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Collaborators: ____________________

CS181 Winter 2019 – Problem Set 2

Due Tuesday, February 5, 11:59 pm

- Please write your student ID **and the names of anyone you collaborated with** in the spaces provided and attach this sheet to the front of your solutions. **Please do not include your name anywhere since the homework will be blind graded.**
- An extra credit of **5%** will be granted to solutions written using LaTeX. Here is one place where you can create LaTeXdocuments for free: `https://www.sharelatex.com/`. The link also has tutorials to get you started. There are several other editors you can use.
- If you are writing solutions by hand, please write your answers in a neat and readable hand-writing.
- Always explain your answers. When a proof is requested, you should provide a rigorous proof.
- If you don't know the answer, write "I don't know" along with a clear explanation of what you tried. For example: "I couldn't figure this out. I think the following is a start, that is correct, but I couldn't figure out what to do next. [[Write down a start to the answer that you are sure makes sense.]] Also, I had the following vague idea, but I couldn't figure out how to make it work. [[Write down vague ideas.]]" At least 20% will be given for such an answer.

  Note that if you write things that do not make any sense, no points will be given.
- The homework is expected to take anywhere between 8 to 14 hours. You are advised to start early.
- Submit your homework online on the course webpage on CCLE. You can also hand it in at the end of any class before the deadline.

> Note: *All questions in the problem sets are challenging; you should not expect to know how to answer any question before trying to come up with innovative ideas and insights to tackle the question. If you want to do some practice problems before trying the questions on the problem set, we suggest trying problems 1.17 and 1.23 from the book. Do not turn in solutions to problems from the book.*

Problem 1

Assume not. That is, both shuffle(L_1, L_2) and shuffle($L_1, \overline{L_2}$) are regular. Since the class of regular languages is closed under the union operation, shuffle(L_1, L_2) $\cup$ shuffle($L_1, \overline{L_2}$) is regular. That is, $L = \{x_1 y_1 x_2 y_2 \ ... \ x_n y_n \mid x_1 \ ... \ x_n \in L_1, \ y_1 ... y_n \in \Sigma^*\} =$ shuffle (L_1, Σ^*). We have proved in the discussion that $L_{alt} = \{x \mid \exists y \in L \ such \ that \ x_1 x_2 x_3 ... = y_1 y_3 y_5 ...\}$ is regular if L is regular. Since $L =$ shuffle (L_1, Σ^*) is regular in this case, L_1 is regular, which is a contradiction as we are given L_1 is not regular. Hence, the languages shuffle(L_1, L_2) and shuffle($L_1, \overline{L_2}$) cannot both be regular. (proven)

Problem 2

(a) for any $w \in L_2 = L_1 \cup b^*$, $|w| \geq 1$, Let $q = 1$

- $w \in L_1$. Let $x = \epsilon,\ y = a,\ z = a^{i-1}b^p$. Thus, for each $w \in L_1$, we can write $w = xyz$, and $|xy| = 1 = q,\ |y| = 1 > 0$.
  $\Rightarrow xy^jz \in L_1$ if $j > 0$, $xy^iz \in b^*$ if $j = 0, i = 1$.
  $\Rightarrow xy^jz \in L_2$
- $w \in b^*$. Let $x = \epsilon,\ y = b,\ z = b^{|w|-1}$. Thus, for each $w \in b^*$, we can write $w = xyz$, and $|xy| = 1 = q,\ |y| = 1 > 0$.
  $\Rightarrow xy^jz \in b^*$ for $j \geq 0$.

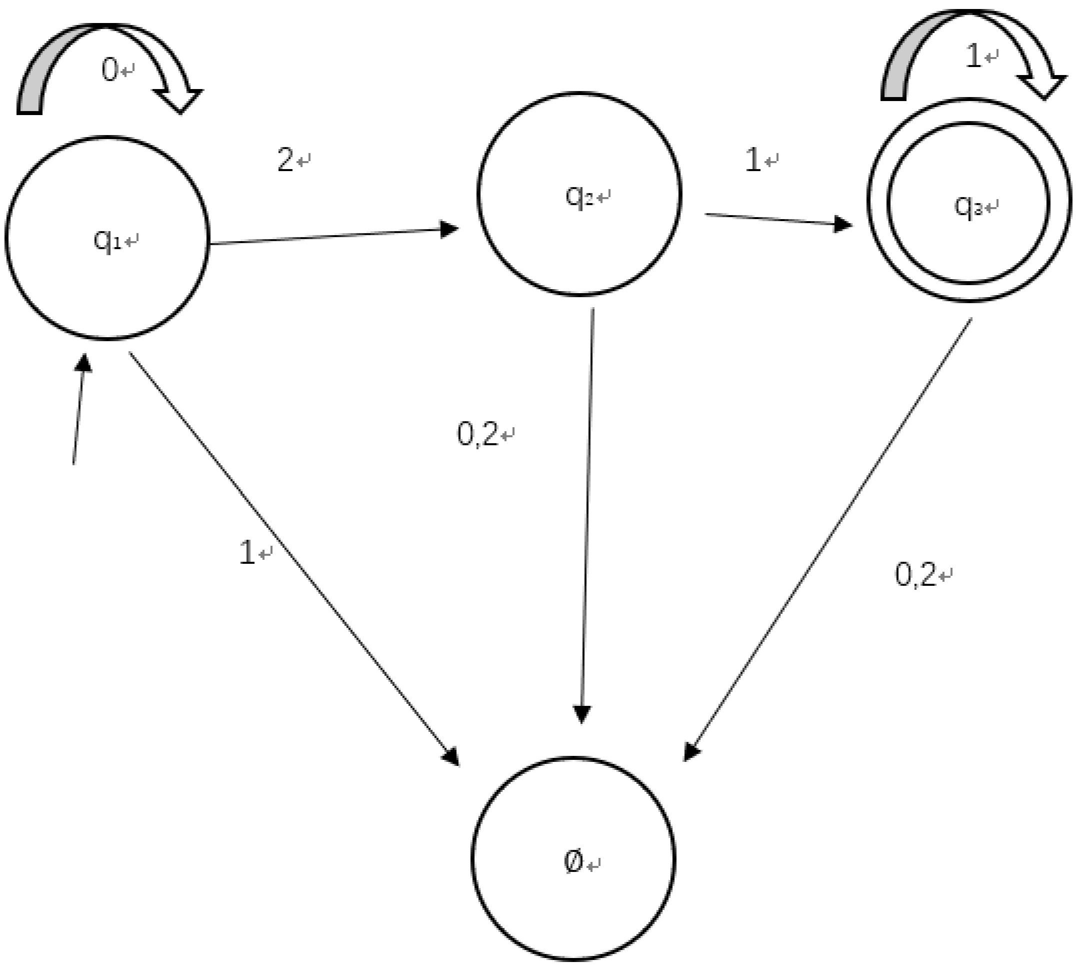
Hence, L_2 satisfies the conditions of the Pumping Lemma. (proven)

(b) Since L is a regular language, there exists a DFA M_1 that accepts L. Assume that there are q states in the DFA. Let $w = w_1w_2w_3...w_n$, where $n \geq q$. Let $r_1r_2...r_nr_{n+1}$ be the sequence of states M_1 enters for the input string w, so $r_{i+1} = \delta(r_i, w_i)$ for $1 \leq i \leq n$. For any partition of $w = xyz,\ y \geq q$, M_1 enters the state of $r_{|x|+1}$ after processing string x. Then M_1 continues to process y which has at least q states. Thus, after finishing processing y, M_1 will go through at least another $q+1$ states including $r_{|x|+1}$. Since there are only q states in the DFA, by Pigeonhole Principle, at least two of the states must be the same. We denote the first of these by r_j, the second by r_l $(j \neq l)$. Let b be the string that makes M_1 enter r_l from r_j $(r_j = r_l)$, a be the string that makes M_1 enter r_j from $r_{|x|+1}$, c be the remaining part of string y. Thus, after process the string xa, M_1 enters the state r_j, and any number of string b input after xa will make M_1 re-enter r_j/r_l. And string cz will make M_1 to enter an accepted state from r_j/r_l, since $w = xyz$ is accepted by L. Hence, we have proved for every $w \in L$, and every partition of w into $w = xyz$ with $|y| \geq q$, there are strings a, b, c such that $y = abc$, $|b| > 0$, and for all $i \geq 0$, $xab^icz \in L$. (proven)

(c) Assume not. Consider a string $w = a^qb^p$, p is a prime and $p > q$. Hence $w \in L_2$. Let $x = a^q,\ y = b^q,\ z = b^{p-q}$ be a partition of w. Clearly if L_2 is a regular language, there exists a partition $y = rst,\ |s| > 0$, such that for all $i \geq 0$, $xrs^itz \in L_2$. Since $y = b^q$, s must be in the form of b^j, $0 < j \leq q$. Thus, $xrs^itz = a^qb^{p+(i-1)j}$, and there exists a j such that $p + (i-1)j$ is a prime regardless the value of i. However, when $i - 1 = p$, $p + (i-1)j = p + pj = p(j+1)$ which is not a prime. This is a contradiction. Hence, L_2 is not a regular language. (proven)

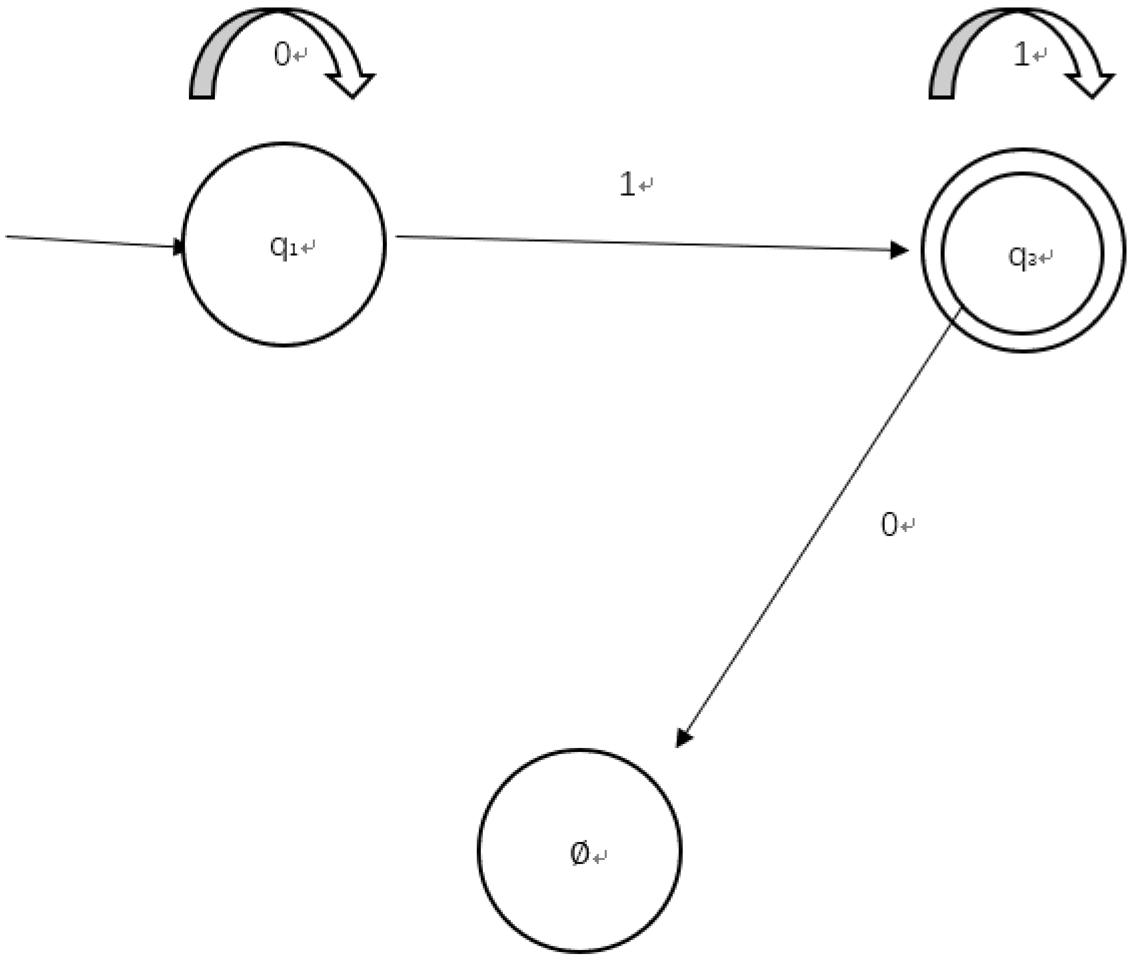
Problem 3

Assume not. Consider $L_1 = 0^*21^*$. L_2 is a regular language as we can easily find out that it is accepted by the following state machine:



Let $L_{\frac{1}{3}-\frac{1}{3}} = \{xz \in \Sigma^* \mid \exists y \in \Sigma^* \; with \; |x| = |y| = |z| \; such \; that \; xyz \in L\}$. In this case, we let $L = L_1, \Sigma = \{0, 1, 2\}$, and $L_{\frac{1}{3}-\frac{1}{3}}$ is a regular language.

Now, consider another language $L_2 = 0^*1^*$, it is also fairly easy to prove that L_2 is a regular language by introducing the following state machine:



Since the class of regular languages is closed under the union operation, $L_3 = L_{\frac{1}{3}-\frac{1}{3}} \cup L_2$ is a regular language by our construction.

Now let's prove $L_3 = 0^i 1^i,\ i > 0$. Assume not. Since $L_3 = L_{\frac{1}{3}-\frac{1}{3}} \cup L_2$, $L_2 = 0^* 1^*$, there is no alphabet "2" in L_3. Since $L_{\frac{1}{3}-\frac{1}{3}} = \{xz \in \Sigma^* \mid \exists y \in \Sigma^*\ with\ |x| = |y| = |z|\ such\ that\ xyz \in L_1\}$, resulting that L_3 must satisfy $L_3 = 0^i 1^j,\ i > 0, j > 0,\ i \neq j, (i + j)\ mod\ 2 = 0$. Without loss of generality, assume $i > j$, since $|x| = |z| = \frac{i+j}{2}$, $x = 0^{\frac{i+j}{2}},\ z = 0^{\frac{i-j}{2}} 1^j$. However, this is a contradiction because no matter what string y is, xyz is not in L_1 because there would be 0 on the right hand side of "2". Hence, $L_3 = 0^i 1^i,\ i > 0$.

However, we have proved in lecture that $L_3 = 0^i 1^i,\ i > 0$ is not a regular language, which is a contradiction. Hence, if L is regular, $L_{\frac{1}{3}-\frac{1}{3}}$ need not be regular. (proven)