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1. $\nabla f = \begin{pmatrix} 4x_1^3 + 6x_1^2 - 4x_1x_2 \\ -2x_1^2 + 8x_2 \end{pmatrix}$ $\nabla^2 f = \begin{pmatrix} 12x_1^2 + 12x_1 - 4x_2 & -4x_1 \\ -4x_1 & 8 \end{pmatrix}$

For stationary points. $\nabla f = 0 \Leftrightarrow 4x_2^2 = x_1^2$ and $3x_1^2(x_1+2)=0$

$\Downarrow$

So stationary points $x_1^*(0,0)$ $x_2^*(-2,1)$

$\nabla^2 f(x_1^*) = \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}$ $\nabla^2 f(x_2^*) = \begin{pmatrix} 20 & 8 \\ 8 & 8 \end{pmatrix}$

$tr(\nabla^2 f(x_2^*)) = 28 > 0$ $\det(\nabla^2 f(x_2^*)) = 160 - 64 > 0$.

$\rightarrow$ Hessian positive define, x_2^* local minimizer.

$f(\pm|t|, 0) = t^4 (2 \pm |t|) t^2$

$\nabla f^2(x_1^*)$ satisfy SONC $= |t|^3(|t| + 2)$

$\rightarrow x_1^*$ a degenarate saddle Point

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2\. (a) $\nabla f(x) = \begin{pmatrix} \frac{1}{4}x_1^2 - x_2^2 - 2 \\ -2x_1x_2 + 4x_2^3 \end{pmatrix}$ $\nabla^2 f(x) = \begin{pmatrix} \frac{1}{2}x_1 & -2x_2 \\ -2x_2 & -2x_1 + 12x_2^2 \end{pmatrix}$

$\nabla f(x) = 0 \rightarrow 2x_2(2x_2^2 - x_1) = 0$

① $x_2 = 0 \rightarrow x_1 = \pm 2\sqrt{2}$

② $x_1 = 2x_2^2$, $\rightarrow x_2^4 - x_2^2 - 2 = 0 \rightarrow x_2^2 = 2$ or -1 $\rightarrow x_2 = \pm\sqrt{2}$, $x_1 = 4$

f has 4 stationary points $\bar{x}_1 = (2\sqrt{2}, 0)$ $\bar{x}_2 = (-2\sqrt{2}, 0)$ $\bar{x}_3 = (4, \sqrt{2})$ $\bar{x}_4 = (4, -\sqrt{2})$

(b) $\nabla^2 f(\bar{x}_1) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & -4\sqrt{2} \end{pmatrix}$ $\nabla^2 f(\bar{x}_2) = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{pmatrix}$

eigenvalue $\sqrt{2}$ $-4\sqrt{2}$ $-\sqrt{2}$ $4\sqrt{2}$ $\longrightarrow \nabla^2 f(\bar{x}_2)$ indefinite, $\bar{x}_2$ saddle point

$\nabla^2 f(\bar{x}_1)$ indefinite $\rightarrow \bar{x}_1$ saddle point.

$\nabla^2 f(\bar{x}_3) = \begin{pmatrix} 2 & -2\sqrt{2} \\ -2\sqrt{2} & 10 \end{pmatrix}$ $\nabla^2 f(\bar{x}_4) = \begin{pmatrix} 2 & 2\sqrt{2} \\ 2\sqrt{2} & 10 \end{pmatrix}$

$tr(\nabla^2 f(\bar{x}_3)) = tr(\nabla^2 f(\bar{x}_4)) = 18 > 0$

$\det(\nabla^2 f(\bar{x}_3)) = \det(\nabla^2 f(\bar{x}_4)) = 32 - 8 = 24 > 0$.

So $\nabla^2 f(\bar{x}_3)$ $\nabla^2 f(\bar{x}_4)$ are positive definite $\rightarrow \bar{x}_3, \bar{x}_4$ are strict local minimizer (second order sufficient condition)

3\. (a) $\nabla g_1(x)=\begin{pmatrix}1\\0\end{pmatrix}$ $\nabla g_2(x)=\begin{pmatrix}0\\1\end{pmatrix}$ $\nabla g_3(x)=\begin{pmatrix}-2x_1\\-2x_2\end{pmatrix}$

$g_1(\bar{x})=g_2(\bar{x})=-1<0$ $g_3(\bar{x})=0$

So $A(\bar{x})=\{3\}$, $\nabla g_3(\bar{x})=(0,-2)^T$ is not 0 $\to$ LICQ satisfy at $\bar{x}$.

(b) $\bar{x}$ is feasible $g_1(\bar{x}), g_2(\bar{x})<0 \to \bar{\lambda}_1=\bar{\lambda}_2=0$

For λ_3, $\nabla f(\bar{x})+\nabla g_3(\bar{x})\bar{\lambda}_3=\begin{pmatrix}3\bar{x}_1^2+2-2\bar{x}_3\\-4\bar{x}_1\bar{x}_2+12\bar{x}_2\end{pmatrix}+\lambda_3\begin{pmatrix}0\\-2\end{pmatrix}=\begin{pmatrix}0\\12-2\bar{\lambda}_3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

So $\lambda_3=6$ while $\bar{\lambda}_1=\bar{\lambda}_2=0$ $\to$ valid for Lagrange multipliers.

Complementarity condition is satisfied obviously

$\to (\bar{x},\bar{\lambda})$ is KKT triple, $\bar{x}$ is KKT point

4\. (a) Largrangian function $L(x_1,x_2,x_3,\lambda)=x_1+2x_2+4x_3+\lambda\left(\frac{4}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}-1\right)$

Main condition $1-\frac{4\lambda}{x_1^2}\geq 0$ $2-\frac{2\lambda}{x_2^2}\geq 0$ $4-\frac{\lambda}{x_3^2}\geq 0$

Dual feasibility $\lambda\geq 0$

Complementarity $x_1\left(1-\frac{4\lambda}{x_1^2}\right)=0$ $x_2\left(2-\frac{2\lambda}{x_2^2}\right)=0$ $x_3\left(4-\frac{\lambda}{x_3^2}\right)=0$ $\lambda\left(\frac{4}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}-1\right)=0$

Primal feasibility $\frac{4}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}\leq 1$, $x_1,x_2,x_3\geq 0$.

(b) From Primal feasibility condition, we have $x_1,x_2,x_3\neq 0$.

So for Complementarity condition

$x_1^2=4\lambda$; $x_2^2=\lambda$ $x_3^2=\frac{\lambda}{4}$

Plug in $x_1=2\sqrt{\lambda}$ $x_2=\sqrt{\lambda}$, $x_3=\frac{\sqrt{\lambda}}{2}$ into $\frac{4}{x_1}+\frac{2}{x_2}+\frac{1}{x_3}=1$

we have $\lambda=36$

So KKT condition has 1 solution

$x_1=12$ $x_2=6$ $x_3=3$

5 (a) dual variable for $\sum_{i=1}^{n} x_i \le p$: $\lambda_0 \ge 0$. for $x_i \ge 0, \forall i$, $\lambda_i \ge 0$. foreach

KKT condition

Main condition $-\frac{1}{\sigma_i + x_i} + \lambda_0 - \lambda_i = 0$ $i = 1, 2, \cdots, n$

Dual feasibility $\lambda_0 \ge 0$. $\lambda_i \ge 0$

Complementarity $\lambda_0 (\sum_{i=1}^{n} x_i - p) = 0$ $\lambda_i x_i = 0$

Primal feasibility $\sum_{i=1}^{n} x_i \le p$ $x_i \ge 0$.

(b) $x_1, x_3 > 0$, by complementarity condition, we have $\lambda_1 = \lambda_3 = 0$

mean condition $\lambda_0 = \frac{1}{\sigma_1 + x_1} = \frac{2}{5}$ $\rightarrow \lambda_2 = -\frac{1}{\sigma_2} + \lambda_0 = \frac{1}{15}$

$\rightarrow$ $\lambda_0 = \frac{2}{5}$ $\lambda_1 = 0$ $\lambda_2 = \frac{1}{15}$ $\lambda_3 = 0$, which all satisfy KKT condition.

$\rightarrow (\frac{1}{2}, 0, \frac{3}{2})$ is a KKT point