Expanding the Horizons of Edge-Irregularity Strength in Graphs: New Bounds and Insights

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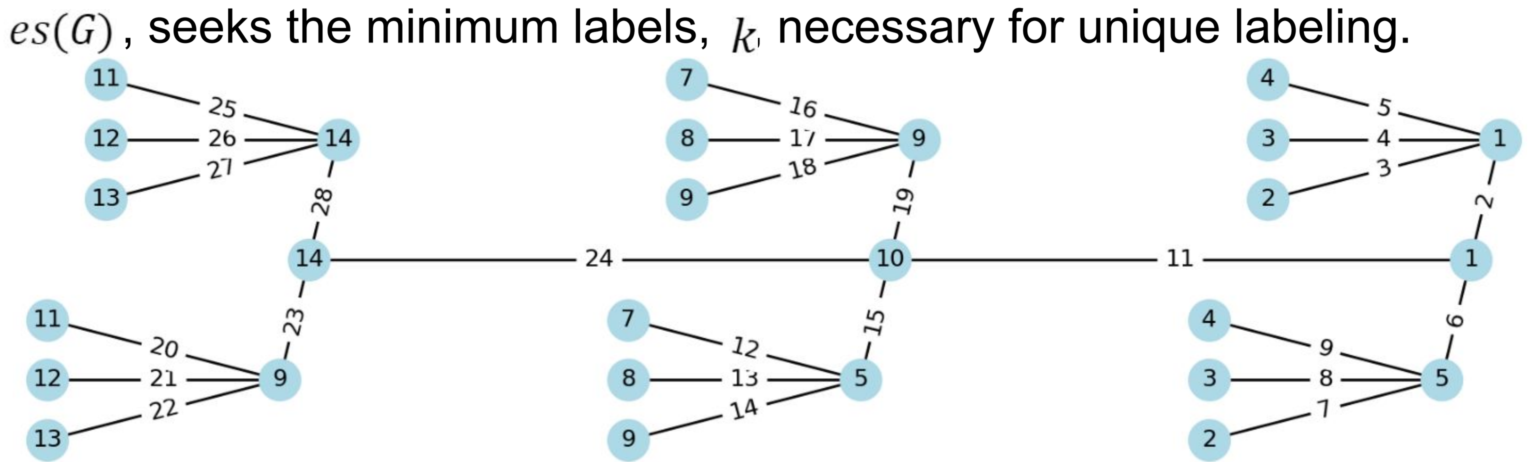
Introduction

In graph theory, the puzzle of how to uniquely label the edges of a graph through strategic labeling of its vertices is captured by the concept of *edge irregular k-labeling*. Initiated by Ahmad et al. in 2014, each edge is labeled with the sum of its incident vertices' labels, aiming for uniqueness across the graph. This measure, known as the edge irregularity strength, $es(G)$, seeks the minimum labels, k, necessary for unique labeling.



Applications:

Network Design: Simplifies routing and data management by ensuring unique path identifiers.
Error Detection and Correction: Enhances data integrity in digital communication through unique error identifiers.
Database Management: Improves query efficiency and data consistency in graph-based databases.

Our study challenges existing boundaries and proposes new insights to refine the bounds of $es(G)$. We marry theoretical exploration with computational analysis to illuminate this intricate domain, revealing its significance beyond abstract mathematics to real-world applications.

Hypothesis/Question

Our work refines edge irregularity strength bounds with two questions:

Upper Bounds: Can we refine or validate the previously proposed upper bounds for $es(G)$ in complete graphs by applying advanced algorithmic optimizations to investigate larger complete graphs?

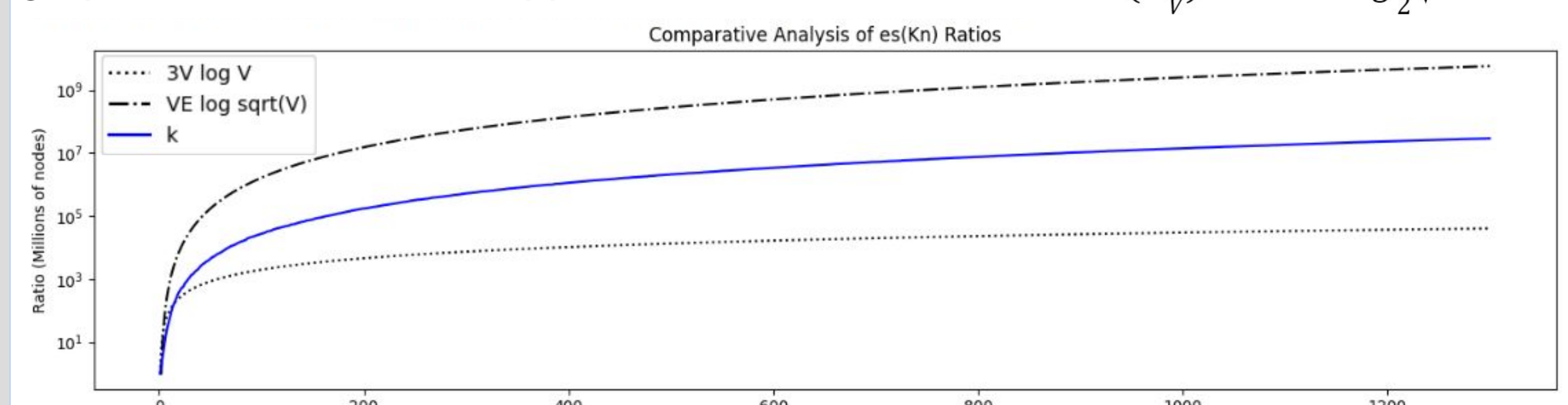
Lower Bounds: Is it possible to derive a more accurate lower bound for $es(G)$ by investigating other graph invariants besides graph order which has been the traditional approach until now?

Tightening these bounds enhances our theoretical understanding of graph properties. It also improves practical algorithms for graph labeling. This has direct impact upon network design, data integrity, and computational efficiency. This effort bridges theoretical insights with real-world applicability to empower future innovations in graph theory.

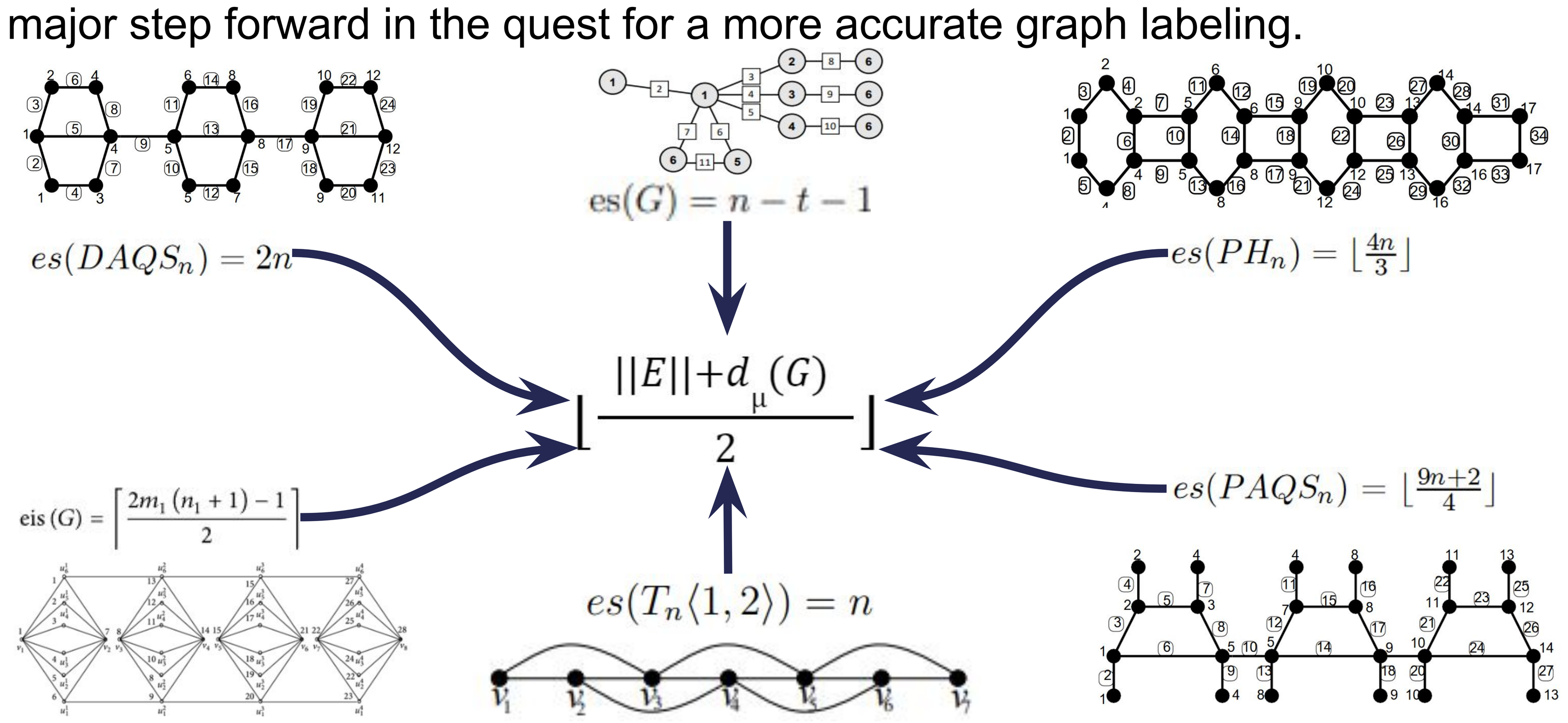
Results

Our study's findings significantly advance the understanding of edge irregularity strength, $es(G)$, in two key areas:

Upper Bound Refutation: Previously, the upper bound for complete graphs, K_V, proposed was $es(G) = 3(E\,log_2 V)$ as obtained by Asim et al in 2018 through computational mathematics on graphs up to 100 nodes, K_{100}. Building upon their work, we optimized the labeling algorithms and strategically narrowed the search space. We found the existing upper bounds lose their validity in the realm of complete graphs of up to 1300 nodes, K_{1300}. Within the scope of the graphs studied, the new upper bound must be at least $es(K_V) = VE\,log_2\sqrt{V}$.



New Lower Bound Establishment: By shifting focus from the sole reliance on graph order (node count), $||V(G)||$, to a comprehensive consideration of graph invariants—specifically, edge count, $||E(G)||$, and average degree (edges per node), $d_\mu(G)$—our analysis has successfully formulated a new lower bound for $es(G)$. This novel approach has led to a unified lower bound, articulated as: $es(G) \geq max(\Delta G, \lfloor\frac{||V||+1}{2}\rfloor, \lfloor\frac{||E||+d_\mu(G)}{2}\rfloor)$. This lower bound is tighter for graphs that are not trees nor have large stars. Additionally, the third term unifies $es(G)$ for a multitude of previously studied sparse graph structures into one, encapsulating formula. This formula, empirically validated against over 100,000 randomly generated graphs without a single violation, represents a significant stride towards refining our theoretical understanding of $es(G)$. The robustness of this new lower bound, supported by exhaustive testing against edge cases, signals a major step forward in the quest for a more accurate graph labeling.



Methodology

Algorithm Optimization for Complete Graphs:
We began by reevaluating existing algorithms for edge irregular k-labeling, with a particular focus on their application to complete graphs, denoted as K_V. Our objective was to enhance algorithmic efficiency and scalability.
Data Structures: Through extensive profiling, we observed that checking for duplicate edges needed the overwhelming majority of computation. To ameliorate this, we employed hashed data structures–renowned for their speed and efficiency. This choice facilitated rapid set comparisons, which was a critical factor when dealing with large graph sizes requiring millions of comparisons.
Search Space Reduction: As we began finding edge irregular k-labelings of complete graphs, we recognized patterns that identified impossible labelings. By eliminating those from the search space of the algorithm, we significantly reduced necessary computation. This step was crucial for analyzing larger graphs efficiently.
Computational Analysis: Our analysis extended to complete graphs of up to 1300 nodes, surpassing the previous limit of 100. This was achieved by leveraging the optimized algorithms and data structures mentioned above. Upon labelling these graphs, the growth of labellings was compared to magnitudes of the graph invariants.

Algorithm 1 label_Kn(n)

```
labs ← [1, 2, 3]
edges ← SET(3, 4, 5)
lastNode = 3
minP = 4
maxP = 5
for nodeNum from 4 to n do
    for newLab from minP to maxP do
        newEdges ← SET()
        for lab in labs do
            newEdges.ADD(newLab + lab)
        if newEdges ∩ edges = ∅ then
            labs.APPEND(newLab)
            edges.UPDATE(newEdges)
            minP ← newLab + 1
            maxP ← newLab + lastNode
            lastNode ← newLab
            break
return labs
```

Empirical Validation for New Lower Bound:
Moving beyond the conventional reliance on graph order, our study incorporated graph size, $||E(G)||$, and average degree, $d_\mu(G)$, into the formulation of a new lower bound for $es(G)$.
Formula Development: The new lower bound formula was developed as $es(G) \geq max(\Delta G, \lfloor\frac{||V||+1}{2}\rfloor, \lfloor\frac{||E||+d_\mu(G)}{2}\rfloor)$, designed to provide a more accurate and universally applicable measure of $es(G)$ across various graph structures.
Testing and Validation: To validate this new formula, we conducted empirical tests involving over 100,000 randomly generated graphs, including a focused examination of edge cases. This exhaustive testing process ensured that the proposed lower bound holds true across a broad spectrum of graph types without exceptions.

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