Exercise 1:

Tutorial Exercise 1:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

$x_t = c\sum_{k=0}^{\infty}\phi^k + \sum_{k=0}^{\infty}\phi^k \varepsilon_{t-k}$. Derive 5a, b, c.

5a $\mu = \frac{c}{1-\phi}$ 5b $\gamma_0 = \frac{\sigma^2}{1-\phi^2}$ 5c = $\gamma_k = \frac{\sigma^2}{1-\phi^2}\phi^k$.

1. $\mu = E(x_t) = E\left(c\sum_{k=0}^{\infty}\phi^k + \sum_{k=0}^{\infty}\phi^k\varepsilon_{t-k}\right)$ $\mu = E(x_t)$, stationarity assumption.

$\mu = c\sum_{k=0}^{\infty}\phi^k + \sum_{k=0}^{\infty}\phi^k 0$ $E(\varepsilon_{t-k}) = 0$, white noise assumption.

$\mu = c\sum_{k=0}^{\infty}\phi^k$ $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

$\therefore \mu = c \times \frac{1}{1-r} = \frac{c}{1-\phi}$

2. $\gamma_0 = var(x_t) = E[(x_t-u)^2]$

$\left(\left(c\sum_{k=0}^{\infty}\phi^k + \sum_{k=0}^{\infty}\phi^k\varepsilon_{t-k}\right) - u\right)^2$

$c\sum_{k=0}^{\infty}\phi^k = u$.

$= \left(u + \sum_{k=0}^{\infty}\phi^k\varepsilon_{t-k} - u\right)^2$

$= E\left[\left(\sum_{k=0}^{\infty}\phi^k\varepsilon_{t-k}\right)^2\right]$

$= E\left[\sum_{k=0}^{\infty}\phi^{k^2}\varepsilon_{t-k}^2\right]$ $E[\varepsilon_t^2] = \sigma^2 > 2$. white noise assumption.

$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ from above.

$\gamma_0 = \frac{1^2}{1^2-r^2} \times \sigma^2 = \frac{\sigma^2}{1-\phi^2}$

3. $\gamma_k = \frac{\sigma^2}{1-\phi^2}\phi^k$

$\gamma_k = cov(x_t, x_{t-k}) = E[(x_t - u)(x_{t-k} - u)]$

$E[x_t - u] = \sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k}$ from variance

$E\left[\sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k} \cdot \sum_{k=0}^{\infty} \phi^{k+1} \varepsilon_{t-k-1}\right]$

$= \sum_{k=0}^{\infty} \phi^k \phi^{k+1} \cdot E(\varepsilon_{t-k}^{\ 2})$

$= \phi^k \sum_{k=0}^{\infty} \phi^{k^2} \cdot \sigma^2$

$= \phi^k \cdot \frac{1^2}{1^2 - \phi^2} \cdot \sigma^2$

$= \frac{\sigma^2}{1-\phi^2}\phi^k$

Exercise 2:

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#Exercise 2:
roots <- Mod(polyroot(c(1, -0.5, 0.8, 0.1, -0.2)))

print(roots)
print("Output of the Modpolyroot function is 1, 1, 2.23, 2.23. Because
one or more of the roots are 1, the process is of unit root.")
```

It is a unit root process because one or more of the roots is equal to 1.

Exercise 3:

Tutorial exercise 3.

$$x_t = 0.6x_{t-1} - 0.9x_{t-2} + 0.4x_{t-3} + \varepsilon_t \qquad \varepsilon_t \sim N(0,1)$$

1. $\mu = \dfrac{c}{1-\phi}$

$$= \frac{0}{1-0.6+0.9-0.4} = 0$$

2. $\gamma_0 = 0.6\gamma_1 - 0.9\gamma_2 + 0.4\gamma_3 + 1$

$$\gamma_1 = 0.6\gamma_{1-1} - 0.9\gamma_{1-2} + 0.4\gamma_{1-3}$$
$$\gamma_2 = 0.6\gamma_{1-2} - 0.9\gamma_{2-2} + 0.4\gamma_{2-3}$$
$$\gamma_3 = 0.6\gamma_{1-3} - 0.9\gamma_{2-3} + 0.4\gamma_{3-3}$$

$\gamma_0 = 1$

$\gamma_k = \gamma_{-k}$

$$\gamma_0 - 0.6\gamma_1 + 0.9\gamma_2 - 0.4\gamma_3 = 1$$
$$\gamma_1 - 0.6\gamma_0 + 0.9\gamma_1 - 0.4\gamma_2 = 0$$
$$\gamma_2 - 0.6\gamma_1 + 0.9\gamma_0 - 0.4\gamma_1 = 0$$
$$\gamma_3 - 0.6\gamma_2 + 0.9\gamma_1 - 0.4\gamma_0 = 0$$

$$\begin{array}{cccc|c} 0 & 1 & 2 & 3 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -0.6 & 0.9 & -0.4 & 1 \\ -0.6 & 1.9 & -0.4 & 0 & 0 \\ 0.9 & -1 & 1 & 0 & 0 \\ -0.4 & 0.9 & -0.6 & 1 & 0 \end{array}\right]$$

Then we use R to solve for γ_0

SEE R CODE FILE

R solves: (4dp)

$$\begin{bmatrix} 3.1928 \\ 0.5109 \\ -2.3627 \\ -0.6003 \end{bmatrix} \quad \therefore \gamma_0 = 3.1928$$

3. $\rho_k = \dfrac{\gamma_k}{\gamma_0}$ $\overline{3.1928}$

$\rho_1 = \dfrac{0.5109}{3.1928} = 0.16$

4. $\rho_2 = \dfrac{-2.3627}{3.1928} = -0.74$

5. $\rho_3 = \dfrac{-0.6003}{3.1928} = -0.188$

6.

$\gamma_4 = 0.6\gamma_1 - 0.9\gamma_2 + 0.4\gamma_3$

$\rho_4 = \dfrac{0.6(0.5109) - 0.9(-2.3627) + 0.4(-0.6003)}{3.1928} = 0.6868$

Solving the gamma matrix:

```
#Exercise 3:
#Solving Gamma Matrix
A <- matrix(data=c(1,-0.6,0.9,-0.4,-0.6,1.9,-0.4,0,0.9,-1,1,0,-
0.4,0.9,-0.6,1), nrow=4, ncol=4, byrow=TRUE)
b <- matrix(data=c(1, 0, 0, 0), nrow=4, ncol=1, byrow=FALSE)
round(solve(A, b), 4)
```

Exercise 4:

```
#Exercise 4:
set.seed(123456)
x <- arima.sim(n = 1e3, list(ar = c(0.6, -0.9, 0.4)), innov=rnorm(1e3))
acf4 <- acf(x, lag.max = 20, main="ACF", ylab="", ylim=c(-1,1), lwd =
3, col="darkblue", ci.col = "coral1")
round(acf4$acf[1:5], 4)
```

The code for correlation values returns:
Rho_0 = 1
Rho_1 = 0.1541
Rho_2 = -0.7492
Rho_3 = -0.1762
Rho_4 = 0.6407

The simulated values are close to the exact values calculated in Exercise 3, however it appears as *k* increases, the simulated value drifts further from the exact value.

Exercise 5:
The process is an AR(4). This is because the ACF is decaying in absolute value and the PACF is spiking. The process is of order 4 because the significant spikes reaches lag 4.

Exercise 6:
The process is an MA(3 or 4) because the ACF is spiking and the PACF is decaying in absolute values. It is hard to define the order because the graph is not too clear. Some further analysis is required to determine the actual order.