Applied Numerical Methods
(MATH 151A, 2019)
Assignment 3

Due day: 3:50 p.m., Feb 22, 2019, Friday.

1. Let $f(x) = 1/x$, $x_i = i + 1$, $0 \leq i \leq 2$, find the Lagrange interpolation polynomial interpolating the points $(x_i, f(x_i))$ using
   (a) Lagrange interpolation formula.
   (b) Neville's method.
   (c) the divided difference interpolation.
2. Find the natural cubic spline passing through $(-1, 1)$, $(0, 1)$, $(1, 2)$.
3. Consider Hermite interpolation problem. Prove the following theorem:
   Let $f \in C^1([a, b])$ and $x_0, x_1, ...x_n$ be n distinct nodes in $[a, b]$, and let
   $$H(x) = \sum_{i=0}^{n} f(x_i)H_{n,j}(x) + \sum_{i=0}^{n} f'(x_i)\hat{H}_{n,j}(x)$$
   where
   $$H_{n,j} = [1 - 2(x - x_j)L'_{n,j}(x_j)]L^2_{n,j}(x) \quad \hat{H}_{n,j} = (x - x_j)L^2_{n,j}(x).$$
   Show that $H(x_i) = f(x_i)$ and $H'(x_i) = f'(x_i)$, for all $0 \leq i \leq n$. (You do not need to prove the uniqueness of H.)

4. Let $f(x)$ be a function defined on the interval $[x_0 - h, x_0 + h]$, and $f \in C^3[x_0 - h, x_0 + h]$.

   (a) Let $P(x)$ be the Lagrange interpolation polynomial of $f(x)$ at the nodes $x = x_0 - h, x_0, x_0 + h$. Write down the expression of $P(x)$.

   (b) Write down the error term $E(x) := f(x) - P(x)$ in terms of the derivatives of $f(x)$.

   (c) Using the fact that $f(x) = P(x)+E(x)$, calculate the derivatives of f, $f'(x)$ at $x = x_0$.

   (d) If we approximate the derivative $f'(x_0)$ by $P'(x_0)$, is it true that the above approximation is exact if f is a polynomial of degree less than or equal to 2 ? Why ?

   (e) Write down an error bound of this approximation rule suggested in (d) for a general function $f(x)$ based on the result in (b).

5. (**Programming problem**)

   Let a number of points $(x_i, f(x_i))$ be given, $0 \leq i \leq n$. Let $P(x)$ be its Lagrange interpolation polynomial interpolating the points $(x_i, f(x_i))$, $0 \leq i \leq n$.

   Write a program which allow inputs $\{(x_i, f(x_i))\}$ and a value a, and calculate the value of $P(a)$.