# Options, caps and floors

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# Constructing a cap contract

One of the most commonly traded interest rate derivatives is the cap and we will begin this section by discussing how this instrument can be priced.

An interest rate cap protects you from paying more in interest rate than a pre-specified cap rate on a floating rate obligation.

There are also floor contracts which guarantees you that the payment on a floating rate asset will never go below some pre-specified floor rate.

An interest rate cap is the sum of a number of basic contracts called caplets and we will begin our treatment by discussing these.

# Caplets

For simplicity, we will assume that present time is $t = 0$ and that the cap works over the time interval $[0, T]$.

The interval $[0, T]$ is divided into equidistant points $0, T_1, T_2, ..., T_N$ and we use the notation $\delta = T_i - T_{i-1}$ for the length of time between two points.

The cap rate is denoted by $R$ and it works on a principal of size $K$.

The floating rate of interest is the LIBOR spot rate $L(T_{i-1}, T_i)$ announced at time $T_{i-1}$ and paid at time $T_i$.

The caplet indexed by $i$ is the following contingent claim paid at time $T_i$

$$\chi_i = K\delta \max\left[L(T_{i-1}, T_i) - R, 0\right] \tag{1}$$

# Caplets

We will now compute the price of a caplet in a complete market where zero coupon bonds are available for all maturities $[T_1, T_2, ..., T_N]$

The main tool we will use is once again that future LIBOR fixings can be replicated and therefore has value at time $0$ given by

$$L(T_{i-1}, T_i) = \frac{1 - P(T_{i-1}, T_i)}{\delta P(T_{i-1}, T_i)} \tag{2}$$

The payment on the caplet $i$ can then be written

$$\begin{aligned}
\chi_i &= K\delta \max\left[L(T_{i-1}, T_i) - R, 0\right] = K\delta\left[L(T_{i-1}, T_i) - R\right]_+ \\
&= K\delta\left[\frac{1 - p(T_{i-1}, T_i)}{\delta p(T_{i-1}, T_i)} - R\right]_+ = K\left[\frac{1}{p(T_{i-1}, T_i)} - (1 + \delta R)\right]_+ \\
&= \frac{K(1 + \delta R)}{p(T_{i-1}, T_i)}\left[\frac{1}{(1 + \delta R)} - p(T_{i-1}, T_i)\right]_+
\end{aligned} \tag{3}$$

# Caplets

The caplet $i$ thus pays

$$\chi_i = \frac{K\left(1+\delta R\right)}{p(T_{i-1}, T_i)}\left[\frac{1}{\left(1+\delta R\right)} - p(T_{i-1}, T_i)\right]_+$$

at time $T_i$ which by a replication argument (or simple discounting) is equivalent to a payment at time $T_{i-1}$ of

$$K\left(1+\delta R\right)\left[\frac{1}{\left(1+\delta R\right)} - p(T_{i-1}, T_i)\right]_+$$

The caplet $i$ is thus the same as long position in $K\left(1+\delta R\right)$ put options with a strike price of $\frac{1}{\left(1+\delta R\right)}$.

To compute prices of cap and floor contracts, we must therefore study the pricing of European options.

# Change of numeraire

Let us briefly revisit how European options can be priced under the risk neutral measure when we do not make the simplifying assumption that the short rate is constant.

Suppose that consider a financial market with a risk free bank account and a stochastic short rate, then the dynamics of the bank account will be

$$dB(t) = r(t)B(t)dt \tag{4}$$

Also assume that we have a traded risky asset with $\mathbb{P}$-dynamics of the form

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW_t \tag{5}$$

# Change of numeraire

The time $t = 0$ price of a European option with $S(t)$ as the underlying and contract function $\chi(S(T))$ can be computed as an expectation under $\mathbb{Q}$

$$\Pi(t=0,\chi) = \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds} \cdot \chi(S(T))\right] \tag{6}$$

The risk neutral measure has the property that if you scale the risky assets including the European option by the bank account, the result becomes a martingale. That is both

$$\frac{S(t)}{B(t)} \quad \text{and} \quad \frac{\Pi(t)}{B(t)}$$

are martingales under $\mathbb{Q}$. Since $B(T) = B(0)e^{\int_0^T r(s)ds}$ and $\frac{\Pi(t)}{B(t)}$ is a martingale

$$\frac{\Pi(0)}{B(0)} = \mathrm{E}^{\mathbb{Q}}\left[\frac{\Pi(T)}{B(T)}\right] \quad \Rightarrow \quad \Pi(0) = \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)ds}\Pi(T)\right] \tag{7}$$

## Change of numeraire

To price European options we therefore first have to change the measure so that $\frac{S_t}{B_t}$ becomes a martingale and then compute an expectation.

In the case of a deterministic short rate, the expectation simply becomes

$$\Pi(t=0,\chi) = e^{-\int_0^T r(s)ds}\, \mathrm{E}^{\mathbb{Q}}\left[\chi(S(T))\right] \tag{8}$$

Computing the expectation 'only' requires knowing the distribution of $S(T)$ and hence involves computing only a single integral.

When the short rate is stochastic, we are however not so lucky. In that case both $\int_0^T r(s)ds$ and $S(T)$ are random variables and we have a triple integral.

# Change of numeraire

How to proceed? What if we could compute prices instead under a different measure where it is not the bank account that serves as the numeraire?

If for example we could use the zero coupon bond as the numeraire and work under a measure let us call it $\mathbb{Q}_T$ where

$$\frac{\Pi(t)}{p(t,T)}$$

is a martingale. Then, since $p(T,T)=1$ we could use that

$$\frac{\Pi(0)}{p(0,T)} = \mathrm{E}^{\mathbb{Q}_T}\left[\frac{\Pi(T)}{p(T,T)}\right] \quad \Rightarrow \quad \Pi(0) = p(0,T)\mathrm{E}^{\mathbb{Q}_T}\left[\Pi(T)\right] \tag{9}$$

and the task at hand would again involve a single integral. As it turns out, the numeraire can be chosen freely as we will now discuss in general.

## Assumption

*We consider an arbitrage-free financial market with $N+1$ assets and prices $S_0, S_1, ..., S_N$ where $S_0$ is assumed to be strictly positive.*

*Under the objective measure $\mathbb{P}$, the price dynamics are of the form*

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\boldsymbol{\sigma}_i(t)' d\mathbf{W}_t^{\mathbb{P}} \tag{10}$$

*where $\alpha_i(t)$ and $\sigma_i(t)$ are adapted processes and $\mathbf{W}_t^{\mathbb{P}}$ is a $d$-dimensional Brownian motion under $\mathbb{P}$.*

We do not necessarily assume the existence of a short rate and a money market account but if they are present, they will be denoted by $r$ and $B$.

The First Fundamental Theorem of Asset Pricing then becomes.

## Theorem (First Fundamental Theorem of Asset Pricing)

*Under the assumptions made above, the following is true.*

*The market is free of arbitrage if there exists a martingale measure $\mathbb{Q}^0$ equivalent to $\mathbb{P}$ such that the processes*

$$\frac{S_1(t)}{S_0(t)}, \frac{S_2(t)}{S_0(t)}, \dots, \frac{S_N(t)}{S_0(t)} \tag{11}$$

*are martingales under $\mathbb{Q}^0$.*

*A contingent claim of European type with maturity $T$ and contract function $\chi(T,\cdot)$ can then be priced according to*

$$\Pi(t,\chi) = S_0(t)\mathrm{E}^{\mathbb{Q}_T}\left[\frac{\chi(T,\cdot)}{S_0(T)}\middle|\mathcal{F}_t\right] \tag{12}$$

# Change of numeraire

In this version of the First Fundamental Theorem of Asset Pricing it was in some sense arbitrary that we chose $S_0$ as our numeraire. We could in principle have chosen any of the $N+1$ assets at our disposal provided the asset is strictly positive.

The theorem then says that the market is arbitrage-free *if* there exists a martingale measure corresponding to our choice of numeraire.

So, the question becomes, which choices of a numeraire guarantee the existence of at least one martingale measure?

The answer to that question is in general that if our numeraire is a traded asset, then there exists a martingale measure with respect to that choice of numeraire and markets are arbitrage-free.

# Change of numeraire

If the numeraire is *not* a traded asset, then further work is needed to insure the existence of a martingale measure.

For example, when we derived the term structure equation, we used the bank account as the numeraire and had to resolve the issue that the bank account is not a traded asset.

We however resolved the issue to arrive at a martingale measure by correcting the drift of all zero coupon bond prices by the market price of risk.

The question of uniqueness of the martingale measure remains one of market completeness and though choosing an appropriate numeraire guarantees existence of a martingale measure, and thus that the market is arbitrage-free, the question of uniqueness still hinges on all contingent claims being replicable. They generally will be if we have at least as many linearly independent assets as we have stochastic components in our model.

# Change of numeraire

Suppose now that we wish to switch from one measure to the other. That is, assume that initially, we have chosen $S_0$ as our numeraire but decide that we would like to switch to having $S_1$ be the numeraire instead. How do we do that?

Suppose that we have a contingent claim $\chi$ with maturity $T$ and that we denote the time 0 price of this claim by $\Pi(0;\chi)$.

Also suppose that we have initially chosen $S_{0t}$ as our numeraire. Then, since $\frac{\Pi_t}{S_{0t}}$ is a martingale under $\mathbb{Q}^0$, we have that

$$\Pi(0,\chi) = S_0(0)\mathrm{E}^0\left[\frac{\chi(T)}{S_0(T)}\right] = S_0(0)\int_\Omega \frac{\chi(T)}{S_0(T)}d\mathbb{Q}^0 \tag{13}$$

Under the martingale measure where $S_1$ is the numeraire, we would have

$$\Pi(0,\chi) = S_1(0)\mathrm{E}^1\left[\frac{\chi(T)}{S_1(T)}\right] = S_1(0)\int_\Omega \frac{\chi(T)}{S_1(T)}d\mathbb{Q}^1 \tag{14}$$

# Change of numeraire

The tool that took us from (13) to (14) will be called the Radon-Nikodym derivative and is defined as

$$L_0^1(T) = \frac{d\mathbb{Q}^1}{d\mathbb{Q}^0} \tag{15}$$

which is adapted to $\mathcal{F}_T$. We then write (14) as

$$\begin{aligned} \Pi(0,\chi) = \; & S_1(0)\mathrm{E}^1\left[\frac{\chi(T)}{S_1(T)}\right] = S_1(0)\int_\Omega \frac{\chi(T)}{S_1(T)} d\mathbb{Q}^1 \\ = \; & S_1(0)\int_\Omega \frac{\chi(T)}{S_1(T)} L_0^1(T) d\mathbb{Q}^0 = S_1(0)\mathrm{E}^0\left[\frac{\chi(T)}{S_1(T)} L_0^1(T)\right] \end{aligned} \tag{16}$$

Equating (13) and (16) then gives us that

$$S_0(0)\mathrm{E}^0\left[\frac{\chi(T)}{S_0(T)}\right] = S_1(0)\mathrm{E}^0\left[\frac{\chi(T)}{S_1(T)} L_0^1(T)\right] \tag{17}$$

# Change of numeraire

Since $S_0(0)$ and $S_1(0)$ are known to us at time 0 we can push them, inside the expectation to get that

$$\mathrm{E}^0\left[\left(\frac{S_0(0)}{S_0(T)} - \frac{S_1(0)}{S_1(T)} L_0^1(T)\right)\chi(T)\right] = 0 \tag{18}$$

and we deduce that

$$\frac{S_0(0)}{S_0(T)} = \frac{S_1(0)}{S_1(T)} L_0^1(T) \quad \Rightarrow \quad L_0^1(T) = \frac{S_0(0)}{S_1(0)} \frac{S_1(T)}{S_0(T)} \tag{19}$$

From this we choose the likelihood process

$$L_0^1(t) = \frac{S_0(0)}{S_1(0)} \frac{S_1(t)}{S_0(t)}, \quad 0 \leq t \leq T \tag{20}$$

and note that $L_0^1(t)$ is, by it's very construction a $\mathbb{Q}^0$ martingale.

# Change of numeraire

## Proposition

*Suppose that $\mathbb{Q}^0$ is a martingale measure corresponding to the numeraire $S_0$ on $\mathcal{F}_T$ and assume that $S_{1t}$ is a positive asset price process such that $\frac{S_{1t}}{S_{0t}}$ is a $\mathbb{Q}^0$ martingale. Define the measure $\mathbb{Q}^1$ on $\mathcal{F}_T$ by the likelihood process*

$$L_0^1(t) = \frac{d\mathbb{Q}^1}{d\mathbb{Q}^0} = \frac{S_0(0)}{S_1(0)} \frac{S_1(t)}{S_0(t)} \tag{21}$$

*Then $\mathbb{Q}^1$ is a martingale measure with $S_1$ as its numeraire.*

# Change of numeraire

The Radon-Nikodym $L_0^1(t)$ follows a stochastic process and is by construction a $\mathbb{Q}^0$ martingale.

If we assume the dynamics of $S_0$ and $S_1$ under $\mathbb{Q}^0$ are of the form

$$\begin{aligned} dS_0(t) &= \alpha_0(t)S_0(t)dt + S_0(t)\boldsymbol{\sigma}_0'(t)d\mathbf{W}_t^0 \\ dS_1(t) &= \alpha_1(t)S_1(t)dt + S_1(t)\boldsymbol{\sigma}_1'(t)d\mathbf{W}_t^0 \end{aligned} \tag{22}$$

we can easily find the dynamics of $L_0^1(t)$ and it will be of the form

$$dL_0^1(t) = (\dots)dt + L_0^1(t)\left(\boldsymbol{\sigma}_1' - \boldsymbol{\sigma}_0'\right)d\mathbf{W}_t^0 = L_0^1(t)\left(\boldsymbol{\sigma}_1' - \boldsymbol{\sigma}_0'\right)d\mathbf{W}_t^0 \tag{23}$$

In the above, we have used that since $\frac{S_1}{S_0}$ is a $\mathbb{Q}^0$ martingale, $L_0^1(t)$ must also be a $\mathbb{Q}^0$ martingale and have zero drift.

The volatility difference $\phi_t = (\boldsymbol{\sigma}_1' - \boldsymbol{\sigma}_0')$ is the so called Girsanov Kernel for the transition from $\mathbb{Q}^0$ to $\mathbb{Q}^1$.

## Change of numeraire

According to a theorem known as the Girsanove theorem, the Girsanov Kernel tells us immediately the relation between the $\mathbb{Q}^0$ Brownian motion and the $\mathbb{Q}^1$ Brownian motion. This relation is given as follows

$$d\mathbf{W}_t^0 = \phi_t dt + d\mathbf{W}_t^1 = \left(\boldsymbol{\sigma}_1' - \boldsymbol{\sigma}_0'\right) dt + d\mathbf{W}_t^1 \tag{24}$$

To see that this is the case we will begin from the measure $\mathbb{Q}^0$ and impose conditions on $\alpha_1$ such that $Z(t) = \frac{S_1(t)}{S_0(t)}$ is a martingale under $\mathbb{Q}^0$.

Then, we will switch from $\mathbb{Q}^0$ to $\mathbb{Q}^1$ by using the relation in (24) and check that under this new measure $\mathbb{Q}^1$, the ratio $Y_t = \frac{S_0(t)}{S_1(t)}$ is a martingale.

# Change of numeraire

The dynamics of $Z_t = \frac{S_{1t}}{S_{0t}}$ under $\mathbb{Q}_0$ are

$$\begin{aligned} dZ_t = & -\frac{S_{1t}}{S_{0t}^2}\left(\alpha_0 S_{0t}dt + S_{0t}\boldsymbol{\sigma}_0' d\mathbf{W}_t^0\right) + \frac{1}{S_{0t}}\left(\alpha_1 S_{1t}dt + S_{1t}\boldsymbol{\sigma}_1' d\mathbf{W}_t^0\right) \\ & + \frac{S_{1t}}{S_{0t}^3}\boldsymbol{\sigma}_{0t}'\boldsymbol{\sigma}_{0t}dt - \frac{1}{S_{0t}^2}S_{0t}S_{1t}\boldsymbol{\sigma}_{0t}'\boldsymbol{\sigma}_{1t}dt \\ = & \, Z_t\left(\alpha_1 - \alpha_0 + \boldsymbol{\sigma}_{0t}'\boldsymbol{\sigma}_{0t} - \boldsymbol{\sigma}_{0t}'\boldsymbol{\sigma}_{1t}\right)dt + Z_t\left(\boldsymbol{\sigma}_1' - \boldsymbol{\sigma}_0'\right)d\mathbf{W}_t^0 \end{aligned} \tag{25}$$

We see, that for $Z_t$ to be a martingale under $\mathbb{Q}^0$, we must impose that

$$\alpha_1 = \alpha_0 - \boldsymbol{\sigma}_{0t}'\boldsymbol{\sigma}_{0t} + \boldsymbol{\sigma}_{0t}'\boldsymbol{\sigma}_{1t} \tag{26}$$

# Change of numeraire

Next, we find the dynamics of $Y_t = \frac{S_{0t}}{S_{1t}}$ still under $\mathbb{Q}^0$ and impose the condition from (26) on $\alpha_1$ to give us

$$
\begin{aligned}
dY_t &= \frac{1}{S_{1t}}(\alpha_0 S_{0t}dt + S_{0t}\boldsymbol{\sigma}_0' d\mathbf{W}_t^0) - \frac{S_{0t}}{S_{1t}^2}(\alpha_1 S_{1t}dt + S_{1t}\boldsymbol{\sigma}_1' d\mathbf{W}_t^0) \\
&\quad + \frac{S_{0t}}{S_{1t}^3}\boldsymbol{\sigma}_{1t}'\boldsymbol{\sigma}_{1t}dt - \frac{1}{S_{1t}^2}S_{1t}S_{0t}\boldsymbol{\sigma}_{1t}'\boldsymbol{\sigma}_{0t}dt \\
&= Y_t(\alpha_0 - \alpha_1 + \boldsymbol{\sigma}_{1t}'\boldsymbol{\sigma}_{1t} - \boldsymbol{\sigma}_{1t}'\boldsymbol{\sigma}_{0t})dt + Y_t(\boldsymbol{\sigma}_0' - \boldsymbol{\sigma}_1')d\mathbf{W}_t^0 \\
&= Y_t(\boldsymbol{\sigma}_{0t}'\boldsymbol{\sigma}_{0t} + \boldsymbol{\sigma}_{1t}'\boldsymbol{\sigma}_{1t} - 2\boldsymbol{\sigma}_{1t}'\boldsymbol{\sigma}_{0t})dt + Y_t(\boldsymbol{\sigma}_0' - \boldsymbol{\sigma}_1')d\mathbf{W}_t^0 \qquad (27)
\end{aligned}
$$

Then, we switch from $\mathbb{Q}^0$ to $\mathbb{Q}^1$ by using (24) and we get

$$
\begin{aligned}
dY_t &= Y_t(\boldsymbol{\sigma}_{0t}'\boldsymbol{\sigma}_{0t} + \boldsymbol{\sigma}_{1t}'\boldsymbol{\sigma}_{1t} - 2\boldsymbol{\sigma}_{1t}'\boldsymbol{\sigma}_{0t})dt + Y_t(\boldsymbol{\sigma}_0' - \boldsymbol{\sigma}_1')(\boldsymbol{\sigma}_1' - \boldsymbol{\sigma}_0')dt + Y_t(\boldsymbol{\sigma}_0' - \boldsymbol{\sigma}_1')d\mathbf{W}_t^1 \\
&= Y_t(\boldsymbol{\sigma}_0' - \boldsymbol{\sigma}_1')d\mathbf{W}_t^1 \qquad (28)
\end{aligned}
$$

And indeed, $Y_t = \frac{S_{0t}}{S_{1t}}$ is a martingale under $\mathbb{Q}^1$.

# Forward measures

Very often, it will be convenient to use the maturity $T$ zero coupon bond as the numeraire and work under the measure $\mathbb{Q}^T$.

The likelihood process for the transformation from $\mathbb{Q}$ to $\mathbb{Q}^T$ is given by

$$L_{\mathbb{Q}}^T(t) = \frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{B(0)}{p(0,T)} \frac{p(t,T)}{B(t)}, \quad 0 \leq t \leq T \tag{29}$$

The drift of the $T$-bond under $\mathbb{Q}$ must be $r(t)$ and the $\mathbb{Q}$ dynamics become

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)\boldsymbol{\sigma}'(t,T)d\mathbf{W}_t^{\mathbb{Q}} \tag{30}$$

The dynamics of $L_{\mathbb{Q}}^T(t)$ then become

$$dL_{\mathbb{Q}}^T(t) = L_{\mathbb{Q}}^T(t)\boldsymbol{\sigma}'d\mathbf{W}_t^{\mathbb{Q}} \tag{31}$$

The Girsanov kernel for the transition from $\mathbb{Q}$ to $\mathbb{Q}^T$ is thus simply the $T$-bond volatility.

# Forward measures

## Theorem

*Consider a $T$-claim with contract function $\chi$ and denote it's time $t$ price by $\Pi(t, \chi)$, then we must have that*

$$\Pi(t, \chi) = p(t, T)\mathrm{E}^T\left[\chi(T)\middle|\mathcal{F}_t\right] \qquad (32)$$

*where $\mathbb{Q}^T$ is the martingale measure with the $T$-bond as the numeraire and $\mathrm{E}^T$ denotes expectation under $\mathbb{Q}^T$.*

# Forward measures

**Proof:**

Let us first note that trivially $p(T,T) = 1$ and also $B(T) = B(t)e^{\int_t^T r(s)ds}$.

The Radon-Nikodym derivative for the transition from $\mathbb{Q}$ to $\mathbb{Q}^T$ is then

$$L_{\mathbb{Q}}^T = \frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{B(t)}{p(t,T)} \frac{p(T,T)}{B(T)} \quad \Rightarrow \quad d\mathbb{Q} = p(t,T)e^{\int_t^T r(s)ds} \cdot d\mathbb{Q}^T \tag{33}$$

We can then change the measure in the pricing formula under $\mathbb{Q}$ to get

$$\begin{aligned}
\Pi(t,\chi) &= \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} \cdot \chi(T) \middle| \mathcal{F}_t\right] = \int e^{-\int_t^T r(s)ds} \cdot \chi(T) d\mathbb{Q} \\
&= \int e^{-\int_t^T r(s)ds} \cdot \chi(T) p(t,T) e^{\int_t^T r(s)ds} d\mathbb{Q}^T = \int \chi(T) p(t,T) d\mathbb{Q}^T \\
&= \mathrm{E}^T\left[p(t,T)\chi(T) \middle| \mathcal{F}_t\right] = p(t,T)\mathrm{E}^T\left[\chi(T) \middle| \mathcal{F}_t\right]
\end{aligned} \tag{34}$$

$\square$

# A general option pricing formula

Now, we will develop a general option pricing formula to compute the price of a European call option.

We consider a financial market with a bank account paying a possibly stochastic short rate and will consider an option with maturity $T$, strike $K$ written on a traded underlying asset with price process $S_t$.

We are thus considering the maturity $T$ claim

$$\chi = \max\left[S(T) - K, 0\right] = \left[S(T) - K\right]\mathbb{1}_{S(T)>K} \tag{35}$$

where as usual, the indicator function $\mathbb{1}$ satisfies

$$\mathbb{1}_{S(T)\geq K} = \begin{cases} 1 & S(T) \geq K \\ 0 & S(T) < K \end{cases} \tag{36}$$

# A general option pricing formula

Computing the option price then gives us

$$\begin{aligned}\Pi(0,\chi) &= \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}(S(T)-K)\mathbb{1}_{S(T)\geq K}\right] \\ &= \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}S(T)\mathbb{1}_{S(T)\geq K}\right] - K\mathrm{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\mathbb{1}_{S(T)\geq K}\right] \\ &= \int_K^{\infty} e^{-\int_0^T r_s ds}S(T)\cdot d\mathbb{Q} - K\int_K^{\infty} e^{-\int_0^T r_s ds}\cdot d\mathbb{Q} = I_1 - K\cdot I_2 \end{aligned} \tag{37}$$

We will compute $I_1$ by switching from $\mathbb{Q}$ to $\mathbb{Q}^S$ where $S$ is the numeraire. In this case, the Radon-Nikodym derivative becomes

$$L_{\mathbb{Q}}^S(T) = \frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{B(t)}{S(0)}\frac{S(T)}{B(T)} \tag{38}$$

and we get that

$$I_1 = \int_K^{\infty} e^{-\int_0^T r_s ds}\frac{B(T)}{B(0)}S(0)d\mathbb{Q}^S = \int_K^{\infty} S(0)d\mathbb{Q}^S = S(0)\mathbb{Q}^S(S(T)\geq K) \tag{39}$$

# A general option pricing formula

To find $I_2$, we will use the $T$-bond as the numeraire and the Radon-Nikodym derivative becomes

$$L_{\mathbb{Q}}^{T}(T) = \frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{B(t)}{p(0,T)} \frac{p(T,T)}{B(T)} = \frac{B(t)}{p(0,T)B(T)} \tag{40}$$

We can then compute $I_2$ to be

$$I_2 = \int_K^\infty e^{-\int_0^T r_s ds} \frac{B(T)}{B(0)} p(0,T) d\mathbb{Q}^T = \int_K^\infty p(0,T) d\mathbb{Q}^T = p(0,T)\mathbb{Q}^T(S(T) \geq K) \tag{41}$$

Putting the pieces together, we get that

$$\Pi(0,\chi) = S(0)\mathbb{Q}^S(S(T) \geq K) - Kp(0,T)\mathbb{Q}^T(S(T) \geq K) \tag{42}$$

# A general option pricing formula

## Proposition

*Consider a financial market with a bank account paying a possibly stochastic short rate, a risky asset with a strictly positive asset price process $S_t$ and a European call option with strike $K$ and maturity $T$. Then, the contract function corresponding to this call option is*

$$\chi = \max\left[S(T) - K, 0\right] \tag{43}$$

*The time $t = 0$ price $\Pi(0, T)$ of this call option is*

$$\Pi(0, T) = S(0)\mathbb{Q}^{S}\left(S(T) \geq K\right) - Kp(0, T)\mathbb{Q}^{T}\left(S(T) \geq K\right) \tag{44}$$

*Here $\mathbb{Q}^S$ denotes a probability computed under the martingale measure where $S_t$ is the numeraire and $\mathbb{Q}^T$ denotes a probability computed under the martingale measure where the $T$-bond is the numeraire.*

# A general option pricing formula

This very general option pricing formula is great news since it reduces the task of computing the price of a European call option to simply computing two probabilities in a very general setting where the short rate is allowed to be stochastic and where we have not imposed restrictions on the volatility of the underlying asset.

The next question that naturally arises is then if there are circumstances under which we are guaranteed to be able to compute these probabilities.

As we will see, the two probabilities can easily be computed in the case when the volatility coefficient of the dynamics of the underlying asset scaled by the numeraire is deterministic.

# A general option pricing formula

We will begin by looking at the probability $\mathbb{Q}^T(S(T) \geq K)$ and since under this measure, $p(t, T)$ is the numeraire, we look at the process $Z_{S,T}(t) = \frac{S(t)}{p(t,T)}$ and assume it has a stochastic differential of the form

$$dZ_{S,T}(t) = m_{S,T}(t)Z_{S,T}(t)dt + Z_{S,T}(t)\boldsymbol{\sigma}'_{S,T}(t)d\mathbf{W}_t \tag{45}$$

and furthermore assume that $\boldsymbol{\sigma}_{S,T}$ is deterministic.

In practice, we have to check that this assumption is true but since the volatility is unaffected by a change in measure from one measure to another, this can be performed under either measure.

Also, since $Z_{S,T}(t)$ has $p(t, T)$ in the denominator, we know that $Z_{S,T}(t)$ is a martingale under $\mathbb{Q}^T$ and hence, under this measure, we have $m_{S,T}(t) = 0$.

We now proceed by reexpressing $\mathbb{Q}^T(S(T) \geq K)$ in terms of $Z_{S,T}(T)$

$$\mathbb{Q}^T(S(T) \geq K) = \mathbb{Q}^T\left(\frac{S(T)}{p(T,T)} \geq K\right) = \mathbb{Q}^T(Z_{S,T}(T) \geq K) \tag{46}$$

To compute this probability, we need the distribution of $Z_{S,T}(T)$ under $\mathbb{Q}^T$.

Since $Z_{S,T}(t)$ is a martingale under $\mathbb{Q}^T$, it has $\mathbb{Q}^T$ dynamics given by

$$dZ_{S,T}(t) = Z_{S,T}(t)\boldsymbol{\sigma}'_{S,T}(t)d\mathbf{W}_t^T \tag{47}$$

and $Z_{S,T}(t)$ follows a GBM with time-varying but deterministic volatility. Applying Ito to $X(t) = \ln Z_{S,T}(t)$, we can find the solution for $Z_{S,T}(T)$

$$Z_{S,T}(T) = Z_{S,T}(0)\exp\left\{-\frac{1}{2}\int_0^T ||\boldsymbol{\sigma}_{S,T}(t)||^2 dt + \int_0^T \boldsymbol{\sigma}'_{S,T}(t)d\mathbf{W}_t^T\right\} \tag{48}$$

# A general option pricing formula

Since $\boldsymbol{\sigma}_{S,T}(t)$ is deterministic, the distribution of $\ln Z_{S,T}(T)$ is

$$\ln Z_{S,T}(T) \sim N\Big(Z_{S,T}(0) - \frac{1}{2}\Sigma^2_{S,T}(T), \Sigma^2_{S,T}(T)\Big), \quad \Sigma^2_{S,T}(T) = \int_0^T \big|\big|\boldsymbol{\sigma}_{S,T}(t)\big|\big|^2 dt$$

We can now compute $\mathbb{Q}^T\big(S(T) \geq K\big)$ through a calculation somewhat similar to that of the Black Scholes formula and we get that

$$\begin{aligned}
\mathbb{Q}^T\big(S(T) \geq K\big) &= \mathbb{Q}^T\big(Z_{S,T}(T) \geq K\big) = \mathbb{Q}^T\big(\ln Z_{S,T}(T) \geq \ln K\big) \\
&= \left(\frac{\ln Z_{S,T}(T) - Z_{S,T}(0) + \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}} \geq \frac{\ln K - Z_{S,T}(0) + \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}\right) \\
&= 1 - \Phi(-d_2) = \Phi(d_2) \qquad (49)
\end{aligned}$$

where

$$d_2 = \frac{\ln\left(\frac{S(0)}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}} \qquad (50)$$

# A general option pricing formula

We can now compute the probability $\mathbb{Q}^S(S(T) \geq K)$ and this time we will express this probability in terms of $Y_{S,T}(T) = \frac{1}{Z_{S,T}(T)}$.

$$\mathbb{Q}^S(S(T) \geq K) = \mathbb{Q}^S\left(\frac{p(T,T)}{S(T)} \leq \frac{1}{K}\right) = \mathbb{Q}^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right) \tag{51}$$

Again, we need to find the dynamics of $Y_{S,T}(t)$ under the measure $\mathbb{Q}^S$ where $Y_{S,T}(t)$ is a martingale

$$dY_{S,T}(t) = -Y_{S,T}(t)\boldsymbol{\sigma}'_{S,T}(t)d\mathbf{W}_t^S \tag{52}$$

$Y_{S,T}(t)$ follows a GBM and the solution for $Y_{S,T}(T)$ is

$$Y_{S,T}(T) = Y_{S,T}(0)\exp\left\{-\frac{1}{2}\int_0^T ||\boldsymbol{\sigma}_{S,T}(t)||^2 dt - \int_0^T \boldsymbol{\sigma}'_{S,T}(t)d\mathbf{W}_t^S\right\} \tag{53}$$

# A general option pricing formula

Since $\boldsymbol{\sigma}_{S,T}(t)$ is deterministic, the distribution of $\ln Y_{S,T}(T)$ is

$$\ln Y_{S,T}(T) \sim N\left(Y_{S,T}(0) - \frac{1}{2}\Sigma^2_{S,T}(T), \Sigma^2_{S,T}(T)\right), \quad \Sigma^2_{S,T}(T) = \int_0^T ||\boldsymbol{\sigma}_{S,T}(t)||^2 dt$$

We can now compute $\mathbb{Q}^S(S(T) \geq K)$ much like before

$$\begin{aligned}
\mathbb{Q}^S(S(T) \geq K) &= \mathbb{Q}^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right) = \mathbb{Q}^S\left(\ln Y_{S,T}(T) \leq -\ln K\right) \\
&= \left(\frac{\ln Y_{S,T}(T) - Y_{S,T}(0) + \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}} \leq \frac{-\ln K - Z_{S,T}(0) + \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}\right) \\
&= \Phi(d_1) \qquad (54)
\end{aligned}$$

where

$$d_1 = \frac{\ln\left(\frac{S(0)}{Kp(0,T)}\right) + \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}} \qquad (55)$$

## Proposition

*Consider a financial market with a bank account paying a possibly stochastic short rate, a risky asset with a strictly positive asset price process* $S_t$ *and a European call option with strike* $K$ *and maturity* $T$*. Also assume that the volatility of* $\frac{S(t)}{p(t,T)}$ *is deterministic and denoted* $\boldsymbol{\sigma}_{S,T}(t)$*.*

*Then, the time* $t=0$ *price* $\Pi(0,T)$ *of this call option is*

$$\Pi(0,T) = S(0)\Phi(d_1) - Kp(0,T)\Phi(d_2)$$

*where*

$$d_1 = \frac{\ln\left(\frac{S(0)}{Kp(0,T)}\right) + \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}, \quad d_2 = d_1 - \sqrt{\Sigma^2_{S,T}(T)}, \quad \Sigma^2_{S,T}(T) = \int_0^T \left|\left|\boldsymbol{\sigma}_{S,T}(t)\right|\right|^2$$

# Option pricing in the Vasicek and Hull-White models

We will now compute European call option prices in the Hull-White model and along the way observe that European call options must be priced in the same manner in the Vasicek model.

In the Hull-White model, the short rate has dynamics

$$dr_t = \left[\Theta(t) - ar_t\right]dt + \sigma dW_t \tag{56}$$

The Hull-White model admits an affine term structure, so we have

$$p(t, T) = e^{A(t,T) - r(t)B(t,T)} \tag{57}$$

where $A(t, T)$ and $B(t, T)$ are deterministic functions and

$$B(t, T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \tag{58}$$

# Option pricing in the Vasicek and Hull-White models

We would like to price a European call option with strike $K$ and maturity $T_1$ on an underlying zero coupon bond with maturity $T_2$.

In the notation of the previous discussion, we have $T = T_1$, $S(t) = p(t, T_2)$ and must use $p(t, T_1)$ as the numeraire.

We begin by checking that the volatility, $\sigma_z$, of

$$Z(t) = \frac{p(t, T_2)}{p(t, T_1)} \tag{59}$$

is deterministic.

Expressing $Z(t)$ in terms of $A(t, T)$ and $B(t, T)$ can be done by inserting (57) into (59), and we get that

$$Z(t) = \exp\left\{A(t, T_2) - A(t, T_1) - \left[B(t, T_2) - B(t, T_1)\right] r(t)\right\} \tag{60}$$

# Option pricing in the Vasicek and Hull-White models

We can now consider $Z$ as a function of $t$ and $r$, and apply Ito to find the dynamics of $Z(t)$ under $\mathbb{Q}$

$$dZ_t = Z_t(\dots)dt - \sigma_z(t)dW_t \tag{61}$$

where

$$\sigma_z(t) = -\sigma\left[B(t,T_2) - B(t,T_1)\right] = \frac{\sigma}{a}e^{at}\left[e^{-aT_2} - e^{-aT_1}\right] \tag{62}$$

The volatility of $Z(t)$ is thus deterministic and we can use the option pricing formula we have just developed. Also, note that

$$\begin{aligned}\Sigma^2 &= \int_0^{T_1} \left|\left|\sigma_z(t)\right|\right|^2 dt = \int_0^{T_1} \frac{\sigma^2}{a^2}e^{2at}\left(e^{-aT_2} - e^{-aT_1}\right)^2 dt \\ &= \frac{\sigma^2}{2a^3}\left(e^{-aT_2} - e^{-aT_1}\right)^2\left(1 - e^{2aT_1}\right) = \frac{\sigma^2}{2a^3}\left(1 - e^{-2aT_1}\right)\left(1 - e^{-a(T_2-T_1)}\right)^2\end{aligned} \tag{63}$$

# Option pricing in the Vasicek and Hull-White models

## Proposition

*In the Hull-White model and in the Vasicek model, the time $t = 0$ price of a European call option with strike $K$ and maturity $T_1$ on a $T_2$ zero coupon bond is given by*

$$\Pi(0;\chi) = p(0,T_2)\Phi(d_1) - Kp(0,T_1)\Phi(d2) \tag{64}$$

*where*

$$d_1 = \frac{\ln\left(\frac{p(0,T_2)}{Kp(0,T_1)}\right) + \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}}, \quad d_1 = \frac{\ln\left(\frac{p(0,T_2)}{Kp(0,T_1)}\right) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}} \tag{65}$$

*and*

$$\Sigma^2 = \frac{\sigma^2}{2a^3}\left[1 - e^{-2aT_1}\right] \cdot \left[1 - e^{-a(T_2 - T_1)}\right]^2 \tag{66}$$

# Option prices with Gaussian forward rates

## Proposition

*Assume that forward rates have dynamics of the form*

$$df(t,T) = \alpha(t,T)dt + \boldsymbol{\sigma}'(t,T)d\mathbf{W}_t$$

*where $\boldsymbol{\sigma}(t,T)$ is deterministic. Then the time $t=0$ price of a European call option with strike $K$ and maturity $T_1$ on a $T_2$ zero coupon bond is*

$$\Pi(0;\chi) = p(0,T_2)\Phi(d_1) - Kp(0,T_1)\Phi(d2)$$

*where*

$$d_1 = \frac{\ln\left(\frac{p(0,T_2)}{Kp(0,T_1)}\right) + \frac{1}{2}\Sigma^2_{T_2,T_1}}{\sqrt{\Sigma^2_{T_2,T_1}}}, \quad d_2 = d_1 - \sqrt{\Sigma^2_{T_2,T_1}}$$

$$\Sigma^2_{T_2,T_1} = \int_0^{T_1} \left|\left|\boldsymbol{\sigma}_{T_2,T_1}(t)\right|\right|^2 dt, \quad \boldsymbol{\sigma}_{T_2,T_1}(t) = -\int_{T_1}^{T_2} \boldsymbol{\sigma}(t,s)ds$$