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Q1 (a) (i) $L(\lambda_0) = \prod_{i=1}^{5} \frac{\lambda_0^{x_i} e^{-\lambda_0}}{x_i!}$ $\quad \log L(\lambda_0) = \sum_{i=1}^{n} (x_i \log(\lambda_0) - \lambda_0 - \log(x_i!))$

$\frac{d}{d\lambda_0} \log L(\lambda_0) = \sum_{i=1}^{n} (\frac{x_i}{\lambda} - 1)$ $\quad \frac{d^2}{d\lambda_0^2} \log L(\lambda_0) = \sum_{i=1}^{n} (-\frac{x_i}{\lambda^2}) < 0$

So $\hat{\lambda}_{MLE} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x} = \frac{80+73+68+56+43}{5} = 65$

(ii) Attendance rate $\mu_{LE} = \frac{\hat{\lambda}}{120} = \frac{13}{24}$

(b) (i) $L(\pi_0) = \prod_{i=1}^{5} \binom{n_i}{x_i} (\pi_0^{x_i}) (1-\pi_0)^{n_i - x_i}$

$\log L(\pi_0) = \sum_{i=1}^{5} \binom{n_i}{x_i} (x_i \log(\pi_0) + (n_i - x_i) \log(1-\pi_0))$

that is $\pi_0 = \frac{\sum_{i=1}^{5} x_i}{\sum_{i=1}^{5} n_i}$ $\Rightarrow \frac{d}{d\pi_0} \log L(\pi_0) = \sum_{i=1}^{5} (\frac{x_i}{\pi_0} - \frac{n_i - x_i}{1-\pi_0}) = 0$ $\quad \frac{d^2}{d^2\pi_0} \log L(\pi_0) = \sum_{i=1}^{5} (-\frac{x_i}{\pi_0^2} + \frac{n_i - x_i}{(1-\pi_0)^2}) < 0$

(ii) So $\hat{\pi}_0 = \frac{\sum_{i=1}^{5} x_i}{\sum_{i=1}^{5} n_i} = \frac{85+73}{100 \times 2} \frac{+68+56+43}{+3\times 120} = \frac{325}{560} \approx 0.58.$

(c) $L(\alpha+\beta t) = \prod_{t=1}^{5} \frac{(\alpha+\beta t)^{x_t} e^{-(\alpha+\beta t)}}{x_t!}$ $\quad L(\alpha, \beta) = \sum_{t=1}^{5} x_t \log(\alpha+\beta t) - 5\alpha - 5\beta t - \sum_{t=1}^{5} \log(x_t!)$

$\log L(\alpha+\beta t) = \sum_{t=1}^{5} (x_t \log(\alpha+\beta t) - (\alpha+\beta t) - \log(x_t)!)$

Q2 (a)

$F_{\hat{\theta}}(x) = P(\hat{\theta} \le x) = P(X_1 \le x, \cdots, X_n \le x)$

$= \prod_{i=1}^{n} P(X_i \le x) = [F_X(x)]^n$

$= (\frac{x}{\theta})^n$ -- cdF

$f_{\hat{\theta}}(x) = \frac{d}{dx} (\frac{x}{\theta})^n = \frac{n}{\theta} (\frac{x}{\theta})^{n-1}$ -- pdF

So $F(x) = \begin{cases} 0 & x < 0 \\ (\frac{x}{\theta})^n & 0 \le x \le \theta \\ 1 & x > \theta \end{cases}$ $\quad f(x) = \begin{cases} \frac{n}{\theta} (\frac{x}{\theta})^{n-1} & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$

(b) $L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda Y_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} Y_i}$

$\log L(\lambda) = n \log \lambda - \lambda \sum_{i=1}^{n} Y_i$

$\frac{d \log L(\lambda)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} Y_i$ $\quad \frac{d^2 \log L(\lambda)}{d\lambda^2} = -\frac{n}{\lambda^2} < 0.$

So $\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} Y_i}$ $\quad$ Since $\sum_{i=1}^{n} Y_i \sim Gamma(n, \lambda)$

$f(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}$ $\quad x \ge 0$

Since $\hat{\lambda} = \frac{n}{\sum Y_i}$, $\hat{\lambda}$ is a transformation of Gamma random var,

$\rightarrow \hat{\lambda}$ follows Inverse Gamma distribution

$\hat{\lambda} \sim$ Inverse Gamma $(n, \frac{n}{\lambda})$

MLE of alpha: 95.79831

MLE of beta: -10.26732

90% CI for lambda: 1.597972 2.227148

The CI contains the true value of lambda.

Coverage rate of 90% CI: 0.92