Fixed Income Derivatives - Problem Set Week 5

Problem 1

Consider the Vasicek model where the short rate r has dynamics

$$
\begin{aligned}
dr_t &= (b - ar_t)dt + \sigma dW_t, \quad t > 0 \\
r_0 &= r
\end{aligned} \tag{1}
$$

Here, present time is denoted by t and the price of a zero coupon bond with maturity T is denoted $p(t,T)$. Now consider a fixed present time t and denote the short rate at time t by r.

a) Use the fact that the Vasicek model possesses an affine term structure to find expressions for:

   i) Zero coupon bond prices, $p(t,T)$, as a function of T for t fixed.

   ii) Spot rates, $R(t,T)$, as a function of T for t fixed.

   iii) Instantaneous forward rates, $f(t,T)$, as a function of T for t fixed.

b) Write three functions in Python that take as input, the parameters a, b and σ, time to maturity T, and the short rate r at present time $t = 0$ and return p, R and f respectively.

c) Use the functions you have written above to plot the term structures of zero coupon bond prices, the term structure of spot rates and the term structure of instantaneous forward rates for maturities from 0 to 10 years in a Vasicek model with $a = 1, b = 0.04, \sigma = 0.03, r = 0.05$.

d) Find the stationary mean of the short rate. Is the current level of the short rate below or above the long-run mean? Is your conclusion also reflected in the shape of the spot- and forward rate curves?

Solution

a) Recall that if the short rate has dynamics of the form

$$
dr_t = \big(\alpha(t)r_t + \beta(t)\big)dt + \sqrt{\gamma(t)r_t + \delta(t)}dW_u, \quad t > 0 \tag{2}
$$

then the model possess an ATS and ZCB prices are of the form

$$
p(t,T) = e^{A(t,T) - B(t,T)r} \tag{3}
$$

where $A(t,T)$ and $B(t,T)$ satisfies the following system of ODE's

$$
B_t = -1 - \alpha(t)B + \frac{1}{2}\gamma(t)B^2, \qquad B(T,T) = 0 \tag{4}
$$

$$
A_t = \beta(t)B - \frac{1}{2}\delta(t)B^2, \qquad A(T,T) = 0 \tag{5}
$$

In the case of Vasicek, we have $\beta(t) = b$, $\alpha(t) = -a$, $\gamma(t) = 0$ and $\delta(t) = \sigma^2$ and the system of ODE's becomes

$$
B_t = -1 + aB, \qquad B(T,T) = 0 \tag{6}
$$

$$
A_t = bB - \frac{1}{2}\sigma^2 B^2, \qquad A(T,T) = 0 \tag{7}
$$

First we solve (6) by first rewriting it as follows

$$
B_t - aB = -1 \tag{8}
$$

The method we will then use is to multiply by a suitable integrating factor that will allow us to rewrite the LHS of (8) into the t derivative of a product of $B(t,T)$ and the integrating factor. The integrating factor we will use is

$$
e^{-\int_0^t a ds} = e^{-at} \tag{9}
$$

giving us that

$$B_t - aB = -1 \;\;\Rightarrow\;\; B_t e^{-at} - aBe^{-at} = -e^{-at} \;\;\Rightarrow\;\; \frac{d}{dt}\Big(e^{-at}B\Big) = -e^{-at}$$
$$\int_t^T d\Big(e^{-at}B(s,T)\Big) = -\int_t^T e^{-as}ds \;\;\Rightarrow\;\; e^{-aT}B(T,T) - e^{-at}B(t,T) = \frac{1}{a}\Big[e^{-as}\Big]_t^T$$
$$-e^{-at}B(t,T) = \frac{1}{a}\Big[e^{-aT} - e^{-at}\Big] \;\;\Rightarrow\;\; B(t,T) = \frac{1}{a}\Big[1 - e^{-a(T-t)}\Big] \tag{10}$$

We have now solved for $B(t,T)$ and to solve for $A(t,T)$, we note that

$$B = \frac{1}{a}\Big[1 - e^{-a(T-t)}\Big] \;\;\Rightarrow\;\; \frac{d}{dt}B = -e^{-a(T-t)} = aB - 1 \;\;\Rightarrow\;\; \frac{d}{dt}\Big[\frac{1}{a}(B+t)\Big] = B$$
$$\frac{d}{dt}B^2 = 2aB^2 - 2B \;\;\Rightarrow\;\; \frac{d}{dt}\Big[\frac{B^2}{2a} + \frac{B}{a^2} + \frac{t}{a^2}\Big] = B^2 \tag{11}$$

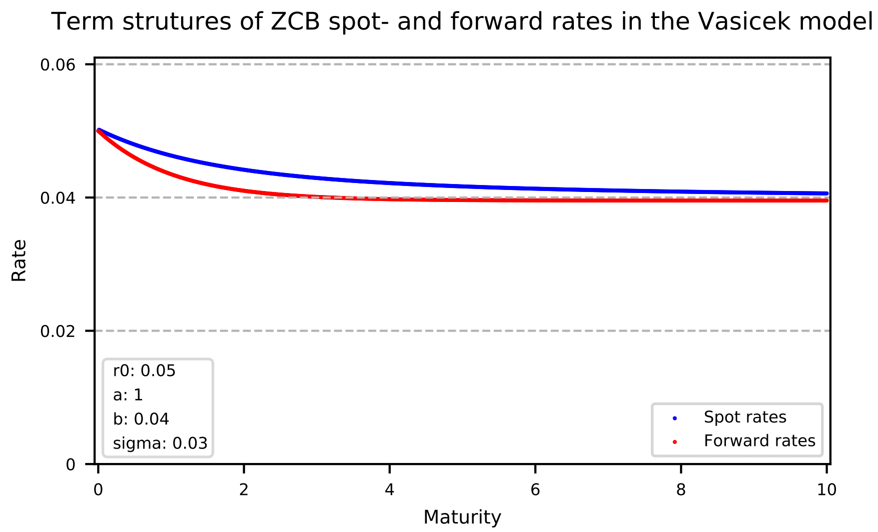
We can now find $A(t,T)$ by separating variables and integrating

$$\frac{d}{dt}A = bB - \frac{1}{2}\sigma^2 B^2 \;\;\Rightarrow\;\; \int_t^T dA(s,T) = \int_t^T \Big(bB(s,T) - \frac{1}{2}\sigma^2 B^2(s,T)\Big)ds \;\;\Rightarrow$$
$$A(T,T) - A(t,T) = \Big[\frac{b}{a}\Big(B(s,T) + s\Big) - \frac{1}{2}\sigma^2\Big(\frac{B^2(s,T)}{2a} + \frac{B(s,T)}{a^2} + \frac{s}{a^2}\Big)\Big]_t^T \;\;\Rightarrow$$
$$A(t,T) = \frac{1}{4a^2}\Big(2\big[\sigma^2 - 2ab\big](T-t) + 2\big[2ab - \sigma^2\big]B(t,T) - a\sigma^2 B^2(t,T)\Big) \tag{12}$$

Spot rates $R(t,T)$ and instantaneous forward rates then become

$$R(t,T) = -\frac{\ln P(t,T)}{T-t} = \frac{-A(t,T) + B(t,T)r_t}{T-t} \tag{13}$$
$$f(t,T) = -\frac{\partial}{\partial T}\ln P(t,T) = A_T(t,T) - B_T(t,T)r_t \tag{14}$$

c) The Term structures of spot- and forward rates for $r_0 = 0.05$, $a = 1$, $b = 0.04$ and $\sigma = 0.03$ look as follows.



d) The stationary mean in the Vasicek model is $\frac{b}{a} = 0.04$ and since the short rate at present time $t = 0$ is 0.05, the short rate is above it's stationary mean which results in a downward-sloping yield curve.

Problem 2

Consider the CIR model where the short rate r has dynamics

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0$$
$$r_0 = r \tag{15}$$

where $a > 0$ and $b > 0$. We will now denote present time by t and proceed in to find explicit formulas for ZCB prices, spot rates and forward rates in the CIR model.

a) In the following, we will compute ZCB prices, spot rates and forward rates in the CIR model by taking a number of steps.

i) Show that ZCB prices in the CIR model are of the form $F^{(T)}(t,r) = A(t,T)e^{-B(t,T)r}$ where $A(t,T)$ and $B(t,T)$ solve the following system of ODE's

$$A_t = abAB, \qquad A(T,T) = 1 \tag{16}$$

$$B_t = -1 + aB + \frac{\sigma^2}{2}B^2, \qquad B(T,T) = 0. \tag{17}$$

ii) Use the substitution $B = -\frac{2}{\sigma^2 V}V(t)$ to transform the ODE for B into the following second order ODE for $V = V(t)$

$$V_{tt} - aV_t - \frac{\sigma^2}{2}V = 0. \tag{18}$$

iii) Use the conjecture that $V(t)$ is of the form $V(t) = e^{yt}$ to show that all solutions for $V(t)$ can be written as

$$V(t) = c_1 e^{\left(\frac{a+\gamma}{2}\right)t} + c_2 e^{\left(\frac{a-\gamma}{2}\right)t}, \quad \gamma = \sqrt{a^2 + 2\sigma^2} \tag{19}$$

where c_1 and c_2 are constants to be found

iv) Use the boundary condition on $B(T)$ to show that

$$B(t,T) = \frac{2e^{\gamma(T-t)} - 2}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)} \tag{20}$$

v) Use the ODE for $A(t,T)$ to show that

$$\ln A(t,T) = -ab\int_t^T B(s,T)ds = \frac{2ab}{\gamma}I, \qquad I = -\gamma\int_t^T \frac{e^{\gamma(T-s)} - 1}{2\gamma + \big(a+\gamma\big)\big(e^{\gamma(T-s)} - 1\big)}ds \tag{21}$$

vi) Use a substitution of the form $u = e^{\gamma(T-s)}$ to put the integral on the form

$$I = \int_{e^{\gamma(T-t)}}^{1} \frac{u-1}{\gamma - a + \big(a+\gamma\big)u}\frac{1}{u}du \tag{22}$$

vii) Show the following rule for partial fractions

$$\frac{a_0 + a_1x}{\big(b_0 + b_1x\big)\big(c_0 + c_1x\big)} = \frac{y}{b_0 + b_1x} + \frac{z}{c_0 + c_1x}, \quad \text{where } y = \frac{a_0b_1 - a_1b_0}{c_0b_1 - c_1b_0} \text{ and } z = \frac{c_0a_1 - c_1a_0}{c_0b_1 - c_1b_0} \tag{23}$$

and use this result to simplify the integral in (22) to

$$I = \int_{e^{\gamma(T-t)}}^{1} \frac{2\gamma}{(\gamma - a)[\gamma - a + \big(a+\gamma\big)u]}du - \int_{e^{\gamma(T-t)}}^{1} \frac{1}{(\gamma - a)u}du \tag{24}$$

viii) Solve the integral in (24) to conclude that

$$A(t,T) = \left(\frac{2\gamma \cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)}\right)^{\frac{2ab}{\sigma^2}} \tag{25}$$

ix) Write down expressions for ZCB prices, spot rates and forward rates in the CIR model.

b) Write three functions in Python that take as input, the parameters a, b and σ, time to maturity T, and the short rate r at present time $t = 0$ and return p, R and f respectively.

c) Use the functions you have written above to plot the term structures of zero coupon bond prices, the term structure of spot rates and the term structure of instantaneous forward rates for maturities from 0 to 10 years in a CIR model with $a = 2, b = 0.05, \sigma = 0.1, r = 0.025$.

d) Find the stationary mean of the short rate. Is the current level of the short rate below or above the long-run mean? Is your conclusion also reflected in the shape of the spot- and forward rate curves?

Solution

a) We work towards computing ZCB prices step by step.

i) We know that ZCB prices $P(t,r;T) = F^{(T)}(t,r)$ in the CIR model must satisfy the term structure equation

$$F_t^{(T)} + a\big(b-r\big)F_r^{(T)} + \frac{1}{2}\sigma^2 r F_{rr}^{(T)} - rF^{(T)} = 0$$
$$F^{(T)}(T,r) = 1 \tag{26}$$

Inserting the functional form $p(t,r;T) = A(t,T)e^{-B(t,T)r}$ of the ZCB price into (26) gives us that

$$\frac{1}{A}PA_t - rPB_t - a\big(b-r\big)BP + \frac{1}{2}r\sigma^2 B^2 P - rP = 0 \tag{27}$$

For the left hand side to equal 0 for all r, both the term independent of r and the term linear in r must be 0. Also, since $p(T,T;T) = 1$, we must have that $A(T,T) = 1$ and $B(T,T) = 0$. This gives us the following system of ODE's to solve

$$A_t = abAB, \qquad A(T,T) = 1 \tag{28}$$
$$B_t = -1 + aB + \frac{\sigma^2}{2}B^2, \qquad B(T,T) = 0 \tag{29}$$

ii) The second of these two equations depends only on $B(t,T)$ and must be solved first. This equation is a Ricatti ODE which can be reduced to a second order linear ODE by defining $B(t,T)$ in terms of a new function $V(t,T)$ as follows

$$B = -\frac{2}{\sigma^2 V}V_t \tag{30}$$

Using this definition we get that

$$B_t = -\frac{2}{\sigma^2}\frac{d}{dt}\left(\frac{V_t}{V}\right) = -\frac{2}{\sigma^2}\left(\frac{V_{tt}}{V} - \frac{V_t^2}{V^2}\right) \tag{31}$$

Inserting (31) into (29) gives us that

$$-\frac{2}{\sigma^2}\left(\frac{V_{tt}}{V} - \frac{V_t^2}{V^2}\right) = -1 - \frac{2a}{\sigma^2 V}V_t + \frac{2}{\sigma^2}\frac{V_t^2}{V^2} \quad \Rightarrow \quad V_{tt} - aV_t - \frac{\sigma^2}{2}V = 0 \tag{32}$$

iii) To solve the second order ODE in (32), we conjecture that the solution is of the form $V(t) = e^{yt}$, where y is some function of the parameters of the model. Inserting the conjecture into (32) gives us that

$$y^2 e^{yt} - aye^{yt} - \frac{\sigma^2}{2}e^{yt} = 0 \quad \Rightarrow \quad y^2 - ay - \frac{\sigma^2}{2} = 0 \quad \Rightarrow \quad y = \frac{a \pm \sqrt{a^2 + 2\sigma^2}}{2} = \frac{a \pm \gamma}{2}. \tag{33}$$

We now have two potential solutions to the ODE in (32) given by

$$V_1(t) = e^{\left(\frac{a+\gamma}{2}\right)t} \quad \text{and} \quad V_2(t) = e^{\left(\frac{a-\gamma}{2}\right)t} \tag{34}$$

However, any linear combination of the two solutions $V_1(t)$ and $V_2(t)$ will also solve (32) and we can write all solutions of (32) as

$$V(t) = c_1 e^{\left(\frac{a+\gamma}{2}\right)t} + c_2 e^{\left(\frac{a-\gamma}{2}\right)t} \tag{35}$$

where c_1 and c_2 are constants to be found.

iv) Inserting (35) into (30) gives us that

$$\begin{aligned} B(t) &= -\frac{1}{\sigma^2}\frac{c_1(a+\gamma)e^{\left(\frac{a+\gamma}{2}\right)t} + c_2(a-\gamma)e^{\left(\frac{a-\gamma}{2}\right)t}}{c_1 e^{\left(\frac{a+\gamma}{2}\right)t} + c_2 e^{\left(\frac{a-\gamma}{2}\right)t}} \\ &= -\frac{1}{\sigma^2}\frac{c_1(a+\gamma)e^{\frac{\gamma}{2}t} + c_2(a-\gamma)e^{-\frac{\gamma}{2}t}}{c_1 e^{\frac{\gamma}{2}t} + c_2 e^{-\frac{\gamma}{2}t}} = -\frac{1}{\sigma^2}\frac{c_3(a+\gamma)e^{\frac{\gamma}{2}t} + (a-\gamma)e^{-\frac{\gamma}{2}t}}{c_3 e^{\frac{\gamma}{2}t} + e^{-\frac{\gamma}{2}t}} \end{aligned} \tag{36}$$

where $c_3 = \frac{c_1}{c_2}$ is just another constant. The solution for $B(t,T)$ must satisfy the boundary condition that $B(t=T)=0$ which will allow us to find c_3

$$B(T)=0 \quad \Rightarrow \quad c_3(a+\gamma)e^{\frac{\gamma}{2}T}+(a-\gamma)e^{\frac{-\gamma}{2}T}=0 \quad \Rightarrow \quad c_3=-\frac{(a-\gamma)}{(a+\gamma)}e^{-\gamma T} \tag{37}$$

Inserting c_3 from (37) into the general solution for B from (36) gives us

$$\begin{aligned} B(t,T) &= \frac{1}{\sigma^2}\frac{(a-\gamma)e^{-\gamma T+\frac{\gamma}{2}t}-(a-\gamma)e^{-\frac{\gamma}{2}t}}{-\frac{(a-\gamma)}{(a+\gamma)}e^{-\gamma T+\frac{\gamma}{2}t}+e^{-\frac{\gamma}{2}t}} = \ldots \quad \Rightarrow \\ B(t,T) &= \frac{2e^{\gamma(T-t)}-2}{2\gamma+(a+\gamma)\big(e^{\gamma(T-t)}-1\big)} \end{aligned} \tag{38}$$

We have now found the solution for $B(t,T)$ in (29).

v) We proceed to find $A(t,T)$ from (28)

$$\begin{aligned} A_t = abAB \quad &\Rightarrow \quad \frac{1}{A}\frac{dA}{dt}=abB \quad \Rightarrow \quad \int_t^T \frac{1}{A(s,T)}dA(s,T)=ab\int_t^T B(s,T)ds \quad \Rightarrow \\ \big[\ln A(s,T)\big]_t^T = ab\int_t^T B(s,T)ds \quad &\Rightarrow \quad \ln A(t,T) = -ab\int_t^T B(s,T)ds = \frac{2ab}{\gamma}I \end{aligned} \tag{39}$$

where I is an integral that we must evaluate

$$I = -\gamma\int_t^T \frac{e^{\gamma(T-s)}-1}{2\gamma+\big(a+\gamma\big)\big(e^{\gamma(T-s)}-1\big)}ds \tag{40}$$

vi) To evaluate this integral, we perform the substitution

$$\begin{aligned} u=e^{\gamma(T-s)} \quad &\Rightarrow \quad du=-\gamma e^{\gamma(T-s)}ds \quad \Rightarrow \quad ds=-\frac{1}{\gamma}e^{-\gamma(T-s)}=-\frac{1}{u\gamma}du \\ s=t \quad &\Rightarrow \quad u=e^{\gamma(T-t)}, \quad s=T \quad \Rightarrow \quad u=1 \end{aligned} \tag{41}$$

and we get that

$$I=\int_{e^{\gamma(T-t)}}^1 \frac{u-1}{2\gamma+\big(a+\gamma\big)(u-1)}\frac{1}{u}du = \int_{e^{\gamma(T-t)}}^1 \frac{u-1}{\gamma-a+\big(a+\gamma\big)u}\frac{1}{u}du \tag{42}$$

vii) In order to evaluate this integral, we will need a trick involving partial fractions.

$$\frac{a_0+a_1x}{\big(b_0+b_1x\big)\big(c_0+c_1x\big)}=\frac{y}{b_0+b_1x}+\frac{z}{c_0+c_1x}, \quad \text{where } y=\frac{a_0b_1-a_1b_0}{c_0b_1-c_1b_0} \text{ and } z=\frac{c_0a_1-c_1a_0}{c_0b_1-c_1b_0} \tag{43}$$

which can be shown by direct computation

viii) Using this little trick gives us that

$$\begin{aligned} I &= \int_{e^{\gamma(T-t)}}^1 \frac{2\gamma}{(\gamma-a)[\gamma-a+\big(a+\gamma\big)u]}du - \int_{e^{\gamma(T-t)}}^1 \frac{1}{(\gamma-a)u}du \\ &= \frac{2\gamma}{\gamma-a}\int_{e^{\gamma(T-t)}}^1 \frac{1}{[\gamma-a+(a+\gamma)u]}du + \frac{1}{a-\gamma}\int_{e^{\gamma(T-t)}}^1 \frac{1}{u}du \\ &= \frac{2\gamma}{(\gamma-a)(a+\gamma)}\Big[\ln\big(\gamma-a+(a+\gamma)u\big)\Big]_{e^{\gamma(T-t)}}^1 + \frac{1}{a-\gamma}\Big[\ln\big(u\big)\Big]_{e^{\gamma(T-t)}}^1 \\ &= \frac{2\gamma}{(\gamma-a)(a+\gamma)}\Big[\ln\big(2\gamma\big)-\ln\big(\gamma-a+(a+\gamma)e^{\gamma(T-t)}\big)+\frac{a+\gamma}{2\gamma}\ln\big(e^{\gamma(T-t)}\big)\Big] \\ &= \frac{2\gamma}{\gamma^2-a^2}\ln\left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{\gamma-a+(a+\gamma)e^{\gamma(T-t)}}\right) = \frac{\gamma}{\sigma^2}\ln\left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{\gamma-a+(a+\gamma)e^{\gamma(T-t)}}\right) \end{aligned} \tag{44}$$

Substituting I from (44) back into (39) gives us that

$$\begin{aligned} \ln A(t,T) &= \frac{2ab}{\gamma}I = \frac{2ab}{\gamma}\frac{\gamma}{\sigma^2}\ln\left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{\gamma-a+(a+\gamma)e^{\gamma(T-t)}}\right) \quad \Rightarrow \\ A(t,T) &= \left(\frac{2\gamma\cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{2\gamma+(a+\gamma)\big(e^{\gamma(T-t)}-1\big)}\right)^{\frac{2ab}{\sigma^2}} \end{aligned} \tag{45}$$

ix) Zero coupon bond prices in the CIR model are in other words given by

$$P(t,T) = \left(\frac{2\gamma \cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)} \right)^{\frac{2ab}{\sigma^2}} \cdot \exp\left(\frac{-2\big(e^{\gamma(T-t)} - 1\big)}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)} r_t \right). \tag{46}$$

Spot rates and forward rates can most easily be expressed by setting be found by setting $A = \left(\frac{N}{D}\right)^c$ and $B = \frac{M}{D}$ so that $B_T = \frac{M_T D - M D_T}{D^2}$. We then have that spot rates $R(t,T)$ are given by

$$R(t,T) = -\frac{\ln P(t,T)}{T-t} = -\ln A - B r_t = -c\big(\ln N - \ln D\big) - \frac{M}{D} r_t. \tag{47}$$

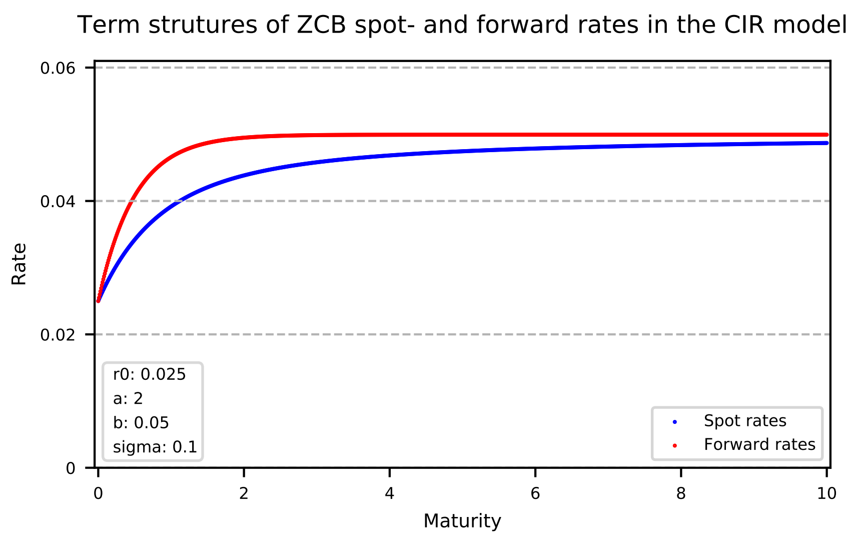
Forward rates are given by

$$f(t,T) = -\frac{\partial log\, P(t,T)}{\partial T} = -\frac{\partial}{\partial T}\Big(\log\, A + r B_T\Big) = -c\Big(\frac{N_T}{N} - \frac{D_T}{D}\Big) + \frac{M_T D - M D_T}{D^2} r_t \tag{48}$$

where

$$\begin{aligned} c &= \frac{2ab}{\sigma^2}, \\ \gamma &= \sqrt{a^2 + 2\sigma^2}, \\ N &= 2\gamma \cdot e^{\frac{(a+\gamma)(T-t)}{2}}, \\ N_T &= \gamma(a+\gamma)e^{\frac{(T-t)(a+\gamma)}{2}}, \\ D &= 2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big), \\ D_T &= \gamma(\gamma + a)e^{\gamma(T-t)}, \\ M &= 2e^{\gamma(T-t)} - 2, \\ M_T &= 2\gamma e^{\gamma(T-t)}. \end{aligned} \tag{49}$$

c) The Term structures of spot- and forward rates for $r_0 = 0.025$, $a = 2$, $b = 0.05$ and $\sigma = 0.1$ look as follows.



d) The long-run stationary mean in the CIR model is simply $b = 0.05$, and we see that the short rate at present time $t = 0$ is only 0.025 and well below the stationary mean resulting in an upward-sloping term structure of zero coupon spot rates.

Problem 3

Consider the Vasicek model where the short rate has dynamics

$$\begin{aligned} dr_t &= (b - ar_t)dt + \sigma dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{50}$$

In this problem, we will first generate zero coupon bond prices using the Vasicek model with known parameters and then seek to recover these parameters by fitting a Vasicek model to the zero coupon bond prices we have generated. In order to do so, we will use the package 'scipy.optimize' in Python. The documentation for this package can be found here and it is advised that you make good use of it.

a) Generate ZCB prices for times to maturity $T = [0, 0.1, 0.2, ..., 9.8, 9.9, 10]$ using an initial value of the short rate of $r = 0.025$ and parameters $a = 2$, $b = 0.1$, $\sigma = 0.02$. Denote these 'empirical' prices by $p^*(t, T)$.

b) Use the function 'minimize' and the method 'nelder-mead' to fit a Vasicek model to the prices $p^*(0, T)$ that you just generated. Do so by minimizing the sum of squared errors as a function of r, a, b, σ and setting the starting values of the parameters in the algorithm to $r_0, a_0, b_0, \sigma_0 = 0.03, 1.8, 0.12, 0.03$. Plot the fitted values $\hat{p}(0, T)$ and the empirical values $p^*(0, T)$. Are the fitted and empirical values close? Also plot the residuals of your fit and find the mean squared error.

c) Try to change the starting values of the parameters and perform the fit again. Which of the four parameters are best recovered by your fit and what does that tell you about the objective function as a function of r, a, b and σ?

d) Now redo the fit but impose that $b = 0.12$. Do this by changing the objective function in your fit so that it only optimizes over r, a and σ. Reproduce the plots from above and investigate the fit you now get.

In the previous, you have performed an unconstrained optimization in the sense that none of the parameters have been restricted to take values in a certain range. Next, we will investigate how to impose, bounds and constraints on the optimization and we will once again optimize over all four parameters r, a, b, σ. You will need to use that method 'trust-constr' also described in the documentation.

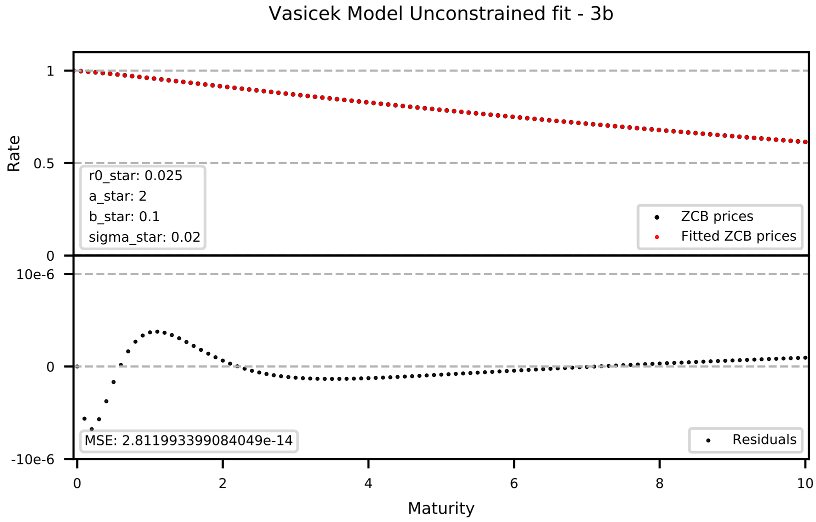
e) Impose the bounds that $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.1$ and perform the fit. Check once again that you recover the true parameters.

f) Now impose the restrictions that $0 \leq r \leq 0.1$, $0 \leq a \leq 1.8$, $0 \leq b \leq 0.08$ and $0 \leq \sigma \leq 0.1$ and perform the fit again. The true parameters are now outside the parameter space of the fit. Where do your fitted parameters now lie and was that to be expected?

g) Now, set the bounds back to the initial values $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.1$ but impose the non-linear constraint that $2ab \geq \sigma^2$ also using the 'trust-constr' method. Again, you will have to consult the documentation to find out, how to impose this non-linear constraint.

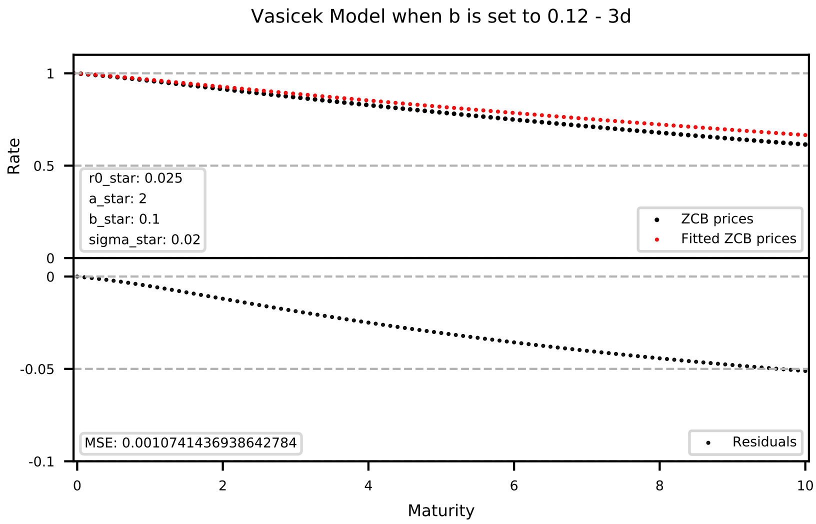
Solution

b) Performing an unconstrained fit of ZCB prices in the Vasicek model results in a near perfect fit illustrated below.

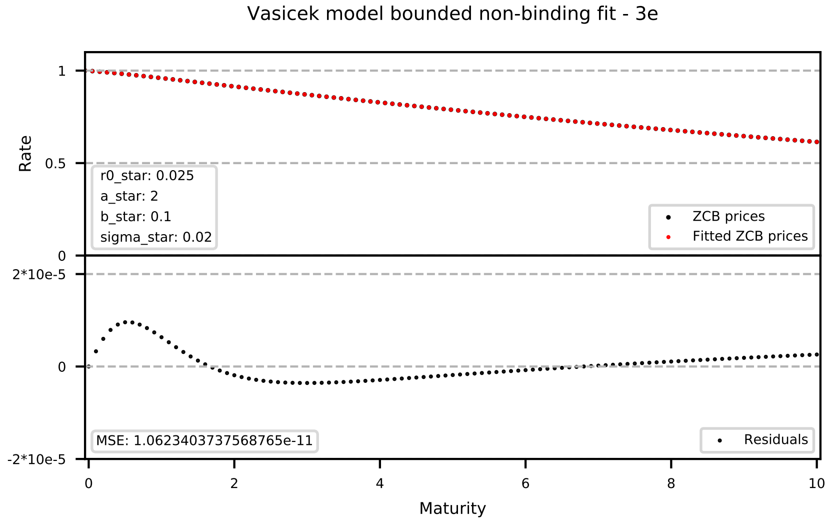


c) Changing the initial values of the parameters, we can see that the algorithm tends to recover the true parameters of the model as long as the initial parameters are not too far from the true values and in particular, that the initial values for a and b are of the right sign (positive). As for σ, it is quite clear that the algorithm is not able to recover the true value of σ but the model fits ZCB prices nonetheless. This reveals that ZCB prices in the Vasicek model are not affected very much by σ and when fitting the model to market data in practice, it is probably best to estimate σ from other sources.

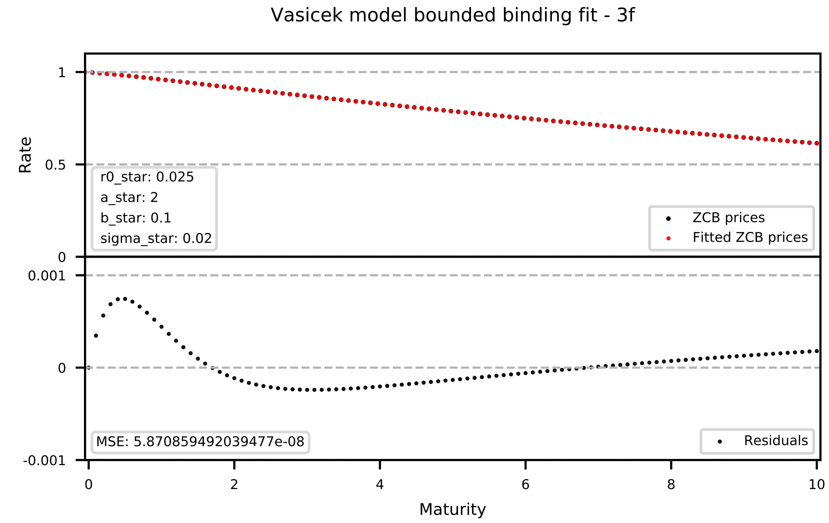
d) Forcing b to take a value which deviates from the true value of b results, not surprisingly, in a poor fit of ZCB prices as illustrated in the plot below.



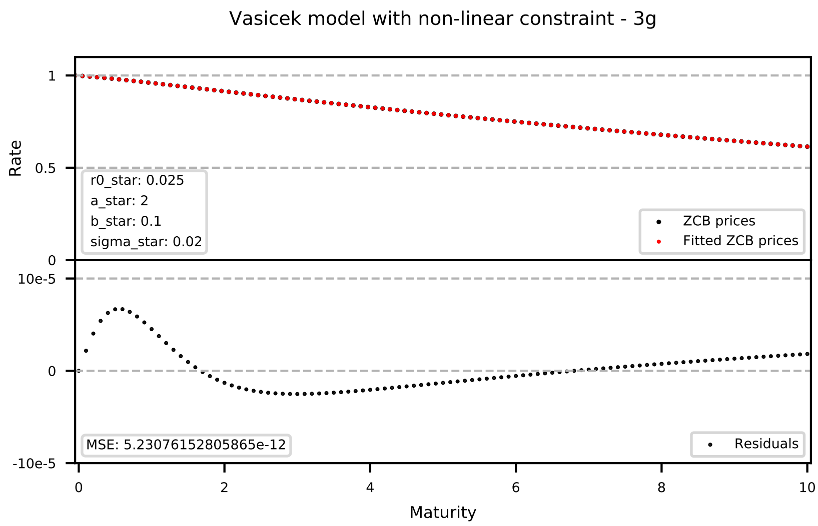
e) When doing a bounded fit but setting the bounds such that the true parameter values are inside the bounds, the algorithm does indeed recover the true parameter values and results in a near perfect fit as illustrated below.



f) Performing a bounded fit where the true parameter values are outside the bounds results in a poorer fit but the true parameter values are not far outside the bounds and the fit is, in this case at least, still pretty decent.



g) Imposing the suggested non-linear constraint and performing the fit once again results in a perfect fit as illustrated in the plot below which is of course due to the fact that the true parameters are consistent with this non-linear constraint



Problem 4

Consider the CIR model where the short rate has dynamics

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{51}$$

where $a > 0$, $b > 0$ and $2ab \geq \sigma^2$. Present time is denoted by t, the short rate at time t is denoted by r and the price of a zero coupon bond with maturity T is denoted $p(t, T)$. In this problem, we will first generate zero coupon bond prices using the CIR model with known parameters and then seek to recover these parameters by fitting a CIR model to the zero coupon bond prices we generated.

a) Generate ZCB prices for times to maturity $\tau = T - t = [0, 0.1, 0.2, ..., 9.8, 9.9, 10]$ using an initial value of the short rate of $r = 0.045$ and parameters $a = 1.5$, $b = 0.06$, $\sigma = 0.08$. Denote these 'empirical' prices by $p^*(t, T)$.

b) Use the function 'minimize' and the method 'nelder-mead' to fit a CIR model to the prices $p^*(t, T)$ that you just generated. Do so by minimizing the sum of squared errors as a function of r, a, b, σ and setting the starting values of the parameters in the algorithm to $r_0, a_0, b_0, \sigma_0 = 0.05, 1.8, 0.08, 0.08$. Plot the fitted values $\hat{p}(t, T)$ and the empirical values $p^*(t, T)$. Are the fitted and empirical values close? Also plot the residuals of your fit and find the mean squared error.

c) Try to change the starting values of the parameters and perform the fit again. Which of the four parameters are best recovered by your fit and what does that tell you about the objective function as a function of r, a, b and σ?

d) Now redo the fit but impose that $b = 0.08$. Do this by changing the objective function in your fit so that it only optimizes over r, a and σ. Reproduce the plots from above and investigate the fit you now get.

In the previous, you have performed an unconstrained optimization in the sense that none of the parameters have been restricted to take values in a certain range. Next, we will investigate how to impose, bounds and constraints on the optimization and we will once again optimize over all four parameters r, a, b and σ. You will need to use that method 'trust-constr' also described in the documentation.

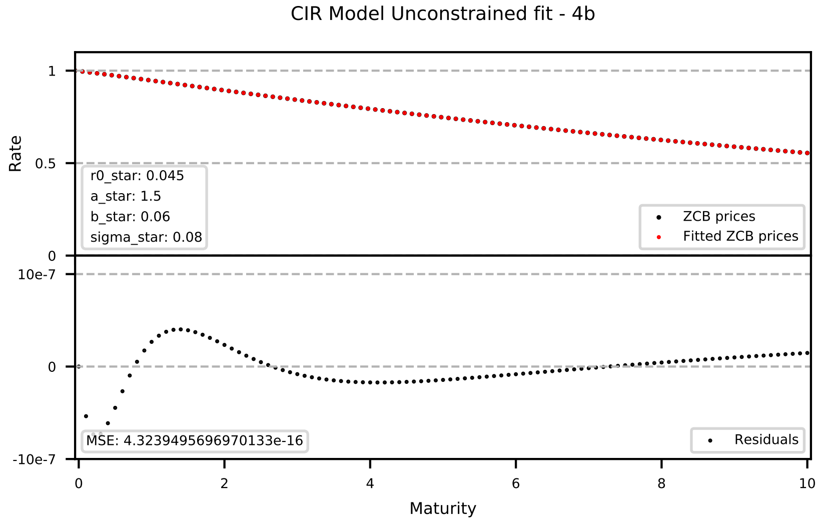
e) Impose the bounds that $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.2$ and perform the fit. Check once again that you recover the true parameters.

f) Now impose the restrictions that $0 \leq r \leq 0.1$, $0 \leq a \leq 1$, $0 \leq b \leq 0.08$ and $0 \leq \sigma \leq 0.1$ and perform the fit again. The true parameters are now outside the parameter space of the fit. Where do your fitted parameters now lie and was that to be expected?

g) Now, set the bounds back to the initial values $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.2$ but impose the non-linear constraint that $2ab \geq \sigma^2$ also using the 'trust-constr' method.

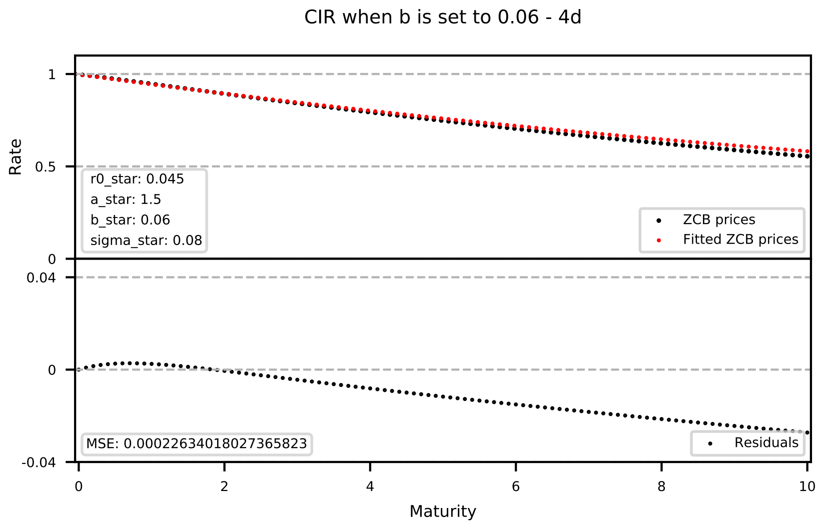
Solution

b) The unconstrained fit of the CIR model results in a near perfect fit as illustrated below.

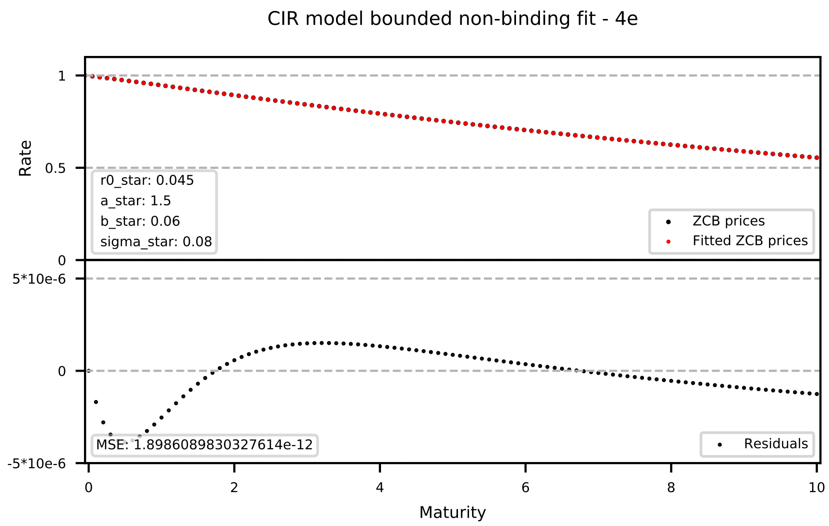


c) In the CIR model as in the Vasicek model, a near perfect fit is obtained as long as the initial parameter values assume reasonable values and in particular met the criteria of the parameters in the model. However, it is also apparent in the CIR model that σ is not well identified reflecting, also in this case, that ZCB prtices are not affected much by σ and that this parameter can be chosen somewhat freely and still, we would be able to achieve a good fit.

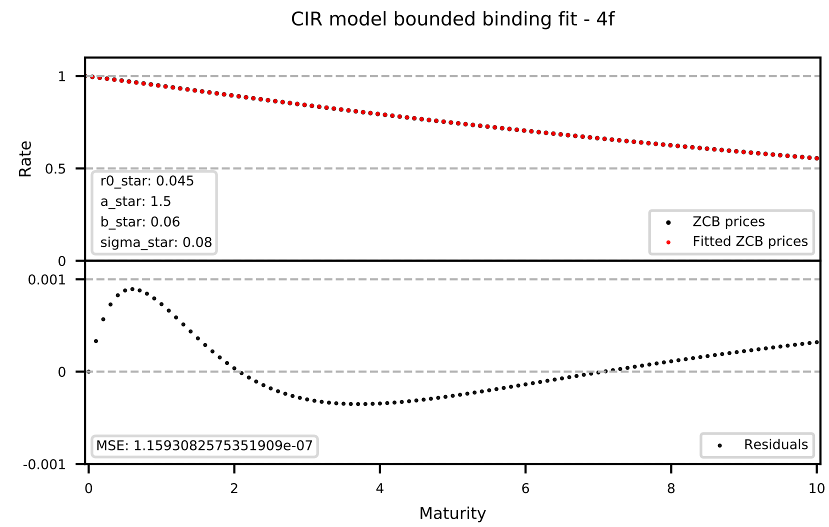
d) Forcing b to taking a values that is not equal to the true value parameter value results, also in this case, results in a poor fit as shown below.



e) Imposing bounds such that the true values of the parameters sit inside the bounds, we recover the true values of the parameters and get a perfect fit once again.



f) Imposing bounds such that the true parameter values are outside the bounds reduces the quality of the fit, but also in the case of these bounds the algorithm is able to compensate and produce a fairly solid fit of ZCB prices.



g) Imposing the Feller condition on the parameters of this version of the CIR model, we are able to get a perfect fit because the Feller condition is met for the parameters used to generate the ZCB prices we are fitting.

CIR model with non-linear constraint - 4g