \in \mathbb{Z}$$