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Extending Defeasible Reasoning Beyond Rational Closure

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ABSTRACT

The KLM framework is a well known extension of classical logic for incorporating defeasible reasoning. Central to the KLM framework is rational closure, recognized as the most conservative approach to defeasible reasoning. Rational closure operates through two complementary paradigms: model-theoretic and formula-theoretic. This paper concentrates on the model-theoretic dimension, known as minimal ranked entailment. Unfortunately, the practical implementation of minimal ranked entailment remains largely impractical due to high computational and storage demands. To address this, we present reduced minimal ranked entailment, an optimization that employs reduced ordered binary decision diagrams to eliminate redundant information, thereby enhancing computational efficiency and reducing memory requirements. We further demonstrate that this optimized approach significantly facilitates the practical application and development of model-theoretic extensions of rational closure. This is illustrated through our own Bayesian refinement of minimal ranked entailment, which conceptualizes defeasible entailment as a form of conditional probability.

CCS CONCEPTS

• Theory of computation → Automated reasoning; **• Computing methodologies → Nonmonotonic, default reasoning and belief revision**.

KEYWORDS

artificial intelligence, knowledge representation and reasoning, defeasible reasoning, rational closure, baysian reasoning

1 INTRODUCTION

The Kraus, Lehmann, and Magidor (KLM) framework [13, 16, 17] serves as a prominent extension of classical logic, designed to accommodate defeasible reasoning. Within this framework, defeasible entailment is facilitated through a ranking system grounded in typicality, operating via two distinct methodologies: model-theoretic and formula-theoretic. The model-theoretic method ranks potential scenarios or "worlds," while the formula-theoretic method ranks individual statements. Presently, two primary forms of defeasible entailment are recognized: rational closure [17] and lexicographic closure [16], each with well-defined definitions in both model-theoretic and formula-theoretic settings.

This paper focuses primarily on the model-theoretic facet of rational closure, known as minimal ranked entailment [8, 11]. Regrettably, the practical implementation of any model-theoretic approach to defeasible entailment poses significant obstacles, chiefly due to elevated computational and memory requirements, rendering it largely impractical. To mitigate this, we present reduced minimal ranked entailment, an optimized version of minimal ranked entailment. This optimization utilizes reduced ordered binary decision diagrams [1, 6, 15] to purge redundant information, thereby achieving marked improvements in computational efficiency and storage capacity.

Rational closure is acknowledged as the most conservative variant of defeasible entailment, making it a compelling foundation for the exploration of more advanced forms, of which lexicographic closure is a primary example [8]. We further demonstrate that reduced minimal ranked entailment can aid substantially in the practical application and advancement of model-theoretic extensions of rational closure. We illustrate this by presenting naive bayesian entailment, a Bayesian refinement of minimal ranked entailment, which treats defeasible entailment as a form of conditional probability.

2 BACKGROUND

2.1 Propositional Logic

2.1.1 Syntax. Propositional logic focuses on fundamental units of knowledge known as propositions or *atoms*, represented here in typewriter text like `c` or `cat`. The set of all atoms is denoted as $\mathcal{P}$. Atoms combine with logical connectives to form *formulas*, symbolized by Greek letters such as α and β. Formulas are recursively defined as $\alpha ::= \top \mid \bot \mid p \mid \neg\alpha \mid \alpha \wedge \beta \mid \alpha \vee \beta \mid \alpha \rightarrow \beta \mid \alpha \leftrightarrow \beta$. The set of all such formulas constitutes the language of propositional logic, $\mathcal{L}$. A knowledge base $\mathcal{K}$ is a collection of statements in $\mathcal{L}$.

2.1.2 Semantics. Truth arises from *valuations*, which assign truth values to atoms in $\mathcal{P}$. Valuations are represented by lowercase letters like v, and visually depicted as sequences of atoms (e.g., p) and barred atoms (e.g., $\overline{\mathsf{p}}$), where the former indicates true and the latter false. When some formula α is true under some valuation v, it is said to be *satisfied* by v, denoted $v \Vdash \alpha$. Such a valuation is termed a *model* of α. The set of models for a formula α is denoted as $[\![\alpha]\!]$.

2.1.3 Classical Entailment. Logical consequence, or entailment, refers to the situation where the truth of one or more statements necessary leads to the truth of one or more other statements.

Definition 1. *A knowledge base $\mathcal{K}$ classically entails a formula α, denoted $\mathcal{K} \models \alpha$, iff $[\![\mathcal{K}]\!] \subseteq [\![\alpha]\!]$ [3].*

Verifying entailment using models is highly inefficient, the standard way to check entailment is known as reduction to unsatisfiability, and allows the use of a regular SAT solver. This works by conjoining all formulas in $\mathcal{K}$ with $\neg\alpha$, if the result is unsatisfiable (has no models), it indicates that $\mathcal{K} \models \alpha$.

2.2 KLM-Style Defeasible Reasoning

Consider the sea squirt. In its youth, this creature uses its basic brain to find a suitable spot—like a rock or coral—to attach to and filter-feed on whatever gunk drifts by. Upon attachment, it digests its brain, keeping only a simple nervous system. † It would be incorrect to categorically assert that sea squirts possess or lack brains. More appropriately, they *typically* do not.

Classical propositional logic, when applied to the above scenario, represented by the knowledge base $\mathcal{K} = \{\mathsf{y} \to \mathsf{s}, \mathsf{s} \to \neg\mathsf{b}, \mathsf{y} \to \mathsf{b}\}$, where y, s, and b correspond to `young sea squirt`, `sea squirt`, and `brain`, respectively, results in y being false in every model of $\mathcal{K}$. Kraus et al. [13], address this by introducing defeasible implication, expressed as $\alpha \mid\sim \beta$, meaning 'if α, then typically β'.

A defeasible knowledge base, $\mathcal{K}$, is composed of both classical propositional formulas and defeasible implications. The semantics of $\mid\sim$ comes from structures called ranked interpretations [17].

Definition 2. *A ranked interpretation is a function $\mathcal{R} : \mathcal{U} \mapsto \mathbb{N} \cup \{\infty\}$ with the property that for any $u \in \mathcal{U}$ where $\mathcal{R}(u) = i$, there must exist a $v \in \mathcal{U}$ such that $\mathcal{R}(v) = j$ for every $j < i$ [8].*

Classical formulas dictate which valuations are matched to either an integer in $\mathbb{N}$, or to ∞ (not possible), while defeasible implications delineate the precise integer ranking that should be assigned to a given, feasible valuation. The general understanding is that valuations with a lower ranking are considered more typical.

Definition 3. *A defeasible implication $\alpha \mid\sim \beta$ is satisfied by a ranked interpretation $\mathcal{R}$, denoted as $\mathcal{R} \Vdash \alpha \mid\sim \beta$, if the lowest ranked models of α are also models of β [8].*

Figure 1 depicts a ranked interpretation for the defeasible knowledge base $\mathcal{K} = \{\mathsf{y} \to \mathsf{s}, \mathsf{s} \mid\sim \neg\mathsf{b}, \mathsf{y} \mid\sim \mathsf{b}\}$, and $\mathcal{P} = \{\mathsf{y}, \mathsf{s}, \mathsf{b}\}$.

2	$\mathsf{ys\overline{b}}$
1	$\mathsf{\overline{y}sb}$ ysb
0	$\mathsf{\overline{y}\overline{s}\overline{b}}$ $\mathsf{\overline{y}\overline{s}b}$ $\mathsf{\overline{y}s\overline{b}}$

Figure 1: Minimal ranked interpretation $\mathcal{R}^{\mathcal{K}}_{RC}$.

2.2.1 Defeasible Entailment.

Defeasible Entailment. In the KLM framework, the concept of defeasible entailment, symbolized by $\mid\approx$, is not strictly defined and can take various forms. In this paper we are concerned with defeasible entailment as defined using ranked interpretations.

Definition 4. *A DI $\alpha \mid\sim \beta$ is defeasibly entailed by a ranked interpretation $\mathcal{R}$ for a particular knowledge base $\mathcal{K}$, denoted as $\mathcal{K} \mid\approx_{\mathcal{R}} \alpha \mid\sim \beta$, iff $\mathcal{R} \Vdash \alpha \mid\sim \beta$ [8].*

2.3 Rational Closure

Rational closure, initially introduced by Lehmann and Magidor [17], serves as the inaugural rational framework for defeasible entailment. It captures the principle of *presumption of typicality* [16], which essentially advocates for assuming information to be as typical as possible. Rational closure can be characterized in two complementary ways: syntactically (formula-theoretic) and semantically (model-theoretic).

†Often likened to the transition in an academic career, before and after securing a permanent university position.

2.3.1 Base Rank.

Base Rank. A DI $\alpha \mid\sim \beta$ implies that the validity of α generally leads to the validity of β. When $\alpha \mid\sim \beta$ is present, it is logical to presume scenarios where α holds true. However, β may conflict with other defeasible information related to α, rendering α classically untenable. In such cases, $\alpha \mid\sim \beta$ is categorized as atypical. The *base rank* of a DI measures its degree of typicality in $\mathcal{K}$ [11].

Being classified as atypical means that $\alpha \mid\sim \beta$ acts as an exception to other DIs in $\mathcal{K}$. In other words, it defeats other defeasible information within $\mathcal{K}$, necessitating a separate evaluation. This separation is performed by the base rank procedure, which initiates at rank 0, identifies all the DIs in $\mathcal{K}$ that are defeated by other DIs, and places them at the current rank. † It then advances the remaining atypical DIs to the next rank and repeats the process. The base rank of $\alpha \mid\sim \beta$ is defined as the lowest rank where α is not recognized as exceptional [11].

The `BaseRank` algorithm [8] employs the base rank procedure to calculate rational closure, denoted as $\mid\approx_{RC}$. It takes a knowledge base $\mathcal{K}$ and DI $\alpha \mid\sim \beta$ as input, and returns **true** if $\mathcal{K} \mid\approx_{RC} \alpha \mid\sim \beta$. Beginning at rank 0, DIs are systematically removed from $\mathcal{K}$ until it is classically consistent with α. The query is then considered defeasibly entailed if the resulting knowledge base classically entails the materialisation of $\alpha \mid\sim \beta$ ($\alpha \to \beta$) [8, 11].

We present a variant of the `BaseRank` algorithm, differing slightly from the version introduced by Casini et al. [8].

Algorithm 1: `BaseRank`

Input: A knowledge base $\mathcal{K}$
Output: An ordered tuple $(\mathcal{R}_0, ..., \mathcal{R}_{n-1}, \mathcal{R}_\infty, n)$

$i := 0$;
$\mathcal{R}_\infty := \mathcal{K} \setminus \{\alpha \mid\sim \beta \in \mathcal{K}\}$;
$\mathcal{R}_i := \{\alpha \to \beta \mid \alpha \mid\sim \beta \in \mathcal{K}\}$;
repeat
    $\mathcal{R}_{i+1} := \{\alpha \to \beta \in \mathcal{R}_i \mid \mathcal{R}_\infty \cup \mathcal{R}_i \models \neg\alpha\}$;
    $\mathcal{R}_i := \mathcal{R}_i \setminus \mathcal{R}_{i+1}$;
    $i := i + 1$;
until $\mathcal{R}_i = \emptyset$ ***or*** $\mathcal{R}_{i-1} = \emptyset$;
if $\mathcal{R}_{i-1} = \emptyset$ **then**
    $\mathcal{R}_\infty := \mathcal{R}_\infty \cup \mathcal{R}_i$;
    $i := i - 1$;
return $(\mathcal{R}_0, ..., \mathcal{R}_{i-1}, \mathcal{R}_\infty, i)$;

Algorithm 2: `RationalClosure`

Input: A knowledge base $\mathcal{K}$ and a DI $\alpha \mid\sim \beta$
Output: **true**, if $\mathcal{K} \mid\approx \alpha \mid\sim \beta$, and **false**, otherwise

$i := 0$;
$(\mathcal{R}_0, ..., \mathcal{R}_{n-1}, \mathcal{R}_\infty, n) :=$ `BaseRank`$(\mathcal{K})$;
$\mathcal{R} := \bigcup_{i=0}^{j<n} \mathcal{R}_j$;
while $\mathcal{R}_\infty \cup \mathcal{R} \models \neg\alpha$ ***and*** $\mathcal{R} \neq \emptyset$ **do**
    $\mathcal{R} := \mathcal{R} \setminus \mathcal{R}_i$;
    $i := i + 1$;
return $\mathcal{R}_\infty \cup \mathcal{R} \models \alpha \to \beta$;

†A DI can also defeat a classical formula, though this would make it redundant.

2.3.2 Minimal Ranked Entailment. The semantic characterization of rational closure, as formulated by Giordano et al. [11], involves the application of a partial ordering across all valid ranked interpretations for $\mathcal{K}$: $\mathcal{R}_1 \preceq_{\mathcal{K}} \mathcal{R}_2$ if for every $u \in \mathcal{U}$, $\mathcal{R}_1(u) \leq \mathcal{R}_2(u)$ [8]. Intuitively, the more the valuations in a ranked interpretation are "pushed down," the greater its overall level of typicality. Giordano et al. [11] demonstrated the existence of a minimal $\mathcal{R}$ in this ranking, denoted as $\mathcal{R}^{\mathcal{K}}_{RC}$.

Definition 5. *A DI $\alpha \mid\sim \beta$ is in the rational closure of $\mathcal{K}$, denoted $\mathcal{K} \mid\approx_{RC} \alpha \mid\sim \beta$, if $\mathcal{R}^{\mathcal{K}}_{RC} \Vdash \alpha \mid\sim \beta$.*

There are two distinct levels of typicality at play here: one among the valuations within each individual ranked interpretation, and another between the individual ranked interpretations themselves. This notion of maximizing typicality resonates with the inherent focus on the presumption of typicality found in rational closure; it consistently uses the lowest-ranked $\mathcal{R}$, thereby adhering to the most conservative pattern of reasoning for any given $\mathcal{K}$. This feature underpins its non-monotonic nature; the introduction of new information into $\mathcal{K}$ may alter the perception of what is most typical. Consequently, the minimal ranked interpretation is subject to change, thereby affecting the conclusions that can be subsequently drawn from it.

The base rank approach further exemplifies the presumption of typicality inherent in rational closure. It minimizes the segregation between the defeasible implications to the extent necessary to ensure that all defeasible implications are satisfied. Giordano et al. demonstrated that the base rank approach and the minimal ranked approach are, indeed, equivalent [11].

Algorithm 3: `MinimalRank`

Input: A knowledge base $\mathcal{K}$
Output: An ordered tuple $(\mathcal{R}_0, ..., \mathcal{R}_{n-1}, n)$

$i := 0$;
$\mathcal{U} := [\![\mathcal{K} \setminus \{\alpha \mid\sim \beta \in \mathcal{K}\}]\!]$;
$\mathcal{B}_i := \{\alpha \rightarrow \beta \mid \alpha \mid\sim \beta \in \mathcal{K}\}$;
repeat
  $\mathcal{R}_i := [\![\mathcal{B}_i]\!] \cap \mathcal{U}$;
  $\mathcal{B}_{i+1} := \{\alpha \rightarrow \beta \in \mathcal{B}_i \mid \mathcal{R}_i \cap [\![\alpha]\!] = \emptyset\}$;
  $\mathcal{U} := \mathcal{U} \setminus \mathcal{R}_i$;
  $i := i + 1$;
until $\mathcal{B}_i = \mathcal{B}_{i-1}$;
return $(\mathcal{R}_0, ..., \mathcal{R}_{i-1}, i)$;

We present `MinimalRank`, an algorithm for generating a standard minimal ranked interpretation for a given defeasible knowledge base $\mathcal{K}$. The algorithm operates by initially computing the models associated with the classical formulas present in $\mathcal{K}$. Subsequently, it iteratively removes the models corresponding to each rank of the associated base ranking. Note that rank ∞ is excluded, as it contains entirely redundant information. Separating classical formulas from DIs diverges from the norm, where a classical formula α is typically represented as $\neg\alpha \mid\sim \bot$ [8]. This key optimization avoids redundant evaluations of classical formulas that would otherwise recur with each rank.

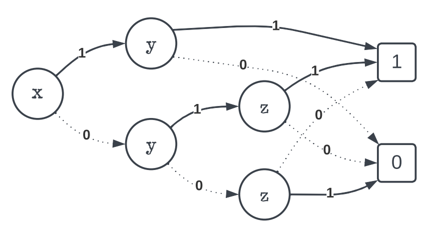
3 REDUCED MINIMAL RANKED ENTAILMENT

3.1 #P-complete

Implementing model-theoretic defeasible entailment requires significant computational resources and memory capacity. The task of identifying all satisfying solutions for a Boolean expression, known as #SAT, has the distinction of being the first problem shown to be #P-complete [19]. Often, the memory requirements of storing every valuation quickly emerge as a bottleneck, posing an even more substantial obstacle than the computational demands. Furthermore, entailment verification involves the individual assessment of each model, a process that can also escalate to exponential complexity in the worst-case scenario. To address these challenges, a compact model representation for each rank is needed, one that balances efficient computation with ease of entailment verification.

3.2 Reduced Ordered Binary Decision Diagrams

Binary Decision Diagrams (BDDs), a Directed Acyclic Graph (DAG) structure, were pioneered by Lee [15], and later Akers [1], to provide compact representations of Boolean expressions. This paper centers on Reduced Ordered Binary Decision Diagrams (ROBDDs), a variant by Bryant [6]. Constructing a ROBDD involves selecting a variable ordering and branching iteratively from the root (first variable) into two paths (0 and 1) for each variable in the ordering. At each decision node, the variable assignment is evaluated against the expression, and accordingly, it branches to the appropriate terminal node (0 or 1). Redundant nodes and isomorphic substructures are eliminated through reduction rules, culminating in a compact, non-redundant representation of the initial expression. Figure 2 illustrates a ROBDD example. Examining the truth table, we observe that when x and y are true, the value of z becomes redundant, leading to the removal of the corresponding node. The same applies to the scenario where x is true, and y is false. In the finalized structure, every path from the root to 1 symbolizes a partial model of the original expression. When expanded, they collectively yield the set of complete models corresponding to the original expression. ROBDDs are vital in solving #SAT, efficiently minimizing computation and memory requirements.[†]



x	y	z	α
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Figure 2: $\alpha = (\neg\mathbf{x} \wedge \neg\mathbf{y} \wedge \neg\mathbf{z}) \vee (\mathbf{x} \wedge \mathbf{y}) \vee (\mathbf{y} \wedge \mathbf{z})$

[†] Many BDD libraries also include optimizations like garbage collection, a process that frees up memory by removing nodes that are no longer in use.

3.2.1 ROBDD Procedures. ROBDDs provide a *canonical* form; under a predetermined variable ordering, equivalent expressions condense to identical ROBDDs [6]. This distinctive property is not only appealing but also highly relevant to this paper, as it greatly improves the efficiency of operations on ROBDDs. Bryant demonstrates that performing a Boolean operation on two ROBDDs has a complexity of $O(N_1 \cdot N_2)$, where N_1 and N_2 represent the respective node counts [6]. In cases where ROBDDs are compact, this offers a powerful method for entailment verification through reduction to unsatisfiability. The conjunction of two ROBDDs yields the intersection of their respective models. Therefore, to verify entailment, one can conjoin a ROBDD with the ROBDD representing the negation of the query in question. If the resulting product simplifies to zero, the entailment is confirmed. There has been extensive research into the use of ROBDDs for symbolic model checking [7].

Bryant also illustrates various other procedures that can be applied to ROBDDs [6]. The `Satisfy-all` procedure returns the set $\mathcal{S}_f$ of partial models for a given ROBDD f, and has a complexity of $O(n \cdot |\mathcal{S}_f|)$. The `Satisfy-one` procedure retrieves a single element from $\mathcal{S}_f$, with a complexity of $O(n)$. Lastly, the `Satisfy-count` procedure returns the total number of complete models represented by f, with a complexity of $O(N_f)$.

3.3 No Free Lunch

The No Free Lunch (NFL) theorem asserts that the average performance of any two optimization algorithms will be identical across a wide set of problems [20], a concept relevant to model-theoretic defeasible entailment (#SAT). The compactness of a ROBDD is directly related to the chosen variable ordering. In some Boolean expressions, a specific ordering may produce a linear number of nodes, while a different order might lead to an exponential number in a worst-case. Figure 3 illustrates an example of a good and bad variable ordering, provided by Bryant [6]. Determining the best variable ordering is a complex task, shown to be NP-hard [4]. A trade-off exists between the construction of ranked interpretations and the verification of entailment. Investing more computational resources in finding the optimal variable ordering may decrease the effort required for entailment checking, and vice versa.

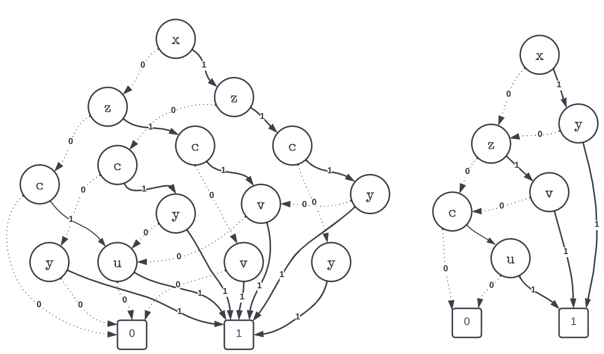


Figure 3: $\alpha = \mathbf{x} \wedge \mathbf{y} \vee \mathbf{z} \wedge \mathbf{v} \vee \mathbf{c} \wedge \mathbf{u}$

Computing optimal or near-optimal variable ordering for Boolean expressions is a subject of extensive research. Key approaches include the sifting algorithm, which starts with an initial variable ordering and iteratively refines it using heuristics [18]. Genetic algorithms apply principles of natural selection to evolve a set of variable orderings over multiple generations, assessing each one's fitness based on the resulting diagram size [10]. Simulated annealing employs a probabilistic approach, exploring the space of possible orderings by making random changes and accepting or rejecting them based on a temperature-dependent probability function [4]. Each of these techniques offers a unique balance between computational efficiency and the quality of the variable ordering.

3.4 Algorithm

`ReducedRank` constructs a distinct ROBDD to represent the models for each rank within the associated minimal ranked interpretation. This ROBDD is then leveraged to identify all DIs whose antecedents are negated by that rank, achieved through reduction to unsatisfiability (where unsatisfiable expressions reduce to 0). In standard practice, a ROBDD is represented as a function, and denoted with the symbol f. Operations on ROBDDs are expressed using the conventional connectives of propositional logic. The `Reduce` function symbolizes the construction of a ROBDD given a specific formula and variable ordering, for which various efficient algorithms are well-established [1, 6, 15].

Utilizing a uniform variable ordering across all ranks yields a canonical form for defeasible knowledge bases (in the context of rational closure). This facilitates an efficient means of assessing formal equivalence. Conversely, if the objective is to attain a compact representation, employing distinct variable orderings for each rank is recommended. The `Order` function serves to specify the algorithm for determining variable ordering for a given expression. The output of this function is at the discretion of the implementer; it may produce a static ordering, generate a random sequence, or implement an established algorithm [4, 10, 18].

Algorithm 4: `ReducedRank`

Input: A knowledge base $\mathcal{K}$ and a set of atoms $\mathcal{P}$
Output: An ordered tuple $((f_0, \mathcal{P}_0), ..., (f_{n-1}, \mathcal{P}_{n-1}), n)$

$i := 0$;
$\Omega := \bigwedge \mathcal{K} \setminus \{\alpha \mathrel{|\!\sim} \beta \in \mathcal{K}\}$;
$\mathcal{B}_i := \{\alpha \rightarrow \beta \mid \alpha \mathrel{|\!\sim} \beta \in \mathcal{K}\}$;
repeat
  $\delta_i := \Omega \wedge (\bigwedge \mathcal{B}_i) \wedge (\bigwedge_{j<i}(\bigvee \neg\mathcal{B}_j))$;
  $\mathcal{P}_i := \mathtt{Order}(\delta_i, \mathcal{P})$;
  $f_i := \mathtt{Reduce}(\delta_i, \mathcal{P}_i)$;
  $\mathcal{B}_{i+1} := \{\alpha \rightarrow \beta \in \mathcal{B}_i \mid f_i \wedge \mathtt{Reduce}(\alpha, \mathcal{P}_i) = 0\}$;
  $i := i + 1$;
until $\mathcal{B}_i = \mathcal{B}_{i-1}$;
return $((f_0, \mathcal{P}_0), ..., (f_{i-1}, \mathcal{P}_{i-1}))$;

Constructing a ROBDD for each rank involves formulating an exact representation of the models for that rank. In this context, Omega, Ω, is used symbolize the constraints from the classical formulas in $\mathcal{K}$. The expression $\bigwedge \mathcal{B}_i$ refers to the constraints imposed by the DIs in the current rank, which all models must satisfy. Meanwhile, $\bigwedge_{j<i}(\bigvee \neg\mathcal{B}_j)$ conveys the constraints from the DIs in previous ranks, each model of the current rank must fail to satisfy at least one DI from each prior rank. The expression is subsequently passed to the `Order` function to ascertain the optimal variable ordering for the current rank, after which the ROBDD is constructed based on the selected ordering.

Algorithm 5: `ReducedEntailment`

Input: A knowledge base $\mathcal{K}$, a set of atoms $\mathcal{P}$, and $\alpha \mid\sim \beta$
Output: **true**, if $\mathcal{K} \mid\approx \alpha \mid\sim \beta$, and **false**, otherwise

$i := 0$;
$((f_0, \mathcal{P}_0), ..., (f_{n-1}, \mathcal{P}_{n-1}), n) :=$ `ReducedRank`$(\mathcal{K}, \mathcal{P})$;
$f_\alpha :=$ `Reduce`$(\alpha, \mathcal{P}_0)$;
while $f_i \wedge f_\alpha = 0$ ***and*** $i < n - 1$ **do**
  $i := i + 1$;
  $f_\alpha :=$ `Reduce`$(\alpha, \mathcal{P}_i)$;
$f_{\neg(\alpha\to\beta)} :=$ `Reduce`$(\neg(\alpha \to \beta), \mathcal{P}_i)$;
return $f_i \wedge f_{\neg(\alpha\to\beta)} = 0$;

`ReducedEntailment` creates a ROBDD for α, represented by f_α, for the purpose of verifying entailment. f_α is consecutively conjoined with the ROBDD of each rank until the outcome does not reduce to 0 (meaning there exists a model in that rank that satisfies α). This ROBDD is then combined with $f_{\neg(\alpha\to\beta)}$. If the resulting ROBDD reduces to 0, it indicates that every model of that rank satisfying α also satisfies β.

3.5 Analysis

Let $\mathcal{K} = \{\mathtt{p} \to \mathtt{b}, \mathtt{b} \mid\sim \mathtt{f}, \mathtt{p} \mid\sim \neg\mathtt{f}, \mathtt{b} \mid\sim \mathtt{w}\}$. Figure 4 illustrates the minimal ranked interpretation for $\mathcal{K}$, a corresponding reduced version, as well as the base ranking. Examining rank 0, p is atypical, mandating $\overline{\mathtt{p}}$ in every partial model. When b is false, the truth of f and w become redundant due to the semantics of implication, leaving only $\overline{\mathtt{b}}\overline{\mathtt{p}}$. However, if b is true, then both f and w must be true, leaving $\mathtt{fwb}\overline{\mathtt{p}}$. Even with a small knowledge base, a significant amount of redundant information can be eliminated.

2	$\mathtt{pbwf}$ $\mathtt{pb}\overline{\mathtt{w}}\mathtt{f}$
1	$\overline{\mathtt{p}}\mathtt{b}\overline{\mathtt{w}}\overline{\mathtt{f}}$ $\overline{\mathtt{p}}\mathtt{b}\overline{\mathtt{w}}\mathtt{f}$ $\overline{\mathtt{p}}\mathtt{bw}\overline{\mathtt{f}}$ $\mathtt{pbw}\overline{\mathtt{f}}$ $\mathtt{pb}\overline{\mathtt{w}}\overline{\mathtt{f}}$
0	$\overline{\mathtt{p}}\overline{\mathtt{b}}\mathtt{w}\overline{\mathtt{f}}$ $\overline{\mathtt{p}}\overline{\mathtt{b}}\mathtt{wf}$ $\overline{\mathtt{p}}\overline{\mathtt{b}}\overline{\mathtt{w}}\overline{\mathtt{f}}$ $\overline{\mathtt{p}}\overline{\mathtt{b}}\overline{\mathtt{w}}\mathtt{f}$ $\overline{\mathtt{p}}\mathtt{bwf}$

2	$\mathtt{fbp}$
1	$\overline{\mathtt{w}}\mathtt{b}\overline{\mathtt{p}}$ $\overline{\mathtt{f}}\mathtt{wb}\overline{\mathtt{p}}$ $\overline{\mathtt{f}}\mathtt{bp}$
0	$\overline{\mathtt{b}}\overline{\mathtt{p}}$ $\mathtt{fwb}\overline{\mathtt{p}}$

∞	$\mathtt{p} \to \mathtt{b}$
1	$\mathtt{p} \mid\sim \neg\mathtt{f}$
0	$\mathtt{b} \mid\sim \mathtt{f}$ $\mathtt{b} \mid\sim \mathtt{w}$

Figure 4: Complete vs reduced $\mathcal{R}^{\mathcal{K}}_{RC}$

ROBDDs are especially effective with expressions containing notable structural regularity, a characteristic often found in ontology-based knowledge bases. As the number of formulas in a knowledge base grows, redundant information also tends to increase, especially in a structurally regular knowledge base. Let $\mathcal{K} = \{\mathtt{p} \to \mathtt{b}, \mathtt{b} \mid\sim \mathtt{f}, \mathtt{p} \mid\sim \neg\mathtt{f}, \mathtt{p} \mid\sim \mathtt{t}, \mathtt{p} \mid\sim \mathtt{y}, \mathtt{p} \mid\sim \mathtt{u}, \mathtt{b} \mid\sim \mathtt{q}, \mathtt{b} \mid\sim \mathtt{w}, \mathtt{b} \mid\sim \mathtt{e}, \mathtt{b} \mid\sim \mathtt{h}, \mathtt{r} \to \mathtt{p}, \mathtt{r} \mid\sim \mathtt{f}, \mathtt{r} \mid\sim \mathtt{z}, \mathtt{r} \mid\sim \mathtt{x}, \mathtt{n} \to \mathtt{r}, \mathtt{n} \mid\sim \neg\mathtt{f}, \mathtt{n} \mid\sim \mathtt{j}, \mathtt{n} \mid\sim \mathtt{k}, \mathtt{n} \mid\sim \mathtt{l}\}$. Figure 5 illustrates a corresponding reduced $\mathcal{R}^{\mathcal{K}}_{RC}$.

∞	$\mathtt{p} \to \mathtt{b}$ $\mathtt{r} \to \mathtt{p}$ $\mathtt{n} \to \mathtt{r}$
3	$\mathtt{n} \mid\sim \neg\mathtt{f}$ $\mathtt{n} \mid\sim \mathtt{j}$ $\mathtt{n} \mid\sim \mathtt{k}$ $\mathtt{n} \mid\sim \mathtt{l}$
2	$\mathtt{r} \mid\sim \mathtt{f}$ $\mathtt{r} \mid\sim \mathtt{z}$ $\mathtt{r} \mid\sim \mathtt{x}$
1	$\mathtt{p} \mid\sim \neg\mathtt{f}$ $\mathtt{p} \mid\sim \mathtt{t}$ $\mathtt{p} \mid\sim \mathtt{y}$ $\mathtt{p} \mid\sim \mathtt{u}$
0	$\mathtt{b} \mid\sim \mathtt{f}$ $\mathtt{b} \mid\sim \mathtt{q}$ $\mathtt{b} \mid\sim \mathtt{w}$ $\mathtt{b} \mid\sim \mathtt{e}$ $\mathtt{b} \mid\sim \mathtt{h}$

4	$\overline{\mathtt{j}}\mathtt{nr}\overline{\mathtt{f}}\mathtt{bp}$ $\overline{\mathtt{k}}\mathtt{jnr}\overline{\mathtt{f}}\mathtt{bp}$ $\overline{\mathtt{l}}\mathtt{kjnr}\overline{\mathtt{f}}\mathtt{bp}$ $\mathtt{n}\overline{\mathtt{z}}\mathtt{rfbp}$ $\mathtt{n}\overline{\mathtt{x}}\mathtt{zrfbp}$ $\mathtt{nxzrfbp}$
3	$\overline{\mathtt{n}}\mathtt{r}\overline{\mathtt{f}}\mathtt{bp}$ $\mathtt{lkjnr}\overline{\mathtt{f}}\mathtt{bp}$ $\overline{\mathtt{n}}\overline{\mathtt{z}}\mathtt{rfbp}$ $\overline{\mathtt{n}}\overline{\mathtt{x}}\mathtt{zrfbp}$
2	$\overline{\mathtt{n}}\overline{\mathtt{t}}\overline{\mathtt{r}}\overline{\mathtt{f}}\mathtt{bp}$ $\overline{\mathtt{n}}\overline{\mathtt{y}}\mathtt{t}\overline{\mathtt{r}}\overline{\mathtt{f}}\mathtt{bp}$ $\overline{\mathtt{n}}\overline{\mathtt{u}}\mathtt{yt}\overline{\mathtt{r}}\overline{\mathtt{f}}\mathtt{bp}$ $\overline{\mathtt{n}}\overline{\mathtt{r}}\mathtt{fbp}$ $\overline{\mathtt{n}}\mathtt{xzrfbp}$
1	$\overline{\mathtt{n}}\overline{\mathtt{r}}\overline{\mathtt{f}}\mathtt{b}\overline{\mathtt{p}}$ $\overline{\mathtt{n}}\overline{\mathtt{r}}\overline{\mathtt{q}}\mathtt{fb}\overline{\mathtt{p}}$ $\overline{\mathtt{n}}\overline{\mathtt{r}}\overline{\mathtt{w}}\mathtt{qfb}\overline{\mathtt{p}}$ $\overline{\mathtt{n}}\overline{\mathtt{r}}\overline{\mathtt{e}}\mathtt{wqfb}\overline{\mathtt{p}}$ $\overline{\mathtt{n}}\overline{\mathtt{r}}\overline{\mathtt{h}}\mathtt{ewqfb}\overline{\mathtt{p}}$ $\overline{\mathtt{n}}\mathtt{uyt}\overline{\mathtt{r}}\overline{\mathtt{f}}\mathtt{bp}$
0	$\overline{\mathtt{n}}\overline{\mathtt{r}}\overline{\mathtt{f}}\overline{\mathtt{b}}\overline{\mathtt{p}}$ $\overline{\mathtt{n}}\overline{\mathtt{r}}\mathtt{f}\overline{\mathtt{b}}\overline{\mathtt{p}}$ $\overline{\mathtt{n}}\overline{\mathtt{r}}\mathtt{hewqfb}\overline{\mathtt{p}}$

Figure 5: Reduced $\mathcal{R}^{\mathcal{K}}_{RC}$.

The corresponding complete version contains 40,960 complete models with 17 atoms each, while the reduced version contains only 24 partial models with at most 9 atoms each. It is clear that this significantly reduces memory requirements, without any loss of information. Importantly, despite the substantial increase in the number of formulas and atoms in $\mathcal{K}$, the count of partial models remains relatively stable. This trend can be expected to continue as more formulas and atoms are added to $\mathcal{K}$, as long as the degree of randomness in the knowledge remains low (with the proviso that substantial computational effort may be necessary to identify the optimal ordering). Furthermore, it is evident that this approach offers a more efficient means of verifying entailment in comparison to its complete counterpart.

An idiosyncrasy arises between the base rank of a DI $\alpha \mid\sim \beta$ and its influence on the overall reducibility of a ranked interpretation. Specifically, the truth value of β becomes redundant in all preceding ranks. This is due to the fact that the only scenario where $\alpha \to \beta$ can be false is when α is true but β is not. Consequently, any information exclusively related to the truth of $\alpha \mid\sim \beta$ can be replaced with simply $\neg\alpha$. This generalizes to other DIs that share the same α. For instance, the base rank of $\mathtt{n} \mid\sim \mathtt{j}, \mathtt{n} \mid\sim \mathtt{k}$, and $\mathtt{n} \mid\sim \mathtt{l}$ is 3. The atoms j, k, and l are solely connected to the rest of the knowledge through n, rendering them redundant in ranks 2, 1, and 0; hence, they are entirely absent. This reveals an important optimisation of the `BaseRank` algorithm.

4 IRREDUNDANT BASE RANK

The `BaseRank` algorithm performs redundant computation by including the defeasible consequences of α in all ranks preceding its base rank, despite the already established falsehood of α in each of those ranks. A more efficient approach would involve retaining this conclusion by including only $\neg\alpha$. Such a modification would significantly enhance the efficiency of verifying entailment, particularly given that α may have multiple defeasible consequences, and that they can be infinitely complex. To address this, we introduce `IrredundantBaseRank` and `IrredundantRationalClosure`, which work by including $\neg\alpha$ rather than $\alpha \to \beta$ in all ranks preceding the base rank of α.

Algorithm 6: `IrredundantBaseRank`

Input: A knowledge base $\mathcal{K}$
Output: An ordered tuple $(\mathcal{R}_0, ..., \mathcal{R}_{n-1}, \mathcal{R}_\infty, n)$

$i := 0$;
$\mathcal{R}_\infty := \mathcal{K} \setminus \{\alpha \mid\!\sim \beta \in \mathcal{K}\}$;
$\mathcal{R}_i := \{\alpha \to \beta \mid \alpha \mid\!\sim \beta \in \mathcal{K}\}$;
repeat
  $\mathcal{R}_{i+1} := \{\alpha \to \beta \in \mathcal{R}_i \mid \mathcal{R}_\infty \cup \mathcal{R}_i \models \neg\alpha\}$;
  $\mathcal{R}_i := (\mathcal{R}_i \setminus \mathcal{R}_{i+1}) \cup \{\neg\alpha \mid \alpha \to \beta \in \mathcal{R}_{i+1}\}$;
  $i := i + 1$;
until $\mathcal{R}_i = \emptyset$ ***or*** $\{\alpha \to \beta \in \mathcal{R}_{i-1}\} = \emptyset$;
if $\{\alpha \to \beta \in \mathcal{R}_{i-1}\} = \emptyset$ **then**
  $\mathcal{R}_\infty := \mathcal{R}_\infty \cup \mathcal{R}_i$;
  $i := i - 1$;
return $(\mathcal{R}_0, ..., \mathcal{R}_{i-1}, \mathcal{R}_\infty, i)$;

Algorithm 7: `IrredundantRationalClosure`

Input: A knowledge base $\mathcal{K}$ and a DI $\alpha \mid\!\sim \beta$
Output: **true**, if $\mathcal{K} \mid\!\approx \alpha \mid\!\sim \beta$, and **false**, otherwise

$i := 0$;
$(\mathcal{R}_0, ..., \mathcal{R}_{n-1}, \mathcal{R}_\infty, n) :=$ `BaseRank`$(\mathcal{K})$;
while $\mathcal{R}_\infty \cup \mathcal{R}_i \models \neg\alpha$ ***and*** $i < n$ **do**
  $i := i + 1$;
if $i < n$ **then**
  return $\mathcal{R}_\infty \cup \mathcal{R}_i \models \alpha \to \beta$;
else
  return $\mathcal{R}_\infty \models \alpha \to \beta$;

Theorem 1. *The irredundant base rank approach is functionally equivalent to the standard base rank approach.*

Proof. Consider $\mathcal{K}$ as an arbitrary subset of $\mathcal{L}$, and let $\alpha \to \beta$ be a formula within $\mathcal{K}$ such that $\mathcal{K} \models \neg\alpha$. To establish the functional equivalence between `IrredundantBaseRank` and `BaseRank`, it suffices to demonstrate that $[\![\mathcal{K}]\!] = [\![(\mathcal{K} \setminus \{\alpha \to \beta\}) \cup \{\neg\alpha\}]\!]$.

(1) First, note that if $\mathcal{K} \models \neg\alpha$, then $[\![\mathcal{K}]\!] = [\![\mathcal{K} \cup \{\neg\alpha\}]\!]$ by the definition of entailment.
(2) Next, consider any valuation $v \in [\![\mathcal{K} \cup \{\neg\alpha\}]\!]$, and any formula $\alpha \to \beta \in \mathcal{L}$. It is guaranteed that $v \Vdash \alpha \to \beta$, because $\alpha \to \beta \equiv \neg\alpha \vee \beta$.
(3) Consequently, the inclusion or exclusion of any formula $\alpha \to \beta \in \mathcal{L}$ from $\mathcal{K} \cup \{\neg\alpha\}$ leaves $[\![\mathcal{K} \cup \{\neg\alpha\}]\!]$ invariant.
(4) Hence, $[\![\mathcal{K} \cup \{\neg\alpha\}]\!] = [\![(\mathcal{K} \setminus \{\alpha \to \beta\}) \cup \{\neg\alpha\}]\!]$.

Therefore, $[\![\mathcal{K}]\!] = [\![(\mathcal{K} \setminus \{\alpha \to \beta\}) \cup \{\neg\alpha\}]\!]$.

□

The irredundant approach is not only an optimization. It also serves to further illuminate the underlying principle of presumption of typicality in rational closure. If typicality is presumed, then the defeasible consequences of atypical information should be set aside unless atypicality is explicitly expressed. We have illustrated that rational closure does exactly this.

5 NAIVE BAYES ENTAILMENT

5.1 Motivation

The application of reduced minimal ranked entailment is not limited to rational closure alone. Representing each rank as a ROBDD introduces a symbolic system for processing and manipulating the models of each rank using ROBDD procedures and Boolean algebra. This has broad utility, such as creating model-theoretic extensions of rational closure, which is an active area of research [8]. In this paper, we illustrate this through our own Bayesian refinement of minimal ranked entailment, which conceptualises defeasible entailment as a form of conditional probability.

Bayesian reasoning is a well known methodology in Statistics and Artificial Intelligence, first introduced by Thomas Bayes in 1763 [2], and later refined by his colleague, Pierre-Simone Laplace, in 1812 [14]. At the core of Bayesian reasoning is Bayes' Rule:

$$P(H|E) = \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|\neg H)P(\neg H)} \tag{1}$$

$P(H|E)$ represents the *posterior* probability of some hypothesis H given some evidence E. $P(E|H)P(H)$ is the *likelihood* of the evidence given the hypothesis, multiplied by the *prior* probability of the hypothesis. $P(E|\neg H)P(\neg H)$ is the likelihood of the evidence given the negation of the hypothesis, multiplied by the prior probability of the negation of the hypothesis.

5.2 Algorithm

The most straightforward approach would be to apply Bayes' Rule directly to $\alpha \mid\!\sim \beta$:

$$P(\beta \mid \alpha) = \frac{\frac{f \wedge f_\alpha \wedge f_\beta}{f \wedge f_\alpha} \cdot \frac{f \wedge f_\beta}{f}}{\frac{f \wedge f_\alpha \wedge f_\beta}{f \wedge f_\alpha} \cdot \frac{f \wedge f_\beta}{f} + \frac{f \wedge f_\alpha \wedge f_{\neg\beta}}{f \wedge f_\alpha} \cdot \frac{f \wedge f_{\neg\beta}}{f}} \tag{2}$$

In this context, f denotes the ROBDD corresponding to the lowest rank that satisfies α, whereas f_α and f_β represent the ROBDDs for α and β, respectively. Each function is used to represent the number of models they contain, which can be computed using the `Satisfy-count` procedure. However, utilizing this approach would diverge too far from the core tenets of defeasible entailment.

In the exploration of defeasible entailment from a statistical standpoint, the aim is to identify the most statistically probable minimal worlds that satisfy α, and evaluate their compliance with β. α can encapsulate any formula in $\mathcal{L}$, thereby representing a variety of scenarios or evidential fragments. These fragments are equivalent to the models of α.[†] In each minimal world compatible with a given fragment, the remaining atoms constitute potential hypotheses. For instance, if $\alpha = \mathsf{a}$, and a minimal world satisfying a is $\mathsf{a}\bar{\mathsf{b}}\mathsf{c}\bar{\mathsf{d}}$, the corresponding probability is $P(\bar{\mathsf{b}} \wedge \mathsf{c} \wedge \bar{\mathsf{d}} \mid \mathsf{a})$. Accurate computation of such conjunctions necessitates understanding the conditional dependencies among atoms. Although it is theoretically possible to derive these dependencies from formulas in $\mathcal{K}$, this task is complex and falls outside the scope of this paper. However, an assumption of conditional independence, as exemplified by the naive

[†]It is crucial to emphasize that this refers to the models of α devoid of extraneous information, meaning they contain exclusively the atoms found in α.

bayes classifier, can often produce useful results. This simplifies the calculation to the product of individual conditional probabilities:

$$P(\overline{\mathsf{b}} \wedge \mathsf{c} \wedge \overline{\mathsf{d}} \mid \mathsf{a}) = P(\overline{\mathsf{b}} \mid \mathsf{a}) * P(\mathsf{c} \mid \mathsf{a}) * P(\overline{\mathsf{d}} \mid \mathsf{a}) \tag{3}$$

For each fragment, the analysis unfolds in three sequential steps:

(1) Identify the minimal partial models that are consistent with the fragment.
(2) Quantify the true, and by implication, false occurrences of each remaining atom within their corresponding complete models.
(3) Employ naive bayesian calculations to ascertain the most probable complete models.

Algorithm 8: `NaiveBayesEntailment`

Input: An ordered tuple $(f_0, ..., f_{n-1}, f_\infty, n)$, a set of propositions $\mathcal{P}$, and a DI $\alpha \mid\!\sim \beta$
Output: **true**, if $\mathcal{K} \mid\!\approx_{NB} \alpha \mid\!\sim \beta$ and **false**, otherwise

$i := 0$;
$P := 0$;
$\mathcal{U}_\alpha := \emptyset$;
$f_\alpha :=$ `Reduce`$(\alpha, \mathcal{P}_i)$;
while $f_i \wedge f_\alpha = 0$ **do**
  $i := i + 1$;
  $f_\alpha :=$ `Reduce`$(\alpha, \mathcal{P}_i)$;
$f_h := f_i \wedge f_\alpha$;
$\mathcal{H} :=$ `Satisfy-all`(f_h);
for $\tilde{e} \in [\![\alpha]\!]$ **do**
  $f_{\tilde{e}} :=$ `Reduce`$(\bigwedge \tilde{e}, \mathcal{P}_i)$;
  $total, count_{\mathcal{P}} :=$ `Count`$(f_{\tilde{e}} \wedge f_h, \mathcal{P}_i)$;
  for $\tilde{w} \in \mathcal{H}$ **do**
    if $\tilde{w} \cap \neg\tilde{e} = \emptyset$ **then**
      $P_{\tilde{w}} := 0$;
      for $p \in \tilde{w} \setminus \tilde{e}$ **do**
        if $\tilde{w} \setminus \tilde{e} \Vdash p$ **then**
          $P_{\tilde{w}} := P_{\tilde{w}} + \log(\frac{count_p}{total})$;
        else
          $P_{\tilde{w}} := P_{\tilde{w}} + \log(\frac{total - count_p}{total})$;
      $P_{\tilde{w}}, \mathcal{U}_{\tilde{w}} :=$ `Expand`$(P_{\tilde{w}}, \tilde{w} \cup \tilde{e}, \mathcal{P} \setminus (\tilde{w} \cup \tilde{e}), total, count_{\mathcal{P}})$;
      if $P_{\tilde{w}} < P$ **then**
        $P := P_{\tilde{w}}$;
        $\mathcal{U}_\alpha := \mathcal{U}_{\tilde{w}}$;
      else if $P_{\tilde{w}} = P$ **then**
        $\mathcal{U}_\alpha := \mathcal{U}_\alpha \cup \mathcal{U}_{\tilde{w}}$;
return $\forall u \in \mathcal{U}_\alpha, u \Vdash \beta$;

ROBDD procedures are highly effective for executing steps 1 and 2. The identification of minimal partial models follows the same procedure employed in `ReducedEntailment`. The ROBDD corresponding to the minimal satisfying rank is then conjoined with f_α, yielding a resultant f_h that delineates the hypothesis space for α. The `Satisfy-all` procedure is then invoked to obtain the set of partial models within this hypothesis space. Partial models are designated with a tilde symbol to differentiate from complete models. In this context, a partial model serves to explicitly encapsulate a collection of literals (negated or non-negated atoms). To facilitate set-based comparisons with partial models derived from a ROBDD, each model of α is likewise represented as a partial model $\tilde{e}$.

The algorithm iterates over each model of α to identify the most probable complete models that they align with. During this procedure, the highest calculated probability is retained in P, and the complete models sharing this probability are stored in $\mathcal{U}_\alpha$. Prior to conducting probabilistic calculations, requisite counting operations must be executed. The current evidence space $f_{\tilde{e}}$ is conjoined with the hypothesis space f_h to isolate the segment of the hypothesis space that aligns with this evidence space. The resultant ROBDD is then passed to `Count`.

Algorithm 9: `Count`

Input: $f, \mathcal{P}_i$

$total :=$ `Satisfy-count`(f);
for $p \in \mathcal{P}_i$ **do**
  $f_p :=$ `Reduce`$(p, \mathcal{P}_i)$;
  $count_p :=$ `Satisfy-count`$(f \wedge f_p)$;
return $total, count_{\mathcal{P}}$;

The `Count` method executes the required counting operations within a specific evidence-hypothesis space, denoted by f. The `Satisfy-count` procedure is directly applied to f to enumerate the total number of complete models. For each atom in $\mathcal{P}$, the method conjoins f with f_p and employs the `Satisfy-count` procedure to tally the total number of true occurrences for each atom. In practical implementation, these values would be stored in an array/dictionary ($count_{\mathcal{P}}$). This detail has been omitted here for brevity.

Algorithm 10: `Expand`

Input: $P, \tilde{u}, \mathcal{P}, total, count_{\mathcal{P}}$

if $\mathcal{P} = \emptyset$ **then**
  return $P, \{\tilde{u}\}$;
$p := p \in \mathcal{P}$;
$P_l, \mathcal{U}_l :=$ `Expand`$(P + \log(\frac{count_p}{total}), \tilde{u} \cup p, \mathcal{P} \setminus p, total, count_{\mathcal{P}})$;
$P_r, \mathcal{U}_r :=$ `Expand`$(P + \log(\frac{total - count_p}{total}), \tilde{u} \cup \neg p, \mathcal{P} \setminus p, total, count_{\mathcal{P}})$;
if $P_l < P_r$ **then**
  return $P_l, \mathcal{U}_l$;
else if $P_l > P_r$ **then**
  return $P_r, \mathcal{U}_r$;
else
  return $P_l, \mathcal{U}_l \cup \mathcal{U}_r$;

The `Expand` method essentially re-expands a given partial model while concurrently executing the requisite probabilistic calculations for each corresponding complete model. To optimize computational efficiency and avert floating-point underflow, these calculations are conducted in logarithmic space. Subsequently, the method returns a set comprising the most probable complete models. This set is then evaluated against $\mathcal{U}_\alpha$ and is either amalgamated with it or supersedes it. Upon completion of the last fragment of evidence, each model within $\mathcal{U}_\alpha$ undergoes evaluation to ascertain its compatibility with the consequent.

Definition 6. *A DI $\alpha \mid\sim \beta$ is naive bayes entailed by a defeasible knowledge base $\mathcal{K}$, denoted $\mathcal{K} \approx_{NB} \alpha \mid\sim \beta$, iff $\forall u \in \mathcal{U}_\alpha, u \Vdash \beta$.*

5.3 Proof Of Properties

In the study of different forms of defeasible entailment, Lehmann and Magidor propose a set of rationality properties known as the KLM postulates [8, 13]. A defeasible entailment relation that adheres to these principles is termed *LM-rational.* Lehmann and Magidor proved that if a defeasible entailment relation can be derived from a ranked interpretation, then it is LM-rational [13].

The fundamental premise of Bayesian reasoning is the dynamic adjustment of beliefs in response to new evidence. By considering the antecedent as evidence, it follows that defeasible consequences are contingent upon the antecedent. In simpler terms, the model rankings are influenced by the query's antecedent. Consequently, naive bayes entailment cannot be encapsulated by a singular ranked interpretation. Nonetheless, we will demonstrate that it adheres to at least a subset of the postulates.

Theorem 2. *Naive bayes entailment is not LM-rational.*

5.3.1 Reflexivity. Reflexivity stipulates that all formulas should be defeasible consequences of themselves, a condition generally met by any defeasible entailment relation.

$$(Ref)\;\; \mathcal{K} \approx \alpha \mid\sim \alpha \tag{4}$$

Lemma 1. *Naive bayes entailment satisfies reflexivity.*

Proof. Naive bayes entailment identifies the most statistically probable minimal models consistent with any evidential fragment in $[\![\alpha]\!]$. It is therefore guaranteed to return a set of models that each satisfy α. □

5.3.2 Left Logical Equivalence. Left logical equivalence posits that if two formulas are logically equivalent, then their respective defeasible consequences must also be identical.

$$(LLE)\;\; \frac{\alpha \equiv \beta,\;\; \mathcal{K} \approx \alpha \mid\sim \beta}{\mathcal{K} \approx \beta \mid\sim \gamma} \tag{5}$$

Lemma 2. *Naive bayes entailment satisfies left logical equivalence.*

Proof. Should $\alpha \equiv \beta$, it follows by definition that $[\![\alpha]\!] = [\![\beta]\!]$, indicating identical evidential fragments. Naive bayes entailment will therefore return the same models for both α and β. □

5.3.3 Right Weakening. Right weakening asserts that if β is a defeasible consequence of α, then any formula that is logically equivalent to β must likewise be a defeasible consequence of α.

$$(RW)\;\; \frac{\mathcal{K} \approx \alpha \mid\sim \beta,\;\; \beta \models \gamma}{\mathcal{K} \approx \alpha \mid\sim \gamma} \tag{6}$$

Lemma 3. *Naive bayes entailment satisfies right weakening.*

Proof. If $\mathcal{K} \approx \alpha \mid\sim \beta$, then all models returned by naive bayes entailment satisfy β. Given that $\beta \models \gamma$, it follows by definition that $[\![\beta]\!] \subseteq [\![\gamma]\!]$, thereby ensuring that each model also satisfies γ. □

5.3.4 And. And dictates that the conjunction of any defeasible consequences of a formula should likewise be a defeasible consequence of that formula.

$$(And)\;\; \frac{\mathcal{K} \approx \alpha \mid\sim \beta,\;\; \mathcal{K} \approx \alpha \mid\sim \gamma}{\mathcal{K} \approx \alpha \mid\sim \beta \land \gamma} \tag{7}$$

Lemma 4. *Naive bayes entailment satisfies and.*

Proof. If every model returned by naive bayes entailment for α satisfies both β and γ, then they are guaranteed to satisfy $\beta \land \gamma$. □

5.3.5 Or. Or states that a defeasible consequence of two individual formulas should also be a defeasible consequence of their disjunction.

$$(Or)\;\; \frac{\mathcal{K} \approx \alpha \mid\sim \gamma,\;\; \mathcal{K} \approx \beta \mid\sim \gamma}{\mathcal{K} \approx \alpha \lor \beta \mid\sim \gamma} \tag{8}$$

Lemma 5. *Naive bayes entailment does not satisfy or.*

Proof. Consider $\mathcal{K} = \{\mathsf{a} \mid\sim \mathsf{b}, \mathsf{c} \rightarrow \mathsf{a}, \mathsf{c} \mid\sim \neg\mathsf{b}, \mathsf{d} \rightarrow \mathsf{c}, \mathsf{d} \mid\sim \mathsf{e}, \mathsf{f} \rightarrow \mathsf{a}, \mathsf{f} \mid\sim \neg\mathsf{b}\}$, figure 6 shows a corresponding reduced $\mathcal{R}^{\mathcal{K}}_{RC}$. Naive bayes entailment identifies $\mathsf{fe\overline{d}c\overline{b}a}$ and $\mathsf{\overline{f}e\overline{d}c\overline{b}a}$ as most probable for c, making $\mathcal{K} \approx_{NB} \mathsf{c} \mid\sim \mathsf{e}$ true. Naive bayes entailment also identifies $\mathsf{fe\overline{d}c\overline{b}a}$ as most probable for f, making $\mathcal{K} \approx_{NB} \mathsf{f} \mid\sim \mathsf{e}$ true. However, the most probable models for $\mathsf{c} \lor \mathsf{f}$ are $\mathsf{fe\overline{d}\overline{c}\overline{b}a}$ and $\mathsf{f\overline{e}\overline{d}\overline{c}\overline{b}a}$, making $\mathcal{K} \approx_{NB} \mathsf{c} \lor \mathsf{f} \mid\sim \mathsf{e}$ false.

□

2	$\mathsf{\overline{e}dc\overline{b}a}$ $\mathsf{f\overline{d}\overline{c}ba}$ cba
1	$\mathsf{\overline{d}\overline{c}\overline{b}a}$ $\mathsf{\overline{d}c\overline{b}a}$ $\mathsf{edc\overline{b}a}$
0	$\mathsf{\overline{f}\overline{d}\overline{c}\overline{b}\overline{a}}$ $\mathsf{\overline{f}\overline{d}\overline{c}b\overline{a}}$ $\mathsf{\overline{f}\overline{d}\overline{c}ba}$

Figure 6: Reduced minimal ranked interpretation $\mathcal{R}^{\mathcal{K}}_{RC}$.

5.3.6 Cautious Monotonicity. Cautious monotonicity posits that the integration of newly deduced information should not negate or undermine any previously inferable conclusions.

$$(CM)\;\; \frac{\mathcal{K} \approx \alpha \mid\sim \beta,\;\; \mathcal{K} \approx \alpha \mid\sim \gamma}{\mathcal{K} \approx \alpha \land \beta \mid\sim \gamma} \tag{9}$$

Lemma 6. *Naive bayes entailment does not satisfy cautious monotonicity.*

Proof. Consider again the knowledge from the previous proof. Both $\mathcal{K} \approx_{NB} \mathsf{c} \mid\sim \mathsf{e}$ and $\mathcal{K} \approx_{NB} \mathsf{c} \mid\sim \neg\mathsf{d}$ are true. However, naive bayes entailment identifies $\mathsf{fe\overline{d}c\overline{b}a}$, $\mathsf{\overline{f}e\overline{d}c\overline{b}a}$, $\mathsf{fedc\overline{b}a}$, and $\mathsf{\overline{f}edc\overline{b}a}$ as most probable for $\mathsf{c} \land \mathsf{e}$, making $\mathcal{K} \approx_{NB} \mathsf{c} \land \mathsf{e} \mid\sim \neg\mathsf{d}$ false. □

5.3.7 Rational Monotonicity. Rational monotonicity asserts that incorporating information which was not previously negated by existing knowledge should not cause the retraction of any prior conclusions.

$$(RM)\;\; \frac{\mathcal{K} \approx \alpha \mid\sim \gamma,\;\; \mathcal{K} \not\approx \alpha \mid\sim \neg\beta}{\mathcal{K} \approx \alpha \land \beta \mid\sim \gamma} \tag{10}$$

Lemma 7. *Naive bayes entailment does not satisfy rational monotonicity.*

Proof. Proving that naive bayes entailment doesn't satisfy cautious monotonicity concurrently establishes its non-compliance with rational monotonicity. □

5.4 Analysis

Failing to adhere to LM-rationality does not preclude the utility of naive bayes entailment. This method is not proposed as a substitute or enhancement of standard rational closure, but rather as a specialized query mechanism applicable in its own distinct contexts. Through a series of examples, we will illustrate the specific contexts in which naive bayes entailment proves useful, highlighting its ability to yield conclusions unattainable via standard rational closure. We will also expose its limitations by demonstrating its potential to generate illogical inferences.

5.4.1 Example 1. Consider the defeasible knowledge base $\mathcal{K} = \{\texttt{penguin} \rightarrow \texttt{bird}, \texttt{bird} \mid\!\sim \texttt{flies}, \texttt{penguin} \mid\!\sim \neg\texttt{flies}, \texttt{bird} \mid\!\sim \texttt{wings}, \texttt{emperor} \rightarrow \texttt{penguin}, \texttt{emperor} \mid\!\sim \texttt{wings}, \texttt{king} \rightarrow \texttt{penguin}, \texttt{king} \mid\!\sim \texttt{wings}, \texttt{chinstrap} \rightarrow \texttt{penguin}, \texttt{chinstrap} \mid\!\sim \neg\texttt{wings}\}$. Figure 7 illustrates a corresponding reduced minimal ranked interpretation. Consider the query $\mathcal{K} \mid\!\approx \texttt{penguin} \mid\!\sim \texttt{wings}$, for which standard rational closure would return false. However, naive bayes entailment discerns $\overline{c}\overline{k}\overline{e}w\overline{f}bp$ as the most probable model, resulting in the evaluation of $\mathcal{K} \mid\!\approx_{NB} \texttt{penguin} \mid\!\sim \texttt{wings}$ as true.

2	$k\overline{e}\overline{w}\overline{f}bp$ $e\overline{w}\overline{f}bp$ $cw\overline{f}bp$ fbp
1	$\overline{c}\overline{k}\overline{e}\overline{w}\overline{f}b\overline{p}$ $\overline{c}\overline{k}\overline{e}w\overline{f}b\overline{p}$ $\overline{c}\overline{k}\overline{e}\overline{w}fb\overline{p}$ $\overline{k}\overline{e}\overline{w}\overline{f}bp$ $\overline{c}w\overline{f}bp$
0	$\overline{c}\overline{k}\overline{e}\overline{w}\overline{b}\overline{p}$ $\overline{c}\overline{k}\overline{e}w\overline{b}\overline{p}$ $\overline{c}\overline{k}\overline{e}wfb\overline{p}$

Figure 7: Reduced minimal ranked interpretation $\mathcal{R}^{\mathcal{K}}_{RC}$.

An intriguing question arises from this: How does employing Bayesian reasoning to logical truth assignments yield any valuable results? The minimal partial penguin models, namely $\overline{k}\overline{e}\overline{w}\overline{f}bp$ and $\overline{c}w\overline{f}bp$, reveal two distinct types of penguin: those with wings (emperor and king) and those without (chinstrap). If a penguin lacks wings, it cannot be emperor or king, thus reducing the scenarios in which a penguin does not have wings by half, twice over. Conversely, if the penguin has wings, it cannot be chinstrap, cutting the scenarios in which a penguin does have wings only once. Consequently, there are twice as many scenarios in which a penguin possesses wings compared to those in which it does not.

5.4.2 Example 2. Let $\mathcal{K} = \{\texttt{dove} \rightarrow \texttt{bird}, \texttt{dove} \mid\!\sim \texttt{wings}, \texttt{dove} \mid\!\sim \texttt{flies}, \texttt{swan} \rightarrow \texttt{bird}, \texttt{swan} \mid\!\sim \texttt{wings}, \texttt{swan} \mid\!\sim \texttt{flies}, \texttt{penguin} \rightarrow \texttt{bird}, \texttt{penguin} \mid\!\sim \neg\texttt{wings}, \texttt{penguin} \mid\!\sim \neg\texttt{flies}, \texttt{chicken} \rightarrow \texttt{bird}, \texttt{chicken} \mid\!\sim \texttt{wings}, \texttt{chicken} \mid\!\sim \neg\texttt{flies}\}$. This knowledge contains information about four specific types of birds: two species that both have wings and can fly (dove and swan), and two species that cannot fly. Among the non-flying birds, chickens possess wings, while penguins do not. Notably, $\mathcal{K}$ does not contain any exceptional information; it does not explicitly state that $\texttt{bird} \mid\!\sim \texttt{wings}$, thus $\mathcal{K} \mid\!\approx \texttt{bird} \mid\!\sim \texttt{wings}$ evaluates to false.

NaiveBayesEntailment identifies $\overline{s}\overline{d}\overline{c}fwb\overline{p}$ as the most probable model for bird, making $\mathcal{K} \mid\!\approx_{NB} \texttt{bird} \mid\!\sim \texttt{wings}$ evaluate to true. It also follows that $\mathcal{K} \mid\!\approx_{NB} \texttt{bird} \mid\!\sim \texttt{flies}$, which might appear counter-intuitive at first glance, but is statistically correct. Since three of the four birds have wings, it is statistically typical for a bird to have wings. However, the number of birds that can and cannot fly are equal, meaning that the ability to fly has no impact on defining what is statistically typical for a bird. Among the statistically typical birds, two can fly, and one cannot.

Consider the addition of $\texttt{bird} \mid\!\sim \texttt{flies}$ to $\mathcal{K}$. Both penguin and chicken are now explicitly recognized as atypical birds, and segregated from those considered typical. Naive bayes entailment identifies the models $sd\overline{c}\overline{w}\overline{p}\overline{f}b$, $\overline{s}d\overline{c}\overline{w}\overline{p}\overline{f}b$, $s\overline{d}\overline{c}\overline{w}\overline{p}\overline{f}b$, $\overline{s}\overline{d}\overline{c}\overline{w}\overline{p}\overline{f}b$, $sd\overline{c}w\overline{p}\overline{f}b$, $\overline{s}d\overline{c}w\overline{p}\overline{f}b$, $s\overline{d}\overline{c}w\overline{p}\overline{f}b$, and $\overline{s}\overline{d}\overline{c}w\overline{p}\overline{f}b$ as the most probable models for $\texttt{bird} \wedge \neg\texttt{flies}$. As a result, both $\mathcal{K} \mid\!\approx_{NB} (\texttt{bird} \wedge \neg\texttt{flies}) \mid\!\sim \texttt{wings}$ and $\mathcal{K} \mid\!\approx_{NB} (\texttt{bird} \wedge \neg\texttt{flies}) \mid\!\sim \neg\texttt{wings}$ will evaluate to false. Lastly, consider the addition of $\texttt{kiwi} \rightarrow \texttt{bird}$, $\texttt{kiwi} \mid\!\sim \texttt{wings}$, and $\texttt{kiwi} \mid\!\sim \neg\texttt{flies}$ to $\mathcal{K}$. NaiveBayesEntailment now identifies $s\overline{c}d\overline{p}\overline{f}wb\overline{k}$, $s\overline{c}\overline{d}\overline{p}\overline{f}wb\overline{k}$, $\overline{s}d\overline{c}w\overline{p}\overline{f}b\overline{k}$, and $\overline{s}\overline{d}\overline{c}w\overline{p}\overline{f}b\overline{k}$ as most probable for $\texttt{b} \wedge \neg\texttt{flies}$, making $\mathcal{K} \mid\!\approx_{NB} (\texttt{bird} \wedge \neg\texttt{flies}) \mid\!\sim \texttt{wings}$ evaluate to true.

5.4.3 Example 3. NaiveBayesEntailment can also form conclusions that may not be desirable. Consider the knowledge base $\mathcal{K} = \{\texttt{bird} \mid\!\sim \texttt{flies}, \texttt{penguin} \rightarrow \texttt{bird}, \texttt{penguin} \mid\!\sim \neg\texttt{flies}, \texttt{emperor} \rightarrow \texttt{penguin}, \texttt{emperor} \mid\!\sim \texttt{wings}, \texttt{insect} \mid\!\sim \texttt{flies}, \texttt{ant} \rightarrow \texttt{insect}, \texttt{ant} \mid\!\sim \neg\texttt{flies}, \texttt{carpenter} \rightarrow \texttt{ant}, \texttt{carpenter} \mid\!\sim \neg\texttt{wings}, \texttt{leafcutter} \rightarrow \texttt{ant}, \texttt{leafcutter} \mid\!\sim \neg\texttt{wings}\}$. Intuitively, one would expect $\mathcal{K} \mid\!\approx_{NB} \texttt{penguin} \mid\!\sim \texttt{wings}$ to be true. However, under rational closure, penguins and ants are assigned the same level of typicality. Given that two ants in the knowledge lack wings, this complicates matters. Logically, there's no evidence to suggest mutual exclusivity between birds and insects. Consequently, the presence of wingless ants interferes with the probabilistic calculations for penguin, for which the most probable models are $\overline{e}\overline{w}b\overline{a}\overline{c}i\overline{l}\overline{f}p$, $\overline{e}\overline{w}ab\overline{c}i\overline{l}\overline{f}p$, $w\overline{e}b\overline{a}\overline{c}i\overline{l}\overline{f}p$ and $w\overline{e}ab\overline{c}i\overline{l}\overline{f}p$, making $\mathcal{K} \mid\!\approx_{NB} \texttt{penguin} \mid\!\sim \texttt{wings}$ false. This issue can be resolved by adding $\texttt{bird} \rightarrow \neg\texttt{insect}$ to the knowledge to explicitly indicate that birds and insects are mutually exclusive, or by using the query $\mathcal{K} \mid\!\approx_{NB} \texttt{penguin} \wedge \neg\texttt{insect} \mid\!\sim \texttt{wings}$, for which the most probable model is $w\overline{e}\overline{i}b\overline{a}\overline{c}\overline{l}\overline{f}p$.

6 DISCUSSION

The initial objective of this project was to develop a Bayesian refinement of rational closure, a goal that has been successfully realized. However, doing so necessitated a tractable method for storing and manipulating the information in each rank. This challenge inadvertently led to the development of reduced minimal ranked entailment, which further gave rise to irredundant base rank. These latter developments are arguably a more significant contribution, prompting a shift in the focus of the paper toward exploring their implications and applications, with naive bayes entailment serving more as an illustrative example.

The development and refinement of reduced entailment took a decent amount of time, leading to unmet objectives due to time constraints. One such objective was to empirically assess algorithmic complexity via benchmark tests using a random knowledge base generator. While these benchmarks could theoretically provide invaluable data on the algorithms' efficiency—particularly in the context of reduced minimal ranked entailment—the results would likely possess limited relevance. We have repeatedly emphasised that the performance of the methodologies presented in this paper is contingent on a complex interplay of various factors. These include the number of atoms, the number of formulas, the complexity of the formulas, variable ordering, redundancy in the knowledge, the number of ranks, the number of DIs sharing the same antecedent, the complexity of the query, the atypicality of the query, and so on. A thorough investigation into computational complexity would be better served by a more strategically selected set of problems, rather than those generated by a simple random knowledge base generator.

The key point is not so much the level of optimization achieved, but rather that using the standard approach to minimal ranked entailment ensures the worst-case scenario. This applies across the board: in constructing ranked interpretations, in storage requirements, in verifying entailment, and in developing model-theoretic extensions such as naive bayes entailment. Without the benefits provided by ROBDDs, the development and implementation of naive bayes entailment would have been entirely impractical.

7 RELATED WORK

The original paper on KLM style defeasible reasoning is [13], after which rational closure was introduced in [17], and lexicographic closure in [16]. The work in this paper is largely inspired by that of Casini et al. in [8]. Other research into model theoretic defeasible entailment includes that by Booth et al. in [5], and that by Cohen et al. in [9]. Other investigations into the optimisation of KLM style defeasible reasoning includes that by Hamilton et al. in [12].

8 CONCLUSION

Although the outcomes of this project diverged from our initial objectives, the endeavor has nonetheless been an insightful and productive exploration of rational closure. We've demonstrated that rational closure is a concept that can be interpreted through multiple lenses, and that probing these various perspectives can yield novel methodologies and understanding, such as the identification of redundancy in the base rank approach.

While naive bayes entailment may not stand out as a particularly sophisticated or useful form of entailment, its consideration has enriched our understanding of rational closure. The formulation of reduced minimal ranked entailment owes its existence to this nuanced understanding of the variances and interconnections between these two aspects of rational closure. Similarly, the emergence of an irredundant base rank was prompted by the imperative to scrutinize the relationships between the base rank methodology and the reduction of ranked interpretations.

The potential benefits of reduced minimal ranked entailment likely extend beyond what has been discussed in this paper. ROBDDs constitute a vibrant research domain with demonstrated efficacy in multiple sectors, including but not limited to digital circuit design and verification, symbolic model checking, data compression, and game theory. Consequently, a variety of other ROBDD procedures and methodologies have emerged that could potentially prove beneficial in the field of defeasible reasoning.

A number of research directions could result from this paper. One evident direction is the creation of a reduced version of lexicographic closure. A thorough exploration of the distinct advantages inherent in different approaches to implementing rational closure could yield a dynamic algorithm proficient in selecting the most efficacious strategy for a given knowledge base or query. Finally, this has limited the use of Bayesian reasoning to data within a ranked interpretation, representing a relatively basic amalgamation of the two methodologies. Investigating the integration of rational closure with more established Bayesian inference techniques, such as Bayesian networks, using real probabilistic data, could lead to more meaningful and robust results.

REFERENCES

[1] Akers. 1978. Binary Decision Diagrams. *IEEE Trans. Comput.* C-27, 6 (1978), 509–516. https://doi.org/10.1109/TC.1978.1675141
[2] Thomas Bayes. 1763. An essay towards solving a problem in the doctrine of chances. *Philosophical Transactions of the Royal Society of London* 53 (1763), 370–418.
[3] Mordechai Ben-Ari. 2012. *Propositional Logic: Formulas, Models, Tableaux*. Springer London, London, 1, 7–47.
[4] B. Bollig and I. Wegener. 1996. Improving the variable ordering of OBDDs is NP-complete. *IEEE Trans. Comput.* 45, 9 (1996), 993–1002. https://doi.org/10.1109/12.537122
[5] Richard Booth, Giovanni Casini, T. Meyer, and Ivan José Varzinczak. 2015. On the Entailment Problem for a Logic of Typicality. In *International Joint Conference on Artificial Intelligence*. https://api.semanticscholar.org/CorpusID:6743019
[6] Bryant. 1986. Graph-Based Algorithms for Boolean Function Manipulation. *IEEE Trans. Comput.* C-35, 8 (1986), 677–691. https://doi.org/10.1109/TC.1986.1676819
[7] Randal E. Bryant. 2019. Binary Decision Diagrams : An Algorithmic Basis for Symbolic Model Checking. https://api.semanticscholar.org/CorpusID:202740601
[8] Giovanni Casini, Thomas Meyer, and Ivan Varzinczak. 2019. Taking defeasible entailment beyond rational closure. In *Logics in Artificial Intelligence: 16th European Conference, JELIA 2019, Rende, Italy, May 7–11, 2019, Proceedings 16*. Springer, 182–197.
[9] Jaron Cohen, Carl Combrinck, and Thomas Meyer. 2022. *Model-based Defeasible Reasoning*. SACAIR 2022 Organising Committee, Online, 150–179. https://2022.sacair.org.za/wp-content/uploads/2023/01/SACAIR_ConferenceProceedings-v7.pdf
[10] Rolf Drechsler, Bernd Becker, and Nicole Göckel. 1996. Genetic algorithm for variable ordering of OBDDs. *Computers and Digital Techniques, IEE Proceedings -* 143 (12 1996), 364 – 368. https://doi.org/10.1049/ip-cdt:19960789
[11] L. Giordano, V. Gliozzi, N. Olivetti, and G.L. Pozzato. 2015. Semantic characterization of rational closure: From propositional logic to description logics. *Artificial Intelligence* 226 (2015), 1–33. https://doi.org/10.1016/j.artint.2015.05.001
[12] Joel Hamilton, Joonsoo Park, Aidan Bailey, and Thomas Meyer. 2021. *An Investigation into the Scalability of Defeasible Reasoning Algorithms*. SACAIR 2021 Organising Committee, Online, 235–251. https://protect-za.mimecast.com/s/OFYSCpgo02fL1l9gtDHUkY
[13] Sarit Kraus, Daniel Lehmann, and Menachem Magidor. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence* 44, 1 (1990), 167–207. https://doi.org/10.1016/0004-3702(90)90101-5
[14] Pierre-Simon Laplace. 1812. *Théorie Analytique des Probabilités*. Courcier.
[15] C. Y. Lee. 1959. Representation of switching circuits by binary-decision programs. *The Bell System Technical Journal* 38, 4 (1959), 985–999. https://doi.org/10.1002/j.1538-7305.1959.tb01585.x
[16] Daniel Lehmann. 1995. Another perspective on default reasoning. *Annals of mathematics and artificial intelligence* 15 (1995), 61–82.
[17] Daniel Lehmann and Menachem Magidor. 1992. What does a conditional knowledge base entail? *Artificial Intelligence* 55, 1 (1992), 1–60. https://doi.org/10.1016/0004-3702(92)90041-U
[18] R. Rudell. 1993. Dynamic variable ordering for ordered binary decision diagrams. In *Proceedings of 1993 International Conference on Computer Aided Design (ICCAD)*. 42–47. https://doi.org/10.1109/ICCAD.1993.580029
[19] Leslie G. Valiant. 1979. The Complexity of Enumeration and Reliability Problems. *SIAM J. Comput.* 8, 3 (1979), 410–421. https://doi.org/10.1137/0208032 arXiv:https://doi.org/10.1137/0208032
[20] D.H. Wolpert and W.G. Macready. 1997. No free lunch theorems for optimization. *IEEE Transactions on Evolutionary Computation* 1, 1 (1997), 67–82. https://doi.org/10.1109/4235.585893