# Fixed Income Derivatives - The Libor Market Model

In this problem, we will construct a Libor market model and use this model to compute the price of a complicated derivative. In the table below, you will find ZCB spot rates $R(0,T_1)$ deduced from market data as well as spot caplet implied volatilies corresponding to caplets on one year Libor with a strike of $K = 0.045$.

| $T_i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $R(0,T_i)$ | 0.03731561 | 0.04106146 | 0.04370112 | 0.04556125 | 0.04687207 |
| $\bar{\sigma}_i$ | - | 0.041 | 0.052 | 0.065 | 0.083 |

The spot implied volatility $\bar{\sigma}_i = \bar{\sigma}(0; T_{i-1}, T_i)$ is the implied volatility of a call option on 1Y Libor announced at time $T_{i-1}$ and paid at time $T_i$.

**Problem 1 - Pricing an interest rate cap from market prices**

a) Plot one year forward Libor rates and plot these along with the spot rates given above.

b) Compute the price of an interest cap on one year Libor that starts today, has a strike of $K = 0.045$ and runs for 5 years. Report the price in terms of an amount in bps paid upfront and also as an annualized premium paid periodically in the future at times $T = 1, 2, 3, 4, 5$.

c) Compute the price of the cap if it is to only cover future Libor rates announced from $T_{i-1} = 2$ and beyond. Report you answer both as an upfront payment and as an amount paid periodically from time $T = 3$.

A client has contacted you because he wants a quote on an interest rate cap of the same type as the one in problem 1b) but with a strike of 0.0475. However, there are no caplets available for that strike but there are caplets available for a strike of $K = 0.05$ and they trade at the following implied volatilities

| $T_i$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\bar{\sigma}_i$ | - | 0.045 | 0.057 | 0.073 | 0.096 |

d) Compute the price of an interest cap on one year Libor that starts today, has a strike of $K = 0.05$ and runs for 5 years. Report the price in terms of an amount in bps paid upfront and also as an annualized premium paid periodically in the future at times $T = 1, 2, 3, 4, 5$.

**Problem 2 - The Libor Market Model**

To quote a price on an interest rate cap with a strike of 0.0475, you could of course take some sort of average between the price in 1b) and the price in 1c) but that is rarely how the problem is solved in practice when more complicated derivatives on Libor or other reference rates for which caplets exist. Instead, market participants typically construct a so called Libor market model. In the following we will consider a set of Libor fixings occuring at $T_0, T_1, ..., T_{N-1}$ corresponding to Libor rate payments at $T_1, ..., T_{]msN}$ and denote the Libor rate $L(t; T_{i-1}, T_i)$ by $L_i(t)$. Also, we will denote the ZCB paying one dollar at time $T_i$ by $p(t, T_i)$.

a) Write the dynamics of the Libor rates $L_i(t)$ when driven by a $K$-dimensional Brownian motion $\mathbf{W}_t^i$ under the $T_i$ forward measure $\mathbb{Q}^i$. Discuss why $L_i(t)$ must be a martingale under $\mathbb{Q}^i$ and also discuss what happens to $L_i(t)$ after time $T_{i-1}$.

b) Impose conditions on the diffusion coeeficients so as to insure that the Libor model is consistent with observed caplet prices.

c) Now give the dynamics of $L_i(t)$ under the unifying measure $\mathbb{Q}^N$ and specify the drift of each of the Libor rates under this new measure.

d) Argue that the Libor market model can also be constructed such that the diffusion coefficients are scalar valued but the Brownian motions driving the Libor rates are correlated.

**Problem 3 - Constructing a Libor market model and using it to price a cap**

In the Libor market model we wish to construct, we have that $T_N = T_5 = 5$ and we will model the Libor rates $L_1(t)$, $L_2(t)$, $L_3(t)$ and $L_4(t)$. In this version of the Libor market model, we will simply assume that the diffusion coefficients are constant so the diffusion coefficient of $L_i(t)$ is the constant $\sigma_i$. Also, assume that you wish the correlation matrix of the Libor rates in your model to be given by the matrix $\mathbf{\Omega}$, where

$$\mathbf{\Omega} = \begin{bmatrix} 1 & 0.95 & 0.9 & 0.85 \\ 0.95 & 1 & 0.95 & 0.9 \\ 0.9 & 0.95 & 1 & 0.95 \\ 0.85 & 0.9 & 0.95 & 1 \end{bmatrix} \tag{1}$$

a) Use the implied volatilities from problem 1b) to find a vector of diffusion coefficients that we must use in our Libor market model for our model to be consistent with the market observed implied volatilities in 1b).

b) Discuss how to create Brownian motion with correlation matrix $\mathbf{\Omega}$.

c) Write a function in Python that computes the drift of the Libor rates in the Libor market model and use this function to write another function that will allow you to simulate Libor rates in the Libor market model. Plot a simulated trajectory for the Libor rates in your model.

d) Compute the price of the interest rate cap with a strike of $K = 0.0475$. Be sure to 'discount' the cashflow at exercise correctly remembering that we are simulating under $\mathbb{Q}^5$. Discuss if it is reasonable to use the market implied volatilities from 1b. Could other choices of implied volatilities have been justified and finally discuss the use of the Libor Market model when caplet prices are available for several strikes.