Homework 1

CSC 445-01: Theory of Computation

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Question I

We will show that $A = B$ using three proofs by contradiction. First, it is helpful to expand the following

$$A \times B = \Big\{(a, b)|a \in A, b \in B\Big\}$$

$$B \times A = \Big\{(b, a)|b \in B, a \in A\Big\}$$

Thus the original condition is

$$A \times B \subseteq B \times A$$

$$\Big\{(a, b)|a \in A, b \in B\Big\} \subseteq \Big\{(b, a)|b \in B, a \in A\Big\}$$

1

Suppose $A \not\subseteq B$. Then $\exists\ a \in A$ where $(a \notin B)$. Then $\exists\ (a, b) \in A \times B$ where $((a, b) \notin B \times A)$. This defies the original condition. We have proven

$$A \subseteq B$$

2

Suppose $B \not\subseteq A$. Then $\exists\ b \in B$ where $(b \notin A)$. Then $\exists\ (a, b) \in A \times B$ where $((a, b) \notin B \times A)$. This defies the original condition. We have proven

$$B \subseteq A$$

3

Suppose $A \neq B$. Then one of the following two statements must be true:

1. $\exists\ a \in A$ where $(a \notin B)$, which violates $A \subseteq B$
2. $\exists\ b \in B$ where $(b \notin A)$, which violates $B \subseteq A$

Thus we have proven

$$A = B$$

Question II

a

b

c

Question III

We will prove

$$P(n) \colon \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \;\Bigg|\; n \geq 0, n \in Z$$

Base case:

$$\begin{aligned}
P(0) \quad &: \quad \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \\
&: \quad \sum_{i=1}^{0} \frac{1}{i(i+1)} = \frac{0}{0+1} \\
&: \quad 0 = 0 \\
&: \quad \text{TRUE}
\end{aligned}$$

Inductive Hypothesis:

$$P(k)\colon \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1} \;\Big|\; k \geq 0, k \in Z$$

Induction:

$$\begin{aligned}
P(k+1) \quad &: \quad \frac{(k+1)}{(k+1)+1} = \sum_{i=1}^{k+1} \frac{1}{i(i+1)} \\
&: \quad \frac{k+1}{k+2} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)((k+1)+1)} \\
&: \quad \frac{k+1}{k+2} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&: \quad \frac{k+1}{k+2} = \frac{k(k+2)+1}{(k+1)(k+2)} \\
&: \quad \frac{k+1}{k+2} = \frac{(k+1)^2}{(k+1)(k+2)} \\
&: \quad \frac{k+1}{k+2} = \frac{k+1}{(k+2)} \\
P(k+1) \quad &: \quad \text{TRUE}
\end{aligned}$$

Conclusion:

$$\Bigg(P(0) \wedge \Bigg(P(k) \to P(k+1)\Bigg)\Bigg) \to P(n)$$

Question IV

Truth Table

p	q	$(p \to q) \wedge (p \to \neg q)$	$p \wedge (p \to q)$	$(p \to q) \wedge (\neg p \to q)$	$p \iff (p \iff q)$	$q \wedge (p \to q)$
T	T	F	T	T	T	T
T	F	F	F	F	F	F
F	T	T	F	T	T	T
F	F	T	F	F	F	F

Equivalence

Problem	*Original*	*Simplified*
a	$(p \to q) \wedge (p \to \neg q)$	$\neg p$
b	$p \wedge (p \to q)$	$p \wedge q$
c	$(p \to q) \wedge (\neg p \to q)$	q
d	$p \iff (p \iff q)$	q
e	$q \wedge (p \to q)$	q

Question V

0.1 1

0.2 2

0.3 3

Question VI

0.4 1

0.5 2

Question VII

0.6 1

0.7 2