Fixed Income Derivatives - The Hull-White Extended Vasicek Model

In this problem, we will consider the Hull-White Extended Vasicek Model model, in which the short rate r_t is assumed to have the following dynamics under the risk-neutral measure $\mathbb{Q}$

$$dr_t = \big(\Theta(t) - ar_t\big)dt + \sigma dW_t \tag{1}$$

where a, $\Theta(t)$ and σ are model parameters.

Problem 1 - Zero coupon bond prices in the Ho-Lee model

Let us recall that the Ho-Lee model possesses an affine term structure and that zero coupon bond prices are of the form

$$\begin{aligned}
P(t,T) &= e^{A(t,T)-B(t,T)r_t}, \\
A(t,T) &= \int_t^T \Big(\frac{1}{2}\sigma^2 B^2(s,T) - \Theta(s)B(s,T)\Big)ds, \\
B(t,T) &= \frac{1}{a}\Big[1 - e^{-a(T-t)}\Big].
\end{aligned} \tag{2}$$

Just as was the case with the Ho-Lee model, the HWEV model can by construction be fitted to the initial term structure and in particular, must be fitted to the observed term structure of forward rates $f^*(0,T)$ and their derivatives $f^*(0,T)$.

a) Show that the solution to the SDE in (1) can be written as

$$r_T = e^{-a(T-t)}r_t + \int_t^T e^{-a(T-s)}\Theta(s)ds + \sigma\int_t^T e^{-a(T-s)}dW_s \tag{3}$$

and use the explicit solution to find the distribution of $r_T|\mathcal{F}_t$.

b) Find an expression for forward rates in the HWEV model and use this this expression to show that fitting the model to observed market data involves setting

$$\Theta(t) = f_T^*(0,t) + g_t(t) + a\big[f^*(0,t) + g(t)\big], \tag{4}$$

where $f_T^*(0,t)$ is the derivative of f^* with respect to the second argument (the maturity T) evaluated at the point $(0,t)$ and where

$$g(t) = \frac{\sigma^2}{2a^2}\big(1 - e^{-at}\big)^2. \tag{5}$$

c) Use the expression you have found for $\Theta(t)$ along with (2) to show that given an initial fit at time 0, ZCB prices at time t are given by

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)}\exp\Big\{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)\big(1 - e^{-2at}\big) - B(t,T)r\Big\}. \tag{6}$$

d) Use the expression for ZCB prices in (2) to show that the dynamics of ZCB prices in the Ho-Lee model are

$$dP(t,T) = \alpha(t,T)P(t,T)dt + \beta(t,T)P(t,T)dW_t. \tag{7}$$

and determine $\alpha(t,T)$ and $\beta(t<T)$.

e) For u such that $t < u < T$ find a solution to (7).

f) Show that $\ln P(u,T)|\mathcal{F}_t$ follows a log-normal distribution and find the mean and variance of $\ln P(u,T)$.

Problem 2 - Option prices in the Hull-White model

Next, we will consider a European call option with time t price denoted $\Pi(t;T_1,T_2)$, strike K and exercise at T_1 on a maturity T_2 zero coupon bond with $t < T_1 < T_2$. Just as in the Ho-Lee model, the easiest way to derive a formula for the price of a European call option on $p(t,T_2)$ is to use a more general result on option pricing.

a) Show that the dynamics of forward rates are of the form

$$df(t,T) = -\frac{\partial}{\partial T}\Big(A_t(t,T)dt - B_t(t,T)r_t dt - B(t,T)dr_t\Big) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW_t \tag{8}$$

and determine $\alpha(t,T)$. Note that $\alpha(t,T)$ is not the same as the one in 1d). The point is however that the exact form of $\alpha(t,T)$ will not matter for option prices.

b) Find the distribution of forward rates and argue that they are Gaussian.

c) Finally use a result we have derived for European call option prices when forward rates follow a Gaussian distribution to show that

$$\Pi(t;T_1,T_2) = p(t,T_2)\Phi(d_1) - Kp(t,T_1)\Phi(d_22) \tag{9}$$

where

$$d_1 = \frac{\ln\left(\frac{p(t,T_2)}{Kp(t,T_1)}\right) + \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}}, \quad d_2 = \frac{\ln\left(\frac{p(t,T_2)}{Kp(t,T_1)}\right) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}} \tag{10}$$

and

$$\Sigma^2 = \frac{\sigma^2}{2a^3}\left[1 - e^{-2a(T_1-t)}\right]\cdot\left[1 - e^{-a(T_2-T_1)}\right]^2 \tag{11}$$

Now and in the coming problems, we will fit a HWEV model to observed market data and compute prices of caplets, a cap and a swaption. We will use the same data as in the Ho-Lee model example and the code used to produce the results will be very close to that of the Ho-Lee model example.

Problem 3 - Fitting the yield curve and the Vasicek model to data

Assume you have the following market data available and that the interest rate swaps pay 6M EURIBOR semiannually against a fixed rate paid annually.

EURIBOR	Fixing	FRA	Midquote	IRS	Midquote
6M	0.0430136	1X7	0.0455066	2Y	0.0558702
		2X8	0.0477436	3Y	0.058811
		3X9	0.0497492	4Y	0.0600937
		4X10	0.0515456	5Y	0.0605263
		5X11	0.0531529	7Y	0.0601899
		6X12	0.0545893	10Y	0.0586669
		7X13	0.0558712	15Y	0.0562267
		8X14	0.0570135	20Y	0.0547351
		9X15	0.0580298	30Y	0.0535523

a) Fit a ZCB spot rate curve to the market data and plot spot and forward rates from your choice of interpolation method. Also discuss if the spot and forward rates from your fit have the properties that we would like fitted spot- and forward rates to have,

b) Using the initial values $r_0 = 0.035$, $a = 0.5$, $b = 0.025$, $\sigma = 0.03$, fit a Vasicek model to the ZCB spot rates you have calibrated from market data and plot the fitted spot- and forward rates from your Vasicek model in a plot that also contains the spot and forward rates you found in a).

c) Using the initial values $r_0 = 0.035$, $a = 0.5$, $b = 0.05$, $\sigma = 0.1$, fit a Vasicek model to the ZCB spot rates you have calibrated from market data and plot the fitted spot- and forward rates from your CIR model in a plot that also contains the spot and forward rates you found in a).

d) You are likely to have found that the Vasicek and CIR model produce almost identical fits but that neither fit the market well. Explain why that is the case.

Problem 4 - Fitting a Hull-White Extended Vasicek model to data

In this problem, we will fit a HWEV model to our market data and you can assume that $a = 0.15$ and $\sigma = 0.01$.

a) Use the function '*interpolate.py*' to first find $\frac{\partial f^*(0,t)}{\partial T}$ and then find $\Theta(t)$.

b) Argue that the Hull-White Extended Vasicek model will by construction fit the initial term structure by appealing to (2).

c) Also argue that the neither a and σ are identified from the ZCB curve fit. That is, argue that the model will fit the data irrespective of the choice of a and σ. How to find a reasonable choice for a will be discussed later.

Problem 5 - Pricing an interest rate cap in the Hull-White Extended Vasicek model

We will now consider the pricing of an interest rate cap on a 10Y receiver swap in which the holder receives a an annual fixed rate in exchange for paying 6M Euribor on the floating leg. As in the previous question, you can assume that $a = 0.15$ $\sigma = 0.01$.

a) Explain the relationship between a caplet on an individual Euribor rate payment and also explain, how a caplet can be seen as a type of European option on a specific underlying asset.

b) Derive an explicit expression for the price of a European put option with the ZCB as the underlying asset.

c) Compute the prices of all caplets on 6M forward Euribor with a strike of $K = 0.06$ corresponding to the floating rate payments on the 10Y receiver swap and use these prices to find.

d) Compute the price of a 10Y interest rate cap with strike $K = 0.06$ on 6M Euribor and express the price of this cap both in terms of an upfront payment as well as a premium to be paid quarterly for 10 years.

e) Investigate how the price of the cap depends on σ and compute the DV01 of changing σ by 0.001 both up and down.

Problem 6 - Simulation of r_t and confidence intervals in the Hull-White Extended Vasicek model

Now we will simulate short rates in the HWEV model using the usual first order Euler scheme on a grid of mesh δ that runs from initial time t_0 to terminal time T. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$, the time points will be $\left[t_0, t_1, ..., t_{M-1}, t_M\right] = \left[0, \delta, 2\delta, ..., T-\delta, T\right]$ and $\delta = \frac{T}{M}$.

a) Develop an Euler scheme to simulate the short rate given $\Theta(t)$ expressed in terms of market observed forward rates.

b) To construct confidence intervals in the HWEV model, we run into the problem that the mean of $r_T|\mathcal{F}_0$ is given by

$$\mathbb{E}\left[r_T|\mathcal{F}_0\right] = e^{-aT}r_0 + \int_0^T e^{-a(T-s)}\Theta(s)ds \tag{12}$$

which is problematic since the integral can not immediately be computed. However, as we will see, a numerical solution can easily be implemented. Argue that the integral in (12) can be approximated by

$$I = \int_0^T e^{-a(T-s)}\Theta(s)ds \approx \frac{1}{2}e^{-aT}\sum_{m=1}^{M}\Big[e^{(m-1)\delta}\Theta\big((m-1)\delta\big) + e^{m\delta}\Theta\big(m\delta\big)\Big] \tag{13}$$

and use this to write a function in Python that return the mean of $r_T|\mathcal{F}_0$ for a vector of T's.

c) Plot simulated values of r_t for $t \in [0, 30]$ along with the mean of r_t, $\Theta(t)$ and a 95% two-sided confidence interval.

d) Assess what happens to the distribution of the short rate in the HWEV model when t grows large. Does the short rate mean-revert in this model and does it make sense to talk about the "asymptotic" properties of r_t in the HWEV model?

e) Redo the plot for the choice of a found when fitting the Vasicek model. What happens and what does that tell you about fitting a Vasicek model to find an appropriate choice for a in a HWEV model?

Problem 7 - Pricing a swaption in the Hull-White Extended Vasicek model

Finally, we will find the price of a 1Y10Y receiver swaption with a strike of $K = 0.06$. The receiver swaption gives the owner the right but not obligation to enter into an 4Y receiver swap at exercise in $T_n = 1$ years. To compute the price of this swaption, you will need to simulate the short rate.

a) Argue that the payoff function $\chi(T_n)$ and the discounted payoff function $\tilde{\chi}(T_n)$ of the payer swaption are.

$$\begin{aligned} \chi(T_n) &= S_n^N(T_n)\big(K - R_n^N(T_n)\big)_+ \\ \tilde{\chi}(T_n) &= \exp\Big\{-\int_0^{T_n} r_t dt\Big\} S_n^N(T_n)\big(K - R_n^N(T_n)\big)_+ \end{aligned} \tag{14}$$

b) Find a method to compute the price at $t = 0$ of the swaption by simulating at least $L = 1000$ trajectories and having at least $M = 1000$ steps in your simulation.

c) Investigate if the price you have computed is accurate by plotting the value of the derivative for various choices of L.

d) Explain how the price of the swaption depends on σ, T_n, T_N and of course K.

e) Finally, we will introduce a digital option that pays one unit of currency if the strike K exceeds the 10Y par swap rate at the time of exercise in 1Y. Find expression for the pay-off function $\chi(T_n)$ and the discounted pay-off function $\tilde{\chi}(T_n)$ in terms of the indicator function $\mathbb{1}$.

f) Compute the price of the digital option and give an interpretation of the price in terms of the behavior of the underlying 10Y par swap rate.