Fixed Income Derivatives E2024 - Problem Set Week 1

Problem 1

In this problem, we will examine an interest rate payer swap where payments to the floating leg are more frequent than payments to the fixed leg. Assume that we consider an interest rate swap that pays coupons of size R to the fixed leg I times a year, pays simple LIBOR rate coupons to the floating leg J times a year, matures in exactly N years and has a principal of K. Also assume that $M = {}^{J}/_{I}$ and N are positive integers. If it helps, you can think of $I = 2$, $J = 4$ and $N = 10$. Initially, we will consider the interest rate swap at time of issuance at $t = T_0 = 0$. Denote the times of coupon payments to the floating leg by $T_1, T_2, ..., T_{JN}$ corresponding to fixings announced at times $T_0, T_1, ..., T_{JN-1}$ and denote the LIBOR rate announced at time T_i to be paid at time T_{i+1} by $L(T_i, T_{i+1})$. Finally, we assume that zero coupon bond prices $p(t,T)$ are available for all $T \geq 0$.

a) Find an expression for the value of the fixed leg.

b) Find an expression for the value of the floating leg.

c) Find an expression for the par swap rate and identify for each of the elements of the expression, if they originate from the fixed or the floating leg.

d) Find an expression for the accrual factor and identify for each of the elements of the expression, if they originate from the fixed or the floating leg.

e) Compare the par swap rate and the accrual factor you found to the case where coupons to both the fixed and floating legs occur at the same time.

Assume now that some time has passed and that t is in-between coupon dates on the floating leg. Specifically assume that a little more than one year has passed so that $T_J < t < T_{J+1}$. The question is now slightly complicated by the fact that the fixing at time T_J has already been announced and that the coupon $L(T_J, T_{J+1})$ to the floating leg at time T_{J+1} is already known. In this case

f) Find an expression for the value of the fixed leg.

g) Find an expression for the value of the floating leg.

h) Find an expression for the par swap rate and the accrual factor.

i) Find an expression for the PnL of the payer swap. Does the expression for the PnL depend on $L(T_J, T_{J+1})$?

Solution

a) The present value of the fixed leg ignoring repayment of the principal is

$$PV_{fixed} = \sum_{i=1}^{IN} p(t, T_{iM})\big(T_{iM} - T_{(i-1)M}\big)RK = S_0 RK. \tag{1}$$

b) The present value of the floating leg ignoring repayment of the principal is

$$\begin{aligned} PV_{float} &= \sum_{i=1}^{JN} p(t, T_i)\big(T_i - T_{i-1}\big)L(T_{i-1}, T_i)K = \sum_{i=1}^{JN} K\big[p(t, T_{i-1}) - p(t, T_i)\big] = K\big[p(t, T_0) - p(t, T_{JN})\big] \\ &= K \times \big(\text{ZCB price } T = \text{time of first fixing after t}\big) - K \times \big(\text{ZCB price } T = \text{time of last cash flow floating leg}\big) \end{aligned} \tag{2}$$

c) The par swap rate then can be found by setting $PV_{float} - PV_{fixed} = 0$

$$R_0^* = \frac{p(t, T_0) - p(t, T_{JN})}{\sum_{i=1}^{IN} p(t, T_{iM})\big(T_{iM} - T_{(i-1)M}\big)}. \tag{3}$$

d) The accrual factor, S_0, is the denominator of the par swap rate and comes solely from the fixed leg

$$S_0 = \sum_{i=1}^{IN} p(t, T_{iM})\big(T_{iM} - T_{(i-1)M}\big). \tag{4}$$

e) The par swap rate and the accrual factors are essentially the same whether or not the cash flows to the fixed and floating legs occur simultaneously or not.

f) Now that time has passed, the first payment to the fixed leg occurs at time $J + M$ and the values of the floating leg becomes

$$PV_{fixed} = \sum_{i=J/M+1}^{IN} p(t, T_{iM})\big(T_{iM} - T_{(i-1)M}\big)RK = S_t RK. \tag{5}$$

g) The floating leg is now a bit different since we have to take into account the fixing and corresponding cash flow announced at time T_J.

$$\begin{aligned}
PV_{float} = &\sum_{i=J+1}^{JN} p(t, T_i)\big(T_i - T_{i-1}\big)L(T_{i-1}, T_i)K = p(t, T_{J+1})\big(T_{J+1} - T_J\big)L(T_J, T_{J+1})K + \sum_{i=J+2}^{JN} K\big[p(t, T_{i-1}) - p(t, T_i)\big] \\
= &\, p(t, T_{J+1})\big(T_{J+1} - T_J\big)L(T_J, T_{J+1})K + K\big[p(t, T_{J+1}) - p(t, T_{JN})\big] \\
= &\, K\big(T_{J+1} - T_J\big) \times (\text{ZCB price of floating coupon received at } T_{J+1}) \times (\text{Floating coupon received at } T_{J+1}) + \\
&K \times (\text{ZCB price } T = \text{time of first fixing after t}) - K \times (\text{ZCB price } T = \text{time of last cash flow floating leg})
\end{aligned} \tag{6}$$

h) The par swap rate is yet again found be setting $PV_{float} - PV_{fixed} = 0$ but now also accounts for the floating coupon received at time T_{J+1}

$$R_t^* = \frac{p(t, T_{J+1})\big(T_{J+1} - T_J\big)L(T_J, T_{J+1}) + p(t, T_{J+1}) - p(t, T_{JN})}{\sum_{i=J/M+1}^{IN} p(t, T_{iM})\big(T_{iM} - T_{(i-1)M}\big)} \tag{7}$$

The accrual factor S_t still depends solely on the fixed leg and stems from the denominator of the par swap rate

$$S_t = \sum_{i=J/M+1}^{IN} p(t, T_{iM})\big(T_{iM} - T_{(i-1)M}\big) \tag{8}$$

i) The PnL of the payer swap becomes

$$\begin{aligned}
PnL = &\, p(t, T_{J+1})\big(T_{J+1} - T_J\big)L(T_J, T_{J+1})K + K\big[p(t, T_{J+1}) - p(t, T_{JN})\big] - \sum_{i=J/M+1}^{IN} p(t, T_{iM})\big(T_{iM} - T_{(i-1)M}\big)R_0^* K \\
= &\, S_t K\big(R_t^* - R_0^*\big)
\end{aligned} \tag{9}$$

The PnL of the payer swap depends on the principal K, the accrual factor S_t and the change in the par swap rate. *But* the PnL does not depend on the first cash flow to the floating leg that was decided at time $T_J < t$ and received at time T_{J+1}. This means that to compute the PnL of a an interest rate swap, all we need is the par swap at time of issuance, $t = 0$, the accrual factor at time t and the par swap rate at time t.

Problem 2

For this problem, assume for simplicity that the year consists of 12 months each with exactly 30 days and that all payments occur at the end of day on the last day of the month. Also assume that there is no credit risk and that the principal of all bonds is 100.

The date today is December 30. 2019, the last day of the year, and the BBA have just at 11 AM announced the 3M LIBOR fixing to be 0.01570161 and the 6M LIBOR fixing to be 0.01980204. In addition, the following bonds are traded in the market.

i) A 3 year fixed rate bullet bond maturing December 30. 2020 paying quarterly simple coupons of 4% annually and a price of 102.33689177.

ii) A 5 year fixed rate bullet bond maturing December 30. 2020 paying semi-annual simple coupons of 5% annually and a price of 104.80430234.

iii) A 10 year fixed rate bullet bond maturing June 30. 2021 paying semi-annual simple coupons of 5% annually and a price of 105.1615306.

iv) An 8 year fixed rate bullet bond maturing June 30. 2021 paying quarterly simple coupons of 6% annually and a price of 105.6581905.

v) A 5 year fixed rate bullet bond maturing December 30. 2021 paying quarterly simple coupons of 5% annually and a price of 104.028999992.

vi) A 30 year fixed rate bullet bond maturing December 30. 2021 paying annual simple coupons of 3% annually and a price of 101.82604116.

Given this information, please solve the following problems

a) Set up the cashflow matrix corresponding to this information.

b) Find the vector of zero coupon prices for all the times that you can based on the above information and find the term structure of continuously compounded zero coupon spot rates (the yield curve). Report the results and plot both curves in an appropriate diagram.

c) Find 3M forward rates and plot these in the diagram from *b*).

d) Find the price of a 2 year floating rate bullet note with principal 100 paying 6M LIBOR issued today.

e) Find the par swap rate for a 2-year interest rate swap paying semi-annual fixed coupons at annual rate R to the 'receiver' and quarterly 3M LIBOR to the 'payer' issued today.

f) Compare the par swap rate to the forward rates you computed in c).

Time now passes and the date becomes January 30. 2020. From a god friend you now know that the price of zero coupon bond maturing on March 30. 2020 is 0.99699147 and that the price of a zero coupon bond maturing on June 30. 2020 is 0.99088748. Also assume that the two corresponding ZCB bonds can be constructed and traded. Due to market fluctuations, the prices of the bonds i)-vi) are now $[101.37241234, 102.33995192, 102.66601781, 104.16399942, 102.75471174, 98.79916103]$ and the price of the receiver swap from e) is now -0.1161878302683732.

g) Set up a system of equations including the interest rate swap to compute zero coupon bond prices. You should now have a system with more equations than unknowns.

h) Solve this system of equations to find zero coupon bond prices, zero coupon bond rates and 3M forward rates as you did in *b*) and plot these. Hint: If $\mathbf{C}$ is an $N \times M$-dimensional matrix where $N > M$ but $\mathbf{C}$ has full rank M, what is the rank of $\mathbf{C}'\mathbf{C}$?

i) You are now in a situation where you have more assets than future states and increasing the risk of arbitrage. Check that the market is arbitrage free.

Now instead assume that the price of the zero coupon bond maturing on March 30. 2020 is 0.99391543, that the price of a zero coupon bond maturing on June 30. 2020 is 0.98379379, that the prices of the 6 bonds are $[100.00015573, 100.95055325, 100.77535024, 100.26763545, 100.48419302, 96.56064083]$ and that the price of the receiver swap is -2.04869321.

j) Use OLS to estimate the zero coupon bond prices from market data.

k) Is the market arbitrage free? Try to find an arbitrage opportunity.

Solution

a) Since there are coupons paid immediately, we will write the cash flow matrix corresponding to times $\mathbf{T}' = [0, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00]$ for the six bonds as follows

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 1 & 1 & 101 & 0 & 0 & 0 & 0 \\ 2.5 & 0 & 2.5 & 0 & 102.5 & 0 & 0 & 0 & 0 \\ 2.5 & 0 & 2.5 & 0 & 2.5 & 0 & 102.5 & 0 & 0 \\ 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 101.5 & 0 & 0 \\ 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 101.25 \\ 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 103 \end{bmatrix} \tag{10}$$

The vector of prices corresponding to the cash flow matrix is then

$$\boldsymbol{\pi}' = [102.33689177, 104.80430234, 105.1615306, 105.6581905, 104.02899992, 101.82604116]. \tag{11}$$

The LIBOR fixings and a row for present time can also be included as a rows in the cash flow matrix by adding the rows

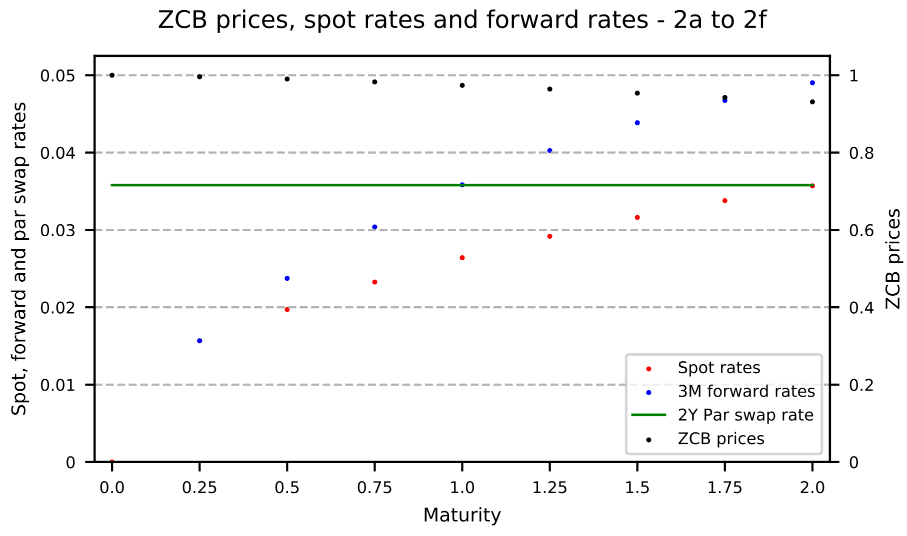
$$\mathbf{C}_{LIBOR} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 + \frac{0.01570161}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 + \frac{0.01980204}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

'on top of' $\mathbf{C}$ and adding three ones as the first three elements of $\boldsymbol{\pi}$. These rows are strictly speaking not necessary. What matters is that you can set up a system of equations $\boldsymbol{\pi} = \mathbf{C} \cdot \mathbf{p}$ that allows you to solve for the vector of ZCB prices $p(t, T)$ corresponding to all T's in $\mathbf{T}'$ above.

bc) The vectors of ZCB prices $\mathbf{p}$, continuously compounded spot rates $\mathbf{R}$ and continuously compounded 3M forward rates $\mathbf{f}$ become

$$\begin{aligned} \mathbf{p}' &= [1, 0.99608995, 0.99019605, 0.98270301, 0.97393963, 0.96418421, 0.95367016, 0.94259165, 0.93110895], \\ \mathbf{R}' &= [-, 0.01567087, 0.01970465, 0.02326445, 0.02640596, 0.02917833, 0.03162494, 0.03378407, 0.03568949], \\ \mathbf{f}' &= [-, 0.01567087, 0.02373843, 0.03038404, 0.03583049, 0.04026782, 0.04385799, 0.04673883, 0.04902745]. \end{aligned}$$

Plots of these are shown below.



d) The price of a newly issued floating rate bond will always be equal to the principal, here $K = 100$.

e) The 2Y swap in question has a par swap rate of $R_{2Y} = 0.035797646654281326$.

f) The par swap rate is a weighted average of the forward rates on the floating leg which in this case is the 3M forward rates computed and shown in the plot above.

g) Now that time has passed, the vector of times becomes $\mathbf{T} = \left[{}^{2}/_{12}, {}^{5}/_{12}, {}^{8}/_{12}, {}^{11}/_{12}, {}^{14}/_{12}, {}^{17}/_{12}, {}^{20}/_{12}, {}^{23}/_{12}\right]$. In the following, it will be helpful to include rows corresponding to the ZCB's and to find the net cash flow to the swap, we will need the 3M LIBOR fixing and the par swap rate when the swap was issued and the cash flow matrix including the 2Y interest rate swap now becomes

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 101 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 102.5 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 2.5 & 0 & 102.5 & 0 & 0 \\ 1.5 & 1.5 & 1.5 & 1.5 & 1.5 & 101.5 & 0 & 0 \\ 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 1.25 & 101.25 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 103 \\ -100.39254019 & 1.78988233 & 0 & 1.78988233 & 0 & 1.78988233 & 0 & 101.78988233 \end{bmatrix}$$

Notice that the -100.39254019 for $T = {}^{2}/_{12}$ and the $+100$ for $T = {}^{23}/_{12}$ originate from the floating leg. The corresponding vector of prices becomes

$$\begin{aligned} \boldsymbol{\pi}' = \big[&0.99699147, 0.99088748, 101.37241234, 102.33995192, 102.66601781, 104.16399942, \\ &102.75471174, 98.79916103, -0.1161878302683732\big] \end{aligned} \tag{13}$$

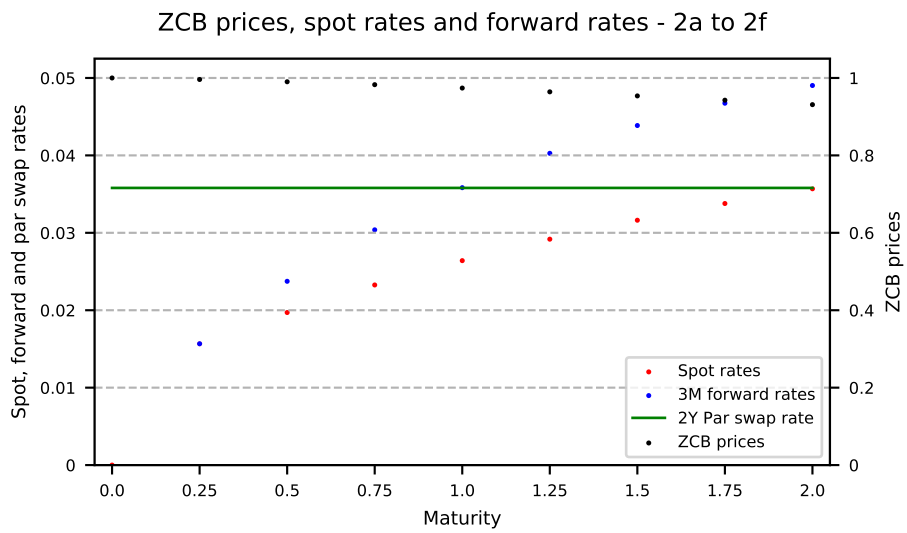
and the system of equations to be solved is again $\boldsymbol{\pi} = \mathbf{C} \cdot \mathbf{p}$.

h) The ZCB prices now have to be 'estimated' by a $\hat{\mathbf{p}}$ given by

$$\hat{\mathbf{p}} = \left(\mathbf{C}'\mathbf{C}\right)^{-1}\mathbf{C}'\boldsymbol{\pi}. \tag{14}$$

and the ZCB prices, spot rates and forward rates become

$$\begin{aligned} \hat{\mathbf{p}}' &= \left[0.99699147, 0.99088748, 0.98320604, 0.97427057, 0.96435494, 0.953689, 0.94246411, 0.93083834\right], \\ \hat{\mathbf{R}}' &= \left[0.0180784, 0.02197029, 0.02540487, 0.02843588, 0.03111073, 0.03347129, 0.03555446, 0.03739286\right], \\ \hat{\mathbf{f}}' &= \left[0.0180784, 0.02456489, 0.03112917, 0.03651857, 0.04091853, 0.04448719, 0.04735914, 0.04964886\right]. \end{aligned}$$



i) Since we are now in a situation where we have more assets and thus restrictions than we have free parameters, and the ZCB prices have to be estimated. This is certainly not implausible in practice and if the market is well-behaved, we should be able to recover the unique ZCB prices used to price the assets. In this case, we see that ZCB prices are positive and since the residuals $\boldsymbol{\pi} - \mathbf{C}\hat{\mathbf{p}}$ are small, we conclude that the market is arbitrage free.

j) Estimating ZCB prices now gives us that

$$\hat{\mathbf{p}}' = \left[0.99391543, 0.98379379, 0.97271045, 0.96088848, -0.38481897, 0.93574287, 2.25603492, 0.90949491\right]. \tag{15}$$

k) We can see that one of the estimated ZCB prices is negative suggesting that there is indeed an arbitrage in the market. An arbitrage can be found in many ways but it is tempting to try to create a portfolio that replicates the ZCB that has a negative price. It turns out that the portfolio

$$\mathbf{h}' = \left[-35.00001891, -0.20972039, -1, 0.98756673, -0.6540747, 0.66666667, 0, 0.3445357, -0.34863167\right]$$

has a cash flow of $\left[0, 0, 0, 0, 1, 0, 0, 0\right]$ and a price of -0.3848189 indeed representing an arbitrage. Note that this portfolio is not very accurately computed and you might get something that looks different. So long as the portfolio you get replicates the ZCB and has roughly the correct price, it should be fine.