Written Exam Economics winter 2023/2024

Fixed Income Derivatives

January 19. 2024 from 09.00 to 21.00

This exam question consists of 7 pages in total

Answers only in English.

A take-home exam paper cannot exceed 10 pages – and one page is defined as 2400 keystrokes

The paper must be uploaded as <u>one PDF document</u>. The PDF document must be named with exam number only (e.g. '1234.pdf') and uploaded to Digital Exam. Please write the exam number on your exam paper as well.

Use of AI tools is permitted. You must explain how you have used the tools. When text is solely or mainly generated by an AI tool, the tool used must be quoted as a source.

Fixed Income Derivatives Final Exam Fall 2023(19.01.2024)

Instructions

- Your answer to the exam must consist of one document only.
- The answer to *all* questions must be clearly stated in the document. Your answers cannot solely be output from Python code.
- Clearly state which question and subquestion you are answering. Problem 1ai) etc.
- Include the complete Python code that you used to generate the answers and do so in a manner so that the code can be run directly and your answers be reconstructed.
- Make it clear by commenting your code which part of the code was used to answer what question.

Problem 1

Your job is to compute the price of a complex derivative and to do that, you have decided to fit the Cox-Ingersoll-Ross model to a set of continuously compounded zero coupon spot rates that your colleague has extracted from market data and which are given in the table below. The spot rates are denoted by $R(0,T)$ and the corresponding maturities by T.

T	0.1	0.25	0.5	0.75	1	1.5	2	3	4	5	7	10
$R(0,T)$	0.0334	0.0352	0.0375	0.0392	0.0405	0.0422	0.0433	0.0445	0.0451	0.0455	0.0459	0.0462

In the CIR model, the dynamics of the short rate r_t under the risk neutral pricing measure $\mathbb{Q}$ are

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r(t=0) &= r_0 \end{aligned} \tag{1}$$

From years of experience, you know that $\sigma = 0.08$, so you will *not* need to estimate σ.

a) Using initial values $\tilde{r}_0 = 0.025$, $\tilde{a} = 1.5$, $\tilde{b} = 0.07$, fit a CIR model to the data from your colleague and estimate r_0, a and b.

   i) Report the estimates $\hat{r}_0$, $\hat{a}$ and $\hat{b}$.

   ii) Demonstrate that a CIR model with your estimated parameter values fits the data.

Next, you will simulate the short rate in the CIR model using the parameter estimates you have just found and a step in time of size δ. Denote by M, the number of steps in your simulation and index the time points in your simulation by m, $m \in \{0, 1, 2, ..., M-1, M\}$ so that the time points will be $\big[t_0, t_1, ..., t_{M-1}, t_M\big] = \big[0, \delta, 2\delta, ..., T-\delta, T = \delta M\big]$ and hence $\delta = \frac{T}{M}$. The scheme you will need to implement is a simple Euler first-order scheme of the form

$$r_m = r_{m-1} + a(b - r_{m-1})\delta + \sigma\sqrt{\delta}\sqrt{r_{m-1}}Z_m, \quad m \in \{1, 2, ..., M\} \tag{2}$$

where $Z_m \sim N(0,1)$, $m \in \{1, ..., M\}$ and all the standard normal random variables are independent. Set the parameters of the model you are simulating to the values you found when fitting the CIR model in a) above. That is, set $r_0 = \hat{r}_0$, $a = \hat{a}$, $b = \hat{b}$ and $\sigma = 0.08$. (If you were not able to fit the CIR model, you can use the initial values $\tilde{r}_0$, $\tilde{a}$ and $\tilde{b}$ from the fit).

b) Now set $t_0 = 0$, $T = 10$ and $M = 10000$ and construct a *single* trajectory for the short rate.

   i) Plot the trajectory of the short rate for the whole simulation from $t = 0$ to $t = 10$.

   ii) Compute a two-sided 99 percent confidence interval for the short rate in 1 year denoted r_1 and report the upper and lower confidence bounds.

   iii) Compute a two-sided 99 percent confidence interval for the short rate under the stationary distribution and report the upper and lower confidence bounds.

You now have to turn your attention to a complex derivative which pays the owner of the derivative the maximum of the short rate over the next $T = 2$ years. That is, the complex derivative will have a maturity of $T = 2$ and a pay-off function $\chi(T)$ given by

$$\chi(T) = \max_{0 \le t \le T} r_t \tag{3}$$

To compute the time $t = 0$ price $\Pi(0)$ of the derivative, you can use that

$$\Pi(0) = \mathrm{E}^{\mathbb{Q}}\big[e^{-\int_0^T r_t dt}\chi(T)\big] \tag{4}$$

To compute the fair price of this complex derivative, simulate the short rate over a two year period and repeat the simulation N times. For each simulation n, compute the approximate discounted pay-off X_n

$$X_n = \exp\Big\{-\frac{T}{M}\sum_{m=0}^{M-1} r_m\Big\} \cdot \Big(\max_{0 \le m \le M} r_m\Big) \tag{5}$$

and store the value in a vector. Once all N simulations have completed, compute the fair value by averaging over the expected discounted payoffs for $n = 1, ..., N$.

c) Set M to 1000 and simulate the short rate from $t_0 = 0$ to $T = 2$ at least $N = 1000$ times.

   i) Report the fair value of the complex derivative.

   ii) Briefly discuss if the method you have chosen will underestimate or overestimate the fair value of the complex derivative.

Problem 2

Your job is to assess the risk of your institution's interest rate swap exposure and therefore, you must first calibrate a zero coupon term structures of interest rates to market data. The 6M EURIBOR fixing has just been announced to be 0.02927 and in addition, you have the data shown in the table below for forward rate agreements and interest rate swaps. The interest rate swaps pay 6M EURIBOR semi-annually against the fixed par swap rates listed below also paid semi-annually.

EURIBOR	Fixing	FRA	Midquote	IRS	Midquote
6M	0.02927	1X7	0.03161	2Y	0.03824
		2X8	0.03295	3Y	0.04083
		3X9	0.03418	4Y	0.04242
		4X10	0.03531	5Y	0.04346
		5X11	0.03635	7Y	0.04468
		6X12	0.03731	10Y	0.04561
		7X13	0.03819	15Y	0.04633
		8X14	0.03900	20Y	0.04667
		9X15	0.03975	30Y	0.04700

a) Calibrate a zero coupon term structure of continuously compounded spot interest rates to the 6M EURIBOR fixing, the forward rate agreement rates and par swap rates given in the table above.

  i) Report zero coupon spot rates for the following six maturities: $T = [0.5, 1, 3, 5, 10, 20, 30]$ based on your calibration.

  ii) Plot the term structures of continuously compounded zero coupon spot rates for maturities ranging from 0 to 30 years.

You will now estimate the continuously compounded *instantaneous* forward rates so that you can assess the properties of your calibration. This should be done by first interpolating, using the same method of interpolation, the spot rates you found above so that you have say 96(or some other multiple of 12) time points per year, then converting spot rates to zcb prices and then converting zcb prices to estimates of instantaneous forward rates.

b) Estimate the term structure of continuously compounded *instantaneous* forward rates.

  i) Plot the instantaneous forward rates in the same type of plot as above for maturities ranging from 0 to 30 years.

  ii) Discuss properties the calibrated and interpolated term structures should posses and whether this can be achieved in practice depending on the choice of interpolation method.

  iii) Discuss whether your calibrated term structures of interest rates has these 'good' properties and relate your conclusion to the choice of interpolation method.

You will now investigate the interest rate risk associated with the $7Y$ Swap by computing the $DV01$ for various changes in the market.

c) Please 'bump' the yield curve in the appropriate ways given below, report the associated $DV01$'s for the $7Y$ swap and answer the question in iii).

  i) The $DV01$ for a 1 bp increase in *all* zero coupon spot rates.

  ii) The $DV01$ for a 1 bp increase in each of the zero coupon spot rates for the following five maturities $T = 1, 2, 3, 5, 7$.

  iii) Compare the values you computed in ii) and briefly discuss whether the 7Y swap is more sensitive to changes to the front end or the back end of the zero coupon yield curve.

Problem 3

Imagine that your task is to help a corporate client decide on the best way to finance his operations. The client has a loan on which he pays a floating rate of 6M EURIBOR starting exactly 6 months from now and ending in exactly $T = 4$ years. The 6M EURIBOR fixing has just been announced and your quant colleague has computed continuously compounded zero coupon bond spot rates based on market prices. The spot rates $R(0, T_i)$ for various choices of maturity T_i are as shown in the table below. Also, you can observe caplet Black implied volatilities for caplets on 6M forward EURIBOR for a strike of $R = 0.05$ and various settlement dates. The caplet black implied volatilities are likewise shown in the table below. For example, the Black implied volatility for a caplet on the 6M EURIBOR fixing announced at time $T_{i-1} = 2.5$ and paid at time $T_i = 3$ is 0.312.

Table 1: Continuously compounded spot rates and caplet Black implied volatilities

T_i	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00
$R(0, T_i)$	0.0385	0.0431	0.0463	0.0486	0.0502	0.0513	0.0521	0.0527
caplet $\bar{\sigma}_i$	-	0.223	0.241	0.260	0.283	0.312	0.355	0.402

Finally, you can observe that the 2Y2Y payer swaption on 6M EURIBOR for a strike of $K = 0.05$ trades at a Black implied volatility of $\bar{\sigma} = 0.39$. Recall that a 2Y2Y payer swaption will give you the option at exercise in two years to enter into a two year payer swap in which you pay the fixed rate $K = 0.05$ and receive 6M EURIBOR.

a) Compute zero coupon bond prices and 6M forward EURIBOR rates $L(0; T_{i-1}, T_i)$ corresponding to the continuously compounded spot rates. Report, the 6M forward EURIBOR rates corresponding to the fixings taking place at T_{i-1} for $T_{i-1} = \{0.5, 1, 1.5, 2, 2.5, 3, 3.5\}$ in a table.

Your client is concerned that future 6M EURIBOR fixings will be very high so you offer three possibilities:

i) Convert *all* future floating rate payments into a known fixed coupon rate using a 4Y swap.

ii) Use an interest rate cap to insure that the clients floating rate payments will never exceed 0.05.

iii) Use a 2Y2Y payer swaption with a strike of 0.05 so that the client can be sure that his payments to be paid at times $T = 2.5, 3.0, 3.5, 4.0$ will be at most 0.05.

b) Your task is now to find the fair value for each of the choices i), ii), iii) and give the client a quote for each of them. For ii) and iii), you will quote the price in terms of a spread in basis points paid on top of the client's coupon payments.

   i) Report the fixed coupon rate the client would need to pay if he converts his floating rate payments into fixed payments using a 4Y interest rate swap.

   ii) Find the spread in bps, the client would pay on top of his regular interest payments to cap these payments at 0.05.

   iii) Find the spread in bps, the client would pay on top of his interest payments by buying the 2Y2Y swaption with a strike of $K = 0.05$.

c) The three different strategies have different risk and uncertainty profiles and also different costs. For each of the three possibilities i), ii) and iii), please *very* briefly discuss their upside, downside and cost.

Problem 4

You work for the options trading desk and your quant has computed zero coupon bond prices for a range of maturities. These prices can be seen in the table below.

Table 2: Zero coupon bond prices

T	0.50	1.00	1.50	2.00	2.50	3.00
$p(0,T)$	0.98322948	0.96455878	0.94449414	0.92344747	0.90175113	0.87967118
T	3.50	4.00	4.50	5.00	5.50	6.00
$p(0,T)$	0.85741902	0.83516131	0.81302835	0.79112104	0.76951663	0.7482734

In addition to zero coupon bond prices, you can also observe prices $\Pi_{swaption}$ of $2Y4Y$ payer swaptions for a notional of 1 and different strikes. The underlying swap pays a fixed rate semi-annually and against 6M EURIBOR received semi-annually and the swaption prices are given in the table below. Recall that a 2Y4Y payer swaption gives you the right but not obligation to enter into a 4 year payer swap at the time of exercise in 2 years. The strikes are to be interpreted such that a K_{offset} of 0 (ATMF) corresponds to a strike equal to the current 2Y4Y forward rate and a K_{offset} of 100 say corresponds to a strike 100 bps above the current 2Y4Y forward rate.

Table 3: $2Y4Y$ Swaption prices

$K_{offset}(bp)$	-300	-250	-200	-150	-100	-50	ATMF
$\Pi_{swaption}$	0.0995524	0.08350629	0.06774531	0.05248227	0.03808218	0.02519355	0.01482874
$K_{offset}(bp)$	50	100	150	200	250	300	-
$\Pi_{swaption}$	0.00785645	0.00404525	0.00219232	0.00128815	0.00081635	0.00054773	-

a) Compute the accrual factor S and the 2Y *forward* par swap rate for the 4Y swap that serves as the underlying asset for the swaptions and then, using the par swap rate you have found, find the strikes K of the swaptions in the table above.

   i) Report the $2Y4Y$ forward par swap rate.

   ii) Compute Black implied volatilities for all strikes and plot these as a function of K_{offset}.

   iii) Interpret the implied volatility plot and assess if the market is pricing swaptions according to Black's model. What can be said about the distribution of the $2Y4Y$ forward par swap rate implied by the pricing measure chosen by the market and how does that distribution compare to the log normal distribution?

You now decide to fit a SABR model to the implied volatilities you have computed from market swaption prices. The SABR model you should consider is of the usual form given below

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta} dW_t^{(1)}, \quad F(0) = F_0 \\ d\sigma_t &= \upsilon \sigma_t dW_t^{(2)}, \quad \sigma(0) = \sigma_0 \\ dW_t^{(1)} dW_t^{(2)} &= \rho \end{aligned} \tag{6}$$

where F_t here is the 2Y4Y forward par swap rate.

b) Using initial values $\tilde{\sigma}_0 = 0.04$, $\tilde{\beta} = 0.5$, $\tilde{\upsilon} = 0.4$ and $\tilde{\rho} = -0.3$, you now decide to fit a SABR model to the implied volatilities you have found above.

   i) Report the fitted parameter values $\hat{\sigma}_0$, $\hat{\beta}$, $\hat{\upsilon}$ and $\hat{\rho}$.

ii) Assess whether the parameter values you have found match the observed market implied volatilities.

c) Your colleague has just taken a large long position in the payer swaption with a strike 100 below ATMF. He is concerned about his exposure, so he asks you to compute how much he would gain/loose in a few different scenarios. Compute your colleagues gain/loss if:

i) υ falls by 0.02,

ii) ρ increases by 0.1,

iii) the entire term structure of zero coupon spot rates fall by 1 bp.