Forward Rate Models

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The Heath-Jarrow-Morton Model

We have now studied short rate models in great detail and seen how this class of models allow us great flexibility when it comes to the dynamics of the very front end of the yield curve.

However, it is somewhat unrealistic to assume that the entire yield curve is driven by the very front end and also, it is hard to construct a model where the back end of the yield curve has realistic dynamics while preserving the tractability of the model.

To create a class of models that allows for more flexibility of the yield curve, it seems natural to create models where dynamics are imposed on more than point of the yield curve.

At the far end of this spectrum we find the class of 'Forward Rate Models' in which dynamics are imposed on *all* forward rates.

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Assumption

We assume that for every fixed $T > 0$, the forward rate has dynamics under the objective probability measure $\mathbb{P}$ of the form

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \boldsymbol{\sigma}'(t,T)d\mathbf{W}_t^{\mathbb{P}}, \\ f(0,T) &= f^*(0,T) \end{aligned} \tag{1}$$

where $W_t^{\mathbb{P}}$ is a d-dimensional Brownian motion, and $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$ are adapted processes.

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Note that we have assumed a stochastic differential equation in t for every fixed $T > t$ and hence, the parameter space in a forward rate model is at it's core infinite dimensional.

The initial condition for this infinite dimensional system of SDE's is the time 0 observed forward rate curve $\{f^*(0,T); T \geq 0\}$ and hence, a froward rate model will by construction fit the initial term structure.

The assumptions underlying forward rate models do not define a single model but rather a class of models. What remains to specify a model is to choose the dynamics of the processes $\alpha(t,T)$ and $\sigma(t,T)$.

Every short rate model can equivalently be formulated in forward rate terms but the converse is not true.

The Heath-Jarrow-Morton Model

In every forward rate model, it remains true that for a European claim

$$\Pi(0,\chi) = G(0,r,\chi;T) = \mathrm{E}^{\mathbb{Q}}\Big[\exp\Big\{-\int_0^T r_s ds\Big\}\cdot\chi(T)\Big] \tag{2}$$

Also, assume that we know the initial forward rate curve $\{f^*(0,T),\ T \geq 0\}$ from market data and that we have specified $\alpha(t,T)$ and $\sigma(t,T)$ for all t, T.

Then we have pinned down the entire term structure of forward rates and zero coupon bond prices can, in principle at least, be computed from

$$p(t,T) = \exp\Big\{-\int_t^T f(t,s)ds\Big\} \tag{3}$$

However, having only d sources of randomness but infinitely many traded assets, we must be very careful to not introduce arbitrages into the model.

The specific tool we need in order to prevent arbitrages in a forward rate model is the HJM drift condition.

Theorem (The HJM drift condition under $\mathbb{P}$)

Assume that bond market is arbitrage free and that the dynamics of the infinite family of forward rates is given by (1). Then there exists a d*-dimensional column-vector process*

$$\boldsymbol{\lambda}(t) = \left[\lambda_1(t), ..., \lambda_d(t)\right]'$$

with the property that for all $T \geq 0$ *and for all* $0 < t \leq T$*, we have*

$$\alpha(t, T) = \boldsymbol{\sigma}'(t, T) \int_t^T \boldsymbol{\sigma}(t, s) ds - \boldsymbol{\sigma}'(t, T)\boldsymbol{\lambda}(t) \tag{4}$$

Proof:

From the dynamics of forward rates, the dynamics of ZCB prices become

$$dp(t,T) = p(t,T)\big(r(t) + A(t,T) + \frac{1}{2}\big|\big|\mathbf{S}(t,T)\big|\big|^2\big)dt + p(t,T)\mathbf{S}'(t,T)d\mathbf{W}_t^{\mathbb{P}} \quad (5)$$

where

$$\begin{aligned} A(t,T) &= -\int_t^T \alpha(t,s)ds \\ \mathbf{S}(t,T) &= -\int_t^T \boldsymbol{\sigma}(t,s)ds \end{aligned} \quad (6)$$

The drift coefficient of zero coupon bond prices is thus

$$r(t) + A(t,T) + \frac{1}{2}\big|\big|\mathbf{S}(t,T)\big|\big|^2$$

The Heath-Jarrow-Morton Model

Now, the forward rate is just as the short rate not a traded asset and we are thus in a similar situation to when we derived a no arbitrage condition for the bond market when imposing dynamics on the short rate.

We could proceed by constructing a risk-free portfolio of $d+1$ zero coupon bonds and impose on that risk-free portfolio that it must have $r(t)$ as it's local rate of return but the result would inevitably be that we must impose a drift condition on the dynamics of forward rates that is similar to that of the drift condition on short rates and involves the market price of risk.

The only slight difference is that we now have d sources of randomness and thus a market price of risk corresponding to each of the driving Brownian motions.

The Heath-Jarrow-Morton Model

We can thus infer that there exists a d-dimensional vector-valued process $\boldsymbol{\lambda}(t)$ such that

$$r(t) + A(t,T) + \frac{1}{2}\left|\left|\mathbf{S}(t,T)\right|\right|^2 = r(t) + \sum_{i=1}^{d} S_i(t,T)\lambda_i(t) = r(t) - \mathbf{S}'(t,T)\boldsymbol{\lambda}(t) \quad (7)$$

Taking a derivative in T on both the LHS and the RHS gives us

$$\frac{\partial}{\partial T}A(t,T) + \mathbf{S}'(t,T)\frac{\partial}{\partial T}\mathbf{S}(t,T) = -\boldsymbol{\lambda}'(t)\frac{\partial}{\partial T}\mathbf{S}(t,T) \quad \Rightarrow$$

$$\alpha(t,T) = \boldsymbol{\sigma}'(t,T)\int_t^T \boldsymbol{\sigma}(t,s)ds - \boldsymbol{\sigma}'(t,T)\boldsymbol{\lambda}(t) \quad (8)$$

which completes the proof.

□

Thus as was the case for a short rate model, we can also choose to rely on so called Martingale modeling and specify the dynamics of the forward rates directly under the risk neutral measure as

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \boldsymbol{\sigma}'(t,T)d\mathbf{W}_t^{\mathbb{Q}} \\ f(0,T) &= f^*(0,T) \end{aligned} \tag{9}$$

where $W_t^{\mathbb{Q}}$ is now a d-dimensional $\mathbb{Q}$ Brownian motion.

The Heath-Jarrow-Morton Model

We now by construction do not have a problem with arbitrage in the model but instead, we have another problem in that we have two different formulas for zero coupon bond prices

$$p(0,T) = \exp\Big\{-\int_0^T f(0,s)ds\Big\} \tag{10}$$

$$p(0,T) = \mathrm{E}^{\mathbb{Q}}\Big[\exp\Big\{-\int_0^T r_s ds\Big\}\Big] \tag{11}$$

where the short rate and the forward rate are connected by $r(t) = f(t,t)$. For these two expressions to hold simultaneously, we have to impose a consistency relation between $\alpha(t,T)$ and $\sigma(t,T)$ knwon as the HJM drift condition on our forward rate model.

Proposition (HJM drift condition under $\mathbb{Q}$)

Consider a forward rate model where the $\mathbb{Q}$-dynamics of the forward rates are of the form

$$df(t,T) = \alpha(t,T)dt + \boldsymbol{\sigma}'(t,T)d\mathbf{W}_t^{\mathbb{Q}}$$

$$f(0,T) = f^*(0,T)$$

for all $T > t$. Then the processes $\alpha(t,T)$ and $\sigma(t,T)$ must for every t and $T \geq t$ satisfy the following condition

$$\alpha(t,T) = \boldsymbol{\sigma}'(t,T)\int_t^T \boldsymbol{\sigma}(t,s)ds$$

The HJM Model

Proof:

The ZCB prices again have dynamics of the form

$$dp(t,T) = p(t,T)\big(r(t) + A(t,T) + \frac{1}{2}\big|\big|\mathbf{S}(t,T)\big|\big|^2\big)dt + p(t,T)\mathbf{S}'(t,T)d\mathbf{W}_t^{\mathbb{Q}} \quad (12)$$

where

$$\begin{aligned} A(t,T) &= -\int_t^T \alpha(t,s)ds \\ \mathbf{S}(t,T) &= -\int_r^T \boldsymbol{\sigma}(t,s)ds \end{aligned} \quad (13)$$

Under $\mathbb{Q}$, the drift of ZCB prices must equal to the short rate giving us

$$A(t,T) + \frac{1}{2}\big|\big|\mathbf{S}(t,T)\big|\big|^2 = 0 \quad (14)$$

Taking a derivative in T gives us

$$\frac{\partial}{\partial T}A(t,T) + \mathbf{S}'(t,T)\frac{\partial}{\partial T}\mathbf{S}(t,T) = 0 \quad \Rightarrow \quad \alpha(t,T) = \boldsymbol{\sigma}'(t,T)\int_t^T \boldsymbol{\sigma}(t,s)ds \quad (15)$$

The HJM Model

The HJM drift condition implies that when specifying an HJM model directly under the risk-neutral measure, we are free to choose the volatility structure, but have no control over the drift of forward rates as these must satisfy the HJM drift condition.

To specify a forward rate model and use it in practice, we can therefore

1) Specify the volatilities $\sigma(t, T)$ for *all* t and $T \geq t$.
2) Compute the drift parameter from $\alpha(t, T) = \boldsymbol{\sigma}'(t, T) \int_t^T \boldsymbol{\sigma}(t, s) ds$.
3) Extract the initial forward rates $\{f^*(0, T); T \geq 0\}$ from market data.
4) Compute forward rates $f(t, T) = f^*(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \boldsymbol{\sigma}'(s, T) d\mathbf{W}_s ds$.
5) Compute ZCB prices $p(t, T) = \exp\Big\{ - \int_t^T f(t, s) ds \Big\}$.
6) Use the results to compute prices of more complicated derivatives.

The Ho-Lee model as a forward rate model

Let us go through the six steps to construct a forward rate model in the simplest case where $\sigma(t, T) = \sigma$ is simply a constant.

Next, we compute the drift from

$$\alpha(t, T) = \sigma \int_t^T \sigma ds = \sigma^2 (T - t) \tag{16}$$

Having observed todays forward rate curve, we obtain forward rates from

$$\begin{aligned} f(t, T) &= f^*(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \sigma(s, T) dW_s \\ &= f^*(0, T) + \int_0^t \sigma^2 (T - s) ds + \int_0^t \sigma dW_s \\ &= f^*(0, T) + \sigma^2 t \Big(T - \frac{1}{2} t \Big) + \sigma W(T) \end{aligned} \tag{17}$$

The Ho-Lee model as a forward rate model

The short rate in this forward rate model with constant volatility can then be found using that $r(t) = f(t,t)$

$$r(t) = f(t,t) = f^*(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma W(t) \tag{18}$$

Taking a differential in t, we get that the dynamics of the short rate is

$$dr_t = \left[f_t(0,t) + \sigma^2 t\right] dt + \sigma dW_t \tag{19}$$

Thus, this very simple forward rate model is indeed equivalent to the Ho-Lee model.