Homework 4

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Problem 1

(a)

$$\begin{aligned}
\lim_{n\to\infty} \frac{|\frac{1}{2}\ln(p_n+1)-0|}{|p_n-0|} &= \lim_{n\to\infty} \frac{|\frac{1}{2}\ln(p_n+1)|}{|p_n|} \\
&= \lim_{n\to\infty} \frac{\frac{1}{2}\ln(p_n+1)}{p_n} \textbf{ Since } P_0 = 1 \\
&= \lim_{p_n\to 0} \frac{\frac{1}{2}\ln(p_n+1)}{p_n} \textbf{ Since } p^* = 0 \\
&= \lim_{p_n\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}p_n}[\frac{1}{2}\ln(p_n+1)]}{\frac{\mathrm{d}}{\mathrm{d}p_n}p_n} \textbf{ Since the limit has the form of } \frac{0}{0} \\
&= \lim_{p_n\to 0} \frac{\frac{1}{2}\frac{1}{p_n+1}}{1} \\
&= \lim_{p_n\to 0} \frac{1}{2p_n+2} \\
&= \frac{1}{2} < 1
\end{aligned}$$

Hence, the sequence is linearly convergent.

(b)

$$\begin{aligned}
\lim_{n\to\infty} \frac{|1+2^{1-(n+1)}+\frac{1}{(n+1+2)^{n+1}}-1|}{|1+2^{1-n}+\frac{1}{(n+2)^n}-1|} &= \lim_{n\to\infty} \frac{|2^{-n}+\frac{1}{(n+3)^{n+1}}|}{|2^{1-n}+\frac{1}{(n+2)^n}|} \\
&= \lim_{n\to\infty} \frac{2^{-n}+\frac{1}{(n+3)^{n+1}}}{2^{1-n}+\frac{1}{(n+2)^n}} < \lim_{n\to\infty} \frac{2^{-n}+\frac{1}{(n+3)^{n+1}}}{2^{-n}+\frac{1}{(n+3)^{n+1}}} = 1 \\
&\textbf{because } 2^{-n} < 2^{1-n},\ \frac{1}{(n+3)^{n+1}} < \frac{1}{(n+2)^n} \\
&\textbf{Also, } \lim_{n\to\infty} \frac{2^{-n}+\frac{1}{(n+3)^{n+1}}}{2^{1-n}+\frac{1}{(n+2)^n}} > \lim_{n\to\infty} \frac{2^{-n}+\frac{1}{(n+3)^{n+1}}}{2^{1-n}+\frac{2}{(n+3)^{n+1}}} = \frac{1}{2} \\
&\textbf{because } \frac{1}{(n+2)^n} \geq \frac{2}{(n+3)^{n+1}} \\
&\textbf{as } \frac{1}{(n+2)^n} - \frac{2}{(n+3)^{n+1}} = \frac{(n+3)^{n+1}-2(n+2)^n}{(n+2)^n(n+3)^{n+1}} > 0
\end{aligned}$$

Hence, the sequence is linearly convergent since

$$\frac{1}{2} < \lim_{n\to\infty} \frac{|p_{n+1}-p^*|}{|p_n-p^*|} < 1$$

Problem 2

$$\begin{aligned}\lim_{n\to\infty} p_n &= \lim_{n\to\infty} 10^{(-2^n)} \\ &= 10^{-\infty} \\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \frac{|p_{n+1} - p^*|}{|p_n - p^*|^2} &= \lim_{n\to\infty} \frac{|10^{(-2^{n+1})}|}{|10^{(-2^n)}|^2} \\ &= \lim_{n\to\infty} \frac{10^{(-2^{n+1})}}{10^{(-2^{n+1})}} \\ &= 1 > 0;\end{aligned}$$

Hence, the sequence is quadratically convergent.

Problem 3

(a)

$$\begin{aligned}
L_1(x) &= \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} = -\frac{x^3-9x^2+26x-24}{6} \\
L_2(x) &= \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} = \frac{x^3-8x^2+19x-12}{2} \\
L_3(x) &= \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} = -\frac{x^3-7x^2+14x-8}{2} \\
L_4(x) &= \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} = \frac{x^3-6x^2+11x-6}{6}
\end{aligned}$$

$$\begin{aligned}
f = P(x) &= \sum_{k=0}^{3} f(x_k)L_k(x) \\
&= -\frac{x^3-9x^2+26x-24}{3} + \frac{x^3-8x^2+19x-12}{2} - 2(x^3-7x^2+14x-8) + \frac{x^3-6x^2+11x-6}{2} \\
&= -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15
\end{aligned}$$

(b)

$$\begin{aligned}
P_{0,1} &= \frac{1}{x_1-x_0}[(x-x_0)P_1 - (x-x_1)P_0] = \frac{1}{2-1}[(x-1)\cdot 1 - (x-2)\cdot 2] = 3-x \\
P_{1,2} &= \frac{1}{x_2-x_1}[(x-x_1)P_2 - (x-x_2)P_1] = \frac{1}{3-2}[(x-2)\cdot 4 - (x-3)\cdot 1] = 3x-5 \\
P_{2,3} &= \frac{1}{x_3-x_2}[(x-x_2)P_3 - (x-x_3)P_2] = \frac{1}{4-3}[(x-3)\cdot 3 - (x-4)\cdot 4] = 7-x \\
P_{0,1,2} &= \frac{1}{x_2-x_0}[(x-x_0)P_{1,2} - (x-x_2)P_{0,1}] = \frac{1}{3-1}[(x-1)\cdot(3x-5) - (x-3)\cdot(3-x)] = 2x^2-7x+7 \\
P_{1,2,3} &= \frac{1}{x_3-x_1}[(x-x_1)P_{2,3} - (x-x_3)P_{1,2}] = \frac{1}{4-2}[(x-2)\cdot(7-x) - (x-4)\cdot(3x-5)] = -2x^2+13x-17 \\
P_{1,2,3,4} &= \frac{1}{x_3-x_0}[(x-x_0)P_{1,2,3} - (x-x_3)P_{0,1,2}] \\
&= \frac{1}{4-1}[(x-1)\cdot(-2x^2+13x-17) - (x-4)\cdot(2x^2-7x+7)] \\
&= -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15 = f
\end{aligned}$$

Hence, we get the same polynomial f as the result of Lagrange interpolation method.