Date HW10

Q1 ① $L(\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X_i-\mu)^2}{2\sigma^2}\right)$

$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{2\sigma^2}\right)$

$\ell(\mu) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{2\sigma^2}$,

② $\frac{\partial L(\mu)}{\partial \mu} = \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} X_i - n\mu\right) = 0$

So $\mu = \frac{\sum_{i=1}^{n} X_i}{n} = \bar{X}$

Since $\mu \geq \mu_0$

$\arg\max_{\mu \geq \mu_0} L(\mu) = \max(\mu_0, \bar{X})$

③. $\Lambda = \frac{\sup_{\mu\geq\mu_0} L(\mu)}{\sup_{\mu\in\mathbb{R}} L(\mu)}$ $\quad L(\bar{X}) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)$

$L(\max(\bar{X},\mu_0)) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \max(\bar{X},\mu_0))^2\right)$

~~So $\Lambda = \exp(-\frac{1}{2\sigma^2}$~~ $\quad = \cancel{(\frac{1}{2}}$ where $\sum_{i=1}^{n}(X_i - \max(\bar{X},\mu_0))^2 = \sum_{i=1}^{n} X_i^2 - 2n\bar{X}\max(\bar{X},\mu_0)$

$+ n\max(\bar{X},\mu_0)^2$

So $\Lambda = \exp\left(-\frac{1}{2\sigma^2}\left[n(\bar{X}-\max(\bar{X},\mu_0))^2\right]\right) \leq c$

$= \sum_{i=1}^{n}(X_i-\bar{X})^2 + n(\bar{X}-\max(\bar{X},\mu_0))^2$

So $-\frac{1}{2\sigma^2} n(\bar{X}-\max(\bar{X},\mu_0))^2 \leq \ln c$.

$(\bar{X}-\max(\bar{X},\mu_0))^2 \geq -\frac{2\sigma^2\ln(c)}{n}$.

$\bar{X} \geq \mu_0$, ~~$\max(\bar{X},\mu_0) = \bar{X}$~~ $\to -\frac{n(\bar{X}-\mu_0)^2}{2\sigma^2} \leq \ln c \Leftrightarrow \bar{X} \leq \mu_0 - \sqrt{\frac{2\sigma^2\ln c}{n}}$

$\to \bar{X} < c$.

④ $P(\frac{1}{n}\sum_{i=1}^{n} X_i < c \mid H_0) = \alpha$ $\quad \bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$

$Z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \sim (0,1)$

$P(\bar{X}<c \mid H_0) = P\left(Z < \frac{c-\mu_0}{\sigma/\sqrt{n}}\right) = \alpha$

$\to c = Z_\alpha \cdot \frac{\sigma}{\sqrt{n}} + \mu_0$.

Q2 ①. $f(Y_i:\beta_0,\beta_1,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(Y_i - X_i\beta_1 - \beta_0)^2}{2\sigma^2}\right)$

$L(\beta_0,\beta_1,\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - X_i\beta - \beta_0)^2$.

So $L(\beta_0,\beta_1) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - X_i\beta_1 - \beta_0)^2$.

② $\frac{L(\beta_0,\beta_1)}{\partial\beta_0} = -\frac{1}{2\sigma^2}\cdot(-2)\sum_{i=1}^{n}(Y_i - X_i\beta_1 - \beta_0) = 0$. $(*)$

$\frac{L(\beta_0,\beta_1)}{\partial\beta_1} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n} X_i(Y_i - X_i\hat{\beta}_1 - \beta_0) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n} X_i(Y_i - X_i\hat{\beta}_1 - (\bar{Y}-\hat{\beta}_1\bar{X})) = 0$.

So $\sum_{i=1}^{n} X_i(Y_i-\bar{Y}) = \hat{\beta}_1\sum_{i=1}^{n} X_i(X_i-\bar{X}) \to \hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}$

Subsitute $\hat{\beta}_1$ into $(*)$ $\quad \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X}$.

③ $\to E(\hat{\beta}_1) = E\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})(X_i\beta_1+\beta_0+\varepsilon_i) - (\bar{X}\beta_1+\beta_0+\bar{\varepsilon})}{\sum_{i=1}^{n}(X_i-\bar{X})^2}\right)$ $\quad < Y_i \sim N(X_i\beta_1+\beta_0, \sigma^2)$

any linear conbination of Y_i is also normal

$\underline{\sum_{i=1}^{n}(X_i-\bar{X})\varepsilon_i = 0}$ $\quad \beta_1$

$E(\hat{\beta}_0) = E(\bar{Y} - \hat{\beta}_1\bar{X}) = E(\bar{Y} - \beta_1\bar{X}) = \beta_0$

$Var(\hat{\beta}_1) = \frac{\sum_{i=1}^{n}(X_i-\bar{X})\varepsilon_i}{\sum_{i=1}^{n}(X_i-\bar{X})^2} = \frac{\sigma^2}{\sum_{i=1}^{n}(X_i-\bar{X})}$ $\quad Var(\hat{\beta}_0) = Var(\bar{Y} - \hat{\beta}_1\bar{X}) = \frac{\sigma^2}{n} + \bar{X}^2\frac{\sigma^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}$

$\to \hat{\beta}_1 \sim N(\beta_1, \frac{\sigma^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2})$ $\hat{\beta}_0 \sim N(\beta_0, \frac{\sigma^2}{n} + \bar{X}^2 \frac{\sigma^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2})$

$t = \frac{\beta_1}{\sigma\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2}}}$

① ~~For $H_0: \beta_0 = 0$ $H_1: \beta_0 \neq 0$.~~

~~$T_0 = \frac{\hat{\beta}_0}{\sqrt{var(\hat{\beta}_0)}} \sim N(0,1)$~~

~~pvalue $p = P(Z > T_0)$~~

② ~~For H_0, $\beta_1 = 0$, H_1 $\beta_1 \neq 0$~~

~~$T_1 = \frac{\hat{\beta}_1}{\sqrt{var(\hat{\beta}_1)}} \sim N(0,1)$~~

~~P-value $p = P(Z < T_1)$~~

$P_1 = P(T > t) = 1 - P(T \leq t)$ $T \sim t_{n-2}$

$P_2 = P(T < t)$ $T \sim t_{n-2}$

Q3.

(i) $f(Y|\beta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{1}{2\sigma^2}(Y-X\beta)^T(Y-X\beta))$

$l(\beta, \sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(Y-X\beta)^T(Y-X\beta)$

We minimize $(Y-X\beta)^T(Y-X\beta)$

$(Y-X\beta)^T(Y-X\beta) = Y^TY - 2\beta^TX^TY + \beta^TX^TX\beta$

$\nabla\beta = -2X^TY + 2X^TX\beta$

$\to X^TX\beta = X^TY$

$\to \hat{\beta} = (X^TX)^{-1}X^TY$

(ii) $H_0: \beta_1 = 0$ H_1 $\beta_1 > 0$. Denote first row of $(X^TX)^{-1}$ by $(v_1, \cdots, v_p)$

$\hat{\beta}_1 = (1, 0, \cdots, 0)\hat{\beta}$ $Var(\hat{\beta}_1) = \sigma^2(1, 0, \cdots, 0)(X^TX)^{-1}(1, 0, \cdots, 0)^T = \sigma^2 v_1$

$T = \frac{\hat{\beta}_1}{\sqrt{\hat{\sigma}^2 v_1}}$

Under H_0, T follow t-distribution with $n-p$ degree of freedom

P-value $= P(T > t_{obs})$

Q4

(i) $f(Y_i: \beta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(Y_i - (\beta_0 + \beta_1 X_i + \beta_2 X_i^2))^2}{2\sigma^2})$

$l(\beta, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - (\beta_0 + \beta_1 X_i + \beta_2 X_i^2))^2$.

$X = \begin{bmatrix} 1 & X_1 & X_1^2 \\ 1 & X_2 & X_2^2 \\ \vdots & \vdots & \vdots \\ 1 & X_n & X_n^2 \end{bmatrix}$ $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$ $Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$

and $\hat{\beta} = (X^TX)^{-1}X^TY$.

~~(ii) $\hat{\beta} \sim N(\beta, \sigma^2(X^TX)^{-1})$~~

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(ii) $Var(\hat{\beta}) = \sigma^2 (X^T X)^{-1}_{[1,1]}$ $\hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1})$

$\hat{\beta}_1 \sim N(\beta_1, V\beta_1)$

So $1-\alpha$ confidence interval is $\hat{\beta}_1 \pm Z_{1-\frac{\alpha}{2}} \sqrt{V\beta_1}$

wher $Z_{1-\frac{\alpha}{2}}$ is critical value from standard normal distribution

(iii) $Z = \frac{\hat{\beta}_1 - 0}{\sqrt{V\beta_1}}$ $Z \sim N(0,1)$

P-value is $P(Z > \text{observed } z)$

(iv) $\hat{\beta} = \begin{bmatrix} -1.262 \\ 9.535 \\ -0.980 \end{bmatrix}$

95% CI for β_1 $(8.\cancel{0}954, \cancel{10.586})$ 10.781)

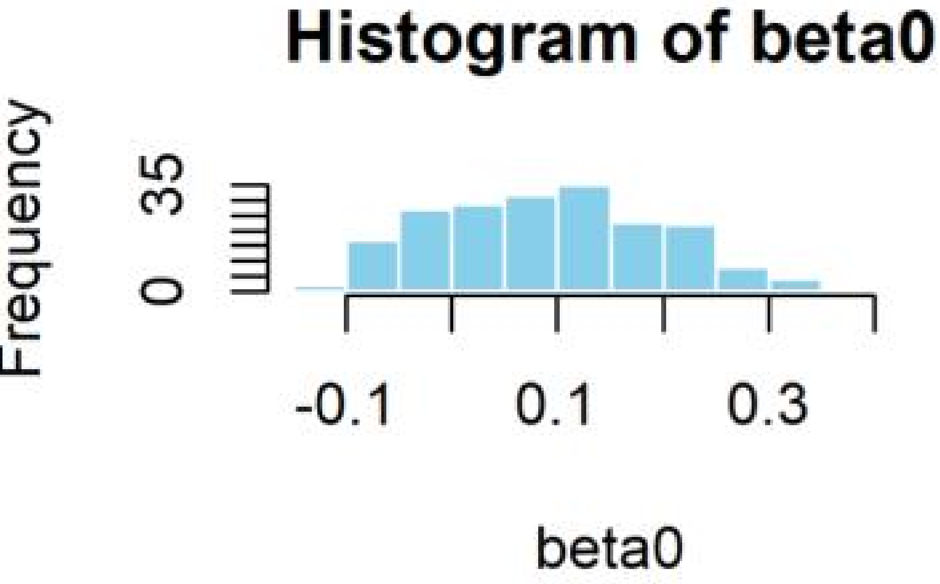
T-statistic for H_0: $T = 19.480$

p-value ≈ 0

We reject H_0 and in favor of H_1

Programe.

b: $\hat{\beta}_0 = -0.02013735$ $\hat{\beta}_1 = 0.4426883$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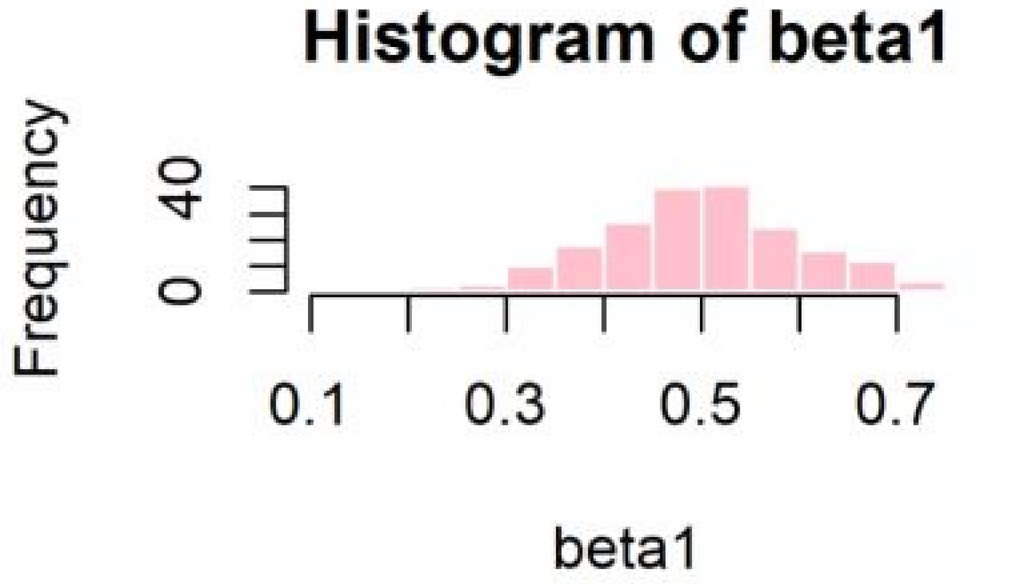
0.1
0.3
beta0


Histogram of beta1
Frequency
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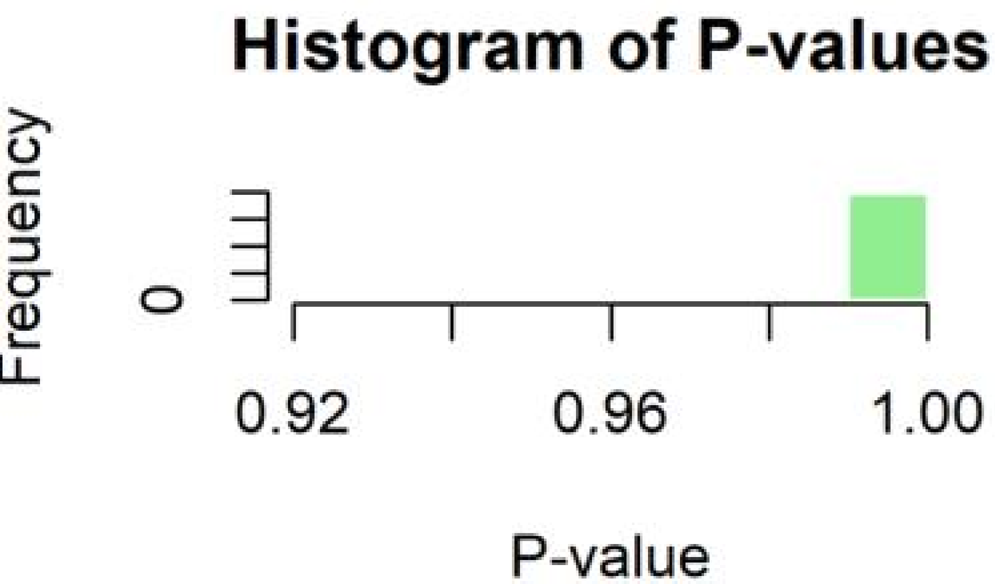
0.3
0.5
0.7
beta1


Histogram of P-values
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1.00
P-value