Fixed Income Derivatives - Problem Set Week 5

Problem 1

Consider the Vasicek model where the short rate r has dynamics

$$\begin{aligned} dr_t &= (b - ar_t)dt + \sigma dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{1}$$

Here, present time is denoted by t and the price of a zero coupon bond with maturity T is denoted $p(t,T)$. Now consider a fixed present time t and denote the short rate at time t by r.

a) Use the fact that the Vasicek model possesses an affine term structure to find expressions for:

   i) Zero coupon bond prices, $p(t,T)$, as a function of T for t fixed.

   ii) Spot rates, $R(t,T)$, as a function of T for t fixed.

   iii) Instantaneous forward rates, $f(t,T)$, as a function of T for t fixed.

b) Write three functions in Python that take as input, the parameters a, b and σ, time to maturity T, and the short rate r at present time $t = 0$ and return p, R and f respectively.

c) Use the functions you have written above to plot the term structures of zero coupon bond prices, the term structure of spot rates and the term structure of instantaneous forward rates for maturities from 0 to 10 years in a Vasicek model with $a = 1, b = 0.04, \sigma = 0.03, r = 0.05$.

d) Find the stationary mean of the short rate. Is the current level of the short rate below or above the long-run mean? Is your conclusion also reflected in the shape of the spot- and forward rate curves?

Problem 2

Consider the CIR model where the short rate r has dynamics

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{2}$$

where $a > 0$ and $b > 0$. We will now denote present time by t and proceed in to find explicit formulas for ZCB prices, spot rates and forward rates in the CIR model.

a) In the following, we will compute ZCB prices, spot rates and forward rates in the CIR model by taking a number of steps.

   i) Show that ZCB prices in the CIR model are of the form $F^{(T)}(t,r) = A(t,T)e^{-B(t,T)r}$ where $A(t,T)$ and $B(t,T)$ solve the following system of ODE's

$$A_t = abAB, \qquad A(T,T) = 1 \tag{3}$$

$$B_t = -1 + aB + \frac{\sigma^2}{2}B^2, \qquad B(T,T) = 0. \tag{4}$$

   ii) Use the substitution $B = -\frac{2}{\sigma^2 V}V(t)$ to transform the ODE for B into the following second order ODE for $V = V(t)$

$$V_{tt} - aV_t - \frac{\sigma^2}{2}V = 0. \tag{5}$$

   iii) Use the conjecture that $V(t)$ is of the form $V(t) = e^{yt}$ to show that all solutions for $V(t)$ can be written as

$$V(t) = c_1 e^{\left(\frac{a+\gamma}{2}\right)t} + c_2 e^{\left(\frac{a-\gamma}{2}\right)t}, \quad \gamma = \sqrt{a^2 + 2\sigma^2} \tag{6}$$

   where c_1 and c_2 are constants to be found

iv) Use the boundary condition on $B(T)$ to show that

$$B(t,T) = \frac{2e^{\gamma(T-t)} - 2}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)} \tag{7}$$

v) Use the ODE for $A(t,T)$ to show that

$$\ln A(t,T) = -ab\int_t^T B(s,T)ds = \frac{2ab}{\gamma}I, \qquad I = -\gamma\int_t^T \frac{e^{\gamma(T-s)} - 1}{2\gamma + (a+\gamma)\big(e^{\gamma(T-s)} - 1\big)}ds \tag{8}$$

vi) Use a substitution of the form $u = e^{\gamma(T-s)}$ to put the integral on the form

$$I = \int_{e^{\gamma(T-t)}}^{1} \frac{u-1}{\gamma - a + (a+\gamma)u}\frac{1}{u}du \tag{9}$$

vii) Show the following rule for partial fractions

$$\frac{a_0 + a_1x}{(b_0 + b_1x)(c_0 + c_1x)} = \frac{y}{b_0 + b_1x} + \frac{z}{c_0 + c_1x}, \quad \text{where } \; y = \frac{a_0b_1 - a_1b_0}{c_0b_1 - c_1b_0} \text{ and } z = \frac{c_0a_1 - c_1a_0}{c_0b_1 - c_1b_0} \tag{10}$$

and use this result to simplify the integral in (9) to

$$I = \int_{e^{\gamma(T-t)}}^{1} \frac{2\gamma}{(\gamma - a)[\gamma - a + (a+\gamma)u]}du - \int_{e^{\gamma(T-t)}}^{1} \frac{1}{(\gamma - a)u}du \tag{11}$$

viii) Solve the integral in (11) to conclude that

$$A(t,T) = \left(\frac{2\gamma \cdot e^{\frac{(a+\gamma)(T-t)}{2}}}{2\gamma + (a+\gamma)\big(e^{\gamma(T-t)} - 1\big)}\right)^{\frac{2ab}{\sigma^2}} \tag{12}$$

ix) Write down expressions for ZCB prices, spot rates and forward rates in the CIR model.

b) Write three functions in Python that take as input, the parameters a, b and σ, time to maturity T, and the short rate r at present time $t = 0$ and return p, R and f respectively.

c) Use the functions you have written above to plot the term structures of zero coupon bond prices, the term structure of spot rates and the term structure of instantaneous forward rates for maturities from 0 to 10 years in a CIR model with $a = 2, b = 0.05, \sigma = 0.1, r = 0.025$.

d) Find the stationary mean of the short rate. Is the current level of the short rate below or above the long-run mean? Is your conclusion also reflected in the shape of the spot- and forward rate curves?

Problem 3

Consider the Vasicek model where the short rate has dynamics

$$\begin{aligned} dr_t &= (b - ar_t)dt + \sigma dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{13}$$

In this problem, we will first generate zero coupon bond prices using the Vasicek model with known parameters and then seek to recover these parameters by fitting a Vasicek model to the zero coupon bond prices we have generated. In order to do so, we will use the package 'scipy.optimize' in Python. The documentation for this package can be found here and it is advised that you make good use of it.

a) Generate ZCB prices for times to maturity $T = [0, 0.1, 0.2, ..., 9.8, 9.9, 10]$ using an initial value of the short rate of $r = 0.025$ and parameters $a = 2$, $b = 0.1$, $\sigma = 0.02$. Denote these 'empirical' prices by $p^*(t,T)$.

b) Use the function 'minimize' and the method 'nelder-mead' to fit a Vasicek model to the prices $p^*(0,T)$ that you just generated. Do so by minimizing the sum of squared errors as a function of r, a, b, σ and setting the starting values of the parameters in the algorithm to $r_0, a_0, b_0, \sigma_0 = 0.03, 1.8, 0.12, 0.03$. Plot the fitted values $\hat{p}(0,T)$ and the empirical values $p^*(0,T)$. Are the fitted and empirical values close? Also plot the residuals of your fit and find the mean squared error.

c) Try to change the starting values of the parameters and perform the fit again. Which of the four parameters are best recovered by your fit and what does that tell you about the objective function as a function of r, a, b and σ?

d) Now redo the fit but impose that $b = 0.12$. Do this by changing the objective function in your fit so that it only optimizes over r, a and σ. Reproduce the plots from above and investigate the fit you now get.

In the previous, you have performed an unconstrained optimization in the sense that none of the parameters have been restricted to take values in a certain range. Next, we will investigate how to impose, bounds and constraints on the optimization and we will once again optimize over all four parameters r, a, b, σ. You will need to use that method 'trust-constr' also described in the documentation.

e) Impose the bounds that $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.1$ and perform the fit. Check once again that you recover the true parameters.

f) Now impose the restrictions that $0 \leq r \leq 0.1$, $0 \leq a \leq 1.8$, $0 \leq b \leq 0.08$ and $0 \leq \sigma \leq 0.1$ and perform the fit again. The true parameters are now outside the parameter space of the fit. Where do your fitted parameters now lie and was that to be expected?

g) Now, set the bounds back to the initial values $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.1$ but impose the non-linear constraint that $2ab \geq \sigma^2$ also using the 'trust-constr' method. Again, you will have to consult the documentation to find out, how to impose this non-linear constraint.

Problem 4

Consider the CIR model where the short rate has dynamics

$$\begin{aligned} dr_t &= a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \quad t > 0 \\ r_0 &= r \end{aligned} \tag{14}$$

where $a > 0$, $b > 0$ and $2ab \geq \sigma^2$. Present time is denoted by t, the short rate at time t is denoted by r and the price of a zero coupon bond with maturity T is denoted $p(t,T)$. In this problem, we will first generate zero coupon bond prices using the CIR model with known parameters and then seek to recover these parameters by fitting a CIR model to the zero coupon bond prices we generated.

a) Generate ZCB prices for times to maturity $\tau = T - t = [0, 0.1, 0.2, ..., 9.8, 9.9, 10]$ using an initial value of the short rate of $r = 0.045$ and parameters $a = 1.5$, $b = 0.06$, $\sigma = 0.08$. Denote these 'empirical' prices by $p^*(t,T)$.

b) Use the function 'minimize' and the method 'nelder-mead' to fit a CIR model to the prices $p^*(t,T)$ that you just generated. Do so by minimizing the sum of squared errors as a function of r, a, b, σ and setting the starting values of the parameters in the algorithm to $r_0, a_0, b_0, \sigma_0 = 0.05, 1.8, 0.08, 0.08$. Plot the fitted values $\hat{p}(t,T)$ and the empirical values $p^*(t,T)$. Are the fitted and empirical values close? Also plot the residuals of your fit and find the mean squared error.

c) Try to change the starting values of the parameters and perform the fit again. Which of the four parameters are best recovered by your fit and what does that tell you about the objective function as a function of r, a, b and σ?

d) Now redo the fit but impose that $b = 0.08$. Do this by changing the objective function in your fit so that it only optimizes over r, a and σ. Reproduce the plots from above and investigate the fit you now get.

In the previous, you have performed an unconstrained optimization in the sense that none of the parameters have been restricted to take values in a certain range. Next, we will investigate how to impose, bounds and constraints on the optimization and we will once again optimize over all four parameters r, a, b and σ. You will need to use that method 'trust-constr' also described in the documentation.

e) Impose the bounds that $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.2$ and perform the fit. Check once again that you recover the true parameters.

f) Now impose the restrictions that $0 \leq r \leq 0.1$, $0 \leq a \leq 1$, $0 \leq b \leq 0.08$ and $0 \leq \sigma \leq 0.1$ and perform the fit again. The true parameters are now outside the parameter space of the fit. Where do your fitted parameters now lie and was that to be expected?

g) Now, set the bounds back to the initial values $0 \leq r \leq 0.1$, $0 \leq a \leq 10$, $0 \leq b \leq 0.2$ and $0 \leq \sigma \leq 0.2$ but impose the non-linear constraint that $2ab \geq \sigma^2$ also using the 'trust-constr' method.