# Fixed Income Derivatives - Problem Set Week 2

**Problem 1**

For this problem, assume that the year consists of 12 months each with exactly 30 days and that all payments occur at the end of day on the last day of the quarter. Also assume that there is no credit risk, that the principal of all bonds is 100 and that all interest rates are annualized. The date today is January 15. 2017 but you know the LIBOR fixings from December 30. 2016 and they were 3M LIBOR: 5.2%, 6M LIBOR: 4.9%, 12M LIBOR: 4.76%. The following three interest rates swaps are available in the market

i) A receiver swap with maturity December 30. 2017 receiving fixed semi-annual coupons of 5.1% and paying floating quarterly 3M LIBOR trading for 0.79492002.

ii) A payer swap with maturity December 30. 2018 paying fixed annual coupons of 4.4% and receiving floating semi-annual 6M LIBOR trading for $-1.02540877$.

iii) A receiver swap with maturity December 30. 2018 receiving fixed annual coupons of 4.9% and paying floating quarterly 3M LIBOR trading for 2.05066409.

In addition to the three interest rate swaps, there are also five fixed rate bullet bonds in the market.

iv) A fixed rate bullet bond maturing December 30. 2017 paying quarterly simple coupons of 7 % and a price of 103.02163487.

v) A fixed rate bullet bond maturing June 30. 2018 paying semi-annual simple coupons of 5 % and a price of 101.80152680.

vi) A fixed rate bullet bond maturing December 30. 2018 paying annual simple coupons of 6 % and a price of 104.48120266.

vii) A fixed rate bullet bond maturing June 30. 2018 paying quarterly simple coupons of 4.5 % and a price of 101.10990798.

viii) A fixed rate bullet bond maturing December 30. 2018 paying quarterly simple coupons of 5.5 % and a price of 103.67216735.

Now that you 8 traded assets in total, please answer the following

a) Set up equations for the value of both the fixed- and floating legs in terms of the LIBOR fixings and zero coupon bond prices for the three interest rate swaps.

b) Construct a cashflow matrix $\mathbf{C}$ with each row corresponding to a traded assets. What is the rank of this matrix? For which maturities can you compute zcb prices?

c) Compute ZCB prices for as many maturities as you can. Is the market complete? Is it arbitrage free?

d) Plot the term structures of ZCB prices, spot rates and 3M forward rates.

e) What would the prices of the 3 swaps and 5 bonds be if all spot rates were to suddenly drop by 10 basispoints (0.1 percentage points)?

f) Find the par swap rate for swap i) today on January 15. 2017. That is, find the size of the fixed coupon such that it would have had a value of 0 today. Can you determine if a trader, who has held this receiver swap since issuance, has made money simply by comparing the par swap rate at issuance to the par swap rate todas

g) Compute the accrual factor of swap i) and use the accrual factor along with the new par swap rate to find the PnL of this swap since the day of issuance.

h) Does your answer conform with other information given in this problem?

Now a trader calls you and offers to sell a fixed rate bullet bond with semi-annual coupons of 5.2% maturing on December 30. 2018 for 100.2.

i) Show that the inclusion of this bond into the market gives rise to an arbitrage?

j) Construct a portfolio that replicates the fixed rate bullet bond the trader is trying to sell you and use this portfolio to construct an arbitrage of type II.

k) Construct a portfolio containing one unit of the bond the trader is trying to sell you such that this portfolio has a price of 0 today, pays a positive amount only on March 30. 2017 and pays exactly 0 at all other future points in time. Check that the price of this portfolio is indeed 0 and hence that you have created an arbitrage of type I. Finally, find the payoff of this portfolio on March 30. 2017.

**Solution**

a) See Problems Set Week 1 and the lecture notes in particular for the case where some time has passed since the previous fixing of the floating rate payment.

b) The cashflow matrix becomes

$$\mathbf{C} = \begin{bmatrix} -101.3 & 2.55 & 0 & 102.55 & 0 & 0 & 0 & 0 \\ 0 & 102.45 & 0 & -4.4 & 0 & 0 & 0 & -104.4 \\ -101.3 & 0 & 0 & 4.9 & 0 & 0 & 0 & 104.9 \\ 1.75 & 1.75 & 1.75 & 101.75 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 2.5 & 0 & 102.5 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 106 \\ 1.125 & 1.125 & 1.125 & 1.125 & 1.125 & 101.125 & 0 & 0 \\ 1.375 & 1.375 & 1.375 & 1.375 & 1.375 & 1.375 & 1.375 & 101.375 \end{bmatrix} \tag{1}$$

The rank of $\mathbf{C}$ is 8 and we can determine ZCB prices for

$$\mathbf{T} = \left[0, \frac{5}{24}, \frac{11}{24}, \frac{17}{24}, \frac{23}{24}, \frac{29}{24}, \frac{35}{24}, \frac{41}{24}, \frac{47}{24}\right] \tag{2}$$

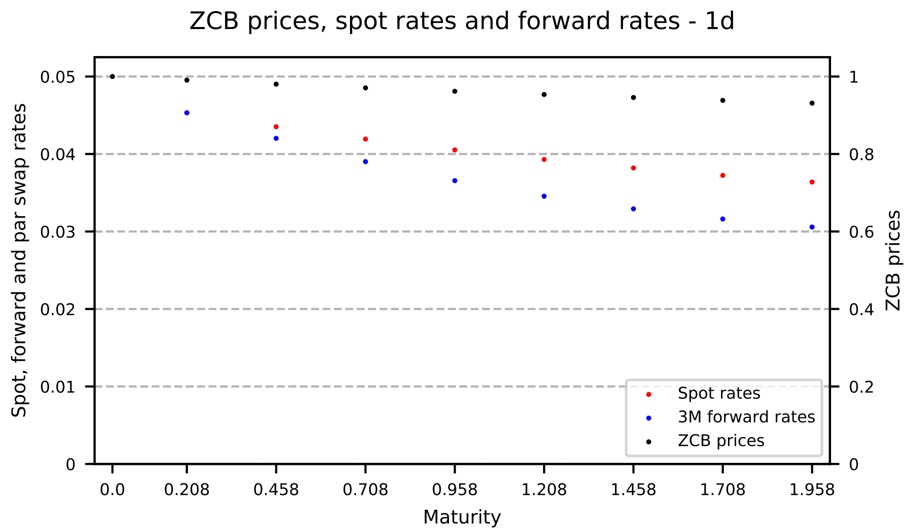
where of course $p(0,0) = 1$.

c) The market is complete since we have 8 linearly independent assets and 8 future states. Zero coupon bond prices become

$$\mathbf{p} = \left[1, 0.99060312, 0.9802517, 0.97073639, 0.96190516, 0.95363276, 0.94581595, 0.93836974, 0.93122426\right]$$

Since all ZCB prices are positive, we conclude that the market is also arbitrage-free.

d) The term structures of ZCB prices, spot rates and 3M forward rates become.




e) The prices of the 3 swaps and 5 bonds after a 10 bps drop in a all spot rates become

$\boldsymbol{\pi}' = [0.86973647, -1.1740026, 2.22576323, 103.11782696, 101.94643885, 104.68023231, 101.25333922, 103.86602576]$

f) The par swap rate of swap i is today on January 15. 2017 $R_i = 0.044693886181534 83$. The par swap rate of swap has fallen from 0.051 to today's value and hence, the owner of the receiver swap will have made money from his position.

g) The accrual factor of swap i is $S_i = 0.930234610936274$ and the PnL of the swap since inception is 0.005866165334439804 per 100 dollar principal.

h) The Pnl computed in g) confirms that the owner of the receiver swap did indeed make money from his position.

i) We know that the market was already complete and arbitrage free before the introduction of this new bond, let us call it bond ix. The bond offered by the trader has cash flows at points in time where the original 3 swaps and 5 bonds had cash flows as well, so the new bond does not increase the state space. To prevent an arbitrage from occurring after the introduction of the new bond, we must require that this bond is priced using the ZCB prices computed in c). However, using these ZCB prices gives a fair value of the new bond of $\pi_{\mathrm{ix}} = 103.0523385081963$. The trader is in other words selling the bond to cheap and he will introduce an arbitrage by doing so.

j) To replicate bond ix, we can use $\mathbf{h}_{ix}$ given by

$$\mathbf{h}_{ix} = \big[-0.0319491, 0.02555447, 0.0319491, 0, 0.02536585, 0.96147572, 0, 0\big] \tag{3}$$

and a type II arbitrage portfolio could look like

$$\mathbf{h}_{II} = \big[0.0319491, -0.02555447, -0.0319491, 0, -0.02536585, 0. - 96147572, 0, 0, 1\big] \tag{4}$$

which would yield a profit today of 2.852338508196297.

k) The portfolio $\mathbf{h}_I$ with a price of 0 today containing one unit of bond ix and a single cash flow on March 30. 2017 offering a type I arbitrage can be found by first finding a portfolio that replicates the March 30. 2017 ZCB and then using the type II arbitrage from the above to finance buying the March 30. 2017 replicating portfolio. We get that

$$\mathbf{h}_I = \big[0.03166424, -0.02554738, -0.06008868, 0, -0.02536585, -0.93362117, 0, 0, 1\big] \tag{5}$$

The payoff on March 30. 2017 from this type I arbitrage portfolio becomes 2.87939584 which is, and should be, nothing other than the size of type II arbitrage from above divided by the Marh 30. 2017 ZCB price.

**Problem 2**

In this problem, we assume that the 6M Euribor rate has just been announced and that the data below for EUR FRA's and EUR denominated swaps has just been recorded. Recall that the coupon on the fixed leg is paid annually and the coupon on the floating leg semi-annually for EUR denominated swaps.

| EURIBOR | Fixing | FRA | Midquote | IRS | Midquote |
|---|---|---|---|---|---|
| 6M | 0.02750 | 1X7 | 0.02980 | 2Y | 0.03782 |
| | | 2X8 | 0.03122 | 3Y | 0.04152 |
| | | 3X9 | 0.03257 | 4Y | 0.04402 |
| | | 4X10 | 0.03384 | 5Y | 0.04577 |
| | | 5X11 | 0.03504 | 7Y | 0.04797 |
| | | 6X12 | 0.03617 | 10Y | 0.04971 |
| | | 7X13 | 0.03724 | 15Y | 0.05105 |
| | | 8X14 | 0.03825 | 20Y | 0.05170 |
| | | 9X15 | 0.03920 | 30Y | 0.05230 |

In addition to these securities, we will also consider a 10Y fixed rate bond paying a simple semi-annual coupon of 0.05 on a principal of 100 EUR. This fixed rate bond has also just been issued.

a) Calibrate a continuously compounded term structure of ZCB spot rates based on this data and plot the resulting spot rates and instantaneous forward rates. Does your calibration meet the criteria of a well-behaved yield curve?

In addition to the FRA's and the swaps above, we will also consider a 10Y fixed rate bond paying a simple semiannual coupon of 0.05 on a principal of 100 EUR. This fixed rate bond has also just been issued.

b) Compute the price, yield-to-maturity, Macauley duration, modified duration and convexity of the fixed rate bond.

c) Use an expansion to estimate the percentage change in the price of the fixed rate bond for a 10 bps increase in the yield-to-maturity first using only duration and second using both duration and convexity. Compare the two numbers and asses whether also using convexity has changed your estimate significantly.

d) Now compute the accrual factor and par swap rate of the 3Y payer swap knowing that the swap has just been issued and that it therefore has a value of 0.

e) Also compute the accrual factor and par swap rate of the 10Y receiver swap again using that the swap has just been issued and has a value of 0.

In the following, we will use the 3Y and 10Y swaps to hedge the risk to a long position in the fixed rate bond. First consider the case of a 10 bps increase in all ZCB spot rates.

f) Compute the PnL of the fixed rate bond, the 3Y payer swap and the 10Y receiver swap resulting from the increase in ZCB yields assuming the swaps have a principal of 100.

g) In order to hedge the long position in the fixed rate bond against a 10 bps increase in ZCB spot rates using only the 3Y payer swap, what should the principal of the swap be?

h) In order to hedge the long position in the fixed rate bond against a 10 bps increase in ZCB spot rates using only the 10Y receiver swap, what should the principal of the swap be?

i) Relate the size of the hedging positions to the duration of the fixed rate bond and the accrual factors of each of the swaps.

Next, consider a 10 bps increase in all market rates. Strictly speaking the 6M LIBOR rate, the FRA rates and the swap rates are of different types but nevertheless simple ad 10 bps to the data in the table above.

j) Compute the PnL of the fixed rate bond, the 3Y payer swap and the 10Y receiver swap resulting from the increase in market rates assuming the swaps have a principal of 100.

k) In order to hedge the long position in the fixed rate bond against a 10 bps increase in market rates using only the 3Y payer swap, what should the principal of the swap be?

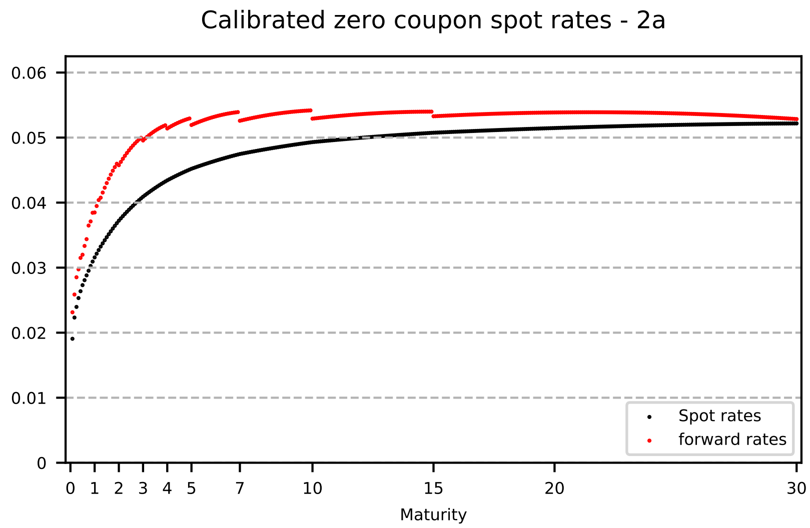
l) In order to hedge the long position in the fixed rate bond against a 10 bps increase in market rates using only the 10Y receiver swap, what should the principal of the swap be?

m) Relate the size of the hedging positions to the duration of the fixed rate bond and the accrual factors of each of the swaps.

n) Discuss pros and cons of hedging the fixed rate bond using the 3Y versus the 10Y swap.

**Solution**

a) Fitting a yield curve with a hermite polynomial of degree 2 results in the follow term structure of spot- and instantaneous forward rates.




The term structure of spot rates looks fairly well-behaved but the term structure of instantaneous forward rates does have jumps reflecting that the term structure of ZCB prices is not differentiable. Later in this course we will develop methods to fix this problem and impose that ZCB prices as a function of $T$ are by construction differentiable.

b) For the 10Y fixed rate bond, we will have that

$$\begin{aligned} &\pi_{fr} = 100.71722924379213, \quad y_{fr} = 0.049686110837155725, \\ &D_{fr} = 7.9979522671590315, \quad MD_{fr} = 7.619375148996139, \quad K_{fr} = 73.46539950572414. \end{aligned} \tag{6}$$

c) The estimated change in the price of the fixed rate bond for a 10 bps increase in the yield-to-maturity is $-0.007619375148996139$ per cent when only duration is used and $-0.007582408219623711$ per cent when both duration and convexity are used. The difference between the two measures is modest which is often the case when only smaller changes in the yield-to-maturity are considered.

d) The accrual factor and par swap rate of the 3Y swap are $S_3 = 2.7816653667378115$ and $R_3 = 0.04152049329502988$.

e) For the 10Y payer swap, we get $S_{10} = 7.829710221439025$ and $R_{10} = 0.04971138643896786$.

f) The Pnl's for a 10 bps increase in all ZCB spot rates become

$$PnL_{fr} = -0.7984214095588982, \quad PnL_3 = 0.28767606176021165, \quad PnL_{10} = -0.804642468989293 \tag{7}$$

g) To hedge the 10Y fixed rate bond using the 3Y payer swap, the principal of the swap must be $K_3 = 277.5418311393637$.

h) To hedge the 10Y fixed rate bond using the 10Y receiver swap, the principal of the swap must be $K_{10} = -99.22685420293449$.

i) The Accrual factors of the two swaps and the modified duration of the fixed rate bond are

$$S_3 = 2.7761927616606616, \quad S_{10} = 7.790099188496136, \quad MD_{fr} = 7.619375148996139 \tag{8}$$

The fixed rate bond falls in value as rates rise whereas the 3Y payer swap rises in value as rates rise. Therefore, a long position in a payer swap is needed to hedge a long position in a fixed rate bond. The ratio of the sizes of the two positions are roughly equal to the ratio of the modified duration of the fixed rate bond to the swap, why the principal in the 3Y payer swap has to be almost three times that of the fixed rate bond. The 10 receiver swap, like the fixed rate bond, falls in value as rates rise and a negative position in the 10Y receiver swap is needed to hedge the fixed rate bond. The modified duration of the fixed rate bond and the 10Y receiver swap are close, why the principal of the 10Y is much closer to the principal of the 10Y fixed rate bond.

j) The Pnl's for a 10 bps increase in all market rates become

$$PnL_{fr} = -0.7731091696715708, \quad PnL_3 = 0.27757220455330683, \quad PnL_{10} = -0.7791227355774284$$

k) To hedge the 10Y fixed rate bond using the 3Y payer swap, the principal of the swap must be $K_3 = 278.52542761467237$.

l) To hedge the 10Y fixed rate bond using the 10Y receiver swap, the principal of the swap must be $K_{10} = -99.22816192735016$.

m) From the results, we see that increasing market rates have roughly the same effect on both the fixed rate bond as well as on the swaps and the conclusions regarding the signs and sizes of the swaps positions from question i) also apply in this case.

n) The 10Y fixed rate bond is exposed to changes in the entire yield curve up to maturities of 10 year and so is the 10Y swap. The 3Y swap however is not exposed to the yield curve beyond maturities up to 10 year. Hedging using the 3 year swap, is not ideal since changes at the back end of the yield curve would only affect the fixed rate bond rendering the hedge ineffective. When hedging interest rate position, it is important to use instruments of equal duration/accrual factor but it is also good practice to use instruments for hedging that have comparable maturity to the maturity of the instrument being hedged.