QP

A PRE–PRINT

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ABSTRACT

This repo explores general formulation of **LP** (linear programs) and **QP** (quadratic programs) with applications in modeling, prediction, and control. These formulations have well-known solution methods that scale and are fully developed technologies integrated into most programming languages (making them readily available for embedded applications). State-of-the-art implamentation uses hardware accelerated distributed optimization specialized for sparse representations and parallelization.

Keywords linprog · **LP** · quadprog · **QP**

Contents

1 Notation

1.1 Sets

$\mathbf{Z}$	: integers (zahlen)
$\mathbf{R}$	: reals
$\mathbf{R}_{+}$	: positive
$\mathbf{R}_{0+}$	: non–negative
$\mathbf{R}^{n}$	: column vector
$\mathbf{R}^{m \times n}$	: matrix
$\mathbf{R}[a,b]$	: $\{x \in \mathbf{R} \mid a \le x \le b\}$
$\mathbf{R}(a,b)$	: $\{x \in \mathbf{R} \mid a < x < b\}$
$\mathbf{R}[a,b]^{n}$	: $\{x_i \in \mathbf{R}[a,b] \mid i \in \mathbf{Z}[0,n-1]\}$
$\{a,b\}^{n}$	: $\{x_i \in \{a,b\} \mid i \in \mathbf{Z}[0,n-1]\}$
$\mathbf{S}^{n}$	: symmetric matrix
$\mathbf{S}^{n}_{+}$	: positive definite matrix
$\mathbf{S}^{n}_{0+}$	: non–negative definite matrix
$\mathbf{P}$	: polyhedron
$\mathbf{D}$	: domain

1.2 Matrices

$\mathbf{1}$	: ones
$\mathbf{0}$	: zeros
$\boldsymbol{I}$	: identity matrix
$\mathbf{e}$	: element vector
$(\cdot)^{\intercal}$	: transpose
$(\cdot)^{+}$	: pseudo inverse
$(\cdot)_{\bullet}$	: index enumeration
diag	: block diagonal
row	: row–wise concatenation
col	: column–wise concatenation

1.3 Probability

$\widetilde{(\cdot)}$	: random variable
$\langle\cdot\rangle$	: expectation

1.4 Optimization

$(\cdot)^{\circ}$	: target solution
$(\cdot)^{*}$	: optimal solution

1.5 Functions

$$\text{Boolean}(x \in \mathbf{D}) = \begin{cases} 1 & \text{if } x \in \mathbf{D} \\ 0 & \text{else} \end{cases} \qquad \text{Boolean}(i = j) =: \delta_{ij}$$

$$\begin{aligned} \text{Indicator}(x \in \mathbf{D}) &= -\log(\text{Boolean}(x \in \mathbf{D})) \\ &= \begin{cases} 0 & \text{if } x \in \mathbf{D} \\ \infty & \text{else} \end{cases} \end{aligned}$$

$$\begin{aligned} \text{Hinge}(x) &= x\,\text{Boolean}(x > 0) \\ &= \begin{cases} x & \text{if } x > 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

$$\text{Deadzone}(x, a, b) = \text{Hinge}(x - b) - \text{Hinge}(-x - a), \quad \text{for} \quad a < 0 < b$$

$$\text{Saturation}(x, a, b) = \begin{cases} b & \text{if } x > b \\ a & \text{if } x < a \\ x & \text{else} \end{cases}, \quad \text{for} \quad a < b$$

$$\text{Window}(x, a, b) = \text{Boolean}(x \in \mathbf{R}[a, b]), \quad \text{for} \quad a < b$$

Note: For the Deadzone(), Saturation(), and Window() functions,

$$\begin{aligned} f(x, c) &= f(x, -c, c), \\ f(x) &= f(x, 1). \end{aligned}$$

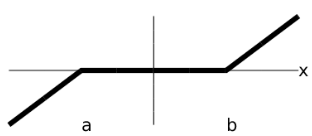


Figure 1: Deadzone

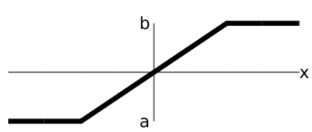


Figure 2: Saturation

Figure 3: Window

2 Introduction

Figure 4 depicts many of the common applications in machine learning. All of these applications can be framed as optimization problems. Ideally, optimization should be convex for reliable performance. Some of these optimization problems can be cast as **LP** or **QP**, e.g., many classical supervised learning problems, or they can be solved with iterative **LP** or **QP** methods, e.g, deep learning.

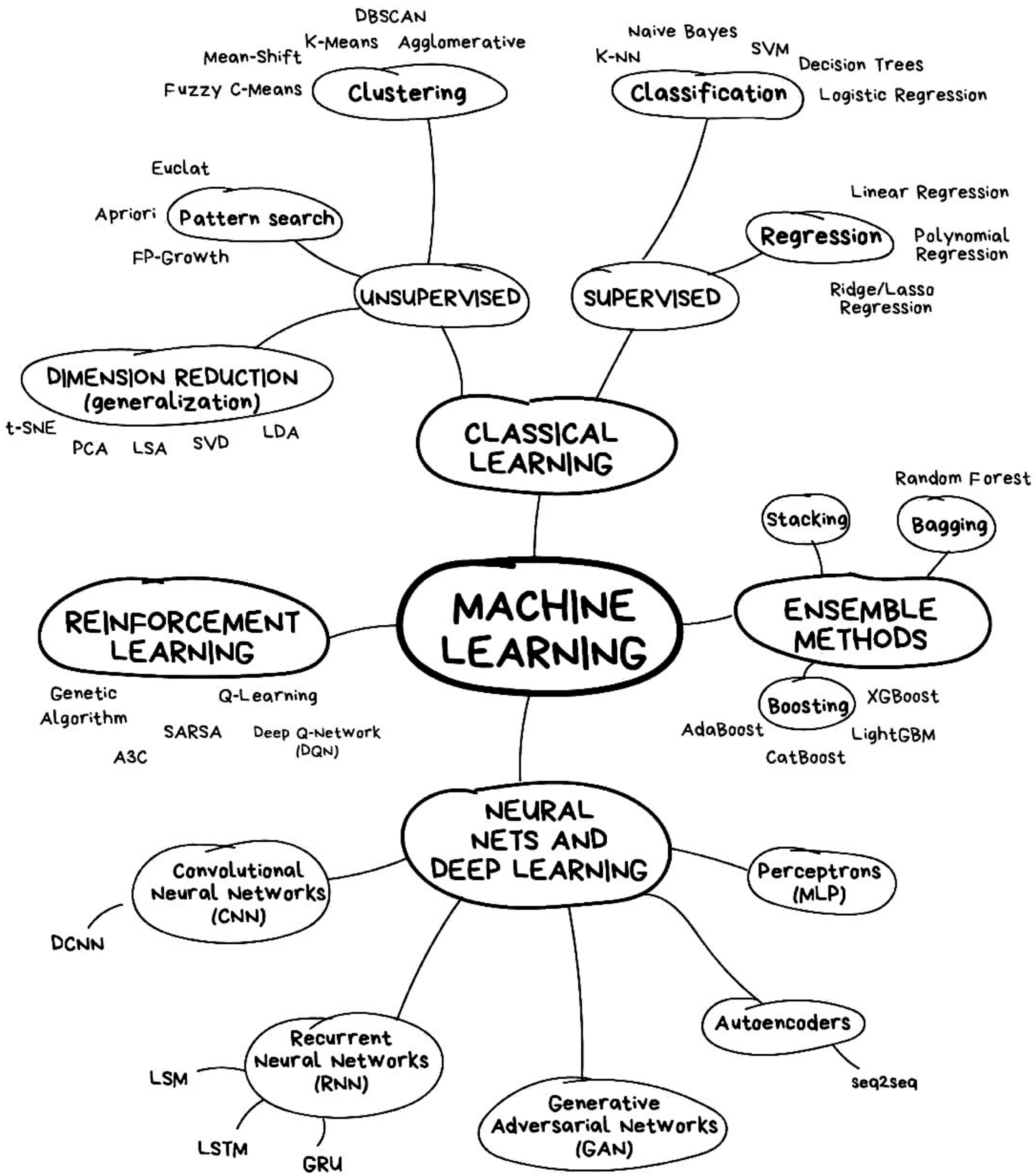


Figure 4: Machine Learning

2.1 Polyhedrons

Polyheron sets are defined by

$$\mathbf{P} := \left\{ x \in \mathbf{R}^n \;\middle|\; \begin{array}{l} A_{\text{ub}} x \leq b_{\text{ub}}, \\ A_{\text{eq}} x = b_{\text{eq}}, \\ x_{\text{lb}} \leq x \leq x_{\text{ub}} \end{array} \right\}. \tag{1}$$

Note: For $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$, $x \leq y$ implies element-wise inequality.
Note: x_{lb} and x_{ub} can be absorbed into A_{ub} and b_{ub}.

2.2 LP : Linear Programming

Linear optimization is formulated as [1, p. 146]

$$\min_{x \in \mathbf{R}^n} \quad J = c^\intercal x \tag{2}$$

$$\text{s.t.} \quad x \in \mathbf{P}.$$

The solver is called with

$$x^* = \ \mathbf{LP}\ (c, \mathbf{P}).\text{solve}().$$

2.3 QP : Quadratic Programming

Quadratic optimization is formulated as [1, p. 152]

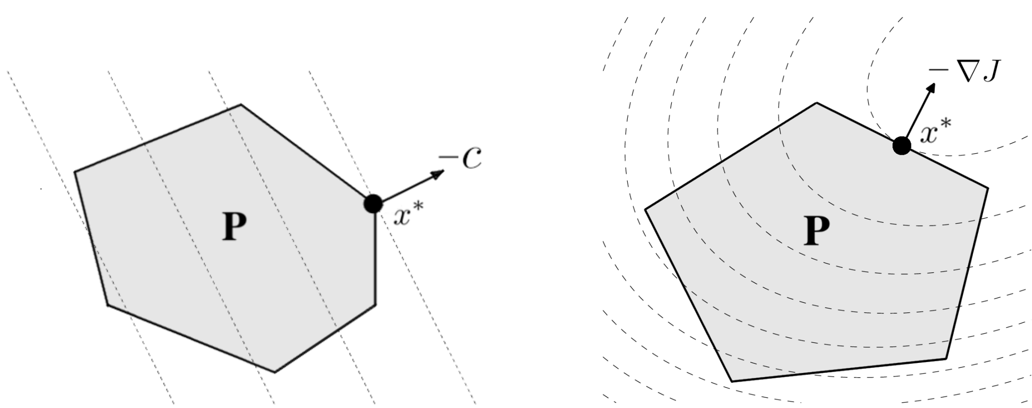
$$\min_{x \in \mathbf{R}^n} \quad J = \frac{1}{2} x^\intercal Q x + c^\intercal x \tag{3}$$

$$\text{s.t.} \quad x \in \mathbf{P}.$$

The solver is called with

$$x^* = \ \mathbf{QP}\ (Q, c, \mathbf{P}).\text{solve}().$$

Note: **LP** is a subproblem of **QP** .



(a) **LP** (b) **QP**

Figure 5: Comparison of **LP** and **QP**

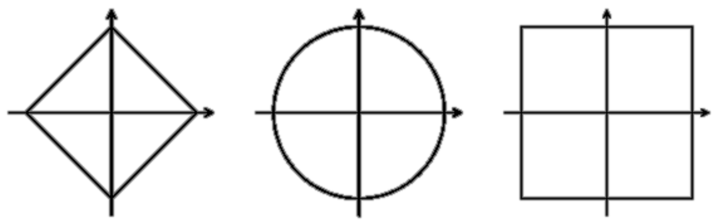


Figure 6: $\|\cdot\|_p =$ constant.

3 Linear p–Norms

The definition and general properties of the p–norm are given in section 16.13.
For $p \in \{1, 2, \infty\}$, consider

$$\min_{x\in\mathbf{R}^n} \quad J = \|Ax-b\|_p, \tag{4}$$

$$\text{s.t.} \quad x \in \mathbf{P}.$$

The ℓ_p contours are depicted in figure 6. The weighted linear p–norm is given by

$$\|Ax-b\|_{p|W} := \|W(Ax-b)\|_p. \tag{5}$$

For $w_i \geq 0$, a typical choice of weighting is

$$W = \text{diag}\{w_i\}_{i=0}^{m-1}.$$

3.1 Linear 2–Norm

For $x \in \mathbf{R}^n$,

$$\|x\|_2^2 := \sum_{i=0}^{n-1} x_i^2. \tag{6}$$

For $A \in \mathbf{R}^{m\times n}$,

$$\min_{x\in\mathbf{R}^n} \quad J = \|Ax-b\|_2^2$$

$$\text{s.t.} \quad x \in \mathbf{P},$$

is the **QP**

$$\begin{aligned}\min_{x\in\mathbf{R}^n} \quad J &= (Ax-b)^\intercal(Ax-b)\\ &= x^\intercal A^\intercal Ax - 2b^\intercal Ax + b^\intercal b\\ &= \frac{1}{2}x^\intercal Qx + c^\intercal x + r\end{aligned}$$

$$\text{s.t.} \quad x \in \mathbf{P},$$

where

$$\begin{aligned}Q &= 2A^\intercal A,\\ c &= -2A^\intercal b,\\ r &= b^\intercal b.\end{aligned}$$

3.1.1 QP to Linear 2-Norm

For the constraint $x \in \mathbf{P}$, consider

$$\begin{aligned}\min_{x\in\mathbf{R}^n} \quad J &= \frac{1}{2}x^\intercal Qx + c^\intercal x \\ &= \|Ax-b\|_2^2 + r,\end{aligned}$$

with

$$Q = \quad 2A^\intercal A,$$

$$c = -2A^\intercal b.$$

If $Q \in \mathbf{S}^n_+$, using SVD, the square root of Q and its inverse can be computed with

$$Q^{1/2} \quad = US^{1/2} \quad U^\intercal,$$

$$Q^{-1/2} = US^{-1/2}U^\intercal.$$

The linear 2-norm can be expressed as

$$A = \quad \frac{1}{\sqrt{2}}Q^{1/2},$$

$$b = -\frac{1}{\sqrt{2}}Q^{-1/2}c.$$

Note: This choice of A gives $A = A^\intercal$.
Note: The constant offset does not affect the solution.

3.1.2 Weighted Linear 2-Norm

For non–negative definite $M = W^\intercal W$,

$$\begin{aligned}\|Ax-b\|_{2|W}^2 &= \|W(Ax-b)\|_2^2 \\ &= (Ax-b)^\intercal W^\intercal W(Ax-b) \\ &= (Ax-b)^\intercal M(Ax-b) \\ &= x^\intercal A^\intercal MAx - 2b^\intercal MAx + b^\intercal Mb,\end{aligned}$$

which gives

$$Q = \quad 2A^\intercal MA,$$

$$c = -2A^\intercal Mb.$$

3.1.3 Expected Linear 2-Norm

Consider the addition of random variables. The expected linear 2–norm is given by

$$\min_{x\in\mathbf{R}^n} \quad J = \langle \|(A+\widetilde{A})(x+\widetilde{x}) - (b+\widetilde{b})\|_2^2 \rangle$$
$$\text{s.t.} \quad x \in \mathbf{P},$$

which can be solved as the **QP**

$$\begin{aligned}\min_{x\in\mathbf{R}^n} \quad J &= x^\intercal \langle (A+\widetilde{A})^\intercal (A+\widetilde{A}) \rangle x \\ &\quad + 2\langle (A+\widetilde{A})^\intercal ((A+\widetilde{A})\widetilde{x} - (b+\widetilde{b}))\rangle^\intercal x \\ &\quad + \langle ((A+\widetilde{A})\widetilde{x} - (b+\widetilde{b}))^\intercal ((A+\widetilde{A})\widetilde{x} - (b+\widetilde{b}))\rangle \\ \\ &= \frac{1}{2} x^\intercal Q x + c^\intercal x + r\end{aligned}$$
$$\text{s.t.} \quad x \in \mathbf{P},$$

where

$$Q = 2A^\intercal A + 2A^\intercal \langle \widetilde{A} \rangle + 2\langle \widetilde{A} \rangle^\intercal A + 2\langle \widetilde{A}^\intercal \widetilde{A} \rangle,$$

$$\begin{aligned}c &= 2\left(A^\intercal A \langle \widetilde{x} \rangle + A^\intercal \langle \widetilde{A}\widetilde{x} \rangle + \langle \widetilde{A}^\intercal A \widetilde{x} \rangle + \langle \widetilde{A}^\intercal \widetilde{A} \widetilde{x} \rangle \right) \\ &\quad - 2\left(A^\intercal b + \langle \widetilde{A} \rangle^\intercal b + A^\intercal \langle \widetilde{b} \rangle + \langle \widetilde{A}^\intercal \widetilde{b} \rangle \right).\end{aligned}$$

3.2 Linear 1–Norm

For $x \in \mathbf{R}^n$,

$$\|x\|_1 := \sum_{i=0}^{n-1} |x_i|. \tag{7}$$

For $A \in \mathbf{R}^{m\times n}$,

$$\min_{x\in\mathbf{R}^n} \quad J = \|Ax-b\|_1$$
$$\text{s.t.} \quad x \in \mathbf{P}$$

is equivalent to [1, p. 294]

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}^m\}} \quad J = \mathbf{1}^\intercal y$$
$$\begin{aligned}\text{s.t.} \quad & x \in \mathbf{P}, \\ & -y \le Ax - b \le y,\end{aligned}$$

which can be expressed as the **LP**

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}^m\}} \quad J = \begin{bmatrix} \mathbf{0}^n \\ \mathbf{1}^m \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{aligned}\text{s.t.} \quad & x \in \mathbf{P}, \\ & \begin{bmatrix} A & -\boldsymbol{I}_m \\ -A & -\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} b \\ -b \end{bmatrix}.\end{aligned}$$

Note: For $r_i = a_i^\intercal x - b_i$,

$$\|Ax-b\|_1 = \textstyle\sum_{i=0}^{m-1} \text{sign}(r_i) r_i.$$

3.3 Linear Inf–Norm

For $x \in \mathbf{R}^n$,

$$\|x\|_\infty := \max_{i \in \mathbf{Z}[0,n-1]} |x_i|. \tag{8}$$

For $A \in \mathbf{R}^{m \times n}$,

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\|_\infty$$

$$\text{s.t.} \quad x \in \mathbf{P}$$

is equivalent to [1, p. 294]

$$\min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}\}} \quad J = y$$

$$\begin{aligned} \text{s.t.} \quad x &\in \mathbf{P}, \\ -y\mathbf{1} \leq Ax - b &\leq y\mathbf{1}, \end{aligned}$$

which can be expressed as the **LP**

$$\min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}\}} \quad J = \begin{bmatrix} \mathbf{0}^n \\ 1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{s.t.} \quad x \in \mathbf{P}$$
$$\begin{bmatrix} -A & -\mathbf{1}^m \\ A & -\mathbf{1}^m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} -b \\ b \end{bmatrix}.$$

3.4 Combined Linear p–Norms

Combinations of $p \in \{1, 2, \infty\}$ result in **LP** or **QP** .

3.4.1 Linear 1–1–Norm

For $A \in \mathbf{R}^{m \times n}$ and $C \in \mathbf{R}^{r \times n}$,

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\|_1 + \|Cx - d\|_1$$

$$\text{s.t.} \quad x \in \mathbf{P}$$

is equivalent to

$$\min_{\{x,y,z\} \in \{\mathbf{R}^n, \mathbf{R}^m, \mathbf{R}^r\}} \quad J = \mathbf{1}^\mathsf{T} y + \mathbf{1}^\mathsf{T} z$$

$$\begin{aligned} \text{s.t.} \quad x &\in \mathbf{P}, \\ -y \leq Ax - b &\leq y, \\ -z \leq Cx - d &\leq z, \end{aligned}$$

which can be expressed as the **LP**

$$\min_{\{x,y,z\} \in \{\mathbf{R}^n, \mathbf{R}^m, \mathbf{R}^r\}} \quad J = \begin{bmatrix} \mathbf{0}^n \\ \mathbf{1}^m \\ \mathbf{1}^r \end{bmatrix}^\mathsf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\text{s.t.} \quad x \in \mathbf{P},$$
$$\begin{bmatrix} -A & -\boldsymbol{I}_m & \mathbf{0}^{m \times r} \\ A & -\boldsymbol{I}_m & \mathbf{0}^{m \times r} \\ -C & \mathbf{0}^{r \times m} & -\boldsymbol{I}_r \\ C & \mathbf{0}^{r \times m} & -\boldsymbol{I}_r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \begin{bmatrix} -b \\ b \\ -d \\ d \end{bmatrix}.$$

3.4.2 Linear 2–2–Norm

For $A \in \mathbf{R}^{m\times n}$ and $C \in \mathbf{R}^{r\times n}$,

$$\begin{aligned}\min_{x\in\mathbf{R}^n} \quad & J = \|Ax-b\|_2^2 + \|Cx-d\|_2^2 \\ \text{s.t.} \quad & x \in \mathbf{P}\end{aligned}$$

is equivalent to

$$\begin{aligned}\min_{x\in\mathbf{R}^n} \quad & J = x^\intercal(A^\intercal A + C^\intercal C)x - 2(b^\intercal A + d^\intercal C)x + b^\intercal b + d^\intercal d \\ \text{s.t.} \quad & x \in \mathbf{P},\end{aligned}$$

which is a **QP** with

$$\begin{aligned}Q &= \quad 2(A^\intercal A + C^\intercal C), \\ c &= -2(A^\intercal b + C^\intercal d).\end{aligned}$$

3.4.3 Linear Inf–Inf–Norm

For $A \in \mathbf{R}^{m\times n}$ and $C \in \mathbf{R}^{r\times n}$,

$$\begin{aligned}\min_{x\in\mathbf{R}^n} \quad & J = \|Ax-b\|_\infty + \|Cx-d\|_\infty \\ \text{s.t.} \quad & x \in \mathbf{P}\end{aligned}$$

is equivalent to

$$\begin{aligned}\min_{\{x,y,z\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}\}} \quad & J = y + z \\ \text{s.t.} \quad & x \in \mathbf{P}, \\ & -y\mathbf{1}^m \le Ax - b \le y\mathbf{1}^m, \\ & -z\mathbf{1}^r \le Cx - d \le z\mathbf{1}^r,\end{aligned}$$

which can be expressed as the **LP**

$$\begin{aligned}\min_{\{x,y,z\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}\}} \quad & J = \begin{bmatrix} \mathbf{0}^n \\ 1 \\ 1 \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \text{s.t.} \quad & x \in \mathbf{P}, \\ & \begin{bmatrix} -A & -\mathbf{1}^m & \mathbf{0}^m \\ A & -\mathbf{1}^m & \mathbf{0}^m \\ -C & \mathbf{0}^r & -\mathbf{1}^r \\ C & \mathbf{0}^r & -\mathbf{1}^r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} -b \\ b \\ -d \\ d \end{bmatrix}.\end{aligned}$$

3.4.4 Linear 1–2–Norm

For $C \in \mathbf{R}^{m\times n}$,

$$\begin{aligned}\min_{x\in\mathbf{R}^n} \quad & J = \|Ax-b\|_2^2 + \|Cx-d\|_1 \\ \text{s.t.} \quad & x \in \mathbf{P}\end{aligned}$$

is equivalent to

$$\begin{aligned}\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}^m\}} \quad & J = x^\intercal A^\intercal Ax - b^\intercal Ax + b^\intercal b + \mathbf{1}^\intercal y \\ \text{s.t.} \quad & x \in \mathbf{P}, \\ & -y \le Cx - d \le y,\end{aligned}$$

which can be expressed as the **QP**

$$\begin{aligned}\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}^m\}} \quad & J = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^\intercal \begin{bmatrix} 2A^\intercal A & \mathbf{0}^{n\times m} \\ \mathbf{0}^{m\times n} & \mathbf{0}^{m\times m} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2A^\intercal b \\ \mathbf{1}^m \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \end{bmatrix} + b^\intercal b \\ \text{s.t.} \quad & x \in \mathbf{P}, \\ & \begin{bmatrix} -C & -\boldsymbol{I}_m \\ C & -\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} -d \\ d \end{bmatrix}.\end{aligned}$$

3.4.5 Linear 1–Inf–Norm

For $A \in \mathbf{R}^{m\times n}$ and $C \in \mathbf{R}^{r\times n}$,

$$\min_{x\in\mathbf{R}^n} \quad J = \|Ax-b\|_1 + \|Cx-d\|_\infty$$

$$\text{s.t.} \quad x \in \mathbf{P}$$

is equivalent to

$$\min_{\{x,y,z\}\in\{\mathbf{R}^n,\mathbf{R}^m,\mathbf{R}\}} \quad J = \mathbf{1}^\intercal y + z$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ -y \le & Ax - b \le y, \\ -z\mathbf{1} \le & Cx - d \le z\mathbf{1}, \end{aligned}$$

which can be expressed as the **LP**

$$\min_{\{x,y,z\}\in\{\mathbf{R}^n,\mathbf{R}^m,\mathbf{R}\}} \quad J = \begin{bmatrix} \mathbf{0}^n \\ \mathbf{1}^m \\ 1 \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\text{s.t.} \quad x \in \mathbf{P},$$
$$\begin{bmatrix} -A & -\boldsymbol{I}_m & \mathbf{0}^m \\ A & -\boldsymbol{I}_m & \mathbf{0}^m \\ -C & \mathbf{0}^{r\times m} & -\mathbf{1}^r \\ C & \mathbf{0}^{r\times m} & -\mathbf{1}^r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} -b \\ b \\ -d \\ d \end{bmatrix}.$$

3.4.6 Linear 2–Inf–Norm

For $C \in \mathbf{R}^{m\times n}$,

$$\min_{x\in\mathbf{R}^n} \quad J = \|Ax-b\|_2^2 + \|Cx-d\|_\infty$$

$$\text{s.t.} \quad x \in \mathbf{P}$$

is equivalent to

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}\}} \quad J = x^\intercal A^\intercal Ax - b^\intercal Ax + b^\intercal b + y$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ -y\mathbf{1} \le & Cx - d \le y\mathbf{1}, \end{aligned}$$

which can be expressed as the **QP**

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}\}} \quad J = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^\intercal \begin{bmatrix} 2A^\intercal A & \mathbf{0}^n \\ \mathbf{0}^{1\times n} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2A^\intercal b \\ 1 \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \end{bmatrix} + b^\intercal b$$

$$\text{s.t.} \quad x \in \mathbf{P},$$
$$\begin{bmatrix} -C & -\mathbf{1}^m \\ C & -\mathbf{1}^m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} -d \\ d \end{bmatrix}.$$

3.4.7 Linear 1–2–Inf–Norm

For $B \in \mathbf{R}^{m\times n}$ and $D \in \mathbf{R}^{r\times n}$,

$$\min_{x\in\mathbf{R}^n} \quad J = \|Ax-b\|_2^2 + \|Bx-c\|_1 + \|Dx-e\|_\infty$$

$$\text{s.t.} \quad x \in \mathbf{P}$$

is equivalent to

$$\min_{\{x,y,z\}\in\{\mathbf{R}^n,\mathbf{R}^m,\mathbf{R}\}} \quad J = x^\intercal A^\intercal Ax - b^\intercal Ax + b^\intercal b + \mathbf{1}^\intercal y + z$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ -y \le{} & Bx - c \le y, \\ -z\mathbf{1} \le{} & Dx - e \le z\mathbf{1}, \end{aligned}$$

which can be expressed as the **QP**

$$\min_{\{x,y,z\}\in\{\mathbf{R}^n,\mathbf{R}^m,\mathbf{R}\}} \quad J = \frac{1}{2}\begin{bmatrix} x \\ y \\ z \end{bmatrix}^\intercal \begin{bmatrix} 2A^\intercal A & \mathbf{0}^{n\times m} & \mathbf{0}^n \\ \mathbf{0}^{m\times n} & \mathbf{0}^{m\times m} & \mathbf{0}^m \\ \mathbf{0}^{1\times n} & \mathbf{0}^{1\times m} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -2A^\intercal b \\ \mathbf{1}^m \\ 1 \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \\ z \end{bmatrix} + b^\intercal b$$

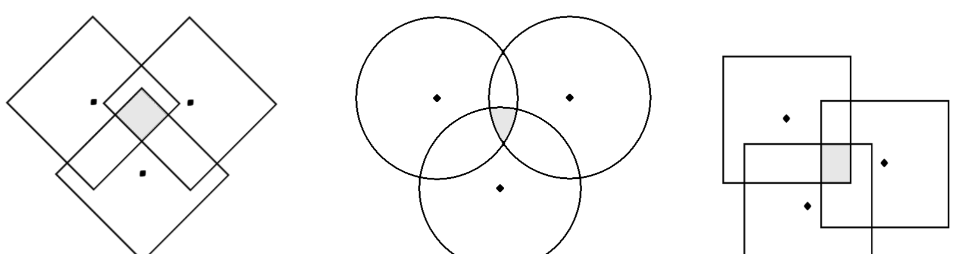
$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & \begin{bmatrix} -B & -\boldsymbol{I}_m & \mathbf{0}^m \\ B & -\boldsymbol{I}_m & \mathbf{0}^m \\ -D & \mathbf{0}^{r\times m} & -\mathbf{1}^r \\ D & \mathbf{0}^{r\times m} & -\mathbf{1}^r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} -c \\ c \\ -e \\ e \end{bmatrix}. \end{aligned}$$

3.5 Linear p–Norm Constraint

For $A_i \in \mathbf{R}^{m_i\times n}$, $p_i \in \{1, 2, \infty\}$, and $t_i > 0$, consider the problem

$$\min_{x\in\mathbf{R}^n} \quad J = f(x)$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & \|A_i x - b_i\|_{p_i} \le t_i. \end{aligned} \tag{9}$$



(a) 1–Norm (b) 2–Norm (c) Inf–Norm

Figure 7: Comparison of p–norm inequality constraints

3.5.1 Linear 1–Norm Constraint

For $p_i = 1$, equation (9) is equivalent to

$$\min_{\{x,y_i\}\in\{\mathbf{R}^n,\mathbf{R}^{m_i}\}} \quad J = f(x)$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & \mathbf{1}^\intercal y_i \leq t_i, \\ & -y_i \leq A_i x - b_i \leq y_i. \end{aligned}$$

Figure 7a depicts the intersection of 1–norm inequality constraints.

If $f(x)$ is linear, this problem is an **LP** .

If $f(x)$ is quadratic, this problem is a **QP** .

3.5.2 Linear 2–Norm Constraint

For $p_i = 2$, equation (9) can be squared to get

$$\|A_i x - b_i\|_2^2 \leq t_i^2,$$

which can be expressed as

$$\frac{1}{2} x^\intercal Q_i x + c_i^\intercal x + r_i \leq 0,$$

where

$$\begin{aligned} Q_i &= 2A_i^\intercal A_i, \\ c_i &= -2A_i^\intercal b_i, \\ r_i &= b_i^\intercal b_i - t_i^2. \end{aligned}$$

Figure 7b depicts the intersection of 2–norm inequality constraints.

If $f(x)$ is quadratic, this problem is a **QCQP** .

3.5.3 Linear Inf–Norm Constraint

For $p_i = \infty$, equation (9) is equivalent to

$$\min_{\{x,y_i\}\in\{\mathbf{R}^n,\mathbf{R}\}} \quad J = f(x)$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & y_i \leq t_i, \\ & -y_i\mathbf{1} \leq A_i x - b_i \leq y_i\mathbf{1}. \end{aligned}$$

Figure 7c depicts the intersection of inf–norm inequality constraints.

If $f(x)$ linear, this problem is an **LP** .

If $f(x)$ quadratic, this problem is a **QP** .

3.6 Piecewise–Linear Minimization

Consider the problem

$$\min_{x\in\mathbf{R}^n} \quad \max_{i\in\mathbf{Z}[0,m-1]} \quad J_i = a_i^\intercal x - b_i,$$
$$\text{s.t.} \quad x \in \mathbf{P}.$$

This problem is equivalent to the **LP** [1, p. 150]

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}\}} y$$
$$\text{s.t.} \quad x \in \mathbf{P},$$
$$a_i^\intercal x - b_i \le y,$$

which can be expressed as

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}\}} \begin{bmatrix} \mathbf{0}^n \\ 1 \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\text{s.t.} \quad x \in \mathbf{P},$$
$$\begin{bmatrix} A & -\mathbf{1}^m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le b,$$

where

$$A = \text{row}\{a_i^\intercal\}_{i=0}^{m-1},$$

$$b = \text{row}\{b_i\}_{i=0}^{m-1}.$$

3.7 Robust p–Norm from a Finite Set

For $A_i \in \mathbf{R}^{q_i\times n}$, consider the problem

$$\min_{x\in\mathbf{R}^n} \quad \max_{i\in\mathbf{Z}[0,m-1]} \quad J_i = \|A_i x - b_i\|_{p_i},$$
$$\text{s.t.} \quad x \in \mathbf{P}.$$

This problem is equivalent to [1, p. 321]

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}\}} y$$
$$\text{s.t.} \quad x \in \mathbf{P},$$
$$\|A_i x - b_i\|_{p_i} \le y. \tag{10}$$

If $p \in \{1, \infty\}$, this is an **LP** .

3.7.1 Robust 1–Norm from a Finite Set

For $p_i = 1$, equation (10) is equivalent to the **LP**

$$\min_{\{x,y,z_i\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}^{q_i}\}} \quad J = y$$
$$\text{s.t.} \quad x \in \mathbf{P},$$
$$\mathbf{1}^\intercal z_i \le y,$$
$$-z_i \le A_i x - b_i \le z_i.$$

3.7.2 Robust Inf–Norm from a Finite Set

For $p_i = \infty$, equation (10) is equivalent to the **LP**

$$\min_{\{x,y,z_i\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}\}} \quad J = y$$
$$\text{s.t.} \quad x \in \mathbf{P},$$
$$z_i \le y,$$
$$-z_i\mathbf{1} \le A_i x - b_i \le z_i\mathbf{1}.$$

4 Penalty Functions

4.1 Deadzone Penalty

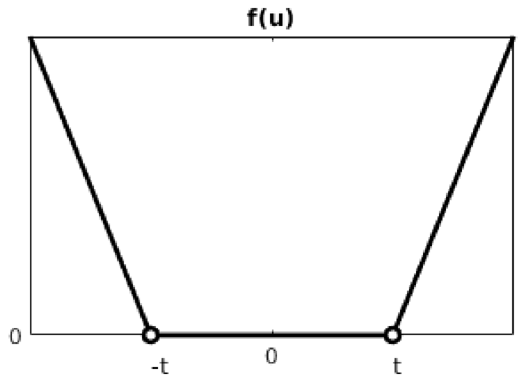


Figure 8: Deadzone Penalty

Consider the problem

$$\min_{x \in \mathbf{R}^n} \quad J = \sum_{i=0}^{m-1} f(a_i^\intercal x - b_i)$$
$$\text{s.t.} \quad x \in \mathbf{P},$$

where $f(u)$ is the deadzone

$$\begin{aligned} f(u) &= \begin{cases} 0 & \text{if } |u| \leq t \\ |u| - t & \text{else} \end{cases} \\ &= \max(\, -u - t,\ 0,\ u - t\,), \end{aligned} \tag{11}$$

$$f'(u) = \begin{cases} 0 & \text{if } |u| \leq t \\ \text{sign}(u) & \text{else} \end{cases} .$$

Note: The deadzone penalty is not a norm because it fails the condition $f(x) = 0$ if and only if $x = 0$.
This problem is equivalent to [1, p. 344]

$$\min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad J = \mathbf{1}^\intercal y$$
$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ -y - t\mathbf{1} \leq & \ Ax - b \leq y + t\mathbf{1}, \\ & y \geq \mathbf{0}, \end{aligned}$$

which can be expressed as the **LP**

$$\min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad J = \begin{bmatrix} \mathbf{0}^n \\ \mathbf{1}^m \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\text{s.t.} \quad x \in \mathbf{P},$$
$$\begin{bmatrix} A & -\boldsymbol{I}_m \\ -A & \boldsymbol{I}_m \\ \mathbf{0}^{n \times m} & -\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} t\mathbf{1}^m + b \\ t\mathbf{1}^m - b \\ \mathbf{0}^m \end{bmatrix}.$$

4.2 Huber Penalty

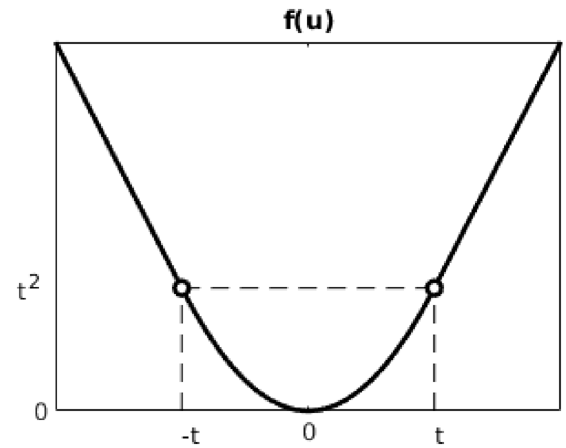


Figure 9: Huber Penalty

Consider the problem

$$\min_{x \in \mathbf{R}^n} \quad J = \sum_{i=0}^{m-1} f(a_i^\mathsf{T} x - b_i)$$

$$\text{s.t.} \quad x \in \mathbf{P},$$

where $f(u)$ is the Huber function

$$\begin{aligned} f(u) &= \begin{cases} u^2 & \text{if } |u| \le t \\ t(2|u| - t) & \text{else} \end{cases}, \\ f'(u) &= \begin{cases} 2u & \text{if } |u| \le t \\ 2t \operatorname{sign}(u) & \text{else} \end{cases} \\ &= 2 \operatorname{Saturation}(u, t). \end{aligned} \tag{12}$$

Note: The Huber penalty is a norm.
This problem is equivalent to [1, p. 190]

$$\min_{\{x,y,z\} \in \{\mathbf{R}^n, \mathbf{R}^m, \mathbf{R}^m\}} \quad J = y^\mathsf{T} y + 2t\mathbf{1}^\mathsf{T} z$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ -y - z \le A&x - b \le y + z, \\ \mathbf{0} \le\ & y \le t\mathbf{1}, \\ & z \ge \mathbf{0}, \end{aligned}$$

which can be expressed as the **QP**

$$\min_{\{x,y,z\} \in \{\mathbf{R}^n, \mathbf{R}^m, \mathbf{R}^m\}} \quad J = \frac{1}{2} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbf{0}^{n \times n} & \mathbf{0}^{n \times m} & \mathbf{0}^{n \times m} \\ \mathbf{0}^{m \times n} & 2\boldsymbol{I}_m & \mathbf{0}^{m \times m} \\ \mathbf{0}^{m \times n} & \mathbf{0}^{m \times m} & \mathbf{0}^{m \times m} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \mathbf{0}^n \\ \mathbf{0}^m \\ 2t\mathbf{1}^m \end{bmatrix}^\mathsf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\text{s.t.} \quad x \in \mathbf{P},$$

$$\begin{bmatrix} A & -\boldsymbol{I}_m & -\boldsymbol{I}_m \\ -A & -\boldsymbol{I}_m & -\boldsymbol{I}_m \\ \mathbf{0}^{m \times n} & \boldsymbol{I}_m & \mathbf{0}^{m \times m} \\ \mathbf{0}^{m \times n} & -\boldsymbol{I}_m & \mathbf{0}^{m \times m} \\ \mathbf{0}^{m \times n} & \mathbf{0}^{m \times m} & -\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \le \begin{bmatrix} b \\ -b \\ t\mathbf{1}^m \\ \mathbf{0}^m \\ \mathbf{0}^m \end{bmatrix}.$$

4.3 Rectified Linear Penalty

Consider the problem

$$\min_{x \in \mathbf{R}^n} \quad J = \sum_{i=0}^{m-1} f(a_i^\intercal x - b_i)$$

$$\text{s.t.} \quad x \in \mathbf{P},$$

where $f(u)$ is the rectified linear penalty function

$$\begin{aligned} f(u) &= \text{Hinge}(u) \\ &= \max(0, u). \end{aligned} \tag{13}$$

This problem can be expressed as the **LP**

$$\min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad J = \mathbf{1}^\intercal y$$

$$\begin{aligned} \text{s.t.} \quad x &\in \mathbf{P}, \\ Ax - b &\leq y, \\ 0^n &\leq y. \end{aligned}$$

Note: Rectified linear penalty is convex and can be used as a constraint function.
Note: Rectified linear penalty is the preffered nonlinearity for deep learning [41].

4.4 Leaky Linear Penalty

$$\min_{x \in \mathbf{R}^n} \quad J = \sum_{i=0}^{m-1} f(a_i^\intercal x - b_i)$$

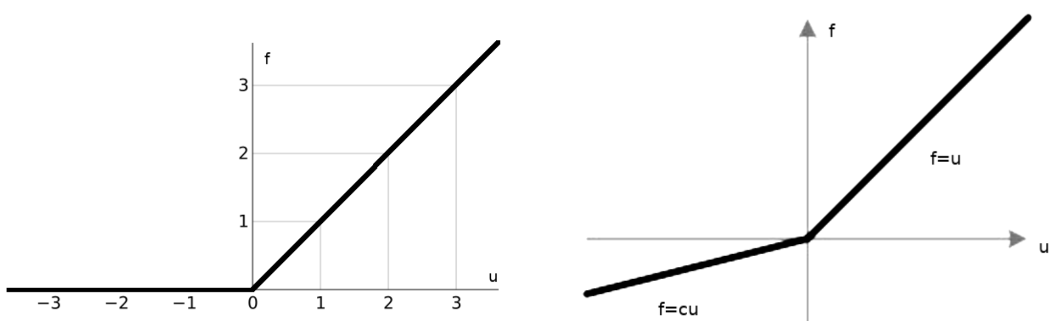
where $f(u)$ is the leaky–linear penalty function

$$\begin{aligned} f(u) &= \text{Hinge}(u) - t\,\text{Hinge}(-u) \\ &= \max(tu, u), \end{aligned} \tag{14}$$

with $t \in \mathbf{R}(0, 1)$. This problem can be expressed as the **LP**

$$\min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad J = \mathbf{1}^\intercal y$$

$$\begin{aligned} \text{s.t.} \quad x &\in \mathbf{P}, \\ Ax - b &\leq y, \\ Ax - b &\leq y/t. \end{aligned}$$



(a) linear (b) leaky–linear

Figure 10: Rectified Linear Penalty

5 Duality

5.1 KKT: Karush–Kuhn–Tucker

Consider the general problem

$$\min_{x \in \mathbf{D}} \quad J = f(x) \tag{15}$$

with the constraint set

$$\mathbf{D} = \{x \mid g_i(x) = 0,\ h_j(x) \leq 0\}.$$

Consider the generalized Lagrangian

$$L(x, y, z) = f(x) + \sum_i y_i g_i(x) + \sum_j z_j h_j(x), \tag{16}$$

which (if there is no duality gap) has the property [57]

$$\max_{y_i} \max_{z_j \geq 0} L(x, y, z) = \begin{cases} f(x) & \text{if } x \in \mathbf{D} \\ \infty & \text{else} \end{cases}. \tag{17}$$

Therefore, (15) can be reformulated as

$$\min_x L(x, y^*, z^*) = \min_x \max_{y_i} \max_{z_j \geq 0} L(x, y, z).$$

The inequality constraint $h_j(x)$ is active if

$$h_j(x^*) = 0.$$

If the inequality constraint is inactive, then $h_j(x^*) < 0$, and maximizing $z_j h_j(x^*)$ such that $z_j \geq 0$ gives

$$z_j^* = 0.$$

Therefore,

$$z^* \circ h(x^*) = 0. \tag{18}$$

Note: This property is known as 'complamentary slackness'.

The first order necessary (but not sufficient) conditions of KKT optimality are:

- the gradient of the generalized Lagrangian is zero, i.e.,

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y_i} = 0, \quad \frac{\partial L}{\partial z_j} = 0,$$

- the constraints $x \in \mathbf{D}$ and $z_j \geq 0$ are satisfied,
- and equation (18) is satisfied.

5.2 Duality with Polyhedron Constraint

Consider the primal problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = f(x) \\ \text{s.t.} \quad & A_{\text{ub}} x \leq b_{\text{ub}}, \\ & A_{\text{eq}} x = b_{\text{eq}}. \end{aligned}$$

Let

$$A = \begin{bmatrix} A_{\text{ub}} \\ A_{\text{eq}} \end{bmatrix}, \quad b = \begin{bmatrix} b_{\text{ub}} \\ b_{\text{eq}} \end{bmatrix}.$$

For $A_{\text{ub}} \in \mathbf{R}^{r \times n}$ and $A_{\text{eq}} \in \mathbf{R}^{m-r \times n}$, the dual problem formulation is

$$\begin{aligned} \max_{y \in \mathbf{R}^m} \quad \inf_{x \in \mathbf{R}^n} \quad & L(x, y) \\ \text{s.t.} \quad & \begin{bmatrix} \boldsymbol{I}_r & \mathbf{0}^{r \times (m-r)} \end{bmatrix} y \geq \mathbf{0}^r, \end{aligned} \tag{19}$$

where the Lagrangian is given by

$$L(x, y) = f(x) + y^{\mathsf{T}}(Ax - b).$$

If strong duality holds, the following steps compute x^*:

- Compute $\inf_{x \in \mathbf{R}^n} L(x, y)$ to get $L(x^*, y)$.
  Note: This step may not give an explicit representation of x^*.
  Implicit constraints for $L(x^*, y) > -\infty$ can be made explicit.
- Solve the dual problem to get y^*.
- The primal problem is solved with $\min_{x \in \mathbf{R}^n} L(x, y^*)$.

Note: The dual problem may be concave even when the primal problem is not convex.

5.3 Norm Duality

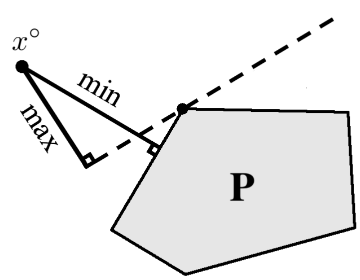


Figure 11: Norm Duality

Consider the problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|x - x^\circ\| \\ \text{s.t.} \quad & x \in \mathbf{P}. \end{aligned}$$

The dual problem is illustrated in figure 11 [36, p. 9].

For $p \in \mathbf{R}[1, \infty)$, conisder the problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|x\|_p \\ \text{s.t.} \quad & Ax = b. \end{aligned}$$

The dual problem is given by

$$\max_{y \in \mathbf{R}^m} \inf_{x \in \mathbf{R}^n} \quad L(x, y) = \|x\|_p + y^\intercal (Ax - b).$$

By Hölder's inequality from section 16.13.2,

$$\begin{aligned} \|x\|_p + \textstyle\sum_i [A^\intercal y]_i x_i \geq \|x\|_p - \sum_i |[A^\intercal y]_i x_i| & \\ \geq \|x\|_p - \|A^\intercal y\|_q \|x\|_p & = (1 - \|A^\intercal y\|_q)\|x\|_p, \end{aligned}$$

where $1/p + 1/q = 1$.

The dual problem becomes [1, p. 221][36, p. 107][36, p. 123]

$$\max_{y \in \mathbf{R}^m} \quad L(x^*, y) = \begin{cases} -b^\intercal y & \text{if} \quad \|A^\intercal y\|_q \leq 1 \\ -\infty & \text{else} \end{cases},$$

where $q = p/(p-1)$.

Note: This does not hold for $p = \infty$.

The dual problem can be stated as

$$\begin{aligned} \min_{y \in \mathbf{R}^m} \quad & -L = b^\intercal y \\ \text{s.t.} \quad & \|A^\intercal y\|_q \leq 1. \end{aligned} \tag{20}$$

5.3.1 1–Norm

For $p = 1$, $q = \infty$. The problem becomes the **LP**

$$\begin{aligned} \min_{\{y,z\} \in \{\mathbf{R}^m, \mathbf{R}\}} \quad & -L = b^\intercal y + z \\ \text{s.t.} \quad & z \leq 1, \\ -z\mathbf{1}^n \leq & A^\intercal y \leq z\mathbf{1}^n, \end{aligned}$$

or

$$\begin{aligned} \min_{\{y,z\} \in \{\mathbf{R}^m, \mathbf{R}\}} \quad & -L = \begin{bmatrix} b \\ 1 \end{bmatrix}^\intercal \begin{bmatrix} y \\ z \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{0}^{1 \times m} & 1 \\ A^\intercal & -\mathbf{1}^n \\ -A^\intercal & -\mathbf{1}^n \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \leq \begin{bmatrix} 1 \\ \mathbf{0}^n \\ \mathbf{0}^n \end{bmatrix}. \end{aligned}$$

5.3.2 2–Norm

For $p = 2$, $q = 2$. The problem becomes the **QCQP**

$$\begin{aligned} \min_{y \in \mathbf{R}^m} \quad & -L = b^\intercal y \\ \text{s.t.} \quad & y^\intercal Q y \leq 1, \end{aligned}$$

where $Q = AA^\intercal$.

5.4 LP Duality

Consider the primal **LP**

$$\min_{x \in \mathbf{R}^n} \quad J = c^\intercal x$$

$$\text{s.t.} \quad \begin{aligned} A_{\text{ub}} x &\leq b_{\text{ub}}, \\ A_{\text{eq}} x &= b_{\text{eq}}. \end{aligned}$$

Let

$$A = \begin{bmatrix} A_{\text{ub}} \\ A_{\text{eq}} \end{bmatrix}, \quad b = \begin{bmatrix} b_{\text{ub}} \\ b_{\text{eq}} \end{bmatrix}.$$

For $A_{\text{ub}} \in \mathbf{R}^{r \times n}$ and $A_{\text{eq}} \in \mathbf{R}^{m-r \times n}$, the dual problem formulation is

$$\max_{y \in \mathbf{R}^m} \quad \inf_{x \in \mathbf{R}^n} \quad L(x, y)$$

$$\text{s.t.} \quad \begin{bmatrix} \boldsymbol{I}_r & \mathbf{0}^{r \times (m-r)} \end{bmatrix} y \geq \mathbf{0}^r,$$

where the Lagrangian is given by

$$\begin{aligned} L(x, y) &= c^\intercal x + y^\intercal (Ax - b) \\ &= (c^\intercal + y^\intercal A)x - y^\intercal b, \end{aligned}$$

which gives

$$L(x^*, y) = \begin{cases} -b^\intercal y & \text{if } c^\intercal + y^\intercal A = 0 \\ -\infty & \text{else} \end{cases}.$$

Making dual constraints explicit gives the **LP**

$$\min_{y \in \mathbf{R}^m} \quad -L(x^*, y) = b^\intercal y \tag{21}$$

$$\text{s.t.} \quad \begin{aligned} \begin{bmatrix} -\boldsymbol{I}_r & \mathbf{0}^{r \times (m-r)} \end{bmatrix} y &\leq \mathbf{0}^r, \\ A^\intercal y &= -c. \end{aligned}$$

5.4.1 Standard Form

If the primal problem is in standard form, i.e., $A_{\text{ub}} = -\boldsymbol{I}_n$ and $b_{\text{ub}} = \mathbf{0}^n$, the dual can be simplified to [1, p. 224]

$$\min_{y_{\text{eq}} \in \mathbf{R}^{m-r}} \quad -L\left(x^*, y = \begin{bmatrix} y_{\text{ub}} \\ y_{\text{eq}} \end{bmatrix}\right) = b_{\text{eq}}^\intercal y_{\text{eq}}$$

$$\text{s.t.} \quad -A_{\text{eq}}^\intercal y_{\text{eq}} \leq c,$$

where

$$y_{\text{ub}}^* = A_{\text{eq}}^\intercal y_{\text{eq}}^* + c.$$

5.5 QP Duality

Consider the primal **QP**

$$
\begin{aligned}
\min_{x\in\mathbf{R}^n} \quad & J = \frac{1}{2}x^\intercal Q x + c^\intercal x \\
\text{s.t.} \quad & A_{\text{ub}} x \le b_{\text{ub}}, \\
& A_{\text{eq}} x = b_{\text{eq}}.
\end{aligned}
$$

Let

$$
A = \begin{bmatrix} A_{\text{ub}} \\ A_{\text{eq}} \end{bmatrix}, \quad b = \begin{bmatrix} b_{\text{ub}} \\ b_{\text{eq}} \end{bmatrix}.
$$

For $A_{\text{ub}} \in \mathbf{R}^{r\times n}$ and $A_{\text{eq}} \in \mathbf{R}^{m-r\times n}$, the dual problem formulation is

$$
\begin{aligned}
& \max_{y\in\mathbf{R}^m} \quad \inf_{x\in\mathbf{R}^n} \quad L(x,y) \\
\text{s.t.} \quad & \begin{bmatrix} \boldsymbol{I}_r & \mathbf{0}^{r\times(m-r)} \end{bmatrix} y \ge \mathbf{0}^r,
\end{aligned}
$$

where the Lagrangian is given by

$$
\begin{aligned}
L(x,y) &= \frac{1}{2}x^\intercal Q x + c^\intercal x + y^\intercal (Ax - b) \\
&= \frac{1}{2}x^\intercal Q x + (c^\intercal + y^\intercal A)x - y^\intercal b.
\end{aligned}
$$

5.5.1 Positive Definite Duality

For $Q \in \mathbf{S}^n_+$,

$$
\frac{\partial L}{\partial x} = x^\intercal Q + c^\intercal + y^\intercal A.
$$

Setting this to zero and solving gives

$$
x^* = -Q^{-1}(A^\intercal y + c) \tag{22}
$$

and

$$
\begin{aligned}
L(x^*,y) &= -\frac{1}{2}(c^\intercal + y^\intercal A)Q^{-1}(A^\intercal y + c) - y^\intercal b \\
&= -\frac{1}{2}y^\intercal A Q^{-1} A^\intercal y - c^\intercal Q^{-1} A^\intercal y - \frac{1}{2}c^\intercal Q^{-1} c - b^\intercal y.
\end{aligned}
$$

The dual problem is the **QP**

$$
\begin{aligned}
\min_{y\in\mathbf{R}^m} \quad & -L(x^*,y) = \frac{1}{2}y^\intercal Q_{\text{dual}}\, y + c_{\text{dual}}^\intercal\, y \\
\text{s.t.} \quad & \begin{bmatrix} -\boldsymbol{I}_r & \mathbf{0}^{r\times(m-r)} \end{bmatrix} y \le \mathbf{0}^r,
\end{aligned} \tag{23}
$$

where

$$
\begin{aligned}
Q_{\text{dual}} &= A Q^{-1} A^\intercal, \\
c_{\text{dual}} &= b + A Q^{-1} c.
\end{aligned}
$$

Note: If $n > m$, the dual problem provides a computationally effecient alternative with m free variables to solve.

5.5.2 Non–Negative Definite Duality

For $Q \in \mathbf{S}^n_{0+}$, SVD gives $Q = U_+ S_+ V_+^\intercal$ with $U_+ = V_+$. Choose

$$x = \begin{bmatrix} V_+ & V_0 \end{bmatrix} \begin{bmatrix} x_+ \\ x_0 \end{bmatrix}.$$

Note: The primal problem can be solved with an ADMM of **LP** and positive definite **QP** .
The Lagrangian becomes

$$\begin{aligned} L(x, y) = \frac{1}{2} x_+^\intercal S_+ x_+ + \begin{bmatrix} c_+^\intercal & c_0^\intercal \end{bmatrix} \begin{bmatrix} x_+ \\ x_0 \end{bmatrix} \\ + y^\intercal \left(\begin{bmatrix} A_+ & A_0 \end{bmatrix} \begin{bmatrix} x_+ \\ x_0 \end{bmatrix} - b_{\text{eq}} \right) \end{aligned}$$

where

$$\begin{aligned} A_+ &:= AV_+, \quad A_0 := AV_0, \\ c_+^\intercal &:= c^\intercal V_+, \quad c_0^\intercal := c^\intercal V_0. \end{aligned}$$

Minimize x_+ with

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_+} \\ &= x_+^\intercal S_+ + c_+^\intercal + y^\intercal A_+, \end{aligned}$$

which gives

$$x_+^* = -S_+^{-1}(A_+^\intercal y + c_+). \tag{24}$$

Minimizing x_0,

$$L(x^*, y) = \frac{1}{2} x_+^{*\intercal} S_+ x_+^* + \begin{bmatrix} c_+^\intercal & c_0^\intercal \end{bmatrix} \begin{bmatrix} x_+^* \\ x_0 \end{bmatrix} + y^\intercal \left(\begin{bmatrix} A_+ & A_0 \end{bmatrix} \begin{bmatrix} x_+^* \\ x_0 \end{bmatrix} - b \right)$$

$$= \begin{cases} \frac{1}{2} x_+^{*\intercal} S_+ x_+^* + \left(c_+^\intercal + y^\intercal A_+ \right) x_+^* - b^\intercal y & \text{if} \quad c_0 + A_0^\intercal y = \mathbf{0} \\ -\infty & \text{else} \end{cases}$$

$$= \begin{cases} -\frac{1}{2} x_+^{*\intercal} S_+ x_+^* - b^\intercal y & \text{if} \quad c_0 + A_0^\intercal y = \mathbf{0} \\ -\infty & \text{else} \end{cases}$$

Making dual constraints explicit gives the **QP**

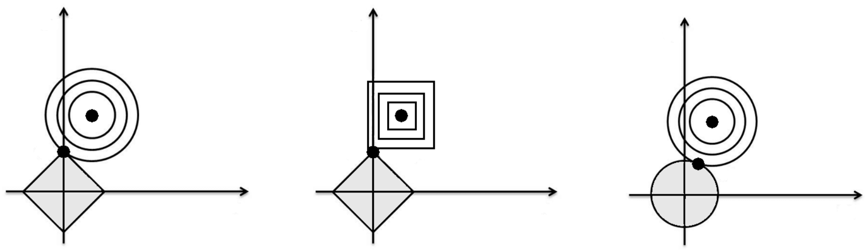
$$\min_{y \in \mathbf{R}^m} \quad -L(x^*, y) = \frac{1}{2} y^\intercal Q_{\text{dual}}\, y + c_{\text{dual}}^\intercal\, y \tag{25}$$

$$\begin{aligned} \text{s.t.} \quad \begin{bmatrix} -\boldsymbol{I}_r & \mathbf{0}^{r \times (m-r)} \end{bmatrix} y \leq \mathbf{0}^r, \\ A_0^\intercal y = -c_0, \end{aligned}$$

where

$$\begin{aligned} Q_{\text{dual}} &= A_+ S_+^{-1} A_+^\intercal, \\ c_{\text{dual}} &= b + A_+ S_+^{-1} c_+. \end{aligned}$$

6 Regularization



(a) $p = 2,\ q = 1$ (b) $p = \infty,\ q = 1$ (c) $p = 2,\ q = 2$

Figure 12: ℓ_p residual with ℓ_q constraint

Consider

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|Ax - b\|_p \\ \text{s.t.} \quad & \|x\|_q \leq t, \end{aligned}$$

where $q \in \{1, 2, \infty\}$. The dual problem is [32]

$$\begin{aligned} \max_{\lambda \in \mathbf{R}} \min_{x \in \mathbf{R}^n} \quad & J = \|Ax - b\|_p + \lambda(\|x\|_q - t) \\ \text{s.t.} \quad & \lambda \geq 0. \end{aligned}$$

The optimal pair $\{x^*, \lambda^*\}$ that meet the threshold t will depend on $\{A, b\}$. Conversely, there exists a t for every fixed selection of $\lambda > 0$. Selecting $\lambda > 0$, the regularized problem is defined as

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\|_p + \lambda \|x\|_q. \tag{26}$$

Figure 12 shows ℓ_1 constraints with ℓ_2 residuals will preferentially seek solutions with ordinates at zero. This also holds true for ℓ_1 constraints with ℓ_∞ residuals, but it does not hold true for ℓ_2 or ℓ_∞ constraints.

6.1 2–Norm Residual with 1–Norm Regularization

This is known as LASSO (Least Absolute Shrinkage and Selection Operator).
Note: This form of regularization seeks solutions with zeros on the ordinates.

The regularized problem is given by

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad J &= \|Ax - b\|_2^2 + \lambda \|x\|_1 \\ &= \frac{1}{2} x^\intercal Q x + c^\intercal x + r + \lambda \|x\|_1, \end{aligned}$$

where

$$\begin{aligned} Q &= 2A^\intercal A, \\ c &= -2A^\intercal b, \\ r &= b^\intercal b, \end{aligned}$$

which can be expressed as the **QP**

$$\begin{aligned} \min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^n\}} \quad & J = \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^\intercal \begin{bmatrix} Q & \mathbf{0}^{n \times n} \\ \mathbf{0}^{n \times n} & \mathbf{0}^{n \times n} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ \mathbf{1}^n \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} -\lambda \boldsymbol{I}_n & -\boldsymbol{I}_n \\ \lambda \boldsymbol{I}_n & -\boldsymbol{I}_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} \mathbf{0}^n \\ \mathbf{0}^n \end{bmatrix}. \end{aligned}$$

6.2 Inf–Norm Residual with 1–Norm Regularization

Note: This form of regularization seeks solutions with zeros on the ordinates.
For $A \in \mathbf{R}^{m \times n}$, the regularized problem is given by

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\|_\infty + \lambda \|x\|_1,$$

which can be expressed as the **LP**

$$\min_{\{x,y,z\} \in \{\mathbf{R}^n, \mathbf{R}^n, \mathbf{R}\}} \quad J = \begin{bmatrix} \mathbf{0}^n \\ \mathbf{1}^n \\ 1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} -\lambda \boldsymbol{I}_n & -\boldsymbol{I}_n & \mathbf{0}^n \\ \lambda \boldsymbol{I}_n & -\boldsymbol{I}_n & \mathbf{0}^n \\ -A & \mathbf{0}^{m \times n} & -\mathbf{1}^m \\ A & \mathbf{0}^{m \times n} & -\mathbf{1}^m \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \begin{bmatrix} \mathbf{0}^n \\ \mathbf{0}^n \\ -b \\ b \end{bmatrix}.$$

6.3 2–Norm Residual with 2–Norm Regularization

The regularized problem is given by the **QP**

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad J &= \|Ax - b\|_2^2 + \lambda \|x\|_2^2 \\ &= \frac{1}{2} x^\mathsf{T} Q x + c^\mathsf{T} x + b^\mathsf{T} b, \end{aligned}$$

where

$$\begin{aligned} Q &= \quad 2A^\mathsf{T} A + \lambda \boldsymbol{I}, \\ c &= -2A^\mathsf{T} b. \end{aligned}$$

Computing $A = U_+ S_+ V_+^\mathsf{T}$ with SVD, the problem can be regularized with uniform circles (as depicted in figure 12). Using

$$x = V_+ S_+^{-1} y + V_0 z,$$

the reformulation is give by

$$\begin{aligned} \min_{\{y,z\} \in \{\mathbf{R}^r, \mathbf{R}^{n-r}\}} \quad J &= \|U_+ y - b\|_2^2 + \lambda \|V_+ S_+^{-1} y + V_0 z\|_2^2 \\ &= \|y - U_+^\mathsf{T} b\|_2^2 + \lambda \|y\|_{2, S_+^{-1}}^2 + \lambda \|z\|_2^2 + b^\mathsf{T} U_0 U_0^\mathsf{T} b \\ &= \frac{1}{2} \begin{bmatrix} y \\ z \end{bmatrix}^\mathsf{T} Q \begin{bmatrix} y \\ z \end{bmatrix} + c^\mathsf{T} \begin{bmatrix} y \\ z \end{bmatrix} + b^\mathsf{T} b, \end{aligned}$$

where

$$\begin{aligned} Q &= \begin{bmatrix} \boldsymbol{I}_r + \lambda S_+^{-2} & \mathbf{0}^{r \times n-r} \\ \mathbf{0}^{n-r \times r} & \lambda \boldsymbol{I}_{n-r} \end{bmatrix}, \\ c &= \begin{bmatrix} -2U_+^\mathsf{T} b \\ \mathbf{0}^{n-r} \end{bmatrix}. \end{aligned}$$

Note: The residual is now circular, and the the regularizaton constraint is elliptical and aligned with the axis.

6.4 Generalized Regularization

A more general form of regularization can be given by

$$\min_{x\in\mathbf{R}^n} \quad J = \|Ax-b\| + \sum_{i=0}^{n-1} f_i(x_i),$$

where $f_i(\cdot)$ is any convex penalty function that can be formulated as an **LP** or **QP** , e.g., deadzone or Huber.

6.5 Non–Convex Regularization

For $p \in \mathbf{R}(0,1)$, consider

$$\begin{aligned}\min_{x\in\mathbf{R}^n} \quad J &= \|Ax-b\|_{1/p} + \|x\|_p \\ &= \left(\sum_{i=0}^{m-1} |a_i^\intercal x - b_i|^{1/p}\right)^p + \left(\sum_{i=0}^{m-1} |x_i|^p\right)^{1/p}.\end{aligned}$$

This problem formulation gives exceptional sparsity for large p. It is equivalent to $1/p$–regularized p–constrained problem illustrated in figure 13a. This problem is non–convex, which means multiple extremum may occur. Figure 13b illustrates non–unique solution.

The gradient is given by

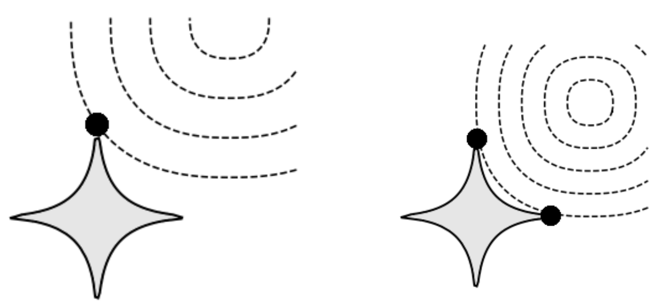
$$\frac{\partial J}{\partial x_j} = \|r\|_{1/p}^{-1}\left(\sum_{i=0}^{m-1} |r_i|^{1/p-1}\text{sign}(r_i)A_{ij}\right) + \|x\|_p^{-1}|x_j|^{p-1}\text{sign}(x_j),$$

where

$$r_i = a_i^\intercal x - b_i,$$

and the Hessian is given by

$$\begin{aligned}\frac{\partial^2 J}{\partial x_j \partial x_k} = &\frac{p-1}{p}\|r\|_{1/p}^{-2}\left(\sum_{i=0}^{m-1} A_{ij}\,\text{sign}(r_i)|r_i|^{1/p-1}\right)\left(\sum_{i=0}^{m-1} A_{ik}\,\text{sign}(r_i)|r_i|^{1/p-1}\right) \\ &+ (1/p-1)\|r\|_{1/p}^{-1}\left(\sum_{i=0}^{m-1} A_{ij}A_{ik}|r_i|^{1/p-2}\right) \\ &+ p(1/p-1)\|x\|_p^{-2}\text{sign}(x_jx_k)|x_jx_k|^{p-1} + (p-1)\|x\|_p^{-1}|x_j|^{p-2}\delta_{jk}.\end{aligned}$$



(a) preference for a single axis (b) possible non–unique solutions

Figure 13: non–convex regularization

7 Nonlinear Programming

Specialized solutions exist for specific forms of nonlinearity, but nonlinear programs do not have a general solution. Convex nonlinear programs expressed in standard form can be solved with interior point methods with gaurantees on determining feasibility and finding a global minimum. Non–convex nonlinear programs may have infinately many local extremum, with no gaurantees on determining feasibility or finding a global minimum. Nonlinear programs are typically solved by computing local first and second derivatives and taking small constrained or regulated steps starting at a random initial guess. For the following section, let $d = \|x - x_0\|_2$.

7.1 Linearizing Nonlinearity Inside Norm

Consider the nonlinear function $f(x) : \mathbf{R}^n \to \mathbf{R}^m$, and consider

$$\begin{aligned} J &= \|f(x)\| \\ &= \|f_0 + g_0^\intercal (x - x_0) + O(d^2)\| \\ &= \|A_0 x + b_0 + O(d^2)\|, \end{aligned}$$

where

$$\begin{aligned} A_0 &= g_0^\intercal, \\ b_0 &= f_0 - g_0^\intercal x_0, \end{aligned}$$

and

$$\begin{aligned} f_0 &= \lim_{x \to x_0} f, \\ g_0 &= \lim_{x \to x_0} \nabla f. \end{aligned}$$

If $d^2 \leq nr^2$ for a sufficiently small $r > 0$, the problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|f(x)\| \\ \text{s.t.} \quad & x \in \mathbf{P} \end{aligned} \tag{27}$$

can be solved with Algorithm 1 or 2. **Note**: Any of the penalty functions that result in **LP** or **QP** can be used.

$x_0 \in \mathbf{P}$, $r > 0$
while x_0 not converged **do**

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|A_0 x + b_0\| \\ \text{s.t.} \quad & x \in \mathbf{P}, \\ -r\mathbf{1}^n \leq & x - x_0 \leq r\mathbf{1}^n. \end{aligned}$$

$x_0 \leftarrow x^*$
end

Algorithm 1: Nonlinear Solver with Step Bounding Box. Alternative: p–norm constraint with equation (9).

$x_0 \in \mathbf{P}$, $\lambda > 0$
while x_0 not converged **do**

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|A_0 x + b_0\| + \lambda \|x - x_0\| \\ \text{s.t.} \quad & x \in \mathbf{P}. \end{aligned}$$

$x_0 \leftarrow x^*$
end

Algorithm 2: Nonlinear Solver with Step Regularization

7.2 Differentiable Objective Functions

Consider

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = f(x) \\ \text{s.t.} \quad & x \in \mathbf{P}, \end{aligned} \tag{28}$$

where $f(x)$ is differentiable. Expansion of the objective gives

$$\begin{aligned} J &= J_0 + g_0^\intercal (x - x_0) + O(d^2) \\ &= J_0 + g_0^\intercal (x - x_0) + \frac{1}{2}(x - x_0)^\intercal H_0 (x - x_0) + O(d^3) \\ &= \frac{1}{2} x^\intercal Q_0 x + c_0^\intercal x + \text{constant} + O(d^3), \end{aligned}$$

with

$$\begin{aligned} Q_0 &= H_0, \\ c_0 &= g_0 - H_0 x_0, \end{aligned}$$

and

$$\begin{aligned} g_0 &= \lim_{x \to x_0} \nabla f, \\ H_0 &= \lim_{x \to x_0} \text{Hessian}(f). \end{aligned}$$

If $d^3 \leq nr^3$ for a sufficiently small $r > 0$, the problem can be solved with Algorithm 3, 4, or 5.

7.2.1 Gradient Search

Consider

$$\min_{x \in \mathbf{R}^n} \quad J = J_0 + g_0^\intercal (x - x_0) + O(d^2).$$

Ignoring the $O(d^2)$ term, stepping in the direction of maximum decline with $s \in \mathbf{R}$ and $M \in \mathbf{S}_+^n$,

$$\begin{aligned} x^*(s) &= x_0 - e^s M g_0, \\ J^*(s) &= J_0 - e^s g_0^\intercal M g_0 + O(e^{2s}). \end{aligned}$$

Algorithm 6 searches for an optimal s and then takes a scaled step.
Note: If $g_0 \neq 0$, $M = g_0 g_0^\intercal / \|g_0\|_2^2$ gives

$$\begin{aligned} x^*(s) &= x_0 - e^s g_0 / \|g_0\|_2, \\ J^*(s) &= J_0 - e^s + O(e^{2s}). \end{aligned}$$

7.2.2 Hessian Search

Consider

$$\min_{x \in \mathbf{R}^n} \quad J = J_0 + g_0^\intercal (x - x_0) + \frac{1}{2}(x - x_0)^\intercal H_0 (x - x_0) + O(d^3).$$

Ignoring the $O(d^3)$ term, the optimal solution is found by solving

$$0 = g_0^\intercal + (x^* - x_0)^\intercal H_0,$$

which gives the pure–Newton iteration

$$x^* = x_0 - H_0^+ g_0.$$

Note: SVD can provide well-conditioned steps using matrix conditioning.
If the pure–Newton iteration is scaled by $s > 0$ with

$$x(s) = x_0 - sH_0^+ g_0,$$

then,

$$J(s) = J_0 + (s^2/2 - s) g_0^\intercal H_0^+ g_0 + O(s^3),$$

where $(s^2/2 - s)$ is negative for $s \in (0, 2)$, equal to zero at $s \in \{0, 2\}$, and minimum at $s = 1$.
Algorithm 7 searches for an optimal s and then takes a scaled step.

7.2.3 Differentiable 2–Norm Objective

Consider a differentiable nonlinear function $f(x) : \mathbf{R}^n \to \mathbf{R}^m$ and the problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|f(x)\|_{2|W}^2 \\ \text{s.t.} \quad & x \in \mathbf{P}. \end{aligned} \tag{29}$$

If $W = \text{diag}\{w_i\}_{i=0}^{m-1}$,

$$J = \sum_{i=0}^{m-1} w_i^2 f_i^2.$$

The gradient is given by

$$\begin{aligned} g &= \nabla J \\ &= 2 \sum_{i=0}^{m-1} w_i^2 f_i \nabla f_i \end{aligned}$$

The Hessian is given by

$$\begin{aligned} H &= \frac{\partial}{\partial x} \nabla J \\ &= 2 \sum_{i=0}^{m-1} w_i^2 (\nabla f_i \nabla^\intercal f_i + f_i \text{Hessian}(f_i)). \end{aligned}$$

If $W \in R^{m \times m}$ with $M = W^\intercal W$,

$$\|f(x)\|_{2|W}^2 = \textstyle\sum_{i=0}^{m-1} \sum_{j=0}^{m-1} M_{ij} f_i f_j.$$

The gradient is

$$\begin{aligned} g &= \nabla J \\ &= 2h^\intercal M f, \end{aligned}$$

where

$$h = \frac{\partial f}{\partial x},$$

and the Hessian is

$$\begin{aligned} H &= 2(\boldsymbol{I}_n \otimes f^\intercal M) \frac{\partial \operatorname{vec}(h^\intercal)}{\partial x} + 2h^\intercal M h \\ &= 2 \text{ row} \left\{ f^\intercal M \frac{\partial h}{\partial x_i} \right\}_{i=0}^{n-1} + 2h^\intercal M h. \end{aligned}$$

$x_0 \in \mathbf{P}$, $r > 0$
while x_0 not converged **do**

$$\min_{x \in \mathbf{R}^n} \quad J = \frac{1}{2}x^\intercal Q_0 x + c_0^\intercal x$$

$$\text{s.t.} \quad x \in \mathbf{P}, \\ -r\mathbf{1}^n \leq x - x_0 \leq r\mathbf{1}^n.$$

$x_0 \leftarrow x^*$
end

Algorithm 3: Nonlinear Solver with Step Bounding Box.

$x_0 \in \mathbf{P}$, $\lambda > 0$
while x_0 not converged **do**

$$\min_{x \in \mathbf{R}^n} \quad J = \frac{1}{2}x^\intercal Q_0 x + c_0^\intercal x + \lambda\|x - x_0\|$$

$$\text{s.t.} \quad x \in \mathbf{P}.$$

$x_0 \leftarrow x^*$
end

Algorithm 4: Nonlinear Solver with Step Regularization

$x_0 \in \mathbf{P}$, $r > 0, p \in \{1, \infty\}$
while x_0 not converged **do**

$$\min_{x \in \mathbf{R}^n} \quad J = \frac{1}{2}x^\intercal Q_0 x + c_0^\intercal x$$

$$\text{s.t.} \quad x \in \mathbf{P}, \\ \|x - x_0\|_p \leq n^{1/p} r.$$

$x_0 \leftarrow x^*$
end

Algorithm 5: Nonlinear Solver with p–norm constraint

$x_0 \in \mathbf{P}$, $M \in \mathbf{S}_+^n$
while x_0 not converged **do**

$$x(s) = x_0 - e^s M g_0,$$

$$\min_{s \in \mathbf{R}} J(x(s)).$$

$x_0 \leftarrow x^*$
end

Algorithm 6: Unconstrained Nonlinear Solver with Gradient Line Search

$x_0 \in \mathbf{P}$
while x_0 not converged **do**

$$x(s) = x_0 - sH_0^+ g_0,$$

$$\min_{s \in \mathbf{R}[0,2]} J(x(s))$$

$x_0 \leftarrow x^*$
end

Algorithm 7: Unconstrained Nonlinear Solver with Hessian Line Search

7.3 Nonlinear Equality Constraints

Consider the nonlinear function $f_{\text{eq}}(x) : \mathbf{R}^n \to \mathbf{R}^m$, and consider the problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|Ax - b\| \\ \text{s.t.} \quad & x \in \mathbf{P}, \\ & f_{\text{eq}}(x) = 0. \end{aligned} \tag{30}$$

7.3.1 Penalty Approach

The nonlinear constraint in equation (30) can be brought into the objective function with

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\| + \|f_{\text{eq}}(x)\|_W$$

$$\text{s.t.} \quad x \in \mathbf{P}.$$

Note: The minimum singular value of W must be much larger than the maximum singular value of A.
If the residual and regularization are ℓ_2, then

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad J &= \|Ax - b\|_2^2 + \|f_{\text{eq}}(x)\|_{2|W}^2 \\ &= x^\intercal A^\intercal A x - 2b^\intercal A x + b^\intercal b + \|f_{\text{eq}}(x_0)\|_{2|W}^2 + g_0^\intercal (x - x_0) + \frac{1}{2}(x - x_0)^\intercal H_0 (x - x_0) + O(d^3) \\ &= \frac{1}{2} x^\intercal Q_0 x + c_0^\intercal x + \text{constant} + O(d^3) \end{aligned}$$

$$\text{s.t.} \quad x \in \mathbf{P},$$

where

$$\begin{aligned} Q_0 &= \quad 2A^\intercal A + H_0 \\ c_0 &= -2A^\intercal b + g_0 - H_0 x_0, \end{aligned}$$

which can be solved with Algorithm 3 or 4.

7.3.2 Linearization

The nonlinear constraint in equation (30) can be linearized with

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\|$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & B_0 x = c_0 + O(d^2), \end{aligned}$$

where

$$\begin{aligned} B &= g^\intercal, \\ c &= g^\intercal x - f_{\text{eq}}, \end{aligned}$$

and

$$g = \nabla f_{\text{eq}}.$$

The constraint $d^2 \leq nr^2$ can be added as a box (Algrothim 8), p–norm (Algorithm 9), or regularization (Algorithm 10).

7.3.3 Nonlinear Inequality

Consider the nonlinear inequality $f_{\text{ub}}(x) \leq 0$. A slack variable $y \in \mathbf{R}^m$ can be used to recover nonlinear equality with

$$\min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad J = \|Ax - b\| \tag{31}$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & f_{\text{ub}}(x) + y = 0, \\ & y \geq 0. \end{aligned}$$

$x_0 \in \mathbf{P}$, $r > 0$
while x_0 not converged **do**

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\|$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & B_0\, x = c_0 \\ & -r\mathbf{1}^n \leq x - x_0 \leq r\mathbf{1}^n. \end{aligned}$$

$x_0 \leftarrow x^*$
end

Algorithm 8: Nonlinear Constrained Solver with Step Bounding Box.

$x_0 \in \mathbf{P}$, $r > 0$, $p \in \{1, \infty\}$
while x_0 not converged **do**

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\|$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & B_0\, x = c_0, \\ & \|x - x_0\|_p \leq n^{1/p} r. \end{aligned}$$

$x_0 \leftarrow x^*$
end

Algorithm 9: Nonlinear Constrained Solver with p–norm constraint solved with equation (9)

$x_0 \in \mathbf{P}$, $\lambda > 0$, $a > 1$
while $\lambda < \lambda^\circ$ **do**
while x_0 not converged **do**

$$\min_{x \in \mathbf{R}^n} \quad J = \|Ax - b\| + \lambda \|B_0\, x - c_0\|$$

$$\text{s.t.} \quad x \in \mathbf{P}.$$

$x_0 \leftarrow x^*$
end
$\lambda \leftarrow a\lambda$
end

Algorithm 10: Nonlinear Constrained Solver with Penalty

7.3.4 Logarithmic Barrier

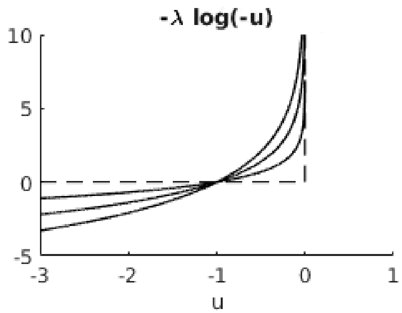


Figure 14: Log–Barrier Penalty

Adding a slack variable, the nonlinar constraint in equation (30) can be re–expressed as the nonlinear inequality problem

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}^m\}} \quad J = \|Ax-b\| + \|y\|_W \tag{32}$$
$$\text{s.t.} \quad -f_{\text{eq}}(x) - y \le 0,$$
$$f_{\text{eq}}(x) - y \le 0.$$

Note: These constraints give $-y \le f_{\text{eq}}(x) \le y$ with $y \ge 0$.

For $\lambda > 0$, the logrithmic barrier problem is given by [1, p. 562]

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}^m\}} \quad J = \|Ax-b\| + \|y\|_W + \lambda h(x,y),$$

where

$$h = -\sum_{i=0}^{m-1} \left(\log(y_i + f_i) + \log(y_i - f_i)\right).$$

The gradient is

$$\nabla^{\mathsf{T}} h = \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix},$$

where

$$\frac{\partial h}{\partial x} = -\sum_{i=0}^{m-1} \left(\frac{1}{y_i + f_i} - \frac{1}{y_i - f_i}\right) \frac{\partial f_i}{\partial x},$$

$$\frac{\partial h}{\partial y} = -\sum_{i=0}^{m-1} \left(\frac{1}{y_i + f_i} + \frac{1}{y_i - f_i}\right) \mathbf{e}_i^{\mathsf{T}}.$$

Note: This problem formulation is equivalent to the regularized problem

$$\min_{x\in\mathbf{R}^n} \quad J = \|Ax-b\| + \|f_{\text{eq}}(x)\|_W.$$

7.4 Generalized Reformulations

There are several possilbe reformulations of the general problem

$$\min_{x\in\mathbf{D}} \quad J = f(x). \tag{33}$$

7.4.1 Projection

If the $\mathbf{D}$ in equation (33) can be expressed in terms of f_{ub} and f_{eq}, then the general problem can be written as

$$\begin{aligned} \min_{x\in\mathbf{R}^n} \quad & \|f(x)\| \\ \text{s.t.} \quad & x \in \mathbf{P}, \\ & f_{\text{ub}}(x) \le b_{\text{ub}}, \\ & f_{\text{eq}}(x) = b_{\text{eq}}. \end{aligned}$$

Consider a one-to-one projective basis $h(x) : \mathbf{R}^n \to \mathbf{R}^m$, and project over the domain $x \in \mathbf{P}$ to get

$$\begin{aligned} f\ \ (x) &\approx A\ \ h(x), \\ f_{\text{ub}}(x) &\approx A_{\text{ub}} h(x), \\ f_{\text{eq}}(x) &\approx A_{\text{eq}} h(x), \end{aligned}$$

where

$$\begin{aligned} A\ \ &= \left(\int_{x\in\mathbf{P}} f\ \ h^\intercal dx\right)\left(\int_{x\in\mathbf{P}} hh^\intercal dx\right)^+, \\ A_{\text{ub}} &= \left(\int_{x\in\mathbf{P}} f_{\text{ub}} h^\intercal dx\right)\left(\int_{x\in\mathbf{P}} hh^\intercal dx\right)^+, \\ A_{\text{eq}} &= \left(\int_{x\in\mathbf{P}} f_{\text{eq}} h^\intercal dx\right)\left(\int_{x\in\mathbf{P}} hh^\intercal dx\right)^+. \end{aligned}$$

The problem can be approximated by

$$\begin{aligned} \min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}^m\}} \quad & \|Ay\| \\ \text{s.t.} \quad & x \in \mathbf{P} \\ & A_{\text{ub}} y \le b_{\text{ub}}, \\ & A_{\text{eq}} y = b_{\text{eq}}, \\ & y = h(x). \end{aligned} \tag{34}$$

7.4.2 Epigraph

The problem given by equation (33) is equivalent to the epigraph problem

$$\begin{aligned} \min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}\}} \quad & J = y \\ \text{s.t.} \quad & f(x) - y \le 0, \\ & x \in \mathbf{D}. \end{aligned} \tag{35}$$

7.4.3 Indicator

The problem given by equation (33) is equivalent to the indicator function problem

$$\begin{aligned} \min_{x\in\mathbf{R}^n} \quad & f(x) \\ \text{s.t.} \quad & \text{Indicator}(x \in \mathbf{D}) \le 0, \end{aligned} \tag{36}$$

where

$$\text{Indicator}(x \in \mathbf{D}) = \begin{cases} 0 & \text{if} \quad x \in \mathbf{D} \\ \infty & \text{else} \end{cases}.$$

7.4.4 Exponential Substitution

With $x_i = y_i e^{z_i}$, the problem given by equation (33) can be solved with

$$\begin{aligned} \min_{y \circ \exp(z) \in \mathbf{D}} \quad & J = f(y \circ \exp(z)) \\ \text{s.t.} \quad & -\mathbf{1} \leq y \leq \mathbf{1}. \end{aligned} \tag{37}$$

The gradient and Hessian of $f(x) : \mathbf{R}^n \to \mathbf{R}$ can be calculated with

$$\begin{aligned}
\frac{\partial}{\partial y_i} f &= g_i e^{z_i} && = [g \circ e^z]_i, \\
\frac{\partial}{\partial z_i} f &= g_i y_i e^{z_i} && = [g \circ y \circ e^z]_i, \\
\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i} f &= H_{ij} e^{z_i + z_j} && = [H \circ (e^z)(e^z)^\intercal]_{ij}, \\
\frac{\partial}{\partial z_j}\frac{\partial}{\partial y_i} f &= H_{ij} y_j e^{z_i + z_j} + g_i e^{z_i} \delta_{ij} && = [H \circ (e^z)(y \circ e^z)^\intercal + \text{diag}(g \circ e^z)]_{ij}, \\
\frac{\partial}{\partial y_j}\frac{\partial}{\partial z_i} f &= H_{ij} y_i e^{z_i + z_j} + g_i e^{z_i} \delta_{ij} && = [H \circ (y \circ e^z)(e^z)^\intercal + \text{diag}(g \circ e^z)]_{ij} \\
\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_i} f &= H_{ij} y_i y_j e^{z_i + z_j} + g_i y_i e^{z_i} \delta_{ij} && = [H \circ (y \circ e^z)(y \circ e^z)^\intercal + \text{diag}(g \circ y \circ e^z)]_{ij}
\end{aligned}$$

where

$$g_i = \frac{\partial}{\partial x_i} f, \quad H_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f.$$

7.4.5 Invertable Nonlinearity

If the problem given by equation (33) has $f(x) = \|h(x) - y\|$, where $h(x)$ is invertable for all $x \in \mathbf{D}$. Then,

$$\underset{x \in \mathbf{D}}{\text{argmin}} \quad \|h(x) - y\| \quad = \quad \underset{x \in \mathbf{D}}{\text{argmin}} \quad \|x - h^{-1}(y)\|. \tag{38}$$

7.5 Convex Programming

Some nonlinear problems are convex, and some of the nonlinear convex problems have well-known solutions. For $i \in \mathbf{Z}[0, m]$ and convex $f_i(x)$, a general convex problem can be expressed as

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \\ & Ax = b. \end{aligned} \tag{39}$$

Note: The equality constraint must be linear [1, p. 137]. For $i \in \mathbf{Z}[1, m]$, the constraints define the set

$$\mathbf{D} = \{x \in \mathbf{R}^n | f_i(x) \leq 0, Ax = b\}.$$

For all $i \in \mathbf{Z}[0, m]$, $x \in \mathbf{D}$, $y \in \mathbf{D}$, and $p \in \mathbf{R}[0, 1]$, the problem is convex if

$$f_i(px + (1 - p)y) \leq p f_i(x) + (1 - p) f_i(y).$$

Any convex problem can be solved with interior point methods.

Several convex problems are given special consideration with specialized simplifications and solutions, e.g.,

- **LP** : Linear Programs,
- **QP** : Quadratic Programs,
- **SOCP** : Second Order Cone Programs [1, p. 158],
- **QCQP** : Quadratic Constrained Quadratic Programs [1, p. 152][42],
- **GP** : Geometric Programs [1, p. 161],
- **LMI** : Linear Matrix Inequalities [1, p. 169],
- **SDP** : Semi Definite Programs [1, p. 168][42].

Solvers for each of these specialized convex problems can be found in many programming languages.
It is often worth the effort to formulate a problem as one of these standard problems.

7.5.1 QCQP : Quadratic Constrained Quadratic Programming

For $i \in \mathbf{Z}[0, m-1]$, any convex **QCQP** can be expressed as

$$\min_{x \in \mathbf{R}^n} \quad J = \frac{1}{2}x^\intercal Q x + c^\intercal x \tag{40}$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & \|A_i x - b_i\|_2^2 \leq 1. \end{aligned}$$

A **QCQP** can also be expressed as

$$\min_{\{x, z_i\} \in \{\mathbf{R}^n, \mathbf{R}^{m_i}\}} \quad J = \frac{1}{2}x^\intercal Q x + c^\intercal x$$

$$\begin{aligned} \text{s.t.} \quad & x \in \mathbf{P}, \\ & \|z_i\|_2^2 \leq 1, \\ & A_i x - z_i = b_i. \end{aligned}$$

If $A_i \in \mathbf{R}^{n \times n}$ with $\text{rank}(A_i) = n$, then

$$\begin{aligned} \|A_i x - b_i\|_2^2 &= \|A_i(x - A_i^{-1} b_i)\|_2^2 \\ &=: (x - y_i)^\intercal R_i^{-1} (x - y_i) \end{aligned}$$

is an ellipse centered at

$$y_i = A_i^{-1} b_i$$

with covariance

$$R_i = (A_i^\intercal A_i)^{-1}.$$

Note: If $\text{rank}(A_i) < n$, SVD can be used to compute the elliptic subspace.

7.5.2 SDP : Semi–Definite Programming

For $A_i \in \mathbf{S}^n$ and $B \in \mathbf{S}^n$, the primal **SDP** is

$$\min_{x \in \mathbf{R}^n} \quad J = c^\intercal x \tag{41}$$

$$\text{s.t.} \quad \sum_{i=0}^{n-1} x_i A_i \preceq B.$$

This form is also known as an **LMI** (linear matrix inequality).
The Lagrangian is

$$L(x, Y) = c^\intercal x + \text{tr}\left(Y \sum_{i=0}^{n-1} x_i A_i\right) - \text{tr}\,(YB)\,,$$

with

$$\inf_{x \in \mathbf{R}^n} L(x, Y) = \begin{cases} -\text{tr}\,(BY) & \text{if} \quad \text{tr}\,(A_i Y) + c_i = 0 \\ -\infty & \text{else} \end{cases} .$$

The dual equals the primal if the primal inequality is strictly feasible. The dual problem is given by

$$\min_{Y \in \mathbf{R}^{n \times n}} \quad J = \text{tr}\,(BY) \tag{42}$$

$$\text{s.t.} \quad \text{tr}\,(A_i Y) = -c_i, \\ Y \succeq 0,$$

For $i \in \mathbf{Z}[0, n-1]$, the positive definite constraint gives inequality constraints of the form

$$-\det(M_i) \leq 0,$$

where M_i are the principle minors of Y, i.e., $M_i = Y[0 : i, 0 : i]$.
If the **SDP** is optimized with the interior point method,

$$\frac{\partial}{\partial Y} \text{tr}(YB) = B^\intercal = B,$$
$$\frac{\partial}{\partial Y} \log\det(M_i) = \text{tr}\left(M_i^{-1} \frac{\partial M_i}{\partial Y}\right).$$

Note: If $B \in \mathbf{S}^n$ and $Y \in \mathbf{S}^n$,

$$\text{tr}\,(BY) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} B_{ij} Y_{ij} = \vec{B}^\intercal \vec{Y}.$$

Note:

$$x^\intercal Q x = \text{tr}\,(Q x x^\intercal) = \text{tr}\,(QX)$$

with $X \succeq 0$ and $\text{rank}(X) = 1$.

8 Sigmoids

Sigmoids are a common form of nonlinearity in generalized machine learning applications.
Many of them can be expressed as derivatives of the convex penalty functions discussed earlier.

8.1 Boolean Sigmoids

Boolean sigmoids map $s : \mathbf{R} \to \mathbf{R}[0, 1]$. Figure 15 shows a sigmoid computed from the gradient of ReLu, and figure 16 shows a sigmoid computed from the gradient of a right-sidded Huber penalty.

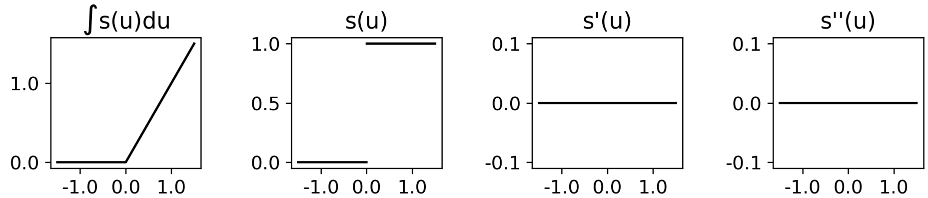


Figure 15: $s(u) = \text{Boolean}(u > 0)$

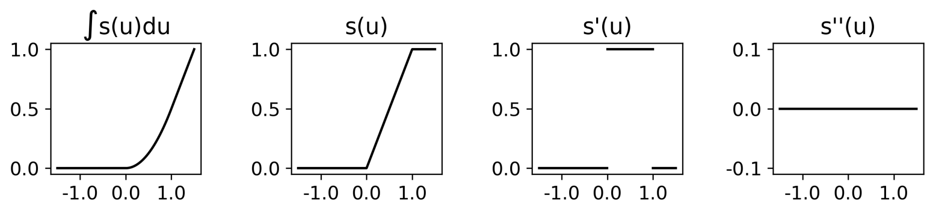


Figure 16: $s(u) = \text{Saturation}(u, 0, 1)$

The logistic sigmoid is defined as

$$s(u) = (1 + e^{-u})^{-1}, \tag{43}$$

with

$$\begin{aligned} s'(u) &= s(1-s), \\ s''(u) &= s(1-s)^2 - s^2(1-s), \\ \int s(u)du &= \log(1+e^u) \qquad \approx \max(0, u). \end{aligned}$$

Note: This function curves primarily for $u \in \mathbf{R}[-2\pi, 2\pi]$.

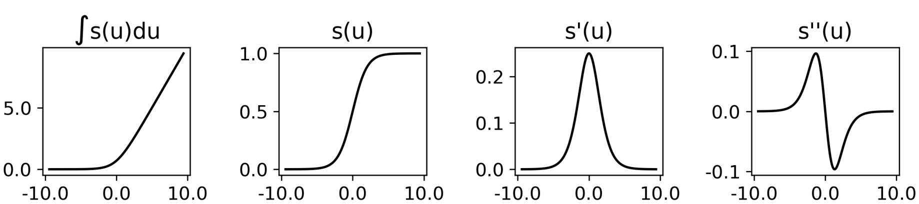


Figure 17: $s(u) = (1 + e^{-u})^{-1}$

In many applications sigmoid functions that map $s : \mathbf{R} \to \mathbf{R}[-1, 1]$ are used instead of $s : \mathbf{R} \to \mathbf{R}[0, 1]$.
It is trivial to re–map in either case.

8.2 Piecewise–Polynomial Sigmoids

There are many ways to obtain an s–shaped curve. Consider a piecewise polynomial given by

$$s_n(u) = \begin{cases} \sum_{i=0}^{2n+1} (a_i/i!)u^i & \text{if} \quad u \in \mathbf{R}[-1,1] \\ \text{sign}(u) & \text{else} \end{cases}, \tag{44}$$

where $s(-1) = -1$, $s(1) = 1$, and all derivatives at $u \in \{-1, 1\}$ are zero. For $n \in \mathbf{Z}[0,3]$ and $|u| < 1$, this gives

$$\begin{aligned} s_0(u) &= u, \\ s_1(u) &= (3u - u^3)/2, \\ s_2(u) &= (15u - 10u^3 + 3u^5)/8, \\ s_3(u) &= (35u - 35u^3 + 21u^5 - 5u^7)/16, \end{aligned}$$

figures 18–21 show $n \in \mathbf{Z}[0,3]$.
Note: The Huber penalty function at $t = 1$ is given by $f_0 = \int s_0(u)du$.

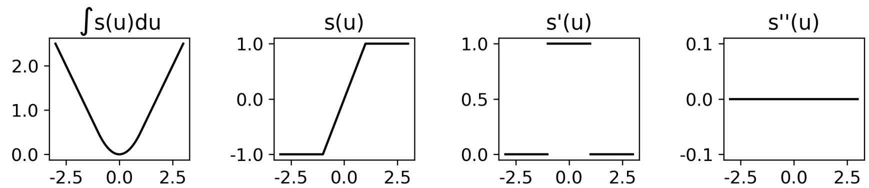


Figure 18: $s_0(u) = \text{Saturation}(u)$

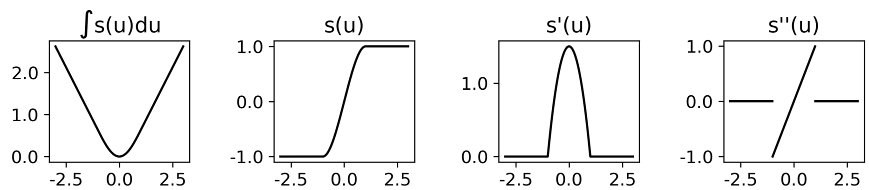


Figure 19: $s_1(u)$

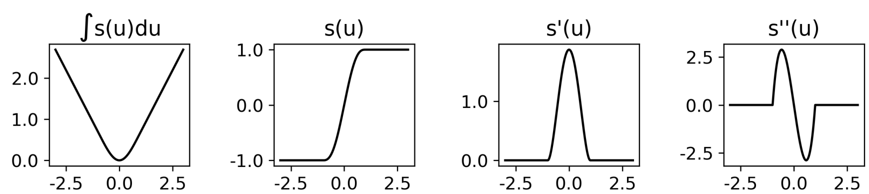


Figure 20: $s_2(u)$

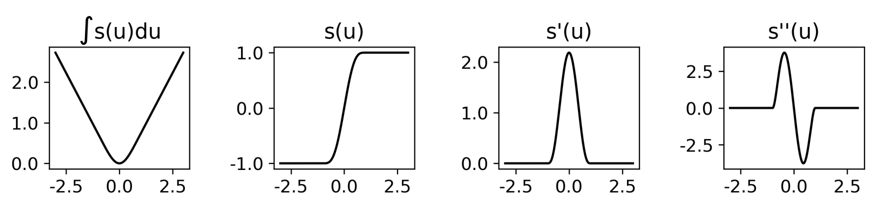


Figure 21: $s_3(u)$

Dividing $f_i = \int s_i(u)du$ by u gives alternative sigmoid functions. For $|u| < 1$,

$$\begin{aligned}
f_0/u &= u/2, \\
f_1/u &= \left(6u - u^3\right)/8, \\
f_2/u &= \left(15u - 5u^3 + u^5\right)/16, \\
f_3/u &= \left(140u - 70u^3 + 28u^5 - 5u^7\right)/128.
\end{aligned}$$

For $|u| > 1$,

$$\frac{f_i}{u} = \text{sign}(u) + \frac{f_i(1) - 1}{u}.$$

The gradient approaches zero asymptotically. Figures 22–25 show $n \in \mathbf{Z}[0, 3]$.

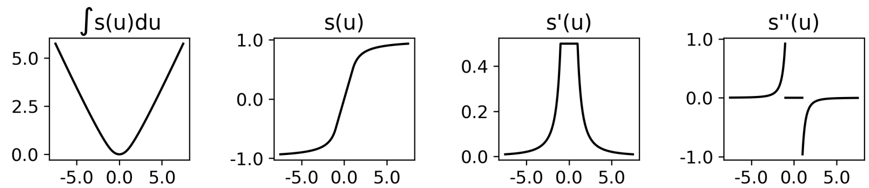


Figure 22: $s_0(u) \leftarrow f_0(u)/u$

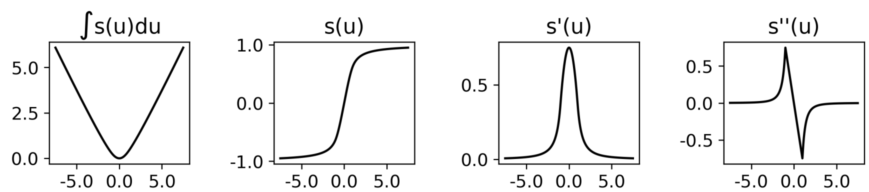


Figure 23: $s_1(u) \leftarrow f_1(u)/u$

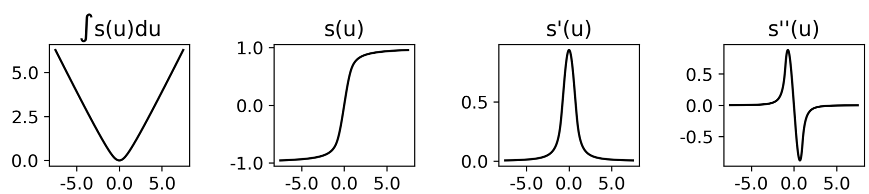


Figure 24: $s_2(u) \leftarrow f_2(u)/u$

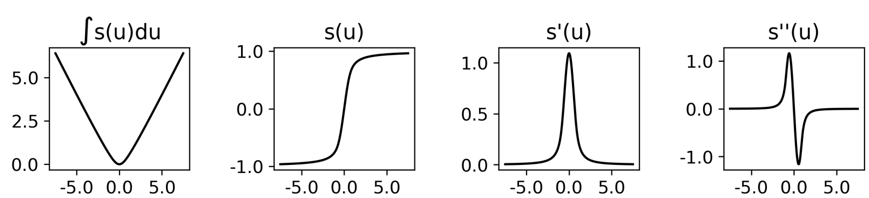


Figure 25: $s_3(u) \leftarrow f_3(u)/u$

Note: For $r_i = a_i^\intercal x - b_i$,

$$\begin{aligned}
\|Ax - b\|_1 &\approx \textstyle\sum_{i=0}^{m-1} r_i s(r_i), \\
\frac{\partial}{\partial x}\|Ax - b\|_1 &\approx \textstyle\sum_{i=0}^{m-1} (s(r_i) + r_i s'(r_i)) a_i^\intercal.
\end{aligned}$$

8.3 Deadzone Sigmoid

For the deadzone penalty $f(u)$, consider the sigmoid function given by

$$\begin{aligned} s(u) &= \frac{f(u)}{u} \\ &= \begin{cases} 0 & \text{if } |u| \le t \\ \text{sign}(u) - t/u & \text{else} \end{cases} . \end{aligned} \tag{45}$$

Figure 26 plots the sigmoid curve. **Note**: In the limit of $|u| \to \infty$, the sigmoid approaches sign(u) asymptotically.
Note: The asymptotic approach is much slower than a logistic sigmoid.
A constraint of the form

$$y = s(u)$$

can be expressed as

$$yu = \begin{cases} 0 & \text{if } |u| \le t \\ |u| - t & \text{else} \end{cases} \quad \text{or} \quad (yu+t)^2 = \begin{cases} t^2 & \text{if } |u| \le t \\ u^2 & \text{else} \end{cases} .$$

The first derivative is

$$s'(u) = \begin{cases} 0 & \text{if } |u| \le t \\ t/u^2 & \text{else} \end{cases} ,$$

and the second derivative is

$$s''(u) = \begin{cases} 0 & \text{if } |u| \le t \\ -2t/u^3 & \text{else} \end{cases} .$$

Note: When $t = 0$, $s(u) = \text{sign}(u)$ and the constraint can be expressed as $yu = |u| = \max(-u, u)$.

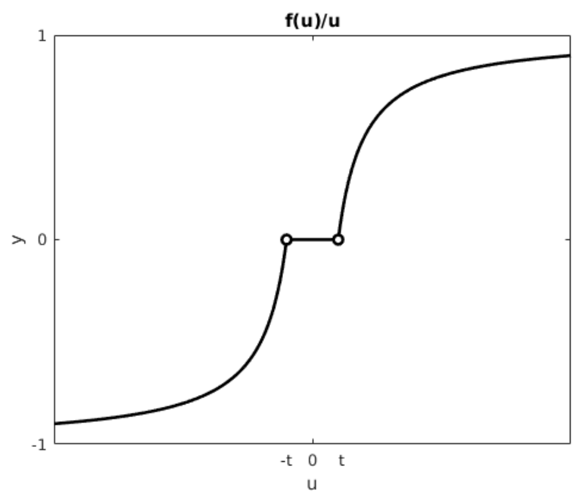


Figure 26: Deadzone Sigmoid

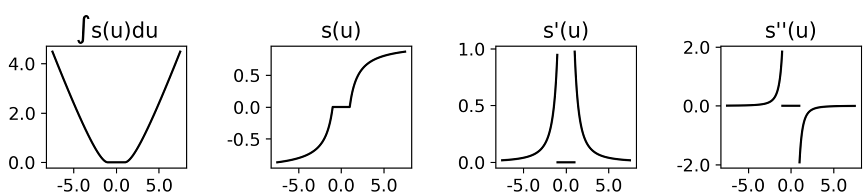


Figure 27: Deadzone Sigmoid

8.4 Huber Sigmoid

For the Huber penalty $f(u)$, consider the sigmoid function given by

$$s(u) = \frac{f(u)}{2u} = \frac{1}{2}\left\{\begin{array}{ll} u & \text{if } |u| \le t \\ 2t\,\text{sign}(u) - t^2/u & \text{else} \end{array}\right. . \tag{46}$$

Figure 28 plots the sigmoid curve.
Note: In the limit of $|u| \to \infty$, the sigmoid approaches $\text{sign}(u)$ asymptotically. **Note**: The asymptotic approach is much slower than a logistic sigmoid.
A constraint of the form

$$y = s(u)$$

can be expressed as

$$2uy = \left\{\begin{array}{ll} u^2 & \text{if } |u| \le t \\ 2t|u| - t^2 & \text{else} \end{array}\right. \quad \text{or} \quad (2yu + t^2)^2 = \left\{\begin{array}{ll} (u^2 + t^2)^2 & \text{if } |u| \le t \\ 4t^2u^2 & \text{else} \end{array}\right. ,$$

which has a unique solution for any choice of $y \in \mathbf{R}[-t, t]$. The Huber sigmoid has a continuous first derivative

$$s'(u) = \frac{1}{2}\left\{\begin{array}{ll} 1 & \text{if } |u| \le t \\ t^2/u^2 & \text{else} \end{array}\right. ,$$

and a discontinuous second derivative

$$s''(u) = \left\{\begin{array}{ll} 0 & \text{if } |u| \le t \\ -t^2/u^3 & \text{else} \end{array}\right. .$$

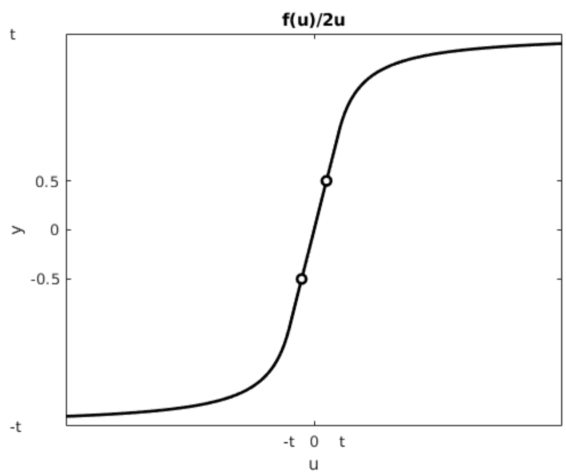


Figure 28: Huber Sigmoid

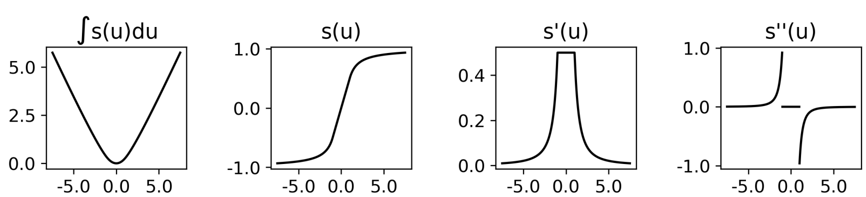


Figure 29: Huber Sigmoid

8.5 Miscellaneous Sigmoids

8.5.1 Trig Sigmoids

Figures 30 and 31 show two common trigonometric sigmoids.

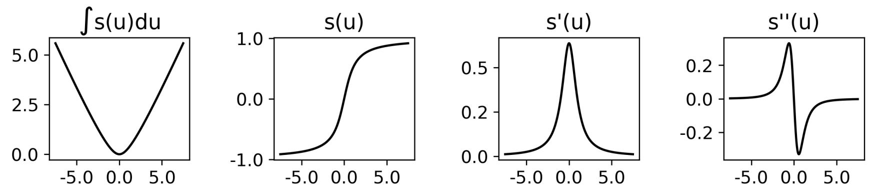


Figure 30: $s(u) = 2\ \text{atan}(u)/\pi$

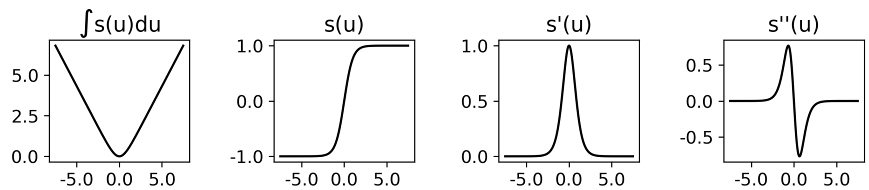


Figure 31: $s(u) = \tanh(u)$

8.5.2 Exponential Sigmoids

Figures 32 and 33 show two common exponential sigmoids.

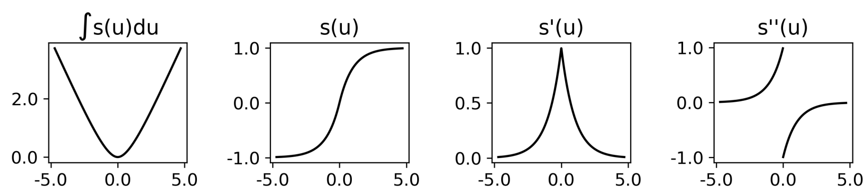


Figure 32: $s'(u) = \exp(-|u|)$

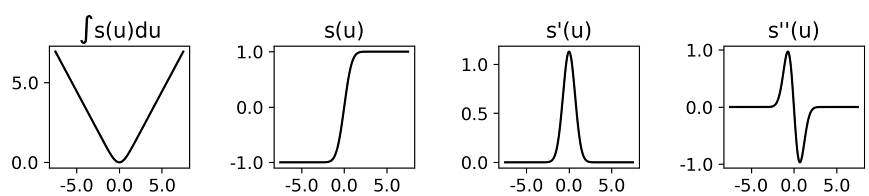


Figure 33: $s(u) = \text{erf}(u), \quad s'(u) = 2\exp(-u^2)/\sqrt{\pi}$

8.5.3 Ramp Sigmoid

Ramp velocity profiles are common in robotics applications, e.g. figure 34.

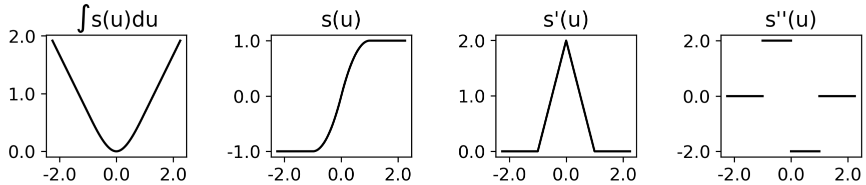


Figure 34: Ramp

8.5.4 Parabolic Sigmoids

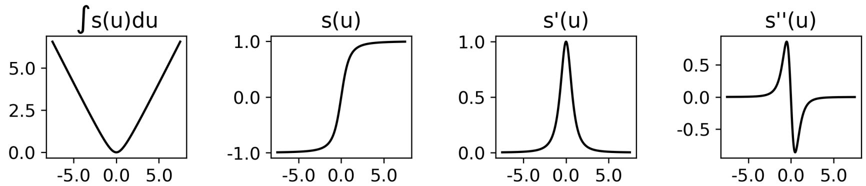


Figure 35: $s(u) = u/\sqrt{1+u^2}$, $\quad \int s(u)du = \sqrt{1+u^2} - 1$

8.5.5 Reciprocal Sigmoids

To obtain sigmoids with slow asymptotic convergence, consider reciprocal functions, e.g., linear–reciprocal (figure 36)

$$s(u) = \begin{cases} (1/u-1)^{-1} & \text{if} \quad u < 0 \\ (1/u+1)^{-1} & \text{else} \end{cases},$$

or log–reciprocal (figure 37)

$$s(u) = \begin{cases} \log(2)\log(2-u)^{-1} - 1 & \text{if} \quad u < 0 \\ -\log(2)\log(2+u)^{-1} + 1 & \text{else} \end{cases}.$$

These sigmoids both have an $s'(u)$ that converges to 0 very slowly.

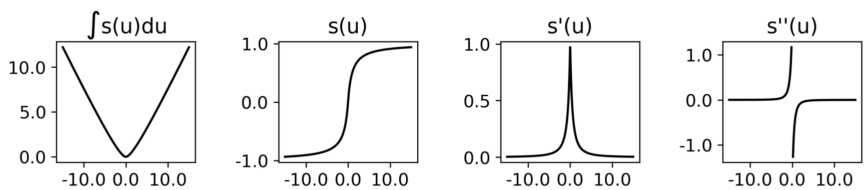


Figure 36: Reciprocal Linear

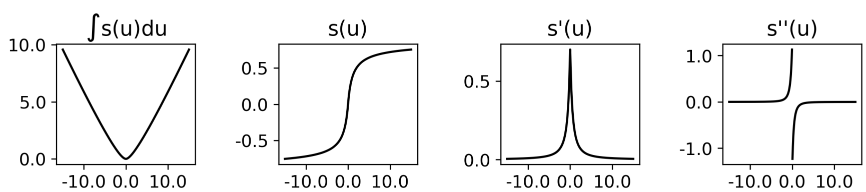


Figure 37: Reciprocal Log

9 NN: Neural Networks

Classical multi–layer perceptrons have been around for several decades. They are well–suited to learning mappings between Boolean tables but suffered early set backs (not living up to the early hype). In recent years, there has a been a resurgence in neural network research with fantastic results in classification [52] and generation of complex data sets from easily interperted low level features [49], [50]. The discoveries that have made this possible are pre– and post–filters that provide appropriately conditioned output for classical multi–layer perceptrons to excel, e.g., VAE (variational auto–encoders) and CNN (convolutional neural networks). These filters convert complex data into easily seperable or predictable features. In the case of CNN, these features are much larger than the input. In the case of VAE, these features are much smaller than the input.

Using only multi–layer perceptrons injected with noise, generative methods (mapping low feature spaces to complex data spaces) have recently proven more powerful than state–of–the–art Markov–based methods. One of my rearch goals is to combine cascading RBST with the machinery of RNN, GAN, and LSTM to find scalable real–world application, similar to the results of [48] but on a much larger scale.

The forward calculation of an n-layer network mapping x to y is shown in Algorithm 11. Neural networks are nonlinear and non–convex, but their optimization is greatly aided with a deeper understanding of **QP** . Using the nonlinear **QP** solvers in this repo, many modifications to vanilla back–propogated ℓ_2 error can be explored, e.g., p–norm regularization, p–norm constraint, upper and lower parameter bounds, relaxations with slack variables, and robust penalties like Huber and dedzone. Enhanced solvers that take advantage of sparsity and parallelization are also of interest.

$z_0 \leftarrow s\left(A_0 x - b_0\right)$
for $i \in \mathbf{Z}[1, n-1]$ **do**
  $z_i \leftarrow s\left(A_i z_{i-1} - b_i\right)$
end
return $y \leftarrow z_{n-1}$

Algorithm 11: Forward NN

9.1 1–Layer Perceptron

For $h : \mathbf{R}^n \to \mathbf{R}^m$, let

$$\begin{aligned} s(h) &= \text{row}\{s(h_i)\}_{i=0}^{m-1}, \\ s'(h) &= \text{diag}\{s'(h_i)\}_{i=0}^{m-1}, \end{aligned}$$

where s could be any of the proposed functions in the sigmoid section.
For $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, consider

$$\min_{\{A,b\} \in \{\mathbf{R}^{m \times n}, \mathbf{R}^m\}} \quad \sum_{\{x,y\}} J(x, y),$$

where

$$\begin{aligned} J(x, y) &= \frac{1}{2} \|s(Ax - b) - y\|_{2|W(x,y)}^2 \\ &= \frac{1}{2} \|z - y\|_{2|W(x,y)}^2, \end{aligned}$$

and

$$\begin{aligned} z &= s(Ax - b) \\ &= s((\boldsymbol{I}_m \otimes x^\intercal)\vec{A} - b). \end{aligned}$$

Let

$$w = W^\intercal W(z - y).$$

The partials are

$$\frac{\partial J}{\partial x} = w^\intercal \frac{\partial z}{\partial x},$$

$$\frac{\partial J}{\partial \vec{A}} = w^\intercal \frac{\partial z}{\partial \vec{A}},$$

$$\frac{\partial J}{\partial b} = w^\intercal \frac{\partial z}{\partial b},$$

where

$$\frac{\partial z}{\partial x} = s'(Ax - b)A,$$

$$\frac{\partial z}{\partial \vec{A}} = s'(Ax - b)(\boldsymbol{I}_m \otimes x^\intercal),$$

$$\frac{\partial z}{\partial b} = -s'(Ax - b).$$

9.2 2–Layer Perceptron

For $z_i \in \mathbf{R}^{m_i}$ with $i \in \mathbf{Z}[0, n-1]$ and $n = 2$, consider

$$\min_{\{A_i, b_i\} \in \{\mathbf{R}^{m_i \times m_{i-1}}, \mathbf{R}^{m_i}\}} \quad \sum_{\{x,y\}} J(x, y),$$

where

$$\begin{aligned} J(x, y) &= \frac{1}{2} \|s(A_1 s(A_0 x - b_0) - b_1) - y\|^2_{2|W(x,y)} \\ &= \frac{1}{2} \|z_1 - y\|^2_{2|W(x,y)}, \end{aligned}$$

and

$$\begin{aligned} z_0 &= s(A_0 x - b_0), \\ z_1 &= s(A_1 z_0 - b_1). \end{aligned}$$

Let

$$w = W^\intercal W(z_1 - y).$$

The partials for A_i and b_i are

$$\begin{aligned} \frac{\partial J}{\partial \vec{A}_1} &= w^\intercal \frac{\partial z_1}{\partial \vec{A}_1}, \\ \frac{\partial J}{\partial \vec{A}_0} &= w^\intercal \frac{\partial z_1}{\partial z_0} \frac{\partial z_0}{\partial \vec{A}_0}, \end{aligned}$$

$$\begin{aligned} \frac{\partial J}{\partial b_1} &= w^\intercal \frac{\partial z_1}{\partial b_1}, \\ \frac{\partial J}{\partial b_0} &= w^\intercal \frac{\partial z_1}{\partial z_0} \frac{\partial z_0}{\partial b_0}, \end{aligned}$$

where

$$\frac{\partial z_1}{\partial z_0} = s'(A_1 z_0 - b_1)A_1,$$

$$\begin{aligned}
\frac{\partial z_1}{\partial \vec{A}_1} &= s'(A_1 z_0 - b_1)(\boldsymbol{I}_{m_1} \otimes z_0^\intercal), \\
\frac{\partial z_1}{\partial b_1} &= -s'(A_1 z_0 - b_1),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z_0}{\partial \vec{A}_0} &= s'(A_0 x - b_0)(\boldsymbol{I}_{m_0} \otimes x^\intercal) \\
\frac{\partial z_0}{\partial b_0} &= -s'(A_0 x - b_0).
\end{aligned}$$

9.3 3–Layer Perceptron

For $z_i \in \mathbf{R}^{m_i}$ with $i \in \mathbf{Z}[0, n-1]$ and $n = 3$, consider

$$\min_{\{A_i, b_i\} \in \{\mathbf{R}^{m_i \times n_i}, \mathbf{R}^{m_i}\}} \sum_{\{x,y\}} J(x, y)$$

where

$$\begin{aligned}
J(x, y) &= \frac{1}{2} \| s(A_2 s(A_1 s(A_0 x - b_0) - b_1) - b_2) - y \|^2_{2|W(x,y)} \\
&= \frac{1}{2} \| z_2 - y \|^2_{2|W(x,y)},
\end{aligned}$$

and

$$\begin{aligned}
z_0 &= s(A_0 x - b_0), \\
z_1 &= s(A_1 z_0 - b_1), \\
z_2 &= s(A_2 z_1 - b_2).
\end{aligned}$$

Let

$$w = W^\intercal W (z_2 - y).$$

The partials for A_i and b_i are

$$\begin{aligned}
\frac{\partial J}{\partial \vec{A}_2} &= w^\intercal s'(A_2 z_1 - b_2)(\boldsymbol{I}_{m_2} \otimes z_1^\intercal), \\
\frac{\partial J}{\partial \vec{A}_1} &= w^\intercal s'(A_2 z_1 - b_2) A_2 s'(A_1 z_0 - b_1)(\boldsymbol{I}_{m_1} \otimes z_0^\intercal), \\
\frac{\partial J}{\partial \vec{A}_0} &= w^\intercal s'(A_2 z_1 - b_2) A_2 s'(A_1 z_0 - b_1) A_1 s'(A_0 x - b_0)(\boldsymbol{I}_{m_0} \otimes x^\intercal),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial J}{\partial b_2} &= -w^\intercal s'(A_2 z_1 - b_2), \\
\frac{\partial J}{\partial b_1} &= -w^\intercal s'(A_2 z_1 - b_2) A_2 s'(A_1 z_0 - b_1), \\
\frac{\partial J}{\partial b_0} &= -w^\intercal s'(A_2 z_1 - b_2) A_2 s'(A_1 z_0 - b_1) A_1 s'(A_0 x - b_0).
\end{aligned}$$

9.4 n–Layer Perceptron

In general, for $z_i \in \mathbf{R}^{m_i}$ with $i \in \mathbf{Z}[0, n-1]$, consider

$$\min_{\{A_i, b_i\} \in \{\mathbf{R}^{m_i \times m_{i-1}}, \mathbf{R}^{m_i}\}} \quad \sum_{\{x,y\}} = J(x, y),$$

where

$$J(x, y) = \frac{1}{2} \|z_{n-1} - y\|^2_{2|W(x,y)}.$$

For $i \in \mathbf{Z}[1, n-1]$,

$$z_i = s(A_i z_{i-1} - b_i),$$

with

$$z_0 = s(A_0 x - b_0).$$

Let

$$w = W^\intercal W (z_{n-1} - y).$$

The partials are given by

$$\begin{aligned}
\frac{\partial J}{\partial \vec{A}_i} &= w^\intercal \frac{\partial z_{n-1}}{\partial z_{n-2}} \frac{\partial z_{n-2}}{\partial z_{n-3}} \cdots \frac{\partial z_{i+1}}{\partial z_i} \frac{\partial z_i}{\partial \vec{A}_i}, \\
\frac{\partial J}{\partial b_i} &= w^\intercal \frac{\partial z_{n-1}}{\partial z_{n-2}} \frac{\partial z_{n-2}}{\partial z_{n-3}} \cdots \frac{\partial z_{i+1}}{\partial z_i} \frac{\partial z_i}{\partial b_i},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial z_j}{\partial z_{j-1}} &= s'(A_j z_{j-1} - b_j) A_j, \\
\frac{\partial z_j}{\partial \vec{A}_j} &= s'(A_j z_{j-1} - b_j)(\boldsymbol{I}_{m_j} \otimes z_{j-1}^\intercal), \\
\frac{\partial z_j}{\partial b_j} &= -s'(A_j z_{j-1} - b_j).
\end{aligned}$$

9.5 n–Layer Perceptron with States

For $z_i \in \mathbf{R}^{m_i}$ with $i \in \mathbf{Z}[0, n-1]$, consider

$$\min_{\{z_i(x), A_i, b_i\} \in \{\mathbf{R}^{m_i}, \mathbf{R}^{m_i \times m_{i-1}}, \mathbf{R}^{m_i}\}} \quad \sum_{\{x,y\}} J(x, y) = \frac{1}{2} \|z_{n-1} - y\|^2_{2|W(x,y)},$$

with

$$f_{\text{eq}}(z, A, b|x) = \begin{bmatrix} z_0 - s(A_0 x - b_0) \\ z_1 - s(A_1 z_0 - b_1) \\ \vdots \\ z_{n-1} - s(A_{n-1} z_{n-2} - b_{n-1}) \end{bmatrix} = \mathbf{0}^M,$$

where

$$r = \sum_{i=0}^{n-1} m_i.$$

This formulation resembles MPC. The nonlinear equality constraint can be implamented with any of the methods presented in seciton 7.3. Adding states adds a degree of freedom in the optimization that might help in cases where the gradient would traditionally vanish. The disadvantage to this approach is, for q sample pair of $\{x, y\}$, there are rq new free variables to optimize.

9.6 ML: Maximum Likelihood

Consider the probability model

$$p = \text{nn}(x),$$

with

$$\begin{aligned} p &= \text{Probability}(\widetilde{y} = 1), \\ 1 - p &= \text{Probability}(\widetilde{y} = 0). \end{aligned}$$

Collect $x \in \{0,1\}^n$ and sort them by $y = 1$ and $y = 0$.
ML formulates the problem

$$\begin{aligned} \min_{\{A_i, b_i\} \in \{\mathbf{R}^{m_i \times m_{i-1}}, \mathbf{R}^{m_i}\}} \quad J &= -l \\ &= -\log\left(\prod_{x \text{ with } y=1} p(x) \prod_{x \text{ with } y=0} (1 - p(x))\right) \\ &= -\sum_{x \text{ with } y=1} \log\left(\text{nn}(x)\right) - \sum_{x \text{ with } y=0} \log\left(1 - \text{nn}(x)\right). \end{aligned}$$

With $\text{nn}(x) = z_{n-1}(x)$ and $z_{n-1} \in \mathbf{R}[0,1]$, the partials are given by

$$\frac{\partial J}{\partial \vec{A}_i} = \begin{cases} -\dfrac{1}{z_{n-1}} \dfrac{\partial z_{n-1}}{\partial z_{n-2}} \dfrac{\partial z_{n-2}}{\partial z_{n-3}} \cdots \dfrac{\partial z_{i+1}}{\partial z_i} \dfrac{\partial z_i}{\partial \vec{A}_i} & \text{if} \quad y = 1 \\ \dfrac{1}{1 - z_{n-1}} \dfrac{\partial z_{n-1}}{\partial z_{n-2}} \dfrac{\partial z_{n-2}}{\partial z_{n-3}} \cdots \dfrac{\partial z_{i+1}}{\partial z_i} \dfrac{\partial z_i}{\partial \vec{A}_i} & \text{if} \quad y = 0 \end{cases},$$

$$\frac{\partial J}{\partial b_i} = \begin{cases} -\dfrac{1}{z_{n-1}} \dfrac{\partial z_{n-1}}{\partial z_{n-2}} \dfrac{\partial z_{n-2}}{\partial z_{n-3}} \cdots \dfrac{\partial z_{i+1}}{\partial z_i} \dfrac{\partial z_i}{\partial b_i} & \text{if} \quad y = 1 \\ \dfrac{1}{1 - z_{n-1}} \dfrac{\partial z_{n-1}}{\partial z_{n-2}} \dfrac{\partial z_{n-2}}{\partial z_{n-3}} \cdots \dfrac{\partial z_{i+1}}{\partial z_i} \dfrac{\partial z_i}{\partial b_i} & \text{if} \quad y = 0 \end{cases},$$

where

$$\frac{\partial z_j}{\partial z_{j-1}} = s'(A_j z_{j-1} - b_j) A_j,$$

$$\frac{\partial z_j}{\partial \vec{A}_j} = s'(A_j z_{j-1} - b_j)(\boldsymbol{I}_{m_j} \otimes z_{j-1}^{\intercal}),$$

$$\frac{\partial z_j}{\partial b_j} = -s'(A_j z_{j-1} - b_j),$$

and

$$s(u) = (1 + e^{-u})^{-1}.$$

10 MPC: Model Predictive Control

Let

$$\begin{aligned} u_t &= \text{input}, \\ y_t &= \text{output}, \\ z_t &= \text{hidden internal dynamic state}, \\ \theta &= \text{hidden internal constant state}. \end{aligned}$$

MPC uses models of the form

$$\begin{aligned} z_{t+1} &= f(z_t, u_t|\theta) \\ y_t &= g(z_t, u_t|\theta) \end{aligned}$$

The models may be nonlinear or generalized deep neural networks, but if they can be expressed as linear, then **QP** can be used to solve them in one iteration. Time will be analyzed on a sliding window where $t = 0$ depicts the present (figure 39). The world takes input and produces output over this moving time window (figure 40). It is assumed that the world has some hidden internel dynamic states that need to be determined. The quantitative expression of these states depends on the model chosen to approximate the world. An estimator uses the model and the observed input and output history to produce a best current guess of the dynamic and constant hidden states (figure 42). Once the current best guess is given for the states, the controller uses the model along with target input and output to find the optimal future input (figure 43). These calculations are done each time step on hardware,

- estimating z_0^* and θ^* from $u[-T, 0)$ and $y[-T, 0]$,
- and computing and applying u_0^* from z_0^* and θ^*.

Lectures on MPC can be found here [15] and here [3]. There are many interesting applications in walking and flying robotics, including [18], [20], and [21]. Additional examples can be found at [24] and [26]. Python packages can be found here [44]. Pytorch extensions can be found here [45].

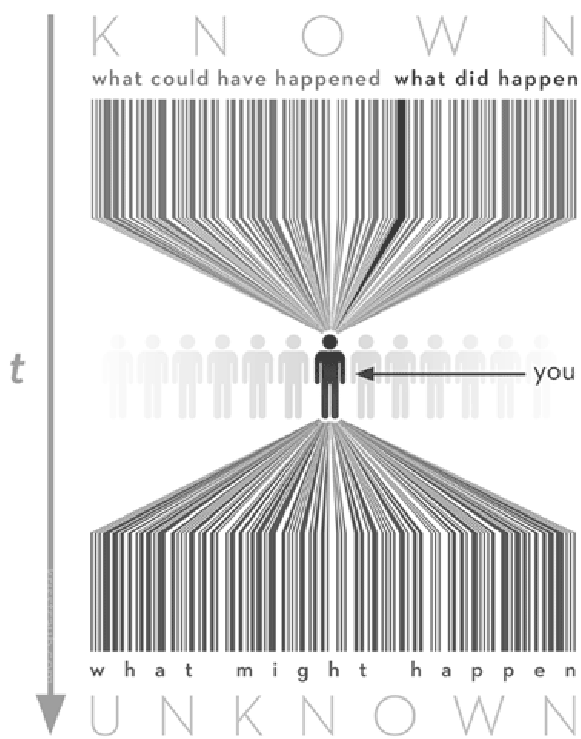


Figure 38: MPC

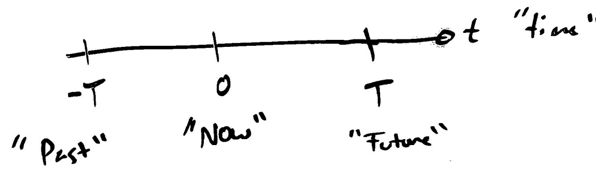


Figure 39: Time

Figure 40: World

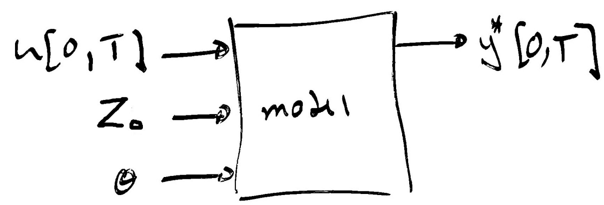


Figure 41: World Model

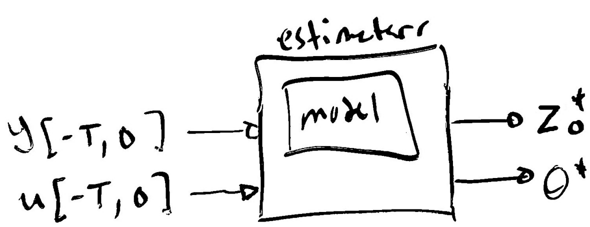


Figure 42: World Model Estimator

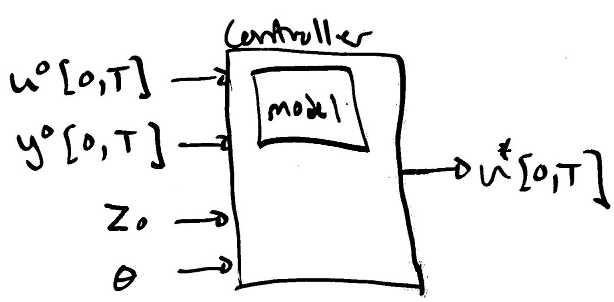


Figure 43: World Model Controller

10.1 LTV: Linear Time Variant

For $i \in \mathbf{Z}[0, q-1]$, consider the discrete-time system

$$\begin{aligned} z_{i+1} &= A_i z_i + B_i u_i, \\ z_0 &= \text{fixed}, \end{aligned}$$

where $u_i \in \mathbf{R}^m$ and $z_i \in \mathbf{R}^n$. For any selected time horizon q, this system can be expressed as

$$A_{\text{eq}} x = b_{\text{eq}},$$

where

$$x_i = \left[\begin{array}{c} u_i \\ z_{i+1} \end{array}\right].$$

For $q = 3$,

$$\begin{aligned} \mathbf{0}^n &= A_0 z_0 + B_0 u_0 - z_1, \\ \mathbf{0}^n &= A_1 z_1 + B_1 u_1 - z_2, \\ \mathbf{0}^n &= A_2 z_2 + B_2 u_2 - z_3, \end{aligned}$$

which gives

$$\left[\begin{array}{cc|cc|cc} B_0 & -\boldsymbol{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_1 & B_1 & -\boldsymbol{I}_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_2 & B_2 & -\boldsymbol{I}_n \end{array}\right] \left[\begin{array}{c} u_0 \\ z_1 \\ \hline u_1 \\ z_2 \\ \hline u_2 \\ z_3 \end{array}\right] = \left[\begin{array}{c} -A_0 z_0 \\ \mathbf{0} \\ \mathbf{0} \end{array}\right].$$

For $q > 0$,

$$\begin{aligned} A_{\text{eq}} &= \left[\begin{array}{cc} \mathbf{0}^{n \times (m+n)(q-1)} & \mathbf{0}^{n \times (m+n)} \\ \text{diag}\{\left[\begin{array}{cc} \mathbf{0}^{n \times m} & A_i \end{array}\right]\}_{i=1}^{q-1} & \mathbf{0}^{n(q-1) \times (m+n)} \end{array}\right] + \text{diag}\{\left[\begin{array}{cc} B_i & -\boldsymbol{I}_n \end{array}\right]\}_{i=0}^{q-1}, \\ b_{\text{eq}} &= \left[\begin{array}{c} -A_0 z_0 \\ \mathbf{0}^{n(q-1)} \end{array}\right]. \end{aligned}$$

10.1.1 Block Components

The u and z components can be accessed with

$$\begin{aligned} u_{0:q-1} &= Gx, \\ z_{1:q} &= Fx, \end{aligned}$$

where

$$\begin{aligned} G &= \text{diag}\left\{\left[\begin{array}{cc} \boldsymbol{I}_m & \mathbf{0}^{m \times n} \end{array}\right]\right\}_{i=0}^{q-1}, \\ F &= \text{diag}\left\{\left[\begin{array}{cc} \mathbf{0}^{n \times m} & \boldsymbol{I}_n \end{array}\right]\right\}_{i=0}^{q-1}. \end{aligned}$$

10.1.2 Rate Regularization

For weight $W_i \in \mathbf{R}^{m \times m}$, the input rate can be regularized by

$$\begin{aligned} \sum_{i=0}^{q-2} \|u_{i+1} - u_i\|_{W_i} &= \|u_{1:q-1} - u_{0:q-2}\|_W \\ &= \|u_{0:q-1}\|_{WD}, \end{aligned}$$

where [1, p. 312]

$$D = \text{diag}(\,\mathbf{0}^{m\times m},\, \boldsymbol{I}_{m(q-1)}\,) - \text{diag}(\,\boldsymbol{I}_{m(q-1)},\, \mathbf{0}^{m\times m}\,).$$

For weight $W_i \in \mathbf{R}^{n\times n}$, the state rate can be regularized by

$$\begin{aligned}\sum_{i=0}^{q-1} \|z_{i+1} - z_i\|_{W_i} &= \|z_{1:q} - z_{0:q-1}\|_W \\ &= \|Az_{1:q} - b\|_{WD},\end{aligned}$$

where

$$A = \begin{bmatrix} \mathbf{0}^{n\times nq} \\ \boldsymbol{I}_{nq} \end{bmatrix}, \quad b = \begin{bmatrix} -z_0 \\ \mathbf{0}^{nq} \end{bmatrix},$$

and

$$D = \text{diag}(\,\mathbf{0}^{n\times n},\, \boldsymbol{I}_{nq}\,) - \text{diag}(\,\boldsymbol{I}_{nq},\, \mathbf{0}^{n\times n}\,).$$

Note: Rate can also be directly constrained with $d_{\text{lb}} \leq Dx \leq d_{\text{ub}}$.

10.1.3 2–Norm Objective

Consider the **QP**

$$\begin{aligned}\min_{x\in\mathbf{R}^{(m+n)q}} \quad J &= \sum_{i=0}^{q-1} \|u_i - u_i^\circ\|^2_{2,R_i^{1/2}} + \|z_{i+1} - z_{i+1}^\circ\|^2_{2,P_{i+1}^{1/2}} \\ &= \|Ax - b\|_2^2 \\ &\text{s.t.} \quad A_{\text{eq}}x = b_{\text{eq}},\end{aligned}$$

where

$$A = \text{diag}\left\{\begin{bmatrix} R_i^{1/2} & \mathbf{0}^{m\times n} \\ \mathbf{0}^{n\times m} & P_{i+1}^{1/2}\end{bmatrix}\right\}_{i=0}^{q-1}, \quad b = \text{row}\left\{\begin{bmatrix} R_i^{1/2}u_i^\circ \\ P_{i+1}^{1/2}z_{i+1}^\circ\end{bmatrix}\right\}_{i=0}^{q-1}.$$

Expanding the objective gives

$$J = \frac{1}{2}x^\mathsf{T}Qx + c^\mathsf{T}x + r,$$

where

$$Q = \quad 2\,\text{diag}\left\{\begin{bmatrix} R_i & \mathbf{0}^{m\times n} \\ \mathbf{0}^{n\times m} & P_{i+1}\end{bmatrix}\right\}_{i=0}^{q-1}, \quad c = -2\,\text{row}\left\{\begin{bmatrix} R_iu_i^\circ \\ P_{i+1}z_{i+1}^\circ\end{bmatrix}\right\}_{i=0}^{q-1}.$$

The diagonal structure has an effecient solution method using the Schur complement [1, p. 552].

10.1.4 Generalized Norm

The objective can be split into state and input with

$$\begin{aligned}\min_{x\in\mathbf{R}^{(m+n)q}} \quad J = \|Gx - u^\circ\|_R + \|Fx - z^\circ\|_Q \\ \text{s.t.} \quad x \in \mathbf{P}, \\ A_{\text{eq}}x = b_{\text{eq}},\end{aligned}$$

where **P** adds additional constraints and $\|\cdot\|$ can be any norm that results in a **LP** or **QP** .

10.2 NTV: Nonlinear Time Variant

For $u_i \in \mathbf{R}^m$ and $z_i \in \mathbf{R}^n$, consider the problem

$$\begin{aligned} \min_{x \in \mathbf{R}^{(n+m)q}} \quad & J = \sum_{i=0}^{q-1} \left\| z_{i+1} - z_{i+1}^{\circ} \right\|^2_{P_{i+1}^{1/2}} + \left\| u_i - u_i^{\circ} \right\|^2_{R_i^{1/2}} \\ \text{s.t.} \quad & z_{i+1} = f(t_i, z_i, u_i), \\ & z_0 = \text{fixed}. \end{aligned}$$

For $i \in \mathbf{Z}[0, q-1]$, let $f_i := f(t_i, z_i, u_i)$ with

$$A_i = \frac{\partial f_i}{\partial z_i}, \quad B_i = \frac{\partial f_i}{\partial u_i},$$

$$x_i = \begin{bmatrix} u_i \\ z_{i+1} \end{bmatrix}.$$

The penalty approach gives

$$\min_{x \in \mathbf{R}^{(m+n)q}} \quad J = \| x - x^{\circ} \|^2_{2|W} + \| f - Fx \|^2_{2|M},$$

where

$$z_{1:q} = Fx.$$

The gradient is

$$\begin{aligned} g^{\intercal} &= \nabla^{\intercal} J \\ &= 2(x - x^{\circ})^{\intercal} W^{\intercal} W + 2(f - Fx)^{\intercal} M^{\intercal} M \left(h - F \right), \end{aligned}$$

where

$$\begin{aligned} h &= \frac{\partial f}{\partial x} \\ &= \left[\begin{array}{cc|cc|cc} B_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_1 & B_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_2 & B_2 & \mathbf{0} \end{array} \right] \\ &= \begin{bmatrix} \mathbf{0}^{n \times (m+n)(q-1)} & \mathbf{0}^{n \times (m+n)} \\ \operatorname{diag}\{ \begin{bmatrix} \mathbf{0}^{n \times m} & A_i \end{bmatrix} \}_{i=1}^{q-1} & \mathbf{0}^{n(q-1) \times (m+n)} \end{bmatrix} + \operatorname{diag}\{ \begin{bmatrix} B_i & \mathbf{0}^{n \times n} \end{bmatrix} \}_{i=0}^{q-1}. \end{aligned}$$

The Hessian is given by

$$\begin{aligned} H &= \frac{\partial g}{\partial x} \\ &= 2W^{\intercal} W + 2 \left(h - F \right)^{\intercal} M^{\intercal} M \left(h - F \right) \\ &\quad + 2 \operatorname{col} \left\{ \left[\operatorname{col} \left\{ \frac{\partial h^{\intercal}}{\partial u_{ij}} \right\}_{j=0}^{m-1} \quad \operatorname{col} \left\{ \frac{\partial h^{\intercal}}{\partial z_{ij}} \right\}_{j=0}^{n-1} \right] \right\}_{i=0}^{q-1} M^{\intercal} M \left(f - Fx \right). \end{aligned}$$

10.3 LTI: Linear Time Invariant

If A and B are constant,

$$A_{\text{eq}} = \begin{bmatrix} \mathbf{0}^{n\times(m+n)(q-1)} & \mathbf{0}^{n\times(m+n)} \\ \boldsymbol{I}_{q-1} \otimes \begin{bmatrix} \mathbf{0}^{n\times m} & A \end{bmatrix} & \mathbf{0}^{n(q-1)\times(m+n)} \end{bmatrix} + \boldsymbol{I}_q \otimes \begin{bmatrix} B & -\boldsymbol{I}_n \end{bmatrix},$$

$$b_{\text{eq}} = \begin{bmatrix} -Az_0 \\ \mathbf{0}^{n(q-1)} \end{bmatrix}.$$

If P and R are constant (as well as the reference points), then

$$\begin{aligned} J &= \sum_{i=0}^{q-1} \|u_i - u^\circ\|_{R^{1/2}}^2 + \|z_{i+1} - z^\circ\|_{P^{1/2}}^2 \\ &= \frac{1}{2} x^\mathsf{T} Q x + c^\mathsf{T} x + r, \end{aligned}$$

where

$$\begin{aligned} Q = &\quad 2\boldsymbol{I}_q \otimes \begin{bmatrix} R & \mathbf{0}^{m\times n} \\ \mathbf{0}^{n\times m} & P \end{bmatrix}, \\ c &= -2\mathbf{1}^q \otimes \begin{bmatrix} Ru^\circ \\ Pz^\circ \end{bmatrix}. \end{aligned}$$

If the bounds are constant with $z_{\text{lb}} \leq z_i \leq z_{\text{ub}}$ and $u_{\text{lb}} \leq u_i \leq u_{\text{ub}}$, then

$$\begin{aligned} \text{lb} &= \mathbf{1}^q \otimes \begin{bmatrix} u_{\text{lb}} \\ z_{\text{lb}} \end{bmatrix}, \\ \text{ub} &= \mathbf{1}^q \otimes \begin{bmatrix} u_{\text{ub}} \\ z_{\text{ub}} \end{bmatrix}. \end{aligned}$$

Figure 44 shows an example application.

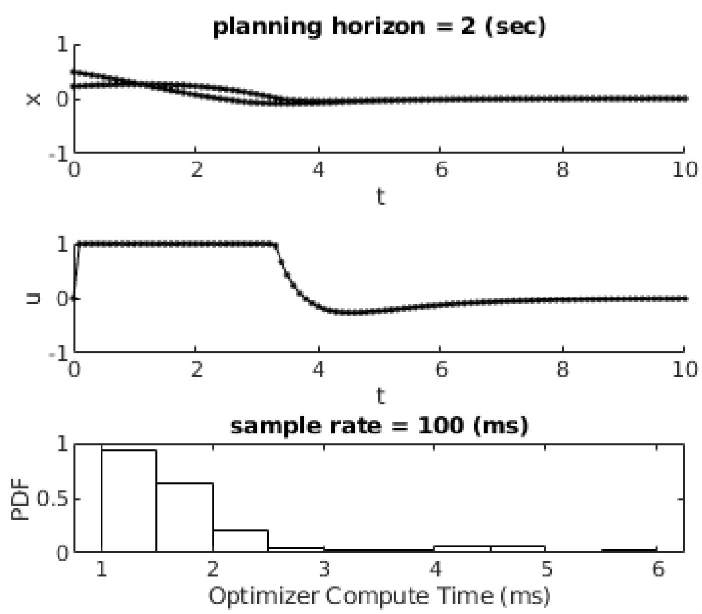


Figure 44: LTI MPC with unstable A and $|u| < 1$

10.4 System Identification

Linear prediction of output from past input and output gives

$$
\begin{aligned}
z_{i+1} &= Az_i + Bu_i \\
&= (\boldsymbol{I}_n \otimes z_i^\intercal)\vec{A} + (\boldsymbol{I}_n \otimes u_i^\intercal)\vec{B} \\
&= \begin{bmatrix} \boldsymbol{I}_n \otimes z_i^\intercal & \boldsymbol{I}_n \otimes u_i^\intercal \end{bmatrix} \begin{bmatrix} \vec{A} \\ \vec{B} \end{bmatrix}.
\end{aligned}
$$

For $i \in \mathbf{Z}[0, q-1]$, optimal paramaterization can be computed with

$$
\min_{x \in \mathbf{R}^{n(n+m)}} \quad J = \|A_{\text{eq}}x - b_{\text{eq}}\|_2^2 + \lambda\|x\|_1,
$$

where

$$
\begin{aligned}
A_{\text{eq}} &= \begin{bmatrix} \boldsymbol{I}_n \otimes z_0^\intercal & \boldsymbol{I}_n \otimes u_0^\intercal \\ \vdots & \vdots \\ \boldsymbol{I}_n \otimes z_{q-1}^\intercal & \boldsymbol{I}_n \otimes u_{q-1}^\intercal \end{bmatrix}, \\
b_{\text{eq}} &= \begin{bmatrix} z_1 \\ \vdots \\ z_q \end{bmatrix}, \\
x &= \begin{bmatrix} \vec{A} \\ \vec{B} \end{bmatrix}.
\end{aligned}
$$

Note: Previous estimates of x can be added to the objective function with

$$
\min_{x \in \mathbf{R}^{n(n+m)}} \quad J = \|A_{\text{eq}}x - b_{\text{eq}}\|_2^2 + \lambda\|x\|_1 + \lambda_{\text{update}}\|x - x_{\text{previous}}\|_p.
$$

If the model paramaters have known ranges, e.g., $\vec{A}_{\text{lb}} \leq \vec{A} \leq \vec{A}_{\text{ub}}$, these can be added with a $x \in \mathbf{P}$ constraint.

10.5 Non-Uniform Time Samples

The accuracy of any model prediction will degrade the farther it gets from the present. Including the prediction uncertainty into the cost weight typically aleviates this. The indexed time sample does not need to be uniform. Figure 45 shows a 3-layer sample. Alternatively, a continuous exponential time sample can be used (depicted in figure 46). Only layerd samples are implementable on hardware.

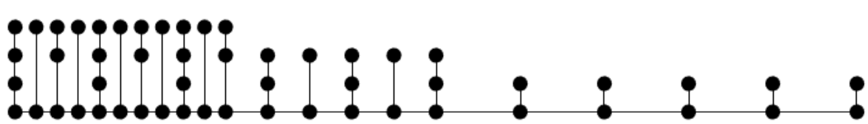


Figure 45: Layerd Time Sample

Figure 46: Exponential Time Sample

10.6 Continuous–Time Problems

Consider the continuous time problem

$$\min_{\{z,u\}:\mathbf{R}\to\{\mathbf{R}^n,\mathbf{R}^m\}} J = \int_{t_0}^{t_0+T} J(t,z,u)dt$$

$$\begin{aligned} \text{s.t.} \quad \begin{bmatrix} z(t) \\ u(t) \end{bmatrix} &\in \mathbf{P}(t), \\ \dot{z} &= f(t,z,u). \end{aligned}$$

10.6.1 Fixed–Time Step

For a fixed $d_i > 0$ with $T = \sum_{i=0}^{q-1} d_i$, the problem can be re–expressed as

$$\min_{x\in\mathbf{R}^{(n+m)q}} \quad J(x) = \sum_{i=0}^{q-1} J(t_i,z_i,u_i)d_i$$

$$\begin{aligned} \text{s.t.} \quad x_i &\in \mathbf{P}_i, \\ x_i &= \begin{bmatrix} u_i \\ z_{i+1} \end{bmatrix}, \\ z_{i+1} &= z_i + f(t_i,z_i,u_i)d_i, \\ t_{i+1} &= t_i + d_i, \\ t_0 &= \text{fixed}, \\ z_0 &= \text{fixed}. \end{aligned}$$

10.6.2 Variable–Time Step

For a variable $d_i \geq 0$, the problem can be re–expressed as the nonlinear problem

$$\min_{x\in\mathbf{R}^{(n+m+1)q}} \quad J(x) = \sum_{i=0}^{q-1} J(t_i,z_i,u_i)d_i$$

$$\begin{aligned} \text{s.t.} \quad x_i &\in \mathbf{P}_i, \\ x_i &= \begin{bmatrix} d_i \\ u_i \\ z_{i+1} \end{bmatrix}, \\ z_{i+1} &= z_i + f(t_i,z_i,u_i)d_i, \\ t_{i+1} &= t_i + d_i, \\ t_0 &= \text{fixed}, \\ z_0 &= \text{fixed}, \\ t_{q-1} &= t_0 + T, \\ d_i &\geq 0. \end{aligned}$$

Note: For $L_i > 0$, an additional Lipshitz constraint can be added with

$$|z_{i+1} - z_i| \leq L_i d_i.$$

10.6.3 Chance-Constrained MPC

Random variables can be added to both the objective function and the constraints. These can include random variables in the time step, input, current state, or model. These random variables may be time dependent. A probablistic formulation of MPC can include constraints that minimize the probability of failure, as defined by the probability of going outside of $x \in \mathbf{D}$. Interesting examples include: [16] and [17]. For $p \in \mathbf{R}(0,1)$, the generalized problem formulation is

$$\begin{aligned} \min_{\{z,u\}\in\{\mathbf{R}^{nq},\mathbf{R}^{mq}\}} \quad & J = \left\langle J_q(z_q) + \sum_{i=0}^{q-1} J_i(z_i,u_i) \right\rangle \\ \text{s.t.} \quad & \text{Probability}\left(\bigwedge_{i=1}^{q} z_i \in \mathbf{D}_i \,\middle|\, z_0 \right) \geq 1-p, \\ & z_{i+1} = f(z_i, u_i, \widetilde{w}_i), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \min_{\{z,u\}\in\{\mathbf{R}^{nq},\mathbf{R}^{mq}\}} \quad & J = \left\langle J_q(z_q) + \sum_{i=0}^{q-1} J_i(z_i,u_i) \right\rangle \\ \text{s.t.} \quad & \left\langle \prod_{i=1}^{q} \text{Boolean}\left(z_i \in \mathbf{D}_i\right) \,\middle|\, z_0 \right\rangle \geq 1-p, \\ & z_{i+1} = f(z_i, u_i, \widetilde{w}_i), \end{aligned}$$

10.7 Smooth Reconstruction

Consider the difference vector

$$\begin{aligned} \begin{bmatrix} x_1 - x_0 \\ x_2 - x_1 \\ \vdots \\ x_{n-2} - x_{n-3} \\ x_{n-1} - x_{n-2} \end{bmatrix} &= \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{bmatrix} \\ &= \underbrace{\left(\begin{bmatrix} \mathbf{0}^{n-1} & \boldsymbol{I}_{n-1} \end{bmatrix} - \begin{bmatrix} \boldsymbol{I}_{n-1} & \mathbf{0}^{n-1} \end{bmatrix} \right)}_{D_n} x, \end{aligned}$$

where $D_n \in \mathbf{R}^{n-1\times n}$. Reconstruction using derivative penalty is computed with

$$\min_{x\in\mathbf{R}^n} \quad J = \|x-y\| + \lambda\|D_n x\|. \tag{47}$$

10.7.1 Quadratic Smoothing

Minimum quadratic smoothing is computed with

$$\min_{x\in\mathbf{R}^n} \quad J = \|x-y\|_2^2 + \lambda\|D_n x\|_2^2,$$

which has the closed form solution

$$x = (\boldsymbol{I}_n + \lambda D_n^\intercal D_n)^{-1} y.$$

10.7.2 Total Varation

Minimum total variation is computed with

$$\min_{x\in\mathbf{R}^n} \quad J = \|x-y\|_2^2 + \lambda\|D_n x\|_1.$$

11 PDE: Partial Differential Equation

PDE problems optimize free variables which are differentially constrained over time and state–space. This differes from MPC only have 1 index corresponding to time. Adding state–space results multiple indicies. There are many potential applications including: eletro–magnetics, thermo–dynamics, surface optimization, classical mechanics, and quantum mechanics. Many interesting PDEs can be expressed as **LP** or **QP** .

Consider optimizing the f in $y = f(x)$. When discretized, x becomes the index and y becomes the free variable to be solved. To converte a PDE to a discrete problem:

- Remap the domain
$$X = \{x \in \mathbf{R}^n \mid x_i \in \mathbf{R}[a_i, b_i]\}$$
to the unit cube $\mathbf{R}[0,1]^n$.
- For $x_i \in \mathbf{R}[0,1]$, let $\Delta x_i = 1/n$.
- Integration of a function of $f(x)$ involves m summations over $\mathbf{Z}[0, n-1]$ divided by n^m.
- The partial derivatives across any of the x_i is n times the difference across that index.

Note: Integration and differentiation are linear operations.

11.1 Optimal Heat Diffusion

Consider $f : \mathbf{R}[0,1]^2 \to \mathbf{R}$ with the Laplace diffusion problem

$$\begin{aligned} \min_f \quad & J = \int_{\mathbf{R}[0,1]^2} g(x, f(x))dx \\ \text{s.t.} \quad & \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} = 0, \\ & \text{Boundary}(f) = \text{fixed}. \end{aligned}$$

Let y_{ij} denote the value of $f(x_{ij})$ at

$$x_{ij} = \begin{bmatrix} i/n \\ j/n \end{bmatrix}, \quad \begin{bmatrix} i \\ j \end{bmatrix} \in \mathbf{Z}[0, n-1]^2.$$

The Laplace diffusion problem can be approximated by

$$\begin{aligned} \min_{y \in \mathbf{R}^{n \times n}} \quad & J = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} g(x_{ij}, y_{ij}) \\ \text{s.t.} \quad & y_{i+1,j} + y_{i,j+1} - 2y_{ij} = 0 \\ & y \in \mathbf{D}. \end{aligned}$$

Note: If $\mathbf{D} = \mathbf{P}$ and $g(\cdot)$ is linear or quadratic, this can be solved as an **LP** or **QP** .

11.2 Minimum Surface Area

Consider $f : \mathbf{R}[0,1]^2 \rightarrow \mathbf{R}$ with surface area [1, p. 159]

$$\begin{aligned} A &= \int_{\mathbf{R}[0,1]^2} \sqrt{1 + \|\nabla f(x)\|_2^2} dx \\ &= \int_{\mathbf{R}[0,1]^2} \left\| \begin{bmatrix} \nabla f(x) \\ 1 \end{bmatrix} \right\|_2 dx. \end{aligned}$$

The minimum surface problem is to find f that minimizes A subject to boundary constraints.
Let y_{ij} denote the value of $f(x_{ij})$ at

$$x = \begin{bmatrix} i/n \\ j/n \end{bmatrix}, \quad \begin{bmatrix} i \\ j \end{bmatrix} \in \mathbf{Z}[0, n-1]^2.$$

An approximate expression for the gradient of f is

$$\nabla f(x) \approx n \begin{bmatrix} y_{i+1,j} - y_{i,j} \\ y_{i,j+1} - y_{i,j} \end{bmatrix},$$

which gives

$$A \approx \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left\| n \begin{bmatrix} y_{i+1,j} - y_{i,j} \\ y_{i,j+1} - y_{i,j} \\ 1/n \end{bmatrix} \right\|_2.$$

The problem can then be stated as

$$\min_{y \in \mathbf{R}^{n \times n}} \quad J = A$$

$$\text{s.t.} \quad y \in \mathbf{D},$$

which, if $\mathbf{D} = \mathbf{P}$, can be recast as the **SOCP** if

$$\min_{\{y,t\} \in \{\mathbf{R}^{n \times n}, \mathbf{R}^{n \times n}\}} \quad J = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} t_{ij}$$

$$\text{s.t.} \quad f \in \mathbf{P},$$
$$\left\| \begin{bmatrix} y_{i+1,j} - y_{i,j} \\ y_{i,j+1} - y_{i,j} \\ 1/n \end{bmatrix} \right\|_2 \leq t_{ij}/n,$$

which can be converted to a **QCQP** by squaring the conic constraints.

Note: At this point, additional linear and quadratic constraints can be added to **D**.

12 SVM: Support Vector Machines

12.1 SVM Models

For $x \in \mathbf{R}^n$, an SVM classifies with

$$\operatorname{svm}(x) = \begin{cases} 1 & \text{if } a^\intercal x - b > 0 \\ -1 & \text{if } a^\intercal x - b < 0 \\ -1 + 2\widetilde{r} & \text{else} \end{cases}, \tag{48}$$

where $\widetilde{r} \sim \text{Bernoulli}(1/2)$, i.e., flip a fair coin. Figure 47 illustrates this classification.

There are many ways to optimizate an SVM. The primary goals are to find **LP** and **QP** formulations. Secondary goals are to take advantage of dimension reduction through duality or sparsity. SVM references can be found at [22] and [23].

12.1.1 SVM as Single–Layer NN

An SVM can be approximated by a single–layer NN. Consider,

$$\operatorname{svm}(x) \approx s\left(a^\intercal x - b\right),$$

where $s(\cdot)$ is a sigmoid function.

This approximation allows gradient calculation with respect to a and b.

If there are m class labels,

$$s\left(Ax - b\right) = \begin{bmatrix} s(a_0^\intercal x - b_0) \\ \vdots \\ s(a_{m-1}^\intercal x - b_{m-1}) \end{bmatrix}.$$

For $i \in \mathbf{Z}[0, q-1]$ samples $x_i \in \mathbf{R}^n$ and $y_i \in \{-1, 1\}$, the SVM can be optimized with

$$\begin{aligned} \min_{\{a,b\} \in \{\mathbf{R}^n, \mathbf{R}\}} J &= \textstyle\sum_{i=0}^{q-1} \left\| \operatorname{svm}\left(a^\intercal x_i - b\right) - y_i \right\| \\ &\approx \textstyle\sum_{i=0}^{q-1} \left\| s\left(a^\intercal x_i - b\right) - y_i \right\|. \end{aligned}$$

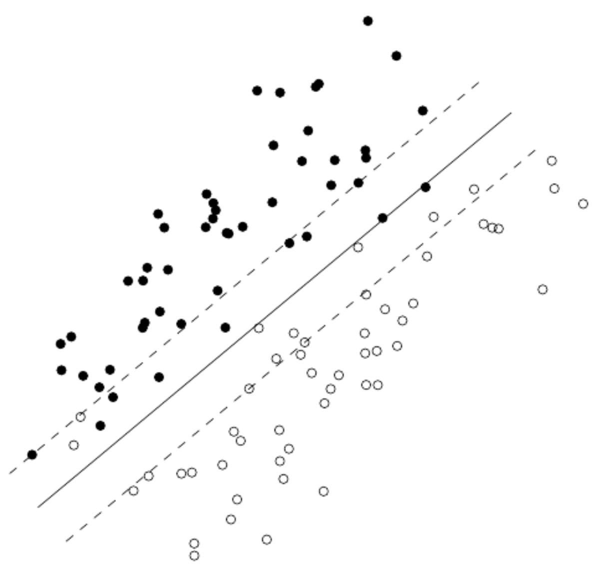


Figure 47: SVM

12.2 Primal Methods

Sort x_i with label $+1$ and y_j with label -1. For the following calculations,

$$x = \text{col}\{x_i\}_{i=0}^{m-1},$$
$$y = \text{col}\{y_j\}_{j=0}^{r-1}.$$

The support vector between $x_i \in \mathbf{R}^n$ and $y_j \in \mathbf{R}^n$ can be comptued with [1, p. 427]

$$\begin{aligned}
\min_{\{a,b,u,v\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}^m,\mathbf{R}^r\}} \quad & J = \|a\| + \lambda(\mathbf{1}^\intercal u + \mathbf{1}^\intercal v) \\
\text{s.t.} \quad & a^\intercal x_i - b \geq \quad 1 - u_i, \quad u_i \geq 0, \\
& a^\intercal y_j - b \leq -1 + v_j, \quad v_j \geq 0.
\end{aligned}$$

12.2.1 1–Norm

If $p = 1$, this is the **LP**

$$\min_{\{a,b,u,v,w\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}^m,\mathbf{R}^r,\mathbf{R}^n\}} \quad J = \begin{bmatrix} \mathbf{0}^n \\ 0 \\ \lambda\mathbf{1}^m \\ \lambda\mathbf{1}^r \\ \mathbf{1}^n \end{bmatrix}^\intercal \begin{bmatrix} a \\ b \\ u \\ v \\ w \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} -x^\intercal & \mathbf{1}^m & -\boldsymbol{I}_m & \mathbf{0}^{m\times r} & \mathbf{0}^{m\times n} \\ y^\intercal & -\mathbf{1}^r & \mathbf{0}^{r\times m} & -\boldsymbol{I}_r & \mathbf{0}^{r\times n} \\ -\boldsymbol{I}_n & \mathbf{0}^n & \mathbf{0}^{n\times m} & \mathbf{0}^{n\times r} & -\boldsymbol{I}_n \\ \boldsymbol{I}_n & \mathbf{0}^n & \mathbf{0}^{n\times m} & \mathbf{0}^{n\times r} & -\boldsymbol{I}_n \\ \mathbf{0}^{m\times n} & \mathbf{0}^m & -\boldsymbol{I}_m & \mathbf{0}^{m\times r} & \mathbf{0}^{m\times n} \\ \mathbf{0}^{r\times n} & \mathbf{0}^r & \mathbf{0}^{r\times m} & -\boldsymbol{I}_r & \mathbf{0}^{r\times n} \end{bmatrix} \begin{bmatrix} a \\ b \\ u \\ v \\ w \end{bmatrix} \leq \begin{bmatrix} -\mathbf{1}^m \\ -\mathbf{1}^r \\ \mathbf{0}^n \\ \mathbf{0}^n \\ \mathbf{0}^m \\ \mathbf{0}^r \end{bmatrix}.$$

12.2.2 2–Norm

If $p = 2$ and the norm is squared, this is the **QP**

$$\min_{\{a,b,u,v\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}^m,\mathbf{R}^r\}} \quad J = \frac{1}{2}\begin{bmatrix} a \\ b \\ u \\ v \end{bmatrix}^\intercal \begin{bmatrix} 2\boldsymbol{I}_n & \mathbf{0}^n & \mathbf{0}^{n\times m} & \mathbf{0}^{n\times r} \\ \mathbf{0}^{1\times n} & 0 & \mathbf{0}^{1\times m} & \mathbf{0}^{1\times r} \\ \mathbf{0}^{m\times n} & \mathbf{0}^m & \mathbf{0}^{m\times m} & \mathbf{0}^{m\times r} \\ \mathbf{0}^{r\times n} & \mathbf{0}^r & \mathbf{0}^{r\times m} & \mathbf{0}^{r\times r} \end{bmatrix} \begin{bmatrix} a \\ b \\ u \\ v \end{bmatrix} + \begin{bmatrix} \mathbf{0}^n \\ 0 \\ \lambda\mathbf{1}^m \\ \lambda\mathbf{1}^r \end{bmatrix}^\intercal \begin{bmatrix} a \\ b \\ u \\ v \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} -x^\intercal & \mathbf{1}^m & -\boldsymbol{I}_m & \mathbf{0}^{m\times r} \\ y^\intercal & -\mathbf{1}^r & \mathbf{0}^{r\times m} & -\boldsymbol{I}_r \\ \mathbf{0}^{m\times n} & \mathbf{0}^m & -\boldsymbol{I}_m & \mathbf{0}^{m\times r} \\ \mathbf{0}^{r\times n} & \mathbf{0}^r & \mathbf{0}^{r\times m} & -\boldsymbol{I}_r \end{bmatrix} \begin{bmatrix} a \\ b \\ u \\ v \end{bmatrix} \leq \begin{bmatrix} -\mathbf{1}^m \\ -\mathbf{1}^r \\ \mathbf{0}^m \\ \mathbf{0}^r \end{bmatrix}.$$

12.2.3 Inf–Norm

If $p = \infty$, this is the **LP**

$$\min_{\{a,b,u,v,w\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}^m,\mathbf{R}^r,\mathbf{R}\}} \quad J = \begin{bmatrix} \mathbf{0}^n \\ 0 \\ \lambda\mathbf{1}^m \\ \lambda\mathbf{1}^r \\ 1 \end{bmatrix}^\intercal \begin{bmatrix} a \\ b \\ u \\ v \\ w \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} -x^\intercal & \mathbf{1}^m & -\boldsymbol{I}_m & \mathbf{0}^{m\times r} & \mathbf{0}^m \\ y^\intercal & -\mathbf{1}^r & \mathbf{0}^{r\times m} & -\boldsymbol{I}_r & \mathbf{0}^r \\ -\boldsymbol{I}_n & \mathbf{0}^n & \mathbf{0}^{n\times m} & \mathbf{0}^{n\times r} & -\mathbf{1}^n \\ \boldsymbol{I}_n & \mathbf{0}^n & \mathbf{0}^{n\times m} & \mathbf{0}^{n\times r} & -\mathbf{1}^n \\ \mathbf{0}^{m\times n} & \mathbf{0}^m & -\boldsymbol{I}_m & \mathbf{0}^{m\times r} & \mathbf{0}^m \\ \mathbf{0}^{r\times n} & \mathbf{0}^r & \mathbf{0}^{r\times m} & -\boldsymbol{I}_r & \mathbf{0}^r \end{bmatrix} \begin{bmatrix} a \\ b \\ u \\ v \\ w \end{bmatrix} \leq \begin{bmatrix} -\mathbf{1}^m \\ -\mathbf{1}^r \\ \mathbf{0}^n \\ \mathbf{0}^n \\ \mathbf{0}^m \\ \mathbf{0}^r \end{bmatrix}.$$

12.3 Dual Methods

There is a computational advantage to using the dual if the m samples are smaller than the n variables of x.

12.3.1 Dual

For $x_i \in \mathbf{R}^n$ and $y_i \in \{-1,+1\}$, if the data is linearly seperable, then

$$\begin{cases} a^\mathsf{T} x_i - b \leq y_i & \text{if} \quad y_i = -1 \\ a^\mathsf{T} x_i - b \geq y_i & \text{if} \quad y_i = +1 \end{cases}.$$

These constraint can be re–expressed as

$$y_i(a^\mathsf{T} x_i - b) \geq 1,$$

which can be written in matrix form as

$$\begin{bmatrix} -\text{row}\{y_i x_i^\mathsf{T}\}_{i=0}^{m-1} & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \leq -\mathbf{1}^m.$$

Consider the primal problem

$$\min_{\{a,b\}\in\{\mathbf{R}^n,\mathbf{R}\}} \quad J = \frac{1}{2}\|a\|_2^2$$

$$\text{s.t.} \quad A_{\text{ub}} \begin{bmatrix} a \\ b \end{bmatrix} \leq b_{\text{ub}}.$$

The dual problem is

$$\max_{z\in\mathbf{R}^m} \quad \inf_{\{a,b\}\in\{\mathbf{R}^n,\mathbf{R}\}} \quad L(a,b,z)$$

$$\text{s.t.} \quad z \geq 0,$$

where

$$L(a,b,z) = \frac{1}{2}\|a\|_2^2 + z^\mathsf{T}\left(A_{\text{ub}} \begin{bmatrix} a \\ b \end{bmatrix} - b_{\text{ub}}\right).$$

Solving

$$\mathbf{0} = \begin{bmatrix} \frac{\partial L}{\partial a} & \frac{\partial L}{\partial b} \end{bmatrix} = \begin{bmatrix} a^\mathsf{T} - z^\mathsf{T}\text{row}\{y_i x_i^\mathsf{T}\}_{i=0}^{m-1} & z^\mathsf{T} y \end{bmatrix}$$

gives

$$\begin{aligned} a &= \text{col}\{x_i y_i\}_{i=0}^{m-1} z, \\ 0 &= y^\mathsf{T} z, \end{aligned}$$

with equality constraint on z.

The dual problem is [34, p. 313]

$$\min_{z\in\mathbf{R}^m} \quad \frac{1}{2} z^\mathsf{T} Q_{\text{dual}}\, z + c_{\text{dual}}^\mathsf{T}\, z$$

$$\begin{aligned} \text{s.t.} \quad z &\geq \mathbf{0}^m, \\ y^\mathsf{T} z &= 0, \end{aligned}$$

where

$$\begin{aligned} [Q_{\text{dual}}]_{ij} &= y_i y_j x_i^\mathsf{T} x_j \\ c_{\text{dual}} &= -\mathbf{1}^m. \end{aligned}$$

To recover the primal solution,

$$a^\mathsf{T} x = \sum_{i=0}^{m-1} z_i^* y_i x_i^\mathsf{T} x.$$

If $z_i^* > 0$, x_i is on the margin (by complamentary slackness) and $b_i^* = y_i - x_i^\mathsf{T} a^*$. Compute $b^* = \text{mean}(b_i^*)$.

12.3.2 Soft Dual

For $c_i \geq 0$, the constraints can be relaxed with

$$y_i(a^\intercal x_i - b) \geq 1 - c_i,$$

where

$$x_i = \begin{cases} \text{correctly classified outside of the margin} & \text{if} \quad c_i = 0 \\ \text{correctly classified inside the margin} & \text{if} \quad c_i \in \mathbf{R}(0,1) \\ \text{missclassified} & \text{if} \quad c_i \geq 1 \end{cases}.$$

Consider the primal problem

$$\begin{aligned} \min_{\{a,b,c\} \in \{\mathbf{R}^n, \mathbf{R}, \mathbf{R}^m\}} \quad & J = \frac{1}{2}\|a\|_2^2 + \lambda \mathbf{1}^\intercal c \\ \text{s.t.} \quad & A_{\text{ub}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \leq b_{\text{ub}}, \end{aligned}$$

where

$$\begin{aligned} A_{\text{ub}} &= \begin{bmatrix} -\text{row}\{y_i x_i^\intercal\}_{i=0}^{m-1} & y & -\boldsymbol{I}_m \\ \mathbf{0}^{m\times n} & \mathbf{0}^{m \times 1} & -\boldsymbol{I}_m \end{bmatrix}, \\ b_{\text{ub}} &= \begin{bmatrix} -\mathbf{1}^m \\ \mathbf{0}^m \end{bmatrix}. \end{aligned}$$

The dual problem is

$$\begin{aligned} \max_{\{z,w\} \in \{\mathbf{R}^m, \mathbf{R}^m\}} \quad & \inf_{\{a,b\} \in \{\mathbf{R}^n, \mathbf{R}\}} \quad L(a,b,z,w) \\ \text{s.t.} \quad & \begin{bmatrix} z \\ w \end{bmatrix} \geq \mathbf{0}^{2m}, \end{aligned}$$

where

$$L(a,b,z) = \frac{1}{2}\|a\|_2^2 + \lambda \mathbf{1}^\intercal c + \begin{bmatrix} z \\ w \end{bmatrix}^\intercal \left(A_{\text{ub}} \begin{bmatrix} a \\ b \\ c \end{bmatrix} - b_{\text{ub}} \right).$$

Solving

$$\mathbf{0} = \begin{bmatrix} \frac{\partial L}{\partial a} & \frac{\partial L}{\partial b} & \frac{\partial L}{\partial c} \end{bmatrix} = \begin{bmatrix} a^\intercal - z^\intercal \text{row}\{y_i x_i^\intercal\}_{i=0}^{m-1} & z^\intercal y & \lambda \mathbf{1}^m - z - w \end{bmatrix}$$

gives

$$\begin{aligned} a &= \text{col}\{x_i y_i\}_{i=0}^{m-1} z, \\ 0 &= y^\intercal z, \\ w &= \lambda \mathbf{1}^m - z. \end{aligned}$$

The dual problem is [34, p. 316]

$$\begin{aligned} \min_{z \in \mathbf{R}^m} \quad & \frac{1}{2} z^\intercal Q_{\text{dual}}\, z + c_{\text{dual}}^\intercal\, z \\ \text{s.t.} \quad & \mathbf{0}^m \leq z \leq \lambda \mathbf{1}^m \\ & y^\intercal z = 0, \end{aligned}$$

where

$$\begin{aligned} [Q_{\text{dual}}]_{ij} &= y_i y_j x_i^\intercal x_j, \\ c_{\text{dual}} &= -\mathbf{1}^m. \end{aligned}$$

To recover the primal solution,

$$a^\intercal x = \sum_{i=0}^{m-1} z_i^* y_i x_i^\intercal x.$$

If $z_i^* \in \mathbf{R}(0, \lambda)$, x_i is on the margin (by complementary slackness) and $b_i^* = y_i - x_i^\intercal a^*$. Compute $b^* = \text{mean}(b_i^*)$.

12.3.3 Soft Dual (Alternative)

For $\lambda \in \mathbf{R}[0,1]$, consider the primal problem

$$\min_{\{a,b,c,d\}\in\{\mathbf{R}^n,\mathbf{R},\mathbf{R}^m,\mathbf{R}\}} \quad J = \frac{1}{2}\|a\|_2^2 + \frac{1}{m}\mathbf{1}^\intercal c - \lambda d$$

$$\begin{aligned} \text{s.t.} \quad y_i(a^\intercal x_i - b) &\geq d - c_i, \\ c &\geq \mathbf{0}^m, \\ d &\geq 0. \end{aligned}$$

The margin is now given by $2d/\|a\|_2$.

The dual problem is [34, p. 319]

$$\min_{z\in\mathbf{R}^m} \quad \frac{1}{2}z^\intercal Q_{\text{dual}}\, z$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{0}^m \leq z &\leq \frac{1}{m}\mathbf{1}^m, \\ \mathbf{1}^\intercal z &\leq \lambda, \\ y^\intercal z &= 0, \end{aligned}$$

where

$$[Q_{\text{dual}}\,]_{ij} = y_i y_j x_i^\intercal x_j.$$

To recover the primal solution,

$$a^\intercal x = \sum_{i=0}^{m-1} z_i^* y_i x_i^\intercal x.$$

12.3.4 Kernels

The kernel trick comes from noting that both

$$[Q_{\text{dual}}\,]_{ij} = y_i y_j K(x_i, x_j)$$

and

$$\begin{aligned} a^\intercal x &= \textstyle\sum_{i=0}^{m-1} z_i^* y_i x_i^\intercal x \\ &= \textstyle\sum_{i=0}^{m-1} z_i^* y_i K(x_i, x). \end{aligned}$$

contain inner products.

If $x = f(u)$ is a mapping from a low dimensional space to a very large feature space, then replacing $x_i^\intercal x_j = f(u_i)^\intercal f(u_j)$ with a kernel $K(u_i, u_j)$ gives [34, p. 321]

$$[Q_{\text{dual}}\,]_{ij} = y_i y_j K(u_i, u_j)$$

and

$$\begin{aligned} a^\intercal x &= \textstyle\sum_{i=0}^{m-1} z_i^* y_i f(u_i)^\intercal f(u) \\ &= \textstyle\sum_{i=0}^{m-1} z_i^* y_i K(u_i, u). \end{aligned}$$

This can significantly reduce computation without imposing restrictive limits on n. In some cases, a feature vector may be near infinite or not even known, but the inner product between $f(u_i)$ and $f(u_j)$ may have a known form or at least a good heuristic.

Some popular kernels include:

- polynomials of degree q, computed with $(u_i^\intercal u_j + 1)^q$,
- radial basis functions computed from $\text{dist}(u_i, u_j)$, e.g., $\|u_j - u_i\|_p$.

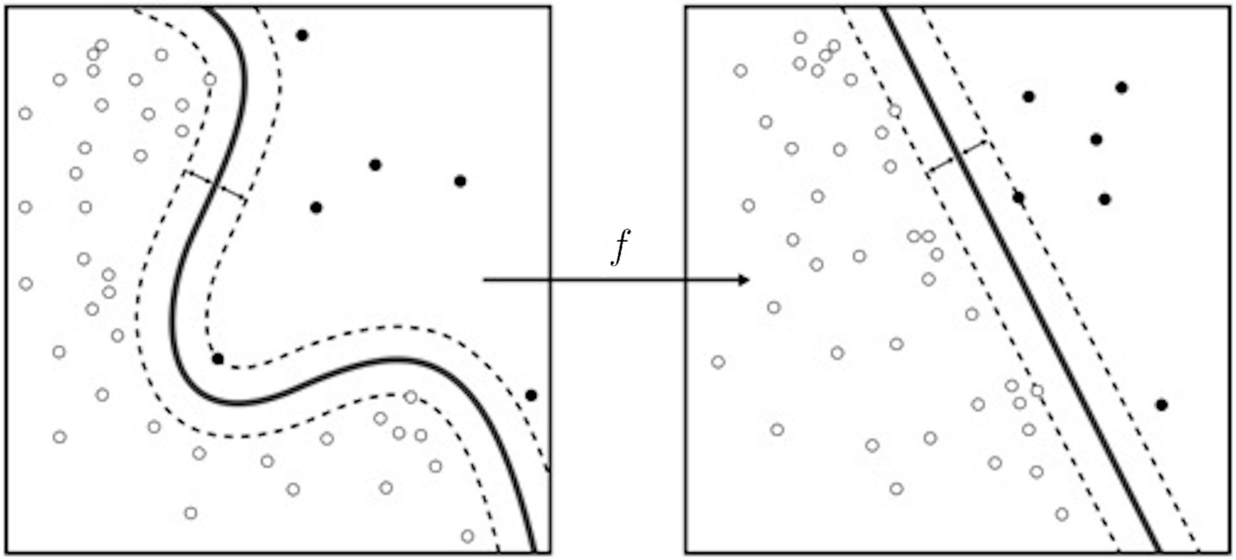


Figure 48: SVM with the kernel trick

12.3.5 Bias

Sort x_i with label $+1$ and y_j with label -1. For the following calculations,

$$
\begin{aligned}
x &= \mathrm{col}\{x_i\}_{i=0}^{m-1}, \\
y &= \mathrm{col}\{y_j\}_{j=0}^{r-1}.
\end{aligned}
$$

Given a^* from the dual problem, the support vector between $x_i \in \mathbf{R}^n$ and $y_j \in \mathbf{R}^n$ can be comptued with

$$
\min_{\{b,u,v\} \in \{\mathbf{R}, \mathbf{R}^m, \mathbf{R}^r\}} \quad J = \mathbf{1}^\intercal u + \mathbf{1}^\intercal v
$$

$$
\begin{aligned}
\text{s.t.} \quad x_i^\intercal a^* - b &\geq 1 - u_i, \quad u_i \geq 0, \\
y_j^\intercal a^* - b &\leq -1 + v_j, \quad v_j \geq 0.
\end{aligned}
$$

These constraints can be re–expressed as

$$
\begin{aligned}
b - u_i &\leq -1 + x_i^\intercal a^*, \quad -u_i \leq 0, \\
-b - v_j &\leq -1 - y_j^\intercal a^*, \quad -v_j \leq 0.
\end{aligned}
$$

The bias can be computed with the primary **LP**

$$
\min_{\{b,u,v\} \in \{\mathbf{R}, \mathbf{R}^m, \mathbf{R}^r\}} \quad J = \begin{bmatrix} 0 \\ \mathbf{1}^m \\ \mathbf{1}^r \end{bmatrix}^\intercal \begin{bmatrix} b \\ u \\ v \end{bmatrix}
$$

$$
\text{s.t.} \quad \begin{bmatrix} \mathbf{1}^m & -\boldsymbol{I}_m & \mathbf{0}^{m\times r} \\ -\mathbf{1}^r & \mathbf{0}^{r\times m} & -\boldsymbol{I}_r \\ \mathbf{0}^m & -\boldsymbol{I}_m & \mathbf{0}^{m\times r} \\ \mathbf{0}^r & \mathbf{0}^{r\times m} & -\boldsymbol{I}_r \end{bmatrix} \begin{bmatrix} b \\ u \\ v \end{bmatrix} \leq \begin{bmatrix} -\mathbf{1}^m \\ -\mathbf{1}^r \\ \mathbf{0}^m \\ \mathbf{0}^r \end{bmatrix} + \begin{bmatrix} \mathrm{row}\{x_i^\intercal a^*\}_{i=0}^{m-1} \\ -\mathrm{row}\{y_j^\intercal a^*\}_{j=0}^{r-1} \\ \mathbf{0}^m \\ \mathbf{0}^r \end{bmatrix}
$$

13 Miscellaneous Applications

13.1 Minimum Complexity Modeling

For samples $i \in \mathbf{Z}[0, q-1]$, find a linear map from $x_i \in \mathbf{R}^n$ to $y_i \in \mathbf{R}^m$ with minimum non–zero paramaters. This objective can be expressed as

$$\min_{A \in \mathbf{R}^{m \times n}} \quad J = \lambda \|\vec{A}\|_1 + \textstyle\sum_{i=0}^{q-1} \|Ax_i - y_i\|^2_{2|W_i}. \tag{49}$$

Vectorizing from left to right and top to bottom and using the Kronecker product, the objective function becomes

$$\begin{aligned} J &= \lambda \|\vec{A}\|_1 + \textstyle\sum_{i=0}^{q-1} \|(\boldsymbol{I}_m \otimes x_i^\intercal)\vec{A} - y_i\|^2_{2|W_i} \\ &= \frac{1}{2}\vec{A}^\intercal Q \vec{A} + c^\intercal \vec{A} + r + \lambda\|\vec{A}\|_1 + \textstyle\sum_{i=0}^{q-1} y_i^\intercal y_i, \end{aligned}$$

where

$$\begin{aligned} Q &= 2\textstyle\sum_{i=0}^{q-1} (\boldsymbol{I}_m \otimes x_i) W_i^\intercal W_i (\boldsymbol{I}_m \otimes x_i^\intercal) \\ &= 2\textstyle\sum_{i=0}^{q-1} (\boldsymbol{I}_m \otimes x_i x_i^\intercal) && \text{if} \quad W_i = \boldsymbol{I}, \\ \\ c &= -2\textstyle\sum_{i=0}^{q-1} (\boldsymbol{I}_m \otimes x_i)\, W_i^\intercal W_i y_i \\ &= -2\textstyle\sum_{i=0}^{q-1} (\boldsymbol{I}_m \otimes x_i)\, y_i && \text{if} \quad W_i = \boldsymbol{I}. \end{aligned}$$

Note: Additional convex paramater constraints such as non–negativity can easily be added with $\vec{A} \in \mathbf{P}$.

13.2 Linear Fractional

Consider

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \frac{a^\intercal x - b}{c^\intercal x - d} \\ \\ \text{s.t.} \quad & x \in \mathbf{P}, \\ & c^\intercal x - d > 0. \end{aligned}$$

The problem can be stated as the **LP** [1, p. 151]

$$\begin{aligned} \min_{\{y,z\} \in \{\mathbf{R}^n, \mathbf{R}\}} \quad & J = a^\intercal y - bz \\ \\ \text{s.t.} \quad & A_{\text{ub}} y - b_{\text{ub}} z \le \mathbf{0}, \\ & A_{\text{eq}} y - b_{\text{eq}} z = \mathbf{0}, \\ & c^\intercal y - dz = 1, \\ & z \ge 0, \end{aligned}$$

which can be expressed as

$$\begin{aligned} \min_{\{y,z\} \in \{\mathbf{R}^n, \mathbf{R}\}} \quad & J = \begin{bmatrix} a \\ -b \end{bmatrix}^\intercal \begin{bmatrix} y \\ z \end{bmatrix} \\ \\ \text{s.t.} \quad & \begin{bmatrix} A_{\text{ub}} & -b_{\text{ub}} \\ \mathbf{0}^{1 \times n} & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \le \mathbf{0}^{m+1}, \\ & \begin{bmatrix} A_{\text{eq}} & -b_{\text{eq}} \\ c^\intercal & -d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{0}^r \\ 1 \end{bmatrix}, \end{aligned}$$

where $x = y/z$. The solution exists if $c^\intercal x > d$.
Note: These problem fomulations are common in projective geometry.

13.3 Sampled Convex Sets

For a convex set $\mathbf{D} \subset \mathbf{R}^n$, consider

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = \|Ax - b\|_p \\ \text{s.t.} \quad & x \in \mathbf{D}. \end{aligned}$$

The constraint $x \in \mathbf{D}$ can be approximated with m sample points $s_i \in \mathbf{D}$ using the convex combination [1, p. 24]

$$x = \textstyle\sum_{i=0}^{m-1} s_i y_i, \quad 0 \leq y_i \leq 1, \quad \textstyle\sum_{i=0}^{m-1} y_i = 1, \quad y_i \in \mathbf{R}.$$

The problem can then be expressed as

$$\begin{aligned} \min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad & J = \|Ax - b\|_p \\ \text{s.t.} \quad & A_{\text{eq}} \begin{bmatrix} x \\ y \end{bmatrix} = b_{\text{eq}}, \\ & A_{\text{ub}} \begin{bmatrix} x \\ y \end{bmatrix} \leq b_{\text{ub}}, \end{aligned}$$

where

$$\begin{aligned} y &= \text{row}\,\{y_i\}_{i=0}^{m-1} \\ A_{\text{eq}} &= \begin{bmatrix} -\boldsymbol{I}_n & s \\ \mathbf{0}^{1\times n} & \mathbf{1}^{1\times m} \end{bmatrix}, \\ b_{\text{eq}} &= \begin{bmatrix} \mathbf{0}^n \\ 1 \end{bmatrix}, \\ A_{\text{ub}} &= \begin{bmatrix} \mathbf{0}^{m\times n} & -\boldsymbol{I}_m \\ \mathbf{0}^{m\times n} & \boldsymbol{I}_m \end{bmatrix}, \\ b_{\text{ub}} &= \begin{bmatrix} \mathbf{0}^m \\ \mathbf{1}^m \end{bmatrix}. \end{aligned}$$

Note: Eliminate interior sample points before optimizing to reduce computation.

13.4 Interset Distance

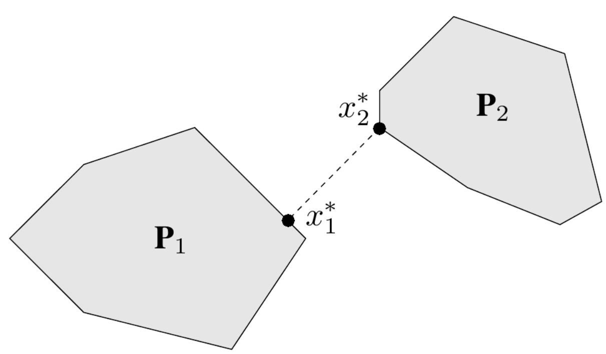


Figure 49: minimum distance between disjoint polyhedron sets

The problem

$$\begin{aligned} \min_{\{x_1,x_2\} \in \{\mathbf{R}^n, \mathbf{R}^n\}} \quad & J = \|x_1 - x_2\|_p \\ \text{s.t.} \quad & x_1 \in \mathbf{P}_1, \\ & x_2 \in \mathbf{P}_2, \end{aligned} \tag{50}$$

is an **LP** for $p \in \{1, \infty\}$ and a **QP** for $p = 2$.

13.5 SPINV: Sparse Pseudo Inverse

For $A \in \mathbf{R}^{m \times n}$ with $m > n$ and $\operatorname{rank}(A) = n$, a sparse representation of $B = A^{+}$ can be computed using [7]

$$\begin{aligned}\min_{B \in \mathbf{R}^{n \times m}} \quad J &= \|BA - \boldsymbol{I}_n\|_2^2 + \lambda \|\vec{B}\|_1 \\ &= \|(\boldsymbol{I}_n \otimes A^{\intercal})\vec{B} - \vec{\boldsymbol{I}}_n\|_2^2 + \lambda \|\vec{B}\|_1.\end{aligned} \tag{51}$$

Note: If $A \in \mathbf{S}_{+}^{n}$, then $B = A^{-1} \in \mathbf{S}_{+}^{n}$, where the symmetry can be enforced with the constraint $B = B^{\intercal}$.

13.6 k–means

Given m sampls of $s_j \in \mathbf{D} \subset \mathbf{R}^n$, k cluster centers can be found using Algorithm 12 [34, p. 149]. See [54, p. 195] for its derivation from the EM algorithm. This is a non-convex optimization, and many potential cluster may exist. Convergence to a particular cluster depends on the initialization of the algorithm. Many different norms may be considered. Different norms create different cell geometries. A 2–norm creates convex polyhedron cells (see figure 50b), but a 1–norm creates non-convex cells (see figure 50a).

For $j \in \mathbf{Z}[0, m-1]$, sample $s_i \in \mathbf{D}$
For $i \in \mathbf{Z}[0, k-1]$, initialize means $x_i \in \mathbf{D}$
while x not converged **do**
  for $j \in \mathbf{Z}[0, m-1]$ **do**

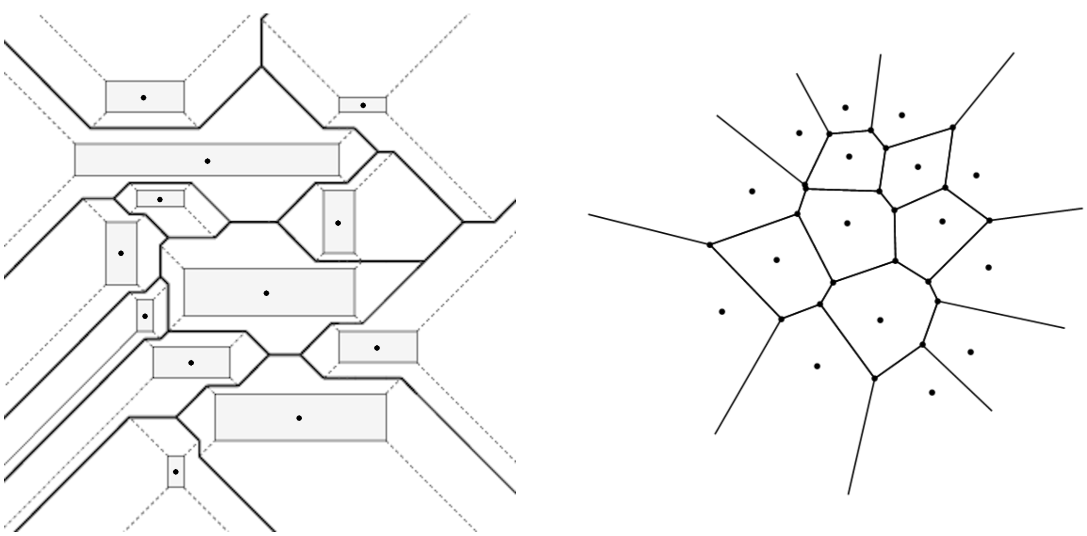
$$b_{ij} \leftarrow \begin{cases} 1 & \text{if} \quad \|s_j - x_i\| = \min_q \|s_j - x_q\| \\ 0 & \text{else} \end{cases}$$

  end
  for $i \in \mathbf{Z}[0, k-1]$ **do**

$$x_i \leftarrow \frac{\sum_{j=0}^{m-1} b_{ij} s_j}{\sum_{j=0}^{m-1} b_{ij}}$$

  end
end

Algorithm 12: k–means



(a) 1–Norm Voronoi

(b) 2–Norm Voronoi

Figure 50: Voronoi Cells

13.7 2–Norm Voronoi

A polyhedron can be expressed as [1, p. 60]

$$\begin{aligned}\mathbf{P}_0 &= \{x \in \mathbf{R}^n | \|x - x_0\|_2 \leq \|x - x_i\|_2, i \in \mathbf{Z}[1, m-1]\} \\ &= \{x \in \mathbf{R}^n | Ax \leq b\}.\end{aligned}$$

Squaring the constraints gives

$$\|x - x_0\|_2^2 \leq \|x - x_i\|_2^2.$$

Expanding the constraints gives

$$x^\intercal x - 2x_0^\intercal x + x_0^\intercal x_0 \leq x^\intercal x - 2x_i^\intercal x + x_i^\intercal x_i.$$

Collecting terms gives

$$2(x_i - x_0)^\intercal x \leq x_i^\intercal x_i - x_0^\intercal x_0,$$

which gives the relationship

$$\begin{aligned}a_i &= 2(x_i - x_0), \\ b_i &= x_i^\intercal x_i - x_0^\intercal x_0.\end{aligned}$$

Therefore,

$$\begin{aligned}x_i &= \frac{1}{2}a_i + x_0, \\ b_i &= \left(\frac{1}{2}a_i + x_0\right)^\intercal \left(\frac{1}{2}a_i + x_0\right) - x_0^\intercal x_0 \\ &= \frac{1}{4}a_i^\intercal a_i + a_i^\intercal x_0.\end{aligned}$$

Let

$$\begin{aligned}A &= \text{row}\{a_i^\intercal\}_{i=1}^{m-1}, \\ b &= \text{row}\{b_i\}_{i=1}^{m-1}, \\ c &= \frac{1}{4}\text{row}\{a_i^\intercal a_i\}_{i=1}^{m-1}.\end{aligned}$$

Then,

$$b = c + Ax_0.$$

For the SVD $A = U_+ S_+ V_+^\intercal$,

$$x_0 = A^+(b - c) + U_0 d,$$

where $d \in \mathbf{R}^{n-r}$ can be freely choosen.

Mutually exclusive polyhedral sets can be computed with

$$\begin{aligned}\mathbf{P}_j &= \{x \in \mathbf{R}^n | \|x - x_j\|_2 \leq \|x - x_i\|_2, i \neq j\} \\ &= \{x \in \mathbf{R}^n | A_j x \leq b_j\}.\end{aligned}$$

If $\text{rank}(A_j) = n$ for all j,

$$\mathbf{R}^n = \bigcup_{j=0}^{m-1} \mathbf{P}_j,$$

and, for $i \neq j$,

$$\text{Interior}(\mathbf{P}_i) \cap \text{Interior}(\mathbf{P}_j) = \emptyset.$$

Note: When only two points are present,

$$\begin{aligned}\|x-x_0\|_2^2 \le \|x-x_1\|_2^2 \quad &\Leftrightarrow \quad (x-x_0)^\intercal(x-x_0) \le (x-x_1)^\intercal(x-x_1)\\ &\Leftrightarrow \quad x^\intercal x - 2x_0^\intercal x + x_0^\intercal x_0 \le x^\intercal x - 2x_1^\intercal x + x_1^\intercal x_1\\ &\Leftrightarrow \quad 2(x_1-x_0)^\intercal x \le x_1^\intercal x_1 - x_0^\intercal x_0,\end{aligned}$$

defines a seperating hyperplane $\{x \in \mathbf{R}^n | a^\intercal x \le b\}$, with $a = 2(x_1 - x_0)$ and $b = x_1^\intercal x_1 - x_0^\intercal x_0$.

13.8 Noisy Unknown Nonlinear Maps

For $i \in \mathbf{Z}[0, m-1]$, consider

$$y_i = f(a_i^\intercal x - b_i + \widetilde{w}_i),$$

where $x \in \mathbf{R}^n$ is a vector to be estimated, $\{y_i, a_i, b_i\} \in \{\mathbf{R}, \mathbf{R}^n, \mathbf{R}\}$ are measured, and $\widetilde{w}_i$ is IID. The function $f : \mathbf{R} \to \mathbf{R}$ is unknown, but it is known that $f' \in \mathbf{R}[c, d]$ with $0 < c < d$. Solving for $\widetilde{w}_i$,

$$\widetilde{w}_i = f^{-1}(y_i) - a_i^\intercal x + b_i^\intercal.$$

The probability of observing a sequence of y_i is then

$$\prod_{i=0}^{m-1} p(f^{-1}(y_i) - a_i^\intercal x + b_i^\intercal).$$

The objective is then to minimize the negative log–likelihood

$$\min_{\{x, z_i\} \in \{\mathbf{R}^n, \mathbf{R}\}} \quad J = -\sum_{i=0}^{m-1} \log p(z_i - a_i^\intercal x + b_i^\intercal).$$

$$\text{s.t.} \quad z_i = f^{-1}(y_i).$$

The constraints can be expressed in terms of the inverse with

$$\frac{\partial}{\partial y_i} f^{-1}(y_i) \in \mathbf{R}[d^{-1}, c^{-1}],$$

which, for $i \in \mathbf{Z}[0, m-1]$ and $j \in \mathbf{Z}[0, m-1]$, gives the constraint

$$\frac{|y_i - y_j|}{d} \le |z_i - z_j| \le \frac{|y_i - y_j|}{c}.$$

This result is easily extended to nested nonlinear models of the form

$$\begin{aligned} x_{j+1} &= f(a_{ij}^\intercal x_j - b_{ij} + \widetilde{w}_{ij}),\\ x_{m-1} &= y_i.\end{aligned}$$

14 Simplifications

14.1 Affine Transforms

Consider $x = Ay - b$. The constraint $x \in \mathbf{P}$ implies

$$Ay - b \in \mathbf{P},$$

which gives

$$\begin{aligned} A_{\text{ub}}(Ay - b) &\leq b_{\text{ub}}, \\ A_{\text{eq}}(Ay - b) &= b_{\text{eq}}, \\ x_{\text{lb}} \leq (Ay - b) &\leq x_{\text{ub}}. \end{aligned}$$

This can be equivalently stated as

$$\begin{bmatrix} A_{\text{ub}}A \\ A \\ -A \end{bmatrix} y \leq \begin{bmatrix} b_{\text{ub}} + A_{\text{ub}}b \\ x_{\text{ub}} + b \\ -x_{\text{lb}} - b \end{bmatrix},$$
$$(A_{\text{eq}}A)y = (b_{\text{eq}} + A_{\text{eq}}b).$$

14.1.1 Full-Rank Inverse

For $A \in \mathbf{R}^{n \times n}$, if $\text{rank}(A) = n$,

$$\min_{x \in \mathbf{R}^n} \ J = f(Ax - b)$$
$$x \in \mathbf{P},$$

is equivalent to

$$\min_{y \in \mathbf{R}^n} \ J = f(y)$$
$$A^{-1}(y - b) \in \mathbf{P},$$

with $x^* = A^{-1}(y^* - b)$.

14.1.2 Full-Rank Right Pseudo Inverse

For $A \in \mathbf{R}^{m \times n}$, if $\text{rank}(A) = m$, then $A^+ = A^\intercal(AA^\intercal)^{-1}$, $AA^+ = \boldsymbol{I}_m$, and

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \ J &= f(Ax - b) \\ &= f(A(x - A^+b)) \\ \text{s.t.} \quad & x \in \mathbf{P} \end{aligned}$$

is equivalent to

$$\min_{y \in \mathbf{R}^n} \ J = f(Ay)$$
$$\text{s.t.} \quad y + A^+b \in \mathbf{P},$$

with $x^* = y^* + A^+b$. **Note**: This problem is ill-conditioned without constraint.

14.1.3 Full-Rank Left Pseudo Inverse

For $A \in \mathbf{R}^{m \times n}$, if $\text{rank}(A) = n$, then $A^+ = (A^\intercal A)^{-1}A^\intercal$, $A^+A = \boldsymbol{I}_n$, and

$$\min_{x \in \mathbf{R}^n} \ J = f(Ax - b).$$
$$x \in \mathbf{P}$$

is equivalent to

$$\min_{y \in \mathbf{R}^m} \ J = f(y)$$
$$\text{s.t.} \quad A^+(y + b) \in \mathbf{P},$$

with $x^* = A^+(y^* + b)$.

14.2 Simplifying Inequality Constriant

Consider the general problem

$$\min_{x\in\mathbf{R}^n} \quad J = f(Ax-b)$$

$$\begin{aligned} \text{s.t.} \quad A_{\text{ub}}x &\le b_{\text{ub}}, \\ A_{\text{eq}}x &= b_{\text{eq}}. \end{aligned}$$

14.2.1 Slack Variables

For $A_{\text{ub}} \in \mathbf{R}^{m\times n}$, introduce a slack vector $y \in \mathbf{R}^m$ to get

$$\min_{x\in\mathbf{R}^n} \quad J = f(Ax-b)$$

$$\begin{aligned} \text{s.t.} \quad A_{\text{ub}}x + y &= b_{\text{ub}}, \\ A_{\text{eq}}x &= b_{\text{eq}}, \\ y &\ge 0, \end{aligned}$$

which can be written as

$$\min_{\{x,y\}\in\{\mathbf{R}^n,\mathbf{R}^m\}} \quad J = f\left(\begin{bmatrix} A & \mathbf{0} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - b\right)$$

$$\begin{aligned} \text{s.t.} \quad \begin{bmatrix} A_{\text{ub}} & \boldsymbol{I}_m \\ A_{\text{eq}} & \mathbf{0} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} b_{\text{ub}} \\ b_{\text{eq}} \end{bmatrix}, \\ y &\ge 0. \end{aligned}$$

14.2.2 Non–Negative Variables

For the problem

$$\min_{x\in\mathbf{R}^n} \quad J = f(Ax-b)$$

$$\begin{aligned} \text{s.t.} \quad A_{\text{ub}}x + y &= b_{\text{ub}}, \\ A_{\text{eq}}x &= b_{\text{eq}}, \\ y &\ge 0, \end{aligned}$$

let $x = x_{\text{pos}} - x_{\text{neg}}$ with $x_{\text{pos}} \ge 0$ and $x_{\text{neg}} \ge 0$, to get

$$\min_{\{x_{\text{pos}},x_{\text{neg}},y\}\in\{\mathbf{R}^n,\mathbf{R}^n,\mathbf{R}^m\}} \quad J = f\left(\begin{bmatrix} A & -A & \mathbf{0} \end{bmatrix}\begin{bmatrix} x_{\text{pos}} \\ x_{\text{neg}} \\ y \end{bmatrix} - b\right)$$

$$\begin{aligned} \text{s.t.} \quad \begin{bmatrix} A_{\text{ub}} & -A_{\text{ub}} & \boldsymbol{I}_m \\ A_{\text{eq}} & -A_{\text{eq}} & \mathbf{0} \end{bmatrix}\begin{bmatrix} x_{\text{pos}} \\ x_{\text{neg}} \\ y \end{bmatrix} &= \begin{bmatrix} b_{\text{ub}} \\ b_{\text{eq}} \end{bmatrix}, \\ \begin{bmatrix} x_{\text{pos}} \\ x_{\text{neg}} \\ y \end{bmatrix} &\ge \mathbf{0}^{2n+m}. \end{aligned}$$

The problem is now in standard form [1, p. 147].

14.3 Removing Equality Constraint

Consider $A_{\text{eq}} = U_+ S_+ V_+^\intercal$. A valid problem has $U_0^\intercal b_{\text{eq}} = \mathbf{0}$. Multiply $U_+^\intercal$ across the equality constraint to get

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad J =& f(Ax - b) \\ \text{s.t.} \quad & f_{\text{ub}}(x) \leq 0 \\ & S_+ V_+^\intercal x = U_+^\intercal b_{\text{eq}}. \end{aligned}$$

If $x = V_+ S_+^{-1} y + V_0 z$, then $y = U_+^\intercal b_{\text{eq}}$, and the problem reduces to

$$\begin{aligned} \min_{z \in \mathbf{R}^{n-r}} \quad J =& f(AV_0 z + AA_{\text{eq}}^+ b_{\text{eq}} - b) \\ \text{s.t.} \quad & f_{\text{ub}}(V_0 z + A_{\text{eq}}^+ b_{\text{eq}}) \leq 0 \end{aligned}$$

14.4 Relaxing Equality Constrains

Consider

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = f(x) \\ \text{s.t.} \quad & f_{\text{ub}}(x) \leq 0, \\ & Ax = b, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = f(x) \\ \text{s.t.} \quad & f_{\text{ub}}(x) \leq 0, \\ & \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}. \end{aligned}$$

However, there is no interior to this set. For a fixed $y > 0$, consider the relaxed problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = f(x) \\ \text{s.t.} \quad & f_{\text{ub}}(x) \leq 0, \\ & Ax - y \leq b, \\ & Ax + y \geq b. \end{aligned}$$

For $W \in \mathbf{S}_+^m$, consider the relaxed problem

$$\begin{aligned} \min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad & J = f(x) + \|y\|_W \\ \text{s.t.} \quad & f_{\text{ub}}(x) \leq 0, \\ & \begin{bmatrix} A & -\boldsymbol{I}_m \\ -A & -\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} b \\ -b \end{bmatrix}. \end{aligned}$$

Note: These constraints force $y \geq 0$.

This is equivalent to the problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad J =& f(x) + \|Ax - b\|_W \\ \text{s.t.} \quad & f_{\text{ub}}(x) \leq 0. \end{aligned}$$

14.5 Partial QP

14.6 Linear 2–Norm Decompostion with SVD

Consider

$$A = U_+ S_+ V_+^\intercal.$$

Let

$$\begin{aligned} x &= \underbrace{\begin{bmatrix} V_+ & V_0 \end{bmatrix}}_{V} \begin{bmatrix} S_+^{-1} y \\ z \end{bmatrix} \\ &= V_+ S_+^{-1} y + V_0 z. \end{aligned}$$

The problem

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \; J &= \|Ax - b\|_2^2 \\ &= x^\intercal V_+ S_+ U_+^\intercal U_+ S_+ V_+^\intercal x - 2b^\intercal U_+ S_+ V_+^\intercal x + b^\intercal b \\ &= y^\intercal y - 2b^\intercal U_+ y + b^\intercal b + \underbrace{b^\intercal U_+ U_+^\intercal b - b^\intercal U_+ U_+^\intercal b)}_{0} \end{aligned}$$

$$\text{s.t.} \quad x \in \mathbf{P}$$

is equivalent to

$$\min_{\{y,z\} \in \{\mathbf{R}^r, \mathbf{R}^{n-r}\}} \quad J = \|y - U_+^\intercal b\|_2^2 + b^\intercal U_0 U_0^\intercal b,$$

$$\text{s.t.} \quad V_+ S_+^{-1} y + V_0 z \in \mathbf{P},$$

with

$$x^* = V_+ S_+^{-1} y^* + V_0 z^*.$$

14.6.1 Polyhedron Decompostion with SVD

Consider the problem

$$\min_{z \in \mathbf{R}^n} \quad J = \frac{1}{2} z^\intercal Q z + c^\intercal z$$

$$\text{s.t.} \quad z \in \mathbf{P},$$

with

$$\begin{bmatrix} A_{\text{ub}} \\ A_{\text{eq}} \end{bmatrix} = U_+ S_+ V_+^\intercal.$$

Let

$$\begin{aligned} z &= V_0 x + V_+ y \\ &= \begin{bmatrix} V_0 & V_+ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

The problem becomes

$$\min_{y \in \mathbf{R}^r} \min_{x \in \mathbf{R}^{n-r}} \quad J = \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^\intercal \begin{bmatrix} V_0^\intercal Q V_0 & V_0^\intercal Q V_+ \\ V_+^\intercal Q V_0 & V_+^\intercal Q V_+ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} V_0^\intercal c \\ V_+^\intercal c \end{bmatrix}^\intercal \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{s.t.} \quad V_+ y \in \mathbf{P}$$

14.6.2 Sum of Squares Decomposition

If $A \in \mathbf{S}^n$, $D \in \mathbf{S}^m$, and $B \in \mathbf{R}^{n\times m}$, then

$$\begin{bmatrix} x \\ y \end{bmatrix}^\intercal \begin{bmatrix} A & B \\ B^\intercal & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (x + A^{-1}By)^\intercal A(x + A^{-1}By) + y^\intercal (D - B^\intercal A^{-1}B)y. \tag{52}$$

This gives a test for global positivity with

$$\begin{bmatrix} A & B \\ B^\intercal & D \end{bmatrix} \in \mathbf{S}_+^{n+m} \quad \Leftrightarrow \quad A \in \mathbf{S}_+^n \quad \text{and} \quad D - B^\intercal A^{-1} B \in \mathbf{S}_+^m.$$

This can be applied recursively to certify non–negativity.

14.6.3 Partial Min

If $A \in \mathbf{S}_+^n$, then partial quadratic optimization can be computed with

$$\begin{aligned} \min_{x\in\mathbf{R}^n} \quad J &= \begin{bmatrix} x \\ y \end{bmatrix}^\intercal \begin{bmatrix} A & B \\ B^\intercal & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= y^\intercal (D - B^\intercal A^{-1}B)y, \end{aligned}$$

where

$$x^* = -A^{-1}By.$$

The problem

$$\begin{aligned} \min_{y\in\mathbf{R}^m} \quad & J(x^*, y) \\ \text{s.t.} \quad & y \in \mathbf{P}, \end{aligned}$$

becomes

$$\begin{aligned} \min_{y\in\mathbf{R}^m} \quad & J = y^\intercal (D - B^\intercal A^{-1}B)y \\ \text{s.t.} \quad & y \in \mathbf{P}. \end{aligned}$$

15 Solvers

There are well–known algorithms for solving **LP** and **QP** in polynomial–time. Many specialized solutions exists for **LP** and **QP** with specific properties, but generalized interior point methods [1, p. 569] typically have similar run–time. Matlab has an **LP** [10] and **QP** [11] solver built into its Optimization Toolbox. Several **QP** solvers are available for Python, e.g., the cvxopt library [35]. There are also **QP** solvers available for C++, e.g., [13], and Java, e.g., [14]. For larger problems or problems that need to be solved quickly, simplifying assumptions and relaxations can be employed as shown in [5]. Further, ADMM [6] can be used for massively parallel optimization. If the problem has sparsity, [25] and [29] can be used. Effecient data structures should be employed wherever there are repeated copies of a sub–matrix or structures with diagonals or zeros. It is also adventageous to decouple problems wherever it is possible to do so. For nonlinear problems, solvers like Matlab's fmincon [12] can be used, which iteratively solve with **QP** . A nonlinear solver library for C++ can be found at [31]. A robot-centeric solver library can be found at [30]

15.1 Interior Point

For $i \in \mathbf{Z}[0, m]$, let $f_i(x)$ be convex functions. Consider

$$\begin{aligned} \min_{x\in\mathbf{R}^n} \quad & J = f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0, \\ & Ax = b, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \min_{x\in\mathbf{R}^n} \quad & J = f_0(x) + \sum_{i=1}^{m} \text{Indicator}(f_i(x) \le 0) \\ \text{s.t.} \quad & Ax = b. \end{aligned} \tag{53}$$

For $t > 0$, this objective function can be approximated by the log–barrier with

$$\begin{aligned} \min_{x\in\mathbf{R}^n} \quad & J = f_0(x) + \sum_{i=1}^{m} -\frac{1}{t}\log(-f_i(x)) \\ \text{s.t.} \quad & Ax = b. \end{aligned} \tag{54}$$

Let

$$h(x) = -\sum_{i=1}^{m} \log(-f_i(x)). \tag{55}$$

Then,

$$\begin{aligned} \nabla h &= -\sum_{i=1}^{m} \frac{1}{f_i}\nabla f_i \\ \text{Hessian}(h) &= -\sum_{i=1}^{m} \frac{1}{f_i}\text{Hessian}(f_i) + \sum_{i=1}^{m} \frac{1}{f_i^2}(\nabla f_i)(\nabla f_i)^\intercal \end{aligned}$$

The approximate problem can be restated as

$$\begin{aligned} \min_{x\in\mathbf{R}^n} \quad & J = tf_0(x) + h(x) \\ \text{s.t.} \quad & Ax = b. \end{aligned}$$

The optimal solution x^* is a function of t and can be computed with Algorithms 13.

$x \in \mathbf{P}$, $t > 0$ $s > 1$, small $c > 0$
while $m/t > c$ **do**
- $x \leftarrow \underset{x\in\mathbf{R}^n}{\text{argmin}} \quad tf_0(x) + h(x) \quad \text{s.t.} \quad Ax = b,$ where h comes from (55),
- $t \leftarrow st$.

end

Algorithm 13: Barrier Method

15.2 Feasible Initial Conditions

A feasible x in the domain

$$\mathbf{D} = \{x \in \mathbf{R}^n | f_i(x) \leq 0, Ax = b\}$$

can be found by introduce a slack variable and minimize it until the the constraints are satisfied, e.g.,

$$\begin{aligned} \min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad & J = \|y\|_1 \\ \text{s.t.} \quad & f_i(x) \leq y_i, \\ & Ax = b, \end{aligned}$$

or

$$\begin{aligned} \min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad & J = \|y\|_2^2 \\ \text{s.t.} \quad & f_i(x) \leq y_i, \\ & Ax = b, \end{aligned}$$

or

$$\begin{aligned} \min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}\}} \quad & J = |y| \\ \text{s.t.} \quad & f_i(x) \leq y, \\ & Ax = b. \end{aligned}$$

Note: This provides a diagnostic of which constraints are not satisfiable.

15.3 Polyhedron Constraints

For $x \in \mathbf{P}$ with $\mathbf{P} = \{x \in \mathbf{R}^n | Ax \leq b\}$,

$$f_i(x) = a_i^\intercal x - b_i.$$

The log-barrier is then computed with

$$h(x) = -\sum_{i=0}^{m-1} \log(b_i - a_i^\intercal x).$$

The gradient and Hessian are computed with

$$\begin{aligned} \nabla h(x) &= \sum_{i=0}^{m-1} \frac{a_i}{b_i - a_i^\intercal x} \\ &= A_{\mathsf{ub}}^\intercal \text{col}\{d_i\}_{i=0}^{m-1}, \end{aligned}$$

$$\begin{aligned} \text{Hessian}(h) &= \sum_{i=0}^{m-1} \frac{a_i a_i^\intercal}{(b_i - a_i^\intercal x)^2} \\ &= A_{\mathsf{ub}}^\intercal \text{diag}\{d_i^2\}_{i=0}^{m-1} A_{\mathsf{ub}}, \end{aligned}$$

where

$$d_i = \frac{1}{b_i - a_i^\intercal x}.$$

The equality constraint can be removed using SVD as described in section 14.3

15.3.1 Positive Quadrant Constraints

The constraint $x \geq 0$, i.e., $-x \leq 0$, simplifies calculations with a log-barrier of

$$h(x) = -\sum_{i=0}^{m-1} \log(x_i).$$

The gradient and Hessian are given by

$$\nabla h(x) = -\sum_{i=0}^{m-1} \frac{\mathbf{e}_i}{x_i} = -\mathrm{col}\{x_i^{-1}\}_{i=0}^{m-1},$$

$$\mathrm{Hessian}(h) = \sum_{i=0}^{m-1} \frac{\mathbf{e}_i \mathbf{e}_i^\intercal}{x_i^2} = \quad \mathrm{diag}\{x_i^{-2}\}_{i=0}^{m-1}.$$

15.4 ADMM: Alternating Direction Method of Multipliers

Consider the convex optimization problem

$$\min_{\{x,y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}} \quad J = f(x) + g(y)$$
$$\text{s.t.} \quad Ax + By = c.$$

The regularized Lagrangian is given by

$$L(x, y, z) = f(x) + g(y) + z^\intercal(Ax + By - c) + \frac{\rho}{2}\|Ax + By - c\|_2^2. \tag{56}$$

The ADMM is computed with Algorithm 14.

$\{x, y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}$
while $\{x, y\}$ not converged **do**
- $x \leftarrow \underset{x \in \mathbf{R}^n}{\operatorname{argmin}} \quad L(x, y, z),$
- $y \leftarrow \underset{y \in \mathbf{R}^m}{\operatorname{argmin}} \quad L(x, y, z),$
- $z \leftarrow z + \rho(Ax + By - c).$

end

Algorithm 14: ADMM

15.5 Scaled ADMM

The regularized Lagrangian can be expanded with

$$\begin{aligned} L(x, y, z) &= f(x) + g(y) + z^\intercal(Ax + By - c) + \frac{\rho}{2}\|Ax + By - c\|_2^2 \\ &= f(x) + g(y) + \frac{\rho}{2}\|Ax + By - c + u\|_2^2 + r, \end{aligned} \tag{57}$$

where

$$u = z/\rho.$$

The scaled ADMM is computed with Algorithm 15.

$\{x, y\} \in \{\mathbf{R}^n, \mathbf{R}^m\}$
while $\{x, y\}$ not converged **do**
- $x \leftarrow \underset{x \in \mathbf{R}^n}{\operatorname{argmin}} \quad f(x) + \frac{\rho}{2}\|Ax + by - c + u\|_2^2,$
- $y \leftarrow \underset{y \in \mathbf{R}^m}{\operatorname{argmin}} \quad g(y) + \frac{\rho}{2}\|Ax + by - c + u\|_2^2,$
- $u \leftarrow u + \rho(Ax + By - c).$

end

Algorithm 15: Scaled ADMM

15.6 Examples

If $A = \boldsymbol{I}_n$,

$$x^* = \underset{x \in \mathbf{R}^n}{\text{argmin}} \quad f(x) + \frac{\rho}{2}\|x - v\|_2^2.$$

- If

  $$\begin{aligned} f(x) &= \text{Indicator}(x \in \mathbf{D}) \\ &= \begin{cases} 0 & \text{if} \quad x \in \mathbf{D} \\ \infty & \text{else} \end{cases}, \end{aligned}$$

  then

  $$\begin{aligned} x^* = \underset{x \in \mathbf{R}^n}{\text{argmin}} & \quad \|x - v\|_2^2 \\ \text{s.t.} & \quad x \in \mathbf{D}. \end{aligned}$$

- If

  $$f(x) = \lambda\|x\|_1,$$

  then

  $$\begin{aligned} x_i^* &= \text{Deadzone}(v_i, \lambda/\rho) \\ &= \text{Hinge}(x - \lambda/\rho) - \text{Hinge}(-x - \lambda/\rho). \end{aligned}$$

  This result can be derived from independently optimizing

  $$\underset{x_i \in \mathbf{R}}{\text{argmin}} \quad \lambda|x_i| + \frac{\rho}{2}(x_i - v_i)^2,$$

  which gives

  $$\begin{aligned} 0 &= \lambda\text{sign}(x_i) + \rho(x_i - v_i), \\ v_i &= \frac{\lambda}{\rho}\text{sign}(x_i) + x_i, \\ x_i &= \text{Deadzone}(v_i, \lambda/\rho). \end{aligned}$$

- If

  $$f(x) = \frac{1}{2}x^\intercal Q x + c^\intercal x + r,$$

  then

  $$x^* = (Q + \rho A^\intercal A)^{-1}(\rho A^\intercal v - c).$$

- If

  $$f(x) = \frac{1}{2}x^\intercal Q x + c^\intercal x + \text{Indicator}(x \in \mathbf{P}),$$

  then

  $$\begin{aligned} \min_{x \in \mathbf{R}^n} & \quad J = \frac{1}{2}x^\intercal Q x + c^\intercal x + \frac{\rho}{2}\|x - v\|_2^2 \\ \text{s.t.} & \quad x \in \mathbf{P}. \end{aligned}$$

15.7 Consensus ADMM

The problem

$$\min_{x \in \mathbf{R}^n} \quad J = \sum_{i=0}^{m} f_i(x)$$

is equivalent to

$$\begin{aligned} \min_{x_i \in \mathbf{R}^n} \quad & J = \sum_{i=0}^{m} f_i(x_i) \\ \text{s.t.} \quad & x_i - z = 0. \end{aligned} \tag{58}$$

This problem can be solved with Algorithm 16.

$x_i \in \mathbf{R}^n$
while x_i not converged **do**
- **for** *parallel* $i \in \mathbf{Z}[0, m]$ **do**
  - $x_i \leftarrow \underset{x_i \in \mathbf{R}^n}{\text{argmin}} \quad f_i(x_i) + y_i^\intercal (x_i - z) + \frac{\rho}{2} \|x_i - z\|_2^2,$
  - $y_i \leftarrow y_i + \rho(x_i - z),$

  end
- $z \leftarrow \frac{1}{1+m} \sum_{i=0}^{m} (x_i + y_i/\rho).$

end

Algorithm 16: Consensus ADMM

15.7.1 Constraint Decomposition

Any convex problem can be expressed as

$$\begin{aligned} \min_{x \in \mathbf{R}^n} \quad & J = f_0(x) \\ \text{s.t.} \quad & x \in \mathbf{P}, \end{aligned}$$

where

$$\mathbf{P} = \bigcap_{i=1}^{m} \mathbf{P}_i,$$

which is equivalent to

$$\min_{x \in \mathbf{R}^n} \quad J = f_0(x) + \sum_{i=1}^{m} \text{Indicator}(x \in \mathbf{P}_i). \tag{59}$$

15.8 Logistic Regression

Logistic regression can be expressed as

$$\begin{aligned} \min_{\{x_i, x_j\} \in \{\mathbf{R}^n, \mathbf{R}^n\}} \quad & J = -\sum_i \log(p(x_i)) - \sum_j \log(1 - p(x_j)) \\ \text{s.t.} \quad & x_i - z = 0, \\ & x_j - z = 0. \end{aligned}$$

16 Appendix

16.1 SVD: Singular Value Decomposition

For $A \in \mathbf{R}^{m\times n}$,

$$
\begin{aligned}
A &= USV^\intercal \\
&= \begin{bmatrix} U_+ & U_0 \end{bmatrix} \begin{bmatrix} S_+ & \mathbf{0}^{r\times n-r} \\ \mathbf{0}^{m-r\times r} & \mathbf{0}^{m-r\times n-r} \end{bmatrix} \begin{bmatrix} V_+^\intercal \\ V_0^\intercal \end{bmatrix} \\
&= U_+ S_+ V_+^\intercal,
\end{aligned}
$$

where $r = \text{rank}(A)$, $S_+ = \text{diag}\{\sigma_i\}_{i=0}^{r-1}$ with $\sigma_i > 0$ for $i \in \mathbf{Z}[0, r-1]$, and

$$
\begin{aligned}
U \in \mathbf{R}^{m\times m}, \quad &\text{s.t.} \quad U^\intercal U = UU^\intercal = \boldsymbol{I}_m, \quad U^{-1} = U^\intercal, \\
V \in \mathbf{R}^{n\times n}, \quad &\text{s.t.} \quad V^\intercal V = VV^\intercal = \boldsymbol{I}_n, \quad V^{-1} = V^\intercal.
\end{aligned}
$$

Orthonormality gives

$$
\begin{aligned}
U_+^\intercal U_+ &= \boldsymbol{I}_r, \\
V_+^\intercal V_+ &= \boldsymbol{I}_r, \\
\\
U_+^\intercal U_0 &= \mathbf{0}^{r\times m-r}, \\
V_+^\intercal V_0 &= \mathbf{0}^{r\times n-r}, \\
\\
U_+ U_+^\intercal &= \boldsymbol{I}_m - U_0 U_0^\intercal, \\
V_+ V_+^\intercal &= \boldsymbol{I}_n - V_0 V_0^\intercal.
\end{aligned}
$$

Note: $U_0^\intercal A = \mathbf{0}^{m-r\times n}$ and $AV_0 = \mathbf{0}^{m\times n-r}$.

16.2 Generalized Pseudo Inverse

For $A \in \mathbf{R}^{m\times n}$, if $\text{rank}(A) < \min(m, n)$, the pseudo inverse is given by

$$
A^+ = \begin{cases} V_+ S_+^{-1} U_+^\intercal + BU_0^\intercal & \text{left} \\ V_+ S_+^{-1} U_+^\intercal + V_0 B & \text{right} \end{cases},
$$

where B can be freely choosen. Unless otherwise specified, assume $B = \mathbf{0}$.

16.3 Matrix Factorization

Adding $A \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{m\times n}$, for SVD $A = U_+ S_+ V_+^\intercal$,

$$
\begin{aligned}
A + B &= \begin{bmatrix} U_+ & U_0 \end{bmatrix} \begin{bmatrix} S_+ & \mathbf{0}^{r\times n-r} \\ \mathbf{0}^{m-r\times r} & \mathbf{0}^{m-r\times n-r} \end{bmatrix} \begin{bmatrix} V_+^\intercal \\ V_0^\intercal \end{bmatrix} + UU^\intercal BV^\intercal V \\
&= U \begin{bmatrix} U_+^\intercal BV_+ + S_+ & U_+^\intercal BV_0 \\ U_0^\intercal BV_+ & U_0^\intercal BV_0 \end{bmatrix} V^\intercal.
\end{aligned}
$$

If $A \in \mathbf{R}^{n\times n}$,

$$
\begin{aligned}
U^\intercal(A+B)U &= \begin{bmatrix} S_+ V_+^\intercal U_+ & S_+ V_+^\intercal U_0 \\ \mathbf{0}^{n-r\times r} & \mathbf{0}^{n-r\times n-r} \end{bmatrix} + \begin{bmatrix} U_+^\intercal BU_+ & U_+^\intercal BU_0 \\ U_0^\intercal BU_+ & U_0^\intercal BU_0 \end{bmatrix}, \\
V^\intercal(A+B)V &= \begin{bmatrix} V_+^\intercal U_+ S_+ & \mathbf{0}^{r\times n-r} \\ V_0^\intercal U_+ S_+ & \mathbf{0}^{n-r\times n-r} \end{bmatrix} + \begin{bmatrix} V_+^\intercal BV_+ & V_+^\intercal BV_0 \\ V_0^\intercal BV_+ & V_0^\intercal BV_0 \end{bmatrix}.
\end{aligned}
$$

16.4 Kronecker Product

For $A \in \mathbf{R}^{m \times n}$,

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{bmatrix}.$$

In general, $A \otimes B \neq B \otimes A$. Some useful properties include:

$$\boldsymbol{I}_q \otimes A = \text{diag}\{A\}_{i=0}^{q-1}, \qquad \mathbf{1}^q \otimes A = \text{row}\{A\}_{i=0}^{q-1},$$

$$\begin{aligned} A \otimes (B + C) &= A \otimes B + A \otimes C, \\ (cA) \otimes B &= A \otimes (cB) = c(A \otimes B), \\ A \otimes B &= C(B \otimes A)D, \end{aligned} \qquad \begin{aligned} (A + B) \otimes C &= A \otimes C + B \otimes C, \\ A \otimes (B \otimes C) &= (A \otimes B) \otimes C, \\ AC \otimes BD &= (A \otimes B)(C \otimes D), \end{aligned}$$

$$\begin{aligned}
(A \otimes B)^\intercal &= A^\intercal \otimes B^\intercal, & \left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right)^\intercal &= \textstyle\bigotimes_{i=0}^{q-1} A_i^\intercal, \\
(A \otimes B)^p &= A^p \otimes B^p, & \left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right)^p &= \textstyle\bigotimes_{i=0}^{q-1} A_i^p, \\
\|A \otimes B\|_2^2 &= \|A\|_2^2 \|B\|_2^2, & \left\|\textstyle\bigotimes_{i=0}^{q-1} A_i\right\|_2^2 &= \textstyle\prod_{i=0}^{q-1} \|A_i\|_2^2, \\
(A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}, & \left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right)^{-1} &= \textstyle\bigotimes_{i=0}^{q-1} A_i^{-1}, \\
(A \otimes B)^+ &= A^+ \otimes B^+, & \left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right)^+ &= \textstyle\bigotimes_{i=0}^{q-1} A_i^+, \\
\text{eig}(A \otimes B) &= \text{eig}(A) \otimes \text{eig}(B), & \text{eig}\left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right) &= \textstyle\bigotimes_{i=0}^{q-1} \text{eig}(A_i), \\
\text{rank}(A \otimes B) &= \text{rank}(A)\text{rank}(B), & \text{rank}\left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right) &= \textstyle\prod_{i=0}^{q-1} \text{rank}(A_i), \\
\text{svd}(A \otimes B) &= \text{svd}(A) \otimes \text{svd}(B), & \text{svd}\left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right) &= \textstyle\bigotimes_{i=0}^{q-1} \text{svd}(A_i), \\
\text{tr}(A \otimes B) &= \text{sum}(\text{eig}(A) \otimes \text{eig}(B)), & \text{tr}\left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right) &= \text{sum}\left(\textstyle\bigotimes_{i=0}^{q-1} \text{eig}(A_i)\right), \\
\det(A \otimes B) &= \det(A)^{\text{rank}(B)} \det(B)^{\text{rank}(A)}, & \det\left(\textstyle\bigotimes_{i=0}^{q-1} A_i\right) &= \textstyle\prod_{i=0}^{q-1} \det(A_i)^{\prod_{j \neq i} \text{rank}(A_j)}, \\
\frac{\partial (A \otimes B)}{\partial s} &= \frac{\partial A}{\partial s} \otimes B + A \otimes \frac{\partial B}{\partial s}, & \frac{\partial}{\partial x_j} \bigotimes_{i=0}^{q-1} f_i(x_i) &= \bigotimes_{i<j} f_i \otimes \frac{\partial f_j}{\partial x_j} \otimes \bigotimes_{i>j} f_i
\end{aligned}$$

For independent $\widetilde{A}$ and $\widetilde{B}$,

$$\langle \widetilde{A} \otimes \widetilde{B} \rangle = \langle \widetilde{A} \rangle \otimes \langle \widetilde{B} \rangle.$$

For $x_i \in \mathbf{R}^{n_i}$, $y_i \in \mathbf{R}^{n_i}$, and $m \in \mathbf{Z}_+$,

$$\begin{aligned} \textstyle\prod_{i=0}^{m-1} x_i^\intercal y_i &= \left(\textstyle\bigotimes_{i=0}^{m-1} x_i^\intercal\right) \left(\textstyle\bigotimes_{i=0}^{m-1} y_i\right) = \left(\textstyle\bigotimes_{i=0}^{m-1} x_i\right)^\intercal \left(\textstyle\bigotimes_{i=0}^{m-1} y_i\right), \\ (x^\intercal y)^m &= \left(\textstyle\bigotimes_{i=0}^{m-1} x\right)^\intercal \left(\textstyle\bigotimes_{i=0}^{m-1} y\right). \end{aligned}$$

16.5 Vectorization

For $A \in \mathbf{R}^{m \times n}$, row-wise vectorization gives

$$\text{vec} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} & \cdots & A_{m1} & \cdots & A_{mn} \end{bmatrix}^\intercal,$$

For $a \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$, $[ab^\intercal]_{ij} = a_i b_j$.
For $a_i \in \mathbf{R}^m$, $b_i \in \mathbf{R}^n$, $A = \text{col}\{a_i\}_i$, $B = \text{col}\{b_i\}_i$,

$$[AB^\intercal]_{ij} = \left[\sum_k a_k b_k^\intercal\right]_{ij} = \sum_k [a_k b_k^\intercal]_{ij} = \sum_k [a_k]_i [a_k]_j = \sum_k A_{ik} B_{jk}.$$

The inverse operation is defined by

$$A = \text{mat}(\ \vec{A},\ m,\ n\ ).$$

For $AB \in \mathbf{R}^{m\times n}$,

$$\text{vec}\,(AB) = (\boldsymbol{I}_m \otimes B^\intercal)\vec{A} = (A \otimes \boldsymbol{I}_n)\vec{B},$$

$$\text{vec}\,(ABC) = (A \otimes C^\intercal)\vec{B}.$$

For $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$,

$$\begin{aligned}
\text{vec}\,(x \otimes y^\intercal) \quad &= \text{vec}\,(xy^\intercal) \quad &&= (\boldsymbol{I}_n \otimes y)x \quad &&= x \otimes y,\\
\text{vec}\,(x^\intercal \otimes y) \quad &= \text{vec}\,(yx^\intercal) \quad &&= (\boldsymbol{I}_m \otimes x)y \quad &&= y \otimes x.
\end{aligned}$$

For row-wise concatenation,

$$\text{vec}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \vec{A} \\ \vec{B} \end{bmatrix}.$$

16.6 Hadamard

The Hadamard Product computes element-wise multiplication with

$$A \circ B = C \quad \Leftrightarrow \quad \text{diag}(\vec{A})\vec{B} = \text{diag}(\vec{B})\vec{A} = \vec{C},$$

$$(A \otimes B) \circ (C \otimes D) = (A \circ C) \otimes (B \circ D).$$

16.7 Trace

For $A \in \mathbf{R}^{n\times n}$,

$$\begin{aligned}
\text{tr}(A) &= \textstyle\sum_{i=0}^{n-1} A_{ii}\\
&= \text{tr}(A^\intercal)\\
&= \vec{\boldsymbol{I}}^\intercal \vec{A},\\
\text{tr}(AB) &= \textstyle\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} A_{ij} B_{ji}\\
&= \text{tr}(BA)\\
&= \text{tr}\left(A \textstyle\sum_{i=0}^{n-1} \lambda_i v_i v_i^{\text{c}\intercal}\right) = \textstyle\sum_{i=0}^{n-1} \lambda_i v_i^{\text{c}\intercal} A v_i,\\
\text{tr}(A+B) &= \text{tr}(A) + \text{tr}(B),\\
\text{tr}(A^\intercal A) &= \text{svd}(A)^\intercal \text{svd}(A),\\
\text{tr}(A \otimes B) &= \text{tr}(A)\text{tr}(B),\\
\text{tr}(A) &= \log(\det(\exp(A))),\\
\text{tr}(ABA^{-1}) &= \text{tr}(B),\\
\text{tr}(A^p) &= \textstyle\sum_{i=0}^{n-1} \lambda_i^p,\\
\frac{\partial}{\partial B}\text{tr}(ABC) &= A^\intercal C^\intercal,\\
\frac{\partial}{\partial A}\text{tr}(ABA^\intercal) &= AB^\intercal + AB.
\end{aligned}$$

For $x \in \mathbf{R}^n$,

$$\text{tr}\,(xx^\intercal) = x^\intercal x.$$

16.8 2–Norm

For $A \in \mathbf{R}^{m\times n}$ and $x \in \mathbf{R}^n$,

$$\begin{aligned}
\|Ax\|_2^2 &= x^\intercal A^\intercal Ax, \\
&= \vec{A}^\intercal(\boldsymbol{I}_m \otimes x)(\boldsymbol{I}_m \otimes x^\intercal)\vec{A}, \\
&= \vec{A}^\intercal(\boldsymbol{I}_m \otimes xx^\intercal)\vec{A}, \\
&= (\vec{A}^\intercal \otimes \vec{A}^\intercal)\mathrm{vec}\left(\boldsymbol{I}_m \otimes xx^\intercal\right), \\
&= \mathrm{vec}\left(A^\intercal A\right)^\intercal \mathrm{vec}\left(xx^\intercal\right), \\
&= \mathrm{tr}(Axx^\intercal A^\intercal), \\
&= \mathrm{tr}(A^\intercal Axx^\intercal), \\
&\leq \max(\mathrm{svd}(A))^2\|x\|_2^2, \\
&\leq \|\vec{A}\|_2^2\|x\|_2^2.
\end{aligned}$$

16.9 Determinant

Some useful properties include:

$$\begin{aligned}
\det(A)\boldsymbol{I}_n &= \mathrm{adj}(A)A \\
\det(\boldsymbol{I}_n) &= 1, \\
\det(U) &= \pm 1, && \text{where} \quad U^\intercal U = UU^\intercal = \boldsymbol{I}_n, \\
\det(A^\intercal) &= \det(A), \\
\det(A^{-1}) &= \det(A)^{-1}, \\
\det(aA) &= a^n \det(A), \\
\det(AB) &= \det(A)\det(B), && \det\left(\prod_{i=0}^{q-1} A_i\right) = \prod_{i=0}^{q-1}\det(A_i), \\
\det(A^p) &= \det(A)^p, \\
\det(A) &= \exp(\mathrm{tr}(\log(A))), \\
\det(A) &= \det(BAB^{-1}) \\
\det(A) &= \prod_{i=0}^{n-1}\lambda_i, && \text{where} \quad \det(A-\lambda_i\boldsymbol{I}_n) = 0, \\
\frac{\partial}{\partial A}\det(A) &= \mathrm{adj}(A)^\intercal, && \frac{\partial}{\partial s}\det(A) = \det(A)\mathrm{tr}\left(A^{-1}\frac{\partial A}{\partial s}\right), \\
\frac{\partial}{\partial A}\log(\det(A)) &= A^{-\intercal} && \frac{\partial}{\partial s}\log(\det(A)) = \mathrm{tr}\left(A^{-1}\frac{\partial A}{\partial s}\right) \\
\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \det(A)\det(D - CA^{-1}B) \\
&= \det(A - BC) \quad \text{if} \quad D = \boldsymbol{I}.
\end{aligned}$$

For $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$,

$$\det(\boldsymbol{I}_n + xy^\intercal) = 1 + x^\intercal y.$$

For $A \in \mathbf{S}^n$,$M_i = A[0:i, 0:i]$, and $i \in \mathbf{Z}[1, n-1]$,

$$\begin{aligned}
A \succeq 0 \quad &\Leftrightarrow \quad \det(M_i) \geq 0, \quad A \preceq 0 \quad \Leftrightarrow \quad -A \succeq 0, \\
A \succ 0 \quad &\Leftrightarrow \quad \det(M_i) > 0, \quad A \prec 0 \quad \Leftrightarrow \quad -A \succ 0.
\end{aligned}$$

For $A \in \mathbf{S}^n$ and $B \in \mathbf{S}^n$,

$$\begin{aligned}
A \succeq B \quad &\Leftrightarrow \quad A - B \succeq 0, \\
A \succ B \quad &\Leftrightarrow \quad A - B \succ 0, \\
\mathrm{tr}(AB) \geq 0 \quad \text{for all} \quad A \succeq 0 \quad &\Leftrightarrow \quad B \succeq 0.
\end{aligned}$$

16.10 Pseudo Inverse

For $x \in \mathbf{R}^n$,

$$x^+ = \begin{cases} (x^\intercal x)^{-1} x^\intercal & \text{if } \|x\| > 0 \\ 0 & \text{else} \end{cases}.$$

For $A \in \mathbf{R}^{m\times n}$ and $p \in \mathbf{R}_+$,

$$\begin{aligned}
A &= U_+ S_+ V_+^\intercal \quad \text{with SVD} && \text{and} \quad A^+ = V_+ S_+^{-1} U_+^\intercal, \\
A^+ &= \lim_{\sigma\to 0} (A^\intercal A + \sigma \boldsymbol{I}_n)^{-1} A^\intercal && \text{and} \quad A^+ = \lim_{\sigma\to 0} A^\intercal (AA^\intercal + \sigma \boldsymbol{I}_m)^{-1}, \\
A &= (A^+)^+ \\
&= A(A^+A)^p && \text{and} \quad A = (AA^+)^p A, \\
A^+ &= (A^\intercal A)^+ A^\intercal && \text{and} \quad A^+ = A^\intercal (AA^\intercal)^+, \\
(A^\intercal)^+ &= (A^+)^\intercal, \\
(aA)^+ &= a^{-1} A^+, \\
(A^\intercal A)^+ &= A^+ A^{+\intercal}, && \text{and} \quad (AA^\intercal)^+ = A^{+\intercal} A^+, \\
(A^+A)^p &= A^+A, && \text{and} \quad (AA^+)^p = AA^+, \\
\mathrm{tr}(AA^+) &= \mathrm{rank}(A), \\
(\boldsymbol{I}_n - A^+A)^p &= \boldsymbol{I}_n - A^+A, && \text{and} \quad (\boldsymbol{I}_m - AA^+)^p = \boldsymbol{I}_m - AA^+, \\
(AB)^+ &= (A^+AB)^+ (ABB^+)^+, \\
\begin{bmatrix} A & B \end{bmatrix}^+ &= \begin{bmatrix} ((\boldsymbol{I} - BB^+)A)^+ \\ ((\boldsymbol{I} - AA^+)B)^+ \end{bmatrix}, \\
\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}^+ &= \begin{bmatrix} A^+ & \mathbf{0} \\ \mathbf{0} & B^+ \end{bmatrix}.
\end{aligned}$$

16.11 Inverse Matrix

For $A \in \mathbf{R}^{n\times n}$ with $\mathrm{rank}(A) = n$,

$$\begin{aligned}
A^+ &= A^{-1}, \\
(A^\intercal)^{-1} &= (A^{-1})^\intercal, \\
(AB)^{-1} &= B^{-1}A^{-1}, \\
\frac{\partial}{\partial s} A^{-1} &= -A^{-1} \frac{\partial A}{\partial s} A^{-1}, \\
\left(\prod_{i=0}^{q-1} A_i\right)^{-1} &= \prod_{j=q-1}^{0} A_j^{-1}.
\end{aligned}$$

The block inverse is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E & -EBD^{-1} \\ -D^{-1}CE & F \end{bmatrix},$$

where

$$\begin{aligned}
E &= (A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}BFCA^{-1}, \\
F &= (D - CA^{-1}B)^{-1} = D^{-1} + D^{-1}CEBD^{-1}.
\end{aligned}$$

For $A \in \mathbf{S}_+^n$, $b \in \mathbf{R}^n$, and $d \in \mathbf{R}$,

$$\begin{bmatrix} A & b \\ b^\intercal & d \end{bmatrix}^{-1} = \begin{bmatrix} E & -Ebd^{-1} \\ -d^{-1}b^\intercal E & f \end{bmatrix},$$

where

$$\begin{aligned}
f^{-1} &= d - b^\intercal A^{-1} b \\
E &= A^{-1} + A^{-1} b b^\intercal A^{-1} f.
\end{aligned}$$

16.12 Derivatives

16.12.1 Scalar to Scalar

For $f : \mathbf{R} \to \mathbf{R}$, the scalar first and second derivative are denoted by f' and f''.
Note: The same notation will be used for scalar to vector derivatives.

16.12.2 Vector to Scalar

For the scalar function $f : \mathbf{R}^n \to \mathbf{R}$,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x_0} & \cdots & \frac{\partial f}{\partial x_{n-1}} \end{bmatrix} \in \mathbf{R}^{1\times n}.$$

The gradient is

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_0} \\ \vdots \\ \frac{\partial f}{\partial x_{n-1}} \end{bmatrix} \in \mathbf{R}^n,$$

where

$$\frac{\partial f}{\partial x} = \nabla^\intercal f \quad = \quad \sum_{i=0}^{n-1} \frac{\partial f}{\partial x_i} \mathbf{e}_i^\intercal$$

The Hessian is

$$\text{Hessian}(f) := \begin{bmatrix} \frac{\partial}{\partial x_0}\frac{\partial f}{\partial x_0} & \cdots & \frac{\partial}{\partial x_{n-1}}\frac{\partial f_0}{\partial x_0} \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_0}\frac{\partial f}{\partial x_{n-1}} & \cdots & \frac{\partial}{\partial x_{n-1}}\frac{\partial f}{\partial x_{n-1}} \end{bmatrix} \in \mathbf{R}^{n\times n},$$

where

$$\text{Hessian}(f) := \frac{\partial}{\partial x}\nabla f \quad = \quad \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{\partial^2 f}{\partial x_i \partial x_j} \mathbf{e}_i \mathbf{e}_j^\intercal.$$

16.12.3 Vector to Vector

For $f : \mathbf{R}^n \to \mathbf{R}^m$, the partial derivative is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_0}{\partial x_0} & \cdots & \frac{\partial f_0}{\partial x_{n-1}} \\ \vdots & & \vdots \\ \frac{\partial f_{m-1}}{\partial x_0} & \cdots & \frac{\partial f_{m-1}}{\partial x_{n-1}} \end{bmatrix} \in \mathbf{R}^{m\times n},$$

where

$$\frac{\partial f}{\partial x} = \text{row}\left\{\nabla^\intercal f_i\right\}_{i=0}^{m-1} \quad = \quad \sum_{i=0}^{n-1} \frac{\partial f_\bullet}{\partial x_i} \mathbf{e}_i^\intercal.$$

Note:

$$\text{vec}\left(\frac{\partial f}{\partial x}\right) = \text{row}\left\{\nabla f_i\right\}_{i=0}^{m-1} \in \mathbf{R}^{mn}.$$

Note:

$$\frac{\partial}{\partial x}\text{vec}\left(\frac{\partial f}{\partial x}\right) = \text{row}\left\{\text{Hessian}(f_i)\right\}_{i=0}^{m-1} \in \mathbf{R}^{mn\times n}.$$

16.12.4 Vector to Scalar to Scalar

For $h = g(f)$, with $f : \mathbf{R}^n \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$,

$$\frac{\partial h}{\partial x} = g'(f)\frac{\partial f}{\partial x}.$$

The gradient is

$$\nabla h = g'(f)\nabla f.$$

The Hessian is

$$\text{Hessian}(h) = g'(f)\text{Hessian}(f) + g''(f)\nabla f \nabla^\intercal f.$$

16.12.5 Vector to Vector to Scalar

For $h = g(f)$, with $f : \mathbf{R}^n \to \mathbf{R}^m$ and $g : \mathbf{R}^m \to \mathbf{R}$,

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial f}\frac{\partial f}{\partial x},$$

where

$$\frac{\partial g}{\partial f} \in \mathbf{R}^{1\times m}, \quad \frac{\partial f}{\partial x} \in \mathbf{R}^{m\times n}.$$

The gradient is

$$\nabla h = \text{col}\{\nabla f_i\}_{i=0}^{m-1}\left(\frac{\partial g}{\partial f}\right)^\intercal.$$

16.12.6 Vector to Vector to Repeated Scalar

For $f : \mathbf{R}^n \to \mathbf{R}^m$ and $g : \mathbf{R} \to \mathbf{R}$, let

$$h = \text{row}\{g(f_i)\}_{i=0}^{m-1}.$$

The partial is

$$\frac{\partial h}{\partial x} = \text{diag}\{g'(f_i)\}_{i=0}^{m-1}\frac{\partial f}{\partial x}$$

with

$$\text{vec}\left(\frac{\partial h}{\partial x}\right) = \text{row}\{\nabla h_i\}_{i=0}^{m-1}.$$

The second partial is

$$\begin{aligned}\frac{\partial}{\partial x}\text{vec}\left(\frac{\partial h}{\partial x}\right) &= \text{row}\{\text{Hessian}(h_i)\}_{i=0}^{m-1}\\ &= \text{row}\{g'(f_i)\text{Hessian}(f_i) + g''(f_i)\nabla f_i \nabla f_i^\intercal\}_{i=0}^{m-1}.\end{aligned}$$

16.12.7 Scalar to Vector to Scalar

For $h = g(f)$, with $f : \mathbf{R} \to \mathbf{R}^m$ and $g : \mathbf{R}^m \to \mathbf{R}$, the gradient is equal to the partial

$$\nabla h = \frac{\partial g}{\partial f}\frac{\partial f}{\partial x} \in \mathbf{R},$$

where $x \in \mathbf{R}$ and

$$\frac{\partial g}{\partial f} \in \mathbf{R}^{1\times m}, \quad \frac{\partial f}{\partial x} \in \mathbf{R}^{m\times 1}.$$

The Hessian is equal to the second partial

$$\text{Hessian}(h) = \frac{\partial f}{\partial x}^\intercal \frac{\partial^2 g}{\partial f^2}\frac{\partial f}{\partial x} + \frac{\partial g}{\partial f}\frac{\partial^2 f}{\partial x^2} \in \mathbf{R}.$$

16.13 p–Norm

For $p \in \mathbf{R}[1, \infty]$, the p–norm is defined by

$$\|x\|_p = \left(\sum_{i=0}^{n-1} |x_i|^p\right)^{1/p}.$$

For $a \in \mathbf{R}$ and $\{x, y\} \in \{\mathbf{R}^n, \mathbf{R}^n\}$, the p–norm obeys the following three axioms

- $\|x + y\|_p \leq \|x\|_p + \|y\|_p$,
- $\|ax\|_p = |a|\|x\|_p$,
- $\|x\|_p = 0 \;\Leftrightarrow\; x = 0$.

Note: If $p \in \mathbf{R}[0, 1)$, $\|x\|_p$ does not satisfy the norm axioms.

Note: If $p \to 0$,

$$\|x\|_0 = \sum_{i=0}^{n-1} \text{Boolean}(|x_i| \neq 0).$$

Note: If $p \to \infty$,

$$\|x\|_\infty = \max_i |x_i|.$$

Note: The set $\|x\|_p \leq r$ is convex for all $p \in \mathbf{R}[1, \infty]$.

16.13.1 p–Norm Volume

If

$$S = \{x \in \mathbf{R}^n \mid \sum_{i=0}^{n-1} |x_i|^{p_i} \leq 1\},$$

the volume of S is given by [58, p. 164]

$$V = \frac{\prod_{i=0}^{n-1} \Gamma(1 + 1/p_i)}{\Gamma(1 + \sum_{i=0}^{n-1} 1/p_i)} 2^n.$$

If

$$S = \{x \in \mathbf{R}^n \mid \|x\|_p \leq r\},$$

the volme of S is proportional to r^n with

$$V = \frac{\Gamma(1/p + 1)^n}{\Gamma(n/p + 1)} (2r)^n.$$

For $p \in \{1, 2, \infty\}$,

$$V_1 = \frac{(2r)^n}{n!}, \quad V_2 = \frac{(\sqrt{\pi} r)^n}{\Gamma(n/2 + 1)}, \quad V_\infty = (2r)^n.$$

Note: $V_1 < V_2 < V_\infty$.
Note: $V_1/r^n \to V_2/r^n \to 0$ as $n \to \infty$.

Note: For all p, the volume is concentrated in the outer shell as n goes to infinity.

16.13.2 Hölder's Inequality

For $p \in \mathbf{R}[1, \infty]$ and $q \in \mathbf{R}[1, \infty]$,
if $1/p + 1/q = 1$, then

$$\sum_{i=0}^{n-1} |x_i y_i| \leq \|x\|_p \|y\|_q.$$

Note: If $p = 2$, then $q = 2$.

Note: If $p = 1$, then $q = \infty$.

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