Linear Algebra

(never ends)

Dhruva V. Raman

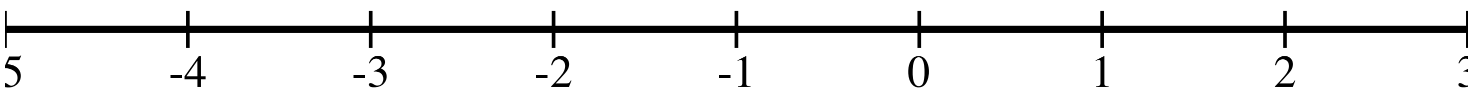
Part 1. More matrix mechanics

- Concepts of size/distance (norm)
- Concepts of length (metric)
- Concepts of angle (metric)

The magnitude of numbers

(Their distance from zero)

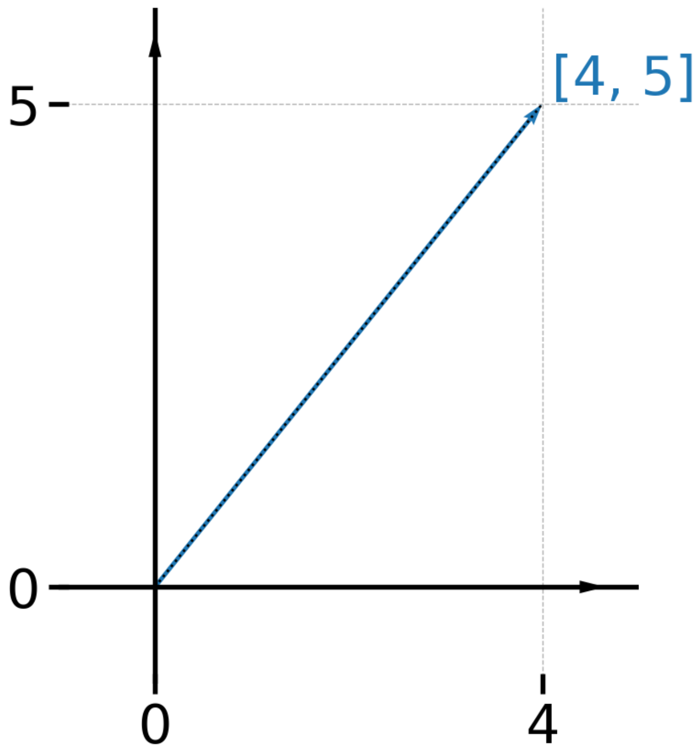
A number line:



- $-5 < 2$
- $|-5| > |2|$
- $|-7+6| = 1$

adding big numbers can give a small number in magnitude

The ~~magnitude~~ norm of vectors



$$\|\underline{v}\|_1 = |4| + |5|$$

$\| \bullet \|_1$ is the $\mathcal{L}_1$ norm.

- Also known as "Taxicab norm". Why?

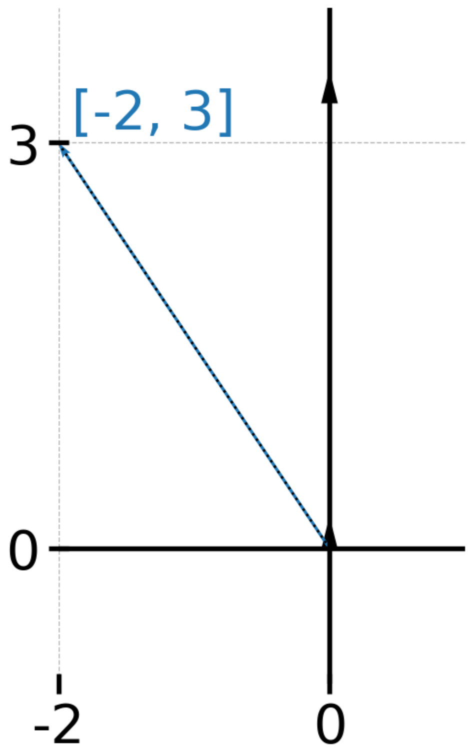
$$\|\underline{v}\|_2 = \sqrt{4^2 + 5^2}$$

$\| \bullet \|_2$ is the $\mathcal{L}_2$ norm.

- Also known as "Euclidean distance". Why?

(Pythagoras theorem)

Norm practice



$$\|\underline{v}\|_1 = |-2| + |3| = 5$$

$$\|\underline{v}\|_2 = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

These are not the only common norms!!

$$\|\underline{v}\|_p = (|-2|^p + |3|^p)^{\frac{1}{p}}$$

- what happens if p=1 or p=2?

$$\|\underline{v}\|_\infty = \max(|-2|, |3|) = 3$$

Build your own norm!

What makes a function $f : V \to \mathbb{R}$ a norm/distance?

Non-negativity

- $||\underline{v}|| \geq 0$
- $||\underline{v}|| = 0 \Rightarrow \underline{v} = 0$

Distance from zero can't be negative.

Only zero vector has zero norm

"Homogeneity"

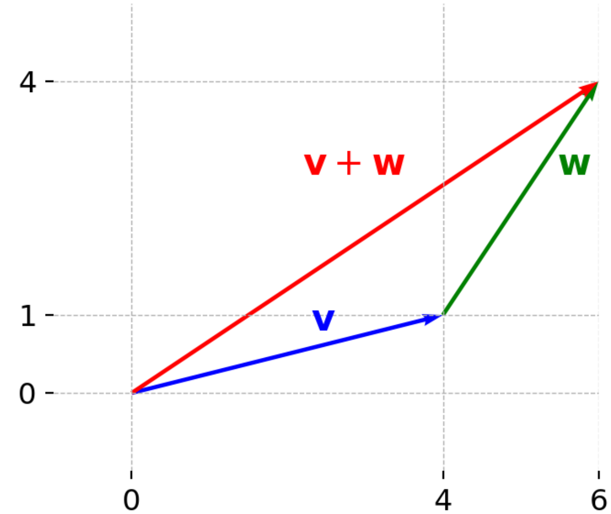
- $||s\underline{v}|| = ||s\underline{v}||$
  $\forall s \in \mathbb{R}$

Twice the vector = twice the distance

$$2\left\|\begin{bmatrix}2\\4\end{bmatrix}\right\| = \left\|2\begin{bmatrix}2\\4\end{bmatrix}\right\|$$
$$= \left\|\begin{bmatrix}4\\8\end{bmatrix}\right\|$$

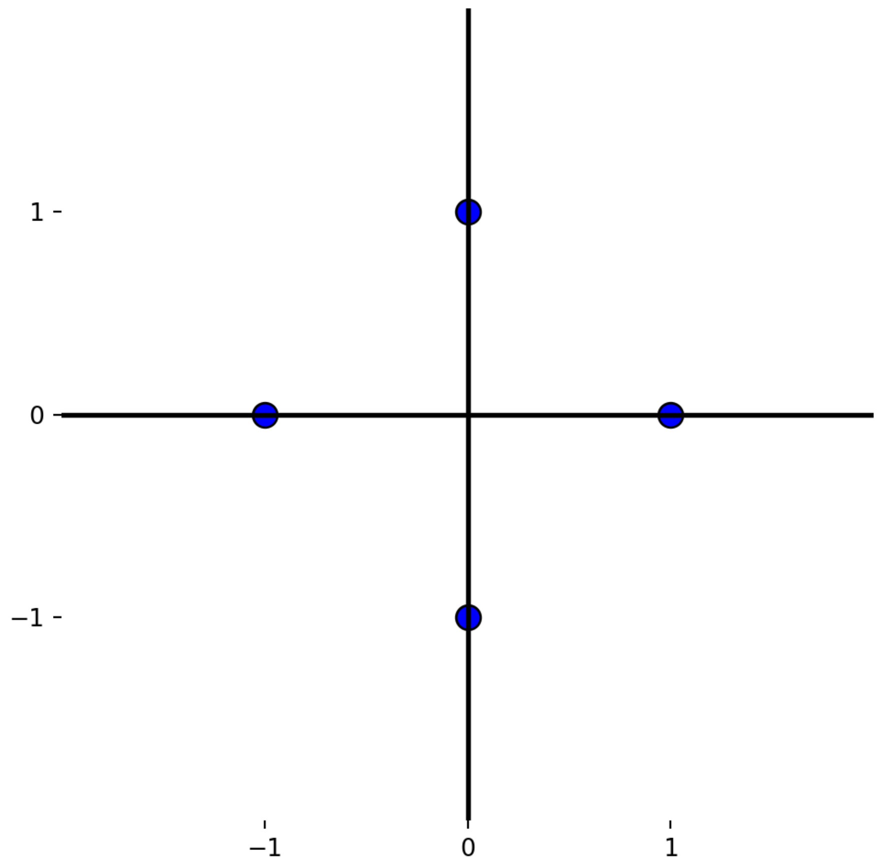
Triangle inequality

- $||\underline{v} + \underline{w}|| \leq ||\underline{v}|| + ||\underline{w}||$
  $\forall v, w \in \mathbb{V}$



- It's always longer to go *via* a waypoint

Unit spheres



Exercise $\times 3$

- Draw a line covering the set of points $\underline{v} = [v_1, v_2]$ satisfying…

$$\|\underline{v}\|_1 = 1$$
$$\|\underline{v}\|_2 = 1$$
$$\|\underline{v}\|_\infty = 1$$

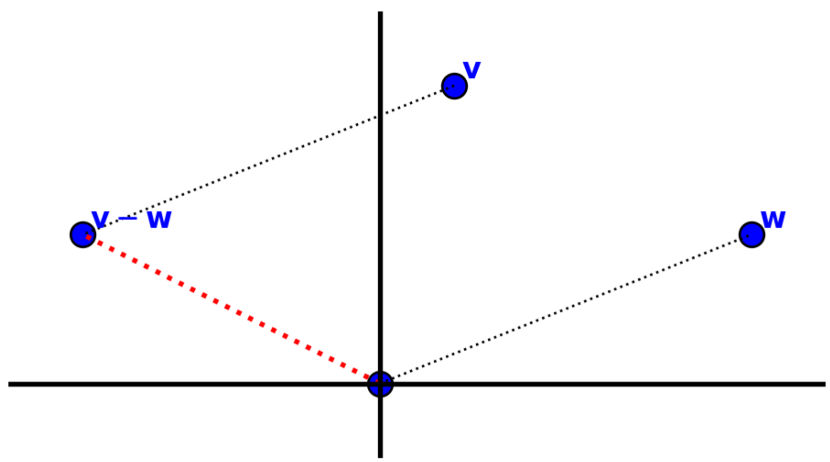
Hint: The four points shown are on *all* of these lines!

Terminology:

Called *unit spheres* in their respective norms. Why? (see L2)

Distance between vectors

- Norms depict distance from zero: $||\underline{v}||$
- Distance between $\underline{v}$ and $\underline{w}$?
- $||\underline{v} - \underline{w}||$!



Translating vectors doesn't change distance between them

Metric

- Notion of distance between vectors
- Each norm induces a metric

EG $\mathcal{L}_2$ metric

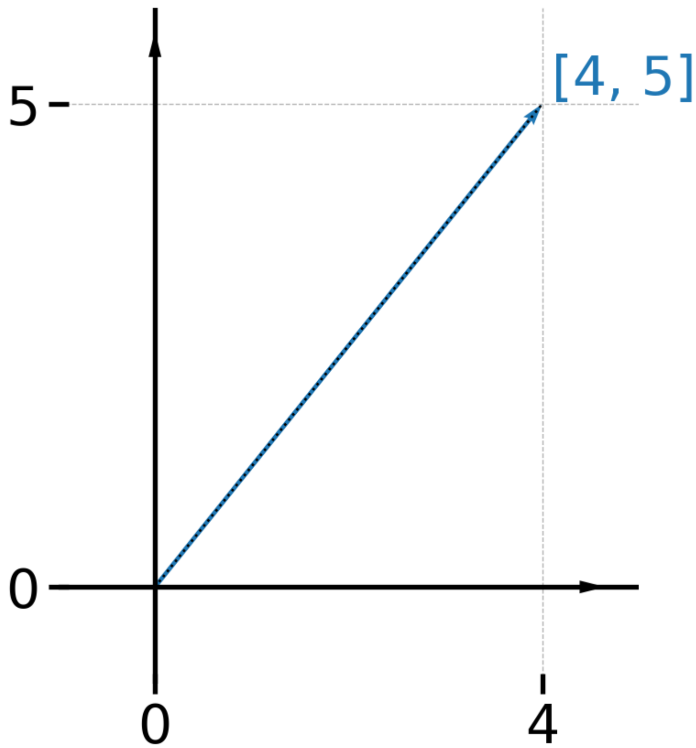
$d(\underline{v}, \underline{w}) = ||v - w||_2$

Maths definition

$$d : V \times V \to \mathbb{R}^+$$

- satisfying what properties?

Vectors live in a vector space



$$v = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \in \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$$

(Cartesian product not multiplication)

$\mathbb{R}^2$ is a vector space

Other vector spaces:

- $\mathbb{R}^3$ (3d vectors)
- $\mathbb{R}^n \quad \forall n \in \mathbb{N}$
- $\mathbb{R}^{n \times m} \quad \forall n, m \in \mathbb{N}$ (matrices of any fixed size)

Real-life vector spaces:

- Images/video/audio signals
- Weights of a neural network

All are shaped collections of numbers

What makes a vector space

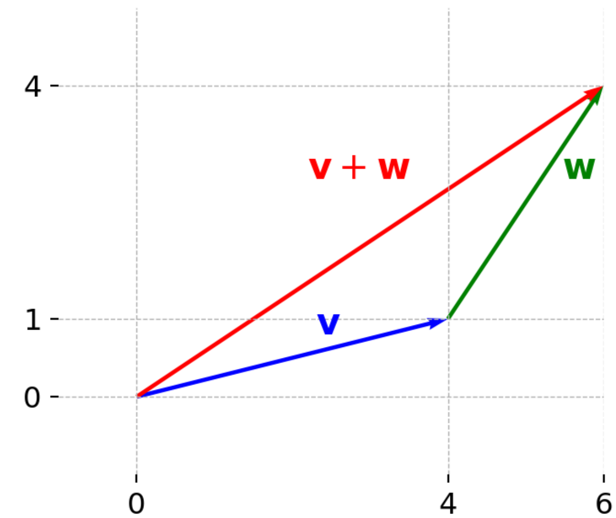
Vector spaces are sets of elements that are

- closed under addition

$$\underline{v}, \underline{w} \in V \Rightarrow \underline{v} + \underline{w} \in V$$

- closed under multiplication by a scalar in the field K (think $K = \mathbb{R}$)

$$\underline{v} \in V, c \in K \Rightarrow c\underline{v} \in V$$



- $2\underline{v}$ is also a vector in $\mathbb{R}^2$
- $c\underline{v}$ is also a vector in $\mathbb{R}^2$ for all $c \in K = \mathbb{R}$

What is a scalar in a field?

Take home:

Read field K as $\mathbb{R}$

- Any vector space rests on a field K of *scalars*
- Scalars are almost always the real numbers
- What makes a more general field? In cheatsheet

What makes a vector space (cont.)

- Additive identity $\underline{0} \in V$

$$\underline{v} \in V \Rightarrow \underline{v} + \underline{0} = \underline{0} + \underline{v} = \underline{v}$$

- Additive inverse:

$$\underline{v} \in V \Rightarrow \exists(-\underline{v}) : \underline{v} + (-\underline{v}) = 0$$

- Other boring stuff in cheatsheet

What's the point of vector space abstraction?

Let's see by example…

Recap: dot product

$$\underline{a} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\underline{a} \bullet \underline{x} = aw + bx + cy + dz$$

- Outputs a scalar

Alternative notation

$$\underline{a}^T \underline{x}$$

$$\langle \underline{a}, \underline{x} \rangle$$

Why? Matrix multiplication!

$$\underline{a}^T \underline{x} = [a, b, c, d] \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

Bilinearity of dot product

Example

$$\left\langle \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle = 7$$

$$\left\langle 2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}, 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle = 2 \times 3 \times 7$$

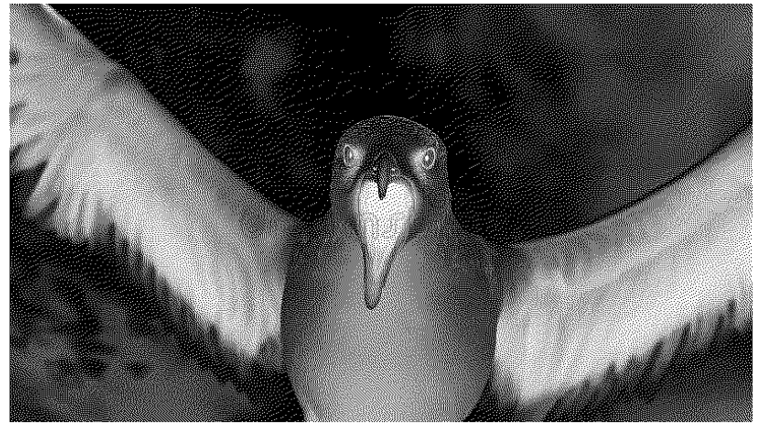
Generality

$$\langle a\underline{v}, b\underline{w} \rangle = ab \langle \underline{v}, \underline{w} \rangle$$
$$\forall a, b \in \mathbb{R}, \quad \forall \underline{v}, \underline{w} \in V$$

Correlation between vectors

Image A

Image B



1. Represent image as matrix of pixels (1 is white, 0 is black)
2. Centre representation:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix}$$

(Mean pixel is zero)

What's the dot product of these images?

- $A_{ij} < 0 \Rightarrow B_{ij} > 0$
- $A_{ij} = -B_{ij}$ actually!
- Anticorrelated images

Uncorrelated images

Image A

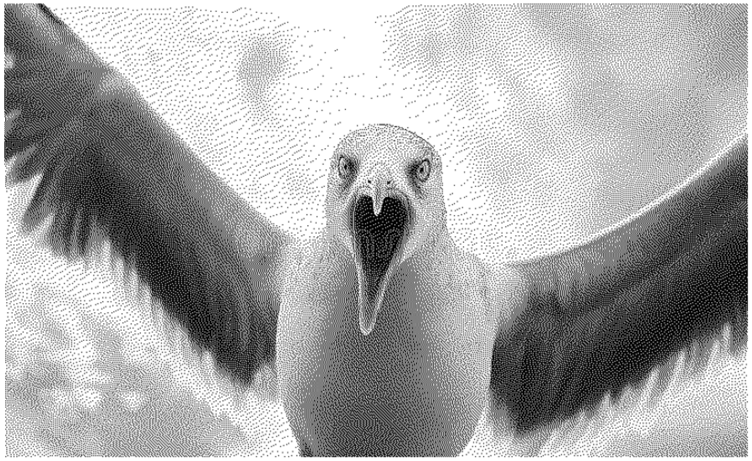
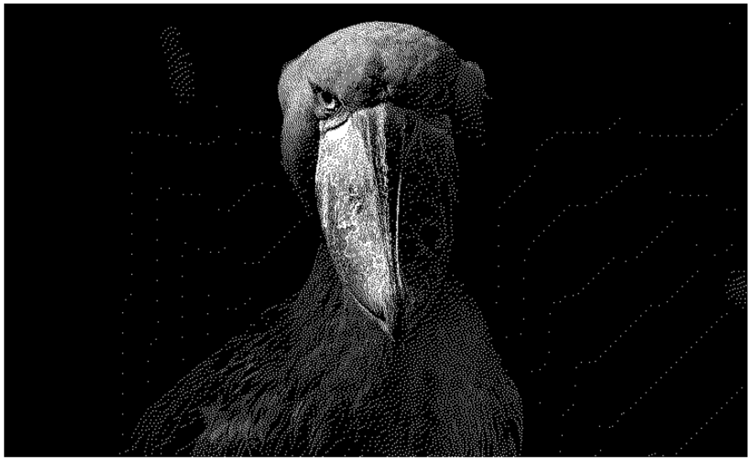
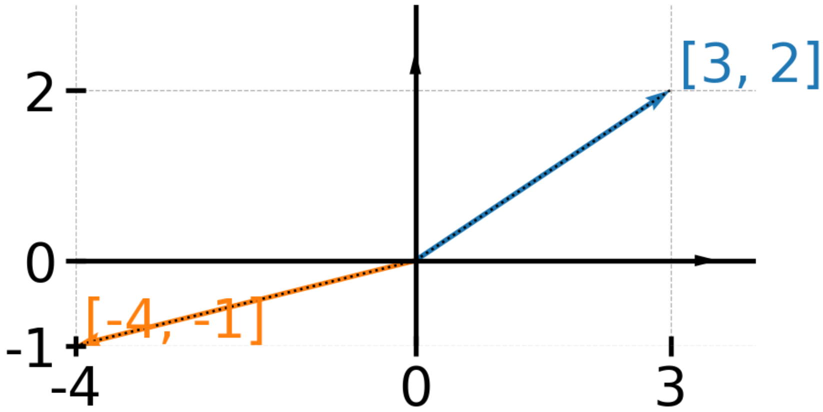


Image B



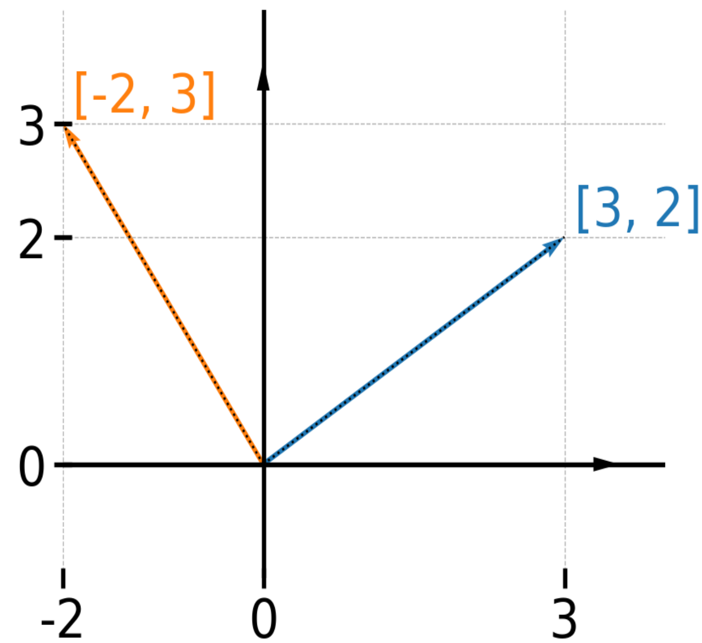
- Image A gives no information about Image B
- They are uncorrelated
- $A_{ij} > 0 \Rightarrow$ nothing

Correlation between vectors in $\mathbb{R}^2$



- $\underline{v}$ is quite negative $\Rightarrow$ $\underline{w}$ is quite positive
- Negative dot product

Correlation between vectors in $\mathbb{R}^2$



$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}^T \begin{bmatrix} -2 \\ 3 \end{bmatrix} = 0$$

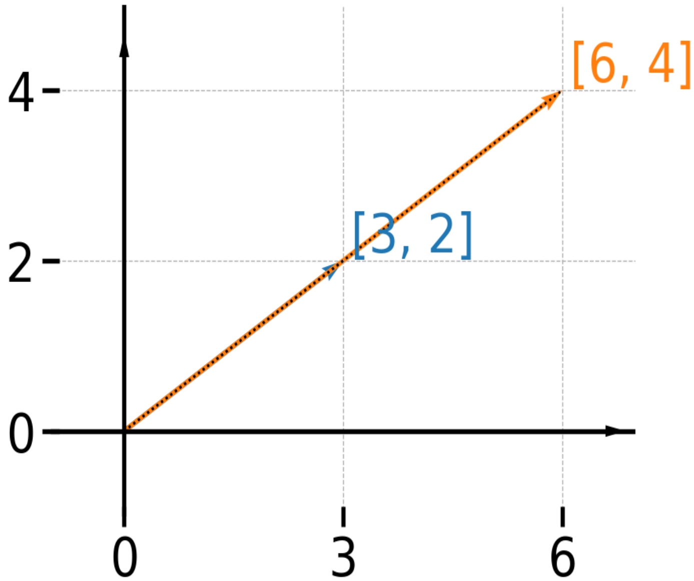
Orthogonal (right-angle) vectors have zero dot product

Uncorrelated vectors are, on average, orthogonal

$\underline{v}$ is quite positive $\Rightarrow$ $\underline{w}$ not much information about $\underline{w}$.

Small dot product

Correlation between vectors in $\mathbb{R}^2$



$\underline{v}$ is quite positive $\Rightarrow$ $\underline{w}$ is quite positive

Positive dot product

$\|\underline{v}\|_2 = \sqrt{\underline{v} \bullet \underline{v}}$?

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}^T \begin{bmatrix} 6 \\ 4 \end{bmatrix} = 26$$

$$\left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\|_2 = \sqrt{3^2 + 2^2} = \sqrt{13}$$

$$\left\| \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\|_2 = ?$$

$= 2\sqrt{13}$ (homogeneity)

$$\frac{\begin{bmatrix} 3 \\ 2 \end{bmatrix}^T \begin{bmatrix} 6 \\ 4 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\|_2 \left\| \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\|_2} = \frac{26}{2\sqrt{13}\sqrt{13}} = 1$$

- Normalised dot product ignores magnitude of vectors

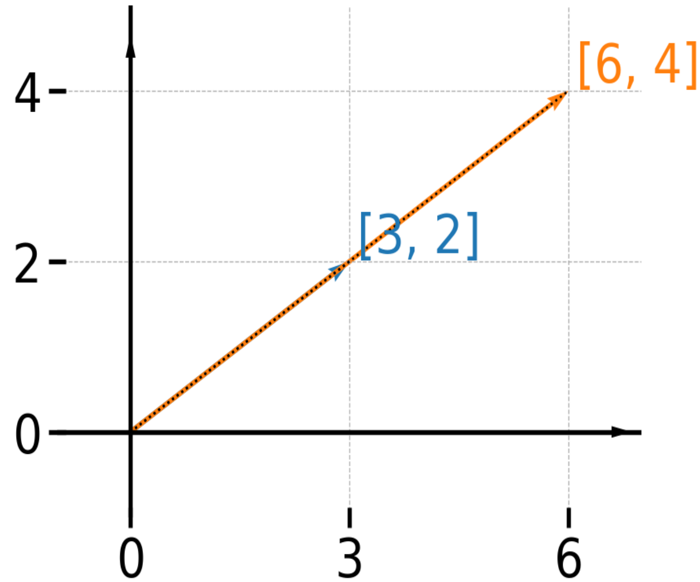
Correlation

$$\mathrm{corr}(\underline{v}, \underline{w}) = \frac{\underline{v}^T \underline{w}}{\|\underline{v}\|_2 \|\underline{w}\|_2} \in [-1, 1]$$

$\underline{v} \propto \underline{w}$: parallel (same direction)

$\underline{v} = c\underline{w}, \quad c \in \mathbb{R}^+$

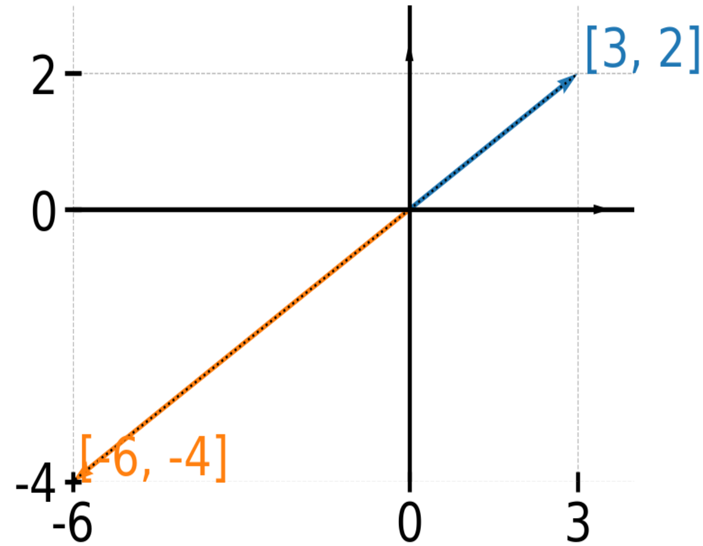
$\Rightarrow \mathrm{corr}(\underline{v}, \underline{w}) = 1$



$\underline{v} \propto \underline{w}$: parallel (opposite direction)

$\underline{v} = c\underline{w}, \quad c \in \mathbb{R}^-$

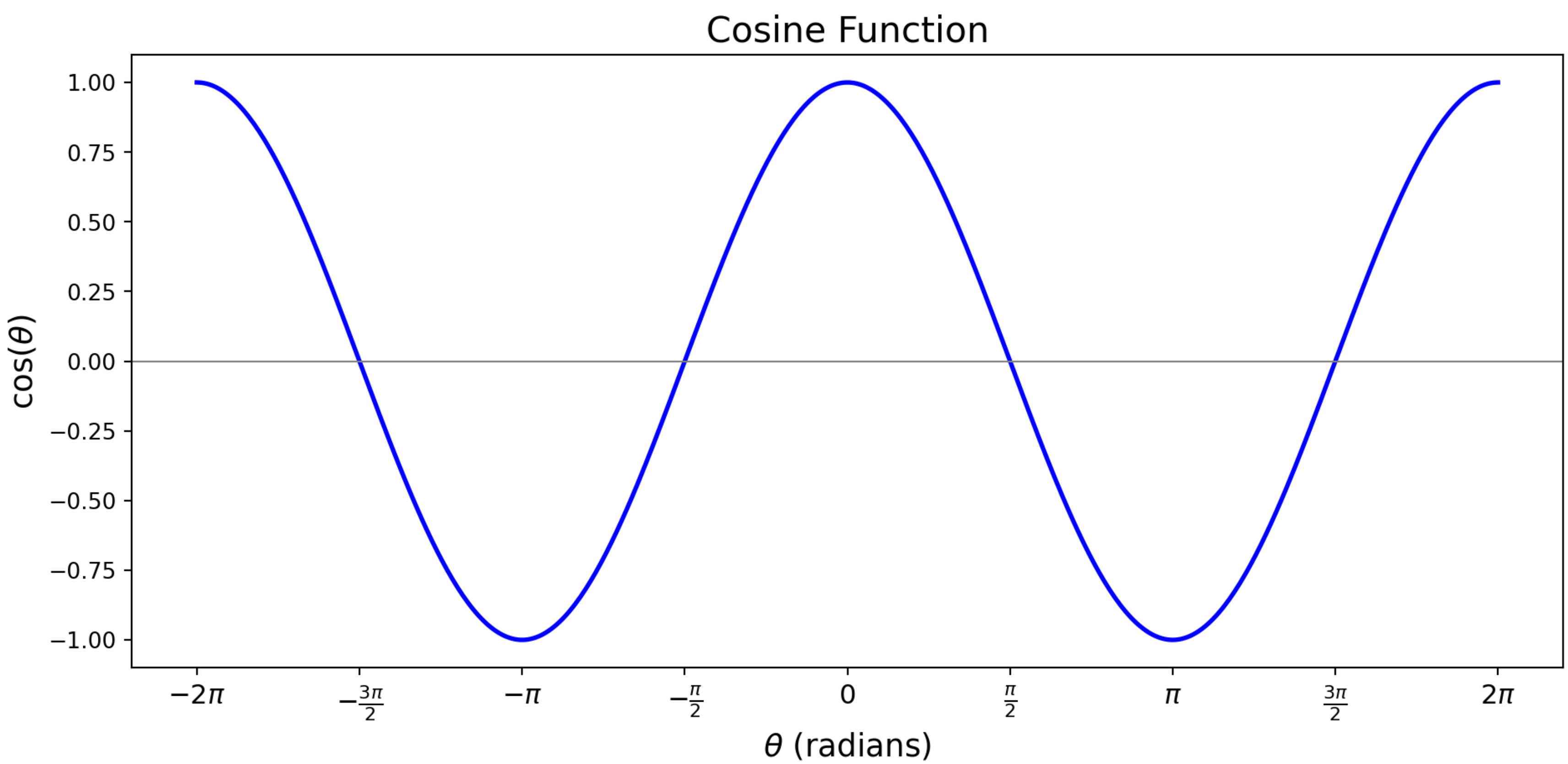
$\Rightarrow \mathrm{corr}(\underline{v}, \underline{w}) = -1$



Exercise at home: Prove it!

Correlation gives angle

$$\operatorname{corr}(\underline{v}, \underline{w}) = \frac{\underline{v}^T \underline{w}}{\|\underline{v}\|_2 \|\underline{w}\|_2} = \cos(\theta)$$

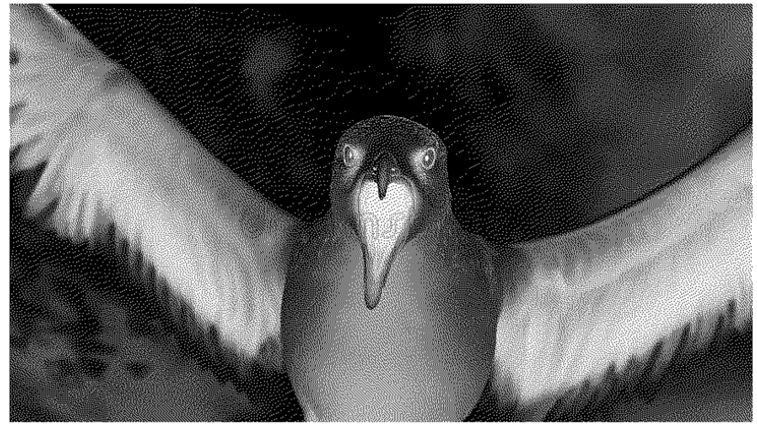


- (Normalised) dot product gives the angle between vectors!

Angle between images?

Image A

Image B



$$v, w \in V$$

$$\underline{v} = -\underline{w}$$

$$\langle \underline{v}, \underline{w} \rangle = -\langle \underline{v}, \underline{v} \rangle = -\|v\|_2^2$$

$$\operatorname{corr}(\underline{v}, \underline{w}) = \cos(\theta) = -1$$

Angle between images?

Image A

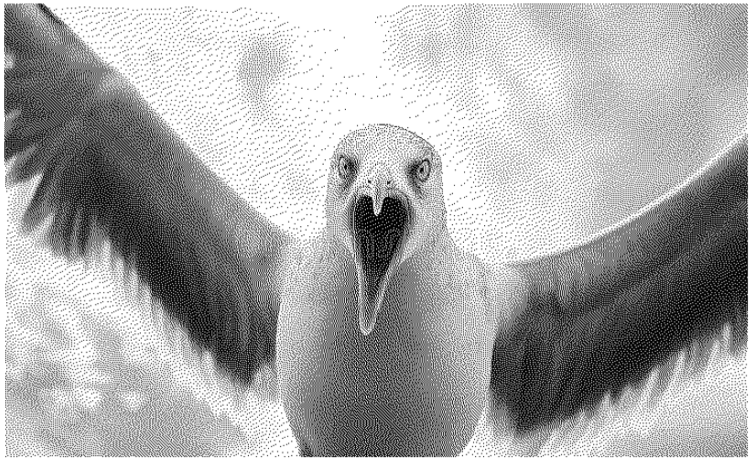
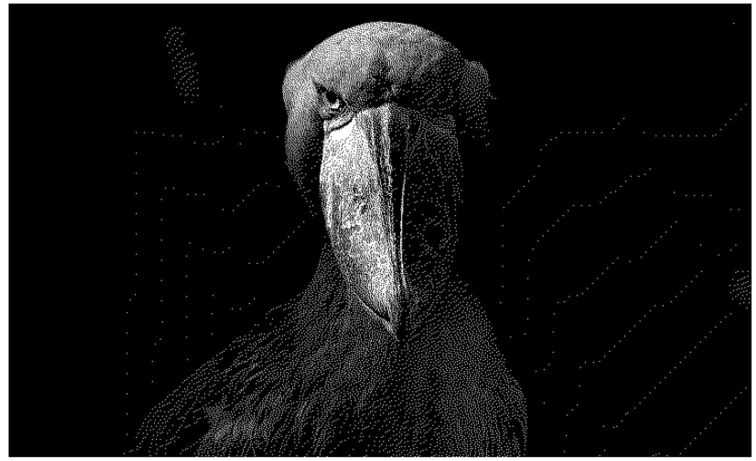


Image B



$$v, w \in V$$
$$\langle \underline{v}, \underline{w} \rangle \approx 0$$

- Images are (roughly) orthogonal
- Brighter pixel in one image gives no info on pixel in other image

Linear (in)dependence

Example of linearly dependent vectors

$$\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}4\\2\\0\end{bmatrix}, \begin{bmatrix}9\\7\\1\end{bmatrix}$$

- $1\underline{v}_1 + 2\underline{v}_2 = \underline{v}_3$
- $1\underline{v}_1 + 2\underline{v}_2 - \underline{v}_3 = 0$

$\{\underline{v}_i \in V\}_{i=1}^{n}$ are linearly dependent if $\exists\{\underline{c}_i \in \mathbb{R}\}_{i=1}^{n}$ such that

$$\sum_{i=1}^{n} c_i \underline{v}_i = 0$$

Example of linearly independent vectors

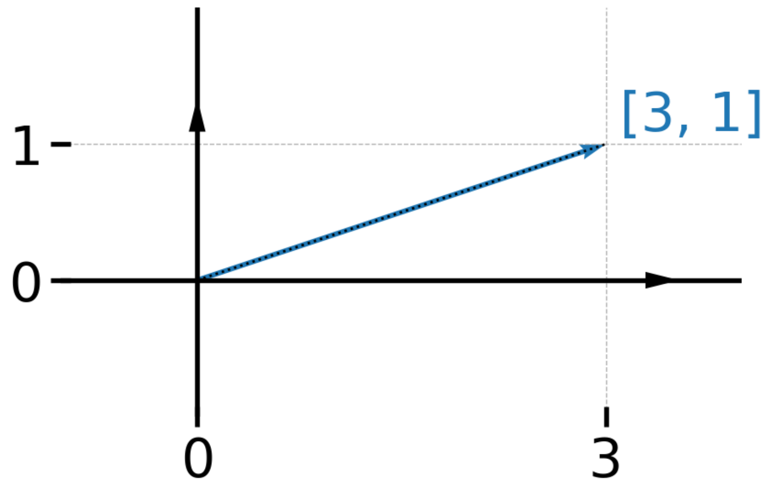
$$\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}4\\2\\0\end{bmatrix}, \begin{bmatrix}9\\7\\12\end{bmatrix}$$

- Could I add a fourth?

N-dimensional vector space:

- Can make at most n linearly independent vectors
- These form a “basis”

Basis of a vector space



Represent $\underline{v}$ in a co-ordinate system

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{v} = 3\underline{e}_1 + 1\underline{e}_2$$

$\{\underline{e}_1, \underline{e}_2\}$ is a basis

- Can reach any vector in V with a linear combination of basis vectors:

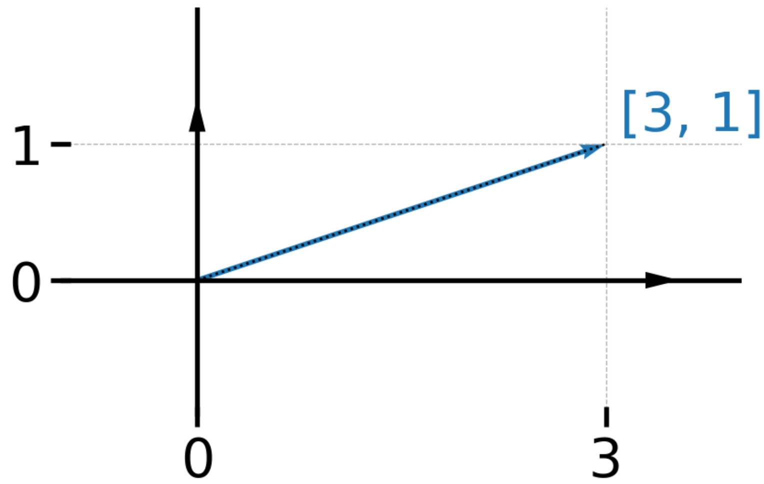
$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\underline{e}_1 + v_2\underline{e}_2$$

Linear combinations means you can add and multiply by scalars

- No redundancy in representation

(Basis elements are linearly independent)

Bases are not unique



$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{v} = 3\underline{e}_1 + 1\underline{e}_2$$

$\{e_i\}$ basis is orthonormal:

- orthogonal: $e_i^T e_j = 0 \ \ \forall i, j$
- normalised $\|e_i\|_2 = 1 \ \ \forall i$

$$\underline{f}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \underline{f}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underline{v} = 3\underline{f}_1 - 2\underline{f}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Can I reach all vectors using linear combinations of f-basis?

$\Leftrightarrow$ Can I reach e-basis with f-basis?

Basis of another vector space

Image A

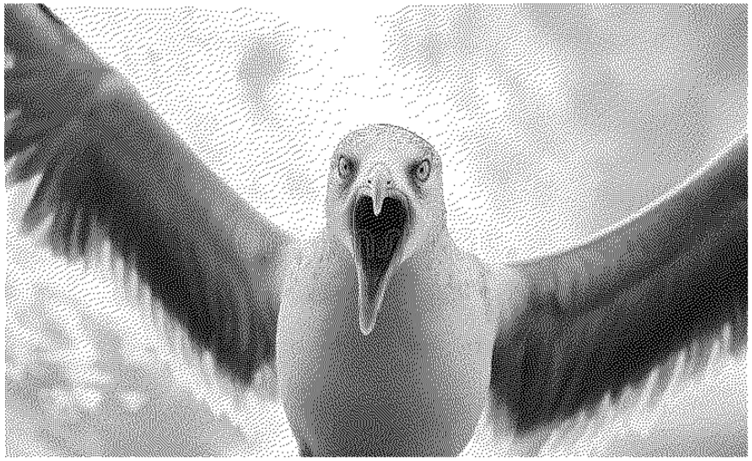
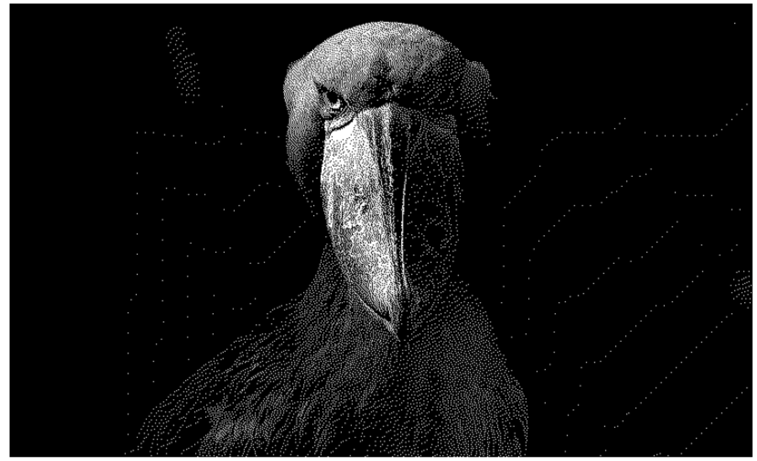


Image B



$$\underline{e}_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\underline{e}_2 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

How many basis elements?

Rows $\times$ columns

Dimension of vector space is the number of basis elements

Bases define vector space dimension

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

Span of these (/any) vectors forms a vector space V

(Span = set of linear combinations)

$$A\underline{v} =$$

$$v_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + v_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + v_3 \begin{bmatrix}1\\1\\0\end{bmatrix}$$

Has a basis of two elements, so vector dimension is 2

Range of A is two-dimensional vector space

- Rank of A is 2
- A is rank-deficient/singular
- $\det(A) = 0$ (squashes volume into area)

Linear (in)dependence

Example of linearly dependent vectors

$$\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}4\\2\\0\end{bmatrix}, \begin{bmatrix}9\\7\\1\end{bmatrix}$$

- $1\underline{v}_1 + 2\underline{v}_2 - \underline{v}_3 = 0$

Range of A is two-dimensional vector space (rank 2)

$$A = \begin{bmatrix}1 & 4 & 9\\3 & 2 & 7\\1 & 0 & 1\end{bmatrix}$$

$$A\begin{bmatrix}1\\2\\-1\end{bmatrix} = 0$$

Example of linearly independent vectors

$$\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}4\\2\\0\end{bmatrix}, \begin{bmatrix}9\\7\\12\end{bmatrix}$$

Range of B is three-dimensional vector space (rank 3)

$$B = \begin{bmatrix}1 & 4 & 9\\3 & 2 & 7\\1 & 0 & 12\end{bmatrix}$$

- B is nonsingular: $B\underline{v} = 0 \Rightarrow \underline{v} = \underline{0}$
- If not, linearly dependent columns

Kernel and image of a matrix $A : V \to W$

Two important vector spaces

$$\mathrm{Im}(A) = \{\underline{w} : \underline{w} = A\underline{v}\} \subseteq W$$

- Spanned by columns
- $\mathrm{Dim}(\mathrm{Im}(A))$ is the rank of A

$$\mathrm{Ker}(A) = \{\underline{v} : A\underline{v} = 0\} \subseteq V$$

- $\mathrm{Dim}(\mathrm{Ker}(A))$ is the nullity of A

Rank-nullity theorem

$$\mathrm{Rank}(A) + \mathrm{Nullity}(\mathrm{A}) = \text{Number of columns}(A)$$

Linear maps between scalars

$$f : \mathbb{R} \to \mathbb{R}$$
$$f(x) = 2x + 4$$

$$g : \mathbb{R} \to \mathbb{R}$$
$$g(x) = 3x$$

- Is f linear?
- Is g linear?

Linearity property

$$T(ax + by) = aT(x) + bT(y)$$

- a, b are scalars
- x, y are vectors
- Both in $\mathbb{R}$ in this case!

$$\begin{aligned} & a(2x+4) + b(2y+4) \\ \neq & 2(ax+by) + 4 \end{aligned}$$

- f is affine but not linear

$$\begin{aligned} & a(3x) + b(3y) \\ = & 3(ax+by) \end{aligned}$$

- g is linear

Linear maps in a vector space

$T : V \to W$ is a linear map if

$$T(a\underline{x} + b\underline{y}) = aT(\underline{x}) + bT(\underline{y})$$

- $a, b \in K$ (the field, ie $\mathbb{R}$)
- $\underline{x}, \underline{y} \in V$

Claim

Know how T transforms basis elements $\Rightarrow$ know how linear map transforms everything.

Example

$T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear map

$\{\underline{e}_1, \underline{e}_2\}$ is some basis of $\mathbb{R}^2$ T(_1)

$\{\underline{f}_1, \underline{f}_2, \underline{f}_3\}$ is some basis of $\mathbb{R}^3$

- $T(\underline{e}_1) = 3\underline{f}_1 + 7\underline{f}_2 + 5\underline{f}_3$
- $T(\underline{e}_2) = 4\underline{f}_1 - 2\underline{f}_2 + 6\underline{f}_3$

Linear maps in a vector space

$$T(a\underline{x} + b\underline{y}) = aT(\underline{x}) + bT(\underline{y})$$

- $T(2\underline{e}_1 + 3\underline{e}_2)$?

= $2T(\underline{e}_1) + 3T(\underline{e}_2)$

$$\begin{bmatrix} 3 & 4 \\ 7 & -2 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}?$$

$$= 2\begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$$

Example

$T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear map

$\{\underline{e}_1, \underline{e}_2\}$ is some basis of $\mathbb{R}^2$ T(_1)

$\{\underline{f}_1, \underline{f}_2, \underline{f}_3\}$ is some basis of $\mathbb{R}^3$

- $T(\underline{e}_1) = 3\underline{f}_1 + 7\underline{f}_2 + 5\underline{f}_3$
- $T(\underline{e}_2) = 4\underline{f}_1 - 2\underline{f}_2 + 6\underline{f}_3$

Matrices are linear maps

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

represents element of vector space V (in e-basis)

Represent elements of vector space through numbers of basis elements

$$\begin{bmatrix} 3 & 4 \\ 7 & -2 \\ 5 & -6 \end{bmatrix}$$

represents linear map $T : V \to W$

Matrices represent linear maps

(All finite-dimensional linear maps are represented by matrices)

$$\begin{bmatrix} 3 & 4 \\ 7 & -2 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

represents $T(\underline{v}) \in W$ (in f-basis)

Matrix multiplication transforms vectors through linear map

The duality of square matrices

$$T : V \to V$$

$$T(\underline{v}) = Av$$

Interpretation 1

Function that transforms input vectors into output vectors

Interpretation 2

Function that represents the same vector in a different basis

Multiplying matrices together composes linear maps

$$T_1 : U \to V \quad T_1\underline{u} = A\underline{u}$$

$$T_2 : V \to W \quad T_2\underline{v} = B\underline{v}$$

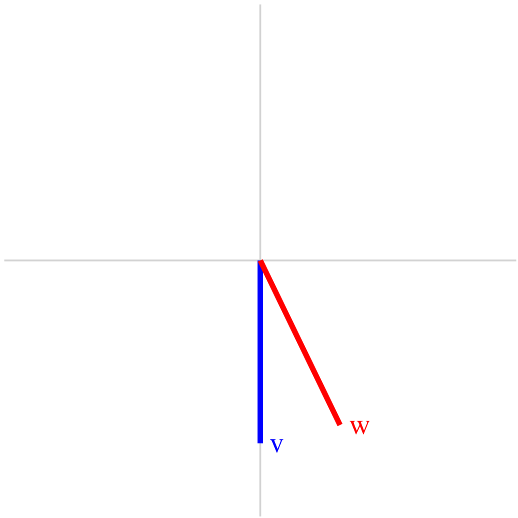
$$T_2 \circ T_1 : U \to W$$

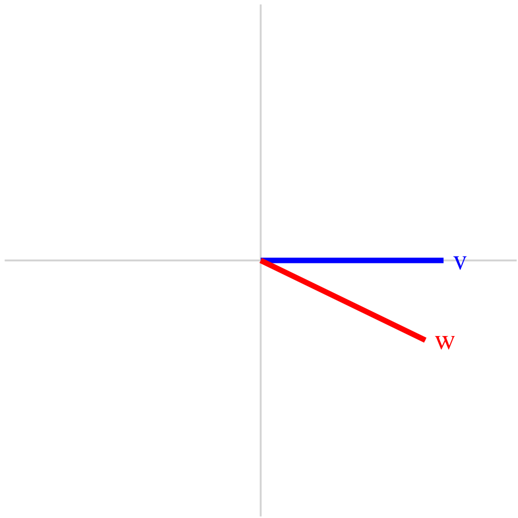
$$T_2 \circ T_1(\underline{u}) = BA\underline{u}$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 9 & 6 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 & 4 \\ 1 & 9 & 6 & 6 \\ 1 & 9 & 6 & 6 \end{bmatrix} : \mathbb{R}^? \to \mathbb{R}^?$$

$$\mathbb{R}^4 \to \mathbb{R}^2$$

What is a rotation?





- vector norm is invariant to rotation
- angle between vectors preserved under rotation
- angle = dot product!

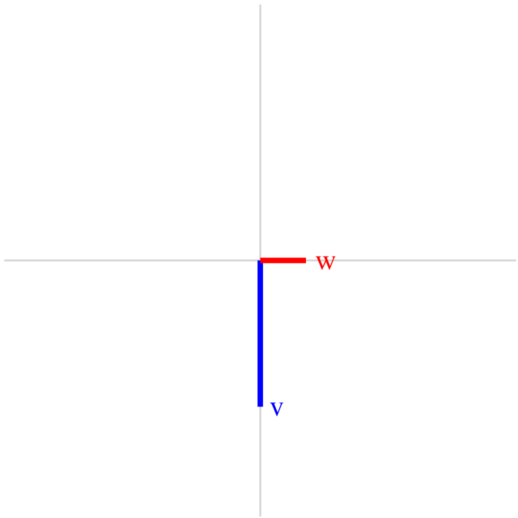
$$\langle A\underline{v}, A\underline{w}\rangle = \langle \underline{v}, \underline{w}\rangle$$

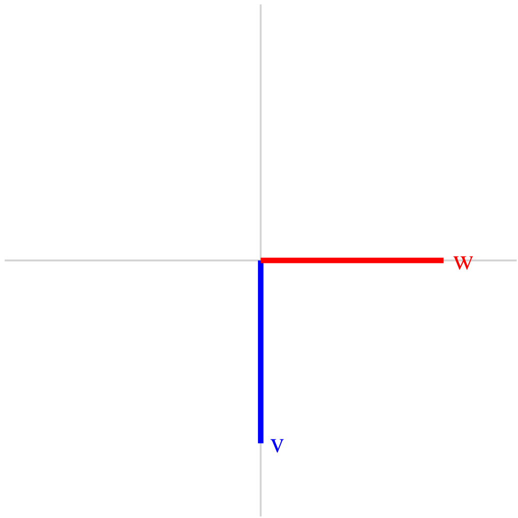
$$(A\underline{v})^T(A\underline{w}) = \langle \underline{v}, \underline{w}\rangle$$

$$\underline{v}^T A^T A\underline{w} = \underline{v}^T\underline{w}$$

Argument applies $\forall \underline{v}, \underline{w}$ so $A^T A = \mathbb{I}_n$!

What is a scaling?





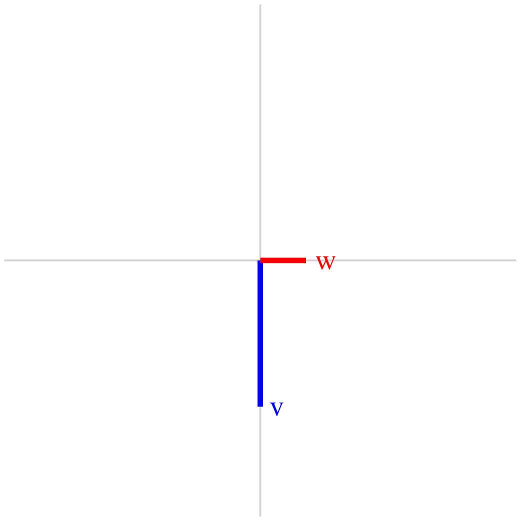
- Basis vectors stretch without changing direction
- Eigenvectors?
- The basis of the vector space!

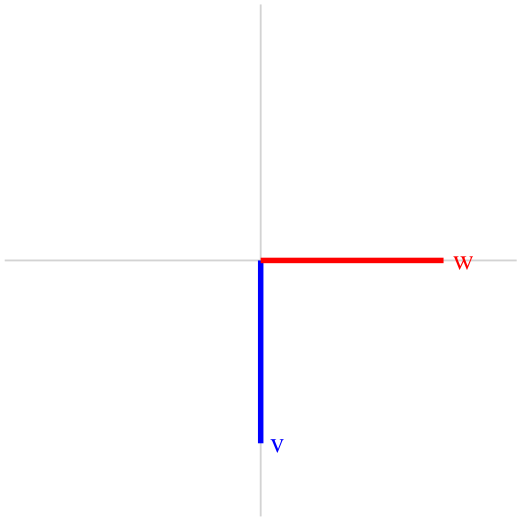
Diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

- d_i is degree of stretch for axis i.

What is a scaling?





- D^k is easy to compute!
- $D^4 = \begin{bmatrix} d_1^4 & 0 & \dots & 0 \\ 0 & d_2^4 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & d_n^4 \end{bmatrix}$

Diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

Singular value decomposition

Every linear map is a composition:

rotate ∘ *stretch* ∘ *rotate*

$$A = U\Sigma V^T \in \mathbb{R}^{m \times n}$$

- $\Sigma \in \mathbb{R}^{m \times n}$ is a scaling (diagonal)
- $V^T \in \mathbb{R}^{n \times n}$ is a rotation: $V^T V = \mathbb{I}_n$
- $U \in \mathbb{R}^{m \times m}$ is a rotation: $U^T U = \mathbb{I}_m$

Always exists. Ubiquitous in algorithms.

Very numerically stable (low approx error)

Example: symmetric matrix

Relationship between U and V?

- $A = U\Sigma U^T$
- $A^k = U\Sigma^k U^T$ (explain yourself)

Why is this better than computing $A \times A \times A \ldots$?

Practice

Nullity?

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}$$

- $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ spans Kernel
- Only one linearly dependent column so nullity=1
- So Rank 2

Left Inverse? Right inverse?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \end{bmatrix}$$

- $A : \mathbb{R}^3 \to \mathbb{R}^2$
- Putative l/r inverse: $\mathbb{R}^2 \to \mathbb{R}^3$
- Image of inverse must be a 2d Vector space. Can't span $\mathbb{R}^3$.

- $A^{-1,l} \circ A : \mathbb{R}^2 \to \mathbb{R}^2$. Possible
- $A \circ A^{-1,r} : \mathbb{R}^3 \to \mathbb{R}^3$. Image must be 2d vector space unlike $\mathbb{I}$.