Interest rate options

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Interest rate options in the LIBOR market model

In practice, market participants use Black's formula and implied volatility to quote prices of interest rate caplets so, inspired by this practice, we constructed the LIBOR market model by imposing dynamics on forward LIBOR rates in such a way that caplet prices are given by Black's formula.

The modeling framework we constructed could be extended to encompass *all* LIBOR rates simultaneously in one unifying model.

Having priced individual caplets, we were able to compute the price of an interest rate cap as the sum of individual caplets.

We also saw how a LIBOR market model can be used to compute prices of more complicated derivatives using simulation or other methods.

Interest rate options in the LIBOR market model

Having a modeling framework in which the price of a fixed income derivative such as caplets, caps and swaptions can be expressed as a call option who's price can be computed using an explicit formula also has the advantage that is allows us to better understand the exposure involved of these types contracts and effectively manage the risk of our positions.

The main tool to mange the risk of interest rate options that are priced using Black's formula are the derivatives of the pricing equation with respect to its various inputs often referred to as the 'greeks'.

The price of a caplet

Let us recall that the time T_{i-1} payoff to a caplet is given by

$$\chi_i = \alpha_i \cdot \max\left[L(T_{i-1}, T_i) - R, 0\right]$$

and the price of this caplet can be computed as

$$\mathbf{Capl}_{\mathbf{i}}^{\mathbf{B}}(t) = \alpha_i \cdot p_i(t)\left[L_i(t)\Phi(d_1) - R\Phi(d_2)\right]$$

where $\Phi(\cdot)$ is the standard normal distribution function and

$$d_1 = \frac{1}{\sigma_i\sqrt{T_{i-1} - t}}\left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\sigma_i^2(T_{i-1} - t)\right], \qquad d_2 = d_1 - \sigma_i\sqrt{T_{i-1} - t}$$

Also, we remember that

$$\alpha_i = T_i - T_{i-1}, \qquad L_i(t) = L(t; T_{i-1}, T_i), \qquad p_i(t) = p(t, T_i)$$

The price of a caplet

Furthermore, we recall that the zero coupon yield curve we use when pricing derivatives is found from LIBOR fixings forward LIBOR rates which implies that $p_i(t)$ also has an implicit dependence on $L_i(t)$ that we should take into account. In particular, we have that

$$p_i(t) = p_{t,T_i} = p(t, T_{i-1})p(t; T_{i-1}, T_i) = \frac{p(t, T_{i-1})}{1 + \alpha_i L_i(t)} \tag{1}$$

The expression, we should use when computing the greeks then becomes

$$\mathbf{Capl_i^B}(t) = \frac{\alpha_i p(t, T_{i-1})}{1 + \alpha_i L_i(t)} \left[L_i(t)\Phi(d_1) - R\Phi(d_2) \right]$$

Taking the derivatives of this function with respect to $L_i(t)$, σ_i and t is slightly messy but certainly doable.

The greeks of a caplet

The greeks of a caplet are

$$
\begin{aligned}
\Delta &= \frac{\partial \mathbf{Capl}_i^B}{\partial L_i} = \alpha_i p_i(t)\left[\Phi(d_1) - \frac{\alpha_i}{1+\alpha_i L_i(t)}\mathbf{Capl}_i^B\right] \\
\Gamma &= \frac{\partial^2 \mathbf{Capl}_i^B}{\partial L_i^2} = \alpha_i p_i(t)\left[\frac{\phi(d_1)}{L_i(t)\sigma_i\sqrt{T_{i-1}-t}} - \frac{2\alpha_i}{(1+\alpha_i L_i(t))^2}\Big(\alpha_i R\Phi(d_2) + \Phi(d_1)\Big)\right] \\
\nu &= \frac{\partial \mathbf{Capl}_i^B}{\partial \sigma_i} = \alpha_i p_i(t) L_i(t)\sqrt{T_{i-1}-t}\,\phi(d_2) \\
\Theta &= \frac{\partial \mathbf{Capl}_i^B}{\partial t} = \alpha_i\left[L_i(t)\Phi(d_1) - R\Phi(d_2)\right]\frac{\partial}{\partial t}p_i(t) - \alpha_i p_i(t)\frac{\sigma_i\phi(d_1)L_i(t)}{2\sqrt{T_{i-1}-t}} \\
&\approx \left[R(t,T) - \frac{\partial R(t,T_i)}{\partial t}(T_i - t)\right]\mathbf{Capl}_i^B - \alpha_i p_i(t)\frac{\sigma_i\phi(d_1)L_i(t)}{2\sqrt{T_{i-1}-t}} \\
&\approx r_t \mathbf{Capl}_i^B - \alpha_i p_i(t)\frac{\sigma_i\phi(d_1)L_i(t)}{2\sqrt{T_{i-1}-t}} \qquad (2)
\end{aligned}
$$

The greeks of a caplet - Example

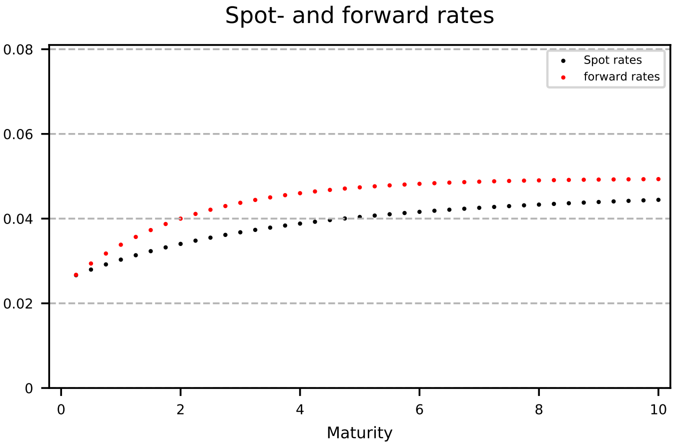
To illustrate the nature of the greeks for a caplet, we will generate spotrates using a Vasicek model with parameters $r_0 = 0.025$, $a = 0.5$, $b = 0.025$ and $\sigma_{r_0} = 0.02$.

The long run mean of the short rate is in this example $\frac{b}{a} = 0.05$.

The short rate is initially $r_0 = 0.025$ why both the spot rate term structure of interest rates as well as the 3M forward term structure of interest rates is increasing.

Having constructed data using this model, we can then plot and analyze the behavior of caplet greeks written on 3M forward LIBOR.

Spot and forward rates - Example



Caplet delta (Δ)

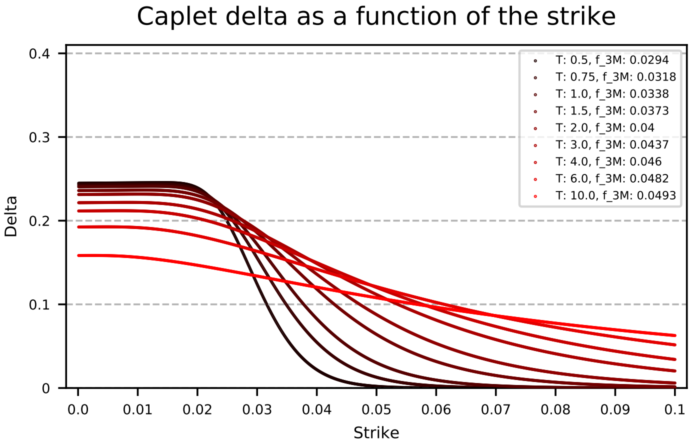
The Δ of a caplet is given by

$$\Delta = \alpha_i p_i(t) \left[\Phi(d_1) - \frac{\alpha_i}{1 + \alpha_i L_i(t)} \mathbf{Capl}_i^B \right] \tag{3}$$

We remember that $\Phi(d_1)$ is the probability of the call option finishing in-the-money and the first term reflects that the expected payoff at maturity increases as $L_i(t)$ increases.

There is however also a second term that is not familiar but arises from the fact that when the forward LIBOR rate increases, the expected future cashflow from the option is discounted back harder.

Caplet delta (Δ) - Example



Caplet gamma (Γ)

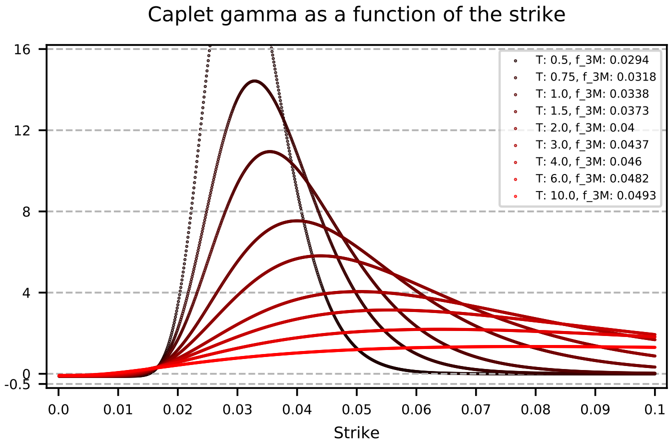
The Γ of a caplet is given by

$$\Gamma = \alpha_i p_i(t)\left[\frac{\phi(d_1)}{L_i(t)\sigma_i\sqrt{T_{i-1}-t}} - \frac{2\alpha_i}{\left(1+\alpha_i L_i(t)\right)^2}\Big(\alpha_i R\Phi(d_2)+\Phi(d_1)\Big)\right] \tag{4}$$

The gamma has a positive and negative component with the negative component being dominant for deep ITM options.

Gamma is typically largest around ATM but also drops off as we move into the OTM territory.

Caplet gamma (Γ) - Example



Caplet vega (ν)

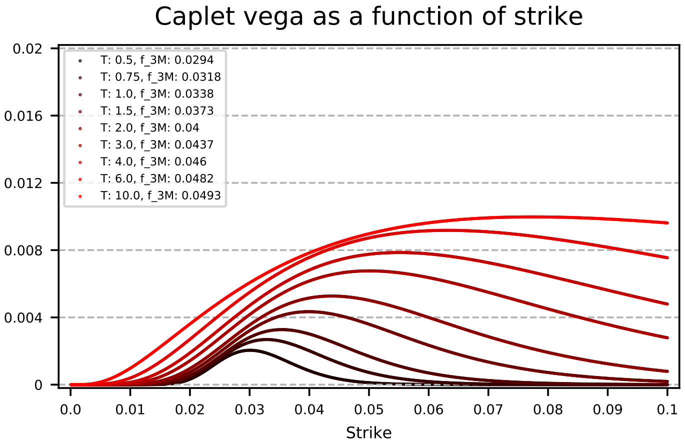
The ν of a caplet is given by

$$\nu = \frac{\partial \mathbf{Capl}_i^B}{\partial \sigma_i} = \alpha_i p_i(t) L_i(t) \sqrt{T_{i-1} - t}\,\phi(d_2) \tag{5}$$

Vega is always positive and hence, option prices are strictly increasing in volatility.

However, the option price increase only slowly for ITM options and also drops to 0 for UTM options.

Caplet vega (ν) - Example



Caplet theta (Θ)

The Θ of a caplet is given by

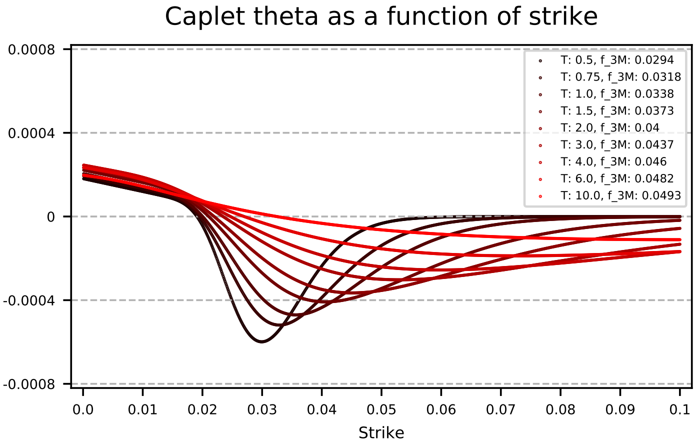
$$\Theta = \frac{\partial \mathbf{Capl}_i^B}{\partial t} = \alpha_i \left[L_i(t)\Phi(d_1) - R\Phi(d_2) \right] \frac{\partial}{\partial t} p_i(t) - \alpha_i p_i(t) \frac{\sigma_i \phi(d_1) L_i(t)}{2\sqrt{T_{i-1} - t}} \tag{6}$$

Theta has a positive component arising from the fact that an option is a financial asset that should over time yield a return to the investor.

Theta also has a nega component arising from the fact that as time increases, the likelihood that the option will expire deep ITM goes down.

Overall, Theta is positive for ITM options but decrases and drops below 0 to assume its smallest value at some point ITM.

Caplet theta (Θ) - Example



The price of a swaption

Let us recall that a $T_n \times (T_N - T_n)$ payer swaption is nothing more than a European call option on the par swap rate $R_n^N(t)$ and maturity T_n.

The price of a payer swaption with strike K then becomes

$$\mathbf{PSN}_n^N(t) = S_n^N(t)\big[R_n^N(t)\Phi(d_1) - K\Phi(d_2)\big] \tag{7}$$

where

$$d_1 = \frac{1}{\sigma_{n,N}\sqrt{T_n - t}}\left[\ln\left(\frac{R_n^N(t)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2(T_n - t)\right]$$
$$d_2 = d_1 - \sigma_{n,N}\sqrt{T_n - t}$$

Also, we recall that the accrual factor $S_n^N(t)$ is given by

$$S_n^N(t) = \sum_{i=n+1}^{N} \alpha_i p(t, T_i) = \frac{p_n(t) - p_N(t)}{R_n^N(t)}$$

The greeks of a payer swaption

The greeks of the payer swaption are

$$\Delta = \frac{\partial \mathbf{PSN}_n^N(t)}{\partial R_n^N(t)} \approx \frac{S_n^N(t)}{R_n^N(t)}\Phi(d_1) - \frac{1}{R_n^N(t)}\mathbf{PSN}_n^N(t)$$

$$\Gamma = \frac{\partial^2 \mathbf{PSN}_n^N(t)}{\partial R_n^{N^2}(t)} \approx \frac{2}{[R_n^N(t)]^2}\mathbf{PSN}_n^N(t) + \frac{S_n^N(t)}{R_n^N(t)}\left[\frac{\phi(d_1)}{\sigma_{n,N}\sqrt{T_n - t}} - 2\Phi(d_1)\right]$$

$$\nu = \frac{\partial \mathbf{PSN}_n^N(t)}{\partial \sigma_i} = S_n^N(t)R_n^N(t)\phi(d_1)\sqrt{T_n - t}$$

$$\begin{aligned}\Theta = \frac{\partial \mathbf{PSN}_n^N(t)}{\partial t} &\approx \left(\sum_{i=n+1}^{N} \alpha_i \frac{\partial}{\partial t} p_i(t)\right)\left[R_n^N(t)\Phi(d_1) - K\Phi(d_2)\right] - S_n^N(t)R_n^N(t)\frac{\sigma_{n,N}\phi(d_1)}{2\sqrt{T_n - t}} \\ &\approx \left(\sum_{i=n+1}^{N} \alpha_i \left[R(t,T_i) - \frac{\partial R(t,T_i)}{\partial t}(T_i - t)\right] p_i(t)\right)\left[R_n^N(t)\Phi(d_1) - K\Phi(d_2)\right] \\ &\quad - S_n^N(t)R_n^N(t)\frac{\sigma_{n,N}\phi(d_1)}{2\sqrt{T_n - t}} \\ &\approx r_t\mathbf{PSN}_n^N(t) - S_n^N(t)R_n^N(t)\frac{\sigma_{n,N}\phi(d_1)}{2\sqrt{T_n - t}} \qquad (8)\end{aligned}$$

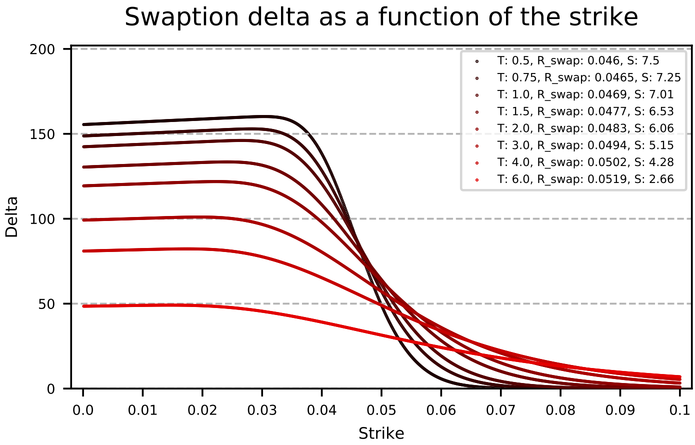
The greeks of a payer swaption

Computing the delta and gamma of a payer swaption, involves taking derivatives of the expression for the price of a payer swaption with respect to the par swap rate $R_n^N(t)$.

The par swap rate however, is a function of the zero coupon bond rates and how these change for a one unit change in the par swap rate is not identified. That is, the term structure of zero coupon bond prices can change in many different ways that all result in a one unit change to the par swap rates.

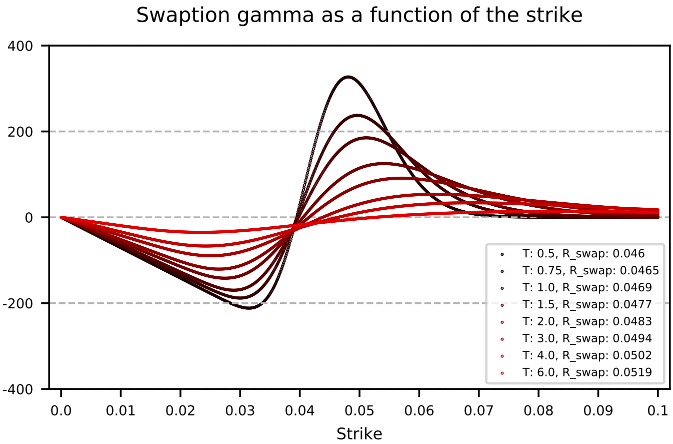
Taking the derivative of $S_n^N(t)$ with respect to is thus somewhat dubious why the greeks of the payer swaption are only approximations.

Swaption delta (Δ) - Example



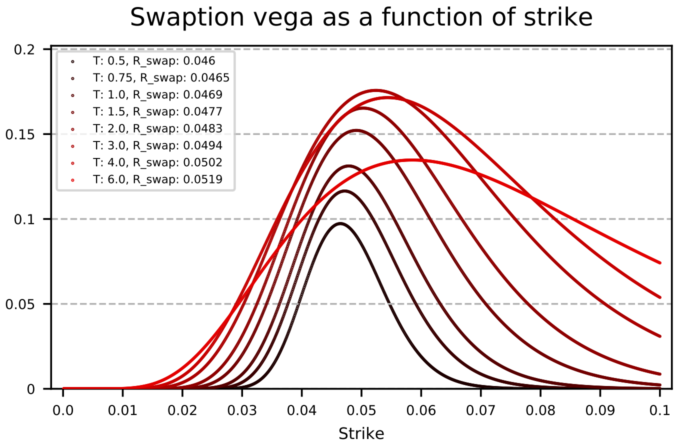
$$\Delta = \frac{\partial \mathbf{PSN}_n^N(t)}{\partial R_n^N(t)} \approx \frac{S_n^N(t)}{R_n^N(t)}\Phi(d_1) - \frac{1}{R_n^N(t)}\mathbf{PSN}_n^N(t)$$

Swaption gamma (Γ) - Example



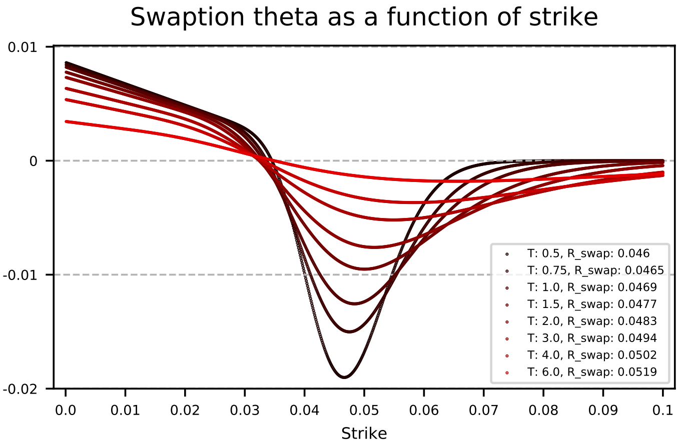
$$\Gamma = \frac{\partial^2 \mathbf{PSN}_n^N(t)}{\partial R_n^{N^2}(t)} \approx \frac{2}{[R_n^N(t)]^2}\mathbf{PSN}_n^N(t) + \frac{S_n^N(t)}{R_n^N(t)}\left[\frac{\phi(d_1)}{\sigma_{n,N}\sqrt{T_n - t}} - 2\Phi(d_1)\right]$$

Swaption vega (ν) - Example



$$\nu = \frac{\partial \mathbf{PSN}_n^N(t)}{\partial \sigma_i} = S_n^N(t) R_n^N(t) \phi(d_1) \sqrt{T_n - t}$$

Swaption theta (Θ) - Example



$$\Theta = \frac{\partial \mathbf{PSN}_n^N(t)}{\partial t} \approx r_t \mathbf{PSN}_n^N(t) - S_n^N(t) R_n^N(t) \frac{\sigma_{n,N} \phi(d_1)}{2\sqrt{T_n - t}}$$

Swaptions for speculation and risk management

Swaptions are OTC products which makes hedging more complicated.

If a trader has taken a position in a given swaption it can be hard, costly or even impossible to unwind the position.

Instead, the trader can manage his exposure by keeping track of the greeks of his position and trade to take on or eliminate risk with respect to delta, gamma and vega.

Swap markets are much more liquid and delta risk is therefore typically managed using swaps and not swaptions.

Gamma and vega are handled using swaptions often in the form of a straddle. A long position in a straddle with a short maturity will expose the trader to gamma risk whereas a long position in a straddle with a long maturity will expose the trader to vega risk.

Calibrating a swap market model to implied volatility

Remember that markets typically quote prices of swaptions in terms of the black implied volatility and we will denote the time t black implied volatility for a $T_n \times (T_{N+1} - T_n)$ payer swaption with strike K and underlying asset R_n^{N+1} by $\bar{\sigma}_n^{N+1}(t, K)$.

Also recall that we were able to construct a swap market model in which the underlying R_n^N has dynamics given by

$$dR_n^{N+1}(t) = \mu_n^{N+1}(t)R_n^{N+1}(t)dt + R_n^{N+1}(t)\boldsymbol{\sigma}'_{n,N+1}(t)d\mathbf{W}_M^{N+1}(t) \tag{9}$$

under the measure $\mathbb{Q}_N^{N+1}$ where R_N^{N+1} is the numeraire and $\mu_n^{N+1}(t)$ a somewhat complicated function of other known quantities.

Calibrating a swap market model to implied volatility

For the swap market model to be consistent with observed market prices, we have to impose a condition on the diffusion and thus drift of the par swap rates in our swap market model. The condition is that

$$\bar{\sigma}_n^{N+1}(t, K) = \int_t^{T_n} ||\boldsymbol{\sigma}_{n,N+1}(s)||^2 ds \tag{10}$$

Now $\boldsymbol{\sigma}_{n,N+1}$ is the deterministic diffusion coefficient of R_n^{N+1} which should of course not depend on the particular choice of the strike K of the swaption implied volatility imposed on $\boldsymbol{\sigma}_{n,N+1}$.

If it does, then what K should we use when we construct our swap market model?

Calibrating a swap market model to implied volatility

If swaption implied volatility observed in the market does not depend on the strike K, we would have no problem but the fact of the matter is that $\bar{\sigma}_n^{N+1}(t, K)$ does vary with K.

This is not a big problem for swaptions that expire far into the future but is very much an issue for short maturity swaptions.

The most common choice is to use the implied volatility of at-the-money swaptions when prescribing a swap market model.

This solves the problem in practice without really addressing the underlying issue.

The implied volatility smile

The phenomenon, that the implied volatility of European options depends not only on maturity but also on the strike, is not solely present in swaption markets but can be found in practically all option markets especially for short maturity options.

In practice, ITM and OTM options trade at higher prices than ATM options and thus higher implied volatilities than what can be explained by a model where the underlying asset has constant volatility.

Often implied volatility as a function of strike has the shape of a smile and hence the name.

The implied volatility smile

The presence of a smile in implied volatilities was detected already in the early days of the Black-Scholes model and lead both practitioners and academics alike to reject the Black-Scholes model as a direct tool for computing prices of European options.

The rejection of the Black-Scholes model has lead to an enormous amount of research and many option pricing models of various types have since been proposed.

After the 1987 Black Monday crash the implied volatility smiles in equity options have become more asymmetric and the smile has turned into something more like a 'smirk' or 'skew'.

The IV smile and the distribution of the underlying

To understand what the presence of an implied volatility smile in observed option prices, we will revisit a famous result due to Breeden and Litzenberger.

Let us recall that the price $\Pi(t)$ of a call option with strike K on an underlying asset S is of the form

$$\Pi(t) = D\mathrm{E}^{\mathbb{Q}^*}\left[(S_T - K)_+\right] \tag{11}$$

where D is a discount factor or the equivalent thereof and $\mathbb{Q}^*$ is the pricing measure chosen by the market.

In the case of a caplet, $D = \alpha_i p_i(t)$ and for a swaption $D = S_n^N(t)$.

The IV smile and the distribution of the underlying

The pricing measure chosen by the market manifests itself in the form of a PDF of the underlying asset price at maturity

$$\frac{\Pi(t)}{D} = \mathrm{E}^{\mathbb{Q}^*}\left[(S_T - K)_+\right] = \int_K^\infty (S_T - K) f^*_{S_T} dS_T \tag{12}$$

where D is often a discount factor or the equivalent thereof and $\mathbb{Q}^*$ is the pricing measure chosen by the market.

Taking a derivative of this expression with respect to K gives us that

$$\frac{1}{D}\frac{\partial \Pi(t)}{\partial K} = -\int_K^\infty f^*_{S_T} dS_T = 1 - P(S_T \leq K) \tag{13}$$

Taking yet another derivative gives us that

$$\frac{1}{D}\frac{\partial^2 \Pi(t)}{\partial K^2} = -\frac{\partial}{\partial K}\int_K^\infty f^*_{S_T} dS_T = f^*_{S_T} \tag{14}$$

The IV smile and the distribution of the underlying

This simple relation tells us that we can recover the distribution of the underlying asset implied by market prices by simply computing or estimating the derivatives of the option prices with respect to the strike price.

Also, notice that we relied on a very general expression for the price of a European call option and did not make any assumptions about the dynamics of the underlying asset making this an entirely model-free result.

If call prices can be computed using Black's formula, (13) becomes

$$\frac{1}{D}\frac{\partial \Pi(t)}{\partial K} = -\Phi(d_2) \tag{15}$$

The IV smile and the distribution of the underlying

The result by Breeden and Litzenberger from (14) can be used to explain the implications of the presence of a smile or smirk in implied volatilities.

The implied volatility smile suggests that the assumption of a constant volatility is not appropriate.

But more specifically, the presence of a smile signifies a rejection of the assumption that the distribution of the underlying asset is log-normal.

So, what can then be said about the distribution of the underlying asset implied by the shape of the volatility smile?

The IV smile and the distribution of the underlying

The fact that ITM and OTM options traded at higher implied volatilities than ATM options tells us that market participants find tail events more likely than what is implied by the log-normal distribution.

The asymmetry of the smile, especially for equity options, tells us that events in the left tail of the distribution are, according to the market, more likely than what is implied by the log-normal distribution.

In order to create a model that is able to reproduce the smile, we have to make assumptions such that tail-events are relatively more likely then what is predicted by the log-normal distribution and for options on some types of underlying assets, events in the left tail have to be even more likely than events in the right tail.

The IV smile and the distribution of the underlying

One class of models that is widely used to capture the implied volatility smile is the class of stochastic volatility models.

In this class of models, the volatility σ_t is assumed to also follow a stochastic process that is often correlated to the stochastic process of the underlying.

To produce fatter tails, volatility is assumed to be correlated over time so that a period of high volatility is likely to be followed by another of high volatility and a period of low volatility likely to be followed by another period of low volatility. This phenomenon is also called volatility *clustering*.

Asymmetry of the distribution of the underlying asset is produced by assuming a negative correlation between changes to the underlying asset and changes to volatility. Periods of low returns are thus more likely to occur when volatility is high.

The SABR model

To be able to reproduce the implied volatility smile in option prices a very popular model is the so called SABR(Stochastic alpha, beta, rho) model introduced by Hagan in 2002.

This model is a stochastic volatility model in which the joint dynamics of the underlying asset denoted F_t and volatility denoted σ_t are given by

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta} dW_t^{(1)}, \qquad F(0) = F_0 \\ d\sigma_t &= \upsilon \sigma_t dW_t^{(2)}, \qquad \sigma(0) = \sigma_0 \\ dW_t^{(1)} dW_t^{(2)} &= \rho \end{aligned} \tag{16}$$

where $0 \leq \beta \leq 1$, $0 < \upsilon$ and $-1 < \rho < 1$. Notice that the notation in the above differs slightly from both that of Linderstrøm(2013) and Hagan et all. (2002).

The SABR model

Prices of European options can not quite be computed explicitly in the SABR model but a singular perturbation can be performed resulting in a very accurate approximation for prices of European call options.

This singular perturbation uses Black's formula as it's point of departure making it easy to compute swaption prices given σ_b.

The approximation is most accurate for ATM implied volatilities but is typically also reliable for a wide range of strikes.

Swaption price in the SABR model

Proposition (SABR swaption price)

The payoff $\chi(T_1)$ *from a call option on the underlying asset* F *with exercise date* T_1 *and strike* K *is given by*

$$\chi(T_1) = \max\left[F(T_1) - K, 0\right] = \left(F(T_1) - K\right)_+ \tag{17}$$

The time t *price of a European call option with exercise* $T_1 < T_2$ *and settlement date* T_2 *then becomes*

$$\Pi(t;K) = P(t,T_2)\mathrm{E}^2\left[\left(F(T_1) - K\right)_+ \middle| \mathcal{F}_t\right] \approx P(t,T_2)\left[F(t)\Phi(d_1) - K\Phi(d_2)\right] \tag{18}$$

where $\mathbb{Q}^2$ *is the measure under which* $p(t,T_2)$ *is numeraire.* d_1 *and* d_2 *are*

$$d_1 = \frac{1}{\sigma_b\sqrt{T_1}}\left[\ln\left(\frac{F_0}{K}\right) + \frac{1}{2}\sigma_b^2 T_1\right], \qquad d_2 = d_1 - \sigma_b\sqrt{T_1}$$

Swaption price in the SABR model

Proposition (SABR swaption price)

The volatility coefficient $\sigma_b = \sigma_b(K, T_n; F_0, \sigma_t)$ *for ITM and OTM options is*

$$\sigma_b = \frac{\sigma_0}{(KF_0)^{\frac{1-\beta}{2}}\left[1 + \frac{(1-\beta)^2}{24}\log_2\left(\frac{F_0}{K}\right) + \frac{(1-\beta)^4}{1920}\log_4\left(\frac{F_0}{K}\right) + \dots\right]} \times \left(\frac{z}{x(z)}\right)$$
$$\times \left(1 + \left[\frac{(1-\beta)^2}{24}\frac{\sigma_0^2}{(F_0K)^{1-\beta}} + \frac{\rho\beta\upsilon\sigma_0}{4(F_0K)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\upsilon^2\right]T_1 + \dots\right) \quad (19)$$

where

$$z = \frac{\upsilon}{\sigma_0}(F_0K)^{\frac{1-\beta}{2}}\log\left(\frac{F_0}{K}\right), \qquad x(z) = \log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right) \quad (20)$$

The volatility coefficient $\sigma_b = \sigma_b(K, T_n; F_0, \sigma_t)$ *for ATM options is*

$$\sigma_b = \frac{\sigma_0}{F_0^{1-\beta}}\left(1 + \left[\frac{(1-\beta)^2}{24}\frac{\sigma_0^2}{F_0^{2(1-\beta)}} + \frac{\rho\beta\upsilon\sigma_0}{4F_0^{1-\beta}} + \frac{2-3\rho^2}{24}\upsilon^2\right]T_1 + \dots\right) \quad (21)$$

Calibration of the SABR model

The SABR model can be calibrated much in the same way that many of the other models we have seen can be calibrated to market data.

It is customary and also most efficient to calibrate a SABR model to swaption market prices quoted in terms of Black implied volatilities.

If we denote black implied volatilities computed from market prices across various strikes K_i by $\sigma_m(K_i, T_1)$, the parameters of the SABR model can be found by solving

$$\min_{\sigma_0,\beta,\upsilon,\rho} \sum_{i=1} \Big(\sigma_m(K_i, T_1) - \sigma_b(\sigma_0, \beta, \upsilon, \rho; K_i, F_0, T_1)\Big)^2 \tag{22}$$

To illustrate how the SABR model can be calibrated, we will consider the following market prices for a 5Y5Y payer swaption with unit notional.

Table: 5Y5Y Swaption market prices

K offset(bp)	Swaption price	K offset(bp)	Swaption price
-300	0.1098731	+50	0.01236822
-250	0.09271305	+100	0.00743971
-200	0.07596615	+150	0.00463353
-150	0.05986245	+200	0.00304943
-100	0.04478005	+250	0.00211939
-50	0.0313187	+300	0.00154348

Calibration of the SABR model

The term structure of zero coupon spotrates can roughly be expressed using a Vasicek model with parameters $r_0 = 0.025$, $a = 0.5$, $b = 0.025$ and $\sigma_{r_0} = 0.02$.

The ATMF par swap rate for the 5Y5Y interest rate swap can be computed from the term structure of interest rates to be roughly 0.046127 and the ATMF swaption price is 0.02029072. We arrive at the parameter estimates:

$$\hat{\sigma_0}, \hat{\beta}, \hat{\upsilon}, \hat{\rho} \approx 0.029,\ 0.52,\ 0.41,\ -0.32 \tag{23}$$

Calibration of the SABR model

If we look at the formula for σ_b in the SABR model, we notice that the model is degenerate for some choices of the β, υ and ρ.

Therefore, it is often wise to impose bounds on the parameters so that they stay within the ranges $0 \leq \beta \leq 1$, $0 < \upsilon$ and $-1 < \rho < 1$.

The parameters β and ρ are often poorly identified and in practice, it is often possible to choose β and still get a satisfying fit.

Practical uses of the SABR model

The SABR model is widely used by practitioners because it allows for an easy and in most case near perfect fit of market prices.

In practice however, the parameters required to fit market prices change over time which is theoretical inconsistent with the underlying modeling assumptions but also implies that the model most be fitted quite frequently.

The SABR model can, and is often, used to compute prices of complex derivatives as usual by fitting the model to market prices and then simulating the dynamics of par swap rates using the dynamics implied by the fitted parameter values.

Hedging in the SABR model

The SABR model is perhaps most appreciated for is value when hedging.

Since the SABR model is more complicated then the Black model, hedging is typically done by 'bumping' the yield curve or relevant parameter and then computing the *DV01* as the change in swaption value before and after the bump.

Notice however that bumping the yield curve also implies that the volatility coefficient σ changes.

Computing changes in swaption prices as a function of the parameters generating the smile σ_0, β, υ and ρ will allow us to understand the risk from changes to the shape of the smile.

Simulating the SABR model

The forward rate, F, and the volatility σ can be simulated in the SABR model using a simple Euler scheme.

Denote by M, the number of steps in the simulation and index the time points in the simulation by m, $m \in \{0, 1, 2, ..., M-1, M\}$ so that the time points will be $[t_0, t_1, ..., t_{M-1}, t_M] = [0, \delta, 2\delta, ..., T-\delta, T = \delta M]$ and hence the step in time will be of size $\delta = \frac{T}{M}$.

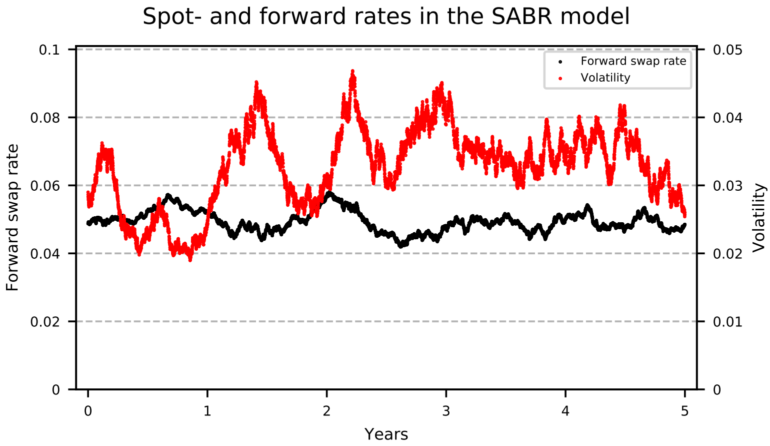
The model can then be simulated using the following equations

$$\begin{aligned} F_m &= F_{m-1} + \sigma_{m-1} F_{m-1}^{\beta} \sqrt{\delta} Z_m^{(1)}, & F(0) &= F_0 \\ \sigma_m &= \sigma_{m-1} + \upsilon \sigma_{m-1} \sqrt{\delta} \left(\rho Z_m^{(1)} + \sqrt{1-\rho^2} Z_m^{(2)} \right), & \sigma(0) &= \sigma_0 \end{aligned} \tag{24}$$

where $Z_m^{(1)}$ and $Z_m^{(2)}$ are independent standard normal random variables.

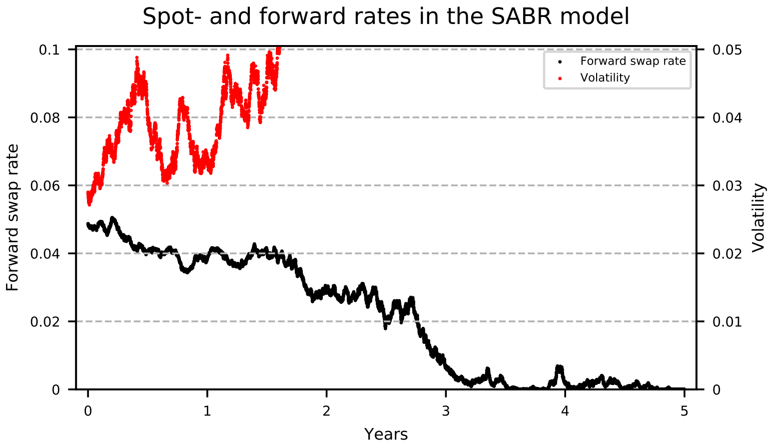
Simulating the SABR model

Simulation of the SABR model resulting in realistic trajectories.



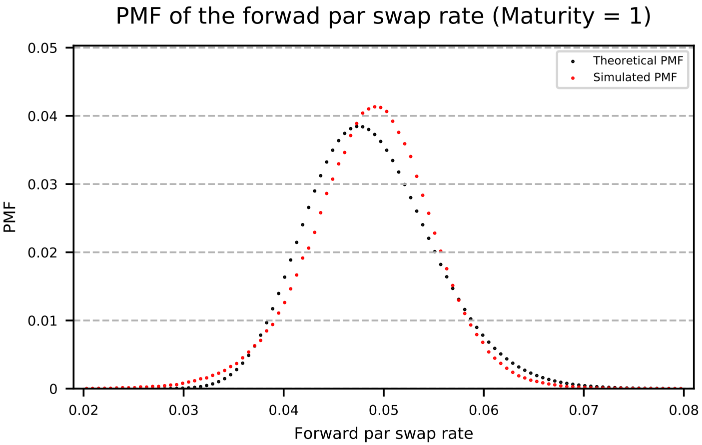
Simulating the SABR model

Simulation of the SABR model resulting in the forward rate freezing at 0.



Simulating the SABR model

Comparison of the PMF of simulated forward par swap rates and theoretical log-normal PMF.



Digital options

European Digital options are options that have a payoff of 1 at maturity if the underlying asset finished above some strike K.

The payoff function of a digital option on an underlying S_T can thus be written as

$$\chi(T) = \mathbb{1}_{S(T)>K} \tag{25}$$

The time t price $\Pi(t)$ of such an asset can be computed under the forward measure as

$$\Pi(t) = p(t,T)\mathrm{E}^T\left[\mathbb{1}_{S(T)>K}\right] = p(t,T)\mathbb{Q}^T\left(S(T)>K\right) \tag{26}$$

where $\mathbb{Q}^T(S(T) > K)$ is the probability that $S(T) > K$ under the forward measure $\mathbb{Q}^T$.

Digital options

Remembering Breeden-Litzenbergers result from (13) allows us to write

$$\Pi(t) = p(t, T)\mathrm{E}^T\left[\mathbb{1}_{S(T)>K}\right] = \frac{\partial C}{\partial K} \tag{27}$$

where $C = (t, K)$ is the time t price of a European call option with strike K.

The value of a digital option can be approximated by a portfolio of two call options with strikes tight below and right above K.

$$\Pi(t) = \frac{C(t, K+\epsilon) - C(t, K-\epsilon)}{2\epsilon} \tag{28}$$

We can create a European derivative with any increasing payoff function from digital options and these in turn can be approximated from traded call options

Butterfly spreads

It is however not only increasing payoffs that can be created using European options. In fact, we can in theory create any payoff we please simply from European call- and put options.

If we consider a portfolio where we are:

- Long 1 European call option with strike $K - \epsilon$,
- Short 2 European put options with strike K,
- Long 1 European call option with strike $K + \epsilon$.

Then this portfolio will have a payoff of 0 for $S(T) \leq K - \epsilon$ and for $S(T) \geq K + \epsilon$ and a payoff of ϵ at $S(T) = K$.

Butterfly spreads can in other words be used to approximate any positive or negative pay-off.