Static Term Structure Models (version 09.05)

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Vector and matrix notation

- Vectors and matrices will, when possible, be boldface and scalar values will not. That is, w is a scalar and $\mathbf{W}$ is either a vector or a matrix.
- $\mathbf{v} \in \mathbb{R}^N$ means that $\mathbf{v}$ is a an N-dimensional (column) vector where all N elements are real-valued. $\forall\, n \in [1, 2, \ldots, N]\colon\ v_n \in \mathbb{R}$.
- $\mathbf{v} = 0$ means that *all* elements of $\mathbf{v}$ are 0. $\forall\, n \in [1, 2, \ldots, N]\colon\ v_n = 0$.
- $\mathbf{v} \geq 0$ or $\mathbf{v} \in \mathbb{R}_+^N$ ($\mathbf{v}$ non-negative) means that all N elements of $\mathbf{v}$ are non-negative. $\forall\, n \in [1, 2, \ldots, N]\colon\ v_n \geq 0$.
- $\mathbf{v} > 0$ ($\mathbf{v}$ positive) means that *all* elements of $\mathbf{v}$ are non-negative and at least one element is strictly positive. $\forall\, n \in [1, 2, \ldots, N]\colon\ v_n \geq 0$ and $\exists\, n\colon\ v_n > 0$.
- $\mathbf{v} \gg 0$ ($\mathbf{v}$ strictly positive) means that *all* elements of $\mathbf{v}$ are strictly positive. $\forall\, n \in [1, 2, \ldots, N]\colon\ v_n > 0$.

Market characteristics

We will consider a discrete time financial market where

1) There is no uncertainty (no credit risk).
2) Present time is denoted $t = T_0 = 0$.
3) There are M distinct future points in time denoted $T_1, T_2, ..., T_M$.
4) There are N traded assets and all assets can be traded in any amount.
5) Asset n can be traded at time $t = 0$ for a price of π_n.
6) If the investor owns asset n, he receives payoff $\mathbf{C}_{n\cdot} = \left[c_{n1}, c_{n2}, ..., c_{nM}\right]$.

The investor is allowed to sell an asset that he does not own (short selling) and the investor can trade any fraction of any asset.

Time will generally in this course be measured as a fraction of one year.

Computing asset prices

Questions:

Under what conditions will the market be arbitrage free?

How should an investor compute the (hopefully unique) price of a stream of future payoffs?

Portfolio

Definition (Portfolio)

A portfolio $\mathbf{h}' = [h_1, .., h_N]$ *is a vector in* R^N *where* h_n *is the number of units of asset* n *owned by the investor. The value,* V_0*, of the portfolio at time* $t = 0$ *is given by*

$$V_0 = \mathbf{h}' \cdot \boldsymbol{\pi} = \sum_{n=1}^{N} h_n \pi_n \tag{1}$$

and the stream of payoffs is $\mathbf{h}'\mathbf{C} \in \mathbb{R}^M$*. The relative portfolio weight,* u_n*, of asset* n *is given by*

$$u_n = \frac{h_n \cdot \pi_n}{V_0} = \frac{h_n \cdot \pi_n}{\mathbf{h}'\boldsymbol{\pi}}, \quad n = 1, ..., N \tag{2}$$

and we have that $\sum_{n=1}^{N} u_n = 1$.

Zero Coupon Bond

Definition (Zero coupon bond)

A zero coupon bond is a financial asset with price $p(t, T)$ *at present time* t *that pays* 1 *dollar at known future point in time* T.

In this course, we will use zero coupon bonds as the main building blocks.

Note that zero coupon bond prices are denoted by lower-case p.

Zero coupon bond

If asset n is a zero coupon bond paying one dollar at time T_m, then

$$\mathbf{C}_{n\cdot} = \big[\underset{T_1}{0}, ..., 0, \underset{T_m}{1}, 0, ..., \underset{T_M}{0}\big]$$

To create a portfolio with a given payoff stream $\mathbf{c} \in \mathbb{R}^M$, all we have to do is buy c_1 maturity T_1 zero coupon bonds, c_2 maturity T_2 zero coupon bonds and so forth. The price, $\pi(t)$, of such a portfolio will simply be

$$\pi(t) = c_1 \cdot p(t, T_1) + c_2 \cdot p(t, T_2) + ... + c_M \cdot p(t, T_M) \tag{3}$$

So, not only do zero coupon bonds serve as building blocks to easily create any desired payoff, prices of zero coupon bonds can also be interpreted as discount factors corresponding to each of the future points in time.

Arbitrage

Definition (Arbitrage)

An arbitrage portfolio of either type I or type II is a portfolio $\mathbf{h}$ *satisfying one of the following two conditions:*

I) $\mathbf{h}'\boldsymbol{\pi} = 0$ and $\mathbf{C}'\mathbf{h} > 0$,

II) $\mathbf{h}'\boldsymbol{\pi} < 0$ and $\mathbf{C}'\mathbf{h} \geq 0$.

A type I arbitrage is a portfolio that costs nothing today but will at some point in the future yield a strictly positive payoff.

A type II arbitrage is a portfolio that the investor can 'sell' for a strictly negative price today, yet the portfolio will not yield a negative payoff at any point in the future.

An arbitrage-free financial market

Definition (Arbitrage-free financial market)

A financial market is said to be arbitrage-free if it does not allow for any arbitrage opportunities.

In this course, we will assume financial markets are arbitrage-free.

In practice, arbitrage opportunities do arise in financial markets from time to time, but typically they either vanish very quickly or only persist due to some technical abnormality preventing investors from exploiting and ultimately eliminating the arbitrage.

Theorem (First Fundamental Theorem of Asset Pricing)

A financial market is arbitrage-free if and only if there exists a strictly positive vector $\mathbf{d} \in \mathbb{R}^N_{++}$ *such that* $\boldsymbol{\pi} = \mathbf{C}\mathbf{d}$.

First Fundamental Theorem of Asset Pricing

Lemma (Stiemke)

Let $\mathbf{A}$ *be an* $N \times K$*-dimensional matrix. Then exactly one of the following two statements must be true:*

I) *There exists an* $\mathbf{x} \in \mathbb{R}^K_{++}$ *such that* $\mathbf{Ax} = 0$

II) *There exists a* $\mathbf{y} \in \mathbb{R}^N$ *such that* $\mathbf{y}'\mathbf{A} > 0$

To prove the First Fundamental Theorem of Asset Pricing we need Stiemke's lemma, but once we have that, it is not difficult to prove the First Fundamental Theorem of Asset Pricing.

First Fundamental Theorem of Asset Pricing

Proof:

Consider the matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} -\pi_1 & c_{11} & c_{12} & \dots & c_{1M} \\ -\pi_2 & c_{21} & c_{22} & \dots & c_{2M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\pi_N & c_{N1} & c_{N2} & \dots & c_{NM} \end{pmatrix}$$

If the first statement of Stiemke's lemma is true and there exists an $\mathbf{x} \in \mathbb{R}^M_{++}$ such that $\mathbf{Ax} = 0$, this is equivalent to the existence of a vector $\mathbf{d}$ of discount factors. Simply divide each equation of $\mathbf{Ax} = 0$ by x_1 and set $d_i = \frac{x_{i+1}}{x_1}$ for $i \geq 1$ and we can write the n'th equation of $\mathbf{Ax} = 0$ as

$$\pi_n - \frac{x_2}{x_1} c_{n1} - \frac{x_3}{x_1} c_{n2} - \dots - \frac{x_M}{x_1} c_{nM} = 0. \tag{4}$$

Define $\mathbf{d} := \left[\frac{x_2}{x_1}, \frac{x_3}{x_1}, \dots, \frac{x_M}{x_1}\right]$ and the system of equations becomes $\boldsymbol{\pi} = \mathbf{Cd}$.

First Fundamental Theorem of Asset Pricing

If instead, the second statement of Stiemke's lemma is true, we have either

$$(\mathbf{y}'\mathbf{A})_1 = 0,\ (\mathbf{y}'\mathbf{A})_i \geq 0 \text{ for all } i > 1, \text{ and } (\mathbf{y}'\mathbf{A})_i > 0 \text{ for some } i \in [2, ..., M+1]$$

or

$$(\mathbf{y}'\mathbf{A})_1 > 0 \text{ and } (\mathbf{y}'\mathbf{A})_i \geq 0 \text{ for all } i \in [2, ..., M+1]$$

The first of these two cases corresponds to the existence of portfolio $\mathbf{y}$ with price $\pi = \mathbf{y}'\boldsymbol{\pi} = 0$ and payoffs $\mathbf{y}'\mathbf{C}$ where all the payoffs are non-negative and at least one is strictly positive. This is an arbitrage of type I.

The second of these two cases corresponds to the existence of a portfolio with negative price $\pi = \mathbf{y}'\boldsymbol{\pi} < 0$ and payoffs $\mathbf{y}'\mathbf{C}$ where all the payoffs are non-negative. This is an arbitrage of type II.

$\square$

Market Completeness

Definition (Market Completeness)

The financial market is complete if for every $\mathbf{y} \in \mathbb{R}^M$*, there exists an* $\mathbf{h}$ *in* $\mathbf{R}^N$ *such that* $\mathbf{h}'\mathbf{C} = \mathbf{y}$.

If the market is complete, any stream of payoffs is reachable. That is, the investor can create a portfolio that gives him any desired sequence of payoffs.

In linear algebra terms, the market is complete if the rank of the payoff matrix $\mathbf{C}$ is equal to the number of future points in time and hence, the market is complete if the rows of $\mathbf{C}$ span $\mathbb{R}^M$.

Second Fundamental Theorem of Asset Pricing

Theorem (Second Fundamental Theorem of Asset Pricing)

Assume the market is arbitrage-free. Then the market is complete if and only if the vector $\mathbf{d}$ *of discount factors is unique.*

Recall that the vector of discount prices is the object we use to compute the price of a given stream of payoffs.

The Second Fundamental Theorem of Asset Pricing tells us that if we assume absence of arbitrage and if the market is complete, there exists a unique vector of discount factors and thus a unique solution to the investors problem of computing the price of a given stream of payoffs.

Second Fundamental Theorem of Asset Pricing

Proof:

First, we prove that (market is complete $\Rightarrow$ $\mathbf{d}$ is unique).

Since the market is complete then for every $\mathbf{y} \in \mathbb{R}^M$, there exists at least one (but if $N > M$ infinitely many) $\mathbf{h} \in \mathbb{R}^N$ such that $\mathbf{h}'\mathbf{C} = \mathbf{y}'$. Thus, the rows of $\mathbf{C}$ span $\mathbb{R}^M$ which, since $\mathbf{h}'\mathbf{C} = \mathbf{y}' \Leftrightarrow \mathbf{C}'\mathbf{h} = \mathbf{y}$, is equivalent to saying that the columns of $\mathbf{C}'$ span $\mathbb{R}^M$. $\mathbf{C}$ therefore has M independent rows and since $Rank(\mathbf{C}) = Rank(\mathbf{C}')$, if follows that $\mathbf{C}$ also has M independent columns. Now, since $\mathbf{C}$ has full rank M, the M columns of $\mathbf{C}$ form a basis of an M-dimensional subspace of $\mathbb{R}^N$ an each element of that subspace has a unique representation. Since we know the market is arbitrage-free and hence there exists a solution to $\boldsymbol{\pi} = \mathbf{C}\mathbf{d}$, we know that $\boldsymbol{\pi} \in \text{span}(\mathbf{C})$ and that $\mathbf{d}$ is unique because it corresponds to the unique representation of $\boldsymbol{\pi}$ in terms of the basis vectors of $\mathbf{C}$.

Second Fundamental Theorem of Asset Pricing

Second, we prove that ($\mathbf{d}$ is unique $\Rightarrow$ market is complete).

We proceed by contraposition and assume that the market is incomplete. Thus, there exists a $\tilde{\mathbf{d}} \in \mathbb{R}^M$ such that $\tilde{\mathbf{d}} \neq 0$ and $\boldsymbol{\pi} = \mathbf{C}\tilde{\mathbf{d}}$. The linear combination $\hat{\mathbf{d}} = (1-\epsilon)\mathbf{d} + \epsilon\tilde{\mathbf{d}}$ also solves $\boldsymbol{\pi} = \mathbf{C}\hat{\mathbf{d}}$ but since all elements $\mathbf{d}$ are strictly positive, we can find a sufficiently small ϵ such that all the elements of $\hat{\mathbf{d}}$ are also strictly positive. Hence, the vector of discount factors is not unique. We have thus proven that (market not complete $\Rightarrow$ $\mathbf{d}$ not unique) and by contraposition, it must be that ($\mathbf{d}$ is unique $\Rightarrow$ market is complete).

$\square$

Interpretation of the fundamental theorems of asset pricing

1) The market is arbitrage-free if and only if prices of the zero coupon bonds associated with *all* future points in time are strictly positive. That is, the time 0 price of one dollar paid in any future state must be strictly positive to prevent arbitrage opportunities from arising.

2) In a complete market, there is only *one* possible arbitrage-free price of any zero coupon bond and thus any portfolio of assets.

3) In an incomplete market there is not generally a unique choice for the vector of zero coupon bond prices and hence, not all future payoff streams have a unique arbitrage free price.

4) If a given payoff stream can be replicated by the portfolio $\mathbf{h}$, then, the payoff stream will have a unique arbitrage free price even in an incomplete market. Not being able to compute prices of certain payoff streams in an incomplete market is thus not 'severe' as it happens only for payoffs we cannot create from a portfolio of the available assets.

Question:

Who determines prices?

Answer:

The market!

Interest rates and rates of return

Throughout this course, we will work with several different rates of return. If we consider an asset bought at time t for the price $\pi(t)$ and sold at time $T > t$ for a price $\pi(T)$, then we can report the rate of return $R(t, T)$ in one of the following three equivalent ways:

Simple compounding:
$$R(t,T) = \frac{\pi(T) - \pi(t)}{(T-t)\pi(t)} \tag{5}$$

Discrete compounding:
$$R(t,T) = \left(\frac{\pi(T)}{\pi(t)}\right)^{1/(T-t)} - 1 \tag{6}$$

Continuous compounding:
$$R(t,T) = \frac{\log \pi(T) - \log \pi(t)}{(T-t)} \tag{7}$$

If, as will very often be the case, time is measured in fractions of a year, the above periodic returns are in fact annualized returns.

The different types of returns are of course all equivalent and from either one of them, we can compute the other two. However, they are certainly *not* the same as can be seen from the following.

Suppose an investor deposits B_0 dollars in a bank at time $t = 0$. The bank promises a simple interest rate of R but allows the investor to choose either simple compounding or a number N representing the number of times a year interest will compound. If for example, the investor chooses quarterly compounding, at the end of each quarter, his deposit will be multiplied by a factor $1 + R/4$.

Interest rates and rates of return

At time T, the value, B_T, of the bank account will be:

Simple compounding $\quad B_T = B_0\left[1+(T-t)R\right]$

Discrete compounding $\quad B_T = B_0\left[1+\dfrac{R}{N}\right]^{N(T-t)}$

Continuous compounding $\quad B_T = \lim_{N\nearrow\infty} B_0\left[1+\dfrac{R}{N}\right]^{N(T-t)} = V_0 e^{R(T-t)}$

If $R = 0.1,\ T-t = 5$ and we compare to quarterly $\left(N=4\right)$, B_T will be

Simple compounding $\quad B_T = B_0\cdot\left(1+5\cdot 0.1\right) = B_0\cdot 1.5$

Discrete compounding $\quad B_T = B_0\left[1+\dfrac{0.1}{4}\right]^{4\cdot 5} \approx B_0\cdot 1.6386$

Continuous compounding $\quad B_T = B_0 e^{0.1\cdot 5} \approx B_0\cdot 1.6487$

Spot rates of return

The investor can use zero coupon bonds to receive a loan 'on the spot' at present time t and the corresponding rate of return will be a 'spot rate'.

Suppose the investor wishes to borrow X dollars to be paid out at present time t and to repay the loan in full at time T, then he can simply do the following.

At time t: Sell $\frac{X}{p(t,T)}$ maturity T zero coupon bonds to receive x dollars.

At time T: Pay $\frac{X}{p(t,T)}$ dollars to the owner of the bonds sold at t.

Spot rates of return

The spot *annualized* rate of return from investing in a zero coupon bond at time t and selling it at time $T > t$ can be expressed in the following three ways.

Simple compounding:
$$R(t,T) = -\frac{p(t,T) - 1}{(T-t)p(t,T)} \tag{8}$$

Discrete compounding:
$$R(t,T) = \left(\frac{1}{p(t,T)}\right)^{1/(T-t)} - 1 \tag{9}$$

Continuous compounding:
$$R(t,T) = -\frac{\log p(t,T)}{(T-t)} \tag{10}$$

Recall that a zero coupon bond consists of only one payment of 1 dollar paid at time T.

Definition (The term structure of spot interest rates)

The term structure of interest rates at time t for maturities $T_1, T_2, ..., T_N$ consists of the collection of continuously compounded annualized zero coupon spot rates $R(t, T_1), R(t, T_2), ..., R(t, T_N)$.

The term structure of interest rates is more loosely termed the 'yield curve'.

Though zero coupon bonds are only very rarely traded, banks and other actors in financial markets devote much attention to inferring the yield curve from traded assets both to compute prices of more complicated derivatives but also because the yield curve contains a great deal information about market expectations.

Forward rates of return

The investor can also use the zero coupon bond market to create a loan agreement that begins in the future.

If the investor at present time t wishes to secure a loan of X dollars to be paid to him at time $S > t$ and to be repaid at time $T > S$, he can:

At time t: Buy X maturity S zero coupon bonds for $X \cdot p(t, S)$ dollars.

At time t: Sell $\frac{X \cdot p(t,S)}{p(t,T)}$ maturity T coupon bonds for $X \cdot p(t, S)$ dollars.

At time S: Receive X dollars.

At time T: Pay $\frac{X \cdot p(t,S)}{p(t,T)}$ to the owner of the maturity T bond.

The price contracted at time t but paid at time $S > t$ for receiving 1 dollar at time $T > S$ is thus $\frac{p(t,T)}{p(t,S)}$.

Forward rates of return

The forward *annualized* rate of return for entering into a loan/investment at present time t for a loan beginning at time $S > t$ and ending at time $T > S$ can thus be expressed as:

Simple compounding:
$$R(t; S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)p(t, T)} \tag{11}$$

Discrete compounding:
$$R(t; S, T) = \left(\frac{p(t, S)}{p(t, T)}\right)^{1/(T-t)} - 1 \tag{12}$$

Continuous compounding:
$$R(t; S, T) = -\frac{\log p(t, T) - \log p(t, S)}{(T - S)} \tag{13}$$

Forward zero coupon bond

Definition (Forward zero coupon bond)

A forward zero coupon bond is a financial asset with price $p(t; S, T)$ *contracted at present time* t *but paid at time* $S > t$ *to receive* 1 *dollar at time* $T > S$.

For all $t < S < T$, we have the following relation

$$p(t, T) = p(t, S)p(t; S, T) \tag{14}$$

In terms of annualized returns discretely compounded returns, (14) can be expressed as

$$\left[1 + R(t, T)\right]^{T-t} = \left[1 + R(t, S)\right]^{S-t}\left[1 + R(t; S, T)\right]^{T-S} \tag{15}$$

London Interbank offered rates (LIBOR)

Fixed income instruments are often tied to some reference rate or benchmark rate meaning that the cash flow of the instrument depends on the reference rate.

Typical examples could be a corporate bond or a mortgage where the sizes of the coupons depend on some floating rate.

Historically, fixed income instruments have been tied to 'London Interbank Offered Rates' or LIBOR for short but that system has now been largely replaced and decentralized.

In this course, we will need a set of reference rates and refer to these as LIBOR or xIBOR rates nonetheless.

London Interbank offered rates (LIBOR)

London interbank rates became widespread as reference rates already in the 1970s and were increasingly being used as the basis of interest rate swaps, foreign currency options and forward rate agreements.

From the early eighties, it became clear that some measure of uniformity was needed to secure the future growth of the London interbank financial market.

In 1984, the British Bankers Association (BBA) working with among others the Bank of England produced the 'BBA standard for interest rate swaps' or 'BBAIRS' terms and from January 1. 1986, LIBOR fixings were announced.

London Interbank offered rates (LIBOR)

LIBOR rates reflected the rate at which prime London based bank were able to borrow from other prime banks without collateral for maturities ranging from 1B (one business day) to 12M (twelve months).

The LIBOR rates were constructed by surveying every day at 11 AM London time a number of prime banks and averaging the responses.

The question asked to participating banks was:

'At what rate could you borrow funds, were you to do so by asking for and then accepting interbank offers in a reasonable market size just prior to 11 am?'

Initially 18 banks were surveyed and average trimmed to exclude the top and bottom four responses.

London Interbank offered rates (LIBOR)

In 1986, LIBOR rates were announced for three currencies: The US dollar, the British pound sterling and the Deutsche mark for maturities ranging from one moth and above.

Over time, the number of currencies grew to 16 including DKK but after the introduction of the EUR in 200, LIBOR rates were announced for the following 10 currencies: USD, EUR, GBP, JPY, CHF, AUD, CAD, NZD, DKK and SEK.

In the US in 2008, 60% of prime adjustable rate mortgages and practically all subprime mortgages were tied to the USD LIBOR.

LIBOR rates were heavily involved in the 2008 financial crisis and the spread between LIBOR rates and corresponding 'safe' assets rose sharply during the 2008 crisis.

London Interbank offered rates (LIBOR)

During and up to the 2008 crisis, rumors began to circulate that member banks were not sincere about their actual borrowing rates suggesting that banks were facing even higher borrowing costs than what was officially reported.

On Thursday May 29. 2008, the Wall Street Journal reported that indeed banks had been underreporting their borrowing costs during the recent credit crunch.

This article in WSJ set of the so called 'LIBOR scandal' and in the wake of the scandal, the system has changed drastically.

The initial study by WSJ suggested that banks understated their borrowing costs to appear stronger than they actually were, but a 2010 study by Connan Snider and Thomas Youle claims that the banks in fact did so to make substantial profits on their LIBOR-linked positions.

London Interbank offered rates (LIBOR)

On September 25. 2012, the BBA announced that it oversight of LIBOR rates would be transferred to UK regulators and since then, money markets globally have been reformed and the LIBOR rate system brought to an end.

From 2013, LIBOR rates were discontinued for AUD, CAD NZD, DKK and SEK, from 2021, LIBOR rates were discontinued for EUR, GBP, JPY and CHF and finally in 2023 the USD LIBOR ended as well.

Though LIBOR rates are no longer reported, interbank lending is still a huge part of the operations of all banks.

Interest rate derivatives still rely on reference rates but the system has become more fragmented and now a wider rage of reference rates are being used.

Which type of reference rate is used now depends more on the region and the currency denomination of the fixed income instrument.

London Interbank offered rates (LIBOR)

To understand the nature of fixed income derivatives, we will need to understand the nature of the reference rates that form the basis of these instruments.

For simplicity, we will refer to reference rates as LIBOR or xIBOR rates in accordance with the textbook.

LIBOR reference rates are reported in annual terms, announced or fixed at present time t, will be denoted by $L(t, T)$ and are assumed to be risk-free.

LIBOR fixings adhere to the *Money Market Convention* and LIBOR rates are *simple* in the sense that the interest paid at time T on a fixing set at time t on a principal K is

$$(T - t)L(t, T)K. \tag{16}$$

Spot rate agreements and LIBOR rates

Major banks lend and borrow vast sums of money from each other on a daily basis over a range of short maturities in the so called 'money market'.

In this course, we will assume that there is no credit risk involved in these transactions and hence that our xIBOR rates reflect risk-free rates.

If such an arrangement involves a transfer of funds at time t and the reverse transaction plus interest at a future point in time, we will refer to this transaction as a 'spot rate agreement'.

Assuming the existence of a set of zero coupon bonds and also assuming that markets are risk and arbitrage free, we can relate spot LIBOR rates to zero coupon bonds.

Spot rate agreements and LIBOR rates

We have already seen how zero coupon bonds can be used to construct a loan of any size, and we have also seen that the simple rate of return $R(t, T)$ on such a loan must be given by

$$R(t, T) = -\frac{p(t, T) - 1}{(T - t)p(t, T)}. \tag{17}$$

To prevent an arbitrage between interbank lending and ZCB's, we must therefore have that

$$L(t, T) = -\frac{p(t, T) - 1}{(T - t)p(t, T)}. \tag{18}$$

Forward rate agreements and LIBOR rates

Suppose now that a bank instead wishes to secure a loan at present time t to begin at time S and to be repaid at time T.

We have seen that also this type of arrangement can be achieved using ZCB's and that the simple forward rate of return $R(t; S, T)$ on such a loan must be given by

$$R(t; S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)p(t, T)}. \tag{19}$$

By a no arbitrage argument, it should therefore be possible for a bank to lock in at time t a future LIBOR rate $L(t; S, T)$ and this rate should be given by

$$L(t; S, T) = -\frac{p(t, T) - p(t, S)}{(T - S)p(t, T)}. \tag{20}$$

Forward rate agreements and LIBOR rates

In practice, mayor banks use so called 'Forward Rate Agreements' (FRA's) to lock in the rate on a future loan agreements.

FRA's are not exactly loan agreements in the traditional sense but can, as we will seen, be used to achieve the same effect.

Forward rate agreements are constructed so that there is no cash flow at time t when the agreement is made, nor at time T.

The cash flow at time S will be tied to the future LIBOR rate $L(S, T)$ and is for a notional of 1 USD given by

$$\frac{(T-S)\big(L(S,T)-F\big)}{1+(T-S)L(S,T)}. \tag{21}$$

The 'price' of the FRA is quoted in terms of F and we will see in the following how F can be interpreted.

Forward rate agreements and LIBOR rates

To get a sense for the quantity F, we will assume that there exists ZCB's for maturities S and T so that we can replicate the FRA as follows.

- Time t: Sell $(T-S)F+1$ maturity T ZCB's for $p(t,T)((T-S)F+1)$
  Buy 1 maturity S ZCB for $p(t,S)$
  Cashflow: $p(t,T)((T-S)F+1) - p(t,S)$
- Time S: Buy $(T-S)F+1$ maturity T ZCB's for $p(S,T)((T-S)F+1)$
  Receive 1 USD from the maturity S ZCB bought at t
  Cashflow: $1 - ((T-S)F+1)p(S,T) = \frac{(T-S)\big(L(S,T)-F\big)}{1+(T-S)L(S,T)}$
- Time T: Pay $(T-S)F+1$ from the time T ZCB bought at t
  Receive $(T-S)F+1$ from the time T ZCB bought at S

Forward rate agreements and LIBOR rates

To insure that the FRA does not provide an arbitrage and knowing the FRA has no upfront payment, we will need to make sure that the cash flow from the replicating strategy at time t equals 0

$$p(t,T)\big((T-S)F+1\big)-p(t,S)=0. \tag{22}$$

The rate F must therefore satisfy

$$F=\frac{p(t,S)-p(t,T)}{(T-S)p(t,T)}. \tag{23}$$

Thus, the quantity F must agree with the forward LIBOR rate we could have achived using ZCB's.

Forward rate agreements and LIBOR rates

The FRA has no cash flow at time 0 and no cash flow at time T. The only time at which money changes hands is at time S where the investor long the FRA will receive

$$\frac{(T-S)\big(L(S,T)-F\big)}{1+(T-S)L(S,T)} \tag{24}$$

The investor can then invest the money from time S to time T and receive the simple return $1+L(T-S)L(S,T)$. The money will then have grown to

$$(T-S)\big(L(S,T)-F\big) \tag{25}$$

If indeed $F=L(t;S,T)$, that cash flow offsets one-to-one any difference between $L(S,T)$ and $L(t;S,T)$ effectively locking in at time t the future LIBOR rate $L(S,T)$.

Forward rate agreements and LIBOR rates

The forward LIBOR rates can, as we have seen, be deduced from quoted rates on FRA's and these rates are simply referred to as the forward rates corresponding to FRA's.

The FRA rate starting at S and maturing at T is referred to as the $S \times T$ FRA rate. So for example, the 6M LIBOR FRA rate starting in 2 months is denoted the 2×8 FRA.

So far, we have treated ZCB's as the building blocks of the market and inferred conditions on spot and forward LIBOR rates. Later on, we will instead move in the opposite direction and deduce ZCB prices from LIBOR and FRA rates using the two relations

forward LIBOR rate: $$L(t; S, T) = \frac{p(t, S) - p(t, T)}{(T - S)p(t, T)}, \tag{26}$$

spot LIBOR rate: $$L(t, T) = \frac{1 - p(t, T)}{(T - t)p(t, T)}. \tag{27}$$

The characteristics of a bond

To determine the cash flows to be exchanged between bond issuer and bond holder, we will need a few pieces of information

- Maturity of the bond
- Principal to be repaid during the life of the bond
- Schedule of principal payments
- Type and size of coupons (fixed versus floating)
- Frequency of coupon payments (annual, semi-annual, quarterly,...)

Fixed rate bullet bond

The simplest type of bond is a bullet bond with a fixed coupon rate.

- 'Fixed' refers to coupons being predetermined and constant.
- 'Bullet' refers to the fact that the entire principal is repaid at maturity.
- Typically, coupon payments are equally spaced in time.
- There is no coupon payment at time $t = 0$ when the bond is issued.

Government bonds and most corporate bonds are bullet bonds.

Fixed rate bullet bond

Let us consider a bullet bond issued at time t with principal K that matures in N years and pays the fixed simple coupon rate R at times $T_1, T_2, \ldots T_N$ where $t < T_1$.

- The coupon dates are $T_1, T_2, \ldots T_N$.
- The times α_i between coupons is $\alpha_1 = T_1 - t$, $\alpha_i = T_i - T_{i-1}$ for $i > 1$.
- The coupons, c_i, are fixed, predetermined and of size $c_i = \alpha_i RK$.
- The entire principal is repaid at time T_N.

The price or present value $PV(t)$ of the fixed rate bond at the date of issuance can be expressed in terms of zero coupon bond prices, $p(t, T_i)$, as

$$PV(t) = \sum_{i=1}^{N} p(t, T_i)\alpha_i RK + p(t, T_N)K \tag{28}$$

Clean price versus dirty price

When trading bonds, you have to careful how prices are quoted. It is often customary that if a bond is traded between coupon dates, the previous owner receives a portion, the accrued interest, of the coupon.

Suppose a fixed rate bond is traded at time t, in-between coupon dates T_{n-1} and T_n. Then the accrued interest rate will be

$$(t - T_{n-1})RK \tag{29}$$

The so called 'dirty' price is most commonly quoted in Europe and given by

$$\text{dirty price} = \text{clean price} + \text{accrued interest.} \tag{30}$$

In the US, the 'clean' price is often used instead and given by

$$\text{clean price} = \text{dirty price - accrued interest.} \tag{31}$$

Fixed rate bullet bond

Even though the future cash flows to a fixed rate bond is know in advance, fixed rate bonds are indeed risky investments in a market with time-varying interest rates.

On one hand, rising interest rates will decrease the present value of future coupon and principal payments.

On the other, future coupons can be reinvested at a higher rate should interest rates go up.

The first of these two effects is almost always the larger of the two, especially if the bond matures far into the future.

For fixed income investors, it is important to be able to quantity and manage interest rate risk and for that purpose, the duration and convexity of bonds are widely used.

Yield-to-maturity

Definition (Yield-to-maturity)

Consider a stream of future positive payments $\mathbf{c} = [c_1, c_2, ..., c_N]$ occurring at times $T_1, ..., T_N$ and assume that the present value of this payment stream at time $t = 0$ is $PV(\mathbf{c})$. Then the yield-to-maturity y solves

$$PV(\mathbf{c}) = \sum_{i=1}^{N} \frac{c_i}{(1+y)^{T_i}} =: PV(\mathbf{c}; y)$$

This is a polynomial equation in y and generally one could and should worry that the solution is not unique. However, the solution will typically be unique provided that the cash flows are positive.

Yield-to-maturity

The yield-to-maturity of a fixed rate coupon bond with principal K, paying the coupon rate R at times $T_1, ..., T_N$ and with present value $PV(0; R)$ at time $t = 0$ is therefore the y that solves

$$PV(R) = \sum_{i=1}^{N} \frac{(T_i - T_{i-1})RK}{(1+y)^{T_i}} + \frac{K}{(1+y)^{T_N}} =: PV(R; y) \tag{32}$$

The yield-to-maturity can be interpreted as the unique level of a flat term structure that would equate the observed price or present value of the fixed rate coupon bond to the discounted value of future cash flows using that flat term structure.

Next, we will need to take derivatives of $PV(\mathbf{c}; y)$ with respect to y and denote

$$PV'(\mathbf{c}; y) := \frac{\partial}{\partial y} PV(\mathbf{c}; y), \quad PV''(\mathbf{c}; y) := \frac{\partial^2}{\partial y^2} PV(\mathbf{c}; y) \tag{33}$$

Duration

Definition (Macauley Duration)

The Macauley duration $D(\mathbf{c}; y)$ *at time* $t = 0$ *of a stream of cash flows* $\mathbf{c} = [c_1, ..., c_N]$ *occurring at times* $T_1, ..., T_N$ *with yield-to-maturity* y *is given by*

$$D(\mathbf{c}; y) = -PV'(\mathbf{c}; y)\frac{1+y}{PV(\mathbf{c}; y)} = \sum_{i=1}^{N} \frac{c_i T_i}{(1+y)^{T_i} PV(\mathbf{c}; y)}.$$

The Macauley duration is, as we see, the ratio of the percentage change in the present value of the cash flow stream to a percentage change in one plus the yield-to-maturity. It is thus the elasticity of the value of the bond with respect to one plus the bonds yield-to-maturity.

Duration

The Macauley Duration can be interpreted as a measure of the sensitivity of the present value to changes in the yield-to-maturity where the yield-to-maturity can be seen as a sort of 'average' of discount rates used when discounting cash flows at different time in the future back to time 0.

Alternatively, the Macauley duration can be interpreted as a measure of the sensitivity of the value of a cash flow stream with respect to changes to a flat term structure of interest rates.

Note however, that this measure does not assume that the term structure of interest rates is in fact flat, it only measures the sensitivity of the bond prices to changes to the term structure had it been flat.

Duration

The Macauley duration can also be seen as a weighted sum of the times of the cash flows s we can see from the below

$$D(\mathbf{c}; y) = \sum_{i=1}^{N} w_i T_i, \quad \text{where } w_i = \frac{c_i}{(1+y)^{T_i} PV(\mathbf{c}; y)}, \quad \sum_{i=1}^{N} w_i = 1. \tag{34}$$

The weight on T_i is the present value of the cash flow at T_i as a fraction of the present value of the entire stream of cash flows.

Cash flows with a larger present value are weighed more heavily.

The Macauley duration has an alternative interpretation as a 'mean waiting time' to receive the future cash flows adjusted for discounting.

Convexity

Definition (Convexity)

The convexity $K(\mathbf{c}; y)$ *at time* $t = 0$ *of a stream of cash flows* $\mathbf{c} = [c_1, ..., c_N]$ *occurring at times* $T_1, ..., T_N$ *with yield-to-maturity* y *is given by*

$$K(\mathbf{c}; y) = PV''(\mathbf{c}; y)\frac{(1+y)^2}{PV(\mathbf{c}; y)} - D(\mathbf{c}; y) = \sum_{i=1}^{N} w_i T_i^2$$

where as before

$$w_i = \frac{c_i}{(1+y)^{T_i} PV(\mathbf{c}; y)}, \quad \sum_{i=1}^{N} w_i = 1.$$

Hedging interest rate risk

The Macauley duration and convexity can be used to manage interest rate risk which we will now explore further.

The present value of the cash flow stream can be seen as a function of y and we can write it's Taylor expansion as

$$PV(\mathbf{c}; y + \Delta y) = PV(\mathbf{c}; y) + PV'(\mathbf{c}; y)\Delta y + \frac{1}{2} PV''(\mathbf{c}; y)(\Delta y)^2 + o(\Delta^3) \tag{35}$$

The percentage change in the present value can then be approximated as

$$\begin{aligned} \frac{PV(\mathbf{c}; y + \Delta y) - PV(\mathbf{c}; y)}{PV(\mathbf{c}; y)} &\approx \frac{PV'(\mathbf{c}; y)\Delta y + \frac{1}{2} PV''(\mathbf{c}; y)(\Delta y)^2}{PV(\mathbf{c}; y)} \\ &= -D(\mathbf{c}; y)\frac{\Delta y}{1+y} + \frac{1}{2}\Big[K(\mathbf{c}; y) + D(\mathbf{c}; y)\Big]\left(\frac{\Delta y}{1+y}\right)^2 \end{aligned} \tag{36}$$

Floating rate bonds

Consider a bullet bond issued at time t with principal K that matures in N years, has resettlement dates at $T_0, T_1, ..., T_{N-1}$ and pays the LIBOR rate corresponding to the previous fixings at times $T_1, ..., T_2, T_N$.

- The resettlement dates are $T_0, T_1, \dots T_{N-1}$.
- The coupon dates are $T_1, T_2, \dots T_N$.
- The time α_i between coupons is $\alpha_i = T_i - T_{i-1}$.
- The coupons are *not* known in advance but determined by the future LIBOR rate. The coupon paid at time T_i is the LIBOR rate $L(T_{i-1}, T_i)$ fixed at time T_{i-1} and at time T_{i-1} we have that

$$c_i = \alpha_i K L(T_{i-1}, T_i) = \alpha_i K \frac{1 - p(T_{i-1}, T_i)}{\alpha_i p(T_{i-1}, T_i)} = K \left(\frac{1}{p(T_{i-1}, T_i)} - 1 \right)$$

- The entire principal K is repaid at time T_N.

Floating rate bullet bonds

We can however replicate the coupon c_i already at time t if we do the following:

- time t: Sell K maturity T_i ZCBs for $Kp(t, T_i)$ dollars
- time t: Buy K maturity T_{i-1} ZCBs for $Kp(t, T_{i-1})$ dollars
- time $T-1$: Receive K dollars
- time $T-1$: Buy $\frac{K}{p(T_{i-1}, T_i)}$ maturity T_i ZCBs for K dollars
- time T: Pay K dollars on the maturity T_i ZCBs bought at time t
- time T: Receive $\frac{K}{p(T_{i-1}, T_i)}$ dollars on the T_i ZCBs bought at T_{i-1}

Floating rate bullet bonds

The cost at time t to replicate the coupon, c_i, at time T_i is thus

$$K\left[p(t, T_{i-1}) - p(t, T_i)\right].$$

The price or present value $PV(t)$ of the floating rate bond then becomes

$$PV(t) = \sum_{i=1}^{N} \left[p(t, T_{i-1}) - p(t, T_i)\right] K + P(t, T_N)K = p(t, T_0)K. \tag{37}$$

If $t = T_0$, we have that $PV(t) = K$.

This very simple result of course stems from the fact that all future coupon bond payments can be replicated.

The interest rate swap

The interest swap is the most simple interest rate derivative and perhaps for that reason, interest rate swaps are very commonly traded in financial markets.

An interest rate swap involves the exchange of a stream of fixed coupon payments for a stream of floating rate coupons.

These two streams of payments are commonly referred to as the 'fixed leg' and the 'floating leg'.

The party who pays the fixed coupon is said to have entered into a 'payer' swap and the party receiving the fixed coupon is said to have entered into a 'receiver' swap.

The interest rate swap

We will consider an interest rate swap with principal K and fixed coupon R. The floating leg coupon rate will be set at times $T_0, \ldots, T_{N-1}$ and coupons on both the fixed and the floating legs are paid at times $T_1, \ldots, T_N$. At time T_i, the payments on the floating and fixed legs are respectively

$$\begin{aligned} &\text{Floating leg}: && \alpha_i L(T_{i-1}, T_i)K \\ &\text{Fixed leg}: && \alpha_i RK \end{aligned}$$

We have previously seen that the time t value of the floating rate coupon paid at T_i is $K\big[p(t, T_{i-1}) - p(t, T_i)\big]$ and the time t value of the fixed coupon paid at T_i is $p(t, T_i)K\delta R$. The time t value of the net cash flow received at time T_i is

$$p(t, T_i)\alpha_i\big[L(T_{i-1}, T_i) - R\big]K = \big[p(t, T_{i-1}) - p(t, T_i)\big]K - p(t, T_i)\alpha_i RK.$$

The interest rate swap

The total time t value, $\mathbf{PS}_0^N(t = T_0, K)$ of the payer swap can be expressed both in terms of zero coupon bond prices and forward LIBOR rates as

$$\begin{aligned}
\mathbf{PS}_0^N(t = T_0, K) &= K \sum_{i=1}^{N} \alpha_i p(t, T_i) \big[L(t; T_{i-1}, T_i) - R \big] \\
&= K \sum_{i=1}^{N} \big[p(t, T_{i-1}) - p(t, T_i) - \alpha_i R p(t, T_i) \big] \\
&= K \big[p(t, T_0) - p(t, T_N) \big] - K \sum_{i=1}^{N} \alpha_i p(t, T_i) \\
&= K \sum_{i=1}^{N} \alpha_i \big[p(t, T_i) L(t; T_{i-1}, T_i) - R \big] \qquad (38)
\end{aligned}$$

It is common practice to set the fixed coupon rate at time of issuance, $t = T_0 = 0$, such that the initial value of the interest swap becomes 0. This choice R_0^N is then referred to as the *par swap rate* and becomes

$$R_0^N = \frac{1 - p(0, T_N)}{\sum_{j=1}^N \alpha_j p(0, T_j)} = \frac{\sum_{i=1}^N \alpha_i p(0, T_i) L(0; T_{i-1}, T_i)}{\sum_{j=1}^N \alpha_j p(0, T_j)} = \sum_{i=1}^N w_i L(0; T_{i-1}, T_i)$$

where

$$w_i = \frac{\alpha_i p(0, T_i)}{\sum_{j=1}^N \alpha_j p(0, T_j)}$$

The denominator denoted S_0^N in the expression for R_0^N is en important quantity and is referred to as the 'accrual factor' or dollar value of one basis point of the swap

$$S_0^N = \sum_{i=1}^N \alpha_i p(0, T_i). \tag{39}$$

The interest rate swap

The par swap rate is a weighted average of the forward LIBOR rates at time of issuance with higher weight on forward rates in the near future.

In practice, the principal is not exchanged at time of maturity and hence the counter-party credit risk of an interest rate swap is limited to the difference between fixed and floating coupons.

Investors can therefore use swaps to take on or hedge interest rate risk while avoiding credit risk.

In the above calculations, coupons were paid on both the fixed and the floating leg at the same time but that might not be the case. Often, the floating leg pays coupons more frequently than the fixed leg.

The interest rate swap

Suppose again we consider an interest rate swap with a principal of K and payments to both the floating and the fixed leg occurring at times $T_1, ..., T_N$. Let us denote the par swap rate of the swap at time of issuance at $t = 0$ by R_0^N.

Further, we will assume that some time has passed end present time t is now $t = T_n$, the exact time that payments to both the floating and fixed legs have just been made.

We will compute the profit-and-loss (PnL) at time $t = T_n$ to the payer swap entered into at time $t = 0$.

It is customary to denote the 'value' of an interest rate swap in terms of the par swap rate, so we will first need to find an expression for the par swap rate of a swap issued at $t = T_n$ with same profile as the original one.

The interest rate swap

An interest rate swap with similar remaining cashflows entered into at time $t = T_n$ will have a value of $\mathbf{PS}_n^N(t = T_n, K) = 0$ since the coupon to the fixed leg will be equal to the par swap rate R_n^N at time $t = T_n$.

$$\begin{aligned}\mathbf{PS}_n^N(t = T_n, K) &= \sum_{i=n+1}^{N} \big[p(t, T_{i-1}) - p(t, T_i)\big] K - \sum_{i=n+1}^{N} p(t, T_i)\alpha_i R_n^N K \\ &= \big[1 - p(t, T_N)\big] K - \sum_{i=n+1}^{N} p(t, T_i)\alpha_i R_n^N K = 0. \end{aligned} \tag{40}$$

The par swap rate R_n^N is therefore given by

$$R_n^N = \frac{1 - p(t, T_N)}{\sum_{i=n+1}^{N} \alpha_i p(t, T_i)} = \frac{1 - p(t, T_N)}{S_n^N}. \tag{41}$$

The interest rate swap

The value $\mathbf{PS}_0^N(t = T_n, K)$ at time $t = T_n$ of the original swap paying R_0^N on the fixed leg will be

$$\mathbf{PS}_0^N(t = T_n, K) = \sum_{i=n+1}^{N} \left[p(t, T_{i-1}) - p(t, T_i)\right] K - \sum_{i=n+1}^{N} p(t, T_i)\alpha_i R_0^N K$$
$$= \left[1 - p(t, T_N)\right] K - S_n^N R_0^N K. \tag{42}$$

Now we can use the expression for R_n^N from (41) to give us that

$$\mathbf{PS}_0^N(t = T_n, K) = S_n^N R_n^N K - S_n^N R_0^N K = S_n^N \left(R_n^N - R_0^N\right) K \tag{43}$$

The interest rate swap

The expression for the profit and loss of the payer swap allows us to reach a number of conclusions.

A payer swap is clearly positioned for rising interest rates and a receiver swap is positioned for falling interest rates.

The accrual factor can indeed also be interpreted as the 'dollar value of a basispoint' since the PnL to the swap will rise by $S_n^N \times K$ dollars per basispoint increase in the par swap rate.

The accrual factor can therefore also be used as a guide when hedging interest rate positions.

This is further eased by the fact that interest rate swaps, by construction, do not cost anything to enter and hence do not directly require much liquidity up front.

The forward starting interest rate swap

Let us now consider an interest rate swap where the first payments occur in the future. We will consider a $T_n \times (T_N - T_n)$ payer swap with first fixing taking place at T_n, payments to the fixed and floating legs will be coinciding and the last payment to both legs will occur at time T_N. Present time is denoted t, the fixed coupon by R and the principal by K.

The value at t of the floating leg will be a telescoping sum and given by

$$\sum_{n+1}^{N} K\left[p(t, T_{i-1}) - p(t, T_i)\right] = K\left[p(t, T_n) - p(t, T_N)\right] \tag{44}$$

The value at t of the fixed leg is

$$\sum_{i=n+1}^{N} p(t, T_i)\alpha_i R \tag{45}$$

The net value at t denoted $\mathbf{PS_n^N}(t; K)$ of the $T_n \times (T_N - T_n)$ payer swap is

$$\mathbf{PS_n^N}(t; K) = K\left[p_n(t) - p_N(t)\right] - \sum_{i=n+1}^{N} \alpha_i p(t, T_i) K \tag{46}$$

The forward starting interest rate swap

The par swap rate for an interest rate swap is then given by

$$R_n^N = \frac{p_n(t) - p_N(t)}{\sum_{i=n+1}^{N} \alpha_i p_i(t)} \tag{47}$$

where the denominator is again the accrual factor

$$S_n^N = \sum_{i=n+1}^{N} \alpha_i p(t, T_i) \tag{48}$$

The value of the payer swap can be written as

$$\mathbf{PS_n^N}(t; K) = \left(R_n^N - R\right) S_n^k \tag{49}$$

The accrual factor measures the dollar value of a one unit change in the par swap rate and is a measure of the *duration* of an interest rate swap.

Calibration of the yield curve

Recall that the so called *yield curve* or *term structure of interest rates* is the collection of continuously compounded zero coupon spot rates for maturities ranging from 0 to as far as fixed income securities are available in the market.

In order to carry out day-to-day operations such as market making, hedging, risk management and pricing of complex derivatives, financial institutions rely heavily on continuously 'knowing' the zero coupon yield curve or term structure of interest rates.

It is common practice to use a combination of LIBOR fixings, *Forward Rate Agreements* (FRA) and possibly also futures to deduce zero coupon bond yields for maturities up to two years and swap market data for maturities beyond 2 years.

Calibration of the yield curve

The yield curve used by financial institutions is typically continuous, that is a continuously compounded zero coupon bond yield is available for *all* maturities.

The yield curve is calibrated according to a number of criteria

- The short rate should be continuous
- The instantaneous forward rate should be positive
- The instantaneous forward rate should be continuous

The curve calibration centers around a set of points, typically corresponding to the maturities of the instruments used to calibrate the curve, and an assumption about the nature of the yield curve between these knot points.

Calibration of the yield curve

To calibrate the yield curve to LIBOR, FRA and swap market data, we will need a few results.

We have the following relationship between continuously compounded spot rates $R(t,T)$, LIBOR rates $L(t,T)$ and zero coupon bond prices $p(t,T)$.

$$1 + L(t,T)(T-t) = e^{R(t,T)(T-t)} = \frac{1}{p(t,T)} \tag{50}$$

and the following relationship between continuously compounded forward rates $R(t;S,T)$, LIBOR rates $L(t;S,T)$ and zero coupon bond prices $p(t,S)$ $p(t,T)$ as seen from present time t

$$1 + L(t;S,T)(T-S) = e^{R(t;S,T)(T-S)} = \frac{p(t,S)}{p(t,T)} \tag{51}$$

Calibration of the yield curve - Example

How exactly the yield curve can be calibrated to market data depends of course on the data available but the typical data looks much like the example in Linderstrøm(2013) reproduced below.

EURIBOR	Fixing	FRA	Midquote	IRS	Midquote
6M	0.00967	1X7	0.00980	2Y	0.01652
		2X8	0.01043	3Y	0.02019
		3X9	0.01130	4Y	0.02319
		4X10	0.01217	5Y	0.02577
		5X11	0.01317	7Y	0.02995
		6X12	0.01399	10Y	0.03395
		7X13	0.01478	15Y	0.03753
		8X14	0.01560	20Y	0.03873
		9X15	0.01637	30Y	0.03975

Calibration of the yield curve - Example

For this example, the knot points become

$$T = \left\{2, 3, 4, 5, 7, 10, 15, 20, 30\right\}$$

The spot rate for $T = 0.5$ can be computed directly and the spot rate for $T = 1$ immediately thereafter from the 6X12 FRA.

The spot rates for the remaining maturities above $T = 0.5$ and below $T = 2$ for which we have FRA data can be chosen according to any reasonable interpolation method we prefer and this section of the curve will pose no challenge.

The remainder of the curve for maturities beyond 2 years is a bit more tricky but can luckily be handled independently of the front end of the yield curve.

Calibration of the yield curve - Example

Recall that for an interest rate swap, it is not given that floating leg and fixed leg payments occur at the same frequency.

In this case, floating rate payments occur semi-annually and fixed rate payments occur annually.

Let us consider an interest rate swap with first floating rate fixing at present time $t = T_0$, fixed rate payments at times $T_1, ..., T_N$ and last payment on both the fixed and the floating leg at time T_N. Then the par swap rate for this interest rate swap is given by

$$R_n^N(t) = \frac{1 - p(t, T_N)}{\sum_{n=1}^{N} (T_n - T_{n-1}) p(t, T_n)} \tag{52}$$

This is formula can be used in our case to calibrate the swap market portion of the yield curve.

Calibration of the yield curve - Example

Using the expression for the par swap rate, we can find an expression for the zero coupon bond prices corresponding to the maturity of the swap as

$$p(t, T_N) = \frac{1 - R_n^N(t) \sum_{n=1}^{N-1} (T_n - T_{n-1}) p(t, T_n)}{1 + R_n^N(t)(T_N - T_{N-1})}. \tag{53}$$

Using the relationship between zero coupon bond prices and spot rates gives us the following expression for the spot rate corresponding to the maturity T_N

$$R(t, T_N) = -\frac{1}{T_N} \log \left[\frac{1 - R_n^N(t) \sum_{n=1}^{N-1} (T_n - T_{n-1}) p(t, T_n)}{1 + R_n^N(t)(T_N - T_{N-1})} \right]. \tag{54}$$

Calibration of the yield curve - Example

We now have the ingredients we need to fit a zero coupon yield curve.

Denote the collection of times where fixed coupon payments on some swap occurs by T_n, $n = 1, ..., N$ and the corresponding spot rate by R_n.

Denote the knot points corresponding to the maturities of the interest rate swaps by T_{k_j}, $j = 1, ..., J$ and the corresponding spot rate by R_{k_j}.

Denote the par swap rate at knot point k_j observed in the market by Y_{k_j}.

Note that the there is an overlap between the two sets of times because there is also a fixed coupon payment when the swaps mature.

An algorithm to find zero coupon bond yields will then look as follows.

Calibration of the yield curve - Example

Initially set the spot rates at the knot points and then iterate over i

1) Given the knot time spot rates R_{k_j}, interpolate the spot rates R_{ni}.

2) Given the spot rates R_{ni} compute the corresponding ZCB prices $p_i(0, T_n)$.

3) Compute estimated values of the par swap rates $\hat{Y}_{k_j i}$ at the knot points

$$\hat{Y}_{k_j i} = \frac{1 - p_i(t, T_{k_j})}{\sum_{n=1}^{k_j} (T_n - T_{n-1}) p_i(t, T_n)}$$

4) Compute the MSE as

$$\sum_{j=1}^{J} \left(\hat{Y}_{k_j i} - Y_{k_j} \right)^2 \tag{55}$$

5) If the MSE is larger then some desired accuracy, choose a new set of spot rates R_{ni+1} and continue the iteration, otherwise stop.

Calibration of the yield curve

Once the zero coupon spot rates have been computed for the knot points, the same interpolation scheme as was used in the algorithm should be used to interpolate spot rates for *all* time points.

The quality of the resulting term structure should then be assessed and in particular, the term structure of the instantaneous forward rate should be computed using.

$$f(t, T) = \frac{\partial}{\partial T}\left[R(t, T)(T - t)\right] \tag{56}$$

How this computation should be carried out, depends highly on the chosen interpolations method but do note that if the spot rate is not differentiable, the term structure of forward rates will be discontinuous.

Risk management of an interest rate swap

The accrual factor of a swap measures the sensitivity of a swap with respect to changes in the par swap rate and is therefore a measure of the risk involved with an interest rate swap.

However, market participants also measure the risk of an interest rate swap by computing how much the value of the swap will change if either zero coupon spot rates or observed market rates change by 1 bps.

Recall that a basispoint (bp) is given by

$$1 \text{ bp} = \left(\frac{1}{10000}\right) \times 1 \text{ percentage point} = 0.0001 \tag{57}$$

Risk management of an interest rate swap

The *Dollar Value of 1 bp* abbreviated *DV01* is commonly computed in one of the following ways for a given interest rate swap or fixed income derivative position.

- Compute the change in monetary value of your position for a 1 bp shift in the entire zero coupon spot curve.
- Compute the change in monetary value of your position for a 1 bp shift in *each* of the zero coupon spot rates and collect these in a Δ-vector (delta vector).
- Compute the change in monetary value of your position for a 1 bp shift in *each* of the observed market rates used to compute the zero coupon yield curve and collect these in a Δ-vector (delta vector).

Risk management of an interest rate swap - DV01

Let us consider a position in a fixed income instruments such as an interest rate swap or a derivative, and denote the value of this position by V.

Let us denote the vector of zero coupon spot rates from our calibration of the zero coupon yield curve by $\mathbf{R}$ and denote the market rates that were used to find $\mathbf{R}$ by $\mathbf{Y}$.

Then the Δ-vector with respect to $\mathbf{R}$ can be found from

$$\boldsymbol{\Delta}_R = \frac{V(\mathbf{R} + 0.0001) - V(\mathbf{R})}{0.0001} \approx \frac{\partial V}{\partial \mathbf{R}} \times 1 \text{ bp} = \frac{\partial V}{\partial \mathbf{R}} \cdot 0.0001 \tag{58}$$

The Δ-vector with respect to $\mathbf{Y}$ can be found from

$$\boldsymbol{\Delta}_Y = \frac{V(\mathbf{Y} + 0.0001) - V(\mathbf{Y})}{0.0001} \approx \frac{\partial V}{\partial \mathbf{Y}} \times 1 \text{ bp} = \frac{\partial V}{\partial \mathbf{R}} \frac{\partial \mathbf{R}}{\partial \mathbf{Y}} \cdot 0.0001 \tag{59}$$

Risk management of an interest rate swap - DV01

The delta vector with respect to zero coupon rates is very easy and fast to compute numerically.

The delta vector with respect to market rates, on the other hand, is typically time-consuming to compute because every time we bump a market rate by 1 bp, we have to recalibrate the zero coupon curve.

However, we can use multivariate calculus and that

$$\left(\frac{\partial \mathbf{R}}{\partial \mathbf{Y}}\right)' = \left(\frac{\partial \mathbf{Y}}{\partial \mathbf{R}}\right)^{-1} \tag{60}$$

This is convenient because the matrix $\frac{\partial \mathbf{Y}}{\partial \mathbf{R}}$ can often be recovered from the optimization procedure when calibrating the zero coupon yield curve.

Risk management of an interest rate swap - DV01

To compute the DV01 of an interest rate swap from bumping either a single point on the spot rate curve or the entire spot rate curve, the following steps must be taken.

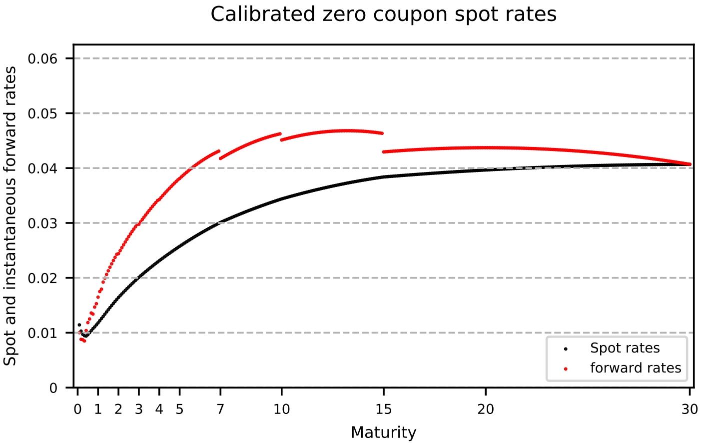
- Bump the appropriate spot rate(s) by adding 0.0001 to it.
- Compute the corresponding ZCB prices.
- Compute the new accrual factor of the swap.
- Compute the new par swap rate of the swap.
- Compute the DV01 by multiplying the new accrual factor and the change in the par swap rate.

Risk management of an interest rate swap - DV01

To compute the DV01 of an interest rate swap from bumping one or more market rates, when calibrating the yield curve is not computationally expensive, the following steps must be taken.

- Bump the appropriate market rate(s) by adding 0.0001 to it.
- Recalibrate the yield curve *or* use the trick described in (60)
- Compute the corresponding ZCB prices.
- Compute the new accrual factor of the swap.
- Compute the new par swap rate of the swap.
- Compute the DV01 by multiplying the new accrual factor and the change in the par swap rate.

In the case where the data looks as described earlier, a fitted ZCB yield curve might look as follows



Risk management of an interest rate swap - DV01

Suppose for example we are interested in knowing our exposure from a position in the 5Y receiver swap.

First, we can compute the accrual factor S_{5Y} of the 5Y swap and we get

$$S_{5Y} = 4.688 \tag{61}$$

Recall that the accrual factor of a swap is much like the duration of a fixed rate bond serving as both a measure of the price-sensitivity and the mean time before the present value of the initial investment is returned to the investor.

The fact that the accrual factor is only a little less than the maturity of the swap reveals that the 5Y swap, in terms of risk, has many of the same properties as a fixed rate bullet bond of same maturity.

Risk management of an interest rate swap - DV01

If we calculate the DV01 when bumping the entire ZCB yield curve by 1 bps up, we get that for a swap with a principal of $K = 100$ is

$$DV01_{\text{bump all}} = -0.0474.$$

If we instead bump *only* the 5Y ZCB rate we get am absolute DV01 that is only slightly smaller

$$DV01_{\text{bump 5Y}} = -0.0451.$$

Despite cash flows accruing throughout the 5Y span of the interest rate swap and the principal not changing hands, the 5Y swap is almost exclusively exposed to changes at the very back end of the term structure of interest rates.

In fact, the X year interest rate swap behaves much like the X year ZCB!