LIBOR Market Models

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LIBOR market models

In the previous sections, we have modeled the market by imposing dynamics on the short rate or alternatively on the term structure of forward rates.

Within these modeling frameworks, we were able to compute prices of zero coupon bonds and European options which could in turn could be used to price interest rate caps and floors.

In short rate models, the dynamics of the entire yield curve is governed by the very front end and to construct realistic models that also allow for realistic behavior of the long end, a very complicated model is needed.

Also, the otherwise more tractable short rate models suffer from the fact that a perfect fit of the initial term structure can not be guaranteed.

Lastly, short rate and forward rate models aim at modeling the yield curve and are typically not designed with prices of derivatives in mind.

LIBOR market models

In this section, we will discuss the methodology commonly used in financial markets and use that as our guide when constructing a so called LIBOR market model.

This approach evolves around the Black-76 formula for the price of a caplet since it is common practice to quote prices of caplets and floorlets in terms of the Black implied volatility.

The Black formula for the price of a caplet has a very familiar form that we will recognize from our treatment of European option prices on an underlying zero coupon bond.

Definitions and market practice

We will consider a set of increasing maturities $T_1, ..., T_N$ and define α_i

$$\alpha_i = T_i - T_{i-1}, \qquad i = 1, ..., N \tag{1}$$

The time between two maturities α_i is typically called the tenor and is thus also the time between two consecutive LIBOR fixings.

Definition (LIBOR forward rates)

We will denote the time t price $p(t, T_i)$ of a ZCB maturing at time T_i by $p_i(t)$ and we will denote by $L_i(t)$ the forward rate contracted at time t for the LIBOR fixing announced at time T_{i-1} and paid at time T_i.

$$L_i(t) = \frac{1}{T_i - T_{i-1}} \frac{p_{i-1}(t) - p_i(t)}{p_i(t)} = \frac{1}{\alpha_i} \frac{p_{i-1}(t) - p_i(t)}{p_i(t)}, \qquad i = 1, ..., N$$

Definitions and market practice

Also, we recall that an interest rate cap with cap rate R and resettlement days $T_0, ..., T_N$ consists of a series of caplets each which gives the holder of the caplet an amount χ_i at time T_i given by

$$\chi_i = \alpha_i \cdot \max\left[L_i(T_{i-1}) - R, 0\right] = \alpha_i \cdot \left[L_i(T_{i-1}) - R\right]_+ \tag{2}$$

The forward rate $L(T_i)$ is of course at time T_{i-1} equal to the spot rate or fixing announced at time T_{i-1}. The amount χ_i is thus known at time T_{i-1} but not paid until time T_i.

A caplet is essentially a call option on the underlying LIBOR spot rate.

However, also remember that owning a caplet with maturity T_i is equivalent to owning $1 + \alpha_i R$ put options with a strike of $\frac{1}{1+\alpha_i R}$ and the zero coupon bond $p_i(T_{i-1})$ as the underlying asset.

Black's formula for the price of a caplet

Definition (Black's formula)

The Black-76 formula for the caplet with payoff χ_i at time T_i where

$$\chi_i = \alpha_i \cdot \max\left[L(T_{i-1}, T_i) - R, 0\right]$$

states that the price of this caplet can be computed as

$$\mathbf{Capl_i^B}(t) = \alpha_i \cdot p_i(t)\left[L_i(t)\Phi(d_1) - R\Phi(d_2)\right]$$

where $\Phi(\cdot)$ is the standard normal distribution function and

$$d_1 = \frac{1}{\sigma_i\sqrt{T_{i-1} - t}}\left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\sigma_i^2(T_{i-1} - t)\right], \qquad d_2 = d_1 - \sigma_i\sqrt{T_{i-1} - t}$$

The constant σ_i is known as the Black volatility for caplet i and to make this dependence clear, we often write the caplet price as $\mathbf{Capl_i^B}(t; \sigma_i)$.

Black's formula for the price of a caplet

Though caplet prices are available in monetary units (USD, EUR, etc), market participants quote caplet prices in terms of Black implied volatility.

There are two types of Black implied volatilities, flat implied volatilities and spot implied volatilities(also known as forward implied volatilities).

To distinguish between the two, we remember that a cap is simply the sum of the individual caplets.

Consider a set of caplet resettlement dates $T_0, T_1, ..., T_{N-1}$ and denote present time by $t < T_0$. For each $i = 1, ..., N$, there is a cap with maturity date T_i and market price $\mathbf{Cap_i^m}(t)$ where

$$\mathbf{Cap_i^m}(t) = \mathbf{Cap_{i-1}^m}(t) + \mathbf{Capl_i^m}(t), \qquad i = 1, ..., N \tag{3}$$

and the convention $\mathbf{Cap_0^m}(t) = 0$.

Black implied volatility

Definition (Black implied volatility)

Given market prices of cap and caplet prices for $T_i \in \{T_0, T_1, ..., T_N\}$*, the Black implied volatilities are defined as follows.*

- *The spot or forward Black implied volatilities* $\bar{\sigma}_1, ..., \bar{\sigma}_N$ *are defined as the solutions to the equations*

$$\mathbf{Capl_i^m}(t) = \mathbf{Capl_i^B}(t; \bar{\sigma}_i), \qquad i = 1, ..., N$$

- *The flat Black implied volatilities* $\bar{\sigma}_1, ..., \bar{\sigma}_N$ *are defined as the solutions to the equations*

$$\mathbf{Cap_i^m}(t) = \mathbf{Cap_i^B}(t; \bar{\sigma}_i) = \sum_{k=1}^{i} \mathbf{Capl_k^B}(t; \bar{\sigma}_i), \qquad i = 1, ..., N$$

The collection $\bar{\sigma}_1, ..., \bar{\sigma}_N$ *is called an implied volatility term structure.*

Black implied volatility

Note that the same notation is used for flat and spot volatilities. Which one is used will be clear from the context.

We see that the flat implied volatility $\bar{\sigma}_i$ arises if you use the same volatility in each of the caplets comprised in the cap with maturity T_i.

The spot implied volatility $\bar{\sigma}_i$ is just the caplet for maturity T_i.

The relation between flat and spot implied volatilities is similar to the relation between spot and forward interest rates and hence, the flat implied volatility is an average of the spot and forward implied volatilities.

Black's formula can not be inverted analytically to give us an expression for the Black implied volatility so in practice, the Black implied volatility must be found numerically. This is however not hard since the price of a caplet is strictly increasing in σ.

LIBOR market models

We have now discussed how the market uses the Black-76 formula and the volatility implied by this formula when quoting prices of caps and caplets.

We will now see if we can develop a theoretical model for which prices of caplets are given by a formula of the Black-76 type and thus reconcile market practice with a well-defined theoretical model.

The price $\mathbf{Capl_i}(t)$ of the caplet with maturity T_i can be found as an expectation under the risk-neutral measure

$$\mathbf{Capl_i}(t) = \alpha_i \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^{T_i} r_s ds} \cdot \left(L_i(T_{i-1}) - R\right)_+ \Big| \mathcal{F}_t \right] \tag{4}$$

However, it is as always easier to use the T_i forward measure denoted $\mathbb{Q}^i$

$$\mathbf{Capl_i}(t) = \alpha_i p_i(t) \mathrm{E}^{T_i}\left[\left(L_i(T_{i-1}) - R\right)_+ \Big| \mathcal{F}_t \right] \tag{5}$$

Lemma

For every $i = 1, ..., N$ the stochastic LIBOR rate with dynamics $L_i(t)$ is a martingale under it's own measure, forward measure $\mathbb{Q}^{T_i}$, on the interval $\left[0, T_{i-1}\right]$.

This result will prove very important in our construction of the LIBOR market model.

Proof:

Once again, we will use that the LIBOR fixing can be replicated and

$$\alpha_i \cdot L_i(t) = \frac{p_{i-1}(t)}{p_i(t)} - 1 \tag{6}$$

To prove that $L_i(t)$ is a martingale under $\mathbb{Q}^i$, it suffices to prove that $\frac{p_{i-1}(t)}{p_i(t)}$ is a martingale. However, under $\mathbb{Q}^i$ it is p_i that serves as the numeraire, and hence the ratio $\frac{p_{i-1}(t)}{p_i(t)}$ is by construction a martingale under $\mathbb{Q}^i$.

$\square$

LIBOR market models

Now, for caplets on an underlying LIBOR rate to be priced by the Black-76 formula in our model, we need to impose on the model that the dynamics of LIBOR rates are lognormal.

In particular, we will define the LIBOR rates such that for each i, $L_i(t)$ will be lognormal under the measure $\mathbb{Q}^i$.

Once we have prescribed the dynamics of each of the LIBOR rates under their own measure, we will need to reconcile the dynamics of *all* the LIBOR rates under one single measure.

Finally, we will discuss how a risk-free asset in the spirit of the bank account in short rate models can be defined in the LIBOR market model.

LIBOR market models

In the construction of the LIBOR market model, assume at $t = 0$ we have

- a set of caplets and caps with resettlement dates $T_0, \dots, T_N$ and resettlement rate R,
- an arbitrage free bond market including ZCB's with maturities $T_0, \dots, T_N$,
- a k-dimensional $\mathbb{Q}^N$ Brownian motion $\mathbf{W}_t^N$,
- for each $i = 1, \dots, N$ a deterministic function of time $\sigma_i(t)$,
- an initial non-negative forward rate term structure $L_1(0), \dots, L_N(0)$,
- for each $i = 1, \dots, N$, we define $\mathbf{W}_t^i$ as the k-dimensional $\mathbb{Q}^i$ Brownian motion generated by $\mathbf{W}_t^N$ under the change of measure (Girsanov transformation) from $\mathbb{Q}^N$ to $\mathbb{Q}^i$.

LIBOR market models

Definition (The LIBOR market model)

If the LIBOR forward rates have the dynamics

$$dL_i(t) = L_i(t)\boldsymbol{\sigma}_i'(t)d\mathbf{W}_t^i, \quad i = 1, ..., N \text{ and } 0 < t \leq T_{i-1}$$

where W_t^i is a k-dimensional Brownian motion under the measure $\mathbb{Q}^i$, the maturity T_i zero coupon bond serves as the numeraire and the volatility coefficient $\boldsymbol{\sigma}_i(t)$ is deterministic. Then we have a discrete tenor LIBOR market model with volatilities $\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_N$.

LIBOR market models

Before we discuss the existence of a LIBOR market model, we will make sure that the LIBOR market model we have just proposed results in a model where caplets can be priced using the Black-76 formula.

Assuming LIBOR rates have dynamics of the form given by the definition, how to price caplets and thus also caps can easily be determined.

According to the definition, LIBOR rates follow a GBM and by applying Ito to the natural log of LIBOR rates and integrating, we can find the solution for $L_i(U)$ for $U \leq T_{i-1}$ as

$$L_i(U) = L_i(t) \cdot \exp\left\{ \int_t^U \boldsymbol{\sigma}_i'(s) d\mathbf{W}_s^i - \frac{1}{2} \int_t^U ||\boldsymbol{\sigma}_i(s)||^2 ds \right\} \tag{7}$$

LIBOR market models

Since we have assumed that $\boldsymbol{\sigma}_i(t)$ is deterministic, it follows that $L_i(U)$ for $U \leq T_{i-1}$ follows a lognormal distribution.

We can under words write that

$$\ln L_i(U)\big|\mathcal{F}_t \sim N\bigg(-\frac{1}{2}\int_t^U ||\boldsymbol{\sigma}_i(s)||^2 ds, \int_t^U ||\boldsymbol{\sigma}_i(s)||^2 ds\bigg) \tag{8}$$

Using this result, we can compute the expectation in the formula for the price of a caplet in the same way as we did for the regular Black-Scholes.

We could also simply note that the situation corresponds to that of the regular Black-Scholes model with the LIBOR rate serving as the underlying, a zero interest rate and $\Sigma_i^2(t, U) = \int_t^U ||\boldsymbol{\sigma}_i(s)||^2 ds$ in place of $\sigma^2(U-t)$.

Proposition

In the LIBOR market model, caplet prices are given by

$$\mathbf{Capl_i^B}(t) = \alpha_i \cdot p_i(t)\big[L_i(t)\Phi(d_1) - R\Phi(d_2)\big]$$

where $\Phi(\cdot)$ *is the standard normal distribution function and*

$$\begin{aligned} d_1 &= \frac{1}{\Sigma_i(t, T_{i-1})}\left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\Sigma_i^2(t, T_{i-1})\right] \\ d_2 &= \frac{1}{\Sigma_i(t, T_{i-1})}\left[\ln\left(\frac{L_i(t)}{R}\right) - \frac{1}{2}\Sigma_i^2(t, T_{i-1})\right] = d_1 - \Sigma_i^2(t, T_{i-1}) \\ \Sigma_i^2(t, T_{i-1}) &= \int_t^{T_{i-1}} ||\boldsymbol{\sigma}_i(s)||^2 ds \end{aligned}$$

We thus conclude that caplet prices can be computed using the Black-76 formula in the LIBOR market model.

LIBOR market models

The LIBOR market model we have constructed is driven by a k-dimensional Brownian motion but it is possible to convert this model into a model where each LIBOR rate is driven by it's own scalar Brownian motion and

$$dL_i(t) = L_i(t)\sigma_i(t)dW_t^i, \qquad i = 1, ..., N \tag{9}$$

The formula for caplet prices still holds with $\sigma_i(s)$ in place of $||\boldsymbol{\sigma}_i(s)||$.

The various scalar Brownian motions driving the LIBOR rates are also allowed to be correlated.

Introducing correlations between the Brownian motions driving the individual LIBOR rates will not affect caplet prices but would affect the pricing of more complicated derivatives.

LIBOR market models

We will now investigate if there always exists a LIBOR market model irrespective of the specifications of the volatilities $\boldsymbol{\sigma}_1(t), .., \boldsymbol{\sigma}_N(t)$.

The approach we will take involves reconciling the Brownian motions driving LIBOR rates under one common measure.

The measure under which *all* LIBOR rates will be specified is the terminal measure $\mathbb{Q}^N$. The program we will pursue falls in two stages

- Specify all LIBOR rates under $\mathbb{Q}^N$ to have dynamics of the form

$$dL_i(t) = L_i(t)\mu_i(t)dt + L_i(t)\boldsymbol{\sigma}_i'(t)d\mathbf{W}_t^N, \quad i = 1, ..., N \tag{10}$$

  where $\mu_i(t)$ is a deterministic function.

- Show that for some suitable choice of $\mu_1, ..., \mu_N$, the $\mathbb{Q}^N$ dynamics in (10) will imply $\mathbb{Q}^i$ dynamics of the form given in the definition of the LIBOR market model.

LIBOR market models

To carry out this program, we need to determine how $\mathbf{W}_t^N$ is transformed into $\mathbf{W}_t^i$ as we change measure from $\mathbb{Q}^N$ to $\mathbb{Q}^i$. We will do this using induction and begin by studying the change of measure (Girsanov transformation) from $\mathbb{Q}^i$ to $\mathbb{Q}^{i-1}$.

In doing so, we will denote the likelihood process not by L since L is used for the LIBOR rate but instead by η.

We will proceed under the assumption that the LIBOR rates $L_i(t)$ have the dynamics given in (10) under $\mathbb{Q}^N$ and are martingales under $\mathbb{Q}^i$ according to the definition of the LIBOR market model.

LIBOR market models

The Radon-Nikodym derivative for the transition from $\mathbb{Q}^i$ to $\mathbb{Q}^{i-1}$ is then

$$\eta_i^{i-1}(t) = \frac{p_i(0)}{p_{i-1}(0)} \frac{p_{i-1}(t)}{p_i(t)} = a_i(1 + \alpha_i L_i(t)) \tag{11}$$

where $a_i = \frac{p_i(0)}{p_{i-1}(0)}$. We then compute the dynamics of $\eta_i^{i-1}(t)$ using (11) and the dynamics of $L_i(t)$ under $\mathbb{Q}^i$

$$\begin{aligned}
d\eta_i^{i-1}(t) &= a_i \alpha_i L_i(t) \boldsymbol{\sigma}_i'(t) d\mathbf{W}_t^i = a_i \alpha_i \frac{1}{\alpha_i}\left(\frac{p_{i-1}(t)}{p_i(t)} - 1\right) \boldsymbol{\sigma}_i'(t) d\mathbf{W}_t^i \\
&= \eta_i^{i-1}(t) \frac{a_i}{\eta_i^{i-1}(t)} \left(\frac{p_{i-1}(t)}{p_i(t)} - 1\right) \boldsymbol{\sigma}_i'(t) d\mathbf{W}_t^i \\
&= \eta_i^{i-1}(t) \left(1 - \frac{1}{1 + \alpha_i L_i(t)}\right) \boldsymbol{\sigma}_i'(t) d\mathbf{W}_t^i \\
&= \eta_i^{i-1}(t) \frac{\alpha_i L_i(t)}{1 + \alpha_i L_i(t)} \boldsymbol{\sigma}_i'(t) d\mathbf{W}_t^i
\end{aligned} \tag{12}$$

The Girsanov Kernel corresponding to $\eta_i^{i-1}(t)$ is in other words

$$\frac{\alpha_i L_i(t)}{1+\alpha_i L_i(t)}\boldsymbol{\sigma}_i(t) \tag{13}$$

and following the steps of a Girsanov transformation, we have that

$$d\mathbf{W}_t^i = \frac{\alpha_i L_i(t)}{1+\alpha_i L_i(t)}\boldsymbol{\sigma}_i(t)dt + d\mathbf{W}_t^{i-1} \tag{14}$$

Using this relation inductively, we get that

$$d\mathbf{W}_t^i = \sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}\boldsymbol{\sigma}_k(t)dt + d\mathbf{W}_t^{N} \tag{15}$$

Now, all that remains is to plug $d\mathbf{W}_t^i$ into the $\mathbb{Q}^i$ dynamics of $L_i(t)$ as given by the definition of the LIBOR market model.

Proposition (Existence of the LIBOR market model)

Consider a given volatility structure $\boldsymbol{\sigma}_0, \ldots, \boldsymbol{\sigma}_N$ *where each* $\boldsymbol{\sigma}_i$ *is assumed to be bounded, a probability measure* $\mathbb{Q}^N$ *and a standard* $\mathbb{Q}^N$ *Brownian motion* $\mathbf{W}_t^N$. *Define the processes* $L_i(t)$ *by*

$$dL_i(t) = -L_i(t)\left(\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)} \boldsymbol{\sigma}_k'(t)\boldsymbol{\sigma}_i(t)\right)dt + L_i(t)\boldsymbol{\sigma}_i(t)d\mathbf{W}_t^N$$

for $i = 1, \ldots, N$. *The* $\mathbb{Q}^i$ *dynamics of* $L_i(t)$ *are as given below and hence, there exists a LIBOR market model with the given volatility structure.*

$$dL_i(t) = L_i(t)\boldsymbol{\sigma}_i'(t)d\mathbf{W}_t^i$$

LIBOR market models

It may be more convenient to work with a LIBOR market model where the Brownian motion is a scalar process

$$dL_i(t) = L_i(t)\sigma_i(t)dW_t^i \tag{16}$$

for $i = 1, ..., N$ and the driving Brownian motions are correlated $dW_t^i dW_t^j = \rho_{ij}$. This can easily be achieved in which case the $\mathbb{Q}^N$ dynamics of $L_i(t)$ become

$$dL_i(t) = -L_i(t)\left(\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}\sigma_k(t)\sigma_i(t)\rho_{ik}\right)dt + L_i(t)\sigma_i(t)dW_t^N$$

Pricing exotic derivatives in a LIBOR market model

When financial institutions need to price some exotic interest derivative such as a Bermudan swaption, they often do so by simulating the LIBOR rates using a LIBOR market model calibrated to current market prices.

In doing so, the financial institution would perform the following steps

- Observe Caplet market prices $\mathbf{Capl}_{\mathbf{i}}^{\mathbf{m}}(t)$ at present time t.
- Convert caplet prices into Black implied volatilities $\sigma_1, \ldots, \sigma_N$.
- Calibrate the deterministic volatilities $\boldsymbol{\sigma}_i(u)$, $u > t$ from the dynamics of LIBOR rates.
- Use the calibrated diffusion coefficients to simulate LIBOR rates.
- For each simulation, compute the value of the derivative and then average over the simulations to find the value of the exotic derivative.

Pricing exotic derivatives in a LIBOR market model

Now, let us discuss some of these steps in detail.

Assume $t = 0$, that we have resettlement dates $T_0, ..., T_N$ and have computed an empirical term structure of the flat Black implied volatilities $\bar{\sigma}_1, ..., \bar{\sigma}_N$.

Assume that we are working under a scalar LIBOR market model.

In order to calibrate a LIBOR market model to the observed term structure of flat Black implied volatilities, $\bar{\sigma}_i$, we have to choose the deterministic LIBOR volatilities $\sigma_1(t, \cdot), ..., \sigma_N(t, \cdot)$ such that

$$\bar{\sigma}_i^2 = \frac{1}{T_{i-1}} \int_0^{T_i - 1} \sigma_i^2(s) ds \tag{17}$$

Pricing exotic derivatives in a LIBOR market model

This system we have constructed is generally not identified since we have N functions of time $\sigma_i(t)$ that each can be any bounded function but only N restrictions of the form in (17) to pin these functions down.

So, further restrictions must be imposed on the $\sigma_i(t)$ functions and here is a list of common choicesof such restrictions.

1) For each $i = 1, ..., N$, assume that the volatility is constant in time

$$\sigma_i(t) = \sigma_i \tag{18}$$

2) For each $i = 1, ..., N$, assume that the volatility is piecewise constant

$$\sigma_i(t) = \sigma_{ij}, \quad \text{for } T_{j-1} < t \leq T_j, \; j = 0, ..., i-1 \tag{19}$$

Pricing exotic derivatives in a LIBOR market model

3) For each $i = 1, ..., N$, assume that the volatility is piecewise constant but assume that the volatility depends on the number of resettlement days left to maturity

$$\sigma_i(t) = \beta_{i-j}, \quad \text{for } T_{j-1} < t \leq T_j,\ j = 0, ..., i-1 \tag{20}$$

4) For each $i = 1, ..., N$, assume that the volatility is piecewise constant but such that

$$\sigma_i(t) = \beta_i \gamma_j, \quad \text{for } T_{j-1} < t \leq T_j,\ j = 0, ..., i-1 \tag{21}$$

5) For each $i = 1, ..., N$, assume some simple functional form for the volatility such as for example

$$\sigma_i(t) = q_i\left(T_{i-1} - t\right) e^{\beta_i (T_{i-1} - t)} \tag{22}$$

where $q_i(\cdot)$ is some polynomial and β_i is a real number.

Assuming that we have calibrated our LIBOR market model, we then have to simulate trajectories of the LIBOR rates.

The simplest scheme to simulate LIBOR rates is a so called Euler scheme where we set the stepsize (mesh) to δ and compute subsequent values of the LIBOR rate from

$$L_i((n+1)\delta) = L_i(n\delta) - L_i(n\delta)\left(\sum_{k=i+1}^{N} \frac{\alpha_k L_k(n\delta)}{1+\alpha_k L_k(n\delta)}\sigma_k(n\delta)\sigma_i(n\delta)\rho_{ik}\right)\delta$$
$$+ L_i(n\delta)\sigma_i(n\delta)\sqrt{\delta} Z_n \qquad (23)$$

for $i = 1, ..., N$, where $Z_n^i \sim N(0,1)$ and $\mathrm{Cor}[Z_n^i, Z_n^j] = \rho_{ij}$.

Pricing exotic derivatives in a LIBOR market model

From a numerical standpoint, it is more accurate to simulate the natural log of LIBOR rates because doing so, the diffusion term becomes deterministic.

Applying Ito to $\ln L_i(t)$ gives us that

$$d \ln L_i(t) = -\left(\frac{1}{2}\sigma_i^2(t) + \sigma_i(t)\sigma_k(t)\sum_{k=i+1}^{N}\frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}\right)dt + \sigma_i(dt)dW_t^N \qquad (24)$$

and the euler scheme then becomes

$$\begin{aligned}\ln L_i((n+1)\delta) = {} & \ln L_i(n\delta) - L_i(n\delta)\left(\frac{1}{2}\sigma_i^2(n\delta) + \sum_{k=i+1}^{N}\frac{\alpha_k L_k(n\delta)}{1+\alpha_k L_k(n\delta)}\sigma_k(n\delta)\sigma_i(n\delta)\rho_{ik}\right)\delta \\ & + \sigma_i(n\delta)\sqrt{\delta}Z_n \end{aligned} \qquad (25)$$

for $i = 1, ..., N$, where $Z_n^i \sim N(0,1)$, $Z_n^j \sim N(0,1)$ and $\mathrm{Cor}[Z_n^i, Z_n^j] = \rho_{ij}$.

The discrete savings account

We have now defined a LIBOR market model, proven that it exists and also seen how such a model can be calibrated to market data and used to price exotic derivatives.

All that now remains is to see if and how a risk-free asset fits in the overall picture.

Now, since we have modeled LIBOR rates over discrete time intervals, it seems unnatural to impose that the risk-free asset is a bank account of the usual form with dynamics

$$dB_t = r_t B_t dt \tag{26}$$

where r_t is the continuously compounded short rate.

The discrete savings account

Instead, we will construct a bank account resettled at time points $T_0, ..., T_N$.

In addition to be resettled at discrete points in time, the bank account should be riskless on a local timescale. That is, the bank account should be riskless from time T_n to time T_N for each n.

The natural way of doing so is by constantly rolling over the bond of shortest maturity at each time $t = T_0, ..., T_{N-1}$ noting that the return to the shortest ZCB available must, to prevent an arbitrage, equal the LIBOR rate for the shortest maturity.

The discrete savings account

The risk-free asset is thus constructed as follows

T_0) At time T_0, invest one dollar in the T_1 bond.

T_n) At each subsequent time T_n, $n = 1, ..., N-1$, reinvest the balance in the bank account into a ZCB maturing at time T_{n+1}.

T_N) At time T_N collect the money accumulated in the bank account.

Denoting the value of this self-financing portfolio by B we have that

$$B(T_N) = \frac{B(T_n)}{p(T_n, T_N)}, \qquad n = 0, 1, ..., N-1 \tag{27}$$

Now, we can use once again that by a replication argument

$$p(T_n, T_N) = \frac{1}{1 + \alpha_N L(T_n, T_N)} \tag{28}$$

The discrete savings account

We then obtain the discrete dynamics of the bank account as

$$\begin{aligned} B(T_0) &= 1, \\ B(T_n) &= \big(1 + \alpha_n L(T_{n-1}, T_n)\big) B(T_{n-1}), \qquad n = 1, ..., N \end{aligned} \tag{29}$$

Where we can also use that

$$B(T_n) = \prod_{k=1}^{n} \frac{1}{p(T_{k-1}, T_k)} = \prod_{k=1}^{n} \big(1 + \alpha_k L(T_{k-1}, T_k)\big) \tag{30}$$

This bank account is indeed locally risk free in that $B(T_N)$ is know already at time T_n and the value in the bank account is entirely predictable despite the fact that the LIBOR rates follow stochastic process. Once again, this result is the product of the fact that future LIBOR fixings can be replicated at present time.

The discrete savings account

Proposition

The Radon-Nikodym derivative for the change of measure from $\mathbb{Q}^N$ to $\mathbb{Q}^B$ is given by

$$\frac{d\mathbb{Q}^B}{d\mathbb{Q}^N} = \frac{B(T_N)}{p(0, T_N)}, \quad \text{on } \mathcal{F}_{T_N}$$

giving us the risk-neutral martingale measure with the discrete bank account as the numeraire.

Proof: The Radon-Nikodym derivative for the change of measure from $\mathbb{Q}^N$ to $\mathbb{Q}^B$ is given by

$$\frac{d\mathbb{Q}^B}{d\mathbb{Q}^N} = \frac{B(0)}{p(0, T_N)} \frac{B(T_N)}{p(T_N, T_N)} \tag{31}$$

But we also have that $B(0) = 1$ and $P(T_N, T_N) = 1$ and the result follows.