Fixed Income Derivatives - Vasicek Model Example

In this problem, we will consider the Vasicek model, in which the short rate r_t is assumed to have the following dynamics under the risk-neutral measure $\mathbb{Q}$

$$dr_t = (b - ar_t)dt + \sigma dW_t \tag{1}$$

where a, $b > 0$ and σ are model parameters.

Problem 1 - Solution and distribution of r_t

a) Find the solution for r_t, $t > 0$, in the Vasicek model.

b) Find the mean $\mathrm{E}[r_t]$ and the variance $\mathrm{Var}[r_t]$ of r_t. There is a 'trick' involved when computing the variance, what is that trick called?

c) Find the distribution of r_t.

d) Find the distribution of r_t as $t \nearrow \infty$. That is, find the stationary distribution of r_∞.

Problem 2 - Solution and distribution of $P(t,T)$

a) Argue that ZCB prices in the Vasicek model possess an affine term structure and are of the form $p(t,T) = F(t,T,r) = e^{A(t,T)-B(t,T)r}$.

b) Show that $A(t,T)$ and $B(t,T)$ solve the following system of ODE's

$$B_t = \ -1 + aB, \qquad B(T,T) = 0 \tag{2}$$

$$A_t = bB - \frac{1}{2}\sigma^2 B^2, \qquad A(T,T) = 0 \tag{3}$$

and solve this system to show that

$$A(t,T) = \frac{1}{4a^2}\Big(2\big[\sigma^2 - 2ab\big](T-t) + 2\big[2ab - \sigma^2\big]B(t,T) - a\sigma^2 B^2(t,T)\Big)$$
$$B(t,T) = \frac{1}{a}\Big[1 - e^{-a(T-t)}\Big]$$

c) Find the distribution of $\log P(t,T)|\mathcal{F}_0$.

d) Consider a European call option with strike K and maturity T_1 written on a ZCB with maturity T_2, $T_1 < T_2$. Argue why there must be a Black-Scholes type of formula for the price, Π of this call option and that it is given by

$$\Pi = p(0,T_2)\Phi(d_1) - Kp(0,T_1)\Phi(d_2) \tag{4}$$

where

$$d_1 = \frac{\ln\Big(\frac{p(0,T_2)}{Kp(0,T_1)}\Big) + \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}}, \quad d_2 = \frac{\ln\Big(\frac{p(0,T_2)}{Kp(0,T_1)}\Big) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}}, \quad \Sigma^2 = \frac{\sigma^2}{2a^3}\Big[1 - e^{-2aT_1}\Big]\cdot\Big[1 - e^{-a(T_2-T_1)}\Big]^2 \tag{5}$$

e) Use the put-call parity to show that the price Ψ of a European put option with same strike K, same maturity T_1 and same time T_2 ZCB as the underlying is given by

$$\Psi = p(0,T_1)K\Phi(-d_2) - p(0,T_2)\Phi(-d_1) \tag{6}$$

f) Show that the $t = 0$ price of a caplet on the Libor rate $L(T_{i-1}, T_i)$ with principal amount K and strike R can be expressed in terms of a European put option. In particular, show that the price of such a caplet can be found as the value of $K\big[1 + (T_i - T_{i-1})R\big]$ European put options with strike $K = \frac{1}{1+(T_i - T_{i-1})R}$.

Problem 3 - Fitting a Vasicek model to data

Assume you have fitted the ZCB yield curve and have gotten the results for continuously compounded spot rates and maturities shown in the file *vasicek_model_example.py*.

a) Using the initial values $r_0 = 0.03$, $a = 0.5$, $b = 0.04$ and $\sigma = 0.04$, fit a Vasicek model to the spot rates given above.

b) Plot the ZCB prices, spot rates and instantaneous forward rates for values of t in $[0, 10]$.

c) Assuming that the market can be described by the Vasicek model with the parameter estimates found in a), compute par swap rates for swaps exchanging three month floating Libor for a fixed rate paid annually. Compute swap rates for maturities $T \in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$ and include these swap rates in your plot from b). You will notice that par swap rates are very close to ZCB spot rates. Is that a 'coincidence' or is there a better explanation?

d) Compute 3M forward rates and plot these in a separate plot where you also plot the 10Y par swap rate as a straight line. What can be said about the relationship between 3M forward rates and the 10 par swap rate?

Problem 4 - Pricing an interest rate cap

We will now consider the pricing of an interest rate cap.

a) Explain the relationship between a caplet on an individual Libor rate payment and also explain, how a caplet can be seen as a type of European option on a specific underlying asset.

b) Compute the prices of all caplets on 3M forward Libor that corresponding to the 10Y receiver swap paying 3m Libor on the floating leg in exchange for an annual fixed rate.

c) Compute the price of a 10Y interest rate cap with strike $K = 0.05$ on 3M Libor and express the price of this cap both in terms of an upfront payment as well as a premium to be paid quarterly for 10 years.

d) Suppose a company has a floating rate loan in which the pay 3M Libor + 200 bps and that the company is afraid that 3M Libor will exceed 0.05 in the future. Explain how they can mitigate that risk using either the 10Y interest rate swap or the 10Y interest rate cap with a strike of 0.05 from the previous question. Explain the pros and cons of each type of agreement and also calculate the total cost to the company in both cases

d) Investigate how the price of the cap depends on σ and compute the DV01 of changing σ by 0.001 both up and down. Is the $DV01$ you have computed a first-order approximation or is it exact? Can both exact and approximated values be computed?

Problem 5 - Simulating the Vasicek model

Next we will simulate short rates in the Vasicek model using the usual first order Euler scheme on a grid of mesh δ that runs from initial time $t_0 = 0$ to terminal time $T = 10$. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$, the time points will be $[t_0, t_1, ..., t_{M-1}, t_M] = [0, \delta, 2\delta, ..., T-\delta, T]$ and $\delta = \frac{T}{M}$. The scheme you will need to implement is a simple Euler first-order scheme of the form

$$r_m = r_{m-1} + (b - ar_{m-1})\delta + \sigma\sqrt{\delta}Z_m, \quad m = 1, 2, ..., M \tag{7}$$

where $Z_m \sim N(0,1)$, $m = 1, ..., M$ and all the standard normal random variables are independent.

a) Simulate one trajectory of the short rate and plot the trajectory up to time $T = 10$.

b) Construct 95 percent two-sided confidence intervals for the short rate and plot these in the same plot.

c) Construct a 95 percent two-sided confidence interval for the short rate under the stationary distribution and plot this confidence interval in the same plot. Based on the plot, how large should t for us to say that the distribution of r_t is roughly the same as that of the stationary distribution? How does this change if you change the parameters of the Vasicek model?

d) Explain the inaccuracies of the proposed simulation scheme and if you can, suggest a different method that would be better.

Problem 6 - Pricing a swaption

We will now introduce a 2Y8Y payer swaption with a strike of $K = 0.05$. That is, we will introduce a swaption that gives the owner the right but not obligation to enter into an 8Y payer swap at exercise in $T_n = 2$ years so that $T_N = 10$. To compute the price of this swaption, you will need to use simulation.

a) Argue that the payoff function $\chi(T_n)$ and the discounted payoff function $\tilde{\chi}(T_n)$ of the payer swaption are.

$$\begin{aligned} \chi(T_n) &= S_n^N(T_n)\big(R_n^N(T_n) - K\big)_+ \\ \tilde{\chi}(T_n) &= \exp\Big\{ - \int_0^{T_n} r_t dt \Big\} S_n^N(T_n)\big(R_n^N(T_n) - K\big)_+ \end{aligned} \tag{8}$$

b) Find a method to compute the price at $t = 0$ of the swaption by simulating at least $L = 1000$ trajectories and having at least $M = 1000$ steps in your simulation.

c) Investigate if the price you have computed is accurate by plotting the value of the derivative for various choices of L.

d) Explain how the price of the swaption depends on σ, T_n, T_N and of course K.

e) Discuss if the payer swaption could potentially have helped the company from Problem 5 manage their interest rate risk.