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8. Stochastic Signal Processing

8.1. Random Variables

- Random variable: mapping of a random event to a value $x \in \mathbb{R}$

- Measuring probability:

  - Cumulative distribution function (cdf)

cdf

$$F_x(\alpha) = \overbrace{P[x \leq \alpha]}^{\text{probability that x takes random values less or eq. to } \alpha}, \quad \alpha \in \mathbb{R}$$

$$\lim_{\alpha \to \infty} F_x(\alpha) = 1$$

  - Probability density function (pdf)
    ↳ differentiate cdf

pdf

$$f_x(\alpha) = \frac{d F_x(\alpha)}{d\alpha}, \quad \alpha \in \mathbb{R}$$

$$F_x(\alpha) = \int_{-\infty}^{\alpha} f(x)\, dx$$

- Expectation
  - some information about rand. var.

$$E[x] = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

calc mean of x

- Expectation theorem

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x)\, dx$$

- Expectation is linear operation
- Moments
  - raw moments:

$$E[x^n] = \int_{-\infty}^{\infty} x^n f_x(x)\, dx$$

  - Special case mean:

$$m_x = E[x] = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

  - central moments

$$E[(x-m_x)^n] = \int_{-\infty}^{\infty} (x-m_x)^n f(x)\, dx$$

  - special case: variance

$$\sigma_x^2 = E[(x-m)^2]$$

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- Relations between random variables
  - cross-correlation:

$$R_{xy} = E[xy]$$

    - Uncorrelated if $E[xy] = E[x]\,E[y]$
  - Covariance

$$C_{xy} = E[(x - m_x)(y - m_y)]$$

(or K_{xy})

↳ simmilar to cross-correlation but each rand var is centered around its mean

  - if zero-mean: $C_{xy} = R_{xy}$
  - two variables are uncorrelated iff their covariance is zero
  - variance:

$$\sigma_x^2 = E[(x - m_x)^2]$$

  - standard deviation (square root of variance)

$$\sigma = \sqrt{\sigma^2}$$

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· Example: Gaussian Random Variable

· pdf:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}, x \in \mathbb{R}$$

↳ = normal distribution

· mean $= m$
· variance $= \sigma^2$
· normalization factor $\frac{1}{\sqrt{2\pi\sigma^2}}$

ensures that for all random variables the integral of the pdf over entire real line is $= 1$

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8.2. Random Vectors

- Probability Distribution
  - Random vector x: collection of N random variables $[x_0 \dots x_{N-1}]^T$

    ↳ $F_x(\alpha) = P[X_i \leq \alpha_i, i = 0, \dots, N-1]$

    where $\alpha = [\alpha_0 \; \dots \; \alpha_{N-1}]^T \in \mathbb{R}^N$

PDF: $f_x(\alpha) = \frac{\partial^N}{\partial \alpha_0, \dots, \partial \alpha_{N-1}} F_x(\alpha_0, \dots \alpha_{N-1})$

- Independent elements:

  a collection of N random variables is independent iff the joint pdf has the form:

  $f_{x_0, \dots, x_{N-1}}(x_0, \dots, x_{N-1}) =$
  $f_{x_0}(x_0) \cdot f_{x_1}(x_1) \cdot \dots \cdot f_{x_{N-1}}(x_{N-1})$

- Independent and identically distributed elements (i.i.d. elements)
  - variables are independent and
  - each rand. var. has same distribution

  $f_{x_i}(x_i) = f(x_i), \; i = 0, \dots, N-1$

- Expectation and Second Order Statistics
  - Definitions for random vars. extend to random vectors
  - Mean of N-element random vector

$$E[X] = [E[X_0], \dots E[X_{N-1}]]^T$$
$$= [m_{X_0} \; m_{X_1} \; \; m_{X_{N-1}}]^T$$
$$= \mathbf{m}_X$$

  - Correlation of 2 N-element rand. vectors:

$$R_{XY} = E[XY^T] \; , \; R_{XY} \in \mathbb{R}^{N \times N}$$

  - Expectation operator is applied individually to all elements of matrix XY^T
  - Covariance

$$K_{XY} = E[(X - m_X)(Y - m_Y)^T]$$

    - equal to correlation for zero-mean random vectors

8.3. Random Processes

- Probability Distribution
  - Random process (stochastic process):
    notation $X[n]$: n-th random variable
    (sample) of the sequence
  - pdf of random process: joint distrib.
    of all samples in the sequence
  - statistical description of random process:
    - joint pdf for all k-tuples of
      time indices i_k, $k \in \mathbb{N}$
  - i.i.d process (independent and
    identically distributed)

$$\text{pdf}: f_{X[i_0]X[i_1]\dots X[i_{k-1}]}(x_0, x_1, \dots, x_{k-1})$$
$$= \prod_{i=0}^{k-1} f(x_i)$$

- Second Order Description
  - mean of process $X[n], n \in \mathbb{Z}$:

$$E[X[n]] \text{ depends on } n$$

  - correlation (autocorrelation)

$$R_X[l,k] = E[X[l]\,X[k]], \quad l,k \in \mathbb{Z}$$

  - covariance (autocovariance)

$$K_X[l,k] = E[(X[l] - m_{X[l]}) \cdot (X[k] - m_{X[k]})] = R_X[l,k] - m_{X[l]} m_{X[k]}, \quad l,k \in \mathbb{Z}$$

  - cross-correlation of two processes

$$R_{XY}[l,k] = E[X[l]\,Y[k]]$$

  - mean and variance of random processes: second order description
    ↳ physically meaningful:
      - mean value
      - mean power

      of random process

more details on stochastic signal processing: see book