HW9

Q1 (i) $L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$

$l(\lambda) = -n\lambda + \sum_{i=1}^{n} x_i \cdot \log\lambda - \log \prod_{i=1}^{n} x_i!$ $\quad \frac{dl(\lambda)}{d\lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} x_i = 0 \to \lambda = \frac{\sum_{i=1}^{n} x_i}{n}$ where $\frac{d^2 l(\lambda)}{d\lambda^2} = -\frac{1}{\lambda^2}\sum_{i=1}^{n} x_i < 0$

$\Lambda = 2(l(\hat{\lambda}) - l(\lambda_0))$ where $\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n}$.

So $\Lambda = 2\left(\sum_{i=1}^{n} x_i \log\frac{\hat{\lambda}}{\lambda_0} - n(\hat{\lambda} - \lambda_0)\right)$

$p = P(\chi_1^2 \geq \Lambda)$

ii $W = \frac{(\hat{\lambda} - \lambda_0)^2}{var(\hat{\lambda})}$ where $var(\hat{\lambda}) = \frac{\hat{\lambda}}{n}$.

So $W = \frac{(\hat{\lambda} - \lambda_0)^2}{\hat{\lambda}/n}$

$p = P(\chi_1^2 \geq W)$

iii $U(\lambda) = \frac{\partial l(\lambda)}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n$

$I(\lambda_0) = -E\left(\frac{\partial^2 l(\lambda)}{\partial \lambda^2}\right) = E\left(\frac{1}{\lambda_0^2}\sum_{i=1}^{n} x_i\right) = \frac{n}{\lambda_0}$.

$S = \frac{U(\lambda_0)^2}{I(\lambda_0)} = \frac{\left(\frac{\sum_{i=1}^{n} x_i}{\lambda_0} - n\right)^2}{n/\lambda_0}$

$p = P(\chi_1^2 \geq S)$

iv. $\hat{\lambda} = \bar{X} = 2$.

So $\Lambda = 2(40\ln 2 - 20) \approx 15.45$

$W = \frac{1}{2/20} = 10.$

$S = 20.$ $\quad$ where $\chi^2_{1,0.95} = 3.8418.$

So. All can reject H_0.

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Q2. i $L(\beta)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(X_i-X_i\beta)^2}{2\sigma^2}\right)=(2\pi\sigma^2)^{-\frac{n}{2}}\exp\left(-\frac{1}{2\sigma^2}\sum(X_i-X_i\beta)^2\right)$

$l(\beta)=-\frac{n}{2}\log(2\pi\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-X_i\beta_0)^2.$

Under H_0, $l(\beta_0)=-\frac{n}{2}\log(2\pi\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-X_i\beta_0)^2$

Under H_1 $\frac{dl(\beta)}{d(\beta)}=-\frac{1}{2\sigma^2}\sum_{i=1}^{n}2X_i(X_i-X_i\beta)$

$=\frac{1}{\sigma^2}\sum_{i=1}^{n}X_i(X_i-X_i\beta)=0 \rightarrow \beta=\sum_{i=1}^{n}\frac{X_iX_i}{X_i^2}$ where $\frac{d^2l(\beta)}{d(\beta)^2}=\frac{1}{\sigma^2}\sum_{i=1}^{n}(-X_i^2)<0.$

So $\Lambda=-2(l(\beta_0)-l(\hat\beta))$

$=\frac{1}{\sigma^2}\left[\sum_{i=1}^{n}(X_i-X_i\beta_0)^2-\sum_{i=1}^{n}(X_i-X_i\hat\beta)^2\right]$

$=\frac{\sum_{i=1}^{n}X_i^2}{\sigma^2}(\hat\beta-\beta_0)^2$ $\quad\Lambda\sim\chi_1^2$

$P\text{value}=P(\chi_1^2\geq\Lambda)$

ii $Var(\hat\beta)=\cancel{\phi^2}\ Var\left(\sum_{i=1}^{n}\frac{X_iX_i}{X_i^2}\right)=\frac{\sigma^2}{\sum_{i=1}^{n}X_i^2}$

So $W=\frac{\sum_{i=1}^{n}X_i^2}{\sigma^2}(\hat\beta-\beta_0)^2$ $\quad W\sim\chi_1^2$

$P\text{-value}=P(\chi_1^2\geq W)$

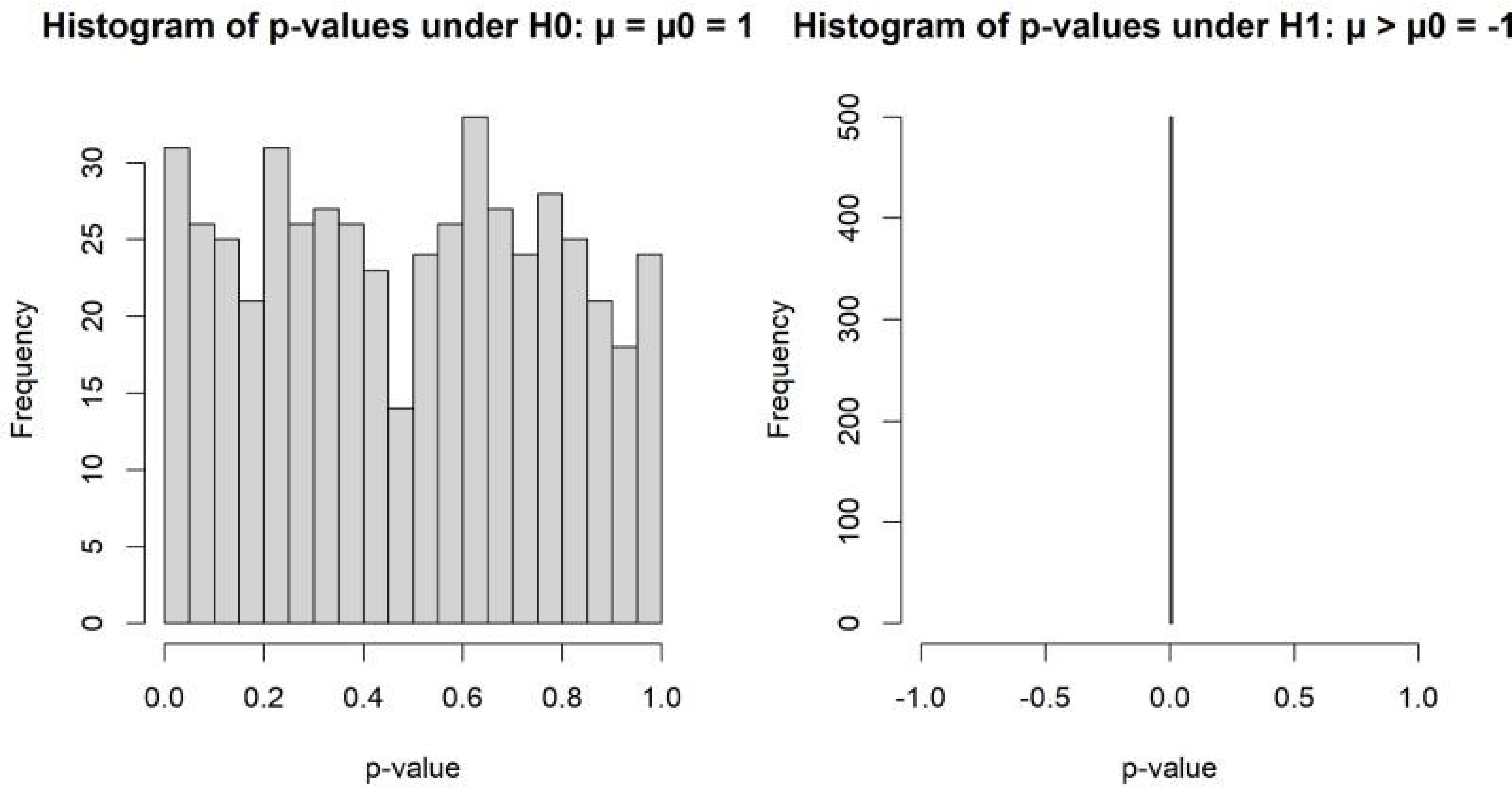
Programming Question

i $Z=\frac{\bar X-\mu_0}{\sqrt{\sigma^2/n}}$ where $\bar X$ is sample mean

$P\text{-value}=P(Z\geq z_{obs})=1-\Phi\left(\frac{\bar x-\mu_0}{\sqrt{\sigma^2/n}}\right)$

ii *

iii see the output


Histogram of p-values under H0: μ = μ0 = 1
Frequency
0
5
10
15
20
25
30
0.0
0.2
0.4
0.6
0.8
1.0
p-value
Histogram of p-values under H1: μ > μ0 = -1
Frequency
0
100
200
300
400
500
-1.0
-0.5
0.0
0.5
1.0
p-value