Homework 1

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Problem 1

(a) Since $f(0.5) = 0.25 - 0.35 = -0.1 < 0$, $f(1) = 1 - 0.7 = 0.3 > 0$,by *Intermediate Value Theorem*, $\exists d \in [0.5, 1]$ s.t. $f(d) = 0$.
$\Rightarrow f$ has at least 1 root. - - - - - ①

Assume there are two roots $a, b \in [0.5, 1]$ s.t. $f(a) = f(b) = 0$.Hence, by *Rolle's Theorem*, $\exists c \in [a, b]$ s.t. $f'(c) = 0$
However, $f'(x) = 2x - 0.7 > 0$ for $x \in [0.5, 1] \Rightarrow$ *contradiction*
$\Rightarrow f$ cannot have two or more roots for $x \in [0.5, 1]$. - - - - - ②

By ① and ②, $f(x) = x^2 - 0.7x, x \in [0.5, 1]$ has exactly one root.

(b) Since $|P_n - P| \leq \dfrac{b-a}{2^n}, n \geq 1$. Let b=1, a=0.5

$$\begin{aligned} \frac{1-0.5}{2^n} &< 10^-5 \\ 2^n &> 5 \times 10^4 \\ n &> 15,29 \\ n &= 16 \end{aligned}$$

Hence, 16 iterations are required.

Problem 2

① If $f(a) = a$ or $f(b) = b$, then there exists at least one fixed point at the endpoint of the interval.

② If not, since $f(x) \in [a, b], f(a) > a, f(b) < b$. Let $h(x) = x - f(x) \Rightarrow h(a) < 0, h(b) > 0$. The *Intermediate Value Theorem* implies $\exists p \in (a, b)$ s.t. $h(p) = 0 \Rightarrow g(p) = p$, there exists at least one fixed point.

$\Rightarrow f$ has at least a fixed point on $[a, b]$

Problem 3

(a)

$$\begin{aligned}
p_1 &= \frac{p_0^2+3}{2p_0} = \frac{9+3}{2\times 3} = 2 \\
p_2 &= \frac{p_1^2+3}{2p_1} = \frac{4+3}{2\times 2} = \frac{7}{4}
\end{aligned}$$

(b)

$$\begin{aligned}
p_n &= \frac{p_n^2+3}{2p_n} \\
2p_n^2 &= p_n^2+3 \\
p_n^2 &= 3 \\
p_n &= \pm\sqrt{3}
\end{aligned}$$

$\Rightarrow$ all possible limits are $\pm\sqrt{3}$.

(c) *Newton's Method* is the functional iteration technique with

$$\begin{aligned}
p_n = g(p_n - 1) = p_{n-1} &- \frac{f(p_{n-1})}{f'(p_{n-1})} \\
&= p_{n-1} - \frac{p_{n-1}^2-3}{2p_{n-1}} \text{ for } f(x) = x^2-3 \\
&= \frac{2p_{n-1}^2 - p_{n-1}^2 + 3}{p_{n-1}} \\
&= \frac{p_{n-1}^2+3}{2p_{n-1}} \text{ (shown)}
\end{aligned}$$

Problem 4

(a)

$$\begin{aligned}
p_n &= p_{n-1} - \frac{f(p_{n-1})(p_{n-1}-p_{n-2}}{f(p_{n-1})-f(p_{n-2})} \\
p_2 &= 3 - \frac{(3^2-3)\cdot(3-\frac{1}{2})}{3^2-3-(\frac{1}{2})^2+3} \\
&= 3 - \frac{15}{8.75} \\
&= \frac{9}{7} \\
p_3 &= \frac{9}{7} - \frac{[(\frac{9}{7})^2-3]\cdot(\frac{9}{7}-3)}{(\frac{9}{7})^2-3-3^2+3} \\
&= \frac{8}{5}
\end{aligned}$$

(b)

$$
\begin{aligned}
q_0 &= (\frac{1}{2})^2 - 3 = -2.75 \\
q_1 &= 3^2 - 3 = 6 \\
p_2 &= p_1 - q_1(p_1 - p_0)/(q_1 - q_0) \\
&= 3 - 6(3 - \frac{1}{2})/(6 - (-2.75)) = \frac{9}{7} \\
q_2 &= (\frac{9}{7})^2 - 3 = -\frac{66}{49} \\
p_3 &= \frac{9}{7} - (-\frac{66}{49})\frac{\frac{9}{7} - 3}{-\frac{66}{49} - 6} = \frac{8}{5}
\end{aligned}
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