Short Rate Models

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Developing a dynamic continuous time model

So far, we have taken a static view of the market. That is, we have described what relations must exist between various fixed income products and how these products can be priced in order to prevent arbitrage opportunities at a given fixed point in time.

In the coming weeks, we will treat the market as a dynamic entity and construct dynamic models of the bond market in order to:

- Explain the shape of the yield curve in terms of market fundamentals and extract what the shape of the yield curve reveals about market expectations.
- Compute prices of more complicated fixed income derivatives.
- Understand and quantify the risk associated with fixed income derivatives as well as develop dynamic strategies to manage risk exposure.

Constructing a dynamic continuous time model

To construct realistic dynamic models for fixed income markets, we will work in continuous time.

The main building block of financial markets will still be zero coupon bonds, but now we will assume that at present time t, we have zero coupon bonds for *all* $T > t$.

We will make additional assumptions such that $p(t, T)$ will be not only continuous as a function of T but also differentiable with respect to T.

These assumptions imply that the term structure of interest rates will now consist of the collection of continuously compounded zero coupon spot rates $R(t, T)$ for *all* $T \geq t$.

The short rate and the instantaneous forward rate

Definition (The instantaneous forward)

The instantaneous forward rate $f(t, T)$ *contracted at present time* t *for maturity* T *is defined as*

$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T} = \frac{\partial}{\partial T}\Big[R(t, T)(T - t)\Big]$$

Definition (The short rate)

The short rate $r(t)$ *at present time* t *is defined as*

$$r_t = f(t, t)$$

The money account

Lemma

For all $t \leq S \leq T$*, we have that*

$$p(t,T) = p(t,S)\exp\left(-\int_S^T f(t,u)du\right)$$

and in particular

$$p(t,T) = \exp\left(-\int_t^T f(t,u)du\right)$$

The short rate and the instantaneous forward rate

We have now defined the following four different rates of return at present time t for future times S and T, $t < S < T$:

Forward rate(continuously compounded): $R(t; S, T) = -\dfrac{\log p(t, T) - \log p(t, S)}{(T - S)}$

Spot rate(continuously compounded): $R(t, T) = -\dfrac{\log p(t, T)}{(T - t)}$

Instantaneous forward rate: $f(t, T) = -\dfrac{\partial \log p(t, T)}{\partial T}$

Short rate: $r_t = f(t, t)$

The relationship between the different rates of return

The spot rate is simply the limit of the forward rate as $S \searrow t$.

The instantaneous forward rate can be interpreted as the limit of the forward rate as $S \nearrow T$ and can be interpreted as the riskless interest rate contracted at time t over the infinitesimal interval from $[T, T + dT]$.

Likewise, the short rate can be interpreted as the riskless rate you can accumulate in the infinitesimal future over the interval $[t, t + dt]$.

Using zero coupon bonds it is thus in principle possible for an investor to create a self-financing strategy that consists of bonds that are just about to reach maturity.

Pursuing such a strategy will yield a return equal to the prevailing short rate at all future points in time and will thus function very much as a bank offering the short rate as the rate of return.

The money account

Definition (The money account)

We will define the process of the money account, $B(t)$*, as*

$$B(t) = b_0 \exp\left(\int_0^t r_s ds\right) \tag{1}$$

Or equivalently, we can define B_t *as the solution to the following differential equation*

$$B(t) = r_t B(t) dt$$

$$B(0) = b_0$$

The money account will also be referred to as simply the bank account.

Link between the short rate, ZCB prices and forward rates

Let the short rate, ZCB prices and forward rates be driven by a d-dimensional Brownian motion $\mathbf{W}_t$ and have dynamics of the form

Short rate dynamics

$$dr_t = a(t)dt + \mathbf{b}'(t)d\mathbf{W}_t \tag{2}$$

ZCB price dynamics dynamics

$$dp(t,T) = m(t,T)p(t,T)dt + p(t,T)\mathbf{v}'(t,T)d\mathbf{W}_t \tag{3}$$

Instantaneous Forward rate dynamics

$$df(t,T) = \alpha(t,T)dt + \boldsymbol{\sigma}'(t,T)d\mathbf{W}_t \tag{4}$$

Part 1 - ZCB price dynamics to forward rate dynamics

If the dynamics of $p(t, T)$ satisfy

$$dp(t, T) = m(t, T)p(t, T)dt + p(t, T)\mathbf{v}'(t, T)d\mathbf{W}_t$$

then the forward rate dynamics are

$$df(t, T) = \alpha(t, T)dt + \boldsymbol{\sigma}'(t, T)d\mathbf{W}_t$$

where

$$\begin{aligned}\alpha(t, T) &= \mathbf{v}_T'(t, T)\mathbf{v}(t, T) - m_T(t, T) \\ \boldsymbol{\sigma}(t, T) &= -\mathbf{v}_T(t, T)\end{aligned} \tag{5}$$

Part 2 - Forward rate dynamics to short rate dynamics

If the dynamics of $f(t, T)$ satisfy

$$df(t, T) = \alpha(t, T)dt + \boldsymbol{\sigma}'(t, T)d\mathbf{W}_t$$

then the short rate dynamics are

$$dr_t = a(t)dt + \mathbf{b}'(t)d\mathbf{W}_t$$

where

$$\begin{aligned} a(t) &= f_T(t, t) + \alpha(t, t) \\ \mathbf{b}(t) &= \boldsymbol{\sigma}(t, t) \end{aligned} \tag{6}$$

Part 3 - Forward rate dynamics to ZCB price dynamics

If the dynamics of $f(t, T)$ satisfy

$$df(t, T) = \alpha(t, T)dt + \boldsymbol{\sigma}'(t, T)d\mathbf{W}_t$$

then the ZCB bond price dynamics satisfy

$$dp(t, T) = \big(r(t) + A(t, T) + \|\mathbf{S}(t, T)\|^2\big)p(t, T)dt + p(t, T)\mathbf{S}'(t, T)d\mathbf{W}_t$$

where $\|\cdot\|$ denotes the Euclidean norm and

$$\begin{aligned} A(t, T) &= -\int_t^T \alpha(t, s)ds \\ \mathbf{S}(t, T) &= -\int_t^T \boldsymbol{\sigma}(t, s)ds \end{aligned} \tag{7}$$

Constructing a dynamic continuous time model

Based on the analysis we have done so far and the introduction of the money account, we now see at least three different approaches to construct a dynamic model that will potentially allow us to price zero coupon bonds.

1) Impose dynamics on the short rate and then seek to price zero coupon bonds by using a no-arbitrage argument.
2) Specify the dynamics of all zero coupon bonds.
3) Specify the dynamics of all forward rates.

In this course, we will see examples all three beginning with number one.

Constructing a dynamic continuous time model

The dynamics we will impose on the short rate will be of the form

$$
\begin{aligned}
dr_t &= \mu dt + \sigma dW_t \\
r_0 &= r
\end{aligned} \tag{8}
$$

The drift $\mu = \mu(t, \cdot)$ and the diffusion $\sigma = \sigma(t, \cdot)$ are allowed to time-varying, and W_t is a Brownian motion.

Notice that r with a lower case denotes both the random short rate as well as its realization. In the following however, it should be clear from the context what is meant.

Market completeness in a short rate model

We have now imposed dynamics on the dynamics of the short rate under the objective probability measure $\mathbb{P}$.

Zero coupon bonds will thus be treated as derivatives with the bank account as the underlying asset.

Question:

Are bond prices uniquely determined by the $\mathbb{P}$ dynamics of the short rate?

Answer:

No!

Market completeness

The fact that specifying the $\mathbb{P}$-dynamics is not sufficient for completeness of the zero coupon bond market can be seen in light of the Meta-theorem.

We have one random source, the Brownian motion driving the short rate.

But the bank account is not a traded asset and can not be used to replicate a zero coupon bond for any maturity.

Prices for zero coupon bonds however still have to satisfy some internal consistency relations to prevent arbitrages from occurring.

If we were to also specify the dynamics of one benchmark zero coupon bond, the market would be complete but we will not do that and instead do something more elegant.

The term structure equation

Assumption

We assume that T-bonds for every maturity are traded in the market and that the market is arbitrage free. Also, we assume that the price of a given T-bond is a function of present time and the short rate at present time. The price of a maturity T bond can thus be written as

$$p(t, T) = F(t, r; T) \tag{9}$$

where F is a smooth function of the real variables.

The term structure equation

We will now try to find out what $F(t, r; T)$ might look like.

Recall that we are treating zero coupon bonds as derivatives with the bank account as the underlying asset.

Also recall that the price of one dollar delivered immediately must be one, so we have the boundary value condition that

$$F(T, r; T) = 1, \quad \text{for all } r \tag{10}$$

Again, be aware that r in lower case is used both as the stochastic process of the short rate *and* its realization.

The term structure equation

To proceed we will now construct a portfolio consisting of h_S maturity S zero coupon bonds and h_T maturity T zero coupon bonds.

We will aim to construct this portfolio such that it has no stochastic component and thus becomes risk-free.

Then, we will impose that this now risk-free portfolio must have local rate of return equal to the short rate.

From this no-arbitrage condition, we will then try to deduce an equation satisfied by $F(t, r; T)$.

The term structure equation

The dynamics of the short rate are of the form

$$dr_t = \mu dt + \sigma dW_t^{\mathbb{P}}, \qquad \mu = \mu(t,\cdot),\ \sigma = \sigma(t,\cdot) \tag{11}$$

Applying Ito's formula, the dynamics of the T zero coupon bond becomes

$$dF^{(T)} = \mu_T F^{(T)} dt + \sigma_T F^{(T)} dW_t^{\mathbb{P}} \tag{12}$$

where

$$\begin{aligned} \mu_T &= \frac{F_t^{(T)} + \mu F_r^{(T)} + \frac{1}{2}\sigma^2 F_{rr}^{(T)}}{F^{(T)}} \\ \sigma_T &= \frac{\sigma F_r^{(T)}}{F^{(T)}} \end{aligned} \tag{13}$$

and similarly for the S bond.

The term structure equation

The replicating portfolio V will consist of h_S S-bonds and h_T T-bonds

$$V_t = h_S F^{(S)} + h_T F^{(T)} = V_t \left(\frac{h_S F^{(S)}}{V_t} + \frac{h_T F^{(T)}}{V_t} \right) = V_t (u_S + u_T) \tag{14}$$

where u_S and u_T denote relative portfolio weights.

Inserting the dynamics of $F^{(S)}$ and $F^{(T)}$ from (12) gives us

$$dV_t = V_t \left[u_T \mu_T + u_S \mu_S \right] dt + V_t \left[u_T \sigma_T + u_S \sigma_S \right] dW_t^{\mathbb{P}} \tag{15}$$

To eliminate the $dW_t^{\mathbb{P}}$ we then have to solve

$$\left. \begin{aligned} u_S \sigma_S + u_T \sigma_T &= 0 \\ u_S + u_T &= 1 \end{aligned} \right\} \Rightarrow \begin{aligned} u_S &= \tfrac{\sigma_T}{\sigma_T - \sigma_S} \\ u_T &= -\tfrac{\sigma_S}{\sigma_T - \sigma_S} \end{aligned} \tag{16}$$

The term structure equation

Substituting the portfolio weights u_S and u_T back into the portfolio dynamics in (15) gives us that

$$dV_t = V_t\left(\frac{\mu_S\sigma_T - \mu_T\sigma_S}{\sigma_T - \sigma_S}\right) \tag{17}$$

To prevent arbitrage, we will impose that the drift of this now risk-free portfolio will be equal to the short rate

$$\frac{\mu_S\sigma_T - \mu_T\sigma_S}{\sigma_T - \sigma_S} = r, \quad \text{for all } t \text{ almost surely} \tag{18}$$

The term structure equation

Rearranging the no-arbitrage condition, we get that

$$\frac{\mu_S - r}{\sigma_S} = \frac{\mu_T - r}{\sigma_T} = \lambda \tag{19}$$

Here, it is important to note that the left-hand side is independent of T and the right hand side is independent of S.

This ratio is universal for *all* zero coupon bonds irrespective of maturity.

The fact that a universal relation between the drift and diffusion coefficients for all zero coupon bonds must be satisfied is a fundamental result and resolves the problem we had with market completeness in our bond market.

Theorem (Universal market price of risk)

If we assume that the market for zero coupon bonds is arbitrage free. Then there exists a process λ_t *such that the relation*

$$\frac{\mu_T - r}{\sigma_T} = \lambda$$

must hold for every maturity T *and thus for every zero coupon bond. Here* $\mu_T = \mu_T(t, \cdot)$ *is the drift and* $\sigma_T = \sigma_T(t, \cdot)$ *the diffusion of the maturity* T *zero coupon bond. Also,* $r = r(t)$ *is the short rate and* $\lambda = \lambda(t)$ *is the market price of risk at time* t*.*

The term structure equation

The market price of risk, $\frac{\mu_T - r}{\sigma_T} = \lambda$, has an appealing interpretation:

- The numerator is the excess return of the T-bond required by the market.
- The denominator is the risk of the T-bond.
- The market price of risk has the flavor of a Sharpe's ratio.
- The relation between the drift and diffusion of a T-bond imposed by the market price of risk must hold for *all* T-bonds simultaneously to prevent arbitrage.

The term structure equation

Inserting the drift and diffusion coefficients of the T-bond from (13) into (19) gives us the term structure equation.

$$F_t^{(T)} + (\mu - \lambda\sigma) F_r^{(T)} + \frac{1}{2}\sigma^2 F_{rr}^{(T)} - rF^{(T)} = 0 \tag{20}$$

where μ and σ are the drift and diffusion coefficients from the dynamics of the short rate.

This PDE together with the boundary condition

$$F^{(T)}(T, r) = 1 \tag{21}$$

gives us a boundary value problem to solve for computing zero coupon bond prices.

The term structure equation

Lemma (Term structure equation)

In an arbitrage free bond market, the price $p(t,T) = F^{(T)}$ of a zero coupon bond with maturity T will satisfy the following boundary value problem

$$
\begin{aligned}
F_t^{(T)} + (\mu - \lambda\sigma)F_r^{(T)} + \frac{1}{2}\sigma^2 F_{rr}^{(T)} - rF^{(T)} &= 0, \\
F^{(T)}(T,r) &= 1,
\end{aligned}
$$

where $\mu = \mu(t,\cdot)$ and $\sigma = \sigma(t,\cdot)$ stem from the dynamics of the short rate and $\lambda = \lambda(t,\cdot)$ is the market price of risk.

Risk neutral valuation of zero coupon bond prices

The term structure equation is obviously closely related to the Black-Scholes equation and therefore, the solution to the term structure equation will very much resemble the solution to the Black-Scholes equation.

Let us fix a time t and a corresponding $r = r_t$, assume that $F^{(T)}$ solves the term structure equation and look at

$$H^{(T)}(s, r_s) = \exp\Big(-\int_t^s r_u du\Big) F^{(T)}(s, r_s), \quad t < s \leq T. \tag{22}$$

The plan is then, exactly as in the case of Black-Scholes, to impose that the short rate has dynamics

$$dr_s = (\mu - \lambda\sigma)ds + \sigma dW_s^{\mathbb{Q}}, \quad s > t, \tag{23}$$

apply Ito's formula to $H^{(T)}$ and use that $F^{(T)}$ by assumption satisfies the term structure equation.

Risk neutral valuation of zero coupon bond prices

Notice that $H^{(T)}$ is a function of s and $r = r_s$ and applying Ito gives us

$$d\Big(H^{(T)}\Big) = \exp\Big(-\int_t^s r_u du\Big)\Big[-rF^{(T)} + F_s^{(T)} + (\mu - \lambda\sigma)F_r^{(T)} + \frac{1}{2}\sigma^2 F_{rr}^{(T)}\Big] + \exp\Big(-\int_t^s r_u du\Big)\sigma F_r^{(T)} dW_s^{\mathbb{Q}} \tag{24}$$

Using that $H^{(T)}$ satisfies the term structure equation and integrating from t to T gives

$$\int_t^T d\Big(H^{(T)}\Big) = H^{(T)}(T, r(T)) - H^{(T)}(t, r(t)) = \int_t^T \exp\Big(-\int_t^s r_u du\Big)\sigma F_r^{(T)} dW_s^{\mathbb{Q}}.$$

Now, $F^{(T)}(T, r(T)) = 1$ and taking expectations will give us that

$$F^{(T)}(t, r(t)) = \mathrm{E}^{\mathbb{Q}}\big[e^{-\int_t^T r_u du}\big] \tag{25}$$

Risk neutral valuation of zero coupon bonds

Lemma (Risk neutral valuation of zero coupon bonds)

Zero coupon bond prices are given by the formula $p(t, T) = F(t, r; T)$

$$F(t, r; T) = \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right] \tag{26}$$

Here the expectation must be taken under the risk neutral measure $\mathbb{Q}$ *where the short rate has dynamics*

$$dr_s = \left(\mu - \lambda\sigma\right) ds + \sigma dW_s^{\mathbb{Q}}, \quad s > t$$

$$r_t = r$$

where $\mu = \mu(s, \cdot)$, $\lambda = \lambda(s, \cdot)$ *and* $\sigma = \sigma(s, \cdot)$.

Risk neutral valuation of contingent claims

The pricing equation has the expected economic interpretation that the price of a zero coupon bond is the expected discounted value of the payoff of one dollar at maturity.

$$F(t, r; T) = \mathrm{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \times 1\right] \tag{27}$$

Also, we now know what the dynamics of the short rate must look like under the risk neutral measure and in particular that the drift of the short rate under the objective probability measure must be corrected by subtracting the market price of risk, λ, times the diffusion of the short rate.

The next question that naturally arises is how to compute prices of more general European style contingent claims and not surprisingly, they too can be priced by taking an expectation under the risk neutral measure.

Risk neutral valuation of contingent claims

Lemma (Risk neutral)

Let χ be a contingent T-claim of the form $\chi = \Phi(r)$. If the market is arbitrage free, then the price, $\Pi(t, \chi)$ of the claim must be

$$\Pi(t, \chi) = F(t, r)$$

where F solves the boundary value problem

$$F_t^{(T)} + (\mu - \lambda\sigma)F_r^{(T)} + \frac{1}{2}\sigma^2 F_{rr}^{(T)} - rF^{(T)} = 0$$

$$F^{(T)}(T, r) = \Phi(r)$$

Risk neutral valuation of contingent claims

Furthermore F has stochastic representation

$$\mathrm{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \times \Phi(r)\right] \tag{28}$$

and the expectation is to be taken under the risk neutral measure where the short rate has dynamics

$$\begin{aligned} dr_s &= (\mu - \lambda\sigma)\, ds + \sigma\, dW_s^{\mathbb{Q}}, \quad s > t \\ r_t &= r. \end{aligned}$$

Choosing the dynamics of the short rate

The drift of short rate under the risk-neutral pricing measure is determined in part by the $\mathbb{P}$-dynamics through μ but also by λ.

The market price of risk, λ, is determined solely by the market and is thus a product of supply and demand for liquidity at various points in time.

Once the market has chosen the market price of risk for any single zero coupon bond, the rest of the bond market will be priced from the term structure equation reflecting once again that in this one factor model, zero coupon bond prices are priced relative to each other.

To extract the drift and diffusion of the short rate as well as the market price of risk we could collect market data and estimate these using time-series econometrics. That would potentially allow us to estimate the parameters of the $\mathbb{P}$-dynamics of the short rate but it remains unclear how we could estimate λ directly from bond prices.

Choosing the dynamics of the short rate

If for a second we assume that σ is known, it does not matter exactly how we choose μ and λ respectively as long as $\mu - \lambda\sigma$ remains the same. We can therefore also simply choose to prescribe our model directly under the risk-neutral measure and that is exactly what we will do.

Recall that $\mu - \lambda\sigma$ is exactly the drift of the short rate and we will thus proceed by specifying the drift and diffusion of the short rate under the risk-neutral measure.

This approach is known as martingale-modelling and typically, the dynamics of the short rate under $\mathbb{Q}$ will be of the form

$$dr_t = \mu(t, r_t)\,dt + \sigma(t, r_t)\,dW_t^{\mathbb{Q}}. \tag{29}$$

Short rate models

Among the most influential short rate models are

Vasicek	$dr_t = (b - ar_t)dt + \sigma dW_t$
Cox-Ingersoll-Ross(CIR)	$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$
Dothan	$dr_t = ar_t dt + \sigma r_t dW_t$
Black-Derman-Toy	$dr_t = \Theta(t) r_t dt + \sigma(t) r_t dW_t$
Ho-Lee	$dr_t = \Theta(t)dt + \sigma dW_t$
Hull-White extended Vasicek	$dr_t = \big(\Theta(t) - a(t)r_t\big)dt + \sigma(t)dW_t,\ \ a(t) > 0$
Hull-White extended CIR	$dr_t = \big(\Theta(t) - a(t)r_t\big)dt + \sigma(t)\sqrt{r_t}dW_t,\ \ a(t) > 0$

Inversion of the yield curve

Now that we have specified a range of short rate models, the next question that naturally arise is how to choose the parameters of a short rate model.

Perhaps the first idea that comes to mind is to estimate the drift and diffusion of the short rate using time-series data. However, when doing so we face the problem that the model prescribes the $\mathbb{Q}$-dynamics of the short rate but we observe the $\mathbb{P}$-dynamics.

Now, it can be shown that the diffusion coefficient does not change when switching the measure so we could potentially estimate the diffusion coefficient from market data, but there is now hope of estimating the drift under $\mathbb{Q}$ from market data.

What we can however do is to 'Invert the yield curve'. The procedure to invert the yield curve will be described next.

Inversion of the yield curve

1) Choose a particular model and collect the parameters of that model in a vector $\boldsymbol{\alpha}$. The dynamics of r_t under $\mathbb{Q}$ can then be written as

$$dr_t = \mu(t, r_t; \boldsymbol{\alpha})dt + \sigma(t, r_t; \boldsymbol{\alpha})dW_t$$

2) Find an expression for the theoretical zero coupon bond prices

$$p(t, T; \boldsymbol{\alpha}) = F^{(T)}(t, r; \boldsymbol{\alpha})$$

as a function of $\boldsymbol{\alpha}$ by solving the term structure equation.

3) Collect bond market data and compute bond prices to construct the empirical term structure $\{p^*(0, T), \geq 0\}$.

Inversion of the yield curve

4) Choose the parameter vector $\boldsymbol{\alpha}$ in such a way that the theoretical term structure fits the empirical as well as possible according to some objective function. This gives us an estimate, $\hat{\boldsymbol{\alpha}}$, of the parameters.

5) Deduce the drift $\mu^* = \mu(\boldsymbol{\alpha}^*)$ and diffusion $\sigma^* = \sigma(\boldsymbol{\alpha}^*)$ from the parameter estimates. We now know which martingale measure market prices of bonds reflect.

6) Now that we have found the martingale measure chosen by the market, we can proceed to compute prices of interest rate derivatives. Say we have an derivative with contract function $\chi = \Gamma(r_T)$, then it's price $\Pi(t, \chi) = G(t, r)$ solves the boundary value problem

$$G_t + \mu^* G_r + \frac{1}{2}{\sigma^*}^2 G_{rr} - rG = 0$$

$$G(T, r) = \Gamma(r)$$

Inversion of the yield curve

The short rate models listed above differ in many ways but one important distinction is whether they might allow us to get a perfect fit of the initial term structure.

The models by Vasicek, Cox-Ingersoll-Ross and Dothan only have a finite parameter space. In fact they all have at most three parameters and hence there is no hope of obtaining a perfect fit of the initial term structure.

This is problematic since prices of derivatives often depend crucially on the initial term structure and a poor fit of portions of the yield curve might lead to incorrect derivatives prices.

The remaining four models all have infinite parameter spaces but this extension does not come without a cost since these models are more difficult to work with and are more prone to parameter instability.

Affine term structure models

To be able to invert the yield curve for a given short rate model, it is vital that we can find an expression for theoretical zero coupon bonds prices.

As of now, we have two ways of computing zero coupon bond prices given a fully parameterized short rate model.

a) Compute ZCB prices as expectations under the risk neutral measure.

b) Solve the term structure equation.

The former requires that we know the distribution of $\int_0^T r_s ds$ and can sometimes be carried out, in particular if the short rate is Gaussian.

The latter seems more demanding as it requires us to solve a PDE but if the model possesses a so called 'Affine Term Structure' it becomes easier.

Definition (Affine Term Structure)

If the term structure $\{p(t, T);\ 0 \le t \le T, T > 0\}$ *has the form*

$$p(t, T) = F(t, r; T)$$

where F *has the form*

$$F(t, r; T) = e^{A(t,T) - B(t,T)r} \tag{30}$$

and where A *and* B *are deterministic functions, then the model is said to possess an affine term structure (ATS).*

Affine term structure models

Affine term structure models will prove very tractable from an analytical point of view so the question is. When does a model possess an ATS?

We will investigate this matter and give a partial answer that will apply to most of the models listed previously.

Let us assume that we have a model where the short rate has dynamics

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t \tag{31}$$

and also assume that the model possesses an Affine Term Structure. We will denote the zero coupon bond price for maturity T by F and we have

$$F(t, r; T) = e^{A(t,T) - B(t,T)r} \tag{32}$$

Computing the partial derivatives of F from (32) and inserting into the term structure equation that F satisfies gives us that

$$A_t - (1 + B_t)r - \mu B + \frac{1}{2}\sigma^2 B^2 = 0 \tag{33}$$

and also, the boundary condition that $F(T, r; T) = 1$ for all r implies that $B(T, T) = 0$ and $A(T, T) = 0$.

The relation in (33) must be satisfied for an ATS to exist and the question becomes, for which choices of μ and σ does there exist functions A and B that solve (33) for *all* choices of r?

Giving a complete answer to this question is not possible, but there are choices of μ and σ that guarantee solutions for A and B.

Affine term structure models

Suppose μ and σ are of the form

$$\begin{aligned} \mu(t,r) &= \alpha(t)r + \beta(t) \\ \sigma(t,r) &= \sqrt{\gamma(t)r + \delta} \end{aligned} \tag{34}$$

Plugging the expressions from (34) into (33) and collecting terms in r gives us that

$$A_t - \beta(t)B + \frac{1}{2}\delta(t)B^2 - \left[1 + B_t + \alpha(t)B - \frac{1}{2}\gamma(t)B^2\right]r = 0 \tag{35}$$

Affine term structure models

To solve the system in (36), we observe that the first equation is free of A and that this equation is a simple Ricatti equation.

Having solved the first equation, we insert the solution for B into the second equation and solve the second equation simply by integrating.

Matching coefficients in r gives us the following two equations to solve for $A = A(t, T)$ and $B = B(t, T)$

$$\begin{aligned} B_t &= -1 - \alpha(t)B + \frac{1}{2}\gamma(t)B^2 \\ A_t &= \beta(t)B - \frac{1}{2}\delta(t)B^2 \end{aligned} \tag{36}$$

Affine term structure models

Proposition (Affine term structure)

Assume that μ and σ are of the form

$$\mu(t,r) = \alpha(t)r + \beta(t)$$
$$\sigma(t,r) = \sqrt{\gamma(t)r + \delta}$$

Then the model the short rate model possesses an affine term structure where A and B solves

$$\begin{aligned} B_t &= -1 - \alpha(t)B + \frac{1}{2}\gamma(t)B^2, & B(T,T) = 0 \\ A_t &= \beta(t)B - \frac{1}{2}\delta(t)B^2, & A(T,T) = 0 \end{aligned}$$

Affine term structure models

Looking at the list of short rate models, we see that except the models by Dothan and the extension by Black-Derman-Toy, they all have dynamics consistent with (34) and possess an affine term structure.

To compute zero coupon bond prices, we could possibly also use that

$$p(t, T) = \mathrm{E}\left[e^{-\int_t^T r_s ds}\right] \tag{37}$$

Doing so rests on knowing the distribution of $-\int_t^T r_s ds$ but as it turns out, even if this distribution can be found, solving (36) is much simpler.

The models by Dothan and Black-Derman-Toy do not possess an affine term structure but the short rate follows a geometric Brownian motion suggesting that solving (37) might be possible for those two. However, in these cases $-\int_t^T r_s ds$ is essentially a sum of log-normals which is not easy to work with.

The Vasicek model

The dynamics of the short rate in the Vasicek model is given by

$$dr_t = (b - ar_t)dt + \sigma dW_t \tag{38}$$

The Vasicek model can be solved explicitly by applying *Ito* to $f(t, r_t) = e^{at} r_t$

$$df(t, r_t) = ae^{at} r_t dt + e^{at}(b - ar_t)dt + e^{at} dW_t = be^{at} dt + \sigma e^{at} dW_t \tag{39}$$

Integrating from t to T gives us that

$$\int_t^T df(u, r_u) = e^{aT} r_T - e^{at} r_t = \int_t^T be^{au} du + \sigma \int_t^T e^{au} dW_u \quad \Rightarrow$$
$$r_T = r_t e^{-a(T-t)} + be^{-aT} \int_t^T e^{au} du + \sigma \int_t^T e^{-a(T-u)} dW_u$$

and hence, the solution becomes

$$r_T = r_t e^{-a(T-t)} + \frac{b}{a}\left[1 - e^{-a(T-t)}\right] + \sigma \int_t^T e^{-a(T-u)} dW_u \tag{40}$$

The distribution of the short rate in the Vasicek model is thus

$$r_T|\mathcal{F}_t \sim N\left(r_t e^{-a(T-t)} + \frac{b}{a}\left[1 - e^{-a(T-t)}\right], \frac{\sigma^2}{2a}\left[1 - e^{-2a(T-t)}\right]\right) \tag{41}$$

The stationary distribution can be found be sending $T \nearrow \infty$

$$r_\infty \sim N\left(\frac{b}{a}, \frac{\sigma^2}{2a}\right) \tag{42}$$

The short rate in the Vasicek model is thus Gaussian which is convenient from a computational point of view but might be problematic from an economic point of view, as it allows for the short rate to become negative.

The Vasicek model

In the Vasicek model, zero coupon bond prices are given by

$$p(t, T) = e^{A(t,T) - B(t,T)r} \tag{43}$$

where

$$\begin{aligned} A(t, T) &= \frac{\left[B(t, T) - (T - t)\right]\left(ab - \frac{1}{2}\sigma^2\right)}{a^2} - \frac{\sigma^2 B^2(t, T)}{4a} \\ B(t, T) &= \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \end{aligned} \tag{44}$$

Spot rates are given by

$$R(t, T) = -\frac{\log P(t, T)}{T - t} = \frac{-A + rB}{T - t} \tag{45}$$

Instantaneous forward rates are given by

$$\begin{aligned} f(t, T) &= -\frac{\partial log\ P(t, T)}{\partial T} = -A_T + rB_T \\ &= \frac{\left(1 - B_T\right)\left(ab - \frac{1}{2}\sigma^2\right)}{a^2} + \frac{\sigma^2 B B_T}{2a} + rB_T \end{aligned} \tag{46}$$

The Cox-Ingersoll-Ross model

Let us try to create a short rate model where the short rate is always positive using what we know about the Vasicek model.

In particular, we will construct a model where the short rate is the sum of squares of independent processes where each of these processes are of the Vasicek form.

Assume X_{jt} has dynamics

$$dX_{jt} = -\frac{a}{2} X_{jt} dt + \frac{1}{2} \sigma dW_{jt} \tag{47}$$

Let us define

$$r_t = \sum_{j=1}^{d} X_{jt}^2 \tag{48}$$

The Cox-Ingersoll-Ross model

The dynamics of r_t then become

$$
\begin{aligned}
dr_t &= -\sum_{j=1}^{d} 2X_{jt}\frac{a}{2}X_{jt}dt + \sum_{j=1}^{d} 2X_{jt}\frac{1}{2}\sigma dW_{jt} + \frac{1}{2}\left(\sum_{j=1}^{d}\frac{1}{2}\sigma dW_{jt}\sigma\right)^2 \\
&= \frac{d\sigma^2}{4}dt - a\left(\sum_{j=1}^{d} X_{jt}^2\right)dt + \sigma\sum_{j=1}^{d} X_{jt}dW_{jt} \\
&= \frac{d\sigma^2}{4}dt - ar_t dt + \sigma\sqrt{r_t}\sum_{j=1}^{d}\frac{X_{jt}}{\sqrt{r_t}}dW_{jt} \\
&= (b - ar_t)dt + \sigma\sqrt{r_t}dB_t
\end{aligned}
\tag{49}
$$

Since it can be shown that

$$
B_t = \sum_{j=1}^{d}\int_0^t \frac{X_{js}}{\sqrt{r_s}}dW_{js} \tag{50}
$$

is indeed a Brownian motion.

The Cox-Ingersoll-Ross model

This construction has resulted in a model where the short rate is the sum of squared Gaussian processes each of which have a stationary distribution. In the CIR model it must therefore be true that the short rate:

i) is non-negative,

ii) follows a non-central chi-squared distribution,

iii) has a stationary distribution.

However, the level to which the short rate mean-reverts is pinned down by the volatility of the process and that is unfortunate.

The Cox-Ingersoll-Ross model

The construction presented above can however be generalized in such a way that all three parameters a, b and σ can be chosen independently of each other.

The dynamics of the short rate in this version of the CIR model are

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{51}$$

where $b > 0$ is the long-run mean and the speed of mean-reversion is governed by a. So, to insure mean-reversion and not explosion, we must choose $a > 0$. Finally, to insure that the short rate is not absorbed at 0, we must choose the parameters of the model according to the so called Feller condition that $2ab \geq \sigma^2$.

The Cox-Ingersoll-Ross model

Let us do as we did in the Vasicek model and apply Ito to $f(t, r_t) = e^{at} r_t$

$$df(t, r_t) = ae^{at} r_t dt + e^{at}[a(b - r_t)]dt + e^{at} dW_t = abe^{at} dt + \sigma e^{at} \sqrt{r_t} dW_t \qquad (52)$$

Integrating from t to T gives us that

$$\int_t^T df(u, r_u) = e^{aT} r_T - e^{at} r_t = ab \int_t^T e^{au} du + \sigma \int_t^T e^{au} \sqrt{r_u} dW_u \quad \Rightarrow$$

$$r_T = r_t e^{-a(T-t)} + ab \int_t^T e^{-a(T-u)} du + \sigma \int_t^T e^{-a(T-u)} \sqrt{r_u} dW_u$$

and hence, we get that

$$r_T = r_t e^{-a(T-t)} + b\Big[1 - e^{-a(T-t)}\Big] + \sigma \int_t^T e^{-a(T-u)} \sqrt{r_u} dW_u \qquad (53)$$

This is exactly as in the Vasicek model *except* for the $\sqrt{r_u}$ in the diffusion term. The appearance of this term implies that the distribution of the short rate in this model is not Gaussian.

The Cox-Ingersoll-Ross model

From r_T in (53), we can compute the mean and variance of $r_T|\mathcal{F}_t$

$$\mathrm{E}\left[r_T \middle| \mathcal{F}_t\right] = r_t e^{-a(T-t)} + b\left[1 - e^{-a(T-t)}\right]$$
$$\mathrm{Var}\left[r_T \middle| \mathcal{F}_t\right] = \frac{\sigma^2 r_t}{a}\left[e^{-a(T-t)} - e^{-2a(T-t)}\right] + \frac{\sigma^2 b}{2a}\left[1 - e^{-a(T-t)}\right]^2 \tag{54}$$

The distribution of $r_T|\mathcal{F}_t$ is that of

$$\frac{\sigma^2}{4a}\left[1 - e^{-a(T-t)}\right] Y \tag{55}$$

where Y follows a non-central chi-squared distribution with k degrees of freedom and non-centrality parameter λ

$$k = \frac{4ab}{\sigma^2}, \quad \lambda = \frac{4ae^{-a(T-t)}}{\sigma^2\left[1 - e^{-a(T-t)}\right]} r_t \tag{56}$$

The stationary distribution is a gamma where

$$r_\infty \sim \text{Gamma}(\alpha, \beta), \quad \alpha = \frac{2ab}{\sigma^2}, \quad \beta = \frac{\sigma^2}{2a}, \quad f_{r_\infty}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \; x > 0$$

The Cox-Ingersoll-Ross model

In the CIR model, zero coupon bond prices are given by

$$p(t, T) = F(t, r; T) = A_0 e^{-Br} \tag{57}$$

where $\gamma = \sqrt{a^2 + 2\sigma^2}$ and

$$\begin{aligned} A_0(t, T) &= \left(\frac{2\gamma e^{(T-t)(a+\gamma)/2}}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma} \right)^{2ab/\sigma^2} \\ B(t, T) &= \frac{2\left(e^{\gamma(T-t)} - 1\right)}{(\gamma + a)\left(e^{\gamma(T-t)} - 1\right) + 2\gamma} \end{aligned} \tag{58}$$

Spot rates are

$$R(t, T) = -\frac{\log P(t, T)}{T - t} = \frac{-\log A_0 + rB}{T - t} \tag{59}$$

The Cox-Ingersoll-Ross model

To compute the instantaneous forward rate, we write $A_0 = \left(\frac{N}{D}\right)^c$

$$f(t,T) = -\frac{\partial log\, P(t,T)}{\partial T} = -\frac{\partial}{\partial T}\left(\log\, A_0 + rB_T\right) = -c\left(\frac{N_T}{N} - \frac{D_T}{D}\right) + rB_T \quad (60)$$

and compute

$$N_T = \gamma(a+\gamma)e^{(T-t)(a+\gamma)/2}, \quad D_T = \gamma(\gamma+a)e^{\gamma(T-t)} \quad (61)$$

Then we write $B = \frac{M}{D}$ and compute

$$B_T = \frac{M_T D - M D_T}{D^2} \quad (62)$$

and compute

$$M_T = 2\gamma e^{\gamma(T-t)} \quad (63)$$

The Ho-Lee model

The dynamics of the short rate in the Ho-Lee model is given by

$$dr_t = \Theta(t)dt + \sigma dW_t \tag{64}$$

The drift parameter $\Theta(t)$ is a function and the parameter space in the Ho-Lee model is infinite dimensional. Therefore, we are, unlike in the Vasicek and CIR mdoels, able to choose the function $\Theta(t)$ in the Ho-Lee model such that we get a perfect fit of the initial term structure.

The Ho/Lee model can be solved directly by integration to give us

$$r_T = r_t + \int_t^T \Theta(u)du + \sigma\left[W(T) - W(t)\right] \tag{65}$$

and the short rate is Gaussian with

$$r_T|\mathcal{F}_t \sim N\Big(r_t + \int_t^T \Theta(u)du, \sigma^2(T-t)\Big) \tag{66}$$

The Ho-Lee model

To be able to the Ho-Lee model perfectly to the initial term structure, we will need to look at zero coupon bond prices.

The Ho-Lee model possesses an affine term structure and we have that

$$p(t,T) = e^{A(t,T)-B(t,T)r} \tag{67}$$

where $A(t,T)$ and $B(t,T)$ solve

$$\begin{aligned} A_t(t,T) &= \Theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), & A(T,T) &= 0 \\ B_t(t,T) &= -1, & B(T,T) &= 0 \end{aligned} \tag{68}$$

The equations for $A(t,T)$ and $B(t,T)$ can easily be solved to give us that

$$\begin{aligned} A(t,T) &= \frac{\sigma^2}{2}\frac{(T-t)^3}{3} + \int_t^T \Theta(s)(s-T)ds \\ B(t,T) &= T-t \end{aligned} \tag{69}$$

The Ho-Lee model

To find the choice of $\Theta(t)$ that insures a perfect fit of the initial term structure, it will turn out that we have to not fit the initial term structure of zero coupon bond prices but rather fit the initial term structure of instantaneous forward rates denoted $\{f^*(0,t), t > 0\}$.

To do so, we must choose $\Theta(t)$ such that

$$\Theta(t) = \frac{\partial f^*(0,t)}{\partial T} + \sigma^2 t \tag{70}$$

where the derivative $\frac{\partial}{\partial T}$ is a derivative in the second variable of f^*. Zero coupon bond prices in the Ho-Lee model then become

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\left\{(T-t)f^*(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r\right\} \tag{71}$$

Hull-White extended Vasicek

The dynamics of the short rate in the Hull-White extended Vasicek is

$$dr_t = \big(\Theta(t) - ar_t\big)dt + \sigma dW_t \tag{72}$$

The extended Vasicek can be solved by applying Ito to $f(t, r_t) = e^{at} r_t$.

$$df(t, r_t) = ae^{at} r_t + e^{at}\big(\Theta(t) - ar_t\big)dt + e^{at}\sigma dW_t = e^{at}\Theta(t)dt + e^{at}\sigma dW_t \tag{73}$$

Integrating then gives us

$$\int_t^T df(u, r_u) = e^{aT} r_T - e^{at} r_t = \int_t^T e^{as}\Theta(s)ds + \sigma \int_t^T e^{as} dW_s \tag{74}$$

and solving for r_T gives us that

$$r_T = e^{-a(T-t)} r_t + \int_t^T e^{-a(T-s)}\Theta(s)ds + \sigma \int_t^T e^{-a(T-s)} dW_s \tag{75}$$

Hull-White extended Vasicek

The distribution of the short rate in the Hull-White extended Vasicek is

$$r_T|\mathcal{F}_t \sim N\left(e^{-a(T-t)}r_t + \int_t^T e^{-a(T-s)}\Theta(s)ds, \frac{\sigma^2}{2a}\left[1 - e^{-2a(T-t)}\right]\right) \tag{76}$$

and the stationary distribution of the short rate is

$$r_\infty \sim N\left(\frac{\Theta(\infty)}{a}, \frac{\sigma^2}{2a}\right) \tag{77}$$

where

$$\Theta(\infty) = \lim_{t \nearrow \infty} \Theta(t) \tag{78}$$

Hull-White extended Vasicek

To find zero coupon bond prices, we will use that the model allows for an ATS and hence

$$p(t, T) = e^{A(t,T) - B(t,T)r} \tag{79}$$

where $A(t, T)$ and $B(t, T)$ solve

$$\begin{aligned} A_t(t, T) &= \Theta(t)B(t, T) - \frac{1}{2}\sigma^2 B^2(t, T), \qquad && A(T, T) = 0 \\ B_t(t, T) &= aB(t, T) - 1, && B(T, T) = 0 \end{aligned} \tag{80}$$

Hull-White extended Vasicek

The solution for $A(t,T)$ and $B(t,t)$ are

$$A(t,T) = \int_t^T \left(\frac{1}{2}\sigma^2 B^2(s,T) - \Theta(s)B(s,T)\right) ds$$
$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \tag{81}$$

Now, we have not chosen $\Theta(t)$ yet but we will do so by fitting the model to observed forward rates and first have to find theoretical forward rates.

$$f(0,T) = -A_T(0,T) + rB(0,T) \tag{82}$$

Hull-White extended Vasicek

Computing the derivatives $A_T(0, T)$ and $B_T(0, T)$, inserting into (82) and solving for $\Theta(T)$ gives us that

$$\Theta(T) = f_T^*(0, T) + g_T(T) + a\big[f^*(0, T) + g(T)\big] \tag{83}$$

where

$$g(T) = \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 \tag{84}$$

Inserting the expression for $\Theta(s)$ from (83) into (80) and then inserting the resulting $A(t, T)$ and $b(t, T)$ into (79), we can compute ZCB prices

$$p(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp\left\{B(t, T)f^*(0, t) - \frac{\sigma^2}{4a}B^2(t, T)\big(1 - e^{-2at}\big) - B(t, T)r\right\} \tag{85}$$