

Fixed Income Derivatives - Problem Set Week 6(Partial Version)

Problem 1

Consider the Hull-White model where the short rate r_t has dynamics

$$dr_t = [\Theta(t) - ar_t]dt + \sigma dW_t \quad (1)$$

a) Argue that ZCB prices are of the form

$$p(t, T) = e^{A(t, T) - r(t)B(t, T)} \quad (2)$$

where

$$\begin{aligned} A(t, T) &= \int_t^T \left[\frac{1}{2} \sigma^2 B^2(s, T) - \Theta(s)B(s, T) \right] ds \\ B(t, T) &= \frac{1}{a} \left[1 - e^{-a(T-t)} \right] \end{aligned} \quad (3)$$

b) Show that forward rates $f(t, T)$ are of the form

$$f(t, T) = -\frac{\partial}{\partial T} A(t, T) + r(t) \frac{\partial}{\partial T} B(t, T) \quad (4)$$

c) Argue that the forward rate dynamics can be found from

$$df(t, T) = -\frac{\partial}{\partial T} \left(A_t(t, T)dt - B_t(t, T)r(t)dt - B(t, T)dr_t \right) \quad (5)$$

where $A_t(t, T) = \frac{\partial}{\partial t} A(t, T)$

d) Show that the forward rate dynamics are

$$df(t, T) = \frac{\sigma^2}{a} e^{-a(T-t)} \left[1 - e^{-a(T-t)} \right] dt + \sigma e^{-a(T-t)} dW_t \quad (6)$$

Now, we will find the forward rate dynamics in a different way. Let us recall that in the Hull-White model, zero coupon bond prices become

$$p(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp \left\{ B(t, T)f^*(0, t) - \frac{\sigma^2}{4a} B^2(t, T)(1 - e^{-2at}) - B(t, T)r(t) \right\} \quad (7)$$

e) Use the above expression to find an expression for forward rates and treat this expression as a function $g = g(t, T, r(t))$.

f) Show from $g(t, T, r(t))$ that the forward rate dynamics are of the form

$$df(t, T) = \alpha(t, T)dt + \sigma e^{-a(T-t)} dW_t \quad (8)$$

where $\alpha(t, T)$ is yet to be determined.

g) Use the HJM drift condition to find $\alpha(t, T)$ and thus show that it is of the same form as in d).

Problem 2

Take as given an HJM model under the risk neutral measure \mathbb{Q} where

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t \quad (9)$$

a) Show that all forward rates and also the short rate are normally distributed.

b) Show that bond prices are log-normally distributed.

Problem 3

Consider the Ho-Lee model where the short rate has dynamics

$$dr_t = \Theta(t)dt + \sigma dW_t \quad (10)$$

- a) Use the procedure outlined in Problem 1 e)-g) to find the forward rate dynamics in this model.
- b) Find the dynamics of zero coupon bond prices under \mathbb{Q} .
- c) Find the distribution of forward rates in this model and argue that they are Gaussian.
- d) Use a result from the chapter 'Change of Numeraire' in Bjork to directly compute the time $t = 0$ price of a European call option with exercise date $T_0 > t$ on a maturity $T > T_1$ zero coupon bond.

Problem 4

Take as given an HJM model under the risk neutral measure \mathbb{Q} of the form

$$df(t, T) = \alpha(t, T)dt + \sigma_1(T - t)dW_{1t} + \sigma_2 e^{-a(T-t)}dW_{2t} \quad (11)$$

- a) Use the HJM drift condition to find $\alpha(t, T)$.
- b) Find the dynamics of zero coupon bond prices under \mathbb{Q} .
- c) Find the distribution of forward rates in this model and argue that they are Gaussian.
- d) Use a result from the chapter 'Change of Numeraire' in Bjork to directly compute the time $t = 0$ price of a European call option with exercise date $T_1 > t$ on a maturity $T_2 > T_1$ zero coupon bond.