Outline of the Dissertation

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Abstract

You should work on points 1 and 2 and choose at least one of the options in point 3. Remember to submit a .pdf file together with the .r files of your simulations via Canvas. Remember to test the programs before submitting them and make sure they do effectively work. Please, read the important notice at the end.

1 Generation of (pseudo-)random numbers

Select the distribution of a random variable X with finite expectation and variance (you can use the book by Devroye as a source) and write an algorithm in R that generates random numbers distributed according to the selected distribution. Plot a histogram of the random numbers and compare it with the theoretical distribution by superimposing a theoretical curve to the histogram. Make your comparison quantitative by using a goodness-of-fit method as a function of sample size. You cannot use the same distributions and programs that are on-line in Canvas.
Marks: Approximately up to 30.

2 Markov Chains and Markov Chain Monte Carlo

Consider a homogeneous Markov chain Z_n whose state space has three states and with the following transition probability matrix

$$P = \begin{bmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \\ \eta & \theta & \iota \end{bmatrix}. \qquad (2.1)$$

Using the definitions and theorems presented during the course, explain why Markov Chain Monte Carlo algorithms work. As an illustration of the theory, choose the values of the parameters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \iota$ so that they are in the interval $[0, 1)$ and the Markov chain is irreducible and aperiodic and clearly explain why your choice leads to an irreducible and aperiodic Markov chain. Compute the invariant distribution of your Markov chain. Write a Monte Carlo program in R that simulates

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your Markov chain and compare the empirical distribution you obtain by Monte Carlo simulations with the theoretical invariant distribution.
Marks: Approximately up to 30.

3 Select at least one of the following projects:

3.1 Ising model

Consider an Ising model with five spins on a graph of your choice in which each spin is connected with at least another spin (please do not use the cyclic graph presented in the lecture notes). Write the Hamiltonian for your model. Compute the partition function, the free energy, the entropy, the energy and the absolute value of the magnetisation of your model. Write a Metropolis Monte Carlo simulation of your model and check that it gives the correct values for the energy per spin and the absolute value of the magnetisation per spin. To this purpose compare the plot the results of the Monte Carlo simulations against your theoretical results as a function of temperature (or inverse temperature).

3.2 Random walk

Use the random variable you introduced in point 1 to define a random walk as follows. Let $\{X_i\}_{i=1}^N$ be a sequence of independent and identically distributed random variables with cumulative distribution function $F_X(u) = \mathbb{P}(X \leq u)$. Consider the random variable

$$Z_N = \sum_{i=1}^{N} X_i, \tag{3.1}$$

Theoretically compute the cumulative distribution function $F_{Z_N}(u) = \mathbb{P}(Z_N \leq u)$ of Z_N. Write a Monte Carlo program in R that generates samples of Z_N for $N \geq 1$. Choose a value of N strictly larger than 2 and plot a histogram of Z_N comparing it with your theoretical calculation. According to the central limit theorem if $\mathbb{E}(X) < \infty$ and $\mathrm{Var}(X) < \infty$, the variable

$$U_N = \frac{Z_N - N\mathbb{E}(X)}{\sqrt{N\mathrm{Var}(X)}} \tag{3.2}$$

should converge in distribution to a standard normal random variable for $N \to \infty$. Illustrate this result using your Monte Carlo simulation.
Marks: Approximately up to 40.

Important notice

Please, in your dissertation, always specify the version of R you have used for your code. I am currently using either R version 3.5.2 (2018-12-20) – “Eggshell Igloo” or R version 3.3.3 (2017-03-06) – “Another Canoe”. If you are using more recent versions, your program may not be compatible with my versions of R and give run-time errors so I need to know what version you were using.