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Collaborators: ______________________

CS181 Winter 2019 – Problem Set 3

Due Monday, February 11, 11:59 pm

- Please write your student ID **and the names of anyone you collaborated with** in the spaces provided and attach this sheet to the front of your solutions. **Please do not include your name anywhere since the homework will be blind graded.**
- An extra credit of **5%** will be granted to solutions written using LaTeX. Here is one place where you can create LaTeXdocuments for free: `https://www.overleaf.com/`. The link also has tutorials to get you started. There are several other editors you can use.
- If you are writing solutions by hand, please write your answers in a neat and readable hand-writing.
- Always explain your answers. When a proof is requested, you should provide a rigorous proof.
- If you don't know the answer, write "I don't know" along with a clear explanation of what you tried. For example: "I couldn't figure this out. I think the following is a start, that is correct, but I couldn't figure out what to do next. [[Write down a start to the answer that you are sure makes sense.]] Also, I had the following vague idea, but I couldn't figure out how to make it work. [[Write down vague ideas.]]" At least 20% will be given for such an answer.

  Note that if you write things that do not make any sense, no points will be given.
- The homework is expected to take anywhere between 8 to 14 hours. You are advised to start early.
- Submit your homework online on Gradescope.

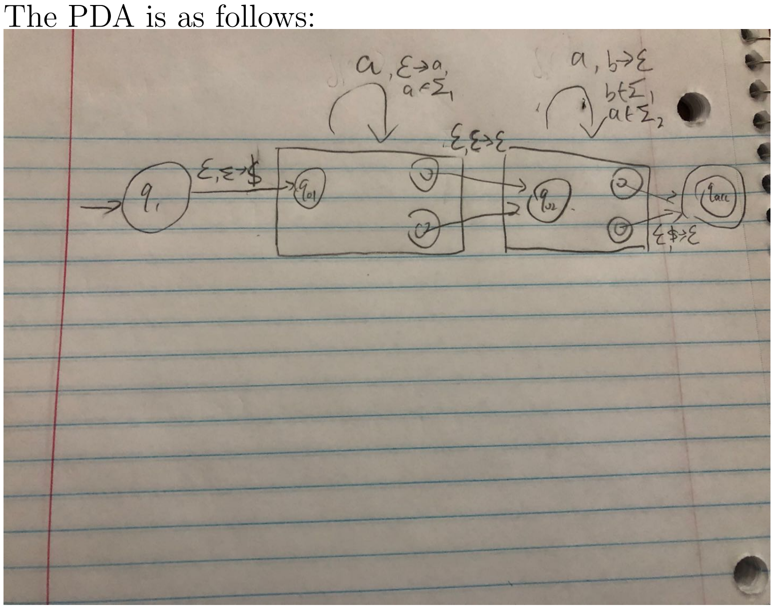
> Note: *Suggested practice problems from the book: 2.4 and 2.5. Please, do not turn in solutions to problems from the book.*

Problem 1

We can solve this problem by constructing a PDA $M = \{Q, \Sigma, \Gamma, \delta, q_1, F\}$. Since, A, B are two regular languages, there exists two DFA $M_1 = \{Q_1, \Sigma_1, \delta_1, q_{01}, F_1\}, M_2 = \{Q_2, \Sigma_2, \delta_2 q_{02}, F_2\}$ that accepts A and B respectively. The idea is, we add ϵ-transitions from all accept states of M_1 to the start state of M_2. For the first half of the PDA, $\delta = \delta_1$ for the state transitions, and push all symbols onto the stack. For the second part of the PDA, $\delta = \delta_2$ for the state transitions, and pop the top element of the stack if the element is not \$. The accepted states of M_2 have ϵ-transitions to the accept state q_{acc} we define for the PDA, the transition will only take place when the top element of the stack is \$. A more formal definition of the PDA is as follows:

- $Q = Q_1 \cup Q_2 \cup \{q_1, q_{acc}\}$, q_1 is the new start state, q_{acc} is the new accept state.
- $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \{\epsilon\}$
- $\Gamma = \Sigma_1 \cup \{\epsilon, \$\}$
- q_1 is the new start state
- $F = q_{acc}$, q_{acc} is the new accept state.
- $\delta(q, a, b) = \begin{cases} (q_{01}, \$) & \textbf{if } q = q_1,\ \ a = \epsilon, b = \epsilon \\ (\delta_1(q, a), a) & \textbf{if } q \in Q_1,\ \ a \in \Sigma_1, b = \epsilon \\ (q_{02}, \epsilon) & \textbf{if } q \in F_1, a = \epsilon, b = \epsilon \\ (\delta_2(q, a), \epsilon) & \textbf{if } q \in Q_2,\ \ a \in \Sigma_2, b \in \Sigma_1 \\ (q_{acc}, \epsilon) & \textbf{if } q \in F_2, a = \epsilon, b = \$ \end{cases}$

The PDA is as follows:



Suppose $s \in A \nabla B$, M will begin in q_1 and we can have ϵ-transition into q_{01} by pushing \$ onto the stack. After processing the first part of the string, we end up in one of the accept states in M_1, with the stack contains the full string x. Then we can have a ϵ-transition into q_{02} without modifying the stack. After processing the remaining part of string s, we end up in one of the accept states of M_2, and since $s \in A\nabla B$, $|x| = |y|$, the stack only contains a \$ symbol. Then the final ϵ-transition will pop the \$ symbol and lead M into its accept state q_{acc}. Hence, $A\nabla B \subseteq L(M)$.

Suppose s is accpeted by M. The only possible transitions to reach M_2 part of the PDA are those with initial input $x \in A$. The only possible transitions to reach states that can have a transition to the accept state q_{acc} are those with subsequent input $y \in M_2$. Now, in order to go to q_{acc}, the PDA must be in one of the accept state of M_2, and the stack in the meantime, must only contain a single \$ symbol. This, means the number of pushes ($|x|$) equals the number of pops ($|y|$). Hence, $s = xy, x \in A, y \in B, |x| = |y|$. Hence, $L(M) \subseteq A\nabla B$.

Hence, $L(M) = A\nabla B$, and hence $A\nabla B$ is a CFG. (Proven)

Problem 2

(a) The intuition is if $x \neq y$, there must be a first position i such that $x_i \neq y_i$ if $|x| = |y|$, or $|x| \neq |y|$. Let G be the CFG shown below. We claim that $L(G) = L_1$.

Grammar:
$S \to A0C \mid B1C \mid E$
$A \to 1C\$ \mid DAD$
$B \to 0C\$ \mid DBD$
$C \to DC \mid \epsilon$
$D \to 0 \mid 1$
$E \to DED \mid \$DC \mid CD\$$

The first two cases of S handles the case when the ith element of x and y are different, $i \in N$. Let's consider when $S = A0C$, the case when $S = B1C$ can be analysed in exactly the same way. When $S = A0C$, S can be written as $S = u1v\$w0x$, where u represents string derived from first $D = 0/1$ in DAD, $1v\$$ represents the string when A generates when A becomes $1C\$$, and v is the string derived from C. w is the right part of DAD, and finally x is the string derived from the second C in $A0C$. Since DAD would expand in a symmetrical manner, $|u| = |w|$, and $x = u1v,\ y = w0x$ must be different in $(|u|+1)th$ element. And from the grammar, we can see $|u|, |v|, |w|, |x|$ can be zero. This case does not require $|x| = |y|$, but we still need to cover when $|x| \neq |y|$ with all the first $min(|x|, |y|)$ element of the two strings are the same.

The third case of S handles the case when $|x| \neq |y|$. We can see that it generates $S = D...D\$D...DCD$ or $S = D...DCD\$D...D$. Since the two $D...D$ in S are generated by the rule DED, both have the same length. Since $|CD| \geq 1$, $|x| \neq |y|$. This case also covers the case when $x/y = \epsilon$: $S = \$DC \mid CD\$$ can serve the purpose.

Suppose $s \in L(G)$. According to the above analysis, a string s generated by G either has x, y differ in ith element or differ in length. Thus, $s \in L_1 \ \Rightarrow\ L(G) \subseteq L_1$.

Suppose $s \in L_1$. Then, either x, y differ in at least one corresponding position, or x, y differ in length, both are covered in the CFG. Thus, $s \in L(G) \ \Rightarrow\ L_1 \subseteq L(G)$.

Hence, we proved our claim and L_1 is a CFL.

(b) The intuition is if $x \neq y$, there must be a first position i such that $x_i \neq y_i$.
Let G be the CFG shown below. We claim that $L(G) = L_2$.

Grammar:
$S \rightarrow AB \mid BA$
$A \rightarrow CAC \mid 0$
$B \rightarrow CBC \mid 1$
$C \rightarrow 0 \mid 1$

Suppose $s \in L(G)$. From the grammar of $L(G)$, we can see that s is divided into two parts with a \$ in between. There are two situations: the first part contains at least a '1' and the second contains at least a '0', or vice versa. Thus, $s = a0bc1d$ or $s = a1bc0d$, $|a| = |b|$, $|c| = |d|$. Also, since A, B are two odd-length strings, s must be able two be divided into two strings of equal length. In fact, the first $(|a| + |c| + 1)$ elements belong to the first substring and the first 0/1 is at the $(|a| + 1)th$ position. The second 1/0 is also at the $(2|a| + 1 + |c| + 1) - (|a| + |c| + 1) = (|a| + 1)th$ position of the second substring. Hence, we can write $s = xy, |x| = |y|, x \neq y$, since the $(|a| + 1)th$ element of x and y must be different. Therefore, $s \in L_2 \;\Rightarrow\; L(G) \subseteq L_1$.

Suppose $s \in L_2$. Hence, there must be a first position i such that $x_i \neq y_i$. Hence, $s = a0ba1c$ or $s = a1ba0c$, where $(|a| + |b| + 1) = \frac{|x|}{2}$, $|b| = |c|$. Since, $|ba| > |c| = |b|$, we can also write $s = a0de1c$, or $s = a1de0c$, where $|e| = |c|$, $|d| = |ba| - |e| = |b| + |a| - |c| = |b| + |a| - |b| = |a|$. This is the form we defined for the strings generated by the above CFL. Hence, s can be generated by the CFLGabove. Therefore, $s \in L(G) \;\Rightarrow\; L_2 \subseteq L(G)$.

Hence, we proved our claim and L_2 is a CFL.