Date. Page. Ω_1 is convex

P1 (a) Proof $\|x-a\| \le \|x-b\|$ That is $\|x-a\|^2 \le \|x-b\|^2$

Or $-2a^Tx + \|a\|^2 \le -2b^Tx + \|b\|^2$

$\Leftrightarrow 2(b^T-a^T)x \le \|b\|^2 - \|a\|^2$.

$\forall \lambda \in [0,1]$

for $x, y \in \Omega$ $\lambda x + (1-\lambda)y \to 2(b^T-a^T)(\lambda x + (1-\lambda)y)$

$= \lambda \cdot 2(b^T-a^T)x + (1-\lambda)2(b^T-a^T)y$

$\le \lambda(\|b\|^2 - \|a\|^2) + (1-\lambda)(\|b\|^2 - \|a\|^2)$

$= \|b\|^2 - \|a\|^2 \to$ So it is a convex set

Ω_2 is convex Proof: let $f(x,t) = x^Tx - t = \|x\|^2 - t$

Convexity via $\nabla f(x,t) = \begin{pmatrix} 2x \\ -1 \end{pmatrix}$ $\nabla^2 f(x,t) = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix} \succeq 0$.

By Hessian, $\nabla^2 g$ is positive for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, g is convex $\to \Omega_2$ is convex level set

(b) $\mathbb{R}^2_+ = \{x \in \mathbb{R}^2, x > 0\}$ So the set is $x_1x_2 \ge 0$ for $x \in \mathbb{R}^2$ $x > 0$

$x_1x_2 \ge 0$ that is. $f(x_1,x_2) := -\log x_1 - \log x_2 \le 0$

gradient $\nabla f(x_1,x_2) = \begin{pmatrix} -\frac{1}{x_1} \\ -\frac{1}{x_2} \end{pmatrix}$ $\nabla^2 f(x_1,x_2) = \begin{pmatrix} \frac{1}{x_1^2} & 0 \\ 0 & \frac{1}{x_2^2} \end{pmatrix}$

$\nabla^2 f > 0$ on $\mathbb{R}^2_+ \to f$ is convex $\to \Omega$ is convex

(c) let $x, y \in \Omega$ $\lambda \in [0,1]$ $(x, f(x)), (y, f(y)) \in S$.

By convex of S. $\lambda(x, f(x)) + (1-\lambda)(y, f(y)) = (\lambda x + (1-\lambda)y, \lambda f(x) + (1-\lambda)f(y)) \in S$.

$\to f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$

Since $x, y \in \Omega$ $\lambda \in [0,1] \to f$ convex on Ω.

P2 (a) False. let $g(x) = x^2$ $f(x) = -x^2$ $f(g(x)) = -x^2$

where $g(x)$, $f(x)$ conovex and $f(g(x))$ concave.

(b) True. $x, y \in \Omega$ $\lambda \in [0,1]$ convex of $g \to g(\lambda x + (1-\lambda)y) \le \lambda g(x) + (1-\lambda)g(y)$

I is convex $g(\Omega) \subseteq \Omega \to \lambda g(x) + (1-\lambda)g(y) \in I$.

f is nondecreasing, $f(g(\lambda x + (1-\lambda)y)) \le f(\lambda g(x) + (1-\lambda)g(y)) \le \lambda f(g(x)) + (1-\lambda)f(g(y))$

So $f \circ g$ is convex.

P3 ① $f(x) = \sqrt{1+x^{-2}}$ $f'(x) = \frac{1}{2}(1+x^{-2})^{-\frac{1}{2}}(-2x)$ $f''(x) = -(1+x^{-2})^{-\frac{1}{2}} - \frac{1}{4}(1+x^{-2})^{-\frac{3}{2}}(4x^2)$

all continuous and differentiable on $\mathbb{R}^{++}$

for $f''(x)$ that is $\frac{2x\sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}}}{x^4(1+x^2)} = \frac{3x^2+2}{x^3(1+x^2)\sqrt{1+x^2}} > 0$ for $x > 0$

f convex on $\mathbb{R}_{++}$

② $f(x) := \frac{1}{2}\|Ax-b\|^2 + \mu\|Lx\|_1$

$\frac{1}{2}\|\cdot\|^2$ is convex $\rightarrow$ $x \rightarrow \frac{1}{2}\|Ax-b\|^2$ is a linear and convex function

By triangle inequality, ℓ_1 norm $\|\lambda x+(1-\lambda)y\|_1 \le \lambda\|x\|_1+(1-\lambda)\|y\|_1$, $\lambda\in[0,1]$, $x,y\in\mathbb{R}^p$

$\rightarrow x \rightarrow \|x\|$ is convex. $\rightarrow x \rightarrow \|Lx\|_1$ is a convex function

$\rightarrow f$ convext

③ let $g(x,y)=\frac{\lambda}{2}\|x\|^2$ $g_i(x,y)=\ln(1+e^{-b_i(a_i^Tx+y)})$

$f=$ ~~g+gi~~ $g+\sum_i g_i$, g and g_i is convex

mapping g_i $z\rightarrow\gamma(z)=\ln(1+e^z)$ affine-linear func. $(x,y)\rightarrow h_i(x,y):=-b_i(a_i^Tx+y)$

h_i convex $\rightarrow g_i$ convex when γ convex

for $\gamma(z)=\left(\frac{e^z}{1+e^z}\right)'=\frac{e^z}{(1+e^z)^2}>0$ $\rightarrow g_i$ convex

Hessian of g $\nabla^2 g(x,y)=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}\succeq 0$ where $\nabla^2 g(x,y)\in\mathbb{R}^{(n+1)(n+1)}$

So f convex.

(b) Constraints are linear and convex. mapping $(x,w)\rightarrow \delta^2\mathbb{1}^Tw$ is lear and convex

So just prove convexity of the function define $\gamma:\mathbb{R}\times\mathbb{R}_+\rightarrow\mathbb{R}$, $\gamma(u,v)=\frac{u^2}{1+v}$

So $\sum_{i=1}^m\frac{(a_i^Tx-b_i)^2}{1+w_i}=\sum_{i=1}^m\gamma(a_i^Tx-b_i, e_i^Tw)$, $e_i=[0 \leftarrow i\text{th}]$

$\nabla\gamma(u,v)=\begin{pmatrix}\frac{2u}{1+v}\\ -\frac{u^2}{(1+v)^2}\end{pmatrix}$, $\nabla^2\gamma(u,v)=\begin{pmatrix}\frac{2}{1+v} & -\frac{2u}{(1+v)^2}\\ -\frac{2u}{(1+v)^2} & \frac{2u^2}{(1+v)^3}\end{pmatrix}$

so $tr(\nabla^2\gamma(u,v))=\frac{2}{1+v}\left(1+\frac{u^2}{(1+v)^2}\right)>0$.

$det(\nabla^2\gamma(u,v))=\frac{4u^2}{(1+v)^4}-\frac{4u^2}{(1+v)^4}=0$ $\forall(u,v)\in\mathbb{R}\times\mathbb{R}_+$

So $\nabla^2\gamma\succeq 0$ on $\mathbb{R}\times\mathbb{R}^+$ so γ is covex

So $(x,w)\rightarrow\gamma(a_i^Tx-b_i, e_i^Tw)$ is convx

$\Rightarrow$ The function convex.

P4 Lagrangian func. is $\mathcal{L}(y,\lambda)=\frac{1}{2}\|y-x\|^2+\lambda(a^Ty-b)$

Primal $a^Ty=b$ Stationary $\frac{\partial\mathcal{L}}{\partial y}=y-x+\lambda a=0$ $y=x-\lambda a$

$\frac{\partial\mathcal{L}}{\partial\lambda}=a^Ty-b$ or $a^T(x-\lambda a)=b\rightarrow\lambda=\frac{a^Tx-b}{a^Ta}$

$\rightarrow y=x-\frac{a^Tx-b}{a^Ta}a$.