Swap Market Models

Jacob Lundbeck Serup

October 30. 2023

Swap market models

We have now studied the LIBOR market model and seen how we were able to construct a model, by imposing dynamics on LIBOR rates in such a that

- Caplet prices within the model conform with the market practice of pricing caplets using the Black-76 formula.
- The LIBOR market model is well-defined and the individual dynamics of LIBOR rates are consistent with each other.

Recall that an interest rate cap can be used by an investor who has a floating rate obligation to hedge against future rises in interest rates.

An alternative way of hedging a floating rate liability is by using a so called swaption and in this section, we will develop a swap market model, much in the same way as we did the LIBOR market model.

Swap market models

To construct a swap market model, we will briefly revisit a basic interest rate swap and introduce some notation.

An interest rate swap is a contract by which one party exchanges a sequence of future fixed rate payments for a sequence of floating rate payments.

The stream of fixed rate payments is referred to as the fixed leg and the stream of floating rate payments are referred to as the floating leg.

The terminology of a swap refers to the fixed leg and hence a receiver swap is one where the holder of the swap receives the fixed rate payments and pays the floating rate payments.

Swaps - Definitions and market practice

We will consider a set of resettlement dates $T_0, ..., T_N$ and define α_i

$$\alpha_i = T_i - T_{i-1}, \quad i = 1, ..., N \tag{1}$$

The holder of a receiver swap with tenor $T_N - T_n$ will at $T_{n+1}, ..., T_N$ receive the payments on the fixed leg and pay the payments on the floating leg.

We will refer to this swap as a $T_n \times (T_N - T_n)$ swap.

Note that the frequency of fixed and floating rate payments might not be the same, in which case we can simply set some of the payments for some of the resettlement dates to zero.

Swaps - Definitions and market practice

Definition (Payer swap)

The payments in a $T_n \times (T_N - T_n)$ payer swap are as follows:

- *Payments will be exchanged at time $T_{n+1}, \ldots T_N$.*
- *For every elementary period $[T_{i-1}, T_i]$, $i = n+1, \ldots, N$ the LIBOR rate $L_i(T_{i-1})$ is set at time T_{i-1} and the floating leg payment*

$$\alpha_i \cdot L_i(T_{i-1})$$

  is received at time T_i.
- *In the same period, the fixed leg payment*

$$\alpha_i \cdot K$$

  is paid at T_i.

Swaps - Definitions and market practice

From our usual replication argument, we know that the time $t < T_n$ arbitrage free value of the floating rate payment is

$$p(t, T_{i-1}) - p(t, T_i) \tag{2}$$

The value at t of the floating leg will be a telescoping sum and given by

$$\sum_{n+1}^{N} p(t, T_{i-1}) - p(t, T_i) = p_n(t) - p_N(t) \tag{3}$$

The value at t of the fixed leg is

$$\sum_{i=n+1}^{N} p(t, T_i)\alpha_i K \tag{4}$$

The net value at t denoted $\mathbf{PS_n^N}(t; K)$ of the $T_n \times (T_N - T_n)$ payer swap is

$$\mathbf{PS_n^N}(t; K) = p_n(t) - p_N(t) - K \sum_{i=n+1}^{N} \alpha_i p_i(t) \tag{5}$$

Swaps - Definitions and market practice

Definition (Par swap rate)

The par or forward swap rate $R_n^N(t)$ *of the* $T_n \times (T_N - T_n)$ *swap is the value of* K *for which* $\mathbf{PS_n^N}(t;K) = 0$ *and is given by*

$$R_n^N(t) = \frac{p_n(t) - p_N(t)}{\sum_{i=n+1}^{N} \alpha_i p_i(t)}$$

The denominator of the par swap rate is the value of a portfolio consisting of zero coupon bonds and is thus itself a traded asset.

This portfolio will play such an important role in both understanding the nature of swaps as well as when pricing more complicated derivatives that we will give it a separate name.

Swaps - Definitions and market practice

Definition (The accrual factor)

For each pair n, k where $n < k$, the process $S_n^k(t)$ defined by

$$S_n^k(t) = \sum_{i=n+1}^{k} \alpha_i p(t, T_i)$$

is known as the accrual factor or as the present value of a basis point.

The accrual factor is always strictly positive.

Also note that the accrual factor is $\mathcal{F}_t$ adapted and hence that the accrual factor is know to us at time t.

Swaps - Definitions and market practice

The price of a payer swap can be expressed in terms of $R_n^N(t)$, K and $S_n^k(t)$.

$$\mathbf{PS_n^N}(t; K) = \Big(R_n^N(t) - K\Big) S_n^k(t) \tag{6}$$

In practice, market participants do not quote prices in monetary values but rather in terms of the par swap rate.

From (6), we see that the accrual factor tells us how much the payer swap increases in value for every basispoint the par swap rate R_n^N rises above the fixed rate K.

Conversely, the accrual factor also tells you how much the receiver swap decreases in value for every basispoint the par swap rate R_n^N rises above the fixed rate K.

Swaptions - Definitions and market practice

Definition (Swaptions)

A $T_n \times (T_N - T_n)$ *payer swaption with swaption strike* K *is a contract which at the exercise date* T_n *gives the holder the right but not the obligation to enter into a* $T_n \times (T_N - T_n)$ *swap with fixed rate* K*.*

The contract function for this contingent claim is therefore

$$\begin{aligned}\chi_n^N &= \max\left[\mathbf{PS_n^N}(T_n;K), 0\right] = \left(\mathbf{PS_n^N}(T_n;K)\right)_+ \\ &= \max\left[R_n^N(T_n) - K, 0\right] S_n^N(T_n) = S_n^N(T_n)\left(R_n^N(T_n) - K\right)_+\end{aligned}$$

Swaptions - Definitions and market practice

A swaption is in other words a call option on R_n^N with a strike price of K.

Since the accrual factor $S_n^N(t)$ is a traded asset, it is the obvious choice of numeraire when pricing swaptions and we will denote the martingale with respect to $S_n^N(t)$ by $\mathbb{Q}_n^N$.

Denoting the time t price of a swaption by $\mathbf{PSN}_n^N(t;K)$, we have that

$$\frac{\mathbf{PSN}_n^N(t;K)}{S_n^N(t)} = \mathbb{E}^{n,N}\left[\frac{S_n^N(T_n)\big(R_n^N(T_n)-K\big)_+}{S_n^N(T_n)}\middle|\mathcal{F}_t\right] \quad \Rightarrow$$

$$\mathbf{PSN_n^N}(t;K) = S_n^N(t)\mathbb{E}^{n,N}\left[\big(R_n^N(T_n)-K\big)_+\middle|\mathcal{F}_t\right] \tag{7}$$

Just as in the case of caplets, market practice is to quote prices of swaptions in terms of implied Black Volatilities

Black-76 swaption price

Definition (Black-76 swaption price)

The Black-76 formula for a $T_n \times (T_N - T_n)$ payer swaption with strike K that can be exercised at time T_n is

$$\mathbf{PSN}_n^N(t; K) = S_n^N(t)\left[R_n^N(t)\Phi(d_1) - K\Phi(d_2)\right] \tag{8}$$

where

$$d_1 = \frac{1}{\sigma_{n,N}\sqrt{T_n - t}}\left[\ln\left(\frac{R_n^N(t)}{K}\right) + \frac{1}{2}\sigma_{n,N}^2(T_n - t)\right]$$

$$d_2 = d_1 - \sigma_{n,N}\sqrt{T_n - t}$$

The constant $\sigma_{n,N}$ is known as the *Black Volatility*.

Black-76 swaption price

Given a market price for a swaption, the choice of $\sigma_{n,N}$ that insures equality between the market price and the price given by Black's formula is known as the Black Implied Volatility.

Exactly as was the case for interest rate caps, the market has used Black's formula and the Black implied volatility to quote prices of swaptions initially without having a theoretical justification for this practice.

Once again, our task is therefore to construct a theoretical model within which prices of swaptions are given by a formula of the Black type.

The numeraire we will use when constructing swap market models is, as mentioned above the accrual factor $S_n^N(t)$.

The swap market models

Lemma

Denote the martingale measure corresponding to the numeraire $S_n^k(t)$ by $\mathbb{Q}_n^k$, then under this measure, the forward swap rate $R_n^k(t)$ is a martingale

Proof:

Recall that the forward swap rate can be written as

$$R_n^k(t) = \frac{p_n(t) - p_k(t)}{\sum_{i=n+1}^{N} \alpha_i p_i(t)} = \frac{p_n(t) - p_k(t)}{S_n^k(t)} \tag{9}$$

In the numerator, we have the price of a traded asset and in the denominator we have $S_n^k(t)$ and recall that the martingale measure is such that the ratio of any asset scaled by $S_n^k(t)$ is a $\mathbb{Q}_n^k$ martingale and the result follows. $\square$

Swap market models

To construct a valid swap market model the idea is now, exactly as when we constructed the LIBOR market, to impose dynamics on $R_n^k(t)$ such that the log of $R_n^k(t)$ is a Gaussian process under $\mathbb{Q}_n^k$.

Once we have imposed such dynamics on $R_n^k(t)$ for k's such that $n < k \leq N$, we will reconcile the dynamics of the par swap rates under *one* common measure and find the dynamics under this new measure implying that $R_n^k(t)$ is a martingale under $\mathbb{Q}_n^k$.

Again, we will begin this process by postulating the model and then verify that the model is indeed valid.

Swap market models

Definition

Take as given the resettlement dates $T_0, T_1, ..., T$ and a fixed subset $\mathcal{N}$ of all pairs of positive integers (n, k) such that $0 \leq n < k \leq N$. Also, define for each pair $(n, k) \in \mathcal{N}$, a deterministic function of time $\sigma_{n,k}(t)$. A swap market model with deterministic volatilities $\boldsymbol{\sigma}_{n,k}(t)$ is then specified by assuming that the par swap rates have dynamics of the following form.

$$dR_n^k(t) = R_n^k(t)\boldsymbol{\sigma}'_{n,k}(t)d\mathbf{W}_n^k(t), \quad (n, k) \in \mathcal{N} \tag{10}$$

where $\mathbf{W}_n^k(t)$ is a possibly multidimensional Brownian motion under $\mathbb{Q}_n^k$.

Swap market models

The par swap rates are functions of zero coupon bond prices and it is not possible to simultaneously model all par swap rates R_n^k for $0 \leq n < k \leq N$ and we have to restrict ourselves to a subset $\mathcal{N}$ of the par swap rates.

In a model with $N - n + 1$ maturity dates for $0 \leq n \leq N$, we can at most model $N - n$ independent swap rates.

- A *Regular Swap Market Model* is specified by modeling for fixed n and N the par swap rates:

$$R_n^N, R_{n+1}^N, \ldots, R_{N-1}^N \text{ and } \mathcal{N} = \{(n, N), (n+1, N), \ldots, (N-1, N)\} \tag{11}$$

- A *Reverse Swap Market Model* is specified by modeling for fixed n and N the par swap rates:

$$R_n^{n+1}, R_n^{n+2}, \ldots, R_n^N \text{ and } \mathcal{N} = \{(n, n+1), \ldots, (n, N)\} \tag{12}$$

Swaption pricing in a swap market model

We already have a general model-independent expression for the price of a swaption from (7) which is restated below

$$\mathbf{PSN_n^N}(t;K) = S_n^N(t)\mathbb{E}^{n,N}\left[\left(R_n^N(T_n)-K\right)_+\middle|\mathcal{F}_t\right] \tag{13}$$

To evaluate the expectation, we need to find the distribution of $R_n^N(T_n)|\mathcal{F}_t$.

This is however not difficult as we are dealing with a Brownian motion and

$$R_n^N(T_n) = R_n^N(t)\cdot\exp\left\{-\frac{1}{2}\int_t^{T_n}||\boldsymbol{\sigma}_{n,N}(s)||^2 ds + \int_t^{T_n}\boldsymbol{\sigma}'_{n,N}(s)d\mathbf{W}_n^N(s)\right\} \tag{14}$$

Since $\boldsymbol{\sigma}_{n,N}(t)$ is deterministic, $\log R_n^N(T_n)$ is Gaussian and we can write

$$R_n^N(T_n) = R_n^N(t)e^{Y_n^N(t)} \tag{15}$$

where $Y_n^N(t)$ is Gaussian random variable.

Swaption pricing in a swap market model

In particular, we have that

$$Y_n^N(t) \sim N\Big(M_n^N(t), \Sigma_{n,N}^2(t)\Big) \tag{16}$$

where

$$\begin{aligned} M_n^N(t) &= -\frac{1}{2}\int_t^{T_n} ||\boldsymbol{\sigma}_{n,N}(s)||^2 ds \\ \Sigma_{n,N}^2(t) &= \int_t^{T_n} ||\boldsymbol{\sigma}_{n,N}(s)||^2 ds \end{aligned} \tag{17}$$

We now have all the ingredients we need and we are in position to give the formula for a swaption in a swap market model noting that the formula is indeed of the Black type.

Swaption pricing in a swap market model

Proposition

In the swap market model, the par swap rate $R_n^N(t)$ has dynamics

$$dR_n^N(t) = R_n^N(t)\boldsymbol{\sigma}'_{n,N}(t)d\mathbf{W}_n^N(t)$$

under $\mathbb{Q}_n^N$ and the price of a $T_n \times (T_N - T_n)$ payer swaption with strike price K and maturity T_n is given by

$$\mathbf{PSN}_n^N(t;K) = S_n^N(t)\left[R_n^N(t)\Phi(d_1) - K\Phi(d_2)\right] \tag{18}$$

where

$$d_1 = \frac{1}{\Sigma_{n,N}(t)}\left[\ln\left(\frac{R_n^N(t)}{K}\right) + \frac{1}{2}\Sigma_{n,N}^2(t)\right], \qquad d_2 = d_1 - \Sigma_{n,N}(t)$$

$$\Sigma_{n,N}(t) = \int_t^{T_n} ||\boldsymbol{\sigma}_{n,N}(s)||^2 ds \tag{19}$$

Swap market models

The Browniam motion driving par swap rates is multi-dimensional but just as in the LIBOR market model, we can also construct exactly the same model by allowing the Brownian motins to be scalar valued but correlated

$$dR_n^N(t) = R_n^N(t)\sigma_{n,N}(t)dW_n^N(t) \tag{20}$$

where $\sigma_{n,N}(t)$ is a scalar deterministic function and $W_n^N(t)$ is a scalar $\mathbb{Q}_n^N$ Brownian motion.

The swaption price formula still holds in the scalar case with $||\boldsymbol{\sigma}_{n,N}||^2$ replaced by $\sigma_{n,N}^2$.

When pricing more exotic derivatives, it will often be necessary to simulate several par swap rates simultaneously under the same measure so therefore, we will find a common measure under which to specify the dynamics of *all* par swap rates.

Drift conditions in the Regular swap market models

In a regular swap market model, we model $R_n^N(t)$ for some fixed N and all n where $0 \leq n \leq N-1$ and the natural choice of a common measure under which to specify the dynamics of *all* par swap rates is seemingly $\mathbb{Q}_{N-1}^N$ which is simply the T_N forward measure $\mathbb{Q}^N$.

However, it turns out that working under the measure $\mathbb{Q}_N^{N+1}$ is more convenient and we will proceed to model $R_0^{N+1}, R_1^{N+1}, \ldots, R_N^{N+1}$ under $\mathbb{Q}_N^{N+1}$.

The task at hand is now to find the drift terms μ_n^{N+1} such that the par swap rates R_n^{N+1} are still martingales under their own measures if we were to switch from the new measure $\mathbb{Q}_N^{N+1}$ back to the old measures $\mathbb{Q}_n^{N+1}$.

Drift conditions in the Regular swap market models

We are now working under the new measure $\mathbb{Q}_N^{N+1}$ but we remember that the diffusion term is unaffected by a change of measure and the $\mathbb{Q}_N^{N+1}$ dynamics of $R_n^{N+1}(t)$ is given by

$$dR_n^{N+1}(t) = \mu_n^{N+1}(t)R_n^{N+1}(t)dt + R_n^{N+1}(t)\sigma'_{n,N+1}(t)d\mathbf{W}_M^{N+1}(t) \tag{21}$$

where $W_M^{N+1}(t)$ is a $\mathbb{Q}_N^{N+1}$ Brownian motion and the drift $\mu_N^{N+1}(t)$ must be chosen carefully to be consistent with the original specification of $R_n^{N+1}(t)$ under $\mathbb{Q}_n^{N+1}$.

It turns out that the drift must be chosen in accordance with the following proposition.

Drift conditions in the Regular swap market models

Proposition

Under the terminal measure $\mathbb{Q}_N^{N+1}$, the dynamics of $R_n^{N+1}(t)$ are of the form

$$dR_n^{N+1}(t) = \mu_n^{N+1}(t)R_n^{N+1}(t)dt + R_n^{N+1}(t)\sigma'_{n,N+1}(t)d\mathbf{W}_M^{N+1}(t)$$

where the drift is given by

$$\mu_N^{N+1}(t) = -\boldsymbol{\sigma}'_{n,N+1}(t)\boldsymbol{\varphi}_n^{N+1}(t)$$

and

$$\boldsymbol{\varphi}_n^{N+1}(t) = \sum_{j=n}^{N-1} \frac{S_{j+1}^{N+1}(t)}{S_n^{N+1}(t)} \left[\prod_{k=n+1}^{j} \left(1 + \alpha_k R_k^{N+1}(t)\right) \right] \alpha_{j+1} R_{j+1}^{N+1}(t)\boldsymbol{\sigma}_{n,N+1}(t)$$