Homework 2

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Problem 1

(a)

$$
\begin{aligned}
L_1(x) &= \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{x^2-5x+6}{2} \\
L_2(x) &= \frac{(x-1)(x-3)}{(2-1)(2-3)} = -x^2+4x-3 \\
L_3(x) &= \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{x^2-3x+2}{2} \\
P(x) &= \frac{x^2-5x+6}{2} - \frac{x^2-4x+3}{2} + \frac{x^2-3x+2}{6} = \frac{x^2-6x+11}{6}
\end{aligned}
$$

(b)

$$
\begin{aligned}
P_{0,1} &= \frac{1}{x_1-x_0}[(x-x_0)P_1-(x-x_1)P_0] = \frac{1}{2-1}[(x-1)\cdot\frac{1}{2}-(x-2)\cdot 1] = -\frac{x-3}{2} \\
P_{1,2} &= \frac{1}{x_2-x_1}[(x-x_1)P_2-(x-x_2)P_1] = \frac{1}{3-2}[(x-2)\cdot\frac{1}{3}-(x-3)\cdot\frac{1}{2}] = -\frac{x-5}{6} \\
P_{0,1,2} &= \frac{1}{x_2-x_0}[(x-x_0)P_{1,2}-(x-x_2)P_{0,1}] = \frac{1}{3-1}[(x-1)\cdot(-\frac{x-5}{6})-(x-3)\cdot(-\frac{x-3}{2})] = \frac{x^2-6x+11}{6}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f[0,1] &= \frac{\frac{1}{2}-1}{2-1} = -\frac{1}{2} \\
f[1,2] &= \frac{\frac{1}{3}-\frac{1}{2}}{3-2} = -\frac{1}{6} \\
f[0,1,2] &= \frac{-\frac{1}{6}+\frac{1}{2}}{3-1} = \frac{1}{6} \\
P(x) &= 1-\frac{1}{2}(x-1)+\frac{1}{6}(x-1)(x-2) \\
&= \frac{x^2-6x+11}{6}
\end{aligned}
$$

Problem 2

We need to determine $a_0, a_1, b_0, b_1, c_0, c_1, d_1, d_2$. From the definition of natural cubic spline , we have:

$$\begin{aligned}
a_0 &= f(-1) = 1 \\
a_1 &= f(0) = 1 = a_0 + b_0 + c_0 + d_0 \\
f(1) &= 2 = a_1 + b_1 + c_1 + d_1 \\
b_0 + 2c_0 + 3d_0 &= b_1 \ (S_0^{'}(0) = S_1^{'}(0)) \\
2c_0 + 6d_0 &= 2c_1 \ (S_0^{(2)}(0) = S_1^{(2)}(0)) \\
2c_0 &= 0 \ (S_0^{(2)}(-1) = 0) \\
2c_1 + 6d_1 &= 0 \ (S_0^{(2)}(1) = 0)
\end{aligned}$$

Solving the system, we have:
$$S(x) = \begin{cases} 1 - \frac{1}{4}(x+1) + \frac{1}{4}(x+1)^3 \\ 1 + \frac{1}{2}x + \frac{3}{4}x^2 - \frac{1}{4}x^3 \end{cases}$$

Problem 3

First, for $H(x_i)$:

$$H_{n,j}(x_i) = \begin{cases} 0 \textbf{ for } i \neq j \textbf{ since } L_{n,j}(x_i) = 0 \textbf{ for } i \neq j \\ [1-0]L_{n,i}^2(x_i) = 1 \textbf{ for } i = j \textbf{ since } L_{n,i}(x_i) = 1 \end{cases}$$

$$\widehat{H}_{n,j}(x_i) = \begin{cases} 0 \textbf{ for } i \neq j \textbf{ since } L_{n,j}(x_i) = 0 \textbf{ for } i \neq j \\ 0 \cdot L_{n,i}^2(x_i) = 0 \textbf{ for } i = j \end{cases}$$

Thus,

$$H(x_i) = \sum_{j=0, j\neq i}^{n} 0 + 1 \cdot f(x_i) + \sum_{j=0}^{n} 0 = f(x_i)$$

Then, for $H^{'}(x_i)$:

$$H_{n,j}^{'}(x_i) = \begin{cases} 0 \textbf{ for } i \neq j \textbf{ since } L_{n,j}(x_i) = 0 \textbf{ for } i \neq j \\ -2L_{n,i}^{'}(x_i)L_{n,i}^2(x_i) + [1 - (x_i - x_i)L_{n,i}^{'}(x_i)]2L_{n,i}^{'}(x_i)L_{n,i}(x_i) = -2L_{n,i}^{'}(x_i) + L_{n,i}^{'}(x_i) = 0 \\ \ \textbf{for } i = j \textbf{ since } L_{n,i}(x_i) = 1 \end{cases}$$

$$\widehat{H}_{n,j}^{'}(x_i) = L_{n,j}^2(x_i)[L_{n,i}^2(x_i) + 2(x_i - x_j)L_{n,i}^{'}(x_i)] = \begin{cases} 0 \textbf{ for } i \neq j \textbf{ since } L_{n,j}(x_i) = 0 \textbf{ for } i \neq j \\ 1 \cdot [1+0] = 1 \textbf{ for } i = j \end{cases}$$

Thus,

$$H(x) = \sum_{j=0, j\neq i}^{n} 0 + 1 \cdot f(x_i) + \sum_{j=0}^{n} 0 + \sum_{j=0, j\neq i}^{n} 0 + 1 \cdot f^{'}(x_i) + \sum_{j=0}^{n} 0 = f(x_i) + f^{'}(x_i)$$

(proven)

Problem 4

(a)

$$\begin{aligned}
L_1(x) &= \frac{(x-x_0)(x-x_0-h)}{(-h)(-2h)} = \frac{x^2-(2x_0+h)x+x_0^2+x_0h}{2h^2}\\
L_2(x) &= \frac{(x-x_0+h)(x-x_0-h)}{(h)(-h)} = \frac{x^2-2x_0x+x_0^2-h^2}{-h^2}\\
L_3(x) &= \frac{(x-x_0+h)(x-x_0)}{(2h)(h)} = \frac{x^2-(2x_0-h)x+x_0^2-x_0h}{2h^2}\\
P(x) &= \frac{x^2-(2x_0+h)x+x_0^2+x_0h}{2h^2}f(x_0-h) - \frac{x^2-2x_0x+x_0^2-h^2}{h^2}f(x_0) + \frac{x^2-(2x_0-h)x+x_0^2-x_0h}{2h^2}f(x_0+h)
\end{aligned}$$

(b)

$$E(x) = \frac{f^{(3)}(\xi(x))}{3!}\prod_{i=0}^{2}(x-x_i)$$

(c)

$$\begin{aligned}
f^{'}(x) &= \frac{2x-2x_0-h}{2h^2}f(x_0-h) - \frac{2x-2x_0}{h^2}f(x_0) + \frac{2x-2x_0+h}{2h^2}f(x_0+h)\\
&\quad + D_x[\frac{\prod_{i=0}^{2}(x-x_i)}{3!}]f^{(3)}(\xi(x)) + \frac{\prod_{i=0}^{2}(x-x_i)}{3!}D_x[f^{(3)}(\xi(x))]\\
f^{'}(x_0) &= \frac{-h}{2h^2}f(x_0-h) - 0 + \frac{h}{2h^2}f(x_0+h) - \frac{f^{(3)}(\xi(x))}{3!}h^2\\
&= \frac{1}{2h}[f(x_0+h)-f(x_0-h)] - \frac{f^{(3)}(\xi(x))}{6}h^2
\end{aligned}$$

(d) Yes. This is because the third derivative of the error term will be zero if f is a polynomial of degree less than or equal to 2. This makes $f^{'}(x) = P^{'}(x)$.

(e) Assume the round-off errors $e(x_0 \pm h)$ are bounded by some constant $\varepsilon > 0$, the third derivative of f is bounded by a number $M > 0$, then:

$$|f^{'}(x_0) - P^{'}(x_0)| \leq \frac{\varepsilon}{h} + \frac{h^2}{M}$$