HW5

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Q1 (1) $n=\left(\frac{1.96\sqrt{P(1-P)}}{0.03}\right)^2 = 4268.444P(1-P) \overset{P=0.5}{=} 1067.11$

(2) $n=\left(\frac{1.96\sqrt{P(1-P)}}{0.02}\right)^2 = 9604P(1-P) \overset{P=0.5}{=} 2401$

(3) $n=\left(\frac{1.645\sqrt{P(1-P)}}{0.03}\right)^2 = 3006.6944P(1-P) \overset{P=0.5}{=} 751.67$

(c) $\sqrt{P(1-P)} \le \frac{P+1-P}{2} = \frac{1}{2}$ & the equilibrum is true when $P=\frac{1}{2}$

and when $P=\frac{1}{2}$, n has its maximum when others are same

So upper bond is when $P=0.5$

Q2

(a) $\log f(X) = k\log\lambda - \lambda - \log(k!)$

$\frac{\partial}{\partial\lambda}\log(f(x)) = \frac{k}{\lambda} - 1$

$\frac{\partial^2}{\partial\lambda^2}\log(f(X)) = -\frac{k}{\lambda^2}$

$I(\lambda) = -E\left(\frac{\partial^2}{\partial\lambda^2}\log f(X)\right) = E\left(\frac{k}{\lambda^2}\right) = \frac{n}{\lambda}$

(b) $\hat{I}_n(\lambda) = \frac{n}{\bar{x}} = \frac{n}{\frac{1}{n}\sum_{i=1}^n X_i}$ Since $\bar{x} = \frac{1}{n}\sum_{i=1}^n X_i$

So $\hat{I}_n(\lambda) = \frac{n^2}{\sum_{i=1}^n X_i}$

(c) Asymptotically we have $\lambda_{MLE} \sim N\left(\lambda, \frac{1}{I(\lambda)}\right)$

Therefore $(\lambda_{MLE}-\lambda)\sqrt{I(\lambda)} \sim N(0,1)$ Setting $\lambda = \bar{x}$

We have $1-\alpha$ interval $\left[\bar{x} - \frac{z_{\alpha/2}}{n}\sqrt{\sum_{i=1}^n X_i},\ \bar{x} + \frac{z_{\alpha/2}}{n}\sqrt{\sum_{i=1}^n X_i}\right]$

(d) $n = \left(\frac{1.96}{0.01}\lambda\right)^2 = 196^2\lambda = 38416\lambda$

Q3 (a) $f(X_i : a) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{x_i^2}{2a}}$ $\rightarrow L(a) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi a}} e^{-\frac{x_i^2}{2a}} = \left(\frac{1}{\sqrt{2\pi a}}\right)^n e^{-\frac{\sum_{i=1}^n x_i^2}{2a}}$

$l(a) = \log L(a) = -\frac{n}{2}\log(2\pi a) - \frac{\sum_{i=1}^n X_i^2}{2a}$

$\frac{\partial l(a)}{\partial a} = -\frac{\cancel{\pi} n}{2\cancel{\pi} a} + \frac{\sum_{i=1}^n X_i^2}{2a^2} = 0.$ $\rightarrow \hat{a} = \frac{\sum_{i=1}^n X_i^2}{n}$

where $\frac{\partial^2 l(a)}{\partial^2 a} = \frac{n}{2a^2} - \frac{\sum_{i=1}^n X_i^2}{a^3} < 0.$

So $\hat{a} = \frac{\sum_{i=1}^n X_i^2}{n}.$

(b) $I(a) = -E\left(\frac{\partial^2}{\partial a^2}\log L(a)\right)$

$= E\left(-\frac{n}{2a^2} + \frac{\sum_{i=1}^n X_i^2}{a^3}\right)$

$= -\frac{n}{2a^2} + \frac{na+0}{a^3} = \frac{n}{2a^2}$

(c) $\hat{I}(a) = \frac{n}{2a^2}$ we use $\hat{a} = \frac{1}{n}\sum_{i=1}^n X_i^2$

$\hat{I}(a) = \frac{n}{\frac{2}{n^2}\left(\sum_{i=1}^n X_i^2\right)^2} = \frac{n^3}{2\left(\sum_{i=1}^n X_i^2\right)^2}.$

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(d) $\frac{1}{\alpha}\sum_{i=1}^{n} X_i^2 \sim \chi_n^2$

So $\frac{n\hat{\alpha}}{\alpha} \sim \chi_n^2$

$P\left(\chi^2_{\alpha/2,n} \le \frac{n\hat{\alpha}}{\alpha} \le \chi^2_{1-\alpha/2,n}\right) = 1-\alpha$

So Confidence Interval is $\left(\cancel{\frac{n\hat{\alpha}}{\chi_{\alpha/2}}}\ \frac{n\hat{\alpha}}{\chi^2_{1-\alpha/2,n}}, \frac{n\hat{\alpha}}{\chi^2_{\alpha/2,n}}\right)$

Confidence Intervals for p
Red intervals do not contain the true p = 0.3
Samples
100
75
50
25
0
0.2
0.4
0.6
p (Sample Proportion)
Contains_p_true
FALSE
TRUE