2: Limits

Limits are the mathematical framework for deciding where a function is going as its argument approaches some value. In the case of algorithmic complexity, we are interested in where $T(n)$, the run time, is going, as n gets large so here we will look at limits at infinity. The limit is a part of mathematics where the intuition is probably more straight forward than the definition, nonetheless here we will look at the formal definition. This is useful not so much for this course, largely we will be able to calculate the limits we need using simple methods, but because it is something that is useful to know in the future and it is interesting because it demonstrates a powerful way of proving things that is common in some types of mathematics.

Informal idea

If we write

$$\lim_{x\to\infty} f(x) = c \tag{1}$$

where $f(x)$ is some function and c a constant, we mean that $f(x)$ heads towards infinity it gets close to c. Take, for example

$$f(x) = \frac{1}{x} \tag{2}$$

it is easy to see that

$$\lim_{x\to\infty} \frac{1}{x} = 0 \tag{3}$$

because the bigger x is, the smaller $1/x$ is, so $1/x$ is heading for zero. Now, lets consider

$$f(x) = \frac{4x^2 + 2x + 1}{2x^2 + 3} \tag{4}$$

Well by dividing above and below by x^2 we have

$$\lim_{x\to\infty} \frac{4x^2 + 2x + 1}{2x^2 + 3} = \lim_{x\to\infty} \frac{4 + 2/x + 1/x^2}{2 + 3/x^2} = \lim_{x\to\infty} \frac{4}{2} = 2 \tag{5}$$

because the $2/x$, $3/x^2$ and so on are getting smaller and smaller as x gets larger so we can replace them by zeros inside the limit.

Of course, many functions just get bigger and bigger, or more and more negative, as x goes to infinity, for these we say the limit is plus or minus infinity, so, for example

$$\lim_{x\to\infty} x = \infty \tag{6}$$

and

$$\lim_{x\to\infty} (-x) = -\infty \tag{7}$$

Formal definition

The informal approach to limits makes sense until you think about it, then it starts to get confusing: what do we mean by 'x approaches infinity', what do we mean by 'gets close to'. The formal definition of the limit explains all of this.

We say

$$\lim_{x\to\infty} f(x) = c \tag{8}$$

if for all $\delta > 0$ there exists an x_0 such that for all $x > x_0$ we have $|f(x) - c| < \delta$.

What this definition is doing is this: no matter what standard of 'close to' you set, that is, if you regard 'close to c' as meaning within δ of c then no matter how small a value of δ you choose, then $f(x)$ will eventually end up that close; the x_0 is the eventually, you can make $f(x)$ lie within δ of c by choosing a high enough x_0.

This style of definition is refered to as an epsilon-delta-box. You might ask where the epsilon is, the delta, of course, is the δ: ϵ is used for limits that aren't at infinity,

$$\lim_{x\to a} f(x) = c \tag{9}$$

which we aren't looking at here. The epsilon plays a similar role to the x_0 here.

There is also a definition of an infinite limit: We say

$$\lim_{x\to\infty} f(x) = \infty \tag{10}$$

if for all $y > 0$ there exists an x_0 such that for all $x > x_0$ we have $f(x) > y$. Thus, no matter what you consider big, that is, no matter how large a y you choose, $f(x)$ will eventually be bigger than it. Finally

$$\lim_{x\to\infty} f(x) = -\infty \tag{11}$$

if for all $y < 0$ there exists an x_0 such that for all $x > x_0$ we have $f(x) < y$.

How to work out limits

Limits actually have nice properties:

$$\lim_{x\to\infty} [f(x) \pm g(x)] = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x) \tag{12}$$

where, following the usual convention, you take the + in both $\pm$s, or the minus in both. If everything is finite:

$$\lim_{x\to\infty} [f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right] \tag{13}$$

and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} \tag{14}$$

Although we didn't mention it, we used these rules earlier when doing

$$\lim_{x\to\infty} \frac{4x^2 + 2x + 1}{2x^2 + 3} = \lim_{x\to\infty} \frac{4 + 2/x + 1/x^2}{2 + 3/x^2} = \lim_{x\to\infty} \frac{4}{2} = 2 \tag{15}$$

What we actually did was

$$\lim_{x\to\infty} \frac{4 + 2/x + 1/x^2}{2 + 3/x^2} = \frac{\lim_{x\to\infty}(4 + 2/x + 1/x^2)}{\lim_{x\to\infty}(2 + 3/x^2)} = \lim_{x\to\infty} \frac{4}{2} \tag{16}$$

and, for example

$$\lim_{x\to\infty} (4 + 2/x + 1/x^2) = \lim_{x\to\infty} 4 + \lim_{x\to\infty} 2/x + \lim_{x\to\infty} 1/x^2 \tag{17}$$

The other method for calculating limits is l'Hôpital's rule. If

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \tag{18}$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} \tag{19}$$

doesn't tell us anything since the right hand side is just infinity over infinity and we did say above the business with products and fractions only applies if everything is finite. However in this case a rule called l'Hôpital's rule.[1] L'Hôpital's rule says that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \tag{20}$$

where

$$f'(x) = \frac{df}{dx}(x) \tag{21}$$

and similar for $g'(x)$. In other words, if the top and the bottom both have infinity limits then differentiating the top and the bottom doesn't change the limit.

You might wonder why you'd want to differenciate the top and the bottom, in fact that can sometimes simplify the limit; this is important for an example we need for algorithmic complexity, calculating

$$\lim_{x\to\infty} \frac{\ln x}{x} \tag{22}$$

Now, $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x = \infty$ so we can apply l'Hôpital's rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0 \tag{23}$$

This tells us that although $\ln x$ goes to infinity it goes to infinity slower than x. This is illustrated in a plot, Fig. 1. Here, by the way, we are using

$$\ln x = \log_e x \tag{24}$$

the natural log because it is more convenient for differenciation, however, in compter science the log to the base two, $\log_2 x$ is more common, in fact changing bases only causes a change of an overall constant, see Table 1 for a reminder of the properties of the log.

Relationship to big-Oh

Recall the definition of $O(g(n)$, called 'big oh' of $g(n)$, is

$$O(g(n)) = \{f(n)|\exists n_0 > 0 \in \mathbf{N} \text{ and } c > 0 \in \mathbf{R} \text{ with } |f(n)| \leq c|g(n)| \forall n \geq n_0\} \tag{34}$$

You can see that the definition of $O(g(n))$ has this wrapped up in it, if says

$$f(n) \in O(g(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \tag{35}$$

Or, put another way, either $f(n)$ goes to infinity slower than $g(n)$, or at the same speed.

Lets look at the example we saw before:

$$T(n) = 5n^2 + n + 6 \tag{36}$$

we can see

$$\lim_{n\to\infty} \frac{5n^2 + n + 6}{n^2} = 5 < \infty \tag{37}$$

so $T(n) \in O(n^2)$.

[1] Which was discovered by Johann Bernoulli; he gave l'Hôpital permission to include it in a book and was then *very* annoyed when he found that people thought it was l'Hôpital's idea.

The logarithm is the opposite of the exponent: if

$$a^b = c \tag{25}$$

then

$$\log_a c = b \tag{26}$$

or, written in one line

$$a^{\log_a c} = c \tag{27}$$

All the laws of logs can be worked out from the laws of exponents. Hence, since $a^0 = 1$ we have $\log_a 1 = 0$. In a similar way, other rules of logs can be deduced like

$$\begin{aligned} \log_a c_1 c_2 &= \log_a c_1 + \log_a c_2 \\ \log_a \frac{c_1}{c_2} &= \log_a c_1 - \log_a c_2 \\ \log_a c^d &= d \log_a c \end{aligned} \tag{28}$$

and so on.
As for the change of base, let $b = \log_{a_1} c$ so $a_1^b = c$. Now take the log to the base a_2 of both sides

$$b \log_{a_2} a_1 = \log_{a_2} c \tag{29}$$

and then solve for b

$$b = \frac{\log_{a_2} c}{\log_{a_2} a_1} \tag{30}$$

and, substituting back the formula for b

$$\log_{a_1} c = \frac{\log_{a_2} c}{\log_{a_2} a_1} \tag{31}$$

Thus we see, that changing bases is just a matter of a multiplicative factor. For example, to change from base e to base two

$$\log_2 x = \frac{\log_e x}{\log_e 2} \approx \frac{\log_e x}{0.6931} \tag{32}$$

Common bases are $\log_2 x$ used in computer science, $\log_e x$ sometimes written $\ln x$ used in mathematics and $\log_{10} x$ used in chemistry. The base two is used because of its link to bits and also, as we will see, because of its relationship with algorithms that divide data into two piles. The natural log $\ln x$ is used where differential equations are common since

$$\frac{d}{dx} \ln x = \frac{1}{x} \tag{33}$$

Table 1: A reminder about logarithms. This is a quick summary of some of the laws of logs.

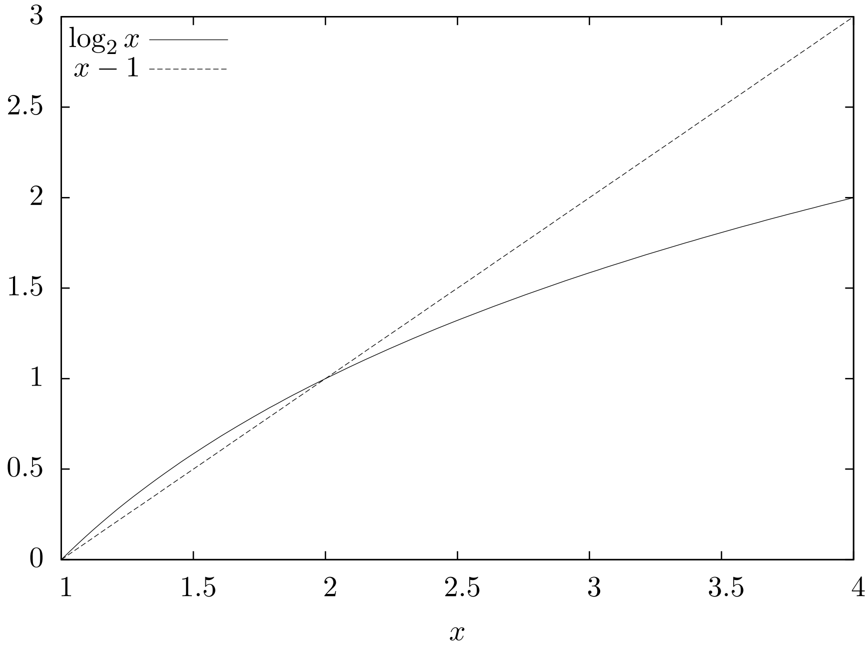

Figure 1: This shows $\log_2 x$ and $x - 1$ plots for $x \in [1, 4]$, the one has been taken from x to make them easier to compare, the key point is that the x grows faster.