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Q1 (a) $M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$

$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$

$= e^{\lambda(e^t - 1)}$

$M_1'(t) = \lambda e^t e^{\lambda(e^t-1)}$ So $EX_1 = M_1'(0) = \lambda$

$\lambda = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X} \rightarrow \hat{\lambda}_{MOM} = \bar{X}$

$< E[\hat{\lambda}_{MOM}] = E(\bar{X}) = \lambda_0$.

(b) $Bias(\hat{\lambda}_{MOM}) = 0$

$Var(\hat{\lambda}_{MOM}) = Var(\frac{1}{n}\sum_{i=1}^n X_i) = \frac{\lambda_0}{n}$

$MSE(\hat{\lambda}_{MOM}) = Bias(\hat{\lambda})^2 + Var(\hat{\lambda}) = \frac{\lambda_0}{n}$

(c) $P(X_1 > 5) = 1 - P(X_1 \le 5)$

$= 1 - \sum_{k=0}^{5} \frac{\lambda_0^k}{k!} e^{-\lambda_0}$.

(d) $L(\lambda; x_1 \cdots x_n) = \prod_{i=1}^n \lambda^{x_i} \frac{\exp(-\lambda)}{x_i!}$

$\log L(\lambda : x_1 \cdots x_n) = \sum_{i=1}^n (x_i \log\lambda - \lambda - \log(x_i!))$

$\frac{d}{d\lambda} \log L(\lambda) = \sum_{i=1}^n (\frac{x_i}{\lambda} - 1) = 0$, $\frac{d^2}{d\lambda^2} \log L(\lambda) = \sum_{i=1}^n (-\frac{x_i}{\lambda^2}) < 0$.

So $\hat{\lambda}_{MLE} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{X}$

(e) let $\theta = \log\lambda_0 \rightarrow \lambda_0 = e^\theta$

$\log L(\theta) = \sum_{i=1}^n (x_i\theta - e^\theta - \log(x_i!))$

$\frac{d}{d\theta} \log L(\theta) = \sum_{i=1}^n (x_i - e^\theta)$, $\frac{d^2}{d\theta^2} \log L(\theta) = \sum_{i=1}^n (-e^\theta) < 0$.

$\hat{\theta}_{MLE} = \log(\frac{1}{n}\sum_{i=1}^n x_i) = \log(\bar{x})$

(f) $\hat{\lambda}_{0MLE} = \bar{X}$ apply the law of the large number, $\forall \lambda_0$, $E(X_i) = \lambda_0$

$\bar{X}_n \xrightarrow{P} \lambda_0$ as $n \to \infty$ Since $\hat{\lambda}_{MLE} = \bar{X}_n$,

$\bar{X}_n$ converges in probability to λ

So, $\hat{\lambda}_{0MLE} = \bar{X}_n \xrightarrow{P} \lambda_0$ as $n \to \infty$

So $\hat{\lambda}_{MLE}$ for parameter λ_0 of Possion distribution is consistent

Q2.

(a) ① $E(X) = \frac{1}{\lambda}$ $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $E(X) = \bar{X}$ → $\frac{1}{\lambda} = \bar{X}$, $\lambda_{MoM} = \frac{1}{\bar{X}}$

② $E(\frac{1}{\hat{\lambda}}) = E(\bar{X})$ $E(X_i) = \frac{a_0}{b_0}$

$E(\bar{X}) = E(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{a_0}{b_0}$

$\rightarrow E(\frac{1}{\lambda}) = \frac{a_0}{b_0}$

(b) $L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda X_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} X_i}$

$\log$ ① $L(\lambda) = n\log(\lambda) - \lambda \sum_{i=1}^{n} X_i$

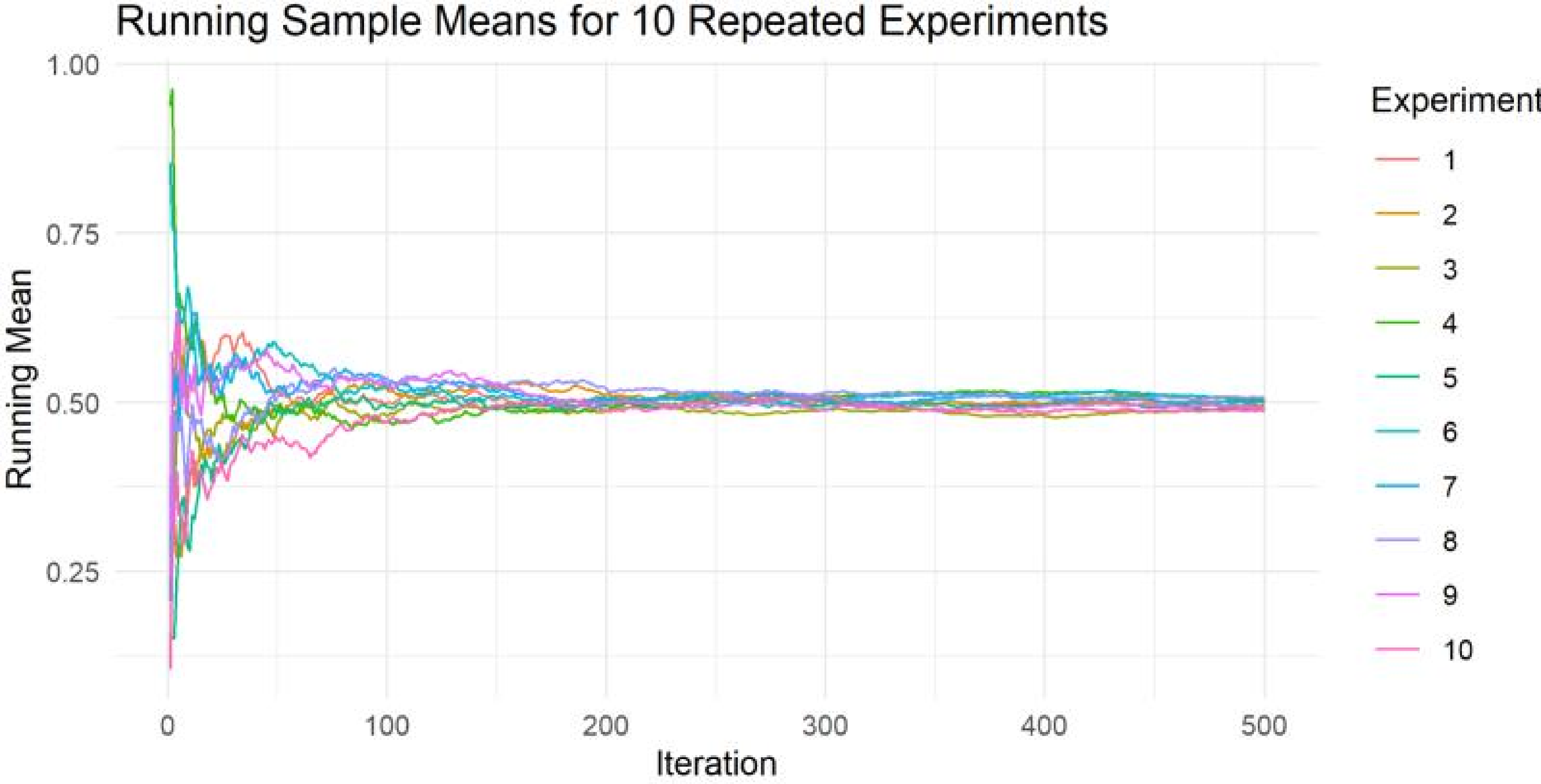
$\frac{d}{d\lambda}\log L(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} X_i = 0$

$\rightarrow \hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}}$ where $\frac{d^2}{d\lambda^2}\log L(\lambda) = -\frac{n}{\lambda^2} < 0$

So $\hat{\lambda}_{MLE} = \frac{1}{\bar{X}}$.

(c) $E(\hat{\lambda}) = \frac{b_0^{a_0}}{\Gamma(a_0)} X^{a_0-1} \exp(-b_0 X)$

$\hat{\lambda} = \frac{1}{\bar{X}}$ will converge to $\frac{1}{\frac{a_0}{b_0}} = \frac{b_0}{a_0}$ which is not equal to λ_0 when $a_0 \neq b_0$, which means inconsistant. But if $a_0 = b_0$, $\hat{\lambda}$ is consistant


Running Sample Means for 10 Repeated Experiments
1.00
0.75
0.50
0.25
Running Mean
0
100
200
300
400
500
Iteration
Experiment
1
2
3
4
5
6
7
8
9
10