Real Analysis Notes

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Notes from textbook *Analysis I* by Terence Tao.

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1 Introduction

Real analysis is the study of real numbers. We will answer questions such as: What is a real number? How do you take the limit of a squenece of real numbers? What is a continuous function? What is a derivative?

Analysis is important to study for both the satisfaction of knowing "how" real numbers work, and to avoid falling into traps of false reasoning. Consider:

Example 1.0.1. *Consider* $\lim_{x\to\infty} \sin(x)$. *Make the change of varialbe* $y = x + \pi$ *and recall that* $\sin(y + \pi) = -\sin(y)$. *Then we get:*

$$\lim_{x\to\infty} \sin(x) = \lim_{y+\pi\to\infty} \sin(y + \pi) = \lim_{y\to\infty} - \sin(y) = - \lim_{y\to\infty} \sin(y)$$

Naturally, $\lim_{x\to\infty} \sin(x) = \lim_{y\to\infty} \sin(y)$. *Thus we get:*

$$\begin{aligned} \lim_{x\to\infty} \sin(x) &= - \lim_{x\to\infty} \sin(x) \\ \lim_{x\to\infty} \sin(x) &= 0 \end{aligned}$$

Another pitfall comes from interchanging sums. As we will see, whilst you can interchance finite sums, it is not always possible to interchange infinite sums. Similarly, we cannot always interchacne intergrals, or limits! What a bother.

Another fun example is when the apply the famous L'Hopital's rule. Written generically, this is:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

But note that we can *only* apply this rule if the limit of $x \to a$ of $f(x)$ and $g(x)$ are both zero. Even apart from this there are circumstances where the rule does not apply.

2 The natural numbers

We start with the natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Note that in this work, following Tao, we include zero in the natural numbers. We start here because once we have $\mathbb{N}$ we can build $\mathbb{Z}$ and $\mathbb{Q}$, and then $\mathbb{R}$.

2.1 The Peano axioms

This is a standard way to define the natural numbers. This is not the only way, and infact you can use the cardinality of sets to do it also.

Definition 2.1.1. *(Informal) The natural numbers are any element of the set:*

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

which is the set of all numbers created by starting with 0 and counting forward indefinitely.

This is an unsatisfactory definition, we don't know how to add multiply etc. What we can say is exponentiation is repeated multipliaction, which is repeated addition, which is repeated *incrementing*.

So do define natural numbers we will use two concepts: the zero number 0, and the increment operation, which we denote with $++$. From this we say that $\mathbb{N}$ contains 0 and everything that can be obtained by incrementing 0.

Axiom 2.1. *(Zero is a natural number)* $0 \in \mathbb{N}$

Axiom 2.2. *If n is a natural number, then $n{++}$ is also a natural number.*

From this we see that $(0{++}){++}$ is a natural number, and so on.

Definition 2.1.2. *We define 1 to be the natural number $0{++}$, 2 to be the number $(0{++}){++}$, etc.*

Note that sets of modular numbers adhere to the previous two axioms. We impose a third axiom to stop this.

Axiom 2.3. *0 is not the successor of any natural number.*

Proposition 2.1.1. $0! = 4$

Proof. *By axiom 2.1,* $4 = 3{++}$. *By axiom 2.3,* $4 = 3{++} \neq 0$ *and thus we have* $0! = 4$. ∎

We still have problems though, take a number system with $1, 2, 3, 4$ and $4{+}{+} = 4$. This adheres to all of the axioms. We add a new axiom to account for this.

Axiom 2.4. *For $n, m \in N$ such that $n \neq m$, then $n{+}{+} \neq m{+}{+}$.*

Finally we may have rogue elements of the number system. For example $1, 2, \ldots, a1, a2, \ldots$ where $an + 1 := an{+}{+}$. We want an axiom that says the only numbers in $\mathbb{N}$ are those that are accessible by incrementing 0. We do so using induxtion.

Axiom 2.5. *(Principle of mathematical induction). Let $P(n)$ be a property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, then $P(n{+}{+})$ is also true. Then $P(n)$ is true for all natural numbers n.*

Note that the above axiom only holds (in particular the last statement that P is true for all natural numbers) if all natrual numbers are "reachable" from 0 by incrementing.

All of these axioms together are called the Peano axioms. We write them again here for completeness:

Definition 2.1.3. *(Peano axioms) The natural numbers are a set $\mathbb{N}$ together with a distinguished element $0 \in \mathbb{N}$ and a successor function $++ : \mathbb{N} \to \mathbb{N}$ which assigns to each natural number $n \in \mathbb{N}$ a natural number $n{+}{+} \in \mathbb{N}$, such that the following axioms are satisfied:*

1. *(Zero is a natural number)* $0 \in \mathbb{N}$
2. *(Successor function is defined on all of $\mathbb{N}$) For every $n \in \mathbb{N}$, $n{+}{+} \in \mathbb{N}$.*
3. *(Zero is not the successor of any natural number) For every $n \in \mathbb{N}$,*
4. *(Successor function is injective) If $n, m \in \mathbb{N}$ and $n{+}{+} = m{+}{+}$, then $n = m$. we have $n{+}{+} \neq 0$.*
5. *(Principle of mathematical induction) Let $P(n)$ be a property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, then $P(n{+}{+})$ is also true. Then $P(n)$ is true for all natural numbers n.*

Proposition 2.1.2. *(Recursive definitions). Suupose for each $n \in \mathbb{N}$ we have a function $f_n : \mathbb{N} \to \mathbb{N}$, Let c be a natural number, then we can assign a unique natural numbers a_n to each $n \in \mathbb{N}$ such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for all $n \in \mathbb{N}$.*

Proof. *We proove this by induction. We note that a_0 is unique because $a_0 := c$, and we define $a_{n++} = f_n(a_n)$ for all $n \in \mathbb{N}$, and by Axiom 2.3 0 is the succesor* *of no natural number as thus will never be redefined. Now let a_n have a unique value. Consider $a_{n++} := f_n(a_n)$. We have a_{n++} will be unqiue with value $f_n(a_n)$ as for all other $m \in \mathbb{N}$ we will have $m{++} \neq n{++}$ because of Axiom 2.4 (that is $++$ is injective). By induction we thus have this holds for all a_n.*

What this says is that we can define a sequence of numbers recursively where each element of the sequence is well defined (that is it takes on a unique value). Also note that we had to use all the axioms. In a number system that wraps around, a recursive definition would not provide unoiquness like this (which you can see trivially).

2.2 Addition of natural numbers

The idea from here is to build up more complex operations from increment.

Definition 2.2.1. *Let m be a natural number. We define $0+m := m$. Assume that we have defined how to add n and m. Then we define $(n{++}) + m := (n+m){++}$. By induction this will give us the definition of all possible sums.*

This is similar to our previous discussion of recursive definitions. In particular we have $a_n = n + m$ with $f_n(a_n) = a_n{++}$.

Proposition 2.2.1. *For any natural number n, $n + 0 = n$.*

Proof. We use induction. Consider the base case $0 + 0 = 0$ by the definition of addition. Now assume that $n + 0 = 0$. We now show that $n{++} + 0 = n{++}$. By the definition of addition we have $(n{++}) + 0 = (n + 0){++} = n{++}$. ∎

We now rattle off a few propoositions (mostly) without proof. In general they require using the above proposition, and the fact that $n + m{++} = (n + m){++}$ (i.e., the symetric versions of the properties given in the definition of additon) and induction to prove.

Proposition 2.2.2. *(Various things about addition). For $a, b, c, d \in \mathbb{N}$ we claim (mostly) without proof:*

1. *(commutative) $a + b = a + b$*
2. *(associative) $(a + b) + c = a + (b + c)$*
3. *(cancellation) If $a + b = a + c$, then $b = c$.*

Proof of cancellation. As before we prove by induction on a. Let b and c be arbitrary. Let $a = 0$. In this case we have $0 + b = 0 + c \implies b = c$ by

Proposition 2.3. Now assume that $a + b = a + c \implies b = c$ for some other a. We now show the same holds for a++. Let $(a{++}) + b = (a{++}) + c$. By the definition of addition we thus have $(a+b){++} = (b+c){++}$. As ++ is injective we must have $a+b = a+c$ and thus $b = c$. By induction, we complete the proof. ■

Definition 2.2.2. *(Positive natural numbers). $n \in \mathbb{N}$ is positive iff it is not equal to 0.*

It quickly foloows that if a is positive and b is a natural number then $a + b$ is positive. We can prove this simply using induction and the fact that 0 is not the successor of any natural number (by Axiom of 2.3). A corollary of this is that if $a + b = 0$, then $a = b = 0$, else we would have a contradiction with the previous statement.

Lemma 2.2.1. *For $a \in \mathbb{N}$ that is positive there exists exactly one $b \in \mathbb{N}$ such that $a = b{++}$. I.e. a is the successor of only one natural number.*

Having defined addition, we can now define a notion of order of the natural numebrs in terms of addition.

Definition 2.2.3. *(Ordering of natural numbers). For $n, m \in \mathbb{N}$ we say that n is less than or equal to m, written as $n \leq m$, iff there exists a natural number a such that $n + a = m$. We say that n is less than m, written as $n < m$, if $n \leq m$ and $n \neq m$.*

Proposition 2.2.3. *(Various properties of order). For $a, b, c \in \mathbb{N}$ we have:*

1. *(Order is reflexive) $a \leq a$.*
2. *(Order is transitive) If $a \leq b$ and $b \leq c$, then $a \leq c$.*
3. *(Order is anti-symmetric) If $a \leq b$ and $b \leq a$, then $a = b$.*
4. *(Addition preserves order) If $a \leq b$, then $a + c \leq b + c$.*
5. *$a < b$ iff $a{++} \leq b$.*
6. *$a < b$ iff $b = a + d$ for some positive number d.*

Proposition 2.2.4. *(Trichotomy of order). For any $a, b \in \mathbb{N}$ exactly one of the following three statements is true: $a < b$, $a = b$, or $a > b$.*

Order allows us to create a *stronger* principle of induction on the natural numbers.

Proposition 2.2.5. *(Strong principle of induction). Let m_0 be some natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m > m_0$ we have the following implication: if $P(m')$ is true for all $m_0 \leq m' < m$, then $P(m)$ is also true. Then we can conclude that $P(m$ is trual for all natural numbers $m \geq m_0$.*

So what is this principle saying? Essentially, if P is true for all $n \in [m_0, m)$ implies that $P(m)$ is true, then P is true for all $n \geq m_0$. This is just like normal induction, except in normal induction we have:

$$[P(0) \wedge P(n) \implies P(n{+\!+})] \implies P(n) \text{ is true for all } n \in \mathbb{N}$$

Now we have:

$$[P(m_0) \wedge P(m') \text{ is true for all } m_0 \leq m' < m \implies P(m)] \implies \\ P(n) \text{ is true for all } n \geq m_0$$

We tend to normally use this with $m_0 = 0$ or $m_0 = 1$.

— Exercises —

Exercise 2.2.6. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+\!+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is true. Prove that $P(m)$ is true for all natural numbers $m \leq n$. This is known as the principle of backwards induction. (Hint: apply induction to the variable n).

Proof. We induct on n. Consider the base case $n = 0$. In this case we know that $P(0)$ is true. There is no $m \in \mathbb{N}$ such that $m < 0$, as this would imply that 0 is the successor of some natural number, which breaks one of the Peano axioms. Thus $P(m)$ is true for all natural numbers $m \leq 0$.

Now assume the claim is true for some n. We now show that it is also true for $n{+\!+}$. We suppose that $P(n{+\!+})$ is true. As everty natual number has a single successor, then n is the single predecessor of $n{+\!+}$. By the induction hypothesis, $P(m)$ is true for all $m \leq n$. As $P(n{+\!+}$ is true by assumption, then $P(m)$ is true for all $m \leq n{+\!+}$. By induction, the claim is true for all $n \in \mathbb{N}$. ■

2.3 Multiplication of natural numbers

From here out we assume we can use all the properties of addition that we know without proof. We now move to multiplication, which is simply the iterated additation operation, in much the same way that addition is simply iterated incrementation.

Definition 2.3.1. *Consider $m \in \mathbb{N}$. We define $0 \times m := 0$. Suppose inductively that we have defined how to multiple n and m. Then we can multiple $n + +$ to m by defining $(n{+\!+}) \times m := (n \times m) + m$. By induction this will give us the definition of all possible products.*

As before we can go ahead and prove lots of lemmas. Such as the distributive law which states that $a(b+c) = ab+ac$, commutativity, associativity, etc. Some useful ones are shown below.

Proposition 2.3.1. *(Useful facts about multiplication). We have:*

1. *(Multiplcation preserves order). If $a, b \in N$ such that $a < b$ and c is positive, then $ac < bc$.*
2. *(Cancellation). If $a, b, c \in \mathbb{N}$ and $a \neq 0$ such that $ab = ac$, then $b = c$.*

Now here comes a big one! Very useful.

Proposition 2.3.2. *(Euclidean algorithm). Let $n \in \mathbb{N}$ and q be positive. Then there exists natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.*

Remark 2.3.1. *In other words, we can divide a natural number n by a positive number q to obtain a quotient m and a remainder r. This was the beginning of* number theory*.*

Going deeper, we can now define exponentiation in terms of multiplicaiton.

Definition 2.3.2. *(Exponentiation). We define $m^0 := 1$ for all $n \in \mathbb{N}$. Suppose we for n and m we have defined m^n. Then we define $m^{n{+}{+}} := m^n \times m$. By induction this will give us the definition of all possible exponentiations.*

3 Set Theory

Here we introduce some of the core aspects of axiomatic set theory, which almost every other branch of mathematics relies on. We leave discussion of more advanced topics, such as infinite sets, and the axiom of choice, to Chapter 8.

3.1 Fundamentals

Just like the natural numbers, we build what a set is using axioms. We start, however, with an informal definition.

Definition 3.1.1. *(Sets, informal). We define a set A to be an unordered colection of objects. If x is an object, we say that x is an element of A, and write $x \in A$, if x belongs to A. If x is not an element of A, we write $x \notin A$.*

This definition is intuitive, but doesn't let us do things like set operations, or say what collections of objects are sets and what aren't.

Axiom 3.1. *(Sets are objects). If A is a set, then A is also an object. Thus given two sets, we can ask is one an element of the other.*

Remark 3.1.1. *There is a special case of set theory called "pure" set theory in which all objects are sets. E.g.* 0 *is the empty set, and 1 is* $\{0\} = \{\emptyset\}$ *and so forth. From a logical point of view, pure set theory is simpler as we only have to deal with one type of object. From a conceptual point of view, it is easier to deal with impure set theory, as we are doing here.*

For us, of all the objects in maths, some are sets, and some are not. If x is an object and A a set then $x \in A$ is true or false. If A is not a set, then $x \in A$ is undefined. E.g., $3 \in 4$ is niether true nor false, but meaningless.

Definition 3.1.2. *(Equality of sets). Two sets A and B are equal iff every element of A is in B and vice versa.*

We informally introduce the axiom of substitution. This states that if two sets are equal, then we can replace one by the other in any expression, and the result will be unchanged. Note that $\in$ respects this axiom, as if $x \in A$ and $A = B$ then $x \in B$. Thus if we build all set operations from $\in$, they too will obey the axiom. Futher note, however, that we do not care about the order of sets, so operations such as "the first in" and "the last in" *would not* obey the axiom of substitution.

We now define what objects are sets in a similar way to defining the natural numbers (where we started with 0 and built up numbers from there).

Axiom 3.2. *(Empty set). There exists a set $\emptyset$, known as the empty set, which contains no elements. That is $\forall x, x \notin \emptyset$.*

Lemma 3.1.1. *(Single choice). If A is a non-empty set, then there exists an object $x \in A$.*

Proof. Assume towards contradiction this is not true. Then $\forall x, x \notin A$, thus $A = \emptyset$ which is a contradiction.

This lemma is trivial, but says something fairly profound. We can always pick an elemnt from a non-empty set. Going further, given a finite set of non-empty sets, we can choose an element from each, known as "finite choice." The extension of this to infinite sets reqiures another axiom, the *axiom of choice*, whic we discuss in Section 8.

Axiom 3.3. *(Singleton and pair sets). For every object a there is a singleton set $\{a\}$ whose only element is a, and for every pair of objects a, b there is a pair set $\{a, b\}$ whose only elements are a and b.*

Note that the singeton set axiom is redundent as it follows from the pair set axiom, as that gives us there exists sets $\{a, a\}$ which by our definition of set equivalence, is the same as $\{a\}$. The next axiom allows us to build bigger sets (and the it + the singleton axiom also gives us the pair set axiom).

Axiom 3.4. *(Pairwaise union). Given sets A, B, there exists a set $A \cup B$ whose elements consist of all the elements belonging to A or B or both. That is:*

$$x \in A \cup B \iff (x \in A \vee x \in B)$$

By usion the definition of set equality and set union, you can see that $\cup$ obeys the axiom of substitution, and so is well defined on sets.

Lemma 3.1.2. *$\cup$ is commutative and associative. That is, for all sets A, B, C we have:*

1. $A \cup B = B \cup A$
2. $(A \cup B) \cup C = A \cup (B \cup C)$

We leave the proof as an exercise to the reader.

Pairwise union lets us build sets with 2 objects, 3 objects etc. Note however, we cannot yes construct sets consisting of n objects for any $n \in \mathbb{N}$ as we have not regierously defined the concept of n-fold iteration. Similarly we cannot create infinite sets. We will introduce axioms later that allow us to do this.

Definition 3.1.3. *(Subsets). Given two sets A, B, we say that A is a subset of B, written $A \subseteq B$, iff every element of A is also an element of B. That is:*

$$A \subseteq B \iff \forall x, x \in A \implies x \in B$$

A is a proper subset of B, written $A \subset B$, iff $A \subseteq B$ and $A \neq B$.

As this definition only uses $\in$ and $=$, it obeys the axiom of substitution and thus is well defined.

Proposition 3.1.1. *(Sets are paritally ordered by set inclusion). For sets A, B, C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Finally, if $A \subset B$ and $B \subset C$ then $A \subset C$.*

Proof. We leave the proof as an exercise to the reader.

Remark 3.1.2. *Note that sets are partially ordered by set inclusion. This is because not all pairs of sets can be related using set inclusion. Thus the relation does not apply to all pairs of sets. In contrast, $<$ as defined on natural numbers is a total order, as it applies to all pairs of natural numbers.*

Axiom 3.5. *(Axiom of specification). Let A be a set, and for each object x let $P(x)$ be a property pertaining to x. Then there exists a set, denoted $\{x \in A : P(x)\}$, whose elements are exactly the elements x in A for which $P(x)$ is true. In other words, for any object y:*

$$y \in \{x \in A : P(x)\} \iff (y \in A \land P(y))$$

The axiom is also known as the axiom of separation.

As before, the axiom of specification does not break the axiom of substitution, and so is well defined. This axiom can be used to define other operations, such as intersections.

Definition 3.1.4. *(Set intersection). The intersection $S_1 \cap S_2$ we define as:*

$$S_1 \cap S_2 := \{x \in S_1 : x \in S_2\}$$

Definition 3.1.5. *(Set difference). We define $A - B$ or $A \setminus B$ as:*

$$A - B := \{x \in A : x \notin B\}$$

Proposition 3.1.2. *(Sets form a boolean algebra). For sets A, B, C, all contained in X, we have:*

1. *(Minimal element) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.*
2. *(Maximal element) $A \cup X = X$ and $A \cap X = A$.*

3. *(Commutative laws)* $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$.

4. *(Associative laws)* $(A \cup B) \cup C = A \cup (B \cup C)$ *and* $(A \cap B) \cap C = A \cap (B \cap C)$.

5. *(Distributive laws)* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

6. *(Partition)* $A \cup (X - A) = X$ *and* $A \cap (X - A) = \emptyset$.

7. *(De Morgan's laws)* $X - (A \cup B) = (X - A) \cap (X - B)$ *and* $X - (A \cap B) = (X - A) \cup (X - B)$.

These all follow fairly simply.

Axiom 3.6. *(Replacement). Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$, there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$, such that for any object z:*

$$z \in \{y : P(x, y) \text{ is true for some } x \in A\} \iff P(x, z) \text{ is true for some } x \in A$$

Example 3.1.1. *Let $A := \{3, 5, 9\}$ and let $P(x, y)$ be $y = x{+}{+}$. For every $x \in A$ there is only one y for which $P(x, y)$ is true as every number has only 1 successor. The above axiom says the the replacement exists, and is the following set:* $\{4, 6, 10\}$.

Using function notation, the axiom can be used to construct sets of the form:

$$\{y : y = f(x) \text{for some} x \in A\}$$

Here we are using the fact that a function maps an element to a single element (something we will define in more detail later, so really using functions here is a bit circular). Note that that is the key feature of $P(x, y)$ in the axiom, x is mapped to at most 1 y. There is a subtle difference here though, we can have no such y that satisfies $P(x, y)$ which doesn't quite mesh with our function definition. We will more commonly write set constructions like this as:

$$\{f(x) : x \in A\}$$

The axiom of replacement can be combined with the axiom of specification to create sets of the form:

$$\{f(x) : x \in A \land P(x)\}$$

We now use sets to formalize natural numbers with our friends the peano axioms.

Axiom 3.7. *(Infiniy). There exists a set $\mathbb{N}$, whose elemetns are called natural numebrs, as well as an object 0 in $\mathbb{N}$, and an object $n{+}{+}$ assigned to every $n \in \mathbb{N}$, such that the Peano axioms hold.*

This is the formal construction of the natural numbers! It is called the axiom of infinity because it introduces the basic example of an infinite set. The axiom of infinity gives us natural numbers, so gives us that numbers are objects in set theory.

— Exercises —

Exercise 3.1.11. Show that the axiom of replacement implies the axiom of specification.

Proof (Informal). Here we do something like take a property $P(x)$ and define a replacement property $Q(x, y) := x = y \wedge P(x)$. Therefore when we apply the replacement, for any x such that $P(x)$ is truen, x is places in the resulting set, and if $P(x)$ is not true, then it is removed. Very nice!

3.2 Russell's Paradox (Optional)

We may be tempted to include the following axiom:

Axiom 3.8. *(Universal specification). (Dangerous!) Suppose for every objet x we have a property $P(x)$ pertaining to x that is true or false. Then there exists a set $\{x : P(x)$ is true$\}$*

This axiom asserts that every property corresponds to a set, and thus that we can make sets of all of a thing. For example the set of all blue things, or the set of all sets. This axiom implies many of our previous axioms. Unfortunately it leads to a **paradox** known as *Russell's Paradox*, and thus cannot be included in set theory.

The paradox is as follows. Define $P(x)$ as:

$$P(x) \iff x \text{ is a set, and } x \notin x$$

So $P(\{1, 2\})$ is true, but if S is the set of all sets, which we can construct from universal specification, then $P(S)$ is false. Now consider, again through universal specification:

$$\Omega := \{x : P(x)\} = \{x : x \text{ is a set, and } x \notin x\}$$

Now is $\Omega \in \Omega$? If it is, then $P(\Omega)$ is false and so it should not be. If it is not, then $P(\Omega)$ is true and so it should be. This is a contradiction, and so we cannot have the axiom of universal specification.

The problem with the axiom is that it creates sets that are too big. One way to resolve this is to put objects into heirarchies. At the bottom are primitive objects which are not sets. One layer up there are sets of primitive objects. Then above this there are sets that contain primitive objects and sets of primitive objects, etc. This means sets at each stage of the heirarchy can only contain things from lower stages, and thus *a set can never contain itself.* Formalizing this is difficult, instead we include an axiom that means we do not run into Russell's paradox.

Axiom 3.9. *(Regularity or foundation). If A is a non-empty set, then there is at least one element of A which is either not a set, or is disjoint from A*

This axiom implies that sets cannot contain themselves. For the purpose of doing analysis, this axiom is never needed, so should be considered a side point.

— Exercises —

Exercise 3.2.2. Use the axiom of regularity and singleton set axiom to show that if A is a set, then $A \notin A$.

Proof. Assume towards contradiction $A \in A$. By the singleton set axion we can make $B := \{A\}$. Not that B breaks the axiom of regularity, as it has 1 element that is a set, and that one element A is not disjoint from B, as $B \cap A = A \neq \emptyset$ as $A \in A$. Thus we have a contradiction, and so $A \notin A$. ■

3.3 Functions

We start with a fareful cumbersome, but precise definition of a function.

Definition 3.3.1. *(Functions). Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and $y \in Y$, such that for every $x \in X$ there is exactly one $y \in Y$ for which $P(x, y)$ is true* [1]*. We define the function $f : X \to Y$ defined by P on the domain X with range Y to be the object which, given any input $x \in X$, assings an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ os true. Thus, for any $x \in X$ and $y \in Y$, we have:*

$$y = f(x) \iff P(x, y) \text{ is true}$$

Functions are also referred to as *maps* and *transformations.* depending on the context. They are also sometimes called *morphisms*, although to be more precise, a morphism refers to a more general class of object, which may or may not be a function.

[1] This is sometimes known as the vertical line test

There are two ways to define functions:

1. Explicitly. In this case say what x gets mapped to, for example $f(x) = x{+}{+}$.
2. Implicitely. In this case we define a property $P(x, y)$ that links x with the output of $f(x)$. When doing this we need to be careful to ensure that the property is well defined (that is there is a single y for each x such that $P(x, y)$ is true).

Functions obey the axiom of substitution, that is $x = x' \implies f(x) = f(x')$, because the property $P(x, y)$ obeys the axiom. We define some useful properties of functions:

Definition 3.3.2. *(Lots of things about functions). We have:*

1. *Equality. For f and g with saem domain and range, $f = g$ iff $\forall x, f(x) = g(x)$.*
2. *Composition. For $f : X \to Y$ and $g : Y \to Z$, the composition $g \circ f : X \to Z$ is defined explicitely as:*

$$(g \circ f)(x) := g(f(x))$$

   If the range of f is not the domain of g, the operation is undefined. This obeys the axiom of substitution. Composition is associative ($f \circ (g \circ h) = (f \circ g) \circ h$).

3. *Injective (one-to-one). A function $f : X \to Y$ is injective iff $f(x) = f(x') \implies x = x'$. Taking the contrapositive, this means that $x \neq x' \implies f(x) \neq f(x')$.*
4. *Surjective (onto). A function $f : X \to Y$ is surjective iff $\forall y \in Y, \exists x \in X$ such that $f(x) = y$.*
5. *Bijective. If f is surjective and injective, then it is bijective.*

3.4 Images and Inverse Images

Definition 3.4.1. *(Images of sets). Consider $f : X \to Y$ and $S \subseteq Y$, we define $f(S)$ as*

$$f(S) := \{f(x) : x \in S\} \subseteq Y$$

and this is called the image of S under f. Sometimes $f(S)$ is called the forward image.

Note that the image is well defined using the axiom of replacement. You can also define $f(S)$ using the axiom of specification (specifying elements of Y to include as opposed to elements of X to replace).

Definition 3.4.2. *(Inverse image). If $U \subseteq Y$, we define $f^{-1}(U)$ to be:*

$$f^{-1}(U) := \{x \in X : f(x) \in U\} \subseteq X$$

We note that functions are objects, and so we should be able to consider sets of *all* functions from a set X to a set Y. Remember from Russell's paradox that we cannot make this using the axiom of universal specification, as we did not introduce this into set theory (due to the paradox it creates). Therefore we introduce a new axiom for this specifically.

Axiom 3.10. *(Power set axiom). Let X and Y be sets. There exists a set, denoted by Y^X, which consists of all the functions from X to Y, thus:*

$$f \in Y^X \iff (f \text{ is a function with domain } X \text{ and range } Y)$$

The reason we use the notation Y^X is if Y has n elements and X has m elements, then Y^X has n^m elements.

Lemma 3.4.1. *Let X be a set. Then the following is a set*

$$\{Y : Y \subseteq X\}.$$

We refer to this as the pwer set of X, denotes by 2^X or $\mathcal{P}(X)$.

To construct this set, we use the Power set axiom to get X^X, and then use the axiom of specification to select all the distinct subsets of X, as for each $A \subseteq X$, there is a function $f : X \to A$ that is in X^X.

Finally we enhance the axiom of pairwise union to create much larger sets.

Axiom 3.11. *(Union). Let A be a set, all of whose elements are sets themselves. Then there exists a set $\bigcup A$ whose elements are precisely those objects that are elements of the elements of A, thus for all objects x we have:*

$$x \in \bigcup A \iff \exists B \in A, x \in B$$

The axiom of union with the axiom of pair set, implies the axiom of pairwise union. Another important consequence is if we hvae an set I, and for all $\alpha \in I$ we have some A_α, then we can form the union set $\bigcup_{\alpha \in I} A_\alpha$ by defining:

$$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha : \alpha \in I\}$$

where $\{A_\alpha : \alpha \in I\}$ is a set by the axiom of replacement. We often call such an I an indexing set. We can also form intersections like this as:

$$\bigcap_{\alpha \in I} A_\alpha := \{x \in A_\beta : x \in A_\alpha \forall \alpha \in I\}$$

where I here is non-empty and $\beta \in I$ is arbitrary. Note this is a set by the axiom of specification (we are simply specifying what elements of A_β to include).

The axioms of set theory that we have introduced here, excluding the dangerous axiom of universal specification, are known as the *Zermelo-Fraenkel axioms of set theory*. The axioms of There is one further axiom, the *axiom of choice*, which gives rise to the *Zermelo-Fraenkel-Choice axioms of set theory*, but we won't need it until much later. We recap the axioms now:

Definition 3.4.3. *(Zermelo-Fraenkel axioms of set theory). We have:*

1. *(Empty set). There exists a set $\emptyset$.*
2. *(Set equality). Two sets are equal iff they have the same elements.*
3. *(Pair set). For any a, b there exists a set $\{a, b\}$.*
4. *(Union). For any set A there exists a set $\bigcup A$.*
5. *(Power set). For any sets X and Y there exists a set Y^X.*
6. *(Axiom of specification). For any set A and property $P(x)$, there exists a set $\{x \in A : P(x)\}$.*
7. *(Replacement). For any set A and property $P(x, y)$, there exists a set $\{y : P(x, y)$ is true for some $x \in A\}$.*
8. *(Infinity). There exists a set $\mathbb{N}$.*
9. *(Regularity). If A is a non-empty set, then there is at least one element of A which is either not a set, or is disjoint from A.*

NOTE: *Technically* Zermelo-Fraenkel set theory is a form of pure set theory, which changes some things. Most notably the axiom of infinity does not use the natural numbers (as in pure set thoery, there are only sets), and instead creates an infinite set of sets that escalate up the heirarchy we described earlier. The minimal set that satisfies this axiom, known as the *von Neumann ordinal*, is equivalent to the natural numbers. For our purposes, we study non-pure set theory, so we define things slightly differently. Additionally, the Power set axiom refers to 2^X not Y^X. Finally, Zermelo-Fraekel set theory only has 2-9 of the above axioms (with some changes like the afformentioned one), and the axiom of the empty set arrises from the axiom of infinity.

3.5 Certesian Products

This is another fundamental operation on sets.

Definition 3.5.1. *(Ordered pairs). If x and y are any object, we define the ordered pair (x, y) to be a new object, consisting of x as its first component and y as its second component. Two ordered pairs (x, y) and (x', y') are equal iff $x = x'$ and $y = y'$.*

Technically this is partly an axiom, because we have postulated that given any two objects x and y, the object (x, y) exists. It is, howeer, possible to define an ordered pair using the axioms of set theory in such a way that we do not need any further axioms (see Exercise 3.5.1).

Definition 3.5.2. *If X and Y are sets, then the cartesian product $X \times Y$ is defined as:*

$$X \times Y = \{(x, y) : x \in X \wedge y \in Y\}$$

or equivalently:

$$a \in (X \times Y) \iff a = (x, y) \wedge x \in X \wedge y \in Y$$

Definition 3.5.3. *(Ordered n-tuple and n-fold cartesian product). An oredered n-tuple $(x_i)_{1 \leq i \leq n}$, also denoted as $(x_1, x_2, \ldots, x_n)$ is a collection of n objects, where x_i is the ith component. Two ordered N-tuples are equal iff each of their components are equal, that is $x_i = y_i$.*

If $(X_i)_{1 \leq i \leq n}$ is an ordered n-tuple of sets, then their cartesian product $\prod X_i$ is defined as

$$\prod X_i := \{(x_i)_{1 \leq i \leq n} : x_i \in X_i \text{ for all } 1 \leq i \leq n\}$$

This means take the first component from the first set, the second from the second set, etc, and make all the ones possible.

This definition simply postulates that an ordered n-tuple and Cartesian product always exist when needed, but using the axioms of set theiry once cna explicitly construct these objects.

Remark 3.5.1. *To get Cartesian product, we just need n-fold version. All wee need to do is show $\prod X_i$ is a set. Here's how we do it. With the power set axiom, we consider the set of functions $i \mapsto x_i$ from $\{1 \leq i \leq n\}$ to $\bigcup X_i$, that is the set $(\bigcup X_i)^{\{1 \leq i \leq n\}}$. Another way of writing this is the set:*

$$(\bigcup X_i)^{\{1 \leq i \leq n\}} = \{f \mid f : [n] \to \bigcup X_i\}$$

Then we can use the axiom of specification to restrict this to the set of functions tha map $i \mapsto x_i$ for $x_i \in X_i$. That is the functions that when given a natural *number i, select and element from the set X_i. Note that each such function f_j defines an n tuple:*

$$(f_j(1), f_j(2), \ldots, f_j(n)) \in \prod X_i$$

With some more work we can show that there is a bijection between the functions and all possible n tuples in $\prod X_i$, and thus using the axiom of replacement, we can construct $\prod X_i$ is a set.

An ordered n-tuple of objects is also called an ordered sequence of n elements, or a finite sequence.

Lemma 3.5.1. *(Finite choice). Consider $n \geq 1$ with $n \in \mathbb{N}$ and for all $1 \leq i \leq n$ let X_i be a non-empty set. Then there exists an n-tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for $\forall i$. In other words, if each X_i is non-empty, then the set $\prod X_i$ is also non-empty.*

Proof (informal). *We prove with induction. When $n = 1$, we have the claim is true thanks to the single choice lemma. Assume true for n. Consider $n + 1$. In this case make the $n + 1$ tuple $(x_i)_{1 \leq i \leq n+1}$ by taking the n tuple $(x_i)_{1 \leq i \leq n}$ and then using the single choice lemma to select $x_{n+1} \in X_{n+1}$. Thus the claim is true for $n + 1$ and so by induction is true for all n.* ■

This lemma is essentially trivial (although requires rigour to prove). It just said given a finite number of non-empty sets, we can pick an element from each. It cannot, however, be extended to an infinite number of sets. For this we need the axiom of choice, which we will introduce later. This is becuase we cannot induct infinitely.

— Exercises —

Exercise 3.5.1. Define an ordered pair (x, y) to be the set $\{\{x\}, \{x, y\}\}$. Show that this definition obeys the definition of an ordered pair.

Proof. Recall the definition of an ordered pair was simply:

$$(x, y) = (x', y') \iff x = x' \wedge y = y'$$

Consider two sets $A = \{\{x\}, \{x, y\}\}$ and $B = \{\{x'\}, \{x', y'\}\}$. We prove both directions in turn. Assume that $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$. Thus the sets contain the same elements (by the Axiom of extensionality). Thus $\{x\} \in B$, but B contains only one element that is a set of one element, thus we must have $\{x\} = \{x'\}$. Similarly we have $\{x, y\} \in B$, and so $\{x, y\} = \{x', y'\}$. Note this implies $x = x'$ and $y = y'$, and thus.

$$A = B \implies \{x\} = \{x'\} \implies \{x, y\} = \{x', y'\} \implies x = x' \wedge y = y'$$

The other direction is simpler, it is trivial that if $x = x'$ and $y = y'$ then $A = B$. Thus we have:

$$A = B \iff x = x' \wedge y = y'$$

as required. ■

Exercise 3.5.12. TODO This exercise looks really interesting.

3.6 Cardinality of sets

In the previous chapter we defined the natural numbers axiomatically. We:

- Assumed there was 0 and an increment operation.
- Assumed 5 axioms of how these interact.

This is philosophically different to thinking of numbers as "how many things there are," or more formally, the cardinality of sets.

The Peano axiom approach treats numbers as more *ordinals* than cardinals. Cardinals are One, Two, Three, etc., and are used to count how many things there are in a set. Ordinals are First, Seond, Third, etc., and are used to order a sequence of objects. *There is a difference between the two*, which arises when considering infinite ordinals and infinite cardinals, although we don't need to worry about this.

In the previous section chapter we did not answer the question "can natural numbers be used to *count* sets." Here we show that they can be used to count the cardinality of sets asssuming the sets are finite.

To start getting at this, we may want to answer a simpler question. Not how many elements does a set have, but when do two sets have the same size. One way to do this is to say they have the same size when they have the same number of elements, but this becomes circular as we have not defined "number of elements" and breaks down if we consider infinite sets.

Definition 3.6.1. *(Equal cardinality). Two sets X and Y have equal cardinaloty iff there exists a bijections $f : X \to Y$.*

Note, interestingly, that we don't know yet if $\{1, 2\}$ and $\{1\}$ are not the same cardinality? One way to do this would be to enumerate all functions between them and show none are bijective. Weirdly, a set can contain another set as a proper subset and still have the same cardinality (only infinite sets), for example the even numbers and the natural numbers.

Proposition 3.6.1. *Equal cardinality is an equivalence relation. Recall this means that the relation is:*

- *Reflexive: X has the same cardinality as X.*
- *Symmetric: If X has the same cardinality as Y, then Y has the same cardinal*
- *Transitive: If X has the same cardinality as Y, and Y has the same cardinality as Z, then X has the same cardinality as Z.*

Consider a natural number n. We now want to define what it *means* for a set to have n elements.

Definition 3.6.2. *Let n be a natural number. A set X is said to have n elements if it has the same cardinality as the set $\{1, 2, \ldots, n\}$. We also say that X has cardinality n iff it has n elements.*

Now lets make sure our definition does not lead to any craziness, such as a set having two different cardinalities.

Proposition 3.6.2. *(Uniqueness of cardinality). If X has cardinality n, then it cannot have another cardinality $m \neq n$.*

Proof (informal). *You start with a lemma that a set with positive cardinality is non-empty, and if $x \in X$ then $X - \{x\}$ has cardinlaity $|X| - 1$ (which here denotes the unique predecessor of n, as we have not defined negation yet). With this lemma, you prove the proposition by inducting on n the cardinality of X. Consider we have the inductive assumption and are looking at a set with cardinality $n{+}{+}$ but also cardinality m with $m \neq n$. Then we have $X - \{x\}$ has cardinality n and $m - 1$, and by the inductive assumption $n = m - 1$. But the Peano axioms say each number has a unique successor, so $n{+}{+} = m$ which is a contradiciton.*

Now we have defined cardinality and with this proposition, we know that $\{1, 2\}$ and $\{1\}$ do not have the same cardinality and we *do not* have to enumerate all fnuctions between them and show none are bijective. Instead we just need to show a function for each to get the cardinality of one as 2 and the other as 1.

Definition 3.6.3. *(Finite sets). A set is finite iff it has a cardinality n for some natural number n. Otherwise the set is called infinite. If X is a finite set, we use $|X|$ to denote its cardinality.*

Theorem 3.6.1. *The set of natural numbers $\mathbb{N}$ is ifinite.*

Proof. *ATC that this is false. Thus $\exists n \in \mathbb{N}$ such that $|\mathbb{N}| = n$. Thus there $\exists$ a bijection $f : \{1, 2, \ldots, n\} \to \mathbb{N}$. Consider $f(1), f(2), \ldots, f(n)$. We can show* *that this is bounded, that is there exists some $M > f(i) \forall i \in \{1, 2, \ldots, n\}$. Then consider $M + 1$. This is not mapped to by f, thus f is not surjective and so is not a bijection, which is a contradiction.*

Now we switch gears for a second. Lets see if we can define airhtmetic in terms of the cardinality of sets as opposed to using the Peano axioms.

Proposition 3.6.3. *(Cardinal arithmetic)*

(a) Let X be finite set and $x \notin X$, then $|X \cup \{x\}| = |X| + 1$.

(b) If X and Y are finite sets then $X \cup Y$ is finite and $|X \cup Y| \leq |X| + |Y|$. If X and Y are disjoint then $|X \cup Y| = |X| + |Y|$.

(c) If X is finite and $Y \subseteq X$ then Y is finite and $|Y| \leq |X|$. If $Y \subset X$ then $|Y| < |X|$.

(d) If X if finite and $f : X \to Y$ then $f(X)$ is finite and $|f(X)| \leq |X|$. If f is injective then $|f(X)| = |X|$.

(e) If X and Y are finite sets then Y^X is finite and $|Y^X| = |Y|^{|X|}$.

(f) If X and Y are finite sets then $X \times Y$ is finite and $|X \times Y| = |X| \times |Y|$.

The above propositions form the basis of arithmetic of natuaral numbers *without using the recursive Peano axioms*. This is the basis of *cardinal arithmetic*. For this work, we won't develop this arithmetic further and instead use the Peano axioms.

— Exercises —

Exercise 3.6.10. Let $A_1, \ldots, A_n$ be finite sets such that $|\bigcup_{i=1}^{n} A_i| > n$. Show thatb there exists $i \in \{1, ..., n\}$ such that $|A_i| \geq 2$. This is known as *the pigeonhole principle*.

Proof. We prove by induction. For $n = 1$ we have $|A_1| > 1$ and thus the claim holds. Assume true for $n \in \mathbb{N}$. Now consider $n+1$. We have $|\bigcup_{i=1}^{n+1} A_i| > n+1$. We have two cases:

1. $|A_{n+1}| \geq 2$ in which case we are done.
2. $|A_{n+1}| = 1$. Thus $A = \{x\}$. Now remove this from the union. Now if $\{x\} \notin \bigcup_{i=1}^{n} A_i$ then we have $|\bigcup_{i=1}^{n} A_i| > n$ and so by the inductive assumption we have $|A_i| \geq 2$ for some i. If $\{x\} \in \bigcup_{i=1}^{n} A_i$ then we have the size of the union is unchanged, so $|\bigcup_{i=1}^{n} A_i| > n$, so again by the inductive assumption we have $|A_i| \geq 2$ for some i.

4 Integers and Rationals

4.1 Integers

We want to introduce a notion of subtraction, on top of addition and multiplicaiton which we already have. Informally, the integers are what we get when subtracting two natural numbers. This is not a complete definition because:

1) It doesn't say when two differences are equal.
2) It doesn'y say how to add and multiply integers (do arithmetic).
3) Its circular, we haven't defined subtrafction yet, and infact need the integers to do this.

We will build the integers by defininging them to follow the algebraic rules we know. For a), if $a - b = c - d$, then this means $a + d = b + c$. So equality can be defined using addition. To answer b), we know that $(a - b) + (c - d) = (a + c) - (b + d)$ and $(a - b)(c - d) = ac - bd$, so we can define addition and multiplication using these rules. Finally for c), we will being by writing integers as a—b instead, where — is simply a placeholder symbol. Later when we define subtraction, we will see that $a - b = a$—b, and we can remove it.

Definition 4.1.1. *(Integers). An integer is an expression of the form a—b, where a and b are natural numbers. Two integers are qual, a—$b = c$—d, iff $a + d = b + c$. We let $\mathbb{Z}$ denote the set of all integers.*

Remark 4.1.1. *This is not the most formal set theoretic definition. What is an "expression"? In the language of set theory, we are imposing an equivalence relation $\sim$ on the space of $\mathbb{N} \times \mathbb{N}$ ordered pairs of natural number, where:*

$$(a, b) \sim (c, d) \iff a + d = b + c$$

Following this, the set theoretic definition of a—b is the equivalence class of (a, b):

$$a\text{—}b := \{(c, d) \in \mathbb{N} \times \mathbb{N} : (a, b) \sim (c, d)\}$$

From this, we can us the normal definition of set equality to say a—$b = c$—d. This interpretation plays no role in how we end up manipulating integers, and infact thinking of integers as a set of equivalent pairs of natural numbers is quite cumbersome.

To check that this is a legitimate notion of equality, we need to make sure that it is reflexive, symmetric, and transitive, and obeys the substitution property. Note that we cannot verify the substiution axiom because we have not defined any binary operations on the integers yet, luckily we only need to do it for the basic operations as more complex operations will be built from these.

Definition 4.1.2. *The asume of two integers is defined as:*

$$a—b + c—d := (a + c)—(b + d)$$

The producwe of two integers is defined as:

$$(a—b)(c—d) := (ac + bd)—(ad + bc)$$

Lemma 4.1.1. *(Addition and multiplication are well defined). We check the axiom of substitution.*

Proof. *For brevity we just show addition. Consider $a—b = a'—b'$. We want to show:*

$$(a—b) + (c—d) = (a'—b') + (c—d)$$

We have:

$$LHS = (a + c)—(b + d)$$
$$RHS = (a' + c)—(b' + d)$$

Now recall the definition of equality on the integers. We have $LHS = RHS \iff a + c + b' + d = a' + c + b + d$. Note that we have

$$a - b = a' - b' \implies a + b' = a' + b$$

By adding $c + d$ to both sides we get:

$$a + c + b' + d = a' + c + b + d \implies LHS = RHS$$

which concludes the proof. ■

The integers $n—0$ behave the same way as the natural numbers (we can show that addition and multiplication works the saem way, and that $n—0 = m—0 \iff n = m$). We say there is an *isomorphism* between $\mathbb{N}$ and the integers of the form $n—0$. This allows us to identify the natural numbers with integers by setting $n = n—0$. We can now define incrementation on the integers by defining $x{+}{+} = x + 1$.

Definition 4.1.3. *(Negation of integers). If $(a—b)$ is an integer, we define the negation $-(a—b)$ to be the integer $(b—a)$. In particular if $n = n—0$ is a positive natural number, we define the negation $-n = n—0$. (We leave showing that this definition is well defined as an exercise for the reader).*

Lemma 4.1.2. *(Trichotomy of integers). Let x be an integer. Then exactly Then exactly one of the following statements is true:*

1. *x is 0.*
2. *x is equal to a positive natural numebr n,*

3. x is the negation $-n$ of a positive natural number n.

Proof. *We first show that at least one of the above is true. By definition $x = a \text{—} b$ for $a, b \in \mathbb{N}$. By the trichotomy of natural numbers, we have $a = b$, $a < b$ or $a > b$. We consider each case in turn:*

- *$a = b$. In this case we have have $x = a \text{—} a$. From the definition of equality $a \text{—} a = 0 \text{—} 0$. From identifying natural numbers with integers $n \text{—} 0$ we have $x = 0$.*
- *$a > b$. In this case we have some $c \in \mathbb{N}$ such that $a = b + c$. Thus we have $x = (b + c) \text{—} b$. Again by equality of integers we have $x = c \text{—} 0 = c$, which is a positive natural number.*
- *$a < b$. In this case we have some $c \in \mathbb{N}$ such that $b = a + c$. Thus we have $x = a \text{—} (a + c)$. This gives us $x = 0 \text{—} c = -(c \text{—} 0) = -c$ by the definition of negation.*

We leave showing that only 1 can happen at a time as an exercise to the reader (basically just look at every possible pair happening and show that they are impossible). ■

If n is a positive natural number, we call $-n$ a negative integer. We now summarize the algebraic properties of the integers.

Proposition 4.1.1. *(Laws of algebra for integers). Let x, y, z be integers. Then we have:*

$$\begin{aligned}
x + y = y + x(x + y) + z &\qquad = x + (y + z) \\
x + 0 = 0 + x &= x \\
x + (-x) = (-x) + x &= 0 \\
xy &= yx \\
(xy)z &= x(yz) \\
x1 = 1x &= x \\
x(y + z) &= xy + xz \\
(y + z)x &= yx + zx
\end{aligned}$$

Proof. *The easiest way to prove this to to define:*

$$x = (a \text{—} b) \quad y = (c \text{—} d) \quad z = (e \text{—} f)$$

then write out each identity in terms of the above, expand using the algebra of *integers we already have and then the algebra of natural numbers. For example:*

$$\begin{aligned}
(xy)z &= (a—b)(c—d)(e—f) \\
&= ((ac+bd)—(ad+bc))(e—f) \\
&= ((ace+bde+adf+bcf)—(acf+bdf+ade+bce)) \\
x(yz) &= (a—b)((ce+df)—(cf+de)) \\
&= (ace+bde+adf+bcf)—(acf+bdf+ade+bce)
\end{aligned}$$

We leave proving the rest as an exercise to the reader. ■

Remark 4.1.2. *The above nine identities assert that the intergers form a commutative ring. This means the set integers forms a commutative additive group, with an additional multiplication operation that is associative, commutative, distributive over addition. If $xy \neq yx$ then it would just be a ring. If the set had multiplicative inverses, then it would also form a multiplicative commutative group, and thus would be a field. The rational $\mathbb{Q}$ will be the first field that we encounter.*

Definition 4.1.4. *(Subtraction). We define the operation of subtraction to be:*

$$x - y := x + (-y)$$

We do not need to verify the substiution axiom for this opertaion, since it is defined in terms of two operations on integers (addition and negation) that already obey this axiom.

Now lets get rid of the pesky — symbol! Let $a, b \in \mathbb{N}$. Then we have:

$$a - b = a + (-b) = (a—0) + (0—b) = (a—b)$$

We now generalize a couple of propositions that we had for the natural numbers to the integers.

Proposition 4.1.2. *(Integers have no 0 divisors). For $a, b \in \mathbb{Z}$ such that $ab = 0$, then either $a = 0$ or $b = 0$.*

Proposition 4.1.3. *(Cancellation law of integers). Let $a, b, c \in \mathbb{Z}$ such that $ac = bc$ and $c \neq 0$. Then $a = b$.*

We now repeat the definition of order (that is defining $<$ and $>$) verbatum for the integers.

Definition 4.1.5. *(Ordering of the integers). Let n and m be integers. $n \geq m$ iff $n = m + a$ for some natural number a. $n > m$ iff $n \geq m$ and $n \neq m$.*

Lemma 4.1.3. *(Properties of order of integers). We summarize some simple facts. Let $a, b, c \in \mathbb{Z}$, then:*

1. *$a > b \iff a - b$ is a positive natural number.*

2. *(Addition preserves order). $a > b \implies a + c > b + c$.*

3. *(Positive ultiplication preserves order). $a > b \wedge c > 0 \implies ac > bc$.*

4. *(Negative multiplication reverses order). $a > b \wedge c < 0 \implies ac < bc$.*

5. *(Order is transitive). $a > b \wedge b > c \implies a > c$.*

6. *(Order is trichotomous). Exactly one of $a > b$, $a = b$, or $a < b$ is true.*

— Exercises —

Exercise 4.1.4. Show that $(-1) \times a = -a$ for any integer a.

Proof. For $a \in \mathbb{Z}$ we have $a = n—m$ for $n, m \in \mathbb{N}$ and $-1 = 0—1$. We get:

$$\begin{aligned}(-1) \times a &= (0—1)(n—m) \\ &= (0n + 1m)—(0m + 1n) \\ &= (m—n) = -a\end{aligned}$$

from the definition of negation. ■

Exercise 4.1.8. Show the the principle of induction does not apply to the integers. That is exhibit a property $P(x)$ for $x \in \mathbb{Z}$ such that $P(0)$ is true, and $P(x) \implies P(x+1)$, but $P(n)$ is not true $\forall n \in \mathbb{Z}$.

Proof. Consider an integer x that is $m - n$ for natural numbers m, n. Let $P(x) = P(m - n)$ be the property that $m \geq n$. We have $P(0) = P(0 - 0)$ is true as $0 \geq 0$. If $P(x)$ is true then $P(x + 1)$ is true as $x + 1 = (m + 1) - n$ and $m + 1 \geq n$ as $m \geq n$. Note that for negative integers, $P(x)$ is untrue however. ■

4.2 Rational Numbers

We defined the integers with addition, multiplication, subtraction, and order and verified all the algebraic and order-theoretic properties. We now add build the rationals, adding division to our list of operations.

As the integers were constructed by subtracting two natural numbers, the rations are constructed by dividing two integers.

We know what we expect, that $a/b = c/d$ iff $ad = bc$. Just like we did with the integers, we create a new meaningless symbol // which will eventually be replaced with the devision symbol, and make the following definition.

Definition 4.2.1. *(Rational numbers). A rational number is an expression of the form $a//b$, where a and b are integers and $b \neq 0$. Two rational numbers $a//b$ and $c//d$ are equal, $a//b = c//d$, iff $ad = bc$. The set of rational numbers is denoted by $\mathbb{Q}$.*

For full rigour we should show this is a valid definition of equality by showing it is reflexive, symmetric, and transitive, and obeys the axiom of substitution. We will not do this here. Now we need to define addition, multiplication, and division, which follow our intution that $a/b + c/d = (ad + bc)/(bd)$, and $a/b \times c/d = ac/bd$ and $-(a/b) = (-a)/b$.

Definition 4.2.2. *(Addition, multiplication, and negation of rationals). Let $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$. We define:*

$$\begin{aligned}(a//b) + (c//d) &= (ad + bc)//(bd)\\ (a//b) \times (c//d) &= (ac)//(bd)\\ -(a//b) &= (-a)//b\end{aligned}$$

Note that if b and d are non zero, then bd is non zero, so addition and multiplication are closed over the rationals.

Lemma 4.2.1. *Addition, product, and negation are well defined on the integers. This means that if one replaces $a//b$ with $a'//b'$ with $a//b = a'//b'$, then the output of the operations remains the same, and the same is true for $c//d$.*

Remark 4.2.1. *At this point when we are in abstract land, why do we not allow dividing by 0? This is because if we did, then $(a/0) \times (0/1) = (a0/0) = (a/1)$ by the definition of rational number equality (as $a0 = a0$), but it would also equal $(0/0) = 0$, which is a contradiction if $a \neq 0$.*

Remark 4.2.2. *The rational numbers $a//1$ behave identically to the integer a:*

$$\begin{aligned}(a//1) + (b//1) &= (a + b)//1\\ (a//1) \times (b//1) &= (ab)//1\\ -(a//1) &= (-a)//1 a//1 = b//1 \iff a = b\end{aligned}$$

Because of this, we will identify a with $a//1$ for all integers a.

We define the reciprocal operation on the rations, which is analogous to the negation operation on the integers.

Definition 4.2.3. *(Reciprocal) For non-zero $x = a//b$ we define the reciprocal to be $x^{-1} = b//a$. This preserves equaity (axiom of substitution).*

Note that an operation such as "numerator" does note respect the axiom of substitution, so we cannot include it. This means we need to be careful in proofs when we say things like "the nubmerator of x is a" and then use this fact. We also have that the reciprocal of 0 is undefined.

Proposition 4.2.1. *(Laws of algebra on the rationals). Let x, y, z be rational numbers, then we have:*

$$
\begin{aligned}
x + y &= y + x \\
(x + y) + z &= x + (y + z) \\
x + 0 = 0 + x &= x \\
x + (-x) = (-x) + x &= 0 \\
xy &= yx \\
(xy)z &= x(yz) \\
x1 = 1x &= x \\
x(y + z) &= xy + xz \\
(y + z)x &= yx + zx
\end{aligned}
$$

If x is non-zero, then:

$$xx^{-1} = x^{-1}x = 1$$

Proof. *The proof is long and invovled, but similar to proving the algebraic properties of the integers, we simply write $x = a//b$, $y = c//d$, etc. and verify each identity in turn. We leave this as an exercise to the reader.* ■

Remark 4.2.3. *Note that the above algebraic properties match that of the integers exactly except for the the additional of the final idnetity involving reicprocals. This identity, which states the existance of multiplicative inverses for all elements of the set except for the addititive identity 0, makes the rationals a field. This is the first field we have encountered (recall that the integers were a commtuative ring).*

Definition 4.2.4. *(Quotient) The quotient of rational numbers x and y, provided that t is non-zero, is defined as:*

$$x/y = x \times y^{-1}$$

For example $(3//4)/(5//6) = (3//4) \times (6//5) = 18//20 = 9//10$. Using this definition, we can see that $a/b = a//b$ for every integer a and non-zero integer b. This is because:

$$a/b = a \times b^{-1} = (a//1) \times (b//1)^{-1} = a//1 \times 1//b = a//b$$

Thus we can discard // and simply use /.

Definition 4.2.5. *(Subtraction on the rations). We define subtraction on the rationals identically as we did for the integers:*

$$x - y = x + (-y)$$

Definition 4.2.6. *A rational number x is positive iff $x = a/b$ for some positive integers a, b. It is egative iff $x = -y$ for some positive rational number y.*

Lemma 4.2.2. *(Trichotemy of rationals). If x is ration, then it is positive, negative, or 0.*

Lemma 4.2.3. *(Ordering of rationals). If x, y are rational, $x > y$ iff $x - y$ is a positive rational, and $x < y$ is a negative rational. We write $x \geq y$ iff either $x > y$ or $x = y$.*

Proposition 4.2.2. *(Propterites of order on the rationals). We've seen these all before for the integers:*

(a) (Order trichotemy). One of $x > y$, $x = y$, or $x < y$ is true.

(b) (Order is anti-symmetric). $x < y$ iff $y > x$.

(c) (Order is transitive). $x > y$ and $y > z$ implies $x > z$.

(d) (Addition preserves order). $x > y$ implies $x + z > y + z$.

(e) (Positive mult preserves order). $x > y$ and $z > 0$ implies $xz > yz$.

The above properties combined with the field algebraic properties combine to make $\mathbb{Q}$ an *ordered field.*

— Exercises —

Exercise 4.2.6. Show that if $x, y, z \in \mathbb{Q}$ such that $x < y$ and z is negative, then $xz > yz$.

Proof. We have $x < y$ so $x - y$ is positive. We have z is negative so $-z$ is positive. Thus $x - y$ and $-z$ are positive so:

$$\begin{gathered} x < y \\ x(-z) < y(-z) \\ -xz < -yz \\ -xy + xz < -yz + xz \\ 0 < xz - yz \\ yz < xz \\ xz > yz \end{gathered}$$

4.3 Absolute Value and exponentiation

We have defined addition, multiplication, subtraction, and division on the rationals, with the latter two being defined in terms of the more primitve negations $x+(-y)$ and reciprocal $x \times y^{-1}$ operations. We can now define other operations. Here we introduce absolute value and exponentiation.

Definition 4.3.1. *(Absolute value). If x is rational, then the absolute value $|x|$ is defined as:*

$$|x| = \begin{cases} x & \text{if } x \text{ is positive} \\ -x & \text{if } x \text{ is negative} \\ 0 & \text{if } x = 0 \end{cases}$$

Definition 4.3.2. *(Distance). Let $x, y \in \mathbb{Q}$. The quantity $|x-y|$ os ca;;ed the distance between x and y, sometimes denoted $d(x,y)$.*

Proposition 4.3.1. *(Basic properties of absolute value and distance). For $x, y, z \in Q$ we have:*

(a) (Non degenerate). $|x| \geq 0$. Also $|x| = 0$ iff $x = 0$.

(b) (Traingle inequality for abs). $|x+y| \leq |x| + |y|$.

(c) $-y \leq x \leq y$ iff $|x| \leq y$. Thus we have $-|x| \leq x \leq |x|$.

(d) (Multiplicity of abs). $|xy| = |x||y|$. Thus $|-x| = |x|$.

(e) (Non gen of distance). $d(x,y) \geq 0$. Also $d(x,y) = 0$ iff $x = y$.

(f) (Symmetry of distance). $d(x,y) = d(y,x)$.

(g) (Triangle inequality for distance). $d(x,z) \leq d(x,y) + d(y,z)$.

Distance (and thus absolute value) are useful for measuring how close two numebrs are.

Definition 4.3.3. *(ε-closeness). Let $\varepsilon > 0$ be a rational number, and x, y be rational. We say that y is ε-close to x iff $d(x,y) < \varepsilon$.*

Note that this definition is not standard in other textbooks. We use it to build scaffolding and then discard it later, much like // and — for the rationals and integers respectively.

Proposition 4.3.2. *(Properties of ε-closeness). Consider $x, y, z, w \in \mathbb{Q}$. We have:*

(a) If $x = y$, then x is ε-close to y for any $\varepsilon > 0$. Conversely, if x is ε-close to y for all $\varepsilon > 0$, then $x = y$.

(b) *(Symmetric). If x is ε-close to y, then y is ε-close*

(c) *(Sort of transititve). If x and y are ε close and y and z are δ close then x and z are $(\varepsilon + \delta)$ close.*

(d) *Let $\varepsilon, \delta > 0$. If x and y are ε close and z and w are δ close, then $x + z$ and $y + w$ are $(\varepsilon + \delta)$ close. Also $x - z$ and $y - w$ are $(\varepsilon + \delta)$ close.*

(e) *If x and y are ε close, then they are ε' close $\forall \varepsilon' > \varepsilon$.*

(f) *If y and z are both ε close to x, and $y \leq w \leq z$ or $z \leq w \leq y$, then w is ε close to x.*

(g) *If x and y are ε close, and $z \neq 0$, then xz and yz are $\varepsilon|z|$ close.*

(h) *If x and y are ε close, and z and w are δ close, then xy and yw are $(\varepsilon|z| + \delta|y| + \varepsilon\delta)$ close.*

Proof. *We prove only (h). We want to say something about $|yw - xz|$ We will write yw in terms of x and z.*

Let $a = x - y$, thus $y = x + a$ with $|a| \leq \varepsilon$. Let $b = w - z$, thus $w = z + b$ with $|b| \leq \delta$. We have:

$$yw = (x + a)(z + b) = xz + az + xb + ab$$

This gives us:

$$\begin{aligned}
|yw - xz| &= |az + xb + ab| \\
&\leq |az| + |xb| + |ab| && \text{By triangle ineq} \\
&\leq |z||a| + |x||b| + |a||b| && \text{By multiplicity} \\
&\leq \varepsilon|z| + \delta|x| + \varepsilon\delta
\end{aligned}$$

Thus we have yw and xz are $(\varepsilon|z| + \delta|y| + \varepsilon\delta)$ close. ∎

Now we define exponentiation for natural numbers recursively as we did with multiplicaiton.

Definition 4.3.4. *(Exponentiation of a natural number). Let x be a rational number. To raise x to the pwer 0 we define $x^0 := 1$, and thus $0^0 = 1$. Now suppose inductively that we have defined x^n for some $n \in \mathbb{N}$. We define $x^{n+1} := x^n \times x$.*

We now do so for negative integers.

Definition 4.3.5. *(Exponentation to a negative numebr). Let x be a non-zero rational number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.*

We now have x^n defined for any integer n, and it is closed over the rationals.

Proposition 4.3.3. *(Properties of exponentiation). For x, y non-zero rational numbers, we have:*

(a) $x^n x^m = x^{n+m}$. $(x^n)^m = x^{nm}$. $(xy)^n = x^n y^n$.

(b) For $n > 0$, $x^n = 0$ iff $x = 0$.

(c) If $x \geq y > 0$ then $x^n \geq y^n > 0$ if n is positive. and $0 < x^n \leq y^n$ if n is negative.

(d) If $x, y > 0$, $n \neq 0$, and $x^n = y^n$, then $x = y$.

(e) We have $|x^n| = |x|^n$.

— Exercises —

Exercise 4.3.5. Prove that $2^N \geq N$ for all positive integers N.

Proof. We do so by induction. For $N = 1$ we have $2^1 = 2 \geq 1$. Assume truen for $N = n$. Now consider $N = n + 1$, we have:

$$\begin{aligned}
2^{n+1} &= 2^n \times 2 \\
&\geq N \times 2 && \text{By inductive assumption} \\
&= N + N \\
&\geq N + 1 && \text{As } N \geq 1
\end{aligned}$$

4.4 Gaps in the rational numbers

This is a non-rigerous argument, but consider lining up all the rational numbers on a line from y to x (for $y < x$). Inside the rationals we have the integers.

Proposition 4.4.1. *(Interspersing of integers by rationals). Consider $x \in \mathbb{Q}$. There exists an $n \in \mathbb{Z}$ such that $n \leq x < n + 1$. Thus there exists and $N \in \mathbb{N}$ such that $N > x$. Thus there is no such thing as a rational number which is larger than all naturals.*

Remark 4.4.1. *In integer n for which $n \leq x < n + 1$ is sometimes called the integer part, and is $n = \lfloor x \rfloor$.*

Proposition 4.4.2. *(Interspersing of rationals by rationals). For $x, y \in \mathbb{Q}$ such that $x < y$, there exists $z \in \mathbb{Q}$ such that $x < z < y$.*

Proof. Start with $z = (x + y)/2$. Sinze $x < y$ and $1/2 = 1//2$ is positive, we have $x/2 < y/2$. Adding $y/2$ to both sides we get $z < y$. Do same for x_2 and we conclude the proof. ■

Despite the rationals having this denseness to them, there are still “gaps” between rationals. The denseness does ensure these “gaps” are infinitesimally small, but they are still there.

Proposition 4.4.3. *There does not exists a rational x such that $x^2 = 2$.*

Proof. *Assume such an x exists. We can assume it is positive (if it were not, then replace with $-x$ as $(-x)^2 = x^2$. So for $p, q \in \mathbb{N}$ we have:*

$$x = p/q \implies x^2 = p^2/q^2 = 2 \implies p^2 = 2q^2$$

Thus we have that p^2 is even. Thus p is even, as otherwise p^2 would be odd. So we have $p = 2k$. Thus $2q^2 = 4k^2 \implies q^2 = 2k^2$. Thus q is even, and $q = 2l$ for some l. Note that $k < p$ and $l < q$, and all are natural numbers. We can repeat this process infinitely, which contradicts the principle of infinite descent.

We can however get rationals that are arbitrarily close to root 2.

Proposition 4.4.4. *For every rational $\varepsilon > 0$, there exists a non-negtaive rational x such that $x^2 < 2 < (x + \varepsilon)^2$.*

What this means is that we can get as close as we want to $\sqrt{2}$. For example:

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

(here and going forward we use terminating decimals which can simply be written as rationals). From the above, it seems like we can make root 2 by taking the “limit” of a sequence of rational numbers. This is how we will construct the real numebrs in the next section.[2]

— Exercises —

Exercise 4.4.2. A sequence $a_0, a_1, a_2, \ldots$ of numbers (natural, interger, rational, real) is said to be in *infinite descent* if we have $a_n > a_{n+1}$ for all natural numbers n. Prove the *principle of infinite descent*: that it is not possible to have a sequence of natural numbers which is in infinite descent.

Proof. Assume towards contradiction that such a sequence did exist. As all a_i are natural numbers we have $a_i \geq 0$. We use induction to show that $a_i \geq k$ for all k. Assume that $a_i \geq n$. Now consider $n + 1$. Assume towards contradiction there is some $a_i < n + 1$. Then $a_{i+1} < n$, which is a contradiction as all the values are greater than n. Thus we have $a_i \geq n + 1$ for all i. Now pick $k = a_1$, and we have that $a_2 > k = a_1$ and thus the sequence is not in infinite descent, which is a contradiction. ■

[2]There are other ways to make the reals, in particular using “Dedekind cuts”, or using infinite decimal expansions.

5 Real Numbers

Recap of what we have done so far:

1. Defined natural numbers using the Peano axioms, and postulated that such a number system exists. Using the axioms, we recursively defined addition and multiplication and showed they obeyed our concepts of algebra on the naturals.
2. We constructed the integers using the notion of difference between two natural numbers $a—b$.
3. We constructed the rationals using the notion of quotient between two integers $a//b$, but excluded dividing by 0 to keep the laws of algebra consistent.

The rationals are useful, but fail in places like geometry and trigonometry. We must thus replace the rational number line with the real number line. We will also need real number for calculus.

We need more machinary to construct the reals than just aiming to add a new operation (like what we did for the integers and rationals). In particular, we need to define a limit.

The real numbers will be similar to the rationals, but with some new operations, in particular *supremum*, that we then use to define limits. When we give the procedure of constructing the reals using limits of sequences of rationals, this is an example of a broader concept known as *completing* one metric space from another.

5.1 Cauch Sequences

Definition 5.1.1. *(Sequences). $m \in \mathbb{Z}$. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function $f : \{n \in Z : n \geq m\} \to \mathbb{Q}$. I.e., a mapping that assigns each integer n greater than or equal to m a rational number a_n. More informally, it is simply a collection of rationals.*

We want to define reals as limits of sequences of rationals. To do this we need to distinguish what sequences converge and what do not.

Definition 5.1.2. *(ε-steadiness). Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be ε-steady iff each pair a_j, a_k of elements is ϵ-close for every natual number j, k.*

In other words, the sequence $a_0, a_1, a_2, \ldots$ is ε-steady iff $d(a_j, a_k) < \varepsilon$ for all $j, k \in \mathbb{N}$.

Remark 5.1.1. *The above definition is not standard in the literature and is just used for scaffolding in this section. The same for the below.*

Definition 5.1.3. *(Eventual ε-steadiness). A sequence $(a_n)_{n=0}^{\infty}$ is said to be eventually ε-steady iff there exists an $N \geq 0$ such that $a_N, a_{N+1}, a_{N+2}, \ldots$ is ε-steady.*

In other words, if $\exists N \geq 0$ such that $d(a_j, a_k) < \varepsilon$ for all $j, k \geq N$.

Example 5.1.1. *The sequence $a_n = 1/n$ is not 0.1-steady but is 0.1-eventual steady. The sequence $10, 0, 0, 0, \ldots$ is not ε-steady for any $\varepsilon < 10$, but is ε-eventual steady for any $\varepsilon > 0$ (not this is strict as the definition of ε closeness uses a strict less than inequality, so nothign can be 0-close.*

We now define the notion of what it means for a sequnce to "want" to converge (this doesn't mean it will).

Definition 5.1.4. *(Cauchy sequence). A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is a Cauchy sequence iff $\forall \varepsilon > 0$, the sequnce is eventually ε-steady.*

That is the sequnce is Cauchy iff $\forall \varepsilon > 0$, there exists and $N \geq 0$ such that $d(a_j, a_k) < \varepsilon$ for all $j, k \geq N$.

Remark 5.1.2. *So far we have ε is rational as we have not defined the reals. Once we have the reals, we will change the definition to allow ε to be real, and show that:*

$$\text{Sequnce is } \varepsilon\text{-eventuallly steady } \forall \varepsilon > 0,\ \varepsilon \in \mathbb{Q} \iff \\ \text{Sequnce is } \varepsilon\text{-eventuallly steady } \forall \varepsilon > 0,\ \varepsilon \in \mathbb{R}.$$

Proposition 5.1.1. *The sequnce $a_n := 1/n$ is Caucy.*

Proof. *Let $\varepsilon > 0$. We want to show that the sequence is eventually ε-steady. We have $\varepsilon = a/b$ for positive a, b. There must exists some $N \in \mathbb{N}$ such that $\varepsilon > 1/N$. For all $n, m > 2N$ we have $1/n, 1/m < /2N$. Thus:*

$$\begin{aligned} |1/n - 1/m| &\leq |1/n| + |1/m| \\ &< 1/2N + 1/2N = && = 1/N \\ &< \varepsilon \end{aligned}$$

Which concludes the proof. ∎

Note that we know such an N exists because for every rational sits between two consequtive reals.

Definition 5.1.5. *(Bounded sequence). Let $M \geq 0$ be rational. A sequence $a_1, \ldots, a_n$ is bounded by M iff $|a_i| \leq M$ for all $1 \leq i \leq n$.*

An infinite sequence is bounded iff $|a_i| \le M$ for all i.

A sequence is bounded iff there exists a rational $M \ge 0$ such that the sequence is bounded by M.

Lemma 5.1.1. *Every finite sequence $a_1, \ldots, a_n$ is bounded.*

Proof (brief). Induct on n. If $n = 1$ then bounded by $|a_1|$. Assume true for n. Consider $n+1$. $a_1, \ldots, a_{n+1}$ is bounded by $M + |a_{n+1}|$. ■

— Exercises —

Exercise 5.1.1. Show that every Cauchy sequence $(a_n)_{n=0}^{\infty}$ is bounded.

Proof. As it is Cauchy, there exists an $N \in \mathbb{N}$ such that for all $j, k \ge N$ we have $|a_j - a_k| < 1$. Now consider $a_n, \ldots, a_{N-1}$. This is finite so by the above lemma is bounded. Whats more $a_N, a_{N+1}, \ldots$ is bounded as it is 1-steady. Thus the entire sequence is bounded. ■

5.2 Eequivalence Cauchy Sequences

Consider two equences:

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

and

$$1.5, 1.42, 1.415, 1.4143, 1.41422, \ldots$$

Inforammly it seems these two sequences are both converging to $\sqrt{2}$. We want to define reals as the limits of Caucy sequences, so we need to know when two sequences give the same limit, but that is circular because a limit will be a real number, which we have not introduced yet.

So we need some other definition to say "these two sequences are the same" or "these two sequences are similar."

Definition 5.2.1. *(ε-close sequences). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$. We say that are ϵ-close iff a_n and b_n are ε-close for all $n \in \mathbb{N}$.*

In other words, $|a_n - b_n| < \varepsilon$ for all n.

Example 5.2.1. *The two sequences*

$$1, -1, 1, -1, \ldots$$

and

$$1.1, -1.1, 1.1, -1.1, \ldots$$

are 0.1-close to each other. Neither however are 0.1-steady.

Definition 5.2.2. *(Eventual ε-closeness). Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$. We say that are eventually ϵ-close iff there exists an $N \in \mathbb{N}$ such that a_n and b_n are ε-close for all $n \geq N$.*

Example 5.2.2. *Consider:*

$$1.1, 1.01, 1.001, 1.0001, \ldots$$

and

$$0.9, 0.99, 0.999, 0.9999, \ldots.$$

The are not 0.1-close, but are 0.1-eventually close.

Definition 5.2.3. *(Equivalent sequences). Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are said to be equivalent iff for all $\varepsilon > 0$ they are eventually ε-close.*

In other words, $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are equivalent iff:

$$\forall \mathbb{Q} \ni \varepsilon > 0, \exists \mathbb{N} \ni N \geq 0 \quad s.t. \quad |a_n - b_n| \leq \varepsilon \quad \forall n \geq N$$

5.3 The construction of the real numbers

We introduce a new formal symbol LIM, similar to the formal notions of — and //. This will eventually match the familiar operation of lim, at which point the formal LIM can be discarded.

Definition 5.3.1. *(Real numbers). A real number is defined to be an object of the form $\mathrm{LIM}_{n\to\infty} a_n$, where $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence of rational numbers.*

Two real numbers $\mathrm{LIM}_{n\to\infty} a_n$ and $\mathrm{LIM}_{n\to\infty} b_n$ are equal iff the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent Cauchy sequences.

The set of all real numbers is denotes by $\mathbb{R}$.

Example 5.3.1. *(Informal). Consider $a_1, \ldots$ to be:*

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

and $b_1, \ldots$ to be:

$$1.5, 1.42, 1.415, 1.4143, 1.41422, \ldots$$

then $\mathrm{LIM}_{n\to\infty} a_n$ is a real number and $\mathrm{LIM}_{n\to\infty} b_n$ is a real number beacuse (a_n) and (b_n) are equivalent Cauchy sequences.

We refer to $\mathrm{LIM}_{n\to\infty} a_n$ as the *formal limit* of $(a_n)_{n=0}^{\infty}$. We check that our definition is valid by ensuring equality obeys the three requiers properties.

Proposition 5.3.1. *(Formal limits are well-defined). The definition of equality of real numbers is reflexive, symmetric, and transitive.*

When we define other ooerations on the reals we need to make sure the law of subtitution. Now we want to define the usual arithmetic operations on the reals.

Definition 5.3.2. *(Addition of reals). $x = \mathrm{LIM}_{\mathrm{n}\to\infty} a_n$ and $y = \mathrm{LIM}_{\mathrm{n}\to\infty} b_n$ are real numbers. We define the sum $x + y$ to be:*

$$x + y := \mathrm{LIM}_{\mathrm{n}\to\infty}(a_n + b_n)$$

Example 5.3.2. *We have:*

$$(\mathrm{LIM}_{\mathrm{n}\to\infty} 1 + 1/n) + (\mathrm{LIM}_{\mathrm{n}\to\infty} 2 + 3/n) = \mathrm{LIM}_{\mathrm{n}\to\infty}(3 + 4/n)$$

We now need to check that this definition is valid. For starters we need to check that the result is still a real number, that is that the sequence $(a_n + b_n)$ is Cauchy.

Lemma 5.3.1. *(Sum of Cauchy sequences). Let $x = \mathrm{LIM}_{\mathrm{n}\to\infty} a_n$ and $y = \mathrm{LIM}_{\mathrm{n}\to\infty} b_n$ be real numbers. Then $x + y$ is also a real number, meaning that $(a_n + b_n)$ is Cauchy.*

Proof. *We need to show that for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|a_i - a_j| < \varepsilon$ for all $i, j \geq N$. We have:*

Let $\varepsilon > 0$. Let $\delta = \varepsilon/2$. We know that there is an N such that $|a_i - a_j| < \delta$ for all $i, j \geq N$ and M such that $|b_i - b_j| < \delta$ for all $i, j \geq M$. Let $K = \max(N, M)$. We have:

$$\begin{aligned} |a_i + b_i - a_j - b_j| &\leq |a_i - a_j| + |b_i - b_j| \\ &< 2\delta \\ &= \varepsilon \end{aligned}$$

for all $i, j \geq K$ which concludes the proof. ∎

We also need to check the axiom of substitution. If we replace x with another real equal to it, this should not change the sum $x + y$.

Lemma 5.3.2. *(Sums of equivalent Cauchy sequnces are equivalent). Let $x = \mathrm{LIM}_{\mathrm{n}\to\infty} a_n$ and $y = \mathrm{LIM}_{\mathrm{n}\to\infty} b_n$ be real numbers, and $x' = \mathrm{LIM}_{\mathrm{n}\to\infty} a'_n$ such that $x = x'$. We have $x + y = x' + y$.*

Proof. *We want to show:*

$$\mathrm{LIM}_{\mathrm{n}\to\infty}(a_n + b_n) = \mathrm{LIM}_{\mathrm{n}\to\infty}(a'_n + b_n)$$

Thus we need to show that for all $\varepsilon > 0$ there exists an N such that $|(a_n + b_n) - (a'_n + b_n)| < \varepsilon$. We simplify this as:

$$|a_n - a'_n|$$

We now that $x = x'$ and thus a_n and a'_n are eventually ε-close, which concludes the proof. ■

Remark 5.3.1. *This verifies the axiom of substitution for x on $x + y$. Proving for y is simply as we can quickly show that addition of reals is commutative.*

Definition 5.3.3. *(Multiplication of reals). Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. We define the product as:*

$$xy := \mathrm{LIM}_{n\to\infty}(a_n b_n)$$

Proposition 5.3.2. *(Multiplication is well defined). Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$ be real numbers. Let $x' = \mathrm{LIM}_{n\to\infty} a'_n$ and $x' = x$. Then we have that xy is a real number and $x'y = xy$.*

Proof. We leave the proof as an exercise to the reader. You need to show that the product of Caucy sequences is Cauchy, that is are eventually ε-steady. That is we want to show:

$$|a_i b_i - a_j b_j| < \varepsilon$$

For axiom of substitution, we want to show that $|a_i b_i - a'_i b_i| < \varepsilon$, that is the sequences are ε close.

We now embed the rationals in the reals, by equating every rational q with the real number $\mathrm{LIM}_{n\to\infty} q$, that is the sequence $q, q, q, \ldots$. This embedding is consistent with our definition of addition and multiplication for the rationals.

This means when you want to add or multiply two rationals, you can think of them as rationals or limits of sequences (i.e. reals). It is also consistent with our notion of equality. We use this to define negation.

Definition 5.3.4. *(Negation of a real). We define negation of a real as:*

$$\begin{aligned}
-x &:= (-1) \times x \\
&= \mathrm{LIM}_{n\to\infty}(-1) \times a_n \\
&= \mathrm{LIM}_{n\to\infty} a_n
\end{aligned}$$

Since -1 is rational and so also real, we already have from above that this product forms a real and obeys the axiom of substitution. Note it is also consistent with out negation of rational numbers.

Definition 5.3.5. *(Subtraction of reals). Let x, y be real, we have:*

$$x - y := x + (-y)$$

We can now verify that the real numbers obey all the laws of algebra (except those concerning division, as we have not yet defined division.

Example 5.3.3. *Say we want to prove $x(y+z) = xy + xz$ for the reals (multiplication dsitributes over addition). We have:*

$$\begin{aligned} xy + xz = \mathrm{LIM}_{\mathrm{n}\to\infty}\, a_n b_n + \mathrm{LIM}_{\mathrm{n}\to\infty}\, a_n c_n \\ &= \mathrm{LIM}_{\mathrm{n}\to\infty}(a_n b_n + a_n c_n) \\ &= \mathrm{LIM}_{\mathrm{n}\to\infty}\, a_n(b_n + c_n) \\ &= x(y+z) \end{aligned}$$

which concludes the proof. Note what we did here. We moved into doing algebra on the rational elements of the sequence in the formal limit, in which we already know the algebra laws hold.

The other algebra laws can be proven similarly.

Finally we need to define reciprocation. This is a little trickier than before though. We may be tempted to have:

$$(\mathrm{LIM}_{\mathrm{n}\to\infty}\, a_n)^{-1} = \mathrm{LIM}_{\mathrm{n}\to\infty}\, a_n^{-1}$$

We have two problems:

1. The section $0.1, 0.01, 0.001, \dots$ becomes $10, 100, 1000, \dots$ which is no longer Cauchy (not even bounded).

   Of course the problem here is that the sequence $0.1, 0.01, 0.001, \dots$ is equal to $0, 0, 0, \dots$ and thus by our embedding of the rels is equal to 0, and we cannot divide by 0.

   So we want to restrict ourselves to non-zero reals.

2. But we still have a problem. A non-zero real could have 0 as part of its Cauchy sequence, and so be undefined under reciprocation. For example

   $$0, 0.9, 0.99, 0.999, \dots$$

   is a real number, and infact the rational 1. But it contains 0 in this form of its Cauchy sequnece.

   To get around this problem we need to keep sequences away from 0.

Definition 5.3.6. *(Sequences bounded away from 0). A sequence $(a_n)_{n=1}^{\infty}$ is said to be bounded away iff there exists $\mathbb{Q} \ni c > 0$ such that $|a_n| \geq c$ for all $n \geq 1$.*

We now want to show that every non-zero real number is the formal limit of a Cauchy sequence that is bounded from 0. That is it has some representation that does not include 0 in the sequence.

Lemma 5.3.3. *Let x be a non-zero real number. Then $x = \mathrm{LIM}_{\mathrm{n}\to\infty} a_n$ for some Cauchy sequence $(a_n)_{n=0}^{\infty}$ that is bounded away from 0.*

Proof. *We sketch the proof here. x is real so $x = \mathrm{LIM}_{\mathrm{n}\to\infty} b_n$ for some Cauchy sequence $(b_n)_{n=0}^{\infty}$. We don't know if b is bounded from 0, but we do know that $x \neq 0$. Thus $(b_n)_{n=0}^{\infty} \neq (0)_{n=0}^{\infty}$. Thus there exists a $\varepsilon > 0$ such that (b_n) is not eventually ε close to 0.*

Fix this ε. (b) is Cauchy so it is eventually ε-steady and also eventually $\varepsilon/2$-steady. Thus there is an $N \geq 1$ such that:

$$|b_i - b_j| < \varepsilon/2 \quad \forall i, j \geq N$$

On the other hand, we cannot have:

$$|b_i - 0| = |b_i| < \varepsilon, \quad \forall i \geq N$$

as this would imply that (b) is eventually ε-close to 0. Thus we must have:

$$\exists n_0 \geq N, \quad |b_{n_0}| \geq \varepsilon$$

Now due to being ε steady above N, we have:

$$|b_{n_0} - b_n| < \varepsilon/2 \quad \forall n \geq N$$

We leave the rest of the proof as an exercise to the reader.

Now we have that if x is a non-zero real number, then x is the formal limit of a Caucy sequence that is bounded away from 0. Now we have this, we can take its reciprocal.

Lemma 5.3.4. *Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence which is bounded away form zero. Then the sequence $(a_n^{-1})_{n=1}^{\infty}$ is also Cauchy.*

Proof. *We have that $\exists \mathbb{Q} \ni c > 0$ such that $|a_n| \geq c, \forall n \geq 1$. Now we must show that (a_n^{-1}) is eventually ε steady.*

Take any $\varepsilon > 0$. We must find an $N \geq 1$ such that:

$$\forall m, n \geq N \quad |a_n^{-1} - a_m^{-1}| \leq \varepsilon.$$

But we have:

$$\begin{aligned} |a_n^{-1} - a_m^{-1}| &= \left| \frac{a_m - a_n}{a_m a_n} \right| \\ &\leq \frac{|a_m - a_n|}{c^2} \end{aligned}$$

as we have that $a_m, a_n \geq c$. Note that we can find an N for which the $|a_m - a_n| \leq c^2\varepsilon$ as (a_n) is Cauchy, and so we are done. ■

From this we can define reciprocation on the reals.

Definition 5.3.7. *(Reciprocation on the real numbers). Let x be a non-zero real number. Let $(a_n)_{n=1}^{\infty}$ be a Cuchy sequence bounded away from 0 such that $x = \mathrm{LIM}_{n\to\infty} a_n$ (which we know exists from the previous lemma). We define the reciprocal x^{-1} as:*

$$x^{-1} := \mathrm{LIM}_{n\to\infty} a_n^{-1}$$

Again from the previous lemma we know that x^{-1} is a real number.

We need to check one more thing. What if x is the limit of two cauchy sequences (a_n) and (b_n) such that the reciprocals are difference. If so then the axiom of substitution would be broken by this operation. We prove this in not the case in the following Lemma.

Lemma 5.3.5. *(Reciprocation is well defined). Let (a_n) and (b_n) be two Cauchy sequences bounded away from 0 such that:*

$$\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$$

then we have:

$$\mathrm{LIM}_{n\to\infty} a_n^{-1} = \mathrm{LIM}_{n\to\infty} b_n^{-1}$$

Proof Consider the product:

$$\begin{aligned} P &:= (\mathrm{LIM}_{n\to\infty} a_n^{-1}) \times (\mathrm{LIM}_{n\to\infty} a_n) \times (\mathrm{LIM}_{n\to\infty} b_n^{-1}) \\ &= \mathrm{LIM}_{n\to\infty} a_n^{-1} a_n b_n^{-1} \\ &= \mathrm{LIM}_{n\to\infty} b_n^{-1} \end{aligned}$$

But as we have $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ we can write P differently by substituting in the middle term:

$$\begin{aligned} P &:= (\mathrm{LIM}_{n\to\infty} a_n^{-1}) \times (\mathrm{LIM}_{n\to\infty} b_n) \times (\mathrm{LIM}_{n\to\infty} b_n^{-1}) \\ &= \mathrm{LIM}_{n\to\infty} a_n^{-1} b_n b_n^{-1} \\ &= \mathrm{LIM}_{n\to\infty} a_n^{-1} \end{aligned}$$

And thus we have the reciprocals are equal as required. ■

Note from the definition that is is clear $z^{-1}x = 1$ as this will just be the formal limit of the sequence of all 1s which we identify with the rational number $1 \in \mathbb{Q}$. So we have all of the field axioms apply to the reals also.

Also note that the reciporcal on reals is consistent with reciprocal on rationas (think about for $q \in \mathbb{Q}$ the reciprocal of the formal limit of the sequence of all q).

Definition 5.3.8. *(Division on the reals). For $x, y \in \mathbb{R}$ we have:*

$$x/y := x \times y^{-1}$$

Hurray! We now have all of the arithmetic operations on the reals and we can do algebra!

— Exercises —

Exercise 5.3.3. Let $a, b \in \mathbb{Q}$. Show if $a = b$ iff $\mathrm{LIM}_{\mathrm{n}\to\infty}\, a = \mathrm{LIM}_{\mathrm{n}\to\infty}\, b$. This allows us to embed the rationals inside the reals in a well-defined manner.

Proof (informal). We proof the right direction first. If $a = b$ then we have:

$$\mathrm{LIM}_{\mathrm{n}\to\infty}\, a = (a, a, a, \dots) = (b, b, b \dots) = \mathrm{LIM}_{\mathrm{n}\to\infty}\, b$$

(Technically to be most formal here we want to say that the two sequences are eventually ε close but this is trivial).

Now the left direction. Assume that:

$$\mathrm{LIM}_{\mathrm{n}\to\infty}\, a = \mathrm{LIM}_{\mathrm{n}\to\infty}\, b$$

Then for all $\varepsilon > 0$ we have $|a - b| < \varepsilon$. This is only possible if $a = b$ (to prove this formally do a quick proof by contradiction). ■.

5.4 Ordering of the reals

We know that all rationals are positive negative or zero. We now do the same for the reals. To define positive and negative we will again have to rely on sequences bounded away from 0.

Definition 5.4.1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of rationals. We say that this sequence is positively bounded away from zero iff we have a positive rational $c > 0$ such that $a_n \geq c$ for all $n \geq 1$. That is the seqeunce is entirely positive.*

The sequence is negatively bounded away form zero iff we have a negative rational $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$. That is the sequence is entirely negative.

Note that any sequence that is positively or negatively bounded away from 0 is bounded away from 0 trivially.

Definition 5.4.2. *A real number x is positive iff it can be written as the limit of a Cauchy sequence that is postively bounded away from zero. It is negative iff it can be written as the limit of a Cauchy seqeunce that is negatively bounded away from zero.*

Proposition 5.4.1. *For all reals x, one of the following is true:*

a x is positive.

b x is negative.

c x is zero.

Additionally, x is negative ii $-x$ is positive. If x and y are positive, then so is $x + y$ and xy.

Note that if q is positive rational, then $\mathrm{LIM}_{n\to\infty} q$ is positively bounded from 0 and so is positive. So this definition preserves our previous notion of positive for the rationals.

Definition 5.4.3. *(Absolute value) Let x be real. We define $|x|$ equal to x if x is positive, abd $-x$ if x is negative, and 0 if x is zero.*

Definition 5.4.4. *(Ordering of the real numbers). Let x and y be real numbers. We say that $x > y$ iff $x - y$ is positive, and $x < y$ iff $x - y$ is negative. We also have $x \geq y$ iff $x > y$ or $x = y$, and analogously $x \leq y$.*

Order on the reals is consistent with the order on the rationals. We inherit all the normal algebraic properties over order, for example that multiplying by a negative number reverses the inequality.

Proposition 5.4.2. *If x is positive then x^{-1} is positive. If y is positive and $x > y$ then $x^{-1} < y^{-1}$.*

Proposition 5.4.3. *(The non-negative reals are closed). Let $a_1, a_2, \ldots$ be a Cuachy sequence of non-negative rational numbers. Then $\mathrm{LIM}_{n\to\infty} a_n$ is a non-negative real number.*

Proof (Informal). Assume towards contradiction that this is false. Then the limit is a negative real numberb So there is $\mathrm{LIM}_{n\to\infty} b_n = x$ with (b_n) negatively bounded away from 0, so we have $b_n < -c$ for all b_n. We also have $a_n \geq 0$ from the hypothesis, so $|a_n - b_n| \geq c/2$ so they will never for ε close for all ε, so the sequences cannot be equal, which is a contradiction. ■

Corollary 5.4.1. *Let (a_n) and $(b_n$ be Cauchy such that $a_n \geq b_n$ for all $n \geq 1$. Then:*

$$\mathrm{LIM}_{n\to\infty} a_n \geq \mathrm{LIM}_{n\to\infty} b_n$$

Proof. Apply the previous proposition to $a_n - b_n$. Basically you have all a_n and b_n are positive, thus we have that $\mathrm{LIM}_{n\to\infty} a_n - b_n$ is non-negative, meaning that $\mathrm{LIM}_{n\to\infty} a_n - \mathrm{LIM}_{n\to\infty} b_n$ is non-negative, and thus by the definition of $\geq$ on the reals we have:

$$\mathrm{LIM}_{n\to\infty} a_n \geq \mathrm{LIM}_{n\to\infty} b_n$$

■

Now we define distance $d(x, y) = |x = y|$ as we did for the rationals.

Proposition 5.4.4. *(Bounding of reals by rationals). Let x be a positive real. Then there exists anpositive rational $q \in \mathbb{Q}$ such that $q \leq x$ and a positive integer $N \in \mathbb{Z}$ such that $x \leq N$*

Corollary 5.4.2. *(Archimedean property). Let x and ε be any positive real numbers. Then there exists a positive integer M such that $M\varepsilon > x$.*

What this says is that no matter how large x is and how small ε, then if we keep adding ε to itself we can eventually get bigger than x.

Proposition 5.4.5. *Given any two real numbers $x < y$ we can find a rational number q such that $x < q < y$.*

We have finished constructing the reals now. We have algebraic operations, order, and have embedded the rationals in them. We have not shown that they are more usfeul than the rationals however. Now we show that they are more useful, because we can do things like take square roots (which remember we could not do with the ratioanl 2).

Remark 5.4.1. *We have not yet adressed the fact that real numbers can be expressed using the decimal system. For instance the formal limit:*

$$1.4, 1.41, 1.414$$

is easiest written as 1.41421*.*

So far we have really just used 0*, and the increment operation to make natural numbers, then used these to make integers, the rationals then reals.*

The basic reason we have not used decimals is because it is not essential to mathematics.

It is useful for computation. See Section 5.7 for a treatment of the decimals.

5.5 The lest upper bound property

We now give one of the most basica advantages of the real numbers over the rationals, the notion of a least upper bound $\sup(E)$ for $E \subset \mathbb{R}$.

Definition 5.5.1. *(Upper bound). Let $E \subset \mathbb{R}$ and let $M \in \mathbb{R}$. We say that M is an upper bound for E iff $x \leq M$ for all $x \in E$.*

Example 5.5.1. $\mathbb{R}^+$ *does not have an upper bound. For $\emptyset$, every M is an upper bound (vacuously true).*

Definition 5.5.2. *(Least upper bound). Consider $E \subset \mathbb{R}$ again, and M real. We say M is a least upper bound of E iff:*

- *M is an upper bound for E.*
- *Any other upper bound M' of E has the property $M' \geq M$.*

Example 5.5.2. $[0, 1]$ *has a least upper bound* 1 *and* $\emptyset$ *has no least upper bound.*

Proposition 5.5.1. *(Uniqueness of the least upper bound). Let $E \subset \mathbb{R}$. Then E can have at most one least upper bound.*

Proof. *Assume towards contradiction untrue. So we have $M_1 \neq M_2$ both real that are least upper bounds. By definition of least upper bound we have:*

$$M_1 \geq M_2 \land M_2 \geq M_1 \implies M_1 = M_2$$

which is a contradiction.

Now comes an **important property of the reals**.

Theorem 5.5.1. *(existance of least upper bound). Let $E \subset \mathbb{R}$ be non-empty. If E has an upper bound, then it must have exactly one least upper bound.*

Proof. *This is a long proof. Consider E. We know it is bounded, so by Proposition 5.5.1 there is at most 1 least upper bound. Thus we have to show that there is at least 1 least upper bound and we are done.*

Let $x_0 \in E$ be arbitrary (we can select an item as E is non-empty).

Let $n \geq 1$. By the Archimedean property (Corollary 5.4.2), we know there exists some integer K such that $K/n \geq M$, and so K/n is an upper bound. Again by Archimedian property, there exists integer L such that $L/n < x_0$. As $x_0 \in E$ we have that L/n is not an upper bound. So:

$$K/n \geq L/n \implies K \geq L$$

We can now find an integer m_n such that $L < m_n \leq K$ with the property that m_n/n is an upper bound for E, but $(m_n - 1)/n$ is not (left to reader). That m_n is unique (left to reader). We subscript by n to show that m depends on n. This gives us a well-defined (and unique) seequence $m_1, m_2, \ldots$ of integers such that all m_n/n are upper bounds and $(m_n - 1)/n$ are not.

Now let $N \geq 1$ be a positive integer, and let $n, n' \geq N$ be integers larger than or equal to N. Since m_n/n is an upper bound for E and $(m_{n'} - 1)/n'$ is not, we must have $m_n/n \geq m_{n'}/n'$. From this we have:

$$\frac{m_n}{m} - \frac{m_{n'}}{n'} > -\frac{1}{n'} \geq -\frac{1}{N}$$

Similarly, we have $m_{n'}/n'$ is an upper bound and $(m_n - 1)/n$ is not giving us:

$$\frac{m_n}{m} - \frac{m_{n'}}{n'} \leq \frac{1}{n} \leq \frac{1}{N}$$

Putting these two together we have:

$$\left|\frac{m_n}{n} - \frac{m_{n'}}{n'}\right| \leq \frac{1}{N} \quad \forall n, n' \geq N \geq 1$$

Thus we have that (m_n/n) is Cauchy! Since m_n/n are rational, we can define the real number formed by taking the limit of this sequence:

$$S := \mathrm{LIM_{n\to\infty}}\,\frac{m_n}{n}$$

We also have:

$$\begin{aligned}
\mathrm{LIM_{n\to\infty}}\,\frac{m_n - 1}{n} &= \mathrm{LIM_{n\to\infty}}\,\frac{m_n}{1} - \mathrm{LIM_{n\to\infty}}\,\frac{1}{n} \\
&= S - 0 \qquad\qquad \text{Easy to show} \mathrm{LIM_{n\to\infty}}\,1/n = 0 \\
&= S
\end{aligned}$$

Now we just need to show that S is the least upper bound for E. Consider any $x \in E$. We know that m_n/n is an upper bound so we have:

$$x \leq \frac{m_n}{n} \quad \forall n \geq 1$$

From this we can conclude that:

$$x \leq \mathrm{LIM_{n\to\infty}}\, m_n/n = S$$

Thus S is indeed an upeer bound for E.

Now we show it is a least upper bound. Suppose that y is an upper bound for E. As $(m_n - 1)/n$ is no an upper bound we have:

$$y \geq (m_n - 1)/n \quad \forall n \geq 1$$

From this we can get

$$y \geq \mathrm{LIM_{n\to\infty}}(m_n - 1)/n = S$$

So we have that S is less than or equal to any upper bound of E, and thus S is the least upper bound. ■

Definition 5.5.3. *(Supremum). Let $E \subset \mathbb{R}$ be non-empty and have some upper bound. We define $\sup(E)$ to be the least upper bound of E, which is well defined by Theorem 5.5.1.*

We introduce two new suymbols. If E is non-empty and has no upper bound, we set $\sup(E) = +\infty$. If E is empty, we set $\sup(E) = -\infty$. (For now these symbols are meaningless. Later we add them to the reals to form the extended real numbers).

Proposition 5.5.2. *There exists a real number x such that $x^2 = 2$. Thus we have that certain numbers are real but not rational.*

Proof. Let:

$$E = \{y \in \mathbb{R} : y \geq 0 \wedge y^2 < 2\}$$

E has an upper bound of 2 and E is non-empty. By the least upper bound property, we have a real number:

$$x = \sup(E)$$

which is the least upper bound of E. Note that x is positive trivially. We now show that $x^2 = 2$.

Assume towards contradiction that this was false. We have two cases:

- Case 1, $x^2 > 2$. Then we have:

$$\begin{aligned}(x-\varepsilon)^2 &= x^2 - 2x\varepsilon + \varepsilon^2 \\ &\geq x^2 - 2x\varepsilon \\ &\geq x^2 - 4\varepsilon\end{aligned}$$

  Now we have $x \leq 2$ and $\varepsilon^2 \geq 0$. Since $x^2 > 2$ from the case, we must have there is some ε such that:

$$x^2 - 4\varepsilon > 2 \implies (x-\varepsilon)^2 > 2$$

  But this gives us:

$$\begin{aligned}&\forall y \in E, y^2 < 2 < (x-\varepsilon)^2 \\ &\implies y < x - \varepsilon\end{aligned}$$

  Thus $x - \varepsilon$ is an upper bound so x is not a least upper bound.

- Case 2, $x^2 < 2$. Then we have:

$$\begin{aligned}(x+\varepsilon)^2 &= x^2 + 2x\varepsilon + \varepsilon^2 \\ &\leq x^2 + 4\varepsilon + \varepsilon \\ &= x^2 + 5\varepsilon\end{aligned}$$

  as $x \leq 2$ and $\varepsilon^2 \leq \varepsilon$. As $x^2 < 2$ we can choose ε such that:

$$x^2 + 5\varepsilon < 2 \implies (x+\varepsilon)^2 < 2$$

  But this means that $x + \varepsilon \in E$, but this contradicts that x is an upper bound of E.

In the next section we will use the concept of least upper bounds to develop the thoery of limits. Note as well as least upper bound, there is also a greatest lower bound, which is $\inf(E)$, and is known as the infimum.

— Exercises —

Exercise 5.5.1 Let E be a subset of the real numbers $\mathbb{R}$ and suppose that E has a least upper bound M which is a real number. Let:

$$-E := \{-x : x \in E\}$$

Show that $-M = \inf(-E)$.

Proof. Assume towards contradiction that this was not true. Then we have two options:

- Case 1. $-M$ is not a lower bound for $-E$. In this case we have $\exists -x \in X$ such that $-x < -M$ and thus $x > M$. As $x \in E$, M would not be an upper bound for E.
- Case 2. $-M$ is not a greatest lower bound. So there exists some $-N > -M$ such that $-N$ is a lower bound for $-E$. Thus N is an upper bound of E (trivially) such that $N < M$, making M not a least upper bound.

This concludes the proof. ■

5.6 Real exponentials, part I

Previously we have defined x^n for $x \in \mathbb{Q}$ and $n \in \mathbb{N}$, or when x is a non-zero rational and n is an integer. Now that we have all the algebra on the reals we can make a similar definition.

Definition 5.6.1. *(Exponentiation a real with a natural). Let x be a real number. To raise x to the power 0, we define $x^0 := 1$. Now suppose recursively that x^n has been defined for some natural number n, we define $x^{n+1} := x^n \times x$.*

Definition 5.6.2. *(Exponentiation of a real by an integer). Let x be a non-zero real number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.*

Proposition 5.6.1. *All of the properties of exponentiation that we proved for the rationals (for example Proposition 4.3.3 remain valid if x and y are real instead of rational.*

Proof (Informal). The previous proofs of these relied on the algebra on the rationals and laws of order. All of these hold for the reals, so we can just substitute rational for real and move the proofs over.

Definition 5.6.3. *Let $x \geq 0$ be a non-negative real, and let $n \geq 1$ be a positive integer. Then:*

$$x^{1/n} := \sup\{y \in \mathbb{R} : y \geq 0 \wedge y^n \leq x\}$$

We refrain from defining the nth root of negative x, to do so we need to define the *complex numbers*, which is complex analysis.

Lemma 5.6.1. *(Existance of nth root). Let $x \geq 0$ be real, and $n \geq 1$ be a positive integer, then the set $E := \{y \in \mathbb{R} : y \geq 0 \wedge y^n \leq x\}$ is non-empty and is bounded above. Thus it has a unique supremum and so $x^{1/n}$ is a real number.*

Lemma 5.6.2. *(Properties of exponentiation). We ommit most, they are all easily known from schol and essentialy are done by raising both sides of an equation to $1/n$. We mention a few here that are less obvious:*

(a) $x > y \iff x^{1/n} > y^{1/n}$

(b) If $x > 1$, then $x^{1/k}$ is a decreasing function of k. If $x < 1$ then it is an increasing function.

Definition 5.6.4. *Let $x > 0$ and let q be a rational number with $q = a/b$ for some integers a and positive integer b. We have:*

$$x^q := (x^{1/b})^a$$

Every rational q can be written in this form, but also for each $q \in \mathbb{Q}$ it can be written in many such forms. Thus we must cehck the definition is well defined.

Lemma 5.6.3. *Consider $a/b = a'/b'$ obeying the above properties. Then we have:*

$$(x^{1/b'})^{a'} = (x^{1/b})^a$$

We ommit some commonly known things about rasing a real to a rational, such as if $x > 1$ then $x^q > x^r \iff q > r$, and the converse if $x < 1$.

Eventually we will have to define x^y where $x > 0$ and y is a real number, but we cannot do this until we define a limit.

5.7 Subquest into decimals (optional)

This is taken from Appendix B of Tao. Note that to complete the proofs in this section, which we do not do, you need some more machinery of the reals that we haven't developed yet, in particular notions of convergent sums and the conventional (as opposed to formal) definition of a limit.

We start with defining decimal representations of natural numbers.

Definition 5.7.1. *(Digits). A digit is one of* $1, 2, \ldots, 9$*. We equate these digits with natural numbers by* $0 \equiv 0$*,* $1 \equiv 0{+}{+}$*, etc. We also define ten as* $:= 9{+}{+}$*. We cannot yet ust decimal notation for ten as that would be circular.*

Definition 5.7.2. *(Positive integer decimals). A positive integer decimal is a string* $a_n a_{n-1} \ldots a_0$ *of digits, where* $n \geq 0$ *is a natural number an* a_n *is non zero. We equate each positive integer with a decimal using:*

$$a_n a_{n-1} \ldots a_0 \equiv \sum_{i=0}^{n} a_i \times \mathrm{ten}^i$$

Note that from this we have:

$$10 = 0 \times \mathrm{ten}^0 + 1 \times \mathrm{ten}^1 = ten = 9{+}{+}$$

and so we can start using 10.

Theorem 5.7.1. *(Uniquness and existance of decimal representation). Every positive integer* m *is equal to exactly one positive integer decimal.*

Proof. We prove this using the principle of strong induction, but leave the proof as an exercise to the reader.

The decimal given by this theorem is called the *decimal representation* of m. Once we have this, we can derive how to do addition and multiplication. We can now represent negative integers using $-$ sign. Now we have decimal representations of all integers. We also have rationals are ratios of decimals, e.g. 335/113.

We now move onto the real numbers. We do so by introducing the decimal point.

Definition 5.7.3. *(Real decimals). A real decimal is any sequence of digits, and a decimal point, arranged as:*

$$\pm a_n \ldots a_0 . a_{-1} a_{-2} \ldots$$

which is finite to the left of the decimal point, but infinite to the right of the decimal point. The decimal is equated to the real number:

$$\pm a_n \ldots a_0 . a_{-1} a_{-2} \cdots \equiv \pm 1 \times \sum_{i=-\infty}^{n} a_i \times 10^i$$

This sequence is always convergent (why?).

Theorem 5.7.2. *(Existance of decimal representations) Every real number* x *has at least one decimal representation:*

$$x = \pm a_n \ldots a_0 . a_{-1} a_{-2} \ldots$$

Proof Surfices to show for positive x, as then we can just put $-$ sign in front, and 0 is a simply case. The proof is fairly long and involved.

Note in the above “at least.” It is possible for one real number to have two decimal representations. We have $0.999\cdots = 1$ (why?). More precisely, if the real is a terminating decimal, it has two decimal representations, (the .9 recuring from the left, and itself if you like), and if not then it has 1.

6 Limits of sequences

6.1 Convergence and limit laws

We now do the work to replace formal limits $\mathrm{LIM}_{\mathrm{n}\to\infty}$ with actual limits $\lim_{\mathrm{n}\to\infty}$. We begin by repeating much of the ε-close machinery for sequences of *real* numbers as opposed to rationals. This distinction will supercede what we did in the previous chapter.

Definition 6.1.1. *(Distance between reals). For $x, y \in \mathbb{R}$ we have $d(x, y) := |x - y|$.*

Definition 6.1.2. *(ε-close real numbers). Let $\varepsilon > 0$ be a real number. We say that $x, y \in \mathbb{R}$ are ε-close iff we have $d(y, x) \leq \varepsilon$.*

Analogous to before, let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers (formally constructed using a function from the naturals to the reals).

Definition 6.1.3. *(Cauchy sequence of reals). Let $\varepsilon > 0$ be a real number. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers starting at integer index N is said to be ε-steady iff a_j and a_k are ε close for every $j, k \geq N$.*

Let $\varepsilon > 0$ be a real number. A sequence $(a_n)_{n=m}^{\infty}$ of real numbers starting at integer index m is said to be eventually ε-steady iff there exists an $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε-steady.

We say that $(a_n)_{n=m}^{\infty}$ is a Cauchy squence iff it is eventually ε-steady for all $\varepsilon > 0$.

These definiitions are consistent with the corresponding ones made for the rationals.

Proposition 6.1.1. *Let $(a_n)_{n=m}^{\infty}$ be a sequence of rational numbers. Then it is Cauchy in the rational sense (Definition 5.1.4) iff it is Cauchy in the real sense (Definition 6.1.3)*

Proof. One direction is easy. Assume that the sequence is Cauchy in the real sense, so it is eventually ε-steady for reals, and as the rationals are embedded in the reals this means that it is eventually ε-steady for all rationals.

We now go in the other direction. Assume that the sequence is eventually ε steady for all rational ε. Consider some positive real δ. Then we know there is some positive rational $\varepsilon < \delta$ (by the bounding of the reals by rationals, Proposition 5.4.4). Thus the sequence is eventually δ-steady.

Remark 6.1.1. *From the above, we will no longer care about the difference between a sequence being Cauchy in the rational or real sense, as they are equivalent. We can think of a Cauchy sequence as a unified concept.*

We now define what it means for a sequence of real number to converge to a limit L!

Definition 6.1.4. *(Convergence of sequences). Let $\varepsilon > 0$ be a real number, and let L be a real number.*

A sequence $(a_n)_{n=N}^{\infty}$ of real numbers is said to be ε-close to L iff a_n is ε-close to L for every $n \geq N$. That is $|a_n - L| \leq \varepsilon$ for every $n \geq N$.

We say that a sequence $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L iff there exsits and $N \geq m$ such that $(a_n)_{n=N}^{\infty}$ is ε-close to L.

We say that a sequnece converged to L iff it is eventually ε-close to L for every real $\varepsilon > 0$.

Definition 6.1.5. *Consider:*

$$0.9, 0.99, 0.999, \ldots$$

It is 0.1 close to 1 but not 0.01-close. However it is eventually 0.01-close. In fact it is eventually ε-close to 1 for all possible $\varepsilon > 0$ and thus we have the limit of this sequence is 1.

Proposition 6.1.2. *(Uniqueness of limits). Consider a sequence (a_n) starting at m, and let $L \neq L'$ be two distinct real numbers. Then it is not possible for the sequence to converge to L and also converge to L'.*

Proof. *Assume towards contradiction that two such different limits. As $L' \neq L$ we know that $|L - L'| > 0$. Let:*

$$\varepsilon = |L - L'|/3$$

As the sequence converges to L and L' we must that they are eventually ε-close to both. Now pick an index n large enogh to kick in both ε-closeness. In this case we have:

$$\begin{aligned}
|a_n - L| &\leq \varepsilon \\
|a_n - L'| &\leq \varepsilon \\
\implies |a_n - L'| + |a_n - L| &\leq 2\varepsilon \\
\implies |a_n - L' - a_n + L| &\leq 2\varepsilon = 2|L - L'|/3 \qquad \text{By the triangle innequality} \\
\implies |L - L'| &< 2|L - L'|/3
\end{aligned}$$

This is only possible if $|L - L'| = 0$ and thus $L = L'$, which is a contradiction. ■

Definition 6.1.6. *(Limits of sequences) If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L, we say that the sequence is convergent and that its limit is L. We write*

$$L = \lim_{\mathrm{n} \to \infty} a_n$$

to denote this fact. If the sequence is not converging, we say it is divergent and leave $\lim_{n\to\infty} a_n$ *undefined.*

Proposition 6.1.3. $\lim_{n\to\infty} 1/n = 0.$

Proof. *We need to show that for each real* $\varepsilon > 0$*, the sqeuence is eventually* ε*-close to* L*.*

Pick some $\varepsilon > 0$*. Now let* $N > 1/\varepsilon$ *(which we can do because of the Archimedean principle). Then for* $n \geq N$ *we have:*

$$|a_n - 0| = |a_n| \qquad = |1/n| \leq |1/N| \qquad < |1/(1/\varepsilon)| = |\varepsilon| \qquad = \varepsilon$$

■

Proposition 6.1.4. *(Convergent sequences are Cauchy). If a sequence of real numbers is convergent then it is also Cauchy.*

Now we show that formal limits can be superceded by actual limits (remember before that formal limits were just a piece of notation that we defined no meaning to). This is the same we did with formal difference for the integers and formal division for the rationals.

Proposition 6.1.5. *(Formal limits are genuine limits). Suppose that* $(a_n)_{n=1}^{\infty}$ *is a Cauchy sequence of rational numbers. Then* $(a_n)_{n=1}^{\infty}$ *converges to* $\mathrm{LIM}_{n\to\infty} a_n$*. That is*

$$\mathrm{LIM}_{n\to\infty} a_n = \lim_{n\to\infty} a_n$$

Definition 6.1.7. *(Bounded sequences). A sequence* $(a_n)_{n=m}^{\infty}$ *of real numbers is bounded by* $M \in R$ *iff we have* $|a_n| \leq M$ *for all* $n \geq m$*.*

We say a sequence is bounded iff it is bounded by some $\mathbb{R} \ni M > 0$*.*

This definition is consistent with what we said about bounded sequences of rationals. We also had a lemma that every Cauchy seqeunce of rational numbers is bounded. We can use the same proof to say that every Cauchy seqeuence of real numbers is bounded.

Proposition 6.1.6. *Every Cauchy sequence of real numbers is bounded.*

Corollary 6.1.1. *Every convergence sequence of real numbers is bounded.*

Proof. *By Proposition 6.1.4 we have that convergent implies the sequence is Cauchy. By Proposition 6.1.6 we have this implies the seqeunce is bounded.*

Taking the contrapositive of the above:

Corollary 6.1.2. *Note bounded implies not cauchy implies not convergent*

Example 6.1.1. *Take* $1, 2, 3, 4, \ldots$. *This is not bounded so we have that it is not convergent using the above corollary.*

We did lots of work to define arithmetic and algebra on the real numbers. Basically saying what happens when I take the quotient of two formal limits, or the difference etc. All of the algebraic rules we proved there cary over to genuine limits as per the following theorem.

Theorem 6.1.1. *(Limit laws). Let* $(a_n)_{n=m}^{\infty}$ *and* $(b_n)_{n=m}^{\infty}$ *be convergent sequences of real numbers. Let* x, y *be the real numbers:*

$$x := \lim_{n\to\infty} a_n, \quad y := \lim_{n\to\infty} b_n$$

(a) The sequence $(a_n + b_n)_{n=m}^{\infty}$ *converges to* $x + y$. *That is:*

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

(b) $(a_n b_n)$ *converges to* xy, *that is:*

$$\lim_{n\to\infty} (a_n b_n) = (\lim_{n\to\infty} a_n) \lim_{n\to\infty} b_n)$$

(c) For any real number c:

$$\lim_{n\to\infty} (c a_n) = c \lim_{n\to\infty} a_n$$

(d) $(a_n - b_n)$ *converges to* $x - y$, *that is:*

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

(e) Suppose that $y \neq 0$ *and that* $b_n \neq 0$ *for all* $n \geq m$. *Then the sequence* $(b_n^{-1})_{n=m}^{\infty}$ *converges to* y^{-1}, *that is:*

$$\lim_{n\to\infty} b_n^{-1} = \left(\lim_{n\to\infty} b_n\right)^{-1}$$

(f) Suppose that $y \neq 0$ *and that* $b_n \neq 0$ *for all* $n \geq m$. *Then the sequence* (a_n/b_m) *converges to* x/y, *that is:*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

(g) The sequence $(\max(a_n, b_n))_{n=m}^{\infty}$ *converges to* $max(x, y)$, *that is:*

$$\lim_{n\to\infty} \max(a_n, b_n) = \max\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right)$$

(h) *The sequence* $(\min(a_n, b_n))_{n=m}^{\infty}$ *converges to max*(x, y)*, that is:*

$$\lim_{\mathrm{n}\to\infty} \min(a_n, b_n) = \min\left(\lim_{\mathrm{n}\to\infty} a_n, \lim_{\mathrm{n}\to\infty} b_n\right)$$

— Exercises —

Exercise 6.1.6 Prove that

$$\mathrm{LIM}_{\mathrm{n}\to\infty}\, a_n = \lim_{\mathrm{n}\to\infty} a_n$$

using the following outline. Let $(a_n)_{n=m}^{\infty}$ be a Cauchy sequence of rational, and write $L = \mathrm{LIM}_{\mathrm{n}\to\infty}\, a_n$. We have to show that (a_n) converges to L.

Let $\varepsilon > 0$. Assume towards contradiction that the sequence is not eventually ε-close to L. Use this and the fact that (a_n) is Cauchy to show that there is an $N \geq m$ such that either $a_n > L + \varepsilon/2$ for all $n \geq N$, or that $a_n < L - \varepsilon/2$ for all $n \geq N$. Then use Exercise 5.4.8

6.2 The Extended real number system

It seems like $1, 2, 4, \ldots$ is trying to converge to $+\infty$ and $-1, -2, -3, \ldots$ is trying to converge to $-\infty$, whilst $1, -1, 1, -1, 1, \ldots$ is not converging to anything. We address these points now.

Definition 6.2.1. *(Extended real number system). The exenteded real number system* $\mathbb{R}^*$ *is the real line* $\mathbb{R}$ *with two additional elements called* $+\infty$ *and* $-\infty$*. These elements are distinct from each other and also distinct from every real number.*

An extended real number x *is called finite iff it is a real number, and infinite iff it is equal to* $+\infty$ *or* $-\infty$*.*

(Note that this is different from the notion of a finite and infinite set which we introduced using bijections to the natural numbers).

We have no operations to manipualte these new symbols other than equality. We change this now.

Definition 6.2.2. *(Negation of extended reals). The operation of negation* $x \mapsto -x$ *on* $\mathbb{R}$ *extends to* $\mathbb{R}^*$ *by defining* $+\infty \mapsto -\infty$ *and* $-\infty \mapsto +\infty$*.*

Definition 6.2.3. *(Ordering of the extended reals). Let* $x, y \in \mathbb{R}^*$*. We say* $x \leq y$ *iff:*

(a) $x, y \in \mathbb{R}$ *and* $x \leq y$

(b) $y = +\infty$

(c) $x = -\infty$

(and trivially extend to all other innequalities).

Proposition 6.2.1. *All the nice properties of order reamin:*

(a) *(Reflexive)* $x \leq x$

(b) *(Trichotemy). One of* $x < y, x = y, x > y$ *is true.*

(c) *(Transitivity)*

(d) *(Negation reverses order).* $x \leq y \iff -y \leq -x$

We do not define other algebraic operations on the extended reals because things start to break down if you do.

Definition 6.2.4. *(Supremum of sets of extended reals). Let* E *be a subset of* $\mathbb{R}^*$. *Then we define* $\sup(E)$, *or the least upper bound of* E, *as:*

(a) *If* $E \subseteq \mathbb{R}$ *then we use Definition 5.5.3.*

(b) *If* $+\infty \in E$ *then* $\sup(E) = \infty$.

(c) *If* E *does not contain* $+\infty$ *but does contain* $-\infty$, *then we set* $\sup(E) := \sup(E - \{-\infty\}$ *(which is handled by case (a)).*

We also define $\inf(E)$ *as:*

$$\inf(E) := -\sup(-E), \quad -E = \{-x : x \in E\}$$

Example 6.2.1. *Consider the set:*

$$E = \{-\infty, -1, -2, -3, -4, \dots\}$$

We have:

$$\sup(E) = \sup(E - \{E\}) = -1$$

and:

$$\inf(E) = -\sup(\{\infty, 1, 2, 3, 4, \dots\}) = -\infty$$

Example 6.2.2. *We ahve* $\sup(\emptyset) = -\infty$, *and so* $\inf(\emptyset) = +\infty$. *This is the only case where the supremum is less than the infimum.*

Remark 6.2.1. *Heres how to imagine the supremum and infimum. Imagine the real line with $-\infty$ on the left and $+\infty$ on the right. For supremum, imagine a piston moving from $+\infty$ left until it hits a set. When it stops is the supremum. Same for infimum except the piston moves from $-\infty$ right.*

For the empty set, it never runs into a set so both pistons travel through each other, making the infimum $+\infty$ and the supremum $-\infty$.

The following theorem justifies the terminology "least upper bound" and "greatest lower bound."

Theorem 6.2.1. *Let $E \subseteq \mathbb{R}^*$. The the following statements are true.*

(a) For every $x \in E$ we have $x \leq \sup(E)$ and $x \geq \inf(E)$

(b) Suppose that $M \in \mathbb{R}^*$ is an upper bound for E (that is $\forall x \in E, x \leq M$) then we have that $\sup(E) \leq M$.

(c) Suppose that $M \in \mathbb{R}^*$ is a lower bound for E (that is $\forall x \in E, x \geq M$) then we have that $\inf(E) \geq M$.

Proof (sketch). *This is a very simple proof. You get most of it for free simply from the original definition of supremum on the reals (Definition 5.5.3), you just need to handle the cases of $+\infty$ and $-\infty$ being in E.*

6.3 Suprema and Infima of sequences