Benchmarking Quantum and Hybrid Algorithms to find the Ground State of Ising Models

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Abstract

The Ising Model, introduced in 1920 by Wilhelm Lenz and Ernst Ising, has become a cornerstone of research, extending far beyond physics to influence numerous other fields, despite its apparent simplicity [2]. The model has proved useful due to its ability to elegantly describe NP-Hard combinatorial optimization problems.

To find the ground state of an Ising Model, or to find the optimal solution to the optimization problem represented by a given Ising Model, we must minimize the following Hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - \sum_i h_i S_i$$

The emergence of quantum computing provides exciting breakthroughs in solving computationally-intensive optimization problems. Cloud-accessible quantum hardware allows for the development and testing of quantum and hybrid (combination of quantum and classical) computing algorithms for efficiently solving or approximating solutions to complex optimizations.

In this project, I benchmark the performance of quantum and hybrid optimization algorithms using cloud-accessible quantum hardware.

Methods

Quantum Approximate Optimization Algorithm (QAOA)

QAOA involves an alternating sequence of running a parameterized circuit on quantum hardware, then using the measurements of the circuit to determine the cost of the function to optimize the circuit parameters using classical optimization techniques [1]. The circuit relies on two unitary operators defined by the parameters γ and β.

$$U(\gamma, C) = e^{i\pi\gamma C(Z)/2} \xrightarrow{\text{Ising Model}} \prod_{\langle i,j \rangle} e^{-i\pi\gamma Z_i Z_j/2} \prod_i e^{-i\pi\gamma h_i Z_i/2}$$

$$U(\beta, B) = e^{i\pi\beta B/2} = \prod_{j=1}^{n} X_j^{\beta}$$

In the former, C represents the cost function $C(z)$ to be minimized, where z represents the collection of variables, $z = z_1 ... z_n$, which in our case represent spins $z_i = \pm 1$. The latter represents a rotation of each qubit about the x-axis by an angle dependent on β, where B represents a mixing Hamiltonian. [3]. After applying an initial Hadamard Gate to place qubits in superposition, QAOA circuits alternate layers of each operator - *Figure 1* displays 2-layered QAOA circuit for a 4-qubit system, transpiled for an IBM device.

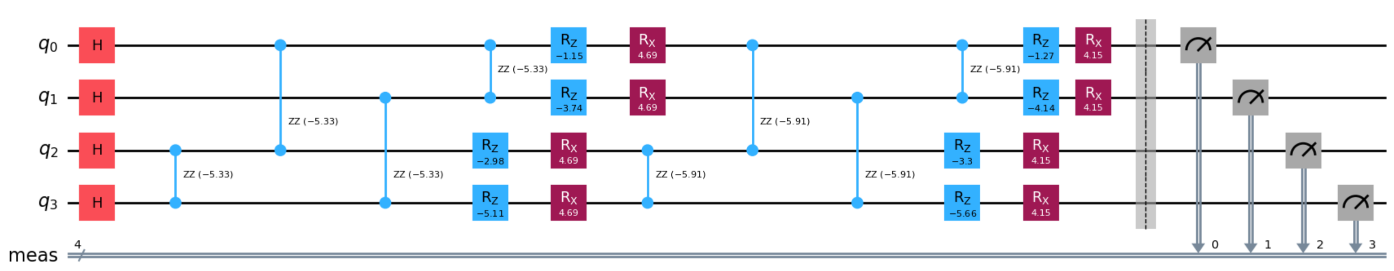


Figure 1. A transpiled QAOA Circuit for a 2 x 2 lattice, 2 iterations.

Quantum Annealing (QA)

QA leverages quantum mechanics to find low-energy configurations corresponding to optimal or near-optimal solutions for complex problems. QA is implemented by D-Wave Systems' hardware. In D-Wave's quantum processing units (QPUs), QA operates by initializing qubits in a superposition of states [4]. As the annealing process progresses, an energy landscape is shaped by programmable parameters, including qubit biases and couplings between qubits. The system evolves towards a minimal energy, and qubits settling into classical states of 0 or 1, representing potential solutions to the problem.

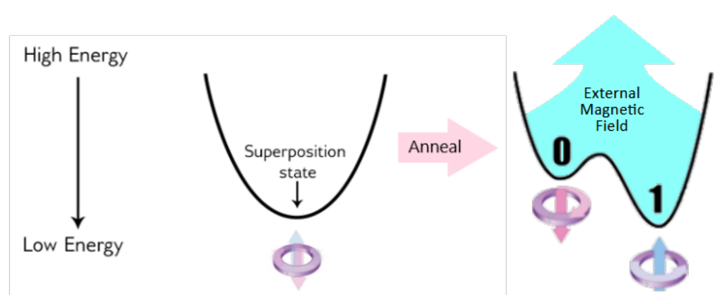


Figure 2. Quantum Annealing via D-Wave [4]

Benchmarking Method

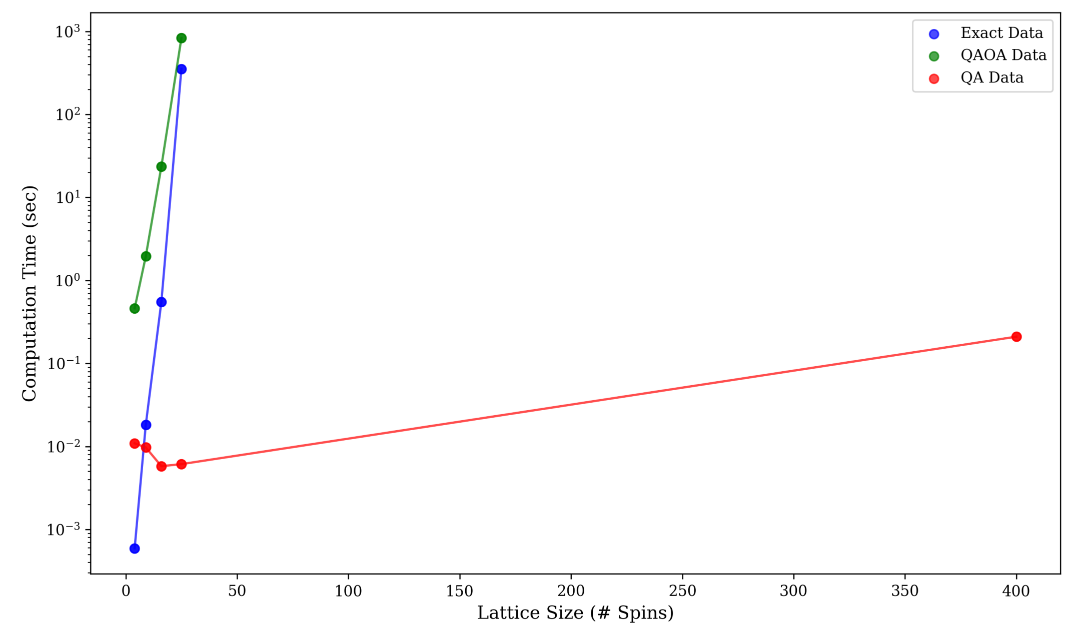
We constructed Ising Models for various lattice sizes, coupling strengths (J) and external magnetic fields (h). Magnetic fields were randomly generate using NumPy. Below is a selection of various Ising configurations of size $N \times N$. Accuracies were determined by comparing QA/QAOA solutions to exact solution.

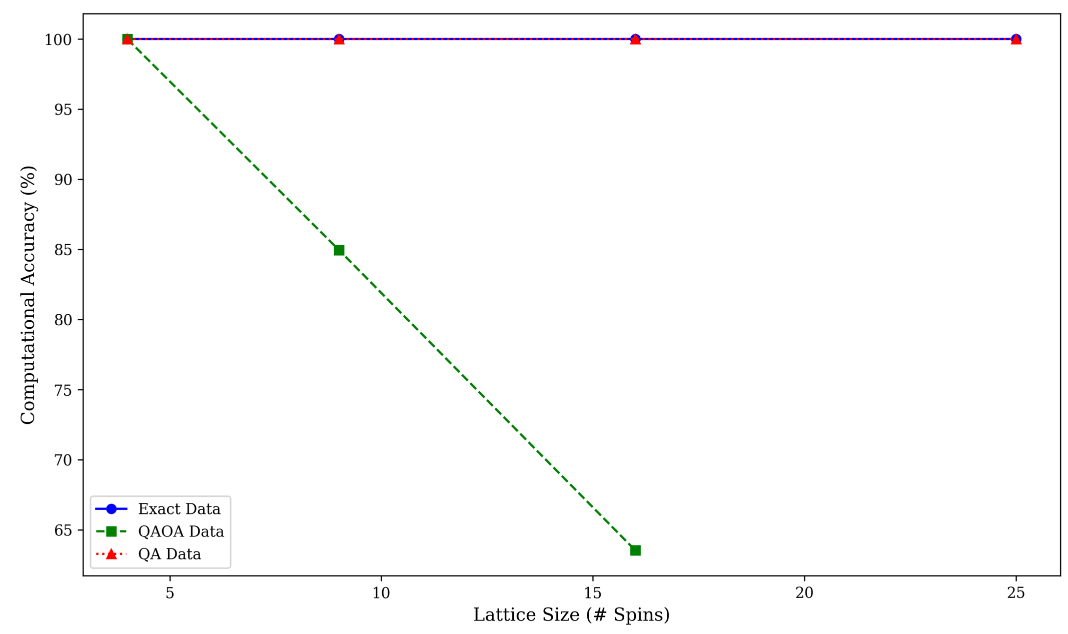
Benchmarking Data

Method	Avg. Duration (s)	Avg. Accuracy	Hardware
		N = 2	
QAOA	$4.6 * 10^{-1}$	100%	IBM Aer Simulator
QA	$1.1 * 10^{-2}$	100%	D-Wave QPU
Exact	$5.9 * 10^{-4}$	100%	Apple M2 Pro
		N = 3	
QAOA	1.95	85.0%	IBM Aer Simulator
QA	$9.7 * 10^{-3}$	100%	D-Wave QPU
Exact	$1.8 * 10^{-2}$	100%	Apple M2 Pro
		N = 4	
QAOA	24.1	63%	IBM Aer Simulator
QA	$5.8 * 10^{-3}$	100%	D-Wave QPU
Exact	$5.5 * 10^{-1}$	100%	Apple M2 Pro
		N = 5	
QAOA*	-	-	-
QA	$6.1 * 10^{-3}$	100%	D-Wave QPU
Exact	354	100%	Apple M2 Pro

Table 1. Performance comparison of different methods for varying values of N. *Unable to perform QAOA for 25 qubits.

Graphs





Analysis

The results of this study provide useful insights into the current state of varying quantum hardware types and algorithms. Inherently, Quantum-Annealing based hardwares, such as the D-Wave device used here, excel in optimizing problems mapped onto Ising models. While both QAOA and exact methods scaled exponentially in computational time, QA scaled linearly, allowing for computation across massive lattices surpassing 500 total spins. QAOA's limitations are derived mainly from the classical optimization component of the algorithm; minimizing cost to find optimal values for γ and β becomes exponentially more expensive computationally with the increase in spin, or qubits, limiting the algorithm to lattices no larger than 16 spins in our case.

References

[1] Kostas Blekos, Dean Brand, Andrea Ceschini, Chiao-Hui Chou, Rui-Hao Li, Komal Pandya, and Alessandro Summer. A review on quantum approximate optimization algorithm and its variants. *Physics Reports*, 1068:1–66, 2023.

[2] Stephen G. Brush. History of the lenz-ising model. *Reviews of Modern Physics*, 39(4):883–85, 1967.

[3] Elijah Pelofske, Andreas Bärtsch, and Stephan Eidenbenz. Short-depth qaoa circuits and quantum annealing on higher-order ising models. *npj Quantum Information*, 10(30), 2024.

[4] D-Wave Systems. What is quantum annealing? `https://docs.dwavesys.com/docs/latest/c_gs_2.html`. Accessed: 2024-12-1.

Project Github Repository