Time Series Econometrics Tutorial 5: Forecasting, Smoothing and Seasonal Models – Spring 2019

Exercise 1:

Exercise 1

Derive an expression for conditional variance of the forecast error for 3 step based on AR(p) model 1.

$e_{T+h} := x_{T+h} - \bar{x}_{T+h}$

$\bar{x}_{T+h} := E(x_{T+h} | I_T)$

$x_t = c + \phi x_{t-1} + \dots + \phi_p x_{t-p} + \varepsilon_t$

$\bar{x}_{T+h} := E(x_{T+h} / I_T)$

$\bar{x}_{T+h} = c + \phi_1 E(x_{T+h-1} | I_T) + \dots + \phi_p E(x_{T+h-p} | I_T) + E(\varepsilon_{T+h} | I_T)$

Forecasted + True

$\bar{x} = c + \sum_{j=1}^{h-1} \phi_j E(x_{T+h-j} | I_T) + \sum_{j=h}^{p} \phi_j x_{T+h-j}$

but $\bar{x}_{T+h} := E(x_{T+h} | I_T)$

so $= c + \sum_{j=1}^{h-1} \phi_j \bar{x}_{T+h-j} + \sum_{j=h}^{p} \phi_j x_{T+h-j}$

Error: $e_{T+h} := x_{T+h} - \bar{x}_{T+h}$

True - Forecasted

where does p go?

cancels

$\left(c + \sum_{j=1}^{p} \phi_j x_{T+h-j} + \varepsilon_{T+h}\right) - \left(c + \sum_{j=1}^{h-1} \phi_j \bar{x}_{T+h-j} + \sum_{j=h}^{p} \phi_j x_{T+h-j}\right)$

$= \sum_{j=1}^{h-1} \phi_j (x_{T+h-j} - \bar{x}_{T+h-j}) + \varepsilon_{T+h}$

but $e_{T+h} := x_{T+h} - \bar{x}_{T+h}$

so $= \sum_{j=1}^{h-1} \phi_j (e_{T+h-j}) + \varepsilon_{T+h}$

error of today = 0 bc we know it.

1-step ahead:

$e_{T+1} = x_{T+1} - \bar{x}_{T+1} = \phi_1 e_{T+1-1} + \varepsilon_{T+1}$

$e_{T+1} = x_{T+1} - \bar{x}_{T+1} = \varepsilon_{T+1}$

2-steps ahead:

$e_{T+2} = x_{T+2} - \bar{x}_{T+2} = \sum_{j=1}^{2-1} \phi_j e_{T+h-j} + \varepsilon_{T+h}$

$= \phi_1 e_{T+2-1} + \varepsilon_{T+2}$

$$= c + \sum_{j=1}^{h-1} \phi_j E(x_{T+h-j})$$

$$e_{T+2} = \phi_1 \varepsilon_{T+1} + \varepsilon_{T+2}$$

from above, $e_{T+1} = \varepsilon_{T+1}$

so $e_{T+2} = \phi_1 \varepsilon_{T+1} + \varepsilon_{T+2}$ $\quad (1+\phi_1^2)\sigma^2$

3-steps ahead:

$$e_{T+3} = x_{T+3} - \hat{x}_{T+2} = \sum_{j=1}^{3-1} \phi_j e_{T+h-j} + \varepsilon_{T+3}$$

* Drop $E[x]$ somewhere?

$$= \phi_1 e_{T+3-1} + \phi_2 e_{T+3-2} + \varepsilon_{T+3}$$

$$e_{T+3} = \phi_1 e_{T+2} + \phi_2 e_{T+1} + \varepsilon_{T+3}$$

but

$e_{T+1} = \varepsilon_{T+1}$ and $e_{T+2} = \phi_1 \varepsilon_{T+1} + \varepsilon_{T+2}$.

$$e_{T+3} = \phi_1(\phi_1 \varepsilon_{T+1} + \varepsilon_{T+2}) + \phi_2(\varepsilon_{T+1}) + \varepsilon_{T+3}$$

$$e_{T+3} = \phi_1^2 \varepsilon_{T+1} + \phi_1 \varepsilon_{T+2} + \phi_2 \varepsilon_{T+1} + \varepsilon_{T+3}$$

$$= (\phi_1^2 + \phi_2)\varepsilon_{T+1} + \phi_1 \varepsilon_{T+2} + \varepsilon_{T+3}$$

$E[\varepsilon_t^2] = \sigma^2 > 0$

$$= (\phi_1^2 + \phi_2)\sigma^2 + \phi_1 \sigma^2 + \sigma^2$$

$$= ((\phi_1^2 + \phi_2)^2 + \phi_1^2 + 1)\sigma^2$$

$$(\phi_1^4 + 2\phi_1^2\phi_2 + \phi_2^2 + \phi_1^2 + 1)\sigma^2$$

$$(\phi_1^4 + \phi_1^2(2\phi_2 + 1) + \phi_2^2 + 1)\sigma^2$$

$$Var(e_{T+3} | I_T) = (1 + \phi_1^2(2\phi_2 + 1) + \phi_1^4 + \phi_2^2)\sigma^2$$

assuming $\varepsilon_t \sim N(0, \sigma^2)$

Exercise 2:

Exercise 2.

AR(4) $x_t = 0.8x_{t-1} - 0.6x_{t-2} - 0.1x_{t-3} + 0.2x_{t-4} + \varepsilon_t \qquad \varepsilon_t \sim N(0,4)$

$x_1 = 0.8639$
$x_2 = -0.8037$
$x_3 = -3.4042$
$x_4 = -4.3961$

$x_{T+h} = 0.8_{T+h-1} - 0.6x_{T+h-2} - 0.1x_{t+h-3} + 0.2x_{T+h-4} + \varepsilon_t$

$x_{T+5} = 0.8(-4.3961) - 0.6(-3.4042) - 0.1(-0.8037) + 0.2(0.8639) + \varepsilon_t$

$x_{T+5} = -1.22121$

$x_{T+6} = 0.8(-1.22121) - 0.6(-4.3961) - 0.1(-3.4042) + 0.2(-0.8037) + \varepsilon_t$

$x_{T+6} = 1.834972$

$x_{T+7} = 0.8(1.834972) - 0.6(-1.22121) - 0.1(-4.3961) + 0.2(-3.4042) + \varepsilon_t$

$x_{T+7} = 1.9594736$

$x_{T+8} = 0.8(1.9594736) - 0.6(1.834972) - 0.1(-1.22121) + 0.2(-4.3961) + \varepsilon_t$

$x_{T+8} = -0.29050332$

forecast
$\bar{x}_5 = -1.22121$
$\bar{x}_6 = 1.834972$
$\bar{x}_7 = 1.9594736$
$\bar{x}_8 = -0.29050332$

95% CI
$\bar{x}_5 = -5.1421, 2.6979$
$\bar{x}_6 = -3.18508, 6.85502$
$\bar{x}_7 = -3.06302, 6.98197$
$\bar{x}_8 = -5.9545, 5.3735$

$x_{T+h} \pm 1.96 \times \text{SDEV}(e_{T+h}|\mathcal{I}_T) \qquad \text{Var}(e_{T+h}|\mathcal{I}_T) = 4$

$\text{SDEV}(e_{T+h}|\mathcal{I}_T) = \sqrt{4}$

$\bar{x}_5$ CI
$-1.22121 \pm 1.96 \times 2 \qquad -5.1421, 2.6979$

$\bar{x}_6$ CI
$= 1.834972 \pm 1.96 \times \sqrt{(1+0.8^2) \times 4} = -3.18508, 6.85502$

$\bar{x}_7$ CI $= 1.9594736 \pm 1.96 \times \sqrt{(1 + 0.8^2(2 \times -0.6 + 1) + 0.8^4 + -0.6^2) \times 4}$

$= -3.06302, 6.98197$

$e_{T+4} = x_{T+3} - \bar{x}_{T+2} = \sum_{j=1}^{4-1} \phi_j e_{T+h-j} + \varepsilon_{T+4}$

$= \phi_1 e_{T+4-1} + \phi_2 e_{T+4-2} + \phi_3 e_{T+4-3} + \varepsilon_{T+4}$

$\phi_1 e_{T+3} + \phi_2 e_{T+2} + \phi_3 e_{T+1} + e_{T+4}$

$= \phi_1\left((\phi_1^2 + \phi_2)\varepsilon_{T+1} + \phi_1 \varepsilon_{T+2} + \varepsilon_{T+3}\right) + \phi_2(\phi_1 \varepsilon_{T+1} + \varepsilon_{T+2}) + \phi_3 \varepsilon_{T+1} + e_{T+4}$

$(\phi_1^3 + \phi_1\phi_2)\varepsilon_{T+1} + \phi_1^2 \varepsilon_{T+2} + \phi_1 \varepsilon_{T+3} + \phi_1\phi_2 \varepsilon_{T+1} + \phi_2 \varepsilon_{T+2} + \phi_3 \varepsilon_{T+1} + \varepsilon_{T+4}.$

$(\phi_1^3 + 2\phi_1\phi_2 + \phi_3)\varepsilon_{T+1} + (\phi_1^2 + \phi_2)\varepsilon_{T+2} + \phi_1 \varepsilon_{T+3} + \varepsilon_{T+4}$

ab^2
$= a^2b^2$

$\text{Var}[e_{T+4}|\mathcal{I}_T] = \left[(\phi_1^3 + 2\phi_1\phi_2 + \phi_3)^2 + (\phi_1^2\phi_2)^2 + \phi_1^2 + 1\right]\sigma^2$

$\phi_1 = 0.8$
$\phi_2 = -0.6$
$\phi_3 = -0.1$
$\phi_4 = 0.2$

$\text{SDev}[e_{T+4}|\mathcal{I}_T] = \sqrt{\left[(0.8^3 + 2 \times 0.8 \times -0.6 + (-0.1))^2 + (0.8^2 \times -0.6)^2 + 0.8^2 + 1\right] \times 4}$

$2\sqrt{1.4449}$ or 2.8898

$\bar{x}_8$ CI $= -0.29050332 \pm 1.96 \times 2\sqrt{1.449}$

$= -5.9545, 5.3735$

Exercise 3:

$$x_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}$$

$$\hat{x}_{T+h} := E(x_{T+h} | \mathcal{I}_T)$$
$$= c + E(\varepsilon_{T+h} | \mathcal{I}_T) + \theta_1 E(\varepsilon_{T+h} | \mathcal{I}_T) + \dots + \theta_q E(\varepsilon_{T+h-q} | \mathcal{I}_T)$$
$$= c + \sum_{j=h}^{q} \theta_j \varepsilon_{T+h-j}$$

$$e_{T+h} := x^{obs}_{T+h} - \hat{x}^{fore}_{T+h}$$
$$= \left(c + \varepsilon_{T+h} + \sum_{j=1}^{q} \theta_j \varepsilon_{T+h-j}\right) - \left(c + \sum_{j=h}^{q} \theta_j \varepsilon_{T+h-j}\right)$$
$$e_{T+h} = \varepsilon_{T+h} + \sum_{j=1}^{\min(h-1,q)} \theta_j \varepsilon_{T+h-j}$$

$$e_{T+3} = \varepsilon_{T+3}^2 + \theta_1 \varepsilon_{T+2}^2 + \theta_2 \varepsilon_{T+1}^2$$
$$\sigma^2 + \theta_1^2 \sigma^2 + \theta_2^2 \sigma^2$$
$$Var(e_{T+3} | \mathcal{I}_T) = (1 + \theta_1^2 + \theta_2^2)\sigma^2$$

Exercise 4:

Exercise 4:

MA(3): $x_t = \varepsilon_t - 0.5\varepsilon_{t-1} + 0.9\varepsilon_{t-2} + 0.5\varepsilon_{t-3}$ $\quad \varepsilon_t \sim N(0,4)$

$x_{T+h} = \varepsilon_{T+h-q} - 0.5\varepsilon_{T+h-1} + 0.9\varepsilon_{T+h-2} + 0.5\varepsilon_{T+h-3}$

$\varepsilon_1 = 1.6675$

$\varepsilon_2 = -0.5521$

$\varepsilon_3 = -0.7100$

$x_{T+4} = 0 - 0.5(-0.7100) + 0.9(-0.5521) + 0.5(1.6675)$

$x_{T+4} = 0.69186$

$\hat{x}_4 =$

$\hat{x}_5 =$

$\hat{x}_6 =$

$\hat{x}_7 =$

$x_{T+5} = 0 - 0 + 0.9(-0.7100) + 0.5(-0.5521)$

$x_{T+5} = -0.91505$

$x_{T+6} = 0 - 0 + 0 + 0.5(-0.7100)$

$x_{T+6} = -0.355$

$x_{T+7} = 0 - 0 + 0 + 0$

$x_{T+7} = 0$

Exercise 5:

Table 1

Month/Year	Forecasted Yield	Actual Yield
December 2010	4.8815	5.02
January 2011	4.8905	5.04
February 2011	4.8982	5.22
March 2011	4.9089	5.13
April 2011	4.9158	5.16
May 2011	4.9182	4.96
June 2011	4.9201	4.99
July 2011	4.9182	4.93
August 2011	4.9201	4.37
September 2011	4.9217	4.09
October 2011	4.9229	3.98
November 2011	4.9239	3.87

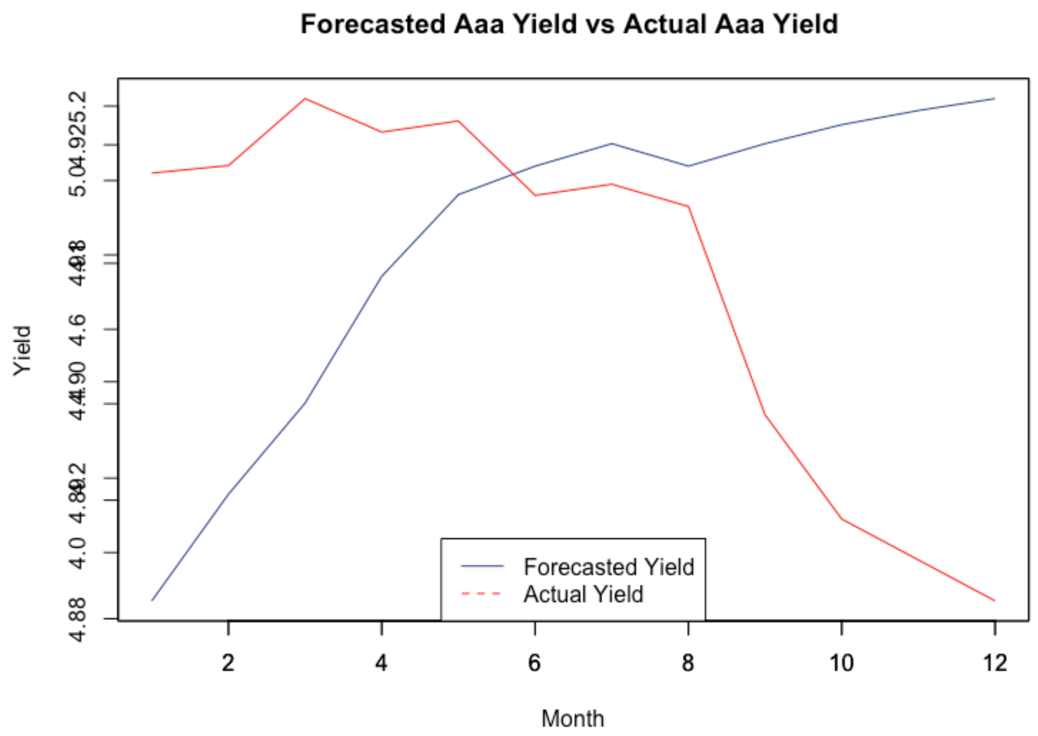
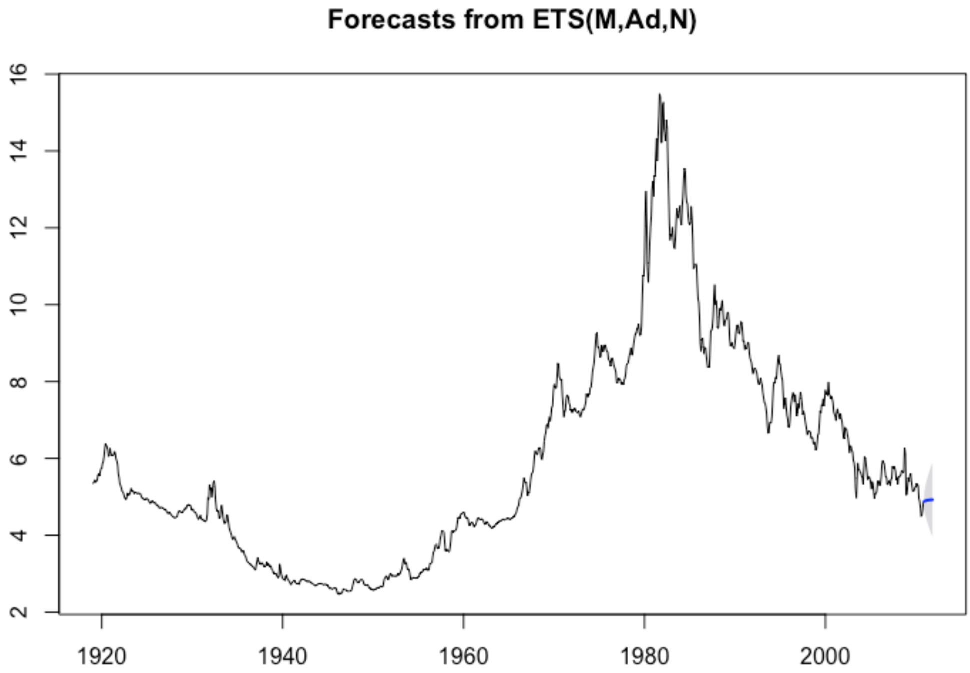


Figure 1



Exponential smoothing forecast with 95% confidence interval

Figure 2