Department of Industrial Engineering

Mechanical Vibrations Project Report

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1 Experiment 1

The first experiment consists in observing a *1 Dof* system composed by a spring, a mass and a damper. Where the mass is physically a cart with two disks on top of it and it is rigidly connected to a rack that is moved by an electric motor. The data we are provided are digital signals registered by an encoder. The file stores in columns the input signal that is a sine sweep voltage (increases frequency over time), the time in seconds and the output cart position. Furthermore we know the total mass to be $m = 1.7kg$. The general equation of motion of the system is:

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

1.1 Sampling Frequency

It is possible to evaluate the sampling frequency of the encoder by calculating:

$$f_s = \frac{n_{samples}}{t_{total}} \simeq 1\ kHz \tag{1}$$

With the given sine-sweep wave input, we get the following mass displacement output:

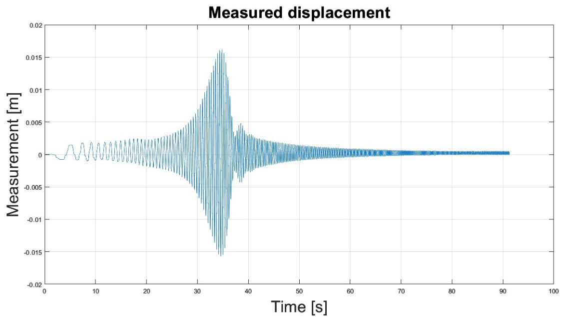

Figure 1: Measured displacements

1.2 Experimental Transfer Function

In order to correctly estimate the Experimental Transfer Function between the sine-sweep wave (input) and the displacement (output), the measurement units have to be converted into $[V]$ and $[m]$. At this point we use the `MATLAB` function `tfestimate` in which we pass the two arrays and a constant, hence the input signal, the output signal and the computed sampling frequency.

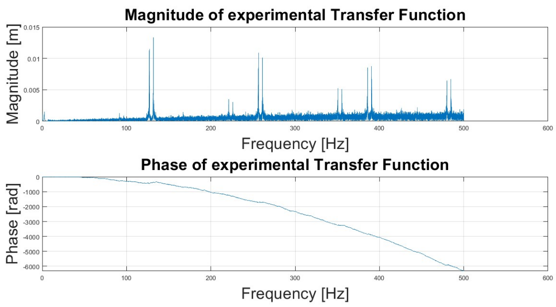

Figure 2: Experimental Transfer Function

The experiment has been conducted by using a sine input voltage with progressive increase of the frequency over time. It is expected that after the Natural Frequency of the system has been exceeded and the effect of the corresponding peak is dissipated is reasonable to ignore the rest of the data, because high frequency is affected by noise that has to be filtered. The relevant part of the experimental transfer function is represented below in Figure 3.

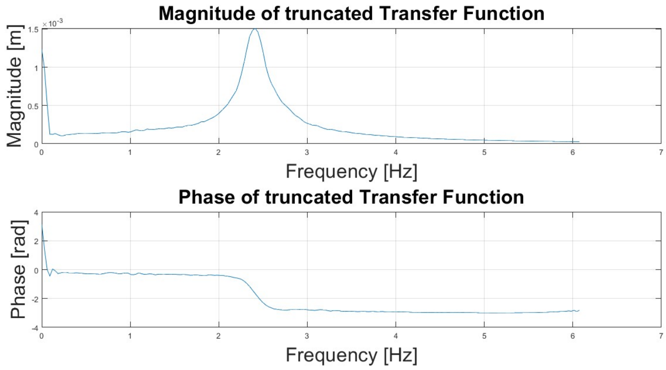

Figure 3: Truncated TF

It is interesting to see the change of the phase exactly where the peak in the magnitude is located.

1.3 Half Power Point Method

Now it is asked to compute the damping ratio of the natural frequency by using the Half Power Point Method. The task can be splitted into smaller steps. At first it is estimated the value (Q) of the peak of the Transfer Function. This value is divided by $\sqrt{2}$ because we are interested in the effective power. As second step, by plotting the graph (Figure 4) it is expected to find 2 values. The function `interp1` that is decided to use performs the interpolation of arrays, but it finds only one solution due to internal software limitation; so it is decided to set two starting points (one for each side of the peak), such that the numerical solver can find also the second intersection.

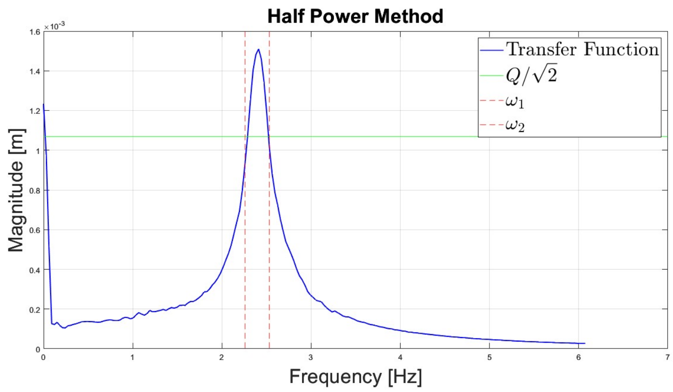

Figure 4: Results

Now that all the constant needed are computed, the last step consists in computing the asked quantities using the following formulas:

$$\xi = \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} = 0.0573 \qquad \omega_n = \frac{\omega_1 + \omega_2}{2} = 2\pi(2.3957)\ rad/s \tag{2}$$

Note that the natural frequency has to be multiplied by 2π to have the right unit of measurement in `rad/s`.

2 Experiment 2

The second experiment is different: two data set are given, and they come from one experiment registered twice. This time it is used an accelerometer instead of the encoder to register the data. The experiment consists in moving the cart in an initial position different from the equilibrium one, and than let it free oscillate until it stops due to small internal viscous forces. The *1 Dof* system is quite similar to the first, expect to the fact it is not moved by the rack, infact in this case the input voltage is null.

2.1 Sampling Frequency and meaningful time window

As first step the reasoning to compute the Sampling Frequency is repeated:

$$f_s = \frac{n_{samples}}{t_{total}} \simeq 12.8\ kHz \tag{3}$$

This value is valid for both the experiments that we call *finger1* and *finger2*. Again the meaningful time window are extracted, and their plots are shown below:

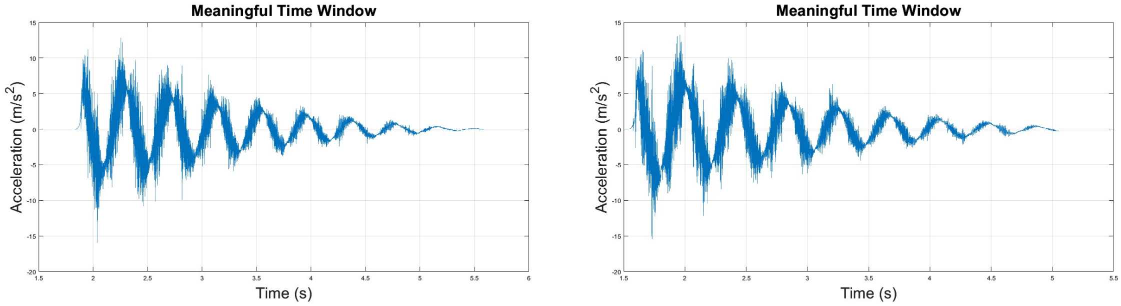

Figure 5: Plot of $\vec{a}$ in a meaningful window for first and second experiments

2.2 Filtering and plotting the data

The Logarithmic Decrement Method is a technique used to analyze and make an estimate on the damping characteristic of a system. It consists in measuring subsequent peaks of an oscillatory response and use this values to compute the Logarithm Decrement.

From the previous plots, it is clear that the signal is highly affected by noise, in order to find and evaluate those peaks it is performed a two-step smoothing operation. The two-step filtering is done by using the function *smooth()* which takes in input the signal to be filtered and the size of the window. The output will be the two-time smoothed signal, that if passed to the function *findpeaks()* it gives an array of the position of the positive peaks as output. This array is used in the one-time smoothed signal to evaluate the right magnitude of the peaks of interest. Below are shown the plots to have a better understanding:

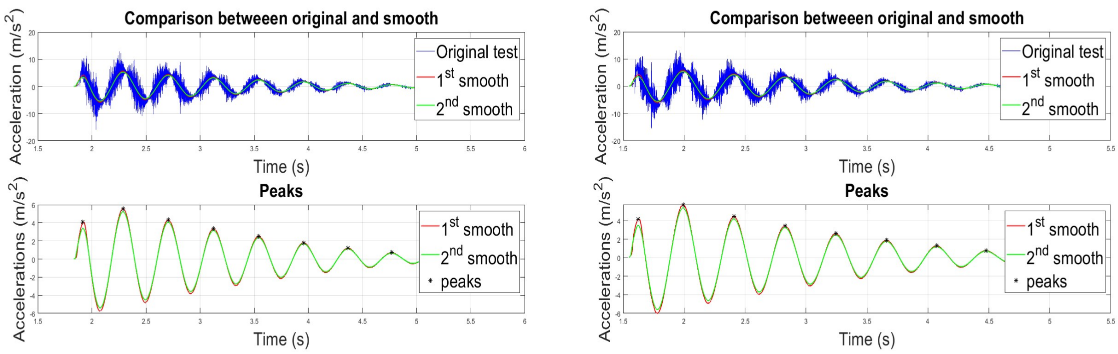

Figure 6: Peaks for *finger1.txt* and *finger2.txt*

2.3 Parameters estimation with Logarithm Decrement Method

At this point the true application of the Logarithm Decrement Method consists in computing the logarithm of the ratios of consecutive elements. The output is a vector δ_i per each peak i. Notice that the first peak is neglected since it does not involve the decrement. Also the last peak is neglected, because, since it has small value, it is affected by noise.

$$\delta_i = \ln \frac{x_i}{x_{i+1}} \tag{4}$$

Now it is possible to estimate the asked quantities:

The period of oscillation T_d has been computed as the mean of the peaks distances in time.
The damping ratio ξ, damped frequency ω_d and the natural frequency ω_n are computed with the following formulas:

$$\xi = \frac{\delta_{\text{mean}}}{\sqrt{4\pi^2 + \delta^2_{\text{mean}}}} \qquad \omega_d = \frac{2\pi}{T_d} \qquad \omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}} \tag{5}$$

Furthermore by knowing the total mass of the cart plus the disks $m = 1.7kg$, it is possible to estimate also the parameters of the system, such as the stiffness k_0 and the damping coefficient c_1.

$$k_0 = \omega_n^2 m \qquad c_1 = 2\xi\omega_n m \tag{6}$$

Since all the values that we have computed are means or come from mean values, it is a good practise to estimate a possible error. In first place the standard deviation is computed with the propagation of errors formula (eq.7), but since the population is small the std may not be a robust estimate. In the face of this it is decided to use a t-distribution to compute the Confidence Interval. Below are reported the general formulas:

$$\sigma = \sqrt{\Big(\frac{\delta f}{\delta x_1}\sigma_{x_1}\Big)^2 + \Big(\frac{\delta f}{\delta x_2}\sigma_{x_2}\Big)^2 + ... + \Big(\frac{\delta f}{\delta x_n}\sigma_{x_n}\Big)^2} \qquad CI = \bar{X} \pm t\Big(\frac{\sigma}{\sqrt{n}}\Big) \tag{7}$$

Where:

- $\bar{X}$ represent the sample mean
- t is the critical value from the t-distribution for the desired confidence level
- σ represent the sample standard deviation
- n represent the sample size

Note that the Margin Of Error (MOE) is represented by the whole term $t\Big(\frac{\sigma}{\sqrt{n}}\Big)$, and its constant are chosen as the following description. It is reasonable to have 95% for the confidence level. Since we have $n = 7$ peaks, the number of degree of freedom is $n - 1 = 6df$ and the corresponding critical value taken from the standard table is $t = 2.447$.
Here are reported the final results for the two parallel experiments:

```
----------------Finger 1-----------------
Period of oscillation (T): 0.40974 s
uncertainty of T: 0.010405 s
Damping ratio (xi): 0.064759
uncertainty of xi: 0.028189
Damped natural frequency (wd): 15.3344 rad/s
uncertainty of wd: 0.3894 rad/s
Natural frequency (wn): 15.3667 rad/s
uncertainty of wn: 0.39123 rad/s
Stiffness constant (k0): 401.4303 N/m
uncertainty of k0: 20.4406 N/m
Damping constant (c1): 3.3834 N s/m
uncertainty of c1: 1.4753 N s/m
```

```
----------------Finger 2-----------------
Period of oscillation (T): 0.41079 s
uncertainty of T: 0.0093137 s
Damping ratio (xi): 0.062211
uncertainty of xi: 0.025772
Damped natural frequency (wd): 15.2953 rad/s
uncertainty of wd: 0.34678 rad/s
Natural frequency (wn): 15.325 rad/s
uncertainty of wn: 0.34833 rad/s
Stiffness constant (k0): 399.2527 N/m
uncertainty of k0: 18.1497 N/m
Damping constant (c1): 3.2415 N s/m
uncertainty of c1: 1.3449 N s/m
```

Figure 7: Results for experiment *finger1.txt* and *finger2.txt*

It is clear the results seems to be pretty similar between the two experiments, but it is possible check the compatibility of the measurements from an objective point of view. If the absolute values of the difference of the computed quantities are within the sum of their margin of errors, they are compatible, infact this is the case for every computed parameter.

$$\text{Compatibility} = \Big|\mu_1 - \mu_2\Big| < (MOE_1 + MOE_2)$$

3 Experiment 3

The experiment is conducted in a similar way to the previous one, i.e. a cart that is made to oscillate freely form an initial position. For this experiment the two disks have been removed from the cart. The target of this experiment is to find the mass of the cart.

Since it will be applied the same Logarithmic Decrement Technique, some passages are repeated.

3.1 Sampling Frequency and Meaningful Time Window

Since it is used the same accelerometer, the Sampling Frequency has the same value as before:

$$f_s = \frac{n_{samples}}{t_{total}} \simeq 12.8\ kHz \tag{8}$$

Once again it is selected the Meaningful Time Window that is showed below:

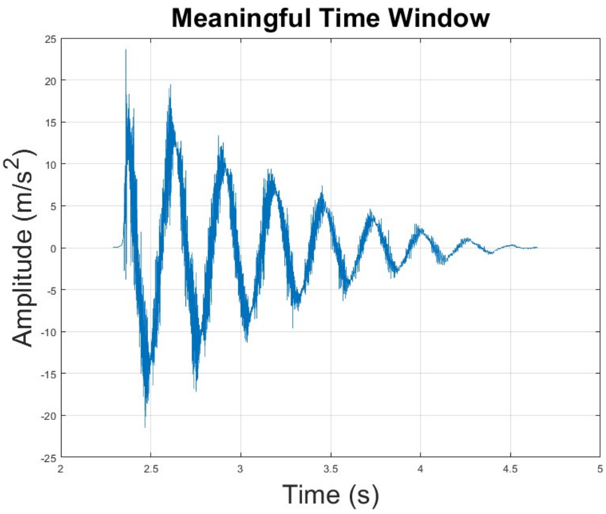

Figure 8: Plot of $\vec{a}$ in a meaningful window

3.2 Filtering and plotting the data

From the previous plot, it is clear that the signal is highly affected by noise. It is performed once again a two-step filtering operation, fist to get the peaks position, and than to evaluate them to the right magnitude. Below are shown the different plots:

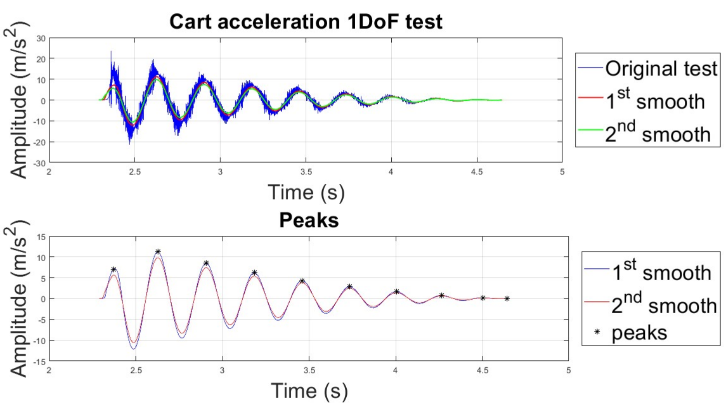

Figure 9: Peaks for *finger only cart* experiment

3.3 Parameters estimation with Logarithm Decrement Method

At this point it is possible to apply once again the Logarithm Decrement Technique. Below are reported the formula to compute the needed parameters:

$$\delta_i = \ln \frac{x_i}{x_{i+1}} \qquad \xi = \frac{\delta_{\text{mean}}}{\sqrt{4\pi^2 + \delta_{\text{mean}}^2}} \qquad \omega_d = \frac{2\pi}{T_d} \qquad \omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}} \tag{9}$$

Remember that δ_{mean} represent the mean value of the considered logarithm ratios δ_i of consecutive peaks x_i, x_{i+1} and T_d represent the mean time difference between consecutive peaks. Furthermore by knowing the total mass of the cart with two disks $m_1 = 1.7kg$ and by considering the stiffness value of the spring as the mean value of the previous experiments $k_0 = 400.34N/m$, it is possible to estimate the mass of the cart alone m_{cart}, the mass of each disk m_{disk} and the damping coefficient c_1.

$$m_{cart} = \frac{k_0}{\omega_n^2} \qquad m_{disk} = \frac{m_1 - m_{cart}}{2} \qquad c_1 = 2\xi\omega_n m_{cart} \tag{10}$$

Below are summed up all computed parameters for this experiment.

```
---------------Finger only cart ----------------
Period of oscillation (T): 0.26814 s
uncertainty of T: 0.013678 s
Damping ratio (xi): 0.090408
uncertainty of xi: 0.052132
Damped natural frequency (wd): 23.4328 rad/s
uncertainty of wd: 1.1954 rad/s
Natural frequency (wn): 23.5292 rad/s
uncertainty of wn: 1.2055 rad/s
Cart mass (m_cart): 0.72313 kg
uncertainty of m_cart: 0.074096 kg
Disk mass (m_disk): 0.48844 kg
uncertainty of m_disk: 0.037048 kg
Damping constant (c1): 3.0765 N s/m
uncertainty of c1: 1.781 N s/m
```

Figure 10: Results for experiment *finger only cart*

Note that the uncertainty has been computed with the same formula as before eq.(7).

4 Experiment 4

The last experiment involves the analysis of the entire *3 Dof* system, that is sketched in the figure below (Figure 11).
The system is composed by three masses that are connected each other by two springs. The first mass is connected to ground through an additional spring, and it is directly connected to a voltage motor that make the system to oscillate. Each mass is supported by ball bearings slides that cause a minimal but non-negligible rolling friction for m_1 and m_2, while for the third mass, since is connected to ground by a damper c_3, the friction becomes negligible. It is suggested to model the friction as a viscous force: $f_{c_i}(t) = c_i\dot{x}_i(t)$.

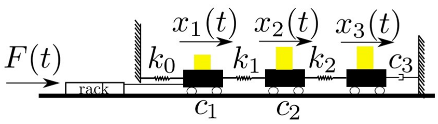

Figure 11: 3 Dof system model

4.1 Equations of motion

The equation of motion are derived by considering the dynamics of the system, the external forces applied by the electric motor and the damping forces. Here are shown in matrix form:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x_1} \\ \ddot{x_2} \\ \ddot{x_3} \end{bmatrix} + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} + \begin{bmatrix} k_0 + k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F(t) \\ 0 \\ 0 \end{bmatrix}$$

4.2 Experimental Transfer Function

In this part the target is to extract the Experimental Transfer Function. The operation conducted are repeated with the difference that in this new loaded dataset there are three accelerometer, one per cart. Below is shown an example of the entire Experimental Transfer Function of the first cart on the left, and the meaningful window extracted for all three Experimental Transfer Function on the right.

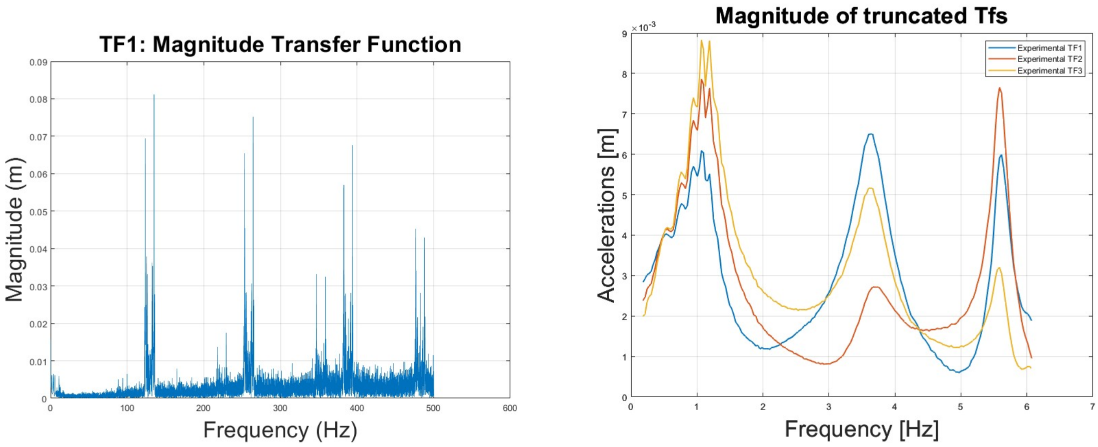

Figure 12: Experimental Transfer Functions

4.3 Analytical Transfer Function

In this part it is requested to compute the Analytical Transfer Function. Notice that the input signal is a voltage. Since the equations consider forces, a conversion is needed. It is assumed the external action to be approximated by the following equation: $F(t) = k_v u(t)$, where there is a conversion constant $k_v = 0.04N/V$ that multiply the input signal $u(t)$ that we remember to be a sine-sweep voltage characterized by increasing frequency. The calculations are based on the Equations of Motion, here is showed the inverted matrix equation after the Laplace Transform is performed:

$$X(s) = \left(\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} s^2 + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} s + \begin{bmatrix} k_0 + k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} F(t) \\ 0 \\ 0 \end{bmatrix}$$

The output of those calculation are long equations that are treated using `MATLAB`.

4.4 Transfer Function Fitting Procedure

The request consists in implement in the software a fitting procedure. It is asked to fit the Analytical Transfer Function to the experimental one, by having the parameters k_1, k_2, c_2 and c_3 as unknowns values. A minimization problem needs to be solved. The `MATLAB` function `fminsearch()` requires an initial guess of the four parameters and a cost functional.

The cost functional was defined as the sum of the root mean square of the differences between the magnitude value of each Experimental Transfer Function and the corresponding Analytical Transfer Function.

In order to direct the solver towards the right solution, it is a good start to define the initial guess point near the expected values. For stiffness values near 800 N/m can be chosen. The damping constant of the second cart is expected to be similar the one of the first cart, while the damper of the third mass is expected to be at least one order of magnitude larger. Below are showed the comparison between the Experimental TF and the Analytical TF for each cart.

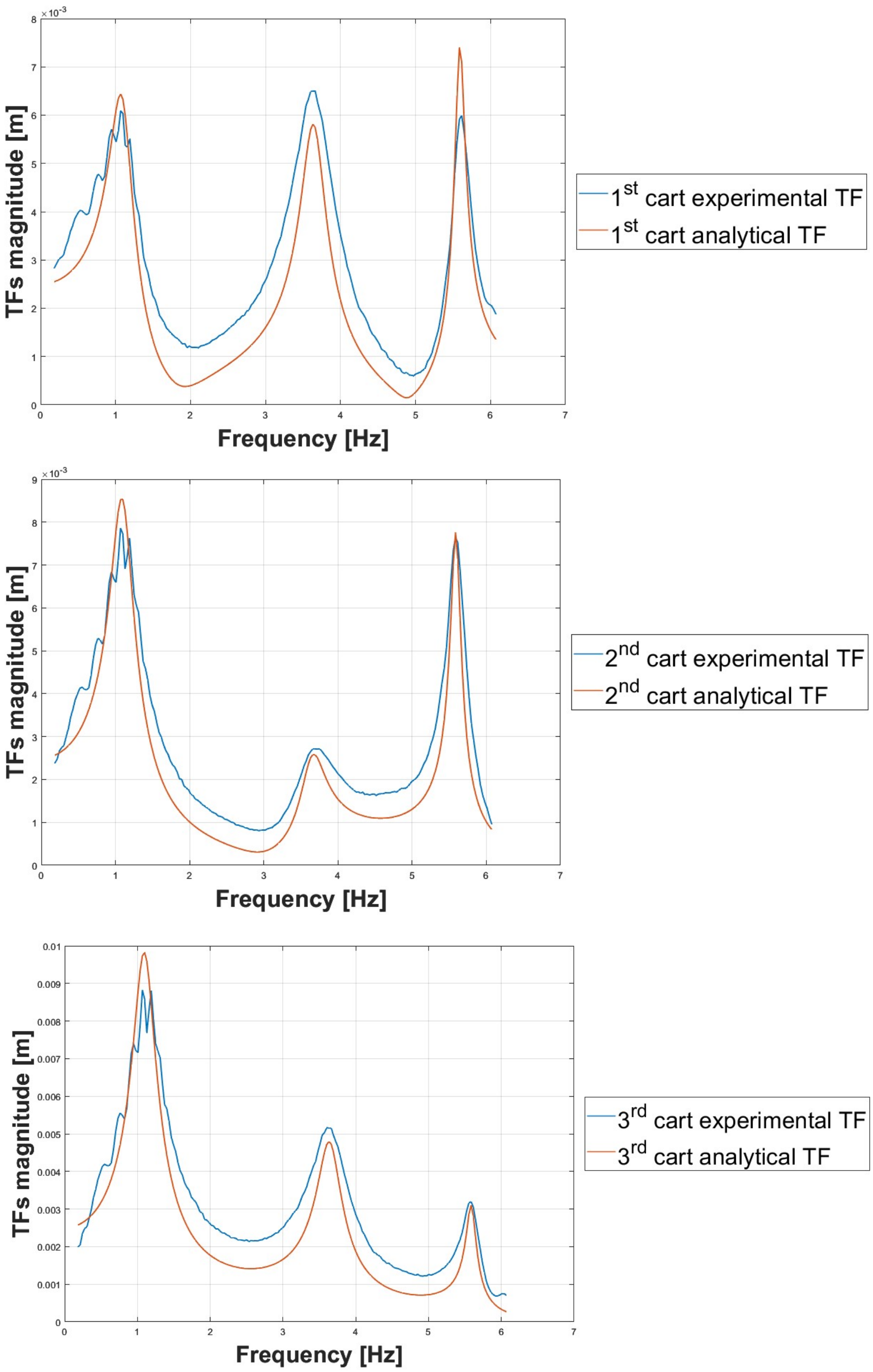

Figure 13: Experimental Transfer Functions

The fitting procedure has been successfully completed, because from the plots it is clear that the peaks of the Analytical TF are superimposed to the peaks of the Experimental TF.
The final values found by the numerical solver are the following:
$k_1 \simeq 829.34 N/m \qquad k_2 \simeq 780.75 N/m \qquad c_2 \simeq 0.44 Ns/m \qquad c_3 \simeq 8.08 Ns/m$
Furthermore, as expected, the stiffness values are similar to the first spring since they have the same magnitude order, the damping constant c_2 is similar to c_1 since they both have the same initial approximation and c_3 has one magnitude more.