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Homework #1

Q1 (i) $M(t)=E[e^{tx}]=\sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x e^{-\lambda}}{x!}$

$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$

$= e^{-\lambda}\cdot e^{\lambda e^t} = e^{\lambda(e^t-1)}$

So $M'(t)=\lambda e^t e^{\lambda(e^t-1)}$ $\quad E(X)=M'(0)=\lambda\cdot1\cdot1=\lambda$

$M''(t)=\lambda e^t e^{\lambda(e^t-1)}+\lambda^2 e^{2t}e^{\lambda(e^t-1)}$ $\quad Var(X)=M''(0)-M'(0)^2=\lambda^2+\lambda-\lambda^2=\lambda.$

(ii) ~~$F(t)=\sum_{x=0}^{t}\frac{\lambda^x e^{-\lambda}}{x!}=e^{-\lambda}\sum_{x=0}^{t}\frac{\lambda^x}{x!}$~~ $\quad F(a)=\sum_{x=0}^{a}\frac{\lambda^x e^{-\lambda}}{x!}$

Markov Inequality Proof: $E(X)=\int_{-\infty}^{\infty}xf(x)dx=\int_0^{\infty}xf(x)dx$ in this case

$E(X)=\int_0^a xf(x)dx+\int_a^{\infty}xf(x)dx\geq\int_a^{\infty}xf(x)dx\geq\int_a^{\infty}af(x)dx$

$=aF(X\geq a)$

So $F(X\geq a)\leq\frac{E(X)}{a}$ in this case, by Markov Inequality

$E(X)=\lambda$ $\quad F(a)=1-F(X\geq a)\geq1-\frac{\lambda}{a}$ $\rightarrow\square$

(iii) $E(Y)=\int_0^1 Y\,dY=\frac{1}{2}$

$Var(Y)=E(Y^2)-E(Y)^2=\int_0^1Y^2dY-\frac{1}{4}=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$

$Sk\ Y=\frac{E(Y^3)-3\cdot\frac{1}{2}\cdot\frac{1}{12}-\frac{1}{8}}{\sigma^3}=\frac{\int_0^1Y^3dY-\frac{1}{4}}{\sigma^3}=\frac{\frac{1}{4}-\frac{1}{4}}{(\frac{1}{12})^3}=0$.

$Kurt(Y)=\frac{E(Y-\frac{1}{2})^4}{(\frac{1}{12})^2}-3=\frac{E(X^4-2X^3+\frac{3}{2}X^2-\frac{1}{2}X+\frac{1}{16})}{(\frac{1}{12})^2}-3=\frac{\frac{1}{80}}{(\frac{1}{12})^2}-3=-\frac{6}{5}$

(iv) By mathmatical induction

when $n=1$ $\quad Var(Z_1)=Var(Z_1)$ is obviously true / stands True.

we suppose when $n=k$ $\quad Var(\sum_{i=1}^{k}Z_i)=\sum_{i=1}^{k}Var(Z_i)$, Just Prove $Var(\sum_{i=1}^{k+1}Z_i)=\sum_{i=1}^{k+1}Var(Z_i)$

$LHD=Var(\sum_{i=1}^{k}Z_i+Z_{k+1})$ $\quad RHD=\sum_{i=1}^{k}Var(Z_i)+Var(Z_{k+1})$

$=Var(\sum_{i=1}^{k}Z_i)+Var(Z_{k+1})$

Since for any function h, $Z_1\cdots Z_k$ Indpendend, $h(Z_1\cdots Z_k)$ and Z_{k+1} is independent

So $\sum_{i=1}^{k}Z_i$, Z_{k+1} are independent $\rightarrow Var(\sum_{i=1}^{k}Z_i+Z_{k+1})=Var(\sum_{i=1}^{k}Z_i)+Var(Z_{k+1})$

$\rightarrow$ Proved $\square$.

Q2. let $Y=\max(X_1,X_2)$ $\quad G(y)=[P(X\leq y)]^2$

$=[\int_1^y\frac{4}{x^5}dx]^2$

$=[1-\frac{1}{y^4}]^2$ $\quad 1<y<\infty.$

$g(y)=G'(y)=\frac{8}{y^5}(1-\frac{1}{y^4})$

$E(Y)=\int_1^{\infty}y\cdot\frac{8}{y^5}(1-\frac{1}{y^4})=8\int_1^{\infty}[\frac{1}{y^4}-\frac{1}{y^8}]dy=\frac{32}{21}$

$Var(Y)=E(Y^2)-E(Y)^2$

$=\int_1^{\infty}y^2\frac{8}{y^5}(1-\frac{1}{y^4})-(\frac{32}{21})^2=\frac{8}{3}-(\frac{32}{21})^2=\frac{152}{441}$

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Q3 $\mathrm{Cov}(XY, \cancel{X}Y) = E(XY \cdot Y\cancel{X}) - E(XY)\,E(\cancel{X}Y)$

$X\,Y$ are independent $E(XY) = E(X)E(Y) = \mu^2$

$E(XY \cdot Y) = E(XY^2) = E(X)\,E(Y^2)$

$= \mu(\sigma^2 + \mu^2)$

So $\mathrm{Cov}(XY, Y) = \mu(\sigma^2 + \mu^2) - \mu^3 = \mu\sigma^2$.

$\mathrm{Var}(XY) = E[(XY)^2] - [E(XY)]^2$

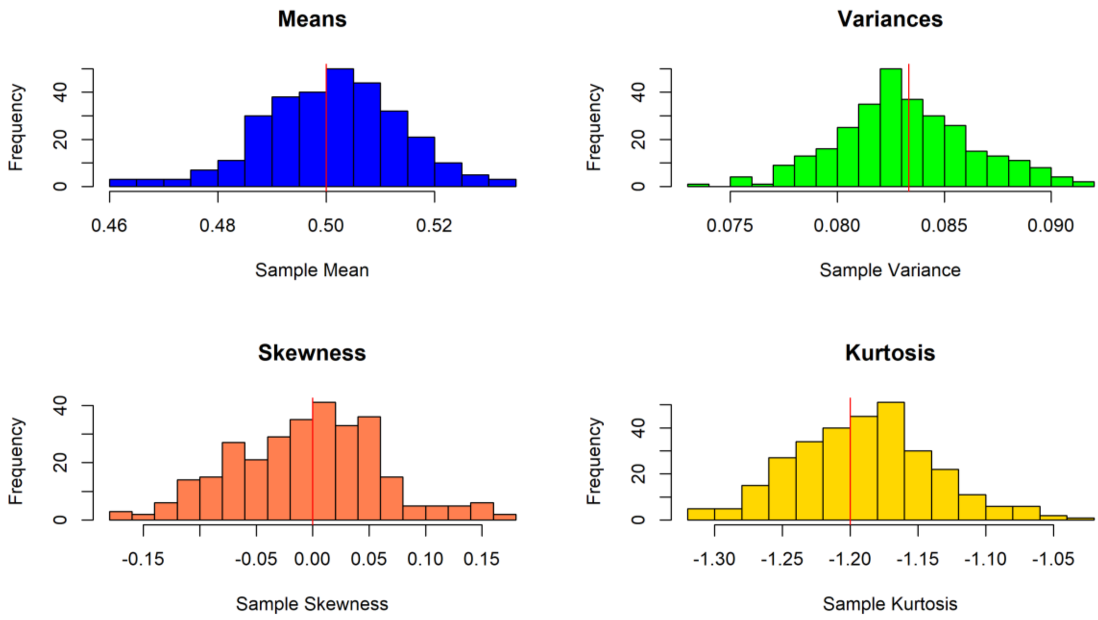
$= E(X^2)E(Y^2) - \mu^4$

$= (\sigma^2 + \mu^2)^2 - \mu^4$

$= \sigma^4 + 2\sigma^2\mu^2$

So $\mathrm{Corr}(XY, Y) = \dfrac{\overset{\mu\sigma^2}{\cancel{\mu(\sigma^2+\mu^2)}}}{\cancel{\sigma}\cdot\sigma\sqrt{\sigma^2+2\mu^2}} = \dfrac{\mu\sigma}{\sqrt{\sigma^2+2\mu^2}}$

```
#Install and load necessary package for skewness and kurtosis
install.packages("moments")
library(moments)

# Initialize vectors to store sample statistics
sample_means <- numeric(300)
sample_variances <- numeric(300)
sample_skewnesses <- numeric(300)
sample_kurtoses <- numeric(300)

# Repeat experiment 300 times
for (i in 1:n_experiments) {
  X <- runif(500, min = 0, max = 1)

  # Calculate sample statistics for each experiment
  sample_means[i] <- mean(X)
  sample_variances[i] <- var(X)
  sample_skewnesses[i] <- skewness(X)
  sample_kurtoses[i] <- kurtosis(X) - 3
}

# Plot  2*2 histograms for each sample statistic
par(mfrow = c(2, 2))

# Histogram for sample mean
hist(sample_means, main = "Means", xlab = "Sample Mean", col = "blue", breaks = 20)
abline(v = 0.5, col = "red")  # True mean

# Histogram for sample variance
hist(sample_variances, main = "Variances", xlab = "Sample Variance", col = "green", breaks =
20)
abline(v = 1/12, col = "red")

# Histogram for sample skewness
hist(sample_skewnesses, main = "Skewness", xlab = "Sample Skewness", col = "coral", breaks
= 20)
abline(v = 0, col = "red")

# Histogram for sample kurtosis
hist(sample_kurtoses, main = "Kurtosis", xlab = "Sample Kurtosis", col = "gold", breaks = 20)
abline(v = -6/5, col = "red")
```


Means
Frequency
0
20
40
0.46
0.48
0.50
0.52
Sample Mean
Variances
Frequency
0
20
40
0.075
0.080
0.085
0.090
Sample Variance
Skewness
Frequency
0
20
40
-0.15
-0.05
0.00
0.05
0.10
0.15
Sample Skewness
Kurtosis
Frequency
0
20
40
-1.30
-1.25
-1.20
-1.15
-1.10
-1.05
Sample Kurtosis