HW4

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Q1(a) False

Primal
minimize x_1+x_2
s.t. $x_1+2x_2=1$
$x_1,x_2\geq 0$
sol $=(1,0)$ opt. value $=1$

Dual
maximize y
$y\leq 1$
$3y\leq 1$
y is free.
unbounded

(b) True. It ail holds, primal basic solution is feasible, by strong duality theorm, primal and dual are feasible and primal solution is optimal, y^* must be a optimal dual solution

let y^* be BFS for dual
$(y^*)^T=C_B^TA_B^{-1}$
$(y^*)^Tb=C_B^TA_B^{-1}b$
$=C^TX^*$
$\to X^*$ is BFS

(c) True. Proof: $Ax=b$. V^* is optimal objective value
$V^*=C^TX^*$, $C\geq A^Ty$, $C^T\geq y^TA$
$\to$ $V^*\geq y^TAX^* \Rightarrow V^*\geq X^TAy$

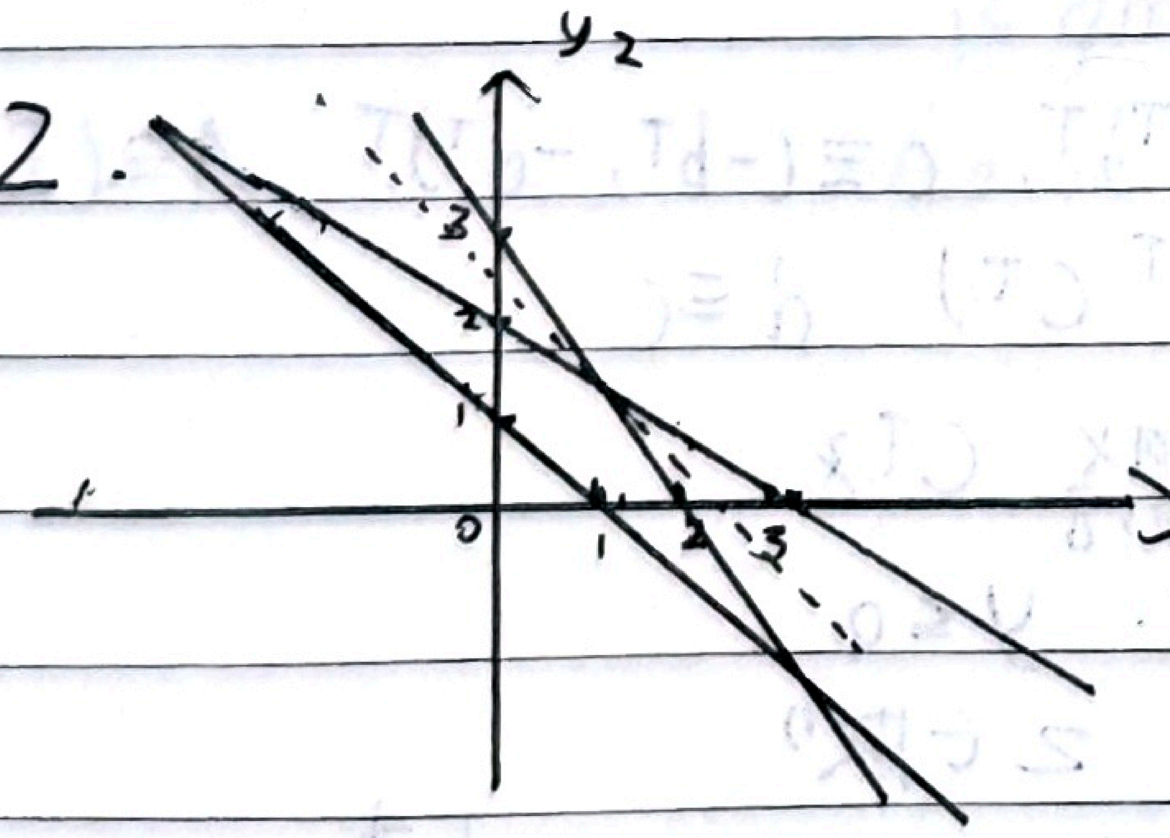
(d) False. Primal minimize x_1+x_2
s.t. $x_1+2x_2=1$
$x_1,x_2\geq 0$

Dual maximize y
s.t. $y\leq 1$
$2y\leq 1$

$\to$ feasible region is nonempty and bounded
$x_1=1$ is only feasible point. also
$\Rightarrow$ feasible region ($y\leq\frac{1}{2}$ is unbounded below ~~bound~~ ed.

Q2 1. $c=[1,6,6]$ $b=\begin{bmatrix}1\\1\end{bmatrix}$ $A=\begin{bmatrix}1&2&3\\1&3&2\end{bmatrix}$

so dual is minimize y_1+y_2
s.t. $y_1+y_2\geq 1$
$2y_1+3y_2\geq 6$
$3y_1+2y_2\geq 6$
$y_1,y_2\geq 0$.

2. 

$\begin{cases} y_2\geq 1-y_1 \\ y_2\geq 2-\frac{2}{3}y_1 \\ y_2\geq 3-\frac{3}{2}y_1 \end{cases}$

opt. sol is $(\frac{6}{5},\frac{6}{5})$
opt. value is $\frac{12}{5}$.

3. $(y_1+y_2-1)x_1=0$ (1)
$(2y_1+3y_2-6)x_2=0$
$(3y_1+2y_2-6)x_3=0$
$(x_1+2x_2+3x_3-1)y_1=0$ (2)
$(x_1+3x_2+2x_3-1)y_2=0$ (3)

$y_1+y_2\neq 1$ by (1) we have $x_1=0$
$y_1,y_2\neq 0$ by (2) (3) we have
$\begin{cases} 2x_2+3x_3=1 \\ 3x_2+2x_3=1 \end{cases} \to \begin{cases} x_2=\frac{1}{5} \\ x_3=\frac{1}{5} \end{cases}$

So $X=(0,\frac{1}{5},\frac{1}{5})$

Date. Page. Dual problem is

Q3. (a) $\max -c^T y$

$\text{s.t. } M^T y \le c$

$y \geq 0.$

Since $M = -M^T$. $M^T y \le c$ equivalent to $My \geq -c$.

So, the dual problem is $\min\ c^T y$

$\text{s.t. } My \geq -c$

$y \geq 0$

R to L So, dual problem equivalent to primal problem.

(b) ~~If~~ The problem ~~must~~ have optmal solution, then it must have feasible solution.

L to R: If problem has a feasible solution x_0, then $y_0 = x_0$ is also feasible to the problem, so dual and primal problems both feasible. By weak duality theorem, they both have finite optimal solutions, Q.E.D.

Q4 that is $\min_x c^T x + \max_{u \le 0, v \in \mathbb{R}^p} u^T(b - Ax) + v^T(d - Cx)$

which is $\max_{u \le 0, v \in \mathbb{R}^p} b^T u + d^T v + \min_x (c - A^T u - C^T v)^T x.$

that is $\max_{u,v} b^T u + d^T v$ → that is $-\min_{u,v} (-b^T, -d^T) \begin{pmatrix} u \\ v \end{pmatrix}$

$\text{s.t. } u \le 0,$ $\quad$ $\text{s.t. } (I\ 0) \begin{pmatrix} u \\ v \end{pmatrix} \le 0$

$v \in \mathbb{R}^p$ $\quad$ $(A^T\ C^T) \begin{pmatrix} u \\ v \end{pmatrix} = c.$

$A^T u + C^T v = c.$

now we let $x \equiv (u^T, v^T)^T$, $c \equiv (-b^T, -d^T)^T$ $A \equiv (I\ 0)$

$b = 0$, $C = (A^T\ C^T)$ $d \equiv c$

So we have dual problem $-\max_{y,z} c^T z$

$\text{s.t. } y \le 0$

$z \in \mathbb{R}^n$

$\begin{pmatrix} I \\ 0 \end{pmatrix} y + \begin{pmatrix} A \\ C \end{pmatrix} z = \begin{pmatrix} -b \\ -d \end{pmatrix}$

let $x = -z$, that is $\min_{x,y} c^T x$

$\text{s.t. } y \le 0$

$Ax - y = b$

$cx = d$

Since $y \le 0$ is a became slack variable. we Remove it
and constraint $Ax \le b$, then it is the same with (1)
Q.E.D

Q5 Consider

primal

$\min 1$

s.t. $Ax \geq \mathbb{1}$

wher $\mathbb{1}$ means $\mathbb{R}^n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \leftarrow n$ 1's.

dual is $\max \mathbb{1}^T y$

s.t. $A^T y = 0$

$y \geq 0$

We have $Ax > 0$ has a solution

$\Longleftrightarrow$ $Ax \geq 1$ has a solution

So P is feasible, which means P has an optimal solution
and optimal value is 0

That is D is feasible, which means D has an optimal
solution and optimal value is 0

~~So $A^T y = 0$, $y \gneq 0$ has no solution~~

~~If it has~~ If $A^T y = 0$ $y \gneq 0$ has a solution y_0,
the objective value of dual problem at y_0
is greater than 0, contradict!

So $A^T y = 0$ $y \gneq 0$ has no solution.