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Ch. 4 Fourier Analysis

Express any repeating (periodic) phenomenon as combined sinusodial "generators".

Any linear time-invariant transformation of a sinusoid is a sinusoid at the same frequency.

Fourier transform: change of basis in appropriate Hilbert Space

4.1. Preliminaries

- Discrete Fourier Transform (DFT):
  - length-N signals → N discrete frequency components
- Discrete Fourier Series (DFS):
  - N-periodic sequences → N discrete frequency comp.
- Discrete-Time Fourier Transform (DTFT):
  - ∞ sequences → 2π periodic functions

4.1.1. Complex Exponentials

Discrete-time complex exponential:
sequence of form:

$$x[n] = A e^{j(\omega n + \varphi)}$$
$$= A\left[\cos(\omega n + \varphi) + j \sin(\omega n + \varphi)\right]$$

- $A \in \mathbb{R}$: Amplitude
- ω : frequency
- φ : initial phase

Discrete-time complex exponential is only periodic if

$$\omega = 2\pi (M/N)$$

- $M, N \in \mathbb{Z}$

Power of complex exponential:
average energy over a period: $|A|^2$
irrespective of frequency

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4.2. The DTF (Discrete Fourier Transform)

N-length signals

↳ find a set of oscillatory signals of length N which contain a whole number of periods (are an integer multiple) over their support.

· Finite length sinusoidal signals (index k)

$w_k[n] = e^{j\omega_k n}$, $n = 0, \dots, N-1$

· ω_k : distinct frequencies

$w_k[N] = w_k[0] = 1$: due to whole number periods

↳ $(e^{j\omega_k})^N = 1$

$\Rightarrow$ $w_k[n] = W_N^{-nk}$, $n = 0, \dots, N-1$

Set of vectors $\{w^{(k)}\}_{k=0,\dots,N-1}$ in $\mathbb{C}^N$

$w^{(k)} = [1 \; W_N^{-k} \; W_N^{-2k} \; \dots \; W_N^{-(N-1)k}]^T$

- Fourier basis for $\mathbb{C}^N$

$$\{w^{(k)}\}_{k=0,\dots,N-1}$$

$\Rightarrow w^{(k)} = [1 \ W_N^{-k} \ W_N^{-2k} \ \dots \ W_N^{-(N-1)k}]^T$

with:

$$W_N = e^{-j\frac{2\pi}{N}}$$

! This is not a orthonormal basis as $\|w^{(k)}\|^2 = N$

$\Rightarrow$ could be made orthonormal by scaling basis vectors by $\frac{1}{\sqrt{N}}$.

Often the normalization factor is kept explicit due to computational efficiency.

Array notation:

$$w_k[n] = e^{+j\frac{2\pi}{N}nk}$$

$k = 0, \dots, N-1$
$n = 0, \dots, N-1$

k: index of signal
n: index of element in signal

$\Rightarrow$ Frequency of signal element: $\omega = \frac{2\pi}{N}k$

4.2.1. Matrix Form

The DFT analysis and synthesis formulas can be expressed in matrix notation:

stacking conjugate of the basis vectors:

$$W_{nk} = e^{-j(2\pi/N)nk} = W_N^{nk} \qquad W \in \mathbb{C}^{N \times N}$$

Analysis:

$$\Rightarrow \quad X = Wx \qquad x \in \mathbb{C}^N$$

synthesis:

$$x = \frac{1}{N} W^H X$$

with:

- X : fourier Coefficients in vector form

DFT preserves energy of finite-length signal. Parseval's relation becomes:

$$\|x\|_2 = \frac{1}{\sqrt{N}} \|X\|_2$$

4.2.2. Explicit Form

Common to write inner product explicitly

Analysis:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}, \quad k = 0, \dots, N-1$$

Synthesis:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk}, \quad n = 0, \dots, N-1$$

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4.2.3 Physical Interpretation

DFT: decomposition of finite N-length signal in N sinusodial components.
Magnitude and phase for each oscillator are given by coefficients $X[k]$

- take array of N complex sinusodial generators
- set frequency of k-th generator to $(2\pi/N)\cdot k$
- set amplitude of k-th generator to $|X[k]|$ (magnitude of k-th DFT coefficient)
- set phase of k-th generator to $\measuredangle X[k]$ (phase of k-th DFT coefficient)
- start generators at same time and sum up their outputs

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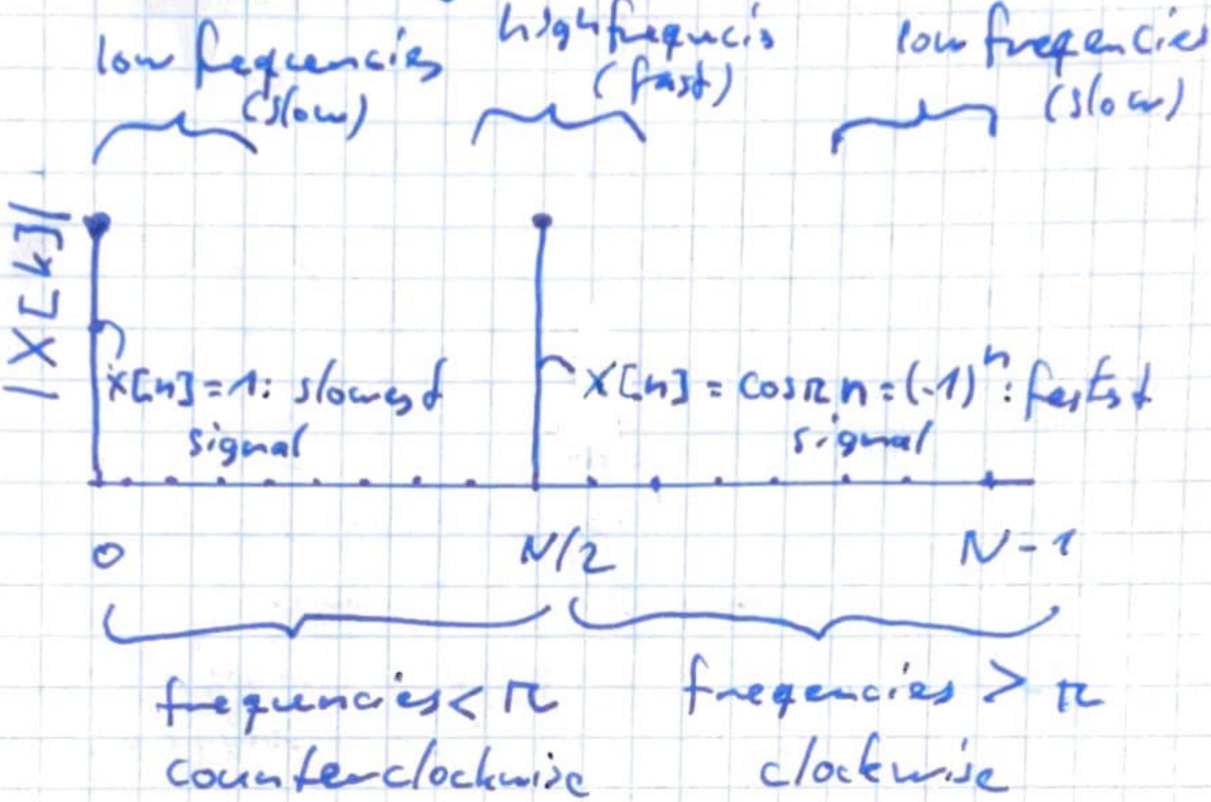
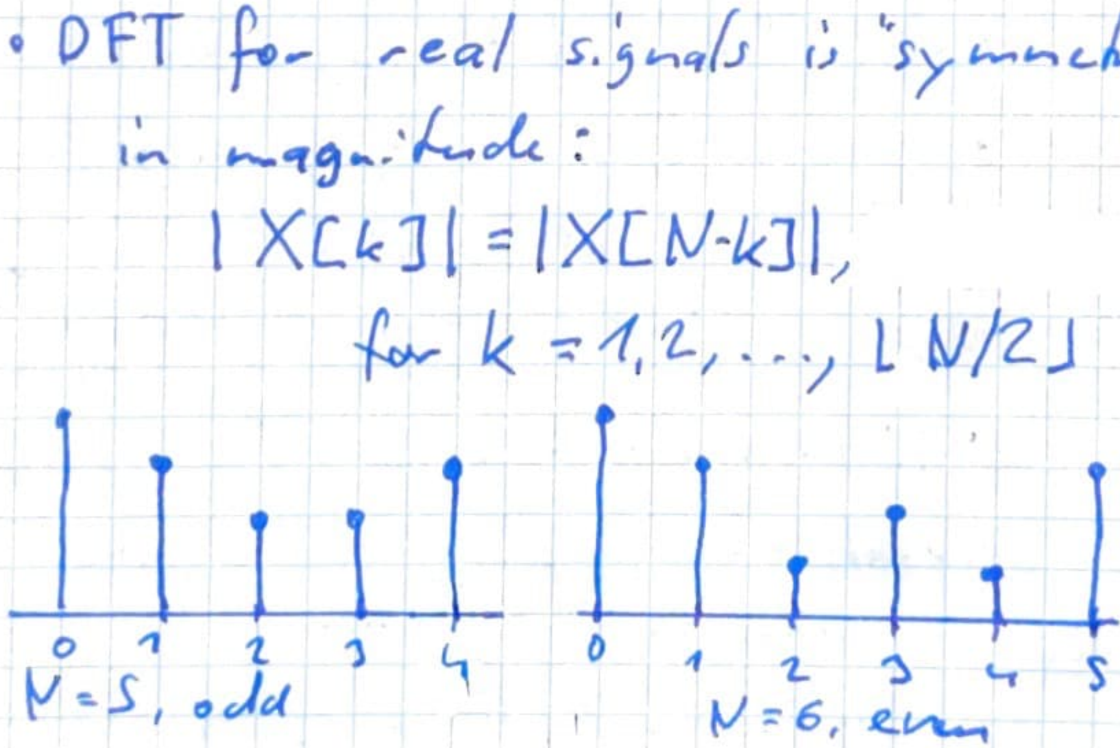
Interpreting a DFT plot



- DFT for real signals is "symmetric" in magnitude:

$$|X[k]| = |X[N-k]|,$$
$$\text{for } k = 1, 2, \ldots, \lfloor N/2 \rfloor$$



For real signals, magnitude plots need only $\lfloor N/2 \rfloor + 1$ points.

Reson for "symmetry":

DFT → complex coefficients which are complex conjugate across $N/2$ index

↳ combine coefficient to get back a real signal (complex conjugates combined eliminate imaginary part)

- Labeling the "frequency" axis:
  - T_s : "clock" of the system
  - fastest (positive) frequency: $\omega = \pi$
  - sinusoid at $\omega = \pi$: needs two samples to do a full revolution

  - time between samples $T_s = 1/F_s$ sec.
  - real-world frequency for fastest sinusoid : $F_s / 2$ Hz
  - continuous frequency corresponding to index k : $\frac{k \cdot F_s}{N}$ (in Hz)

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4.3. The DFS (Discrete Fourier Series)

For discrete, infinite N-periodic signals

Reconstruction formula (DFT):

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk}, n = 0, \dots, N-1$$

with $W_N = e^{-j\frac{2\pi}{N}}$

↳ wrap around for n outside of interval $[0, N-1]$, since

$$W_N^{(n+iN)k} = W_N^{nk}, \forall i \in \mathbb{Z}$$

↳ $x[n+iN] = x[n]$

⇒ Periodic sequences are natural way to embed finite-length signals into sequences

DFS:
- $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk}, k \in \mathbb{Z}$
- $\tilde{x}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{X}[k] W_N^{-nk}, k \in \mathbb{Z}$

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The short-time Fourier Transform (STFT)

- time domain obfuscates frequency
- frequency domain obfuscates time

↳ STFT: combines both

Algorithm:

- take small signal pieces of length L
- look at DFT of each piece

$$X[m, k] = \sum_{n=0}^{L-1} x[m+n]\, e^{-j\frac{2\pi}{L}nk}$$

k: DFT index for that chunk

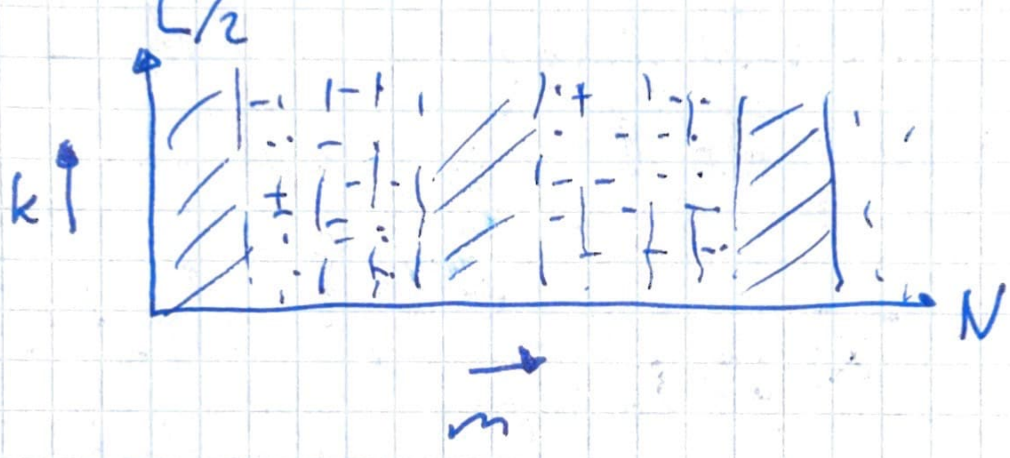
m: starting point for localized DFT

Spectrogram

- showing time varying spectral information of STFT

- color code: · dark → small
  · light → large

- $10 \log_{10}(|X[m;k]|)$: power in dBs
- plot spectral slices one after other



- if we know sampling rate (system clock)

$$F_s = 1/T_s$$

label axes

- Highest frequency (positive): $F_s/2$ Hz
- frequency resolution F_s/L Hz
- Width of time slices: LT_s seconds

Spectrogram / STFT

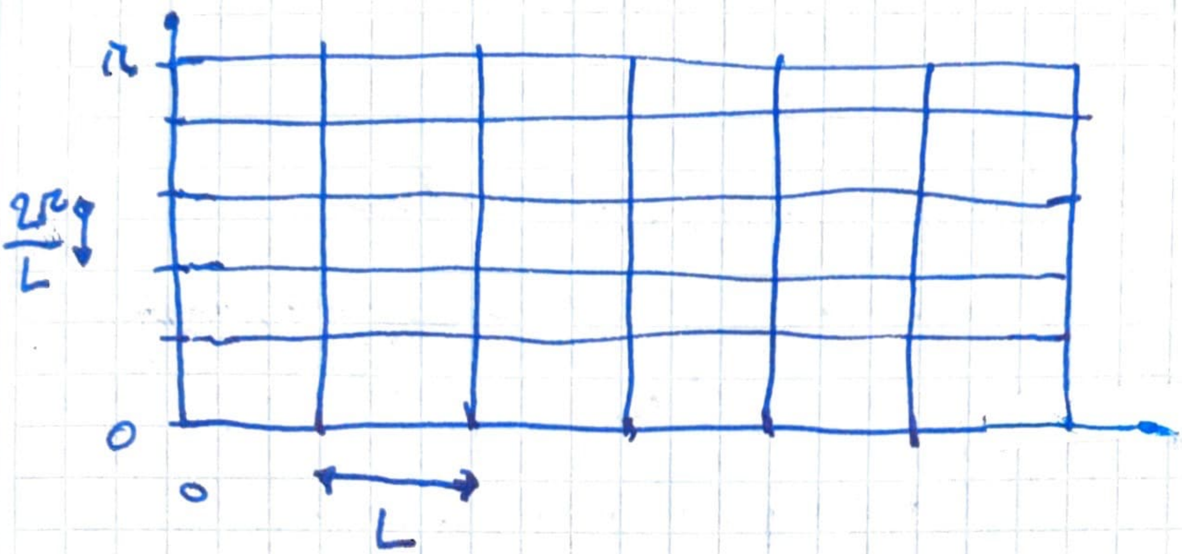
- Design choices
  - width of analysis window
  - position of window (overlapping?)
  - shape (windowing)
    ↳ weighting the samples

- Long window:
  - narrowband spectrogram
  - more DTF points
    ↳ more frequency resolution
  - more "things can happen"
    ↳ less precision in time

- Shord window:
  - wideband spectrogram
  - many time slices (chunks)
    ↳ precise location of transitions
  - Fewer DFT points
    ↳ poor frequency resolution

Uncertainty principle

Time - Frequency tiling



- L: size of window
- Shape of tiles changes with different L but size of each tile stay the same
- short time window (small L)
  - → higher tiles
  - → narrower tiles
- long time window (big L)
  - → lower tiles
  - → broader tiles

more sophisticated tilings
→ wavelet transform

- time "resolution": $\Delta t = L$
- frequency "resolution" $\Delta f = 2\pi / L$
- $\Delta t \cdot \Delta f = 2\pi$

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4.4. The DTFT (Discrete-Time Fourier Transform)

- infinite non-periodic sequences

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

  - $\omega \in \mathbb{R}$: real valued frequency
  - $X(e^{j\omega})$: standard notation in DSP
    - free variable is $\omega \in \mathbb{R}$
    - periodic in ω with period 2π $[-\pi, \pi]$
      since $e^{j(\omega+2\pi)} = e^{j\omega}$
    - immediately identifies function as Fourier transform of a discrete time sequence
    - similarity to z-transform
  - $x[n] \in l_2(\mathbb{Z})$: square summable
    $\rightarrow$ finite energy
- Inversion (iDTFT):

  when $X(e^{j\omega})$ exists:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

$n \in \mathbb{Z}$

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4.4.1. The DTFT as Limit of a DFS

DFS of periodic sequence with larger and larger periods

→ number of basis vectors in $\tilde{X}[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$

grow with length N: frequencies become dense between 0 and 2π

→ in the limit: reconstruction formula for FTFT

4.4.2 The DTFT as a Formal Change of Basis

$$\text{DTFT} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- for any given ω_0 the DTFT is the inner product in $\ell_2(\mathbb{Z})$ of sequence $x[n]$ with sequence $e^{j\omega_0 n}$

$$X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle$$

DTFT operator maps

$l_2(\mathbb{Z})$ onto $L_2([-\pi, \pi])$

↳ preserves physical meaning of inner product

- measure content of every frequency in signal
- number of oscillators: ∞
- frequency separation: infinitesimally small

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Review: DFT / DFS

$$X[k] = \langle e^{j\frac{2\pi}{N}nk}, x[n] \rangle$$

$$\mathbb{C}^N \rightleftarrows \mathbb{C}^N$$

$$x[n] = 1/N \sum X[k] e^{j\frac{2\pi}{N}nk}$$

basis: $\{e^{j\frac{2\pi}{N}nk}\}_k$

Review: DTFT

$$X(e^{j\omega}) = \langle e^{j\omega n}, x[n] \rangle$$

$$\ell_2(\mathbb{Z}) \rightleftarrows L_2([-\pi, \pi])$$

$$x[n] = 1/2\pi \int X(e^{j\omega}) e^{j\omega n} d\omega$$

"basis": $\{e^{j\omega n}\}_\omega$

4.6. Fourier Transform Properties

4.6.1. DTFT Properties

- Symmetries and structure
  - DTFT: $X(e^{j\omega})$
  - Time-reversed sequence

    $$x[-n] \overset{DTFT}{\longleftrightarrow} X(e^{-j\omega})$$

  - Complex conjugate of sequence

    $$x^*[n] \overset{DTFT}{\longleftrightarrow} X^*(e^{-j\omega})$$

  - Real sequence $x[n] \in \mathbb{R}$:
    - conjugate symmetric:

      $$X(e^{j\omega}) = X^*(e^{-j\omega})$$

      $\Rightarrow$ for real signals:
      - symmetric magnitude

        $$|X(e^{j\omega})| = |X(e^{-j\omega})|$$

      - antisymmetric phase

        $$\measuredangle X(e^{j\omega}) = -\measuredangle X(e^{-j\omega})$$

      - symmetric real part

        $$Re\{X(e^{j\omega})\} = Re\{X(e^{-j\omega})\}$$

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· antisymmetric imaginary part

$$\text{Im}\{X(e^{j\omega})\} = -\text{Im}\{X(e^{-j\omega})\}$$

· if $x[n] \in \mathbb{R}$ (is real) and symmetric

$$x[n] \in \mathbb{R},\ x[-n] = x[n]$$
$$\Longleftrightarrow X(e^{j\omega}) \in \mathbb{R}$$: DTFT is real

· if $x[n] \in \mathbb{R}$ (is real) and antisymmetric

$$x[n] \in \mathbb{R},\ x[-n] = -x[n]$$
$$\Longleftrightarrow \text{Re}\{X(e^{j\omega})\} = 0$$: DTFT is purely immaginary

· Linearity and Shifts

· DTFT is linear operator

$$\alpha x[n] + \beta y[n] \overset{\text{DTFT}}{\Longleftrightarrow} \alpha X(e^{j\omega}) + \beta X(e^{j\omega})$$

· shift in discrete-time → multiplication by phase in freq. domain

$$x[n-n_0] \overset{\text{DTFT}}{\Longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

- Modulation by complex "carrier"
  ↳ multiplication at freq ω_0

$$e^{j\omega_0 n} x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$$

↳ spectrum is shifted by ω_0

- Energy conservation: see book

4.6.2. DFS Properties

- Symmetries and Structure
  - time reversed sequence

    $$\tilde{x}[-n] \xleftrightarrow{DFS} \tilde{X}[-k]$$

  - complex conjugate of sequence

    $$\tilde{x}^*[n] \xleftrightarrow{DFS} \tilde{X}^*[-k]$$

  - real periodic sequences → symmetries:

    $$\tilde{X}[k] = \tilde{X}^*[-k]$$

    $$|\tilde{X}[k]| = |\tilde{X}[-k]|$$

    $$\measuredangle \tilde{X}[k] = -\measuredangle \tilde{X}[k]$$

    $$Re\{\tilde{X}[k]\} = Re\{\tilde{X}[-k]\}$$

    $$Im\{\tilde{X}[k]\} = -Im\{\tilde{X}[-k]\}$$

  if $\tilde{x}[n]$ is real and symmetric the DFS is real

  $$\tilde{x}[n] = \tilde{x}[-n] \Leftrightarrow \tilde{X}[k] \in \mathbb{R}$$

Linearity and Shifts (DFS)

- Shift in discrete-time
  → multiplication by phase in freq. domain

$$\tilde{x}[n-n_0] \overset{DFS}{\longleftrightarrow} W_N^{kn_0}\tilde{X}[k]$$

- Multiplication of signal by complex exponential of frequency multiple of $\frac{2\pi}{N}$
  → shift in frequency

$$W_N^{-nk_0}\tilde{x}[n] \overset{DFS}{\longleftrightarrow} \tilde{X}[k-k_0]$$

Energy Conservation

$$\sum_{n=0}^{N-1} |\tilde{x}[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |\tilde{X}[k]|^2$$

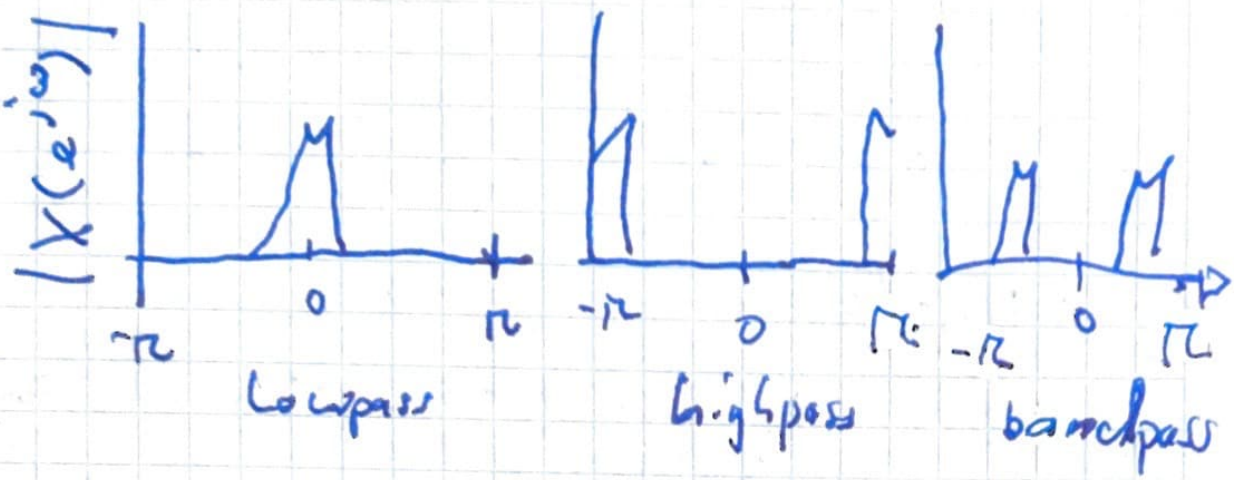
4.6.3 DFT Properties

Analogous to DFS with shifts as modulo N.

See book for details.

Sinusodial Modulation 9.3.24

- Three broad categories of signals:
  lowpass-, highpass-, bandpass-signals
  "baseband"

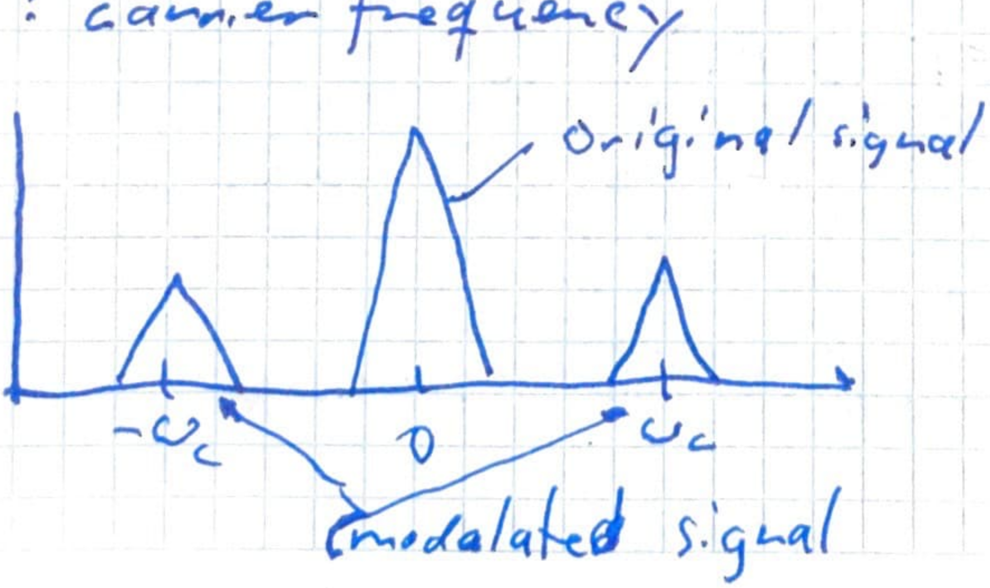


- Sinusodial Modulation

$$\mathrm{DTFT}\{x[n]\cos(\omega_c n)\}$$
$$= \mathrm{DTFT}\left\{\frac{1}{2}e^{j\omega_c n}x[n] + \frac{1}{2}e^{-j\omega_c n}x[n]\right\}$$
$$= \frac{1}{2}\left[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})\right]$$

usually $x[n]$: baseband

ω_c: carrier frequency



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- Sinusodial Demodulation
  just multiply the received signal by the carrier again

  - Modulation

    $y[n] = x[n] \cos(\omega_c n)$

    $\hookrightarrow Y(e^{j\omega}) = \frac{1}{2}[X(e^{j(\omega-\omega_c)}) + X(e^{j(\omega+\omega_c)})]$

  - Demodulation

    $\text{DTFT}\{y[n] \cdot 2\cos(\omega_c n)\}$

    $= Y(e^{j(\omega-\omega_c)}) + Y(e^{j(\omega+\omega_c)})$

    $= X(e^{j(\omega)}) + \frac{1}{2}[X(e^{j(\omega-2\omega_c)}) + X(e^{j(\omega+2\omega_c)})]$

  $\hookrightarrow$
  - recovered baseband signal exactly
  - spurious additional high-freq. components
  - use filters to remove spurious high-freq. components

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4.9. Digital Frequency vs. Real Frequency

- Discrete-time signals:
  - dimensionless "time", indicated by index n
  - just sequences of numbers
  - π : highest digital frequency

- Link between real-world signal and discrete-time signal: Sampling Theorem

- associating time duration between entries in discrete-time signal
- smallest phase increment between samples of complex exponential $e^{j\omega_0 n}$ is ω_0 rad.
- oscillation: full cycle in $n_0 = 2\pi/\omega_0$ samples

- T_s : time between samples
  → full cycle in $n_0 T_s$ seconds
- "real-world" frequency: $f_0 = 1/n_0 T_s$ Hz

- $f_0 \xleftrightarrow{T_s} \frac{1}{2\pi} \frac{\omega_0}{T_s}$

f_0: "real" freq.
ω_0: digital freq.
$F_s = 1/T_s$

- Highest "real" freq ($\omega = \pi$): $F_{max} = \frac{F_s}{2}$

$$\boxed{\omega_0 = 2\pi \frac{f_0}{F_s}}$$