Extending Defeasible Reasoning Beyond Rational Closure Literature Review

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ABSTRACT

The use of formal logic for problem-solving in artificial intelligence has a long history. Monotonic algorithms have traditionally been used for AI reasoning, which means that adding new knowledge does not retract any previous inferences. However, when knowledge contains exceptions to stated facts, the entire knowledge base may become unsatisfiable. Nonmonotonic reasoning, which includes defeasible reasoning, can withdraw conclusions upon the addition of new information. The Kraus, Lehmann, and Magidor (KLM) framework is a prominent example of nonmonotonic reasoning. This review aims to discuss important aspects of the KLM framework, including propositional logic, preferential semantics, defeasible connectives, and nonmonotonic entailment relations. Additionally, the review describes recent extentions to the framework, which define a new class of defeasible entailment. By the end of this review, the reader should have a high level understanding of various approaches to defeasible entailment.

CCS CONCEPTS

• Theory of computation → Automated reasoning; **• Computing methodologies → Nonmonotonic, default reasoning and belief revision**.

KEYWORDS

artificial intelligence, knowledge representation and reasoning, defeasible reasoning

1 INTRODUCTION

Reasoning is a cognitive process that allows humans to form beliefs and make decisions based on available information. The field of knowledge representation and reasoning attempts to formalize these processes to better model and simulate human reasoning in machines.

Classical reasoning is a type of reasoning that uses logical inference to draw conclusions from a set of premises. However, it has limitations in the presence of exceptions, contradictions, and uncertainty, which are common in real-world knowledge. This is because classical reasoning assumes that every statement is either true or false, and the world is consistent. In reality, the world is complex and full of exceptions.

Defeasible reasoning acknowledges these exceptions by rejecting the concept of absolute truth in information. There are several approaches to defeasible reasoning, the first being *belief revision*, which involves revising beliefs based on new information, and identifying reasons to doubt such information. Another approach, *circumscription*, was developed by John McCarthy to capture the common-sense notion that things are as expected unless otherwise specified. *Default logic* is another approach that formalizes reasoning with default assumptions, such as "by default, something is true." A more recent approach, *propositional typicality logic*, extends classical propositional logic by allowing the notion of typicality to occur anywhere in a propositional statement, not just the antecedent. Lastly, *autoepistemic logic* permits the expression of not only knowledge, but also a lack thereof.

In the late 20th century, Kraus, Lehmann, and Magidor made significant contributions to the field of preferential reasoning, first introduced by Shoham [13, 14]. Building on this foundation, they expanded the scope of preferential reasoning by defining additional approaches to defeasible reasoning. Their work culminated in the development of the KLM framework, which has become a widely recognized framework for defeasible reasoning [2], and will be the primary focus of this review.

In this review, we will begin by providing an overview of propositional logic, which forms a foundation for many reasoning systems, including the KLM framework. We will then delve into the framework's contributions to preferential reasoning. After which we will explore two extensions that the KLM framework introduces to preferential reasoning, namely rational and lexicographic closure. Lastly, we will discuss recent refinements to the framework proposed by T Meyer, G Casini, and I Varzinczak [2].

2 PROPOSITIONAL LOGIC

Natural languages, such as English, are expressive, but not suitable for formal reasoning. They require long and complex paragraphs that are hard to handle mathematically. Propositional logic offers a way of representing complex ideas with simple symbols. It converts statements, called propositions, in natural language into symbols that follow a set of rules. This allows us to symbolize and deduce the relationships between propositions and verify them mathematically.

2.1 Syntax

At the center of propositional logic are *atoms*, or statements, which serve as fundamental units of knowledge [12]. Atoms are indivisible, and are expressed here in both typewriter text and as letters of the Latin alphabet. In other words, "Birds can fly" might be symbolized as `bird`, or b, while retaining the same meaning as the original statement. The symbol P will be used to denote the set of all propositional atoms [1].

Atoms alone are insufficient for performing logical reasoning, and so a mechanism that demonstrates their relationships is required. This is achieved through the use of Boolean operators, also known as connectives. The set of Boolean operators relevant to this review are, in order of precedence: negation, conjunction, disjunction, implication, and equivalence. These operators are respectively denoted as $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.

Atoms are combined with connectives to create formulas, denoted by lower case Greek letters such as α, β, and γ. A formula can be defined recursively as p, $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, or $\alpha \leftrightarrow \beta$. Where p is a propositional statement and α, β are formulas in $\mathcal{L}$, the set of all *well formed formulas* [1]. Formulas allow us to create complex propositions for logical reasoning and inference.

2.2 Semantics

The semantics of propositional logic provide a precise definition of truth. In ordinary arithmetic, a statement such as $x + y = z$ is neither true nor false until specific values are assigned to x, y, and z. Propositional logic relies on assignments that map to truth values, where satisfaction describes the situation where a statement is true in that specific assignment.

Valuations, also known as worlds or interpretations, assign truth values to atoms [1]. These valuations are represented by the Latin letters u, v, and w. A valuation can be expressed as a sequence of atoms such as $\mathsf{p\bar{q}r}$, where p and r are true and q is false. When an atom, or formula, α is true in some valuation u, it is said to be satisfied. This is expressed as $u \Vdash \alpha$.

Let $\mathcal{U}$ be the set of all valuations. To refer only to those valuations that satisfy a formula α, we use the notation $\hat{\alpha}$, where any valuation $u \in \hat{\alpha}$ is called a *model* of α. A *tautology*, denoted $\top$, is a formula α such that $u \Vdash \alpha$ holds for all valuations $u \in \mathcal{U}$. α is said to be *satisfiable* if there exists *any* valuation $u \in \mathcal{U}$, such that $u \Vdash \alpha$.

If for every valuation $u \in \mathcal{U}$ such that $u \Vdash \alpha$, then $u \Vdash \beta$, we say that β is a logical consequence of α, denoted by $\alpha \models \beta$. We use $\alpha \equiv \beta$ to denote that two formulas α and β are logically equivalent if they are both logical consequences of each other, that is, if both $\alpha \models \beta$ and $\beta \models \alpha$ [7].

2.3 Object and Meta Levels

An important distinction between the object-level and meta-level must be made. The former is any part of the language used to model knowledge, and the latter is anything that operates over the object-level. The symbols used to define object-level concepts are different from those used for meta-level concepts, with connectives and propositional atoms being object-level, and symbols like logical consequence and equivalence being meta-level. Mistaking meta-level concepts as being object-level can cause errors, such as adding meta-level inferences to object-level knowledge.

3 DEFEASIBLE REASONING

Let $\mathcal{K}$ be a knowledge base with the following statements:

- `bird` $\rightarrow$ `flies`
- `bird` $\rightarrow$ `wings`
- `penguin` $\rightarrow$ `bird`
- `penguin` $\rightarrow$ `¬flies`

Using classical deduction, one might infer that penguins both fly and do not fly, or that penguins do not exist. This is because penguins are birds, and birds fly by definition, so penguins should also fly by modus ponens. This shows the problem of handling exceptional information in classical logic, which is due to its monotonic nature.

3.0.1 *Monotonicity and Nonmonotonicity*.

Monotonicity is a key concept in defeasible reasoning, and essentially states that any inference that follows from a set of statements also follows from any superset of those statements [15]. In other words, adding more statements to a knowledge base can only increase, but never retract, the set of possible inferences. This violates the common sense intuition that penguins do not fly, and that some statements can be overridden or revoked by other statements in certain situations. This notion is what we call defeasible entailment, and demonstrates why any valid approach to defeasible reasoning must be nonmonotonic in nature.

3.1 Preferential Reasoning

According to Kraus, Lehmann and Magidor, nonmonotonic logics should be able to state “an x is typically a y”, where “typically” means “in normal cases, y is a reasonable conclusion from x”. This idea forms the basis for a class of reasoning in which preferences or rankings are assigned to statements, according to how exceptional or plausible they are.

3.1.1 *Preferential Consequence Relations*.

Consequence relations are structures that attempt to capture patterns of logical reasoning by describing relationships that hold true when one statement logically follows from another. A *preferential* consequence relation, $\mid\!\sim$, is a set of ordered pairs: $\{(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n)...\}$. Where each pair is a *defeasible implication*, and can be understood as "α implies β, in the absence of refuting evidence". For a preferential consequence relation $\mid\!\sim$, $\alpha \mid\!\sim \beta$ iff (α, β) belongs to it.

3.1.2 *KLM Postulates*.

The KLM postulates are a set of well-known structural inference rules used to define preferential consequence relations. Each rule contains a number of premises (above), and a single conclusion (below). Each postulate states that if the premises hold, and are in the consequence relation, then the same must apply for the conclusion. For a consequence relation $\mid\!\sim$ to be classified as preferential, it must satisfy each of the following postulates [9]:

$$\text{(RW)}\ \frac{\top \models \alpha \rightarrow \beta, \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta}\ \text{(CM)}\ \frac{\alpha \mid\!\sim \gamma, \alpha \mid\!\sim \beta}{\alpha \wedge \beta \mid\!\sim \gamma}\ \text{(And)}\ \frac{\alpha \mid\!\sim \beta, \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma}$$

$$\text{(LLE)}\ \frac{\top \models \alpha \leftrightarrow \beta, \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}\ \text{(Or)}\ \frac{\alpha \mid\!\sim \gamma, \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}\ \ \text{(Ref)}\ \alpha \mid\!\sim \alpha$$

Right weakening (RW), implies that the classical logic underlying a preferential consequence relation dominates the defeasible reasoning. RW ensures that defeasible conclusions are logically consistent, and necessitates that their classical consequences can also be defeasibly inferred from the original premises.

Cautious monotonicity (CM), ensures that any conclusion previously derived remains valid if redundant information is introduced.

And implies that a defeasible consequence follows from the combination of two defeasible consequences.

Left logical equivalence (LLE), asserts that if two statements are clasically equivalent, then another statement should follow from either both of them or neither of them. LLE reflects the impact that classical logic has on a preferential consequence relation.

Or states that a formula is a defeasible consequence of the disjunction of two formulas that can defeasibly entail it.

Reflexivity (Ref), is a classical property of reasoning stating that any formula is a preferential consequence of itself.

3.1.3 ***Preferential interpretations***.

Preferential reasoning orders valuations by preference or typicality: an agent considers a more preferred or typical valuation u before a less preferred or typical valuation v. Preference is captured by states, which are distinct from valuations. A state maps to a valuation and indicates its typicality level. Hence, multiple states can map to the same valuation.

A preferential interpretation $\mathcal{P}$, is defined as a triple $\langle S, l, \prec \rangle$, where S represents a set of possible states and l is a function that maps each state in S to a valuation in $\mathcal{U}$ [8]. The symbol $\prec$ denotes a strict partial order on S, which dictates how states are ranked according to preference. The notation $[\![\alpha]\!]^{\mathcal{P}}$ represents the set of all states in $\mathcal{P}$ for which the associated valuation satisfies α.

A state s is said to be minimal in $\mathcal{P}$ iff there exists no other state s' such that $s' \prec s$. $\mathcal{P}$ then defines a preferential consequence relation, $\mid\!\sim_{\mathcal{P}}$, where $\alpha \mid\!\sim_{\mathcal{P}} \beta$, iff every minimal state in $[\![\alpha]\!]^{\mathcal{P}}$ satisfies β.

3.1.4 ***Ranked Interpretations***.

Ranked interpretations are a specific type of preferential interpretation where $\prec$ must satisfy certain conditions. If $\prec$ is a strict partial order on some set $\mathcal{V}$, then for any $x, y, z \in \mathcal{V}$:

- If x and y are incomparable, and $z \prec x$, then $z \prec y$.
- If $x \prec y$, then either $z \prec y$, or $x \prec z$.

These conditions imply the existence of a totally ordered set Ω, with strict order $<$, and ranking function $r : \mathcal{V} \mapsto \Omega$ that preserves the order between elements in $\mathcal{V}$ [9]. As a result, in ranked interpretations (also known as *modular* interpretations), any two states that cannot be ranked relative to each other will share the same rank.

Modularity introduces a useful property to ranked interpretations. If two states map to the same valuation, then one of the following can occur without losing or gaining any inferences: if one state is preferred over the other according to the preference ordering, it can be removed; if both states have the same rank, either can be removed.

2	pbf
1	$\overline{p}b\overline{f}$ $pb\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$ $\overline{p}\overline{b}f$ $\overline{p}bf$

Figure 1: Ranked interpretation for $\mathcal{P} = \{\mathrm{p}, \mathrm{b}, \mathrm{f}\}$

This indicates that ranked interpretations need not allow duplicate states, as is the case with preferential interpretations. This allows us to build a more formal definition: A ranked interpretation is a function $\mathcal{R}\colon \mathcal{U} \mapsto \mathbb{N} \cup \{\infty\}$, which ensures that ranks are assigned in a convex manner (without gaps) [2]. Lower ranks are more typical, while valuations with an infinite rank are impossible.

Ranked interpretations also classify as preferential interpretations, and should therefore inherit the same properties. For any formula α, a valuation $u \in [\![\alpha]\!]^{\mathcal{R}}$ is minimal iff there exists no other valuation $v \in [\![\alpha]\!]^{\mathcal{R}}$, such that $\mathcal{R}(v) < \mathcal{R}(u)$.

3.1.5 ***Rational Consequence Relations***.

Ranked interpretations define *rational* consequence relations, which are similar to preferential consequence relations, in that they must satisfy all KLM postulates. However, rational consequence relations must satisfy an additional property, *rational monotonicity* [9]:

$$(\mathrm{RM})\ \frac{\alpha \mid\!\sim \gamma, \alpha \not\mid\!\sim \neg\beta}{\alpha \wedge \beta \mid\!\sim \gamma}$$

Rational monotonicity is a principle stating that new information should not invalidate a conclusion unless it directly contradicts existing information or inferences. This differs from cautious monotonicity, which holds that conclusions should never be invalidated by already inferred knowledge. The key difference between the two lies in the fact that rational monotonicity allows for more speculative inferences. In other words, it permits the addition of new information without invalidating existing conclusions, provided that it does not directly contradict them.

Rational monotonicity is a useful alternative to monotonicity in nonmonotonic systems, and allows for more intuitive inferences. For example, let's say we have a defeasible implication $\alpha \mid\!\sim \beta$. According to rational monotonicity, if we add new information γ to our system, our original inference should remain unchanged unless there is evidence to suggest otherwise. In other words, we can infer $\alpha \wedge \gamma \mid\!\sim \beta$.

A ranked interpretation defines a rational consequence relation, $\mid\!\sim_{\mathcal{R}}$, where $\alpha \mid\!\sim_{\mathcal{R}} \beta$ iff every minimal valuation in $[\![\alpha]\!]^{\mathcal{R}}$ satisfies β.

3.1.6 ***Preferential Entailment***.

Defining entailment with preferential semantics requires determining what inferences follow from defeasible information. This involves treating $\mid\!\sim$ as an object-level connective for specifying defeasible information, instead of a meta-level consequence relation. The connective $\mid\!\sim$, which represents the defeasible counterpart to $\rightarrow$, will now be included in $\mathcal{L}$. Note that $\mid\!\sim$ can not be nested.

A preferential interpretation $\mathcal{P}$ satisfies a defeasible implication $\alpha \mid\!\sim \beta$ iff for every minimal state s in $[\![\alpha]\!]^{\mathcal{P}}$, $s \Vdash \beta$. In this case, $\mathcal{P}$ is considered a *model* of $\alpha \mid\!\sim \beta$. $\mathcal{P}$ satisfies a defeasible knowledge base $\mathcal{K}$ iff for every $\alpha \mid\!\sim \beta \in \mathcal{K}$, $\mathcal{P} \Vdash \alpha \mid\!\sim \beta$. When $\mathcal{P} \mid\!\sim \mathcal{K}$, it is considered a model of $\mathcal{K}$. A formula α is satisfied by $\mathcal{P}$ iff for all states $s \in \mathcal{P}$, $s \Vdash \alpha$.

Entailment, also known as logical consequence, is a meta-level connective that was previously represented by the symbol $\models$. In defeasible logic, it will be represented by the symbol $\mid\!\approx$. *Preferential* entailment, denoted by $\mid\!\approx_P$, is defined as follows: $\mathcal{K} \mid\!\approx_P \alpha \mid\!\sim \beta$ iff for every preferential model of $\mathcal{K}$, $\mathcal{P} \Vdash \alpha \mid\!\sim \beta$.

It is now necessary to reformulate the KLM postulates using $\mid\!\approx$ as the meta-level notion of consequence due to $\mid\!\sim$ being an object-level connective [7]:

$$(\mathrm{RW})\ \frac{\mathcal{K} \mid\!\approx \alpha \rightarrow \beta, \mathcal{K} \mid\!\approx \gamma \mid\!\sim \alpha}{\mathcal{K} \mid\!\approx \gamma \mid\!\sim \beta} \qquad (\mathrm{CM})\ \frac{\mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma, \mathcal{K} \mid\!\approx \alpha \mid\!\sim \beta}{\mathcal{K} \mid\!\approx \alpha \wedge \beta \mid\!\sim \gamma}$$

$$(\mathrm{LLE})\ \frac{\mathcal{K} \mid\!\approx \alpha \leftrightarrow \beta, \mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma}{\mathcal{K} \mid\!\approx \beta \mid\!\sim \gamma} \qquad (\mathrm{Or})\ \frac{\mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma, \mathcal{K} \mid\!\approx \beta \mid\!\sim \gamma}{\mathcal{K} \mid\!\approx \alpha \vee \beta \mid\!\sim \gamma}$$

$$(\mathrm{And})\ \frac{\mathcal{K} \mid\!\approx \alpha \mid\!\sim \beta, \mathcal{K} \mid\!\approx \alpha \mid\!\sim \gamma}{\mathcal{K} \mid\!\approx \alpha \mid\!\sim \beta \wedge \gamma} \qquad (\mathrm{Ref})\ \mathcal{K} \mid\!\approx \alpha \mid\!\sim \alpha$$

Accordingly, rational monotonicity also holds:

$$\text{(RM)}\ \frac{\mathcal{K} \approx \alpha \mid\!\sim \gamma, \mathcal{K} \not\approx \alpha \mid\!\sim \neg\beta}{\mathcal{K} \approx \alpha \wedge \beta \mid\!\sim \gamma}$$

These properties provide a framework for deductive reasoning in preferential entailment. This means that given a knowledge base $\mathcal{K}$, we can use them as inference rules to derive all statements that are preferentially entailed by $\mathcal{K}$. In other words, every statement that is preferentially entailed by $\mathcal{K}$ can be proven using the KLM postulates [9]. Any defeasible entailment relation satisfying each of these properties is referred to as *LM-rational*.

3.2 Rational Closure

Rational closure is an extension of preferential reasoning, and provides a more comprehensive approach to defeasible entailment. It captures the idea that we can reason with general rules unless there are specific exceptions or contradictions. For example, we can infer that a bird flies unless it is a penguin or it has a broken wing. Rational closure allows us to handle such cases without giving up on the usefulness of general rules.

3.2.1 ***Minimal Ranked Entailment.*** Defining entailment based on ranked interpretations yields an interesting result. Specifically, a knowledge base $\mathcal{K}$ rank entails a defeasible implication $\alpha \mid\!\sim \beta$ iff every ranked interpretation $\mathcal{R}$ that satisfies $\mathcal{K}$ also satisfies $\alpha \mid\!\sim \beta$ [7]. This is exactly the same as preferential entailment.

Ranked entailment is therefore not LM-rational, which was shown by Lehmann [9], and is consequently also monotonic. This leads to the question of how to define semantics for a nonmonotonic entailment relation. The solution is *minimal* ranked entailment.

Let $\preceq_{\mathcal{K}}$ be a partial ordering on all ranked models of $\mathcal{K}$ such that: $\mathcal{R}_1 \preceq_{\mathcal{K}} \mathcal{R}_2$ if for every valuation u, $\mathcal{R}_1(u) \leq \mathcal{R}_2(u)$ [2]. This places the more typical ranked models of $\mathcal{K}$ lower down in the ordering. It has been shown that there is a unique $\preceq_{\mathcal{K}}$-minimal element [6], denoted here as $\mathcal{R}^{\mathcal{K}}_{RC}$.

The minimal ranked interpretation satisfying $\mathcal{K}$ gives rise to a rational entailment relation known as minimal ranked entailment, denoted as $\approx_{RC}$. More formally, $\mathcal{K} \approx_{RC} \alpha \mid\!\sim \beta$ holds iff $\mathcal{R}^{\mathcal{K}}_{RC} \Vdash \alpha \mid\!\sim \beta$. Alternatively, we can say that $\alpha \mid\!\sim \beta$ belongs to the rational closure of $\mathcal{K}$ [2]. Minimal ranked entailment therefore provides a semantic characterization of rational closure. In the subsequent sections, we will explore a complementary, syntactic approach to defining rational closure.

3.2.2 ***Materialization.*** In a defeasible knowledge base $\mathcal{K}$, each defeasible implication has a corresponding propositional formula $\alpha \rightarrow \beta$, known as its *material counterpart*. The set of all such counterparts for $\mathcal{K}$ is denoted as $\overrightarrow{\mathcal{K}}$. Materialization is used to find classical formulas satisfied by preferred valuations of $\mathcal{K}$: $\overrightarrow{\mathcal{K}} \models \alpha$ iff $\mathcal{K} \approx_P \top \mid\!\sim \alpha$. A propositional formula is then exceptional for $\mathcal{K}$, iff its negation is entailed by $\mathcal{K}$.

3.2.3 ***Exceptionality.*** An exceptional formula represents a statement that is false in more typical worlds described by $\mathcal{K}$, but may be true in those that are less typical. We define ϵ, as a function that returns a subset of $\mathcal{K}$, containing all defeasible statements with exceptional antecedents: $\epsilon(\mathcal{K}) := \{\alpha \mid\!\sim \beta \mid \mathcal{K} \approx_P \top \mid\!\sim \neg\alpha\}$ [9]. This function can identify statements that deviate from the norm, but it does not measure how much they deviate.

3.2.4 ***Base Rank.*** Starting with the original knowledge base $\mathcal{K}$, which in this context will be denoted as $\mathcal{E}^{\mathcal{K}}_0$. Each statement that is not considered exceptional is removed using the function ϵ, the result is denoted as $\mathcal{E}^{\mathcal{K}}_1$. This process is repeated until the resulting knowledge base contains only exceptional statements, which is denoted as $\mathcal{E}^{\mathcal{K}}_\infty$.

In other words, for each i, where $0 < i < n$, we have $\mathcal{E}^{\mathcal{K}}_i = \epsilon(\mathcal{E}^{\mathcal{K}}_{i-1})$. And for n, where n is the smallest value of k such that $\mathcal{E}^{\mathcal{K}}_k = \mathcal{E}^{\mathcal{K}}_{k+1}$, we have $\mathcal{E}^{\mathcal{K}}_n = \mathcal{E}^{\mathcal{K}}_\infty$. Given that the knowledge base contains no universally exceptional formulas, $\mathcal{E}^{\mathcal{K}}_\infty = \emptyset$.

The base rank of α, $br_{\mathcal{K}}(\alpha)$, is the lowest r for which $\mathcal{E}^{\mathcal{K}}_r$ does not make α exceptional. The base rank of any defeasible implication is equivalent to that of its antecedent: $br_{\mathcal{K}}(\alpha \mid\!\sim \beta) \equiv br_{\mathcal{K}}(\alpha)$. For two formulas $\alpha, \beta \in \mathcal{L}$, if $br_{\mathcal{K}}(\alpha) < br_{\mathcal{K}}(\beta)$, then α is more general and defeasible than β [6]. Defeasible implications with infinite rank are classical and true in all valuations and ranked models of $\mathcal{K}$.

3.2.5 ***Defeasible Entailment.*** Rational closure ranks statements in a knowledge base by typicality. It checks if a defeasible implication is true by comparing two ranks: the condition, and the condition with the consequence being negated. If the condition is lower than the other (or infinite), then the implication is true. This deals with exceptions and contradictions.

Rational closure relies on the base rank function to measure the exceptionality of a defeasible implication, $\alpha \mid\!\sim \beta$, in $\mathcal{K}$. If $br_{\mathcal{K}}(\alpha) < br_{\mathcal{K}}(\alpha \wedge \neg\beta)$ or $br_{\mathcal{K}}(\alpha) = \infty$, then $\alpha \mid\!\sim \beta$ is entailed by $\mathcal{K}$ under rational closure [6]. Which is formally expressed as $\mathcal{K} \approx_{RC} \alpha \mid\!\sim \beta$.

3.3 Lexicographic Closure

Lexicographic closure is an alternative extension to preferential reasoning, and is a subsumption of rational closure. While rational closure considers only the minimal rank of possible worlds. Lexicographic closure provides a more sophisticated form of defeasible entailment by selecting the most preferred possible worlds, based on their degree of normality.

Lexicographic closure was introduced by Lehmann as a method for capturing a pattern of reasoning that we have not yet encountered in this review, namely, *default* reasoning [3, 5, 10, 11].

3.3.1 ***Properties of Default Reasoning.*** Default reasoning can be characterized by a set of informal and intuitive properties [7]:

(1) Presumption of typicality. This principle complements rational monotonicity, which *guides* us in choosing among multiple possible conclusions from incomplete information. Presumption of typicality *restricts* our inferences to the most typical situations that are consistent with the available information. It does this by asserting that typicality should be assumed unless there is evidence to the contrary.

(2) Presumption of independence. Presumption of typicality implies a proclivity toward monotonicity. Presumption of independence supports this by assuming that statements are unrelated unless there is evidence of a connection. This is achieved by presuming typicality for each consequent until proven otherwise.

(3) Priority to typicality. Sometimes a term can have several properties that are not related to each other. This can create a problem when we have to choose between what is typical and what

is independent. To resolve this conflict, priority to typicality favors presumption of typicality, over presumption of independence.

(4) Respect for specificity. When two inferences clash due to different defeasible implications, respect for specificity states that we should choose the one with the more specific antecedent.

3.3.2 ***Seriousness.*** In rational closure, the base rank function is used to measure the degree of exceptionality for each statement in a knowledge base. We now define an additional concept, called the *order* of a knowledge base, denoted by k. Which is the highest rank assigned to any statement in $\mathcal{K}$, excluding ∞. *Seriousness* is another closely related concept, and will be described next.

The size of the knowledge base, and the specificity of its statements, are used as a measure of seriousness [10]. Violating a smaller subset is less serious than violating a larger subset. Violating a statement with a lower base rank, which measures specificity, is less serious than violating a statement with a higher base rank. Lexicographic closure composes these two measures, where specificity overrides size if they conflict.

To measure the seriousness of any subset of defeasible implications, D, in $\mathcal{K}$, a $k+1$ tuple can be assigned to it, based on its order. This tuple counts how many statements in D have a certain base rank. The first element, n_0, is the number of statements with a base rank of ∞. While the ith element, n_i, is the number of statements with base rank $k-i$.

3.3.3 ***Ranked Interpretation.*** The ranked interpretation for lexicographic closure, $\mathcal{R}_{LC}$, requires finding a modular ordering, $\prec_{LC}$, over a set of valuations. However, some valuations may violate certain subsets of defeasible implications [7]. Lexicographic closure ranks these subsets by measuring their seriousness.

We denote the tuple corresponding to subset D_1 as n_{D_1}, and compare it with any other subset D_2 in $\mathcal{K}$. This gives a strict modular partial ordering, $\prec_s$, based on lexicographic order: we compare each element from left to right. If at any point n_{D_1} has a higher value than n_{D_2}, then we prefer D_1 over D_2, written as $D_2 \prec_s D_1$ [10].

The resulting ranking corresponds to the properties of seriousness mentioned earlier because it compares two subsets by looking at how many formulas they have at each rank. A subset is considered more serious if it has more formulas at a lower rank, and the same number of formulas at higher ranks.

We can now use this seriousness ordering to construct a ranked interpretation that corresponds to lexicographic closure, which will be denoted as $\mathcal{R}^{\mathcal{K}}_{LC}$. Given two valuations m and n, the preference order $\prec_{LC}$ can be defined as follows: $m \prec_{LC} n$ iff $V(m) \prec_s V(n)$, where $V(m)$ is a function that returns the set of defeasible implications violated by m.

5	$\mathrm{pbf}\overline{\mathrm{w}}$				
4	pbfw				
3	$\mathrm{pb}\overline{\mathrm{f}}\overline{\mathrm{w}}$	$\overline{\mathrm{p}}\mathrm{b}\overline{\mathrm{f}}\overline{\mathrm{w}}$			
2	$\mathrm{pb}\overline{\mathrm{f}}\mathrm{w}$	$\overline{\mathrm{p}}\mathrm{b}\overline{\mathrm{f}}\mathrm{w}$			
1	$\overline{\mathrm{p}}\mathrm{bf}\overline{\mathrm{w}}$				
0	$\overline{\mathrm{p}}\overline{\mathrm{b}}\overline{\mathrm{f}}\overline{\mathrm{w}}$	$\overline{\mathrm{p}}\overline{\mathrm{b}}\overline{\mathrm{f}}\mathrm{w}$	$\overline{\mathrm{p}}\overline{\mathrm{b}}\mathrm{f}\overline{\mathrm{w}}$	$\overline{\mathrm{p}}\overline{\mathrm{b}}\mathrm{fw}$	$\overline{\mathrm{p}}\mathrm{bfw}$

Figure 2: Lexicographic ranked model, $\mathcal{R}^{\mathcal{K}}_{LC}$, for $\mathcal{K}$.

3.3.4 ***Defeasible Entailment.*** The ranked interpretation $\mathcal{R}^{\mathcal{K}}_{LC}$ can be used to define an entailment relation for lexicographic closure: given a defeasible implication $\alpha \mid\sim \beta$, it holds that $\mathcal{K} \mid\approx_{LC} \alpha \mid\sim \beta$ iff $\mathcal{R}^{\mathcal{K}}_{LC} \Vdash \alpha \mid\sim \beta$.

An alternative theorem for lexicographic closure can be defined by introducing an additional concept, that follows from the seriousness ordering described earlier. The *basis*, denoted B, of some formula α, is a subset of $\mathcal{K}$, where its material counterpart is consistent with α, $\overrightarrow{B} \models \neg\alpha$. Additionally, there must not exist any B' such that $B \prec_s B'$ [10].

Given a defeasible implication $\alpha \mid\sim \beta$, $\mathcal{K} \mid\approx_{LC} \alpha \mid\sim \beta$ iff for any basis of α, $\overrightarrow{B} \cup \{\alpha\} \models \beta$.

3.4 Basic Defeasible Entailment

We have shown that both rational and lexicographic closure can be defined by a ranked interpretation, and as such are both considered LM-rational. Various other defeasible entailment relations have also been shown to be LM-rational, however, the KLM postulates are not restrictive enough to determine which ranked models of a knowledge base are useful. Basic defeasible entailment aims to extend the postulates so that any common sense pattern of reasoning can be formalized as a defeasible entailment relation.

3.4.1 ***Basic Properties of Defeasible Entailment.*** Basic defeasible entailment introduces three additional properties that should be inferred by a defeasible entailment relation, $\mid\approx$, at the very least [2]:

(1) Inclusion. The inclusion property indicates that if a knowledge base $\mathcal{K}$ has a defeasible implication, then $\mathcal{K}$ should also have a defeasible entailment of that implication. In other words, for every $\alpha \mid\sim \beta \in \mathcal{K}, \mathcal{K} \mid\approx \alpha \mid\sim \beta$.

(2) Classic Preservation. Classic preservation requires that the defeasible implications corresponding to classical sentences should be the same as those preferentially entailed in $\mathcal{K}$. In other words, $\mathcal{K} \mid\approx \alpha \mid\sim \bot$ iff $\mathcal{K} \mid\approx_P \alpha \mid\sim \bot$.

(3) Classic Consistency. Classic consistency states that if a knowledge base is satisfiable with respect to preferential entailment, then it is also satisfiable with respect to classic preservation. In other words, $\mathcal{K} \mid\approx \top \mid\sim \bot$ iff $\mathcal{K} \mid\approx_P \top \mid\sim \bot$.

A basic defeasible entailment relation is any relation that satisfies LM-rationality, Inclusion, Classic Preservation, and Classic Consistency [2].

3.4.2 ***$\mathcal{K}$-faithfulness.*** A ranked model, $\mathcal{R}$, of a defeasible knowledge base $\mathcal{K}$, is *$\mathcal{K}$-faithful* if the set of valuations in $\mathcal{R}$ with a non-infinite rank is equivalent to the set of possible valuations for $\mathcal{K}$. In other words, for every $u \in \mathcal{U}$, $\mathcal{R}(u) \neq \infty$ iff $\mathcal{R}^{\mathcal{K}}_{R}(u) \neq \infty$.

$\mathcal{K}$-faithfulness forms a semantic characterisation of basic defeasible entailment, its entailment relations are defined by the set of $\mathcal{K}$-faithful ranked models [2]. The minimal $\mathcal{K}$-faithful ranked interpretation satisfying $\mathcal{K}$ is its rational closure, $\mathcal{R}^{\mathcal{K}}_{RC}$. This means that $\mathcal{R}^{\mathcal{K}}_{RC}$ also classifies as a basic defeasible entailment relation.

3.4.3 ***Ranking Function.*** We can define basic defeasible entailment by ranking formulas in $\mathcal{K}$ using a generalised version of the base rank function. $r : \mathcal{L} \mapsto \mathbb{N} \cup \{\infty\}$ assigns ranks in a convex manner, and gives a rank of 0 to any tautology [3].

r is called *entailment preserving* if it respects classical logic in ranking formulas: if β is a logical consequence of α, then $r(\beta)$ must not be greater than $r(\alpha)$.

For a ranking function to be $\mathcal{K}$-faithful, it must *(1)* be entailment preserving. *(2)* For any $\alpha \in \mathcal{L}$ such that $\neg\alpha$ is a tautology in every ranked model of $\mathcal{K}$, then $r(\alpha) = \infty$. *(3)* For every defeasible implication $\alpha \mid\!\sim \beta \in \mathcal{K}$, then either $r(\alpha) < r(\alpha \wedge \neg\beta)$ or $r(\alpha) = \infty$.

2	$\mathrm{pbf}\overline{\mathrm{w}}$ pbfw
1	$\overline{\mathrm{p}}\mathrm{b}\overline{\mathrm{f}}\overline{\mathrm{w}}$ $\overline{\mathrm{p}}\mathrm{b}\overline{\mathrm{f}}\mathrm{w}$ $\mathrm{pb}\overline{\mathrm{f}}\overline{\mathrm{w}}$ $\mathrm{pb}\overline{\mathrm{f}}\mathrm{w}$
0	$\overline{\mathrm{p}}\overline{\mathrm{bf}}\overline{\mathrm{w}}$ $\overline{\mathrm{p}}\overline{\mathrm{bf}}\mathrm{w}$ $\overline{\mathrm{p}}\overline{\mathrm{b}}\mathrm{f}\overline{\mathrm{w}}$ $\overline{\mathrm{p}}\overline{\mathrm{b}}\mathrm{fw}$ $\overline{\mathrm{p}}\mathrm{bf}\overline{\mathrm{w}}$ $\overline{\mathrm{p}}\mathrm{bfw}$

Figure 3: Minimal $\mathcal{K}$-faithful ranked model, $\mathcal{R}^{\mathcal{K}}_{RC}$, of $\mathcal{K}$.

3.4.4 *Defeasible Entailment.*

The ranking of a formula in lexicographic closure is a reflection of its base rank. The lower the rank, the more specific the statement it represents. However, while the base rank from rational closure represents a result, the general rank from some r is more declarative. It encodes a pattern of reasoning that directly results in an entailment relation [7].

3.5 Rational Defeasible Entailment

Rational closure represents the smallest set of conclusions that can be drawn from a knowledge base, and therefore reflects the minimum entailment by a nonmonotonic system. This strongly argues that rational closure should be the nonmonotonic core of defeasible entailment. However, the defeasible implications entailed by rational closure are not guaranteed to be inferred by basic defeasible entailment. This leads to a new class of defeasible entailment, *rational defeasible entailment*, formulated around rational closure.

3.5.1 *Properties of Rational Defeasible Entailment.*

Rational defeasible entailment can be described by three addititional properties [7]:

(1) RC Extension. Rational closure extension states that a defeasible entailment relation should extend rational closure. A defeasible entailment relation, $\approx$, satisfies RC extension iff for every situation that $\mathcal{K} \models\!\approx_{RC} \alpha \mid\!\sim \beta$, then $\mathcal{K} \models\!\approx \alpha \mid\!\sim \beta$.

(2) Rank Preservation. Rank preservation indicates that for a ranked model to be rank preserving, it must respect the relative rankings between each pair of valuations defined in rational closure. In other words, a ranked interpretation, $\mathcal{R}$, is rank preserving if for all $v, u \in \mathcal{U}$, it is the case that if $\mathcal{R}^{\mathcal{K}}_{RC}(v) < \mathcal{R}^{\mathcal{K}}_{RC}(u)$, then also $\mathcal{R}^{\mathcal{K}}(v) < \mathcal{R}^{\mathcal{K}}(u)$.

(3) Base Rank Preservation. A *base rank preserving* ranking function must respect the relative rankings between formulas assigned by the base rank function. In other words, r is said to be base rank preserving if for all $\alpha, \beta \in \mathcal{L}$, $br_{\mathcal{K}}(\alpha) < br_{\mathcal{K}}(\beta)$, then also $r(\alpha) < r(\beta)$ [2].

3.5.2 *Defeasible Entailment.*

A basic defeasible entailment relation that satisfies rational closure extension can also be classified as a rational defeasible entailment relation.

For a rational defeasible entailment relation $\models\!\approx$, there must exist a rank preserving $\mathcal{K}$-faithful ranked interpretation $\mathcal{R}$, and a base rank preserving $\mathcal{K}$-faithful rank function r, such that $r(\alpha) = \min\{i | v \in [\![\alpha]\!]$ and $\mathcal{R}(v) = i\}$ [6], and $\models\!\approx$ can be generated from either $\mathcal{R}$ or r.

7	$\mathrm{pbf}\overline{\mathrm{w}}$
6	pbfw
5	$\mathrm{pb}\overline{\mathrm{f}}\overline{\mathrm{w}}$
4	$\overline{\mathrm{p}}\mathrm{b}\overline{\mathrm{f}}\overline{\mathrm{w}}$
3	$\mathrm{pb}\overline{\mathrm{f}}\mathrm{w}$
2	$\overline{\mathrm{p}}\mathrm{b}\overline{\mathrm{f}}\mathrm{w}$
1	$\overline{\mathrm{p}}\mathrm{bf}\overline{\mathrm{w}}$
0	$\overline{\mathrm{p}}\overline{\mathrm{bf}}\overline{\mathrm{w}}$ $\overline{\mathrm{p}}\overline{\mathrm{bf}}\mathrm{w}$ $\overline{\mathrm{p}}\overline{\mathrm{b}}\mathrm{f}\overline{\mathrm{w}}$ $\overline{\mathrm{p}}\overline{\mathrm{b}}\mathrm{fw}$ $\overline{\mathrm{p}}\mathrm{bfw}$

Figure 4: Ranked model, $\mathcal{R}$, of $\mathcal{K}$.

A rational defeasible entailment relation is therefore a basic defeasible entailment relation that satisfies the additional properties of being generated from a rank preserving $\mathcal{K}$-faithful ranked interpretation, or a base rank preserving $\mathcal{K}$-faithful rank function.

4 DISCUSSION

In this review, an important differentiation must be made between the various approaches to defeasible entailment. Each corresponds to one of two paradigms in defeasible reasoning: *prototypical reasoning*, and *presumptive reasoning*, as identified by Lehmann [10].

Figure 2 depicts the lexicographic closure of the knowledge base $\mathcal{K}$, defined earlier. This model illustrates presumptive reasoning, which states that the properties of a class are assumed to hold for all members of that class unless there is evidence to the contrary.

Figure 3 depicts the corresponding rational closure. This model exemplifies prototypical reasoning, which assumes that typical members of a class inherit the properties of that class, but atypical members may not.

Note that $\mathsf{p} \mid\!\sim \mathsf{w}$ is not part of the rational closure, but is included in the lexicographic closure. This reflects that penguins, being atypical birds, have no wings according to prototypical reasoning.

Although we have covered both semantic and syntactic definitions for various forms of defeasible entailment, it is worth noting that practical algorithms are necessary for computing ranked models and entailment. However, these algorithms have been omitted from this review as they are not specific to any particular approach. Moreover, there often exist multiple possible algorithms for certain computations. The correct choice of algorithm depends on the system being implemented.

The `BaseRank` and `RationalClosure` algorithms are frequently utilized for computing rational closure [2, 4]. On the other hand, the `Rank` and `DefeasibleEntailment` algorithms are commonly used for computing lexicographic closure, as well as basic and rational defeasible entailment [2, 7].

5 CONCLUSIONS

This review has explored the topic of logical reasoning, and focused primarily on the challenges associated with exceptional information. We presented propositional logic as a foundational mathematical system for reasoning, as well as its limitations in dealing with defeasible arguments. We then introduced the KLM framework, which strengthens preferential reasoning, a pattern of reasoning that ranks statements according to plausibility. We explained how this framework allows for rational closure, which preserves consistency and minimality, as well as lexicographic closure, which accounts for specificity and relevance. Finally, we discussed a new class of reasoning, basic and rational defeasible entailment, which aims to strengthen the KLM framework by addressing some of its shortcomings.

Through careful analysis, this review has demonstrated logical reasoning as a complex and ever-evolving field, that demands constant refinement and adaptation to tackle novel challenges and scenarios. While the approaches we have discussed are distinct, they are not mutually exclusive; rather, they are complementary and interconnected.

There is no universal solution, each approach presents a fundamental idea in defeasible reasoning, which must be modified and adapted to incorporate new ideas and alternative strategies. In doing so, we gain a more comprehensive understanding for what works best in specific situations, and grasp the strengths and limitations of each approach. Additionally, by merging the best aspects of each approach, we create a more robust toolset for analyzing and evaluating arguments in diverse domains and contexts.

Overall, the approaches we have discussed offer diverse perspectives and tools for comprehending, and tackling, challenges in logical reasoning. They emphasize the crucial role of continuous learning, personal development, and adaptation. These are crucial aspects of human cognition, which, as the most intricate reasoning systems, we are highly familiar with.

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