Herbert Amann

Vector-Valued Distributions and Fourier Multipliers

This manuscript was originally planned to be Chapter VI, that is, the ørst chapter of Volume II of the treatise whose ørst volume is [Ama95]. In the meantime I have changed the concept of that volume and do not plan to include this material. For this reason, and since some of the results presented here have already been cited by several authors being in possession of preliminary versions, I have decided to make it publicly available by putting it on my home page.

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Contents

Preface

In the study of evolution equations by functional-analytical techniques one is led quite naturally to the study of pseudodiœerential operators with operator-valued symbols. If one wants to avoid being restricted to a Hilbert space setting ¯ that would be much too narrow for nonlinear problems ¯ one has to be able to handle eŒciently distributions with values in general Banach spaces. In particular, one has to be in possession of a good theory of convolutions for the case that both factors are Banach-space-valued distributions. For this reason, specializing the much more general results of L. Schwartz to a Banach space setting, we develop in the ørst section the theory of vector-valued distributions. Since there seems to be no other complete exposition of this theory easily available we try to be rather complete, thus going beyond the minimum we really need.

Vector-valued distributions are most useful in connection with techniques of harmonic analysis in a Banach space setting. In particular, we prove some Fourier multiplier theorems involving operator-valued symbols. These results, combined with dyadic decompositions in the Fourier image, are of fundamental importance. They are developed in Section 2. It is an important fact that, unlike the Banach space version of the Mikhlin multiplier theorem, our multiplier theorems do not require any restriction on the class of Banach spaces, like the UMD property, for example.

In the last section we ørst give a simple application of vector-valued distribution theory to linear evolution equations. Namely, we introduce the concept of distributional solutions and give suŒcient conditions for their existence and uniqueness. The rest of that section is then essentially devoted to a detailed study of the Gau-Weierstra semigroup and related subjects. Its results are fundamental for an in-depth study of function spaces.

1 Vector-Valued Distributions

The theory of vector-valued distributions has been created and almost completely developed by L. Schwartz about forty years ago. His results are very general since he considered distributions with values in LCSs. For our purposes it suŒces to consider distributions with values in Banach spaces only. This allows for some simpliøcations of the general theory, and we shall develop the relevant parts of that theory in this more restricted setting in the present section.

It will be crucial for our purposes to be in possession of a good theory of convolutions and 'point-wise multiplications' of vector-valued distributions. These are special cases of bilinear operations on vector-valued distributions. In order to handle such bilinear operations we prove in Subsection 1.5 a basic extension theorem that allows us to carry the desired operations from scalar distributions over to the vector-valued setting. To prove the extension theorem we need a considerable amount of preparation, namely rather deep results from the theory of LCSs as well as approximation and subtle continuity theorems for concrete spaces of distributions and related operations. These preparations occupy Subsections 1.1 to 1.4.

Having established the basic extension theorem in Subsection 1.5, it is not too diŒcult to deøne point-wise multiplications, tensor products, and convolutions for vector-valued distributions. This is done in Subsections 1.6˘1.9. Finally, in the last subsection we introduce Laplace transforms for vector-valued distributions and prove the basic rules for dealing with them, as well as an important characterization theorem.

Although the proofs and techniques of this section are rather heavy, the ønal results are very simple to state and very satisfactory. Indeed, it can be said that, given the correct interpretation, all known rules for dealing with distributions carry over to the vector-valued setting.

In [Ama95, Remark III.4.2.1] we have already commented on the possibility of deøning a convolution for two vector-valued distributions and claimed that we will hardly have an occasion to use such a generality. In the meantime we became convinced, however, that the theory of distributions with values in Banach spaces is a very powerful tool with a high potential for nontrivial applications.

1.1 Locally Convex Topologies

For the reader's convenience we collect in this subsection some more advanced topics from the theory of LCSs that we shall need below. Hereafter, we often use these facts without further mention.

The Uniform Boundedness Principle

An LCS is barreled if each absolutely convex, closed, and absorbing subset is a neighborhood of zero. Every Frchet space, hence every Banach space, is barreled ([Hor66, III.6]). Every reÆexive LCS is barreled. In fact, an LCS is reÆexive iœ it is barreled and bounded subsets are relatively weakly compact.

Let E and F be LCSs. A set $\mathcal{A} \subset \mathcal{L}(E,F)$ is equicontinuous if for each neighborhood V of zero in F there exists a neighborhood U of zero in E such that $A(U) \subset V$ for all $A \in \mathcal{A}$. Equivalently, $\mathcal{A}$ is equicontinuous iœ to each continuous seminorm q on F there exists a continuous seminorm p on E such that

$$q(Ae) \le p(e) , \qquad e \in E , \quad A \in \mathcal{A} .$$

The set $\mathcal{A}$ is uniformly bounded if for each bounded subset B of E there exists a bounded set $C \subset F$ such that $A(B) \subset C$ for all $A \in \mathcal{A}$. It is easily seen that each equicontinuous set is uniformly bounded. Finally, $\mathcal{A}$ is point-wise bounded if $\mathcal{A}(e) := \{\, Ae \ ;\ A \in \mathcal{A} \,\}$ is bounded in F for each $e \in E$, that is, if $\mathcal{A}$ is bounded in $\mathcal{L}_s(E,F)$. Since $\mathcal{L}(E,F) \hookrightarrow \mathcal{L}_s(E,F)$, every bounded subset of $\mathcal{L}(E,F)$ is point-wise bounded. The UNIFORM BOUNDEDNESS PRINCIPLE asserts that each point-wise bounded subset of $\mathcal{L}(E,F)$ is equicontinuous, hence uniformly bounded, if E is barreled (e.g., [Jar81, Proposition 11.1.1] or [Sch71, Theorem III.4.2]). In particular, it follows that

$$\sup_{A\in\mathcal{A}} \|Ae\| < \infty \text{ for each } e \in E \quad \Longrightarrow \quad \sup_{A\in\mathcal{A}} \|A\| < \infty , \tag{1.1.1}$$

provided E and F are Banach spaces.

Suppose that E is barreled. Let (A_α) be a net in $\mathcal{L}(E,F)$ that is bounded in $\mathcal{L}_s(E,F)$, and assume that $Ae := \lim A_\alpha e$ exists in F for each $e \in E$. Then the BANACH-STEINHAUS THEOREM asserts that $A \in \mathcal{L}(E,F)$ and that $A_\alpha \to A$, uniformly on compact sets (e.g., [Jar81, Theorem 11.1.3]). If E and F are Banach spaces and (A_j) is a sequence in $\mathcal{L}(E,F)$ such that $(A_j e)$ converges in F for each $e \in E$, it follows that

$$\|A\| \le \liminf_{j\to\infty} \|A_j\| < \infty \tag{1.1.2}$$

(e.g., [Yos65, Corollary II.1.2]).

Hypocontinuity

Let G be a third LCS and let $b :\ E \times F \to G$ be bilinear. Clearly, b is separately continuous if $b(\cdot, f) \in \mathcal{L}(E,G)$ for each $f \in F$ and $b(e,\cdot) \in \mathcal{L}(F,G)$ for each $e \in E$. Of course, every continuous bilinear map is separately continuous. A separately continuous bilinear map is continuous if E and F are Frchet spaces (e.g., [Rud73, Theorem 2.1.7] or [Hor66, Theorem 4.7.1]).

The map b is hypocontinuous if $b(\cdot, f) :\ E \to G$ is continuous, uniformly with respect to f in bounded subsets of F, and $b(e,\cdot) :\ F \to G$ is continuous, uniformly with respect to e in bounded subsets of E. If b is hypocontinuous, it is separately continuous and $b(A,B)$ is bounded in G if A is bounded in E and B is bounded in F. Moreover, b is continuous on $A \times F$ and on $E \times B$, and uniformly continuous on $A \times B$ (e.g., [Hor66, Propositions 4.7.2 and 4.7.3] or [Sch71, III.5.3]). Consequently, every hypocontinuous bilinear map is sequentially continuous.

Every separately continuous bilinear map is hypocontinuous if E and F are barreled (e.g., [Hor66, Theorem 4.7.2] or [Sch71, III.5.2]). Of course, a continuous bilinear map is hypocontinuous.

The following lemma will be useful for proving the hypocontinuity of certain bilinear maps.

1.1.1 Lemma Suppose that F is barreled. Then the bilinear map

$$\mathcal{L}(E,F) \times \mathcal{L}(F,G) \to \mathcal{L}(E,G) \ , \quad (S,T) \mapsto TS$$

is hypocontinuous.

Proof Let q be a continuous seminorm on G and let B be a bounded subset of E. Let $\mathcal{T} \subset \mathcal{L}(F,G)$ be bounded. It follows from the uniform boundedness principle that there exists a continuous seminorm p on F such that

$$q(Tf) \le p(f) \ , \qquad f \in F \ , \quad T \in \mathcal{T} \ .$$

Consequently,

$$\sup_{e \in B} q(TSe) \le \sup_{e \in B} p(Se) \ , \qquad S \in \mathcal{L}(E,F) \ , \quad T \in \mathcal{T} \ ,$$

which shows that

$$(S \mapsto TS) \in \mathcal{L}\big(\mathcal{L}(E,F), \mathcal{L}(E,G)\big) \ ,$$

uniformly with respect to T in bounded subsets of $\mathcal{L}(F,G)$.

Let $\mathcal{S} \subset \mathcal{L}(E,F)$ be bounded. Then $\mathcal{S}$ is uniformly bounded. Hence there exists a bounded subset C of F such that $S(B) \subset C$ for $S \in \mathcal{S}$. Consequently,

$$\sup_{e \in B} q(TSe) \le \sup_{f \in C} q(Tf) \ , \qquad T \in \mathcal{L}(F,G) \ , \quad S \in \mathcal{S} \ ,$$

which shows that

$$(T \mapsto TS) \in \mathcal{L}\big(\mathcal{L}(F,G), \mathcal{L}(E,G)\big) \ ,$$

uniformly with respect to S in bounded subsets of $\mathcal{L}(E,F)$. ÷

Montel Spaces

Recall that an LCS is a Montel space if it is barreled and bounded subsets are relatively compact. Since the unit-ball of a Banach space is relatively compact iœ it is ønite-dimensional, there are no inønite-dimensional Banach-Montel spaces. Every Montel space is reÆexive (e.g., [Hor66, 3, 9]). The dual of a Montel space is a Montel space as well (e.g, [Hor66, Proposition 9 in Section 3, 9]).

Strict Inductive Limits

Let E be a vector space and let $\{ E_\alpha \ ; \ \alpha \in \mathsf{A} \}$ be a family of subspaces such that:

(i) Each E_α is a Frchet space.

(ii) If $E_\alpha \supset E_\beta$ then E_α induces the original topology on E_β.

(iii) There exists a coønal increasing sequence (E_n) in $\{ E_\alpha \ ; \ \alpha \in \mathsf{A} \}$, that is, $E_n \subset E_{n+1}$ and to each E_α there is an E_n with $E_n \supset E_\alpha$.

(iv) $\bigcup_\alpha E_\alpha = E$.

Then there exists a ønest locally convex Hausdorœ topology τ on E such that $E_\alpha \hookrightarrow E$ for each $\alpha \in \mathsf{A}$, the (strict) inductive limit topology or LF-topology induced by $\{ E_\alpha \ ; \ \alpha \in \mathsf{A} \}$. The LCS (E, τ) is denoted by

$$\varinjlim E_\alpha = \varinjlim_\alpha E_\alpha$$

and said to be an LF-space. Every LF-space is complete and barreled, and induces on each E_α its original topology. A subset B of E is bounded iœ $B \subset E_\alpha$ for some $\alpha \in \mathsf{A}$ and B is bounded in E_α. Let F be an LCS. Then a linear map $T : \ E \to F$ is continuous iœ $T \,|\, E_\alpha$ is continuous from E_α into F for each $\alpha \in \mathsf{A}$. This is the case iœ each $T \,|\, E_\alpha$ is bounded. Thus an LF-space E is bornological, that is, every bounded linear map from E into an LCS is continuous. Clearly, every Frchet space, hence every Banach space, is an LF-space (e.g., [Jar81, IV], [Hor66, II.12], [Sch71, II.6˘II.8]).

It follows from (iii) that our inductive limits are countable. Thus, in principle, we could have restricted ourselves to the consideration of sequences (E_k) only, instead of admitting possibly uncountable families $\{ E_\alpha \ ; \ \alpha \in \mathsf{A} \}$. However, the given formulation is well adapted to the concrete spaces we have in mind. In those cases, uncountable families of Frchet spaces occur naturally. Then we do not have to select a particular sequence and keep repeating that the topology is independent of that particular sequence.

Smooth Functions

Let X be a nonempty open subset of $\mathbb{R}^n$ and let E be a Banach space. Recall from [Ama95, Subsection III.4.1] that

$$\mathcal{E}(X, E) := \Big(C^\infty(X, E), \ \{ p_{m,K} \ ; \ m \in \mathbb{N}, \ K \subset\subset X \} \Big)$$

is a Frchet space, where the seminorms $p_{\ m,K}$ are deøned by

$$p_{m,K}(u) := \max_{|\alpha| \le m} \|\partial^\alpha u\|_{\infty,K} \ . \tag{1.1.3}$$

Test Functions

Given $K \subset\subset X$, let

$$\mathcal{D}_K(X,E) := \left\{ u \in \mathcal{E}(X,E) \; ; \; \operatorname{supp}(u) \subset \overline{K} \right\} . \tag{1.1.4}$$

Then $\mathcal{D}_K(X,E)$ is a closed linear subspace of $\mathcal{E}(X,E)$, and

$$\bigcup_{K \subset\subset X} \mathcal{D}_K(X,E) = \left\{ u \in \mathcal{E}(X,E) \; ; \; \operatorname{supp}(u) \subset\subset X \right\} .$$

Let

$$X_k := \left\{ x \in X \; ; \; \operatorname{dist}(x, X^c) > 1/k \right\} \cap k\mathbb{B}^n , \qquad k \in \mathbb{N} , \tag{1.1.5}$$

where $\operatorname{dist}(x, \emptyset) := \infty$. Then

$$X_k \subset\subset X_{k+1} , \qquad \bigcup_k X_k = X . \tag{1.1.6}$$

Hence $\left(\mathcal{D}_{X_k}(X,E)\right)$ is a coønal increasing sequence in $\left\{ \mathcal{D}_K(X,E) \; ; \; K \subset\subset X \right\}$. Thus

$$\mathcal{D}(X,E) := \varinjlim_{K \subset\subset X} \mathcal{D}_K(X,E) ,$$

the space of E-valued test functions, is an LF-space (cf. Subsection III.1.1 of [Ama95]).

Rapidly Decreasing Smooth Functions

Also recall from [Ama95, Subsection III.4.1] that the Schwartz space of smooth rapidly decreasing E-valued functions on $\mathbb{R}^n$ is deøned by

$$\mathcal{S}(\mathbb{R}^n, E) := \Big(\mathcal{S}(\mathbb{R}^n, E), \left\{ q_{k,m} \; ; \; k,m \in \mathbb{N} \right\} \Big) ,$$

where

$$q_{k,m}(u) := \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \le m}} (1+|x|^2)^k \, |\partial^\alpha u(x)| . \tag{1.1.7}$$

Slowly Increasing Smooth Functions

Finally, we recall that $\mathcal{O}_M(\mathbb{R}^n, E)$ is the space of slowly increasing smooth functions on $\mathbb{R}^n$. This means that $u \in \mathcal{O}_M(\mathbb{R}^n, E)$ iœ $u \in \mathcal{E}(\mathbb{R}^n, E)$ and, given $\alpha \in \mathbb{N}^n$, there exist $m_\alpha \in \mathbb{N}$ and $c_\alpha > 0$ such that

$$|\partial^\alpha u(x)| \le c_\alpha (1+|x|^2)^{m_\alpha} , \qquad x \in \mathbb{R}^n . \tag{1.1.8}$$

Moreover, $\mathcal{O}_M(\mathbb{R}^n, E)$ is given the topology induced by the family of seminorms[1]

$$u \mapsto \|\varphi \partial^\alpha u\|_\infty , \qquad \varphi \in \mathcal{S}(\mathbb{R}^n) , \quad \alpha \in \mathbb{N}^n , \tag{1.1.9}$$

so that it is an LCS.

[1] For a proof of the fact that these seminorms are well-deøned we refer to Proposition 1.6.1.

Spaces of Vector-Valued Distributions

To simplify the writing we agree to put

$$\mathbf{F}(X, E) := \mathbf{F}(\mathbb{R}^n, E) \qquad \text{if} \quad \mathbf{F} \in \{\mathcal{S}, \mathcal{O}_M\} \ . \tag{1.1.10}$$

Then, as usual,

$$\mathbf{F}(X) := \mathbf{F}(X, \mathbb{K}) \ , \qquad \mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\} \ ,$$

if no confusion seems likely, and

$$\mathbf{F}'(X, E) := \mathcal{L}\big(\mathbf{F}(X), E\big) \tag{1.1.11}$$

so that

$$\mathbf{F}'(X) = \mathbf{F}(X)'$$

for $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\}$. (Recall that $\mathcal{L}$ is always given the bounded convergence topology.)

If $u \in \mathcal{D}'(X, E)$ then, as a rule, we write $u(\varphi)$ for the value of u at $\varphi \in \mathcal{D}(X)$. However, if u is a scalar distribution, that is, $u \in \mathcal{D}'(X)$, then we continue to denote $u(\varphi)$ by $\langle u, \varphi\rangle$.

1.1.2 Theorem Let $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\}$. Then $\mathbf{F}(X)$ and $\mathbf{F}'(X)$ are complete Montel spaces, hence reÆexive.

Proof Every LF-space is complete and bornological. The dual of a bornological LCS is complete (e.g., [Sch71, IV.6.1]). Hence $\mathbf{F}(X)$ and $\mathbf{F}'(X)$ are complete for $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$. The fact that $\mathbf{F}(X)$ is a Montel space for $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$ is well-known (e.g., [Hor66, Examples 3, 4, and 6 in Section 3, 9]). Thus $\mathbf{F}\ '(X)$ is a Montel space in these cases as well. The assertion for $\mathbf{F} = \mathcal{O}_M$ follows from [Gro55, II.4.4]. ÷

1.2 Convolutions

In [Ama95, Subsection III.4.2] we have already stated the deønition of the convolution of a vector-valued distribution and a scalar test function, as well as some of its basic properties. In this subsection we present proofs for some of the less obvious properties. In addition, we extend the deønition to include convolutions of a vector-valued and a scalar distribution.

Convolutions of Distributions and Test Functions

Let $E := (E, |\cdot|)$ be a Banach space. Recall from [Ama95, Subsection III.4.2] that, given

$$(u, \varphi) \in \mathbf{F}'(\mathbb{R}^n, E) \times \mathbf{F}(\mathbb{R}^n) \ , \qquad \mathbf{F} \in \{\mathcal{D}, \mathcal{E}\} \ ,$$

the convolution, $u * \varphi$, of u and φ, is deøned by

$$u * \varphi(x) := u(\tau_x \check{\varphi}) \ , \qquad x \in \mathbb{R}^n \ . \tag{1.2.1}$$

As already noted, the usual scalar proof (e.g., [Hr83, Theorem 4.1.1]) carries over to the present situation to show that $u * \varphi \in \mathcal{E}(\mathbb{R}^n, E)$ and

$$\partial^\alpha (u * \varphi) = \partial^\alpha u * \varphi = u * \partial^\alpha \varphi \ , \qquad \alpha \in \mathbb{N}^n \ . \tag{1.2.2}$$

Moreover,

$$\operatorname{supp}(u * \varphi) \subset \operatorname{supp}(u) + \operatorname{supp}(\varphi) \tag{1.2.3}$$

so that

$$u * \varphi \in \mathcal{D}(\mathbb{R}^n, E) \qquad \text{if} \quad (u, \varphi) \in \mathcal{E}'(\mathbb{R}^n, E) \times \mathcal{D}(\mathbb{R}^n) \ . \tag{1.2.4}$$

It is obvious that convolution is bilinear. For the reader's convenience we prove now that it is hypocontinuous.

1.2.1 Proposition Convolution is a bilinear and hypocontinuous mapping:

(i) $\mathcal{D}'(\mathbb{R}^n, E) \times \mathcal{D}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n, E)$;

(ii) $\mathcal{E}'(\mathbb{R}^n, E) \times \mathcal{E}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n, E)$;

(iii) $\mathcal{E}'(\mathbb{R}^n, E) \times \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n, E)$.

Proof To simplify the notation we put $\mathcal{D} := \mathcal{D}(\mathbb{R}^n)$ and $\mathcal{E} := \mathcal{E}(\mathbb{R}^n)$.

(i) Let $K_r := r\mathbb{B}^n$ for $r > 0$. Then $u \in \mathcal{D}'(\mathbb{R}^n, E)$ if and only if $u \in \mathcal{L}(\mathcal{D}_{K_r}, E)$ for each $r > 0$.

Let A be a bounded subset of $\mathcal{D}'(\mathbb{R}^n, E)$. Given $r > 0$, the uniform boundedness principle implies the existence of $k \in \mathbb{N}$ such that

$$|u(\varphi)| \le c p_{k,K_r}(\varphi) \ , \qquad \varphi \in \mathcal{D}_{K_r} \ , \quad u \in A \ .$$

Observe that

$$x \in K_\rho \ , \quad \varphi \in \mathcal{D}_{K_r} \quad \Longrightarrow \quad \tau_x \check{\varphi} \in \mathcal{D}_{K_{r+\rho}} \tag{1.2.5}$$

and

$$p_{k,K_{r+\rho}}(\tau_x \check{\varphi}) = p_{k,x-K_r}(\tau_x \check{\varphi}) = p_{k,K_r}(\varphi) \tag{1.2.6}$$

for $\rho > 0$. Consequently,

$$p_{K_\rho}(u * \varphi) = \sup_{x \in K_\rho} |u(\tau_x \check{\varphi})| \le c p_{k,K_{r+\rho}}(\tau_x \check{\varphi}) = c p_{k,K_r}(\varphi) \tag{1.2.7}$$

for $\varphi \in \mathcal{D}_{K_r}$ and $u \in A$. Now we obtain from (1.2.2) that

$$p_{j,K_\rho}(u * \varphi) \le c p_{k+j,K_r}(\varphi) \ , \qquad \varphi \in \mathcal{D}_{K_r} \ , \quad u \in A \ , \quad j \in \mathbb{N} \ , \quad \rho > 0 \ .$$

This shows that

$$(\varphi \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{D}_{K_r}, \mathcal{E}(\mathbb{R}^n, E)\big) \ , \qquad u \in A \ , \quad r > 0 \ . \tag{1.2.8}$$

Hence $(\varphi \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{D}, \mathcal{E}(\mathbb{R}^n, E)\big)$, uniformly with respect to u in bounded subsets of $\mathcal{D}'(\mathbb{R}^n, E)$.

Now let B be a bounded subset of $\mathcal{D}$. Then there exists $r > 0$ such that B is contained and bounded in $\mathcal{D}_{K_r}$. It follows from (1.2.7) and (1.2.2) that

$$p_{j,K_\rho}(u * \varphi) = \max_{|\alpha|\le j} \sup_{x\in K_\rho} |u * \partial^\alpha \varphi(x)| \le c \sup_{\psi\in C} |u(\psi)| \ , \tag{1.2.9}$$

where

$$C := \left\{ \tau_x(\partial^\alpha \varphi)\check{} \ ; \ x \in K_\rho,\ |\alpha| \le j,\ \varphi \in B \right\}$$

is a bounded subset of $\mathcal{D}$, thanks to (1.2.5) and (1.2.6). Consequently,

$$(u \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{D}'(\mathbb{R}^n, E), \mathcal{E}(\mathbb{R}^n, E)\big) \ ,$$

uniformly with respect to φ in bounded subsets of $\mathcal{D}$. This proves the hypocontinuity of (i).

(ii) Let A be a bounded subset of $\mathcal{E}'(\mathbb{R}^n, E) = \mathcal{L}(\mathcal{E}, E)$. Then the uniform boundedness principle implies the existence of $r > 0$ and $k \in \mathbb{N}$ such that

$$|u(\varphi)| \le c p_{k,K_r}(\varphi) \ , \qquad \varphi \in \mathcal{E} \ , \quad u \in A \ . \tag{1.2.10}$$

Thus, given $\rho > 0$ and $x \in K_\rho$,

$$|u(\tau_x \check{\varphi})| \le c p_{k,K_r}(\tau_x \check{\varphi}) \le c p_{k,K_r+\rho}(\varphi) \ , \qquad \varphi \in \mathcal{E} \ , \quad u \in A \ .$$

Hence, by (1.2.2), given $\rho > 0$,

$$p_{j,K_\rho}(u * \varphi) \le p_{k+j,K_{r+\rho}}(\varphi) \ , \qquad j \in \mathbb{N} \ , \quad \varphi \in \mathcal{E} \ , \quad u \in A \ ,$$

which shows that $(\varphi \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{E}, \mathcal{E}(\mathbb{R}^n, E)\big)$, uniformly with respect to u in bounded subsets of $\mathcal{E}'(\mathbb{R}^n, E)$.

If B is a bounded subset of $\mathcal{E}$, estimate (1.2.9) is valid, where C is now a bounded subset of $\mathcal{E}$. Hence $(u \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{E}'(\mathbb{R}^n, E), \mathcal{E}(\mathbb{R}^n, E)\big)$, uniformly with respect to φ in bounded subsets of $\mathcal{E}$. Thus (ii) is hypocontinuous as well.

(iii) Suppose that A is a bounded subset of $\mathcal{E}'(\mathbb{R}^n, E)$. Then we infer from (1.2.10) that $u(\varphi) = 0$ for $\varphi \in \mathcal{D}(\mathbb{R}^n \backslash K_r)$ and $u \in A$. Thus $\mathrm{supp}(u) \subset K_r$ for $u \in A$. Hence we deduce from (1.2.3) and (1.2.8) that

$$(\varphi \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{D}_{K_\rho}, \mathcal{D}_{K_{r+\rho}}(\mathbb{R}^n, E)\big) \ , \qquad u \in A \ , \quad \rho > 0 \ .$$

Since $\mathcal{D}_{K_{r+\rho}}(\mathbb{R}^n, E) \hookrightarrow \mathcal{D}(\mathbb{R}^n, E)$, we see that

$$(\varphi \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{D}_{K_\rho}, \mathcal{D}(\mathbb{R}^n, E)\big) \ , \qquad u \in A \ , \quad \rho > 0 \ .$$

Consequently, $(\varphi \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{D}, \mathcal{D}(\mathbb{R}^n, E)\big)$, uniformly for u in bounded subsets of $\mathcal{E}'(\mathbb{R}^n, E)$.

If B is a bounded subset of $\mathcal{D}$, then we deduce from (1.2.9) and (1.2.3) that

$$(u \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{E}'(\mathbb{R}^n, E), \mathcal{D}(\mathbb{R}^n, E)\big) ,$$

uniformly with respect to $\varphi \in B$. Hence (iii) is also hypocontinuous. ÷

Translation-Invariant Operators

It is easy to verify that

$$u(\varphi) = u * \check{\varphi}(0) = \check{u} * \varphi(0) \tag{1.2.11}$$

and

$$(u * \varphi)\check{} = \check{u} * \check{\varphi} \tag{1.2.12}$$

for $u \in \mathbf{F}'(\mathbb{R}^n, E)$ and $\varphi \in \mathbf{F}(\mathbb{R}^n)$, where $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}\}$.

The following theorem gives an important characterization of convolutions.

1.2.2 Theorem Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}\}$ and $T \in \mathcal{L}\big(\mathbf{F}(\mathbb{R}^n), C(\mathbb{R}^n, E)\big)$ and that T commutes with translations:

$$T(\tau_x \varphi) = \tau_x T(\varphi) , \qquad x \in \mathbb{R}^n , \quad \varphi \in \mathbf{F}(\mathbb{R}^n) .$$

Then there exists a unique $u \in \mathbf{F}'(\mathbb{R}^n, E)$ such that

$$T(\varphi) = u * \varphi , \qquad \varphi \in \mathbf{F}(X) .$$

Proof It is trivial that reÆection is a toplinear automorphism of $\mathbf{F}(\mathbb{R}^n)$. Thus the continuity hypothesis implies that

$$u := \big[\varphi \mapsto (T\check{\varphi})(0)\big] \in \mathbf{F}'(\mathbb{R}^n, E) .$$

Consequently, we infer from the commutativity hypothesis that

$$(T\varphi)(x) = \tau_{-x}(T\varphi)(0) = T(\tau_{-x}\varphi)(0) = u\big((\tau_{-x}\varphi)\check{}\big) = u(\tau_x\check{\varphi}) = (u * \varphi)(x)$$

for $x \in \mathbb{R}^n$; thus $T\varphi = u * \varphi$ for $\varphi \in \mathbf{F}(X)$. ÷

Convolutions of Two Distributions

Now suppose that $u \in \mathcal{D}'(\mathbb{R}^n, E)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$ and u or v has compact support. Then

$$(\varphi \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{D}(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n, E)\big)$$

and

$$(\varphi \mapsto v * \varphi) \in \mathcal{L}\big(\mathcal{D}(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)\big) \qquad \text{if} \quad \operatorname{supp}(v) \subset\subset \mathbb{R}^n$$

by Proposition 1.2.1. The same arguments show that

$$\big(\varphi \mapsto u * (v * \varphi)\big) \in \mathcal{L}\big(\mathcal{D}(\mathbb{R}^n), \mathcal{E}(\mathbb{R}^n, E)\big) \ .$$

Thus (III.4.2.15) of [Ama95] and Theorem 1.2.2 guarantee the existence of a unique distribution $u * v \in \mathcal{D}'(\mathbb{R}^n, E)$, the convolution of u and v, such that

$$(u * v) * \varphi = u * (v * \varphi) \ , \qquad \varphi \in \mathcal{D}(\mathbb{R}^n) \ . \tag{1.2.13}$$

It is clear that convolution is bilinear. The next proposition shows that it is hypocontinuous as well.

1.2.3 Proposition Convolution is a bilinear and hypocontinuous mapping:
(i) $\mathcal{E}'(\mathbb{R}^n, E) \times \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n, E)$;
(ii) $\mathcal{D}'(\mathbb{R}^n, E) \times \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n, E)$;
(iii) $\mathcal{E}'(\mathbb{R}^n, E) \times \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n, E)$.

Proof (i) Let A and B be bounded subsets of $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D} := \mathcal{D}(\mathbb{R}^n)$, respectively. Since reÆection is obviously a toplinear automorphism of $\mathcal{D}$ and of $\mathcal{D}'$, it follows from Proposition 1.2.1 and the boundedness properties of hypocontinuous maps that

$$C := \big\{ \, (u * \check{\varphi})\check{} \ ; \ u \in A, \ \varphi \in B \, \big\}$$

is a bounded subset of $\mathcal{E}$. Hence, given $v \in \mathcal{E}'(\mathbb{R}^n, E)$,

$$\begin{aligned} \sup_{\varphi \in B} |v * u(\varphi)| = \sup_{\varphi \in B} |(v * u) * \check{\varphi}(0)| &= \sup_{\varphi \in B} |v * (u * \check{\varphi})(0)| \\ &= \sup_{\varphi \in B} \big| v\big((u * \check{\varphi})\check{}\,\big)\big| = \sup_{\psi \in C} |v(\psi)| \ . \end{aligned}$$

This shows that $(v \mapsto v * u) \in \mathcal{L}\big(\mathcal{E}'(\mathbb{R}^n, E), \mathcal{D}'(\mathbb{R}^n, E)\big)$, uniformly with respect to u in bounded subsets of $\mathcal{D}'$.

Let (u_α) be a net in $\mathcal{D}'$ converging to zero. Then $u_\alpha * \check{\varphi} \to 0$ in $\mathcal{E}$, uniformly with respect to φ in bounded subsets of $\mathcal{D}$, thanks to Proposition 1.2.1(i). Consequently, Proposition 1.2.1(ii) guarantees that $v * (u_\alpha * \check{\varphi}) \to 0$ in $\mathcal{E}(\mathbb{R}^n, E)$, uniformly with respect to φ in bounded subsets of $\mathcal{D}$ and to v in bounded subsets of $\mathcal{E}'(\mathbb{R}^n, E)$. Hence $(v * u_\alpha)(\varphi) = (v * u_\alpha) * \check{\varphi}(0) \to 0$ in E, uniformly with respect to φ in bounded subsets of $\mathcal{D}$ and to v in bounded subsets of $\mathcal{E}'(\mathbb{R}^n, E)$. This shows that $(u \mapsto v * u) \in \mathcal{L}\big(\mathcal{D}', \mathcal{D}'(\mathbb{R}^n, E)\big)$, uniformly with respect to v in bounded subsets of $\mathcal{E}'(\mathbb{R}^n, E)$. Hence (i) is hypocontinuous.

The hypocontinuity of (ii) and (iii) follows by modifying the above arguments in the obvious way. ÷

Elementary Properties of Convolutions

1.2.4 Remarks (a) Let $\delta \in \mathcal{E}'(\mathbb{R}^n)$ be the Dirac distribution, that is,

$$\langle \delta, \varphi \rangle := \varphi(0) \ , \qquad \varphi \in \mathcal{E}(\mathbb{R}^n) \ .$$

Then $u * \delta = u$ for $u \in \mathcal{D}'(\mathbb{R}^n, E)$.

Proof From (1.2.1) it is obvious that $\delta * \varphi = \varphi$ for $\varphi \in \mathcal{E}(\mathbb{R}^n)$. Hence

$$(u * \delta) * \varphi = u * (\delta * \varphi) = u * \varphi \ , \qquad \varphi \in \mathcal{D} \ ,$$

by (1.2.13). ÷

(b) Suppose that $u \in \mathcal{D}'(\mathbb{R}^n, E)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$ and that u or v has compact support. Then

$$\partial^{\alpha+\beta}(u * v) = \partial^\alpha u * \partial^\beta v \ , \qquad \alpha, \beta \in \mathbb{N}^n \ .$$

Proof By applying repeatedly (1.2.2) we see that

$$\begin{aligned}\partial^{\alpha+\beta}(u * v) * \varphi &= (u * v) * \partial^{\alpha+\beta}\varphi = u * (v * \partial^{\alpha+\beta}\varphi) \\ &= u * \partial^\alpha(\partial^\beta v * \varphi) = \partial^\alpha u * (\partial^\beta v * \varphi) = (\partial^\alpha u * \partial^\beta v) * \varphi\end{aligned}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$. ÷

(c) Let $u \in \mathcal{D}'(\mathbb{R}^n, E)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$ such that u or v has compact support. Then

$$\tau_a(u * v) = \tau_a u * v = u * \tau_a v \ , \qquad a \in \mathbb{R}^n \ ,$$

and

$$(u * v)\check{} = \check{u} * \check{v} \ .$$

Proof This follows easily from (1.2.13) and (1.2.12), or (III.4.2.15) of [Ama95], respectively.

(d) If $\{\, \varphi_\varepsilon \ ; \ \varepsilon > 0 \,\}$ is a molliøer then $\varphi_\varepsilon \to \delta$ in $\mathcal{E}'(\mathbb{R}^n)$ as $\varepsilon \to 0$. Consequently, $\varphi_\varepsilon * u \to u$ in $\mathcal{D}'(\mathbb{R}^n, E)$ as $\varepsilon \to 0$ for $u \in \mathcal{D}'(\mathbb{R}^n, E)$.

Proof Given $\psi \in \mathcal{E}(\mathbb{R}^n)$,

$$\langle \varphi_\varepsilon - \delta, \psi \rangle = \int_{\mathcal{R}^n} \varphi_\varepsilon(x)\psi(x)\, dx - \psi(0) = \int_{\mathcal{R}^n} \varphi(y)\big(\psi(\varepsilon y) - \psi(0)\big)\, dy \ .$$

Hence, by the mean-value theorem,

$$|\langle \varphi_\varepsilon - \delta, \psi \rangle| \le \sup_{|y| \le 1} |\psi(\varepsilon y) - \psi(0)| \le \varepsilon \sup_{|y| \le 1} |\partial\psi(y)| \le \varepsilon p_{1,\mathcal{B}^n}(\psi)$$

for $0 < \varepsilon \le 1$. This shows that $\langle \varphi_\varepsilon - \delta, \psi \rangle \to 0$ as $\varepsilon \to 0$, uniformly with respect to ψ in bounded subsets of $\mathcal{E}(\mathbb{R}^n)$, that is, $\varphi_\varepsilon \to \delta$ in $\mathcal{E}'(\mathbb{R}^n)$ as $\varepsilon \to 0$. The second part of the assertion now follows from Proposition 1.2.3 and (a). ÷

Translation Groups on $\mathcal{S}$ and $\mathcal{S}'$

Next we turn to the case of temperate distributions. For this we need some preparation.

1.2.5 Lemma The translation group acts strongly continuously on $\mathcal{S}(\mathbb{R}^n, E)$ and $\mathcal{S}'(\mathbb{R}^n, E)$.

Proof Given $\varphi \in \mathcal{S}(\mathbb{R}^n, E)$ and $k, m \in \mathbb{N}$, we ønd

$$q_{k,m}(\tau_x \varphi) = \sup_{\substack{y \in \mathbb{R}^n \\ |\alpha| \le m}} (1 + |x+y|^2)^k |\partial^\alpha \varphi(y)| \le 2^k (1 + |x|^2)^k q_{k,m}(\varphi) \tag{1.2.14}$$

for $x \in \mathbb{R}^n$, thanks to the trivial inequality $1 + |x+y|^2 \le 2(1+|x|^2)(1+|y|^2)$. Hence $\tau_x \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E)\big)$. Thus, by the mean-value theorem,

$$q_{k,m}(\tau_x \varphi - \varphi) = q_{k,m}\Big(\int_0^1 \sum_{j=1}^n x_j \tau_{tx} \partial_j \varphi \, dt\Big) \le 2^k (1+|x|^2)^k \sum_{j=1}^n |x_j| \, q_{k,m}(\partial_j \varphi) \ .$$

Consequently, $\tau_x \varphi \to \varphi$ in $\mathcal{S}(\mathbb{R}^n, E)$ as $x \to 0$. Since $x \mapsto \tau_x$ is bounded on bounded sets, as follows from (1.2.14), we see that $(x \mapsto \tau_x) \in C\big(\mathbb{R}^n, \mathcal{L}_s(\mathcal{S}(\mathbb{R}^n, E))\big)$.

We deduce from the Banach-Steinhaus theorem and the fact that $\mathcal{S}(\mathbb{R}^n)$ is a Montel space that

$$(x \mapsto \tau_x) \in C\big(\mathbb{R}^n, \mathcal{L}(\mathcal{S}(\mathbb{R}^n))\big) \ . \tag{1.2.15}$$

Now $(\tau_x u)(\varphi) = u(\tau_{-x}\varphi)$ for $u \in \mathcal{S}'(\mathbb{R}^n, E)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, and (1.2.15) imply that the translation group is also strongly continuous on $\mathcal{S}'(\mathbb{R}^n, E)$. ÷

Convolutions of Temperate Distributions

It is clear that $(\varphi \mapsto \check{\varphi}) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E)\big)$. Thus, given $u \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we can again deøne the convolution, $u * \varphi$, by (1.2.1), since the right-hand side deønes a continuous E-valued function of $x \in \mathbb{R}^n$ by Lemma 1.2.5.

In order to prove continuity properties of this convolution map we need the following technical result.

1.2.6 Lemma $\mathcal{O}_M(\mathbb{R}^n, E)$ is a complete LCS.

Proof Let (a_β) be a Cauchy net in $\mathcal{O}_M(\mathbb{R}^n, E)$. From $\mathcal{O}_M(\mathbb{R}^n, E) \hookrightarrow \mathcal{E}(\mathbb{R}^n, E)$ it follows that (a_β) is a Cauchy net in $\mathcal{E}(\mathbb{R}^n, E)$. Since the latter space is complete, there exists $a \in \mathcal{E}(\mathbb{R}^n, E)$ such that $a_\beta \to a$ in $\mathcal{E}(\mathbb{R}^n, E)$. Given $\alpha \in \mathbb{N}^n$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $\varepsilon > 0$, there exists β_0 such that

$$\|\varphi \partial^\alpha a_\beta - \varphi \partial^\alpha a_\gamma\|_\infty < \varepsilon \ , \qquad \beta, \gamma \ge \beta_0 \ .$$

Thus, since $a_\gamma \to a$ in $\mathcal{E}(\mathbb{R}^n, E)$,

$$\|\varphi \partial^\alpha a_\beta - \varphi \partial^\alpha a\|_\infty \le \varepsilon \ , \qquad \beta \ge \beta_0 \ .$$

Similarly,

$$\|\varphi \partial^\alpha a_\gamma\|_\infty \le \|\varphi \partial^\alpha a_{\beta_0}\|_\infty + \varepsilon \ , \qquad \gamma \ge \beta_0 \ ,$$

implies $\|\varphi \partial^\alpha a\|_\infty < \infty$. Hence $a \in \mathcal{O}_M(\mathbb{R}^n, E)$ and $a_\beta \to a$ in $\mathcal{O}_M(\mathbb{R}^n, E)$. ÷

1.2.7 Proposition Convolution is a bilinear and

(i) continuous map: $\mathcal{S}(\mathbb{R}^n, E) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n, E)$;

(ii) hypocontinuous map: $\mathcal{S}'(\mathbb{R}^n, E) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{O}_M(\mathbb{R}^n, E)$.

Proof (i) Given $u \in \mathcal{S}(\mathbb{R}^n, E)$, it follows from $\partial^\alpha u \in BUC(\mathbb{R}^n, E)$ for $\alpha \in \mathbb{N}^n$ and from [Ama95, (III.4.2.10), (III.4.2.19)] that $u * v \in \mathcal{E}(\mathbb{R}^n, E)$ for $v \in \mathcal{S}(\mathbb{R}^n) \subset L_1$. Note that

$$|x|^{2\ell} \le (|x-y| + |y|)^{2\ell} = \sum_{j=0}^{2\ell} \binom{2\ell}{j} |x-y|^j \, |y|^{2\ell-j}$$

implies

$$\begin{aligned}|x|^{2\ell} \, |u * v(x)| &\le \sum_{j=0}^{2\ell} \binom{2\ell}{j} \int |x-y|^j \, |u(x-y)| \, |y|^{2\ell-j} \, (1+|y|^2)^n \, |v(y)| \, \frac{dy}{(1+|y|^2)^n} \\ &\le c_\ell q_{\ell,0}(u) q_{\ell+n,0}(v)\end{aligned}$$

for $u, v \in \mathcal{S}(\mathbb{R}^n)$. This proves (i).

(ii) If $u \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then $u * \varphi \in \mathcal{E}(\mathbb{R}^n, E)$ by Proposition 1.2.1. Let U be a bounded subset of $\mathcal{S}'(\mathbb{R}^n, E)$. Then there exist $k, m \in \mathbb{N}$ with

$$|u(\varphi)| \le c q_{k,m}(\varphi) \ , \qquad u \in U \ , \quad \varphi \in \mathcal{S}(\mathbb{R}^n) \ .$$

Thus, for $\alpha \in \mathbb{N}^n$, we deduce from (1.2.14) that

$$\begin{aligned}|\partial^\alpha (u * \varphi)(x)| = |(u * \partial^\alpha \varphi)(x)| &= \bigl|u\bigl(\tau_x (\partial^\alpha \varphi)\check{}\bigr)\bigr| \\ &\le c q_{k,m}\bigl(\tau_x(\partial^\alpha \varphi)\check{}\bigr) \le c(1+|x|^2)^k q_{k,m+|\alpha|}(\varphi)\end{aligned}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $u \in U$. Consequently, given a bounded subset B of $\mathcal{S}(\mathbb{R}^n)$,

$$\sup_{\psi \in B} \|\psi \partial^\alpha (u * \varphi)\|_\infty \le c_{k,m} \sup_{\psi \in B} q_{k,0}(\psi) q_{k,m+|\alpha|}(\varphi)$$

for $u \in U$ and $\varphi \in B \cap \mathcal{D}(\mathbb{R}^n)$. This shows that, for $u \in \mathcal{S}'(\mathbb{R}^n, E)$, the linear map

$$\mathcal{D}(\mathbb{R}^n) \to \mathcal{O}_M(\mathbb{R}^n, E) \ , \quad \varphi \mapsto u * \varphi \ ,$$

is continuous with respect to the topology induced on $\mathcal{D}(\mathbb{R}^n)$ by $\mathcal{S}(\mathbb{R}^n)$, uniformly with respect to u in bounded subsets of $\mathcal{S}'(\mathbb{R}^n, E)$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n, E)$ is complete, it follows that

$$(\varphi \mapsto u * \varphi) \in \mathcal{L}\bigl(\mathcal{S}(\mathbb{R}^n), \mathcal{O}_M(\mathbb{R}^n, E)\bigr) \ ,$$

uniformly with respect to u in bounded subsets of $\mathcal{S}'(\mathbb{R}^n, E)$.

Now let B and C be bounded subsets of $\mathcal{S}(\mathbb{R}^n)$ and observe that, given $\alpha \in \mathbb{N}^n$, the image of the map

$$\mathbb{R}^n \times B \times C \to \mathcal{S}(\mathbb{R}^n)\ , \quad (x, \psi, \varphi) \mapsto \psi(x)\tau_x(\partial^\alpha \varphi)\check{}$$

is contained in a bounded subset D of $\mathcal{S}(\mathbb{R}^n)$, since (1.2.14) implies

$$q_{k,m}\big(\psi(x)\tau_x(\partial^\alpha\varphi)\check{}\,\big) \le c_{k,m} q_{k,m}(\psi) q_{k,m+|\alpha|}(\varphi)\ , \qquad x \in \mathbb{R}^n\ , \quad k \in \mathbb{N}\ .$$

Thus

$$\sup_{\psi \in B} \|\psi \partial^\alpha (u * \varphi)\|_\infty = \sup_{\psi \in B} \sup_{x \in \mathcal{R}^n} \big|u\big(\psi(x)\tau_x(\partial^\alpha\varphi)\check{}\,\big)\big| \le \sup_{\chi \in D} |u(\chi)|\ ,$$

which shows that

$$(u \mapsto u * \varphi) \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E), \mathcal{O}_M(\mathbb{R}^n, E)\big)\ ,$$

uniformly with respect to φ in bounded subsets of $\mathcal{S}(\mathbb{R}^n)$. This proves the asserted hypocontinuity. ÷

1.3 Approximations

It is the main purpose of this subsection to show that tensor products of the form $\mathcal{D}(X) \otimes E$ are dense in $\mathbf{F}(X, E)$, where $\mathbf{F}$ stands for one of the letters $\mathcal{D}$, $\mathcal{E}$, $\mathcal{S}$, $\mathcal{O}_M$, $\mathcal{D}'$, $\mathcal{E}'$, and $\mathcal{S}'$. This is a very useful approximation result that allows to reduce many results for vector-valued distributions to the corresponding scalar versions. It will be of particular importance in later subsections for extending the operations of point-wise multiplication or of convolution to the case of two vector-valued factors.

Multiplications

Let E and E_j, $j = 0, 1, 2$, be Banach spaces. If no confusion seems likely, we denote the norms in these spaces simply by $|\cdot|$. We also suppose that

$$E_1 \times E_2 \to E_0\ , \quad (x_1, x_2) \mapsto x_1 \bullet x_2 \tag{1.3.1}$$

is a multiplication. Recall that this means that (1.3.1) is a continuous bilinear map of norm at most 1.

1.3.1 Examples The following maps are multiplications:

(a) Ordinary multiplication in a Banach algebra.

(b) Multiplication with scalars: $\mathbb{K} \times E \to E$, $(\alpha, x) \mapsto \alpha x$.

(c) The duality pairing $E' \times E \to \mathbb{K}$, $(x', x) \mapsto \langle x', x\rangle$.

(d) The evaluation map $\mathcal{L}(E_1, E_0) \times E_1 \to E_0$, $(A, x) \mapsto Ax$.

(e) Compositions $\mathcal{L}(E_1, E_2) \times \mathcal{L}(E_0, E_1) \to \mathcal{L}(E_0, E_2)$, $(S, T) \mapsto ST$.

(f) Convolution in each one of the cases (III.4.2.18)ˇ(III.4.2.22) of [Ama95].

(g) If $b \in \mathcal{L}(E_1, E_2; E_0)$ and $b \neq 0$ then

$$E_1 \times E_2 \to E_0 \ , \quad (x_1, x_2) \mapsto \frac{1}{\|b\|} b(x_1, x_2)$$

is a multiplication. This shows that it is no restriction assuming that the norm of a multiplication is bounded by 1. ÷

Leibniz' Rule

We also recall that, given any nonempty set S, point-wise multiplication

$$E_1^S \times E_2^S \to E_0^S \ , \quad (a_1, a_2) \mapsto a_1 \bullet a_2$$

induced by (1.3.1) is deøned by

$$a_1 \bullet a_2(s) := a_1(s) \bullet a_2(s) \ , \qquad s \in S \ .$$

1.3.2 Lemma Let $p \in \mathbb{K}[X_1, \ldots, X_n]$ be a polynomial of degree at most k in n indeterminates and let $X \subset \mathbb{R}^n$ be a nonempty open subset of $\mathbb{R}^n$. Then, putting

$$p^{(\beta)} := \partial^\beta p \ , \qquad \beta \in \mathbb{N}^n \ ,$$

the generalized Leibniz rule

$$p(\partial)(a_1 \bullet a_2) = \sum_\beta \frac{1}{\beta!} (\partial^\beta a_1) \bullet p^{(\beta)}(\partial) a_2 \ ,$$

holds for $a_j \in C^k(X, E_j)$, $j = 1, 2$.

Proof From the obvious 'product rule'

$$\partial_j (a_1 \bullet a_2) = \partial_j a_1 \bullet a_2 + a_1 \bullet \partial_j a_2 \ , \qquad 1 \leq j \leq n \ , \tag{1.3.2}$$

we deduce by induction that

$$p(\partial)(a_1 \bullet a_2) = \sum_\beta \partial^\beta a_1 \bullet q_\beta(\partial) a_2 \ , \qquad a_j \in C^k(X, E_j) \ , \quad j = 1, 2 \ , \tag{1.3.3}$$

where $q_\beta \in \mathbb{K}[X_1, \ldots, X_n]$ and $q_\beta = 0$ for $|\beta| > k$. Given $y_j \in E_j$ and $\xi, \eta \in \mathbb{R}^n$, we put $a_1 := e^{\langle \xi, \cdot \rangle} y_1$ and $a_2 := e^{\langle \eta, \cdot \rangle} y_2$. Since

$$e^{-\langle \zeta, \cdot \rangle} q(\partial) e^{\langle \zeta, \cdot \rangle} = q(\zeta) \ , \qquad \zeta \in \mathbb{R}^n \ , \quad q \in \mathbb{K}[X_1, \ldots, X_n] \ , \tag{1.3.4}$$

it follows from (1.3.3) that

$$p(\xi+\eta)y_1 \bullet y_2 = \sum_\beta \xi^\beta q_\beta(\eta) y_1 \bullet y_2 \ .$$

Since this is true for every choice of y_1 and y_2,

$$p(\xi+\eta) = \sum_\beta \xi^\beta q_\beta(\eta) \ , \qquad \xi, \eta \in \mathbb{R}^n \ .$$

By diœerentiating this identity with respect to ξ and putting $\xi = 0$ it follows that $\partial^\beta p(\eta) = \beta!\, q_\beta(\eta)$. This proves the assertion. ÷

Letting $p(\xi) := \xi^\alpha$ in Lemma 1.3.2, we obtain the standard Leibniz rule:

$$\partial^\alpha(a_1 \bullet a_2) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta a_1 \bullet \partial^{\alpha-\beta} a_2 \ , \qquad a_j \in C^{|\alpha|}(X, E_j) \ , \quad j = 1, 2 \ . \quad (1.3.5)$$

Approximation by Test Functions

After these preparations we can prove a useful approximation theorem for vector-valued distributions. For this we recall convention (1.1.10).

1.3.3 Proposition Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$ and $u \in \mathbf{F}'(X, E)$. Then there exists a sequence (u_j) in $\mathcal{D}(X, E)$ such that $u_j \to u$ in $\mathbf{F}'(X, E)$.

Proof First suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}\}$. Denote by (X_k) a sequence of nonempty relatively compact open subsets of $\mathbb{R}^n$ such that $\overline{X}_k \subset X_{k+1}$ and $\bigcup X_k = X$. For each $k \in \mathbb{N}$ øx $\chi_k \in \mathcal{D}(X)$ with $\chi_k \,|\, X_k = 1$. Then $\chi_k u \in \mathcal{E}'(X, E) \subset \mathcal{E}'(\mathbb{R}^n, E)$. Denote by $\{\, \psi_\varepsilon \ ; \ \varepsilon > 0 \,\}$ a molliøer such that ψ_1 is even and put

$$u_k := (\chi_k u) * \psi_{1/k} \ , \qquad k \in \mathbb{N} \ .$$

Then $u_k \in \mathcal{D}(X, E)$ by Proposition 1.2.1. Given $\varphi \in \mathcal{D}(X)$,

$$\begin{aligned} u_k(\varphi) &= \big((\chi_k u) * \psi_{1/k}\big) * \check{\varphi}(0) = (\chi_k u) * (\psi_{1/k} * \check{\varphi})(0) \\ &= \chi_k u * (\check{\psi}_{1/k} * \varphi)\check{}\,(0) = u\big(\chi_k(\psi_{1/k} * \varphi)\big) \ , \end{aligned}$$

where we used (1.2.11), (1.2.12), and the evenness of $\psi_{1/k}$. Since $\mathrm{supp}(\varphi) \subset X_j$ for some $j \in \mathbb{N}$, it follows from [Ama95, (III.4.2.10) and (III.4.2.25)] that $\psi_{1/k} * \varphi \to \varphi$ in $\mathcal{D}(X)$. Also $\chi_k \to 1$ in $\mathcal{E}(X)$. Consequently,

$$\chi_k(\psi_{1/k} * \varphi) \to \varphi \quad \text{in } \mathcal{D}(X) \qquad \text{if } \varphi \in \mathcal{D}(X) \ , \qquad (1.3.6)$$

so that $u_k(\varphi) \to u(\varphi)$ in E. This means that

$$u_k \to u \quad \text{in } \mathcal{L}_s\big(\mathcal{D}(X), E\big) \ . \qquad (1.3.7)$$

Next we choose $\varphi \in \mathcal{E}(X)$. Then the estimate

$$\begin{aligned}|\partial^\alpha(\psi_\varepsilon * \varphi)(x) - \partial^\alpha\varphi(x)| &= |\psi_\varepsilon * \partial^\alpha\varphi(x) - \partial^\alpha\varphi(x)| \\ &= \Big|\int \psi_\varepsilon(y)\big[\partial^\alpha\varphi(x-y) - \partial^\alpha\varphi(x)\big]\,dy\Big| \\ &= \Big|\int \psi(y)\big[\partial^\alpha\varphi(x-\varepsilon y) - \partial^\alpha\varphi(x)\big]\,dy\Big| \\ &\le \sup_{|y|<1} |\partial^\alpha\varphi(x-\varepsilon y) - \partial^\alpha\varphi(x)|\end{aligned} \tag{1.3.8}$$

for $x \in X$ and $0 < \varepsilon < \operatorname{dist}(x, \partial X)$ shows that $\psi_{1/k} * \varphi \to \varphi$ in $\mathcal{E}(X)$. Thus

$$\chi_k(\psi_{1/k} * \varphi) \to \varphi \quad \text{in } \mathcal{E}(X) \qquad \text{if } \varphi \in \mathcal{E}(X)\ , \tag{1.3.9}$$

so that $u_k(\varphi) \to u(\varphi)$ in E for $u \in \mathcal{E}'(X, E)$. Consequently,

$$u_k \to u \quad \text{in } \mathcal{L}_s\big(\mathcal{E}(X), E\big) \qquad \text{if } u \in \mathcal{E}'(X, E)\ . \tag{1.3.10}$$

Now suppose that $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Choose any sequence (χ_k) in $\mathcal{D}(\mathbb{R}^n)$ satisfying $\chi_k\,|\,(k\bar{\mathbb{B}}^n) = 1$ and $\operatorname{supp}\chi_k \subset (k+1)\mathbb{B}^n$ such that

$$\sup_k \|\partial^\alpha\chi_k\|_\infty < \infty\ , \qquad \alpha \in \mathbb{N}^n\ . \tag{1.3.11}$$

For example, put

$$\chi_k := \psi_\varepsilon * \chi_{(k+1/2)\bar{\mathbb{B}}^n}\ , \qquad k \in \mathbb{N}\ ,$$

for a øxed $\varepsilon \in (0, 1/4)$. Then, using Leibniz' rule and (1.3.8), it is easily seen that

$$\chi_k(\psi_{1/k} * \varphi) \to \varphi \quad \text{in } \mathcal{S}(\mathbb{R}^n)\ . \tag{1.3.12}$$

Consequently, $u_k(\varphi) \to u(\varphi)$ in E if $u \in \mathcal{S}'(\mathbb{R}^n, E)$, which means that

$$u_k \to u \quad \text{in } \mathcal{L}_s\big(\mathcal{S}(\mathbb{R}^n), E\big) \qquad \text{if } u \in \mathcal{S}'(\mathbb{R}^n, E)\ . \tag{1.3.13}$$

Altogether, (1.3.7), (1.3.10), and (1.3.13) show that

$$u_k \to u \quad \text{in } \mathcal{L}_s\big(\mathbf{F}(X), E\big) \qquad \text{if } u \in \mathbf{F}'(X, E)$$

for $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$. Now, by the Banach-Steinhaus theorem, $u_k(\varphi) \to u(\varphi)$ in E, uniformly for φ in compact subsets of $\mathbf{F}(X)$. Since $\mathbf{F}(X)$ is a Montel space it follows that $u_k \to u$ in $\mathcal{L}\big(\mathbf{F}(X), E\big) = \mathbf{F}'(X, E)$. This proves the assertion. ÷

1.3.4 Remark The arguments leading to (1.3.9) and (1.3.12) remain valid if $\mathbf{F}(X)$ is replaced by $\mathbf{F}(X, E)$. This proves that

$$\mathcal{D}(X, E) \overset{d}{\hookrightarrow} \mathcal{E}(X, E) \quad \text{and} \quad \mathcal{D}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^n, E)\ ,$$

with sequential density, since the continuity of these injections is clear. This has already been stated in [Ama95, (III.4.1.2) and (III.4.1.5)], respectively. ÷

Our next proposition shows that $\mathcal{D}(\mathbb{R}^n, E)$ is dense in $\mathcal{O}_M(\mathbb{R}^n, E)$ as well.

1.3.5 Proposition $\quad \mathcal{D}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{O}_M(\mathbb{R}^n, E)$.

Proof Suppose that $K \subset\subset \mathbb{R}^n$. Then, given $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$,

$$\|\varphi \partial^\alpha u\|_\infty \leq \|\varphi\|_\infty \, \|\partial^\alpha u\|_\infty \leq \|\varphi\|_\infty \, p_{|\alpha|,K}(u) \ , \qquad u \in \mathcal{D}_K(\mathbb{R}^n, E) \ .$$

This shows that $\mathcal{D}_K(\mathbb{R}^n, E) \hookrightarrow \mathcal{O}_M(\mathbb{R}^n, E)$. Hence $\mathcal{D}(\mathbb{R}^n, E) \hookrightarrow \mathcal{O}_M(\mathbb{R}^n, E)$.

Now øx $u \in \mathcal{O}_M(\mathbb{R}^n, E)$, as well as $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$. There exist $m \in \mathbb{N}$ and $c_\alpha > 0$ such that

$$|\partial^\beta u(x)| \leq c_\alpha (1 + |x|^2)^m \ , \qquad x \in \mathbb{R}^n \ , \quad \beta \leq \alpha \ . \tag{1.3.14}$$

Let (χ_k) be a sequence as in (1.3.11). Then, by Leibniz' rule,

$$\partial^\alpha(\chi_k u) - \partial^\alpha u = \sum_{\beta < \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\chi_k \partial^\beta u + (\chi_k - 1)\partial^\alpha u \ .$$

Hence we infer from (1.3.11) and (1.3.14) that

$$\left\| \varphi \big(\partial^\alpha(\chi_k u) - \partial^\alpha u \big) \right\|_\infty \leq c q_{m+1,0}(\varphi)(1 + k^2)^{-1} \ , \qquad k \in \mathbb{N} \ .$$

Since $\chi_k u \in \mathcal{D}(\mathbb{R}^n, E)$, it follows that $\mathcal{D}(\mathbb{R}^n, E)$ is dense in $\mathcal{O}_M(\mathbb{R}^n, E)$. ÷

The preceding approximation results can be used to prove a number of useful density results. For this we make use of the following elementary facts that will also often be used without further mention:

Density by Iteration

Let A, B, and C be topological spaces. It is obvious that

$$A \overset{d}{\hookrightarrow} B \overset{d}{\hookrightarrow} C \quad \Longrightarrow \quad A \overset{d}{\hookrightarrow} C \tag{1.3.15}$$

and

$$A \overset{d}{\subset} C \ , \quad A \subset B \subset C \quad \Longrightarrow \quad B \overset{d}{\subset} C \ . \tag{1.3.16}$$

Implication (1.3.16) is also true if d stands for 'sequentially dense'.

Approximation by Tensor Products

We recall from [Ama95, Subsection V.2.4] that, given $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\}$, the tensor product $\mathbf{F}(X) \otimes E$ is deøned by putting ørst

$$\varphi \otimes e(x) := \varphi(x)e \ , \qquad \varphi \in \mathbf{F}(X) \ , \quad e \in E \ , \quad x \in X \ , \tag{1.3.17}$$

and then

$$\mathbf{F}(X) \otimes E := \operatorname{span}\{\, \varphi \otimes e \;;\; \varphi \in \mathbf{F}(X),\ e \in E \,\} \;, \tag{1.3.18}$$

where the linear hull is taken in E^X. Analogously, we deøne the tensor product $\mathbf{F}'(X) \otimes E$ by

$$(u \otimes e)(\varphi) := u(\varphi)e \;, \qquad u \in \mathbf{F}'(X) \;, \quad e \in E \;, \quad \varphi \in \mathbf{F}(X) \;, \tag{1.3.19}$$

and

$$\mathbf{F}'(X) \otimes E := \operatorname{span}\{\, u \otimes e \;;\; u \in \mathbf{F}'(X),\ e \in E \,\} \;, \tag{1.3.20}$$

now the span being taken in $E^{F(X)}$. It is obvious that $\mathbf{F}(X) \otimes E \subset \mathbf{F}(X, E)$ and $\mathbf{F}'(X) \otimes E \subset \mathbf{F}'(X, E)$. In the following approximation theorem these tensor products are given the corresponding subspace topologies.

1.3.6 Theorem

(i) $\mathcal{D}(X) \otimes E \overset{d}{\hookrightarrow} \mathcal{D}(X, E) \overset{d}{\hookrightarrow} \mathcal{E}'(X, E) \overset{d}{\hookrightarrow} \mathcal{D}'(X, E)$;

(ii) $\mathcal{D}(X) \otimes E \overset{d}{\hookrightarrow} \mathcal{E}(X) \otimes E \overset{d}{\hookrightarrow} \mathcal{E}(X, E) \overset{d}{\hookrightarrow} \mathcal{D}'(X, E)$;

(iii) $\mathcal{E}'(X) \otimes E \overset{d}{\hookrightarrow} \mathcal{E}'(X, E)$;

(iv) $\mathcal{D}'(X) \otimes E \overset{d}{\hookrightarrow} \mathcal{D}'(X, E)$;

(v) $\mathcal{D}(\mathbb{R}^n) \otimes E \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^n) \otimes E \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{D}'(\mathbb{R}^n, E)$;

(vi) $\mathcal{S}(\mathbb{R}^n) \otimes E \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n) \otimes E \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n, E)$;

(vii) $\mathcal{S}(\mathbb{R}^n) \otimes E \overset{d}{\hookrightarrow} \mathcal{O}_M(\mathbb{R}^n) \otimes E \overset{d}{\hookrightarrow} \mathcal{O}_M(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n, E)$;

(viii) $\mathcal{S}(\mathbb{R}^n) \otimes E \overset{d}{\hookrightarrow} W_p^k(\mathbb{R}^n) \otimes E \overset{d}{\hookrightarrow} W_p^k(\mathbb{R}^n, E)$, $\quad k \in \mathbb{N}, \quad 1 \le p < \infty$;

(ix) $\mathcal{D}(X) \otimes E \overset{d}{\hookrightarrow} C^k(X) \otimes E \overset{d}{\hookrightarrow} C^k(X, E)$, $\quad k \in \mathbb{N}$;

(x) $\mathcal{D}(X) \otimes E \overset{d}{\hookrightarrow} C_0(X) \otimes E \overset{d}{\hookrightarrow} C_0(X, E)$.

Proof From Remark 1.3.4 and [Ama95, Proposition V.2.4.1] we know that

$$\mathcal{D}(X) \otimes E \overset{d}{\hookrightarrow} \mathcal{D}(X, E) \overset{d}{\hookrightarrow} \mathcal{E}(X, E) \;. \tag{1.3.21}$$

Proposition 1.3.3 implies $\mathcal{D}(X, E) \overset{d}{\hookrightarrow} \mathcal{E}'(X, E)$ and $\mathcal{D}(X, E) \overset{d}{\hookrightarrow} \mathcal{D}'(X, E)$, since the continuity of these injections is obvious. This and (1.3.16) give (i). From $\mathcal{D}(X) \hookrightarrow \mathcal{E}(X)$ it follows that $\mathcal{D}(X) \otimes E \hookrightarrow \mathcal{E}(X) \otimes E$. Now (ii) is entailed by (i) and (1.3.16). Since $\mathcal{D}(X) \overset{d}{\hookrightarrow} \mathcal{E}'(X)$ by (i), $\mathcal{D}(X) \otimes E \overset{d}{\hookrightarrow} \mathcal{E}'(X) \otimes E$. Thus (iii) follows from (1.3.15) and (i). Analogous arguments prove (iv)˘(viii), where one has to use Proposition 1.3.5 for (vii), and [Ama95, Theorem V.2.4.3] and $\mathcal{S}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} W_p^k(\mathbb{R}^n, E)$, $1 \le p < \infty$, for (viii). Details are left to the reader.

(ix) Given $K = \overline{K} \subset\subset L = \overline{L} \subset\subset X$, choose $\varphi \in \mathcal{D}(X)$ with $\varphi \,|\, L = 1$. Then $\varphi u \in C^k(X, E)$ for $u \in C^k(X, E)$, and $\operatorname{supp}(\varphi u) \subset\subset X$ with $\varphi u \,|\, L = u$. By mollifying φu we find $v \in \mathcal{D}(X, E)$ satisfying $v \,|\, K = u$. This gives $\mathcal{D}(X, E) \overset{d}{\hookrightarrow} C^k(X, E)$. Now the assertion is obvious, as is (x). ÷

Approximation by Polynomials

Finally, we use this approximation theorem to prove the separability of the spaces under consideration, provided E is separable. For this we need a preparatory lemma that will be useful in Subsection 1.8 as well.

1.3.7 Lemma Every element of $\mathcal{D}(X, E)$ is the limit in $\mathcal{E}(X, E)$ of a sequence of (restrictions of) polynomials (in n indeterminates and with coefficients in E).

Proof Define $w \in \mathcal{S}(\mathbb{R}^n)$ by

$$w(x) := (4\pi)^{-n/2} e^{-|x|^2/4} , \qquad x \in \mathbb{R}^n . \tag{1.3.22}$$

Then w is analytic and everywhere positive. Since $w(x) = \prod_{j=1}^n w(x_j)$, it follows from Fubini's theorem that $\|w\|_1 = \|w\|_{L_1(\mathbb{R})}^n$. But, again by Fubini's theorem,

$$\begin{aligned}\Bigl(\int_{\mathbb{R}} e^{-x^2/4}\,dx\Bigr)^2 &= \int_{\mathbb{R}^2} e^{-(x^2+y^2)/4}\,d(x,y) \\ &= 2\pi \int_0^\infty r e^{-r^2/4}\,dr = -4\pi e^{-r^2/4}\Big|_0^\infty = 4\pi .\end{aligned}$$

Hence $\|w\|_1 = 1$, and, putting

$$w_t(x) := t^{-n/2} w\bigl(x/\sqrt{t}\bigr) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)} , \qquad x \in \mathbb{R}^n , \quad t > 0 , \tag{1.3.23}$$

it follows that $\{\, w_t \ ; \ t > 0 \,\}$ is an approximate identity. Thus, given $\varphi \in \mathcal{D}(X, E)$, we infer from [Ama95, (III.4.2.10) and (III.4.2.25)] that $w_{1/k} * \varphi \to \varphi$ in $\mathcal{E}(X, E)$.

Let $k \in \mathbb{N}^{\times}$ be fixed and put

$$f(z) := \int_{\mathbb{R}^n} w_{1/k}(z - y)\varphi(y)\,dy , \qquad z \in \mathbb{C}^n .$$

It is easily verified that f is well-defined and analytic. Hence f can be represented by its Taylor series, that is easily seen to converge uniformly on compact subsets of $\mathbb{C}^n$ towards f. This implies that the sequence of Taylor polynomials of $w_{1/k} * \varphi = f \,|\, \mathbb{R}^n$ converges in $\mathcal{E}(\mathbb{R}^n, E)$ towards $w_{1/k} * \varphi$, which entails the assertion. ÷

Separability

1.3.8 Proposition If E is separable then $\mathcal{D}(X, E)$ is also separable.

Proof It is an obvious consequence of Theorem 1.3.6(i) that it suŒces to prove that $\mathcal{D}(X)$ is separable. Thus let $\mathcal{P}$ be the set of all polynomials in n indeterminates with rational coeŒcients (in $\mathbb{K}$). Deøne (X_k) by (1.1.5). If $X_k \neq \emptyset$, øx $\varphi_k \in \mathcal{D}(X_{k+1})$ with $\varphi_k \,|\, X_k = 1$, and put $\varphi_k := 0$ if $X_k = \emptyset$. Then $S := \{\, \varphi_k p \ ; \ p \in \mathcal{P}, \ k \in \mathbb{N} \,\}$ is a countable subset of $\mathcal{D}(X)$. It is an easy consequence of Lemma 1.3.7 and the properties of the LF-topology of $\mathcal{D}(X)$ that S is dense in $\mathcal{D}(X)$. ÷

1.3.9 Corollary Suppose that

$$\mathbf{F} \in \{\, \mathcal{D}', \mathcal{E}, \mathcal{E}', \mathcal{S}, \mathcal{S}', \mathcal{O}_M, C_0, C^k, W_p^k \ ; \ 1 \le p < \infty, \ k \in \mathbb{N} \,\} \ ,$$

and that $X = \mathbb{R}^n$ if $\mathbf{F} = W_p^k$. Then $\mathbf{F}(X,E)$ is separable if E is separable.

Proof Theorem 1.3.6 implies $\mathcal{D}(X,E) \overset{d}{\hookrightarrow} \mathbf{F}(X,E)$. Hence the assertion follows from Proposition 1.3.8. ÷

1.4 Topological Tensor Products and the Kernel Theorem

In the ørst part of this subsection we collect some facts about tensor products of LCSs. This theory is easily accessible in standard books on linear functional analysis and topological vector spaces, in particular, in [Jar81], [Sch71], and [Tre67]. Thus we are rather brief and do not give proofs. In the second part we prove a version of the kernel theorem. This general abstract theorem will be of fundamental importance for deøning bilinear operations on vector-valued distributions.

Algebraic Tensor Products

Let V and W be vector spaces. A tensor product of V and W is a pair (T,β) consisting of a vector space T and a bilinear map $\beta : \ V \times W \to T$ such that

(i) $T = \operatorname{span}\big(\operatorname{im}(\beta)\big)$,

(ii) $\beta(\mathcal{V} \times \mathcal{W})$ is linearly independent in T, if $\mathcal{V}$ and $\mathcal{W}$ are linearly independent in V and W, respectively.

It can be shown that there exists a tensor product of V and W and that it is unique, except for linear isomorphisms. If (T,β) is a tensor product of V and W, we put

$$V \otimes W := T \ , \quad v \otimes w := \beta(v,w) \ , \qquad (v,w) \in V \times W \ ,$$

which is justiøed by 'uniqueness'.

1.4.1 Remarks (a) The tensor product has the following important universality property: if U is a vector space and $b : \ V \times W \to U$ is bilinear, there exists a unique linear map $B:\ V\otimes W\to U$ such that the diagram

$$\begin{tikzcd} V\times W \arrow[rr, "\beta"] \arrow[dr, "b"'] & & V\otimes W \arrow[dl, "B"] \\ & U & \end{tikzcd}$$

is commutative.

Proof Let $\mathcal{V}$ and $\mathcal{W}$ be bases of V and W, respectively. Then $\beta(\mathcal{V},\mathcal{W})$ is a basis of $V\otimes W$ by (i) and (ii) above. Deøne B on $\beta(\mathcal{V},\mathcal{W})$ by

$$B(v\otimes w):=b(v,w)\ ,\qquad (v,w)\in\mathcal{V}\times\mathcal{W}\ ,$$

and extend it linearly. Then B has the desired property. ÷

(b) Let V_j and W_j be vector spaces and $A_j\in\mathrm{Hom}(V_j,W_j)$ for $j=1,2$. Then there exists a unique

$$A_1\otimes A_2\in\mathrm{Hom}(V_1\otimes V_2,W_1\otimes W_2)\ ,$$

the tensor product of A_1 and A_2, such that

$$(A_1\otimes A_2)(v_1\otimes v_2)=A_1v_1\otimes A_2v_2\ ,\qquad (v_1,v_2)\in V_1\times V_2\ .$$

Proof Since $(v_1,v_2)\mapsto A_1v_1\otimes A_2v_2$ is a bilinear map from $V_1\times V_2$ into $W_1\otimes W_2$, the assertion follows from (a). ÷

(c) Let V_1, V_2, and V_3 be vector spaces. There exists a linear isomorphism

$$V_1\otimes(V_2\otimes V_3)\to(V_1\otimes V_2)\otimes V_3 \tag{1.4.1}$$

such that

$$v_1\otimes(v_2\otimes v_3)\mapsto(v_1\otimes v_2)\otimes v_3\ . \tag{1.4.2}$$

This means that tensor products are (canonically) associative (so that parentheses can be omitted).

Proof Let $\mathcal{V}_j$ be a basis of V_j. Deøne (1.4.1) by (1.4.2) on the basis $\mathcal{V}_1\otimes(\mathcal{V}_2\otimes\mathcal{V}_3)$ of $V_1\otimes(V_2\otimes V_3)$ and extend it linearly. ÷

Basic Examples

1.4.2 Examples (a) Let $\mathbb{K}^{m\times n}$ be the vector space of all $(m\times n)$-matrices with entries in $\mathbb{K}$ and let

$$\mathbb{K}^m\times\mathbb{K}^n\to\mathbb{K}^{m\times n}\ ,\qquad (x,y)\mapsto[x_jy_k]_{\substack{1\le j\le m\\ 1\le k\le n}}\ .$$

Then $\mathbb{K}^m \otimes \mathbb{K}^n = \mathbb{K}^{m\times n}$. Note that

$$x \otimes y = xy^\top \ , \qquad x, y \in \mathbb{K}^m \times \mathbb{K}^n \ ,$$

if $\mathbb{K}^m$ and $\mathbb{K}^n$ are identiøed with $\mathbb{K}^{m\times 1}$ and $\mathbb{K}^{n\times 1}$, respectively, and $a^\top$ denotes the transposed of $a \in \mathbb{K}^{r\times s}$ for $r, s \in \ddot{\mathbb{N}}$. Furthermore,

$$x \otimes y = \langle y, \cdot\rangle x \ , \qquad (x, y) \in \mathbb{K}^m \times \mathbb{K}^n \ ,$$

if $\mathbb{K}^{m\times n}$ and $(\mathbb{K}^n)'$ are identiøed with $\mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)$ and $\mathbb{K}^n$, respectively.

(b) Let S be a nonempty set, V a vector space, and $\Phi(S)$ a vector subspace of $\mathbb{K}^S$. Deøne a bilinear map

$$\Phi(S) \times V \to V^S \ , \qquad (\varphi, v) \mapsto \varphi \otimes v$$

by $\varphi \otimes v(s) := \varphi(s)v$. Then

$$\Phi(S) \otimes V := \operatorname{span}\big\{\, \varphi \otimes v \ ; \ \varphi \in \Phi(S), \ v \in V \,\big\}$$

is the tensor product of $\Phi(S)$ and V in V^S, that is, the span lies in V^S (cf. Subsections V.2.4 of [Ama95] and 1.3).

Proof Let $\varphi_1, \dots, \varphi_n$ be linearly independent in $\Phi(S)$ and $v_1, \dots, v_n$ linearly independent in V. Suppose that there exist $\xi_{jk} \in \mathbb{K}$ such that $\sum_{jk} \xi_{jk}\varphi_j \otimes v_k = 0$. Put $\psi_k := \sum_j \xi_{jk}\varphi_j$ so that $\sum_k \psi_k \otimes v_k = 0$. Then $\sum \psi_k(s)v_k = 0$ for each $s \in S$. This implies $\psi_k = 0$ for $k = 1, \dots, n$ by the linear independence of $v_1, \dots, v_n$. Since the φ_j are linearly independent in $\Phi(S)$, we see that each ξ_{jk} is zero. ÷

(c) Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\}$. Then $\mathbf{F}(X) \otimes E$, deøned by (1.3.17) and (1.3.18), is the tensor product of $\mathbf{F}(X)$ and E in $\mathbf{F}(X, E)$. Similarly, if $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$ then $\mathbf{F}'(X) \otimes E$, deøned by (1.3.19) and (1.3.20), is the tensor product of $\mathbf{F}'(X)$ and E in $\mathbf{F}'(X, E)$.

Proof This follows from (b) by putting $S := X$ and $S := \mathbf{F}(X)$, respectively. ÷

(d) Let E be a Banach space, S and T nonempty sets, and $\Phi(S)$ and $\Psi(T, E)$ vector subspaces of $\mathbb{K}^S$ and E^T, respectively. Deøne a bilinear map

$$\Phi(S) \times \Psi(T, E) \to E^{S\times T} \ , \quad (\varphi, \psi) \mapsto \varphi \otimes \psi$$

by $\varphi \otimes \psi(s, t) := \varphi(s)\psi(t)$. Then

$$\Phi(S) \otimes \Psi(T, E) := \operatorname{span}\big\{\, \varphi \otimes \psi \ ; \ \varphi \in \Phi(S), \ \psi \in \Psi(T, E) \,\big\}$$

is the tensor product of $\Phi(S)$ and $\Psi(T, E)$ in $E^{S\times T}$.

Proof Identify $E^{S\times T}$ and $(E^T)^S$ by means of

$$\big[(s, t) \mapsto u(s, t)\big] \longleftrightarrow \big[s \mapsto u(s, \cdot)\big] \ .$$

Then the assertion follows from (b). ÷

Projective Tensor Products

Let now $F := (F, \mathcal{P})$ and $G := (G, \mathcal{Q})$ be LCSs. For $(p, q) \in \mathcal{P} \times \mathcal{Q}$ we put

$$p \otimes_\pi q(z) := \inf\left\{ \sum p(f_j)q(g_j) \ ; \ z = \sum f_j \otimes g_j \right\} , \qquad z \in F \otimes G \ .$$

This defines a seminorm on $F \otimes G$, the tensor product of the seminorms p and q. In fact, it is the Minkowski functional of the absolutely convex hull of $\mathbb{B}_p \otimes \mathbb{B}_q$ where $\mathbb{B}_p := [p < 1]$. It has the property

$$p \otimes_\pi q(f \otimes g) = p(f)q(g) \ , \qquad (f, g) \in F \times G \ . \tag{1.4.3}$$

Thus the family of the tensor product seminorms is separating. Consequently, it defines a locally convex Hausdorff topology on $F \otimes G$, the projective topology, and

$$F \otimes_\pi G := \bigl(F \otimes G, \ \{ p \otimes_\pi q \ ; \ p \in \mathcal{P}, \ q \in \mathcal{Q} \}\bigr)$$

is the projective tensor product of F and G. We denote by $F \ \widetilde{\otimes} \ G$ the completion of the LCS $F \otimes_\pi G$.

Nuclear Maps and Spaces

A linear map N from the LCS F into a Banach space E is nuclear if there exist an equicontinuous sequence (f_j') in F', a bounded sequence (b_j) in E, and a summable sequence (λ_j) in $\mathbb{K}$ such that

$$Nf = \sum_j \lambda_j \langle f_j', f \rangle b_j \ , \qquad f \in F \ . \tag{1.4.4}$$

The LCS F is nuclear if each continuous linear map from F into a Banach space is nuclear. It is conuclear if its dual is nuclear.

For the following theorem we recall convention (1.1.10).

1.4.3 Theorem Let X be a nonempty open subset of $\mathbb{R}^n$ and suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\}$. Then $\mathbf{F}(X)$ is nuclear and conuclear.

Proof If $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$, this can be found in [Tre67] (also cf. [Jar81] and [Sch71]). As for $\mathbf{F} = \mathcal{O}_M$, we refer to [Gro55, II.4.4]. ÷

Projective Tensor Products and Maps of Finite Rank

Lastly, we define an injective linear map by

$$\tau : \ F \otimes G \to \mathcal{L}(F', G) \ , \quad f \otimes g \mapsto \langle \cdot, f \rangle g \ , \tag{1.4.5}$$

the canonical injection. Then we prepare for the proof of the abstract kernel theorem by deriving a series of lemmas.

1.4.4 Lemma Let F be reÆexive and nuclear. Then τ is a toplinear isomorphism from $F \otimes_\pi G$ onto the subspace of $\mathcal{L}(F', G)$ of maps of ønite rank.

Proof Given a continuous seminorm q on G and a bounded subset B' of F',

$$\sup_{f' \in B'} q\bigl(\tau z(f')\bigr) \le \sum_j \sup_{f' \in B'} |\langle f', f_j\rangle| \, q(g_j) \ , \qquad z = \textstyle\sum f_j \otimes g_j \in F \otimes G \ . \tag{1.4.6}$$

Since

$$f \mapsto p(f) := \sup_{f' \in B'} |\langle f', f\rangle|$$

deønes a continuous seminorm on $F'' = F$, it follows from (1.4.6) that

$$\sup_{f' \in B'} q\bigl(\tau z(f')\bigr) \le p \otimes_\pi q(z) \ , \qquad z \in F \otimes G \ .$$

Hence

$$\tau \in \mathcal{L}\bigl(F \otimes_\pi G, \mathcal{L}(F', G)\bigr) \ .$$

It is obvious that τz has ønite rank for $z \in F \otimes G$. Conversely, suppose that $T \in \mathcal{L}(F', G)$ has ønite rank, and let $g_1, \ldots, g_m$ be a basis of $\operatorname{im}(T)$. Then

$$Tf' = \sum_{j=1}^m \xi_j(f') g_j \ , \qquad f' \in F' \ ,$$

where

$$\xi_j(f') = \langle g_j', Tf'\rangle_G = \langle T' g_j', f'\rangle_{F'} \ , \qquad f' \in F' \ ,$$

the $g_j' \in G'$ satisfying $\langle g_j', g_k\rangle = \delta_{jk}$. Since $T' \in \mathcal{L}(G', F)$ by the reÆexivity of F, it follows that $f_j := T' g_j' \in F$ and $T = \sum_{j=1}^m \langle \cdot, f_j\rangle_F g_j$. Hence

$$T = \tau z \qquad \text{with} \quad z := \textstyle\sum_{j=1}^m f_j \otimes g_j \in F \otimes G \ .$$

This shows that τ is a bijection from $F \otimes G$ onto the subspace of $\mathcal{L}(F', G)$ of ønite rank operators.

Now let p and q be continuous seminorms on F and G, respectively. Given $T \in \mathcal{L}(F', G)$ of ønite rank, let $z := \tau^{-1} T$, and let $\sum f_j \otimes g_j$ be a representation of z in $F \otimes G$. Denote by

$$\mathbb{B}_p^\circ := \bigl\{\, f' \in F' \ ; \ |\langle f', f\rangle| \le 1 \text{ for } f \in \mathbb{B}_p \,\bigr\}$$

the polar of $\mathbb{B}_p$. Then

$$\begin{aligned} p \otimes_\varepsilon q(z) &:= \sup_{(f', g') \in \mathbb{B}_p^\circ \times \mathbb{B}_q^\circ} \Bigl| \sum_j \langle f', f_j\rangle \langle g', g_j\rangle \Bigr| \\ &= \sup_{f' \in \mathbb{B}_p^\circ} \sup_{g' \in \mathbb{B}_q^\circ} \bigl|\bigl\langle g', \tau z(f')\bigr\rangle\bigr| = \sup_{f' \in \mathbb{B}_p^\circ} q\bigl(\tau z(f')\bigr) \end{aligned} \tag{1.4.7}$$

by the bipolar theorem. Since F is reÆexive, $\mathbb{B}_p^\circ$ is weakly compact by the Alaoglu-Bourbaki theorem. Hence it is weakly bounded and thus, thanks to Mackey's theorem, bounded in F' (cf. [Hor66], [Jar81], [Tre67], or [Sch71] for these standard theorems from the theory of LCSs).

Observe that $p \otimes_\varepsilon q$, as deøned in (1.4.7), is a seminorm on $F \otimes G$. In the theory of LCSs it is shown that the family $\{\, p \otimes_\varepsilon q \;;\; p \in \mathcal{P},\; q \in \mathcal{Q} \,\}$ is a separating family of seminorms on $F \otimes G$ deøning the 'injective tensor product topology' on $F \otimes G$. Since F is nuclear, this injective tensor product topology coincides with the projective tensor product topology of $F \otimes G$ (e.g., [Tre67, Theorem 50.1(f)]). Thus we infer from (1.4.7) that $\tau^{-1} \in \mathcal{L}(\mathcal{R}, F \otimes_\pi G)$, where $\mathcal{R}$ is the linear subspace of $\mathcal{L}(F', G)$ of ønite rank operators. ÷

Approximation by Maps of Finite Rank

Let p be a continuous seminorm on F. Then $\ker(p)$ is a closed linear subspace of F. Hence $F_p := F/\ker(p)$ is a normed vector space with respect to the quotient norm

$$\widehat{x} := \widehat{x}_p := x + \ker(p) \mapsto \widehat{p}(\widehat{x}) := \inf\{\, p(y) \;;\; y \in \widehat{x} \,\} \ . \tag{1.4.8}$$

Observe that $\widehat{p}(\widehat{x}) = p(y)$ for $y \in \widehat{x}$. Let q be a second continuous seminorm on F such that $q \geq p$. Then $\ker(q) \subset \ker(p)$ implies that $\widehat{x}_q \mapsto \widehat{x}_p$ is a well-deøned continuous linear map from F_q into F_p of norm at most one. Let $\widetilde{F}_p$ be the completion of F_p. Then we consider $\widehat{x}_q \mapsto \widehat{x}_p$ to be a linear map from F_q into $\widetilde{F}_p$, the canonical map $F_q \to \widetilde{F}_p$.

1.4.5 Lemma Suppose that F is nuclear and p is a continuous seminorm on F. Then there exists a continuous seminorm $q \geq p$ on F such that the canonical map $F_q \to \widetilde{F}_p$ is nuclear.

Proof Since F is nuclear, the quotient map $\pi_p :\; F \to \widetilde{F}_p$ is nuclear. Hence π_p has a representation of the form (1.4.4) where (b_j) is a bounded sequence in $\widetilde{F}_p$. Consequently,

$$p(f) = \widehat{p}(\pi_p f) \leq c \sup_j |\langle f_j', f \rangle| =: q(f) \ , \qquad f \in F \ , \tag{1.4.9}$$

with $c := \sum_j |\lambda_j| \sup_j \|b_j\|$. The equicontinuity of the sequence (f_j') implies that q is a continuous seminorm on F. By (1.4.8) and (1.4.9) each f_j' deønes naturally a continuous linear form h_j' on F_q of norm at most $1/c$. Thus, since the canonical map $F_q \to \widetilde{F}_p$ is given by

$$\widehat{f}_q \mapsto \sum_j \lambda_j \langle h_j', \widehat{f}_q \rangle_{F_q} b_j \ ,$$

it is nuclear. ÷

By means of the preceding lemma we can give suŒcient conditions for linear operators to be approximable by operators of ønite rank.

1.4.6 Lemma If F is nuclear, the maps of ønite rank are dense in $\mathcal{L}(F,G)$.

Proof Let p be a continuous seminorm on F. Lemma 1.4.5 guarantees the existence of a continuous seminorm $q \geq p$ on F such that the canonical map $S : F_q \to \widetilde{F}_p$ is nuclear. Hence there exist a summable sequence (λ_j) in $\mathbb{K}$ and bounded sequences $\bigl(\widehat{f}'_j\bigr)$ and $\bigl(\widehat{f}_j\bigr)$ in $(F_q)'$ and $\widetilde{F}_p$, respectively, such that

$$S\widehat{f} = \sum_j \lambda_j \bigl\langle \widehat{f}'_j, \widehat{f} \bigr\rangle \widehat{f}_j \ , \qquad \widehat{f} \in F_q \ .$$

Thus, letting

$$S_n := \sum_{j=0}^{n} \lambda_j \bigl\langle \widehat{f}'_j, \cdot \bigr\rangle \widehat{f}_j \ , \qquad n \in \mathbb{N} \ ,$$

it is obvious that

$$S_n \to S \qquad \text{in } \mathcal{L}\bigl(F_q, \widetilde{F}_p\bigr) \ . \tag{1.4.10}$$

Observe that the diagram

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}"] \arrow[d, "\pi_q"'] & F \arrow[d, "\pi_p"] \\ F_q \arrow[r, "S"] & \widetilde{F}_p \end{tikzcd} \tag{1.4.11}$$

is commutative, where π_q and π_p are the quotient maps. Choose $f_j \in \widehat{f}_j$ and put

$$T_n f := \sum_{j=0}^{n} \lambda_j \bigl\langle \widehat{f}'_j, \pi_q f \bigr\rangle f_j \ , \qquad f \in F \ .$$

Then, thanks to (1.4.10) and (1.4.11),

$$p(T_n f - f) \to 0 \ , \qquad n \to \infty \ , \tag{1.4.12}$$

uniformly with respect to f in bounded subsets of F.

Now let $R \in \mathcal{L}(F,G)$ and a continuous seminorm r on G be given. Then $p := r \circ R$ is a continuous seminorm on F. Put $R_n := R \circ T_n$. It follows from (1.4.12) that $r\bigl((R_n - R)f\bigr) \to 0$ as $n \to \infty$, uniformly with respect to f in bounded subsets of F. Since $R_n \in \mathcal{L}(F,G)$ has ønite rank, the assertion follows. ÷

Completeness of Spaces of Linear Operators

In our last preparatory lemma we give conditions for $\mathcal{L}(F',G)$ to be complete.

1.4.7 Lemma Let F and G be complete and let F be reÆexive. Then $\mathcal{L}(F',G)$ is complete.

Proof Let (T_α) be a Cauchy net in $\mathcal{L}(F', G)$. Since G is complete and $(T_\alpha f')$ is a Cauchy net in G, there exists $T \in \mathrm{Hom}(F', G)$ such that $T_\alpha f' \to Tf'$ in G for each $f' \in F'$. Given $g' \in G'$,

$$\langle f', T'_\alpha g' \rangle_F = \langle g', T_\alpha f' \rangle_G \to \langle g', Tf' \rangle_G = g' \circ T(f') \tag{1.4.13}$$

for each $f' \in F'$, thanks to $F'' = F$. Observe that $(T'_\alpha g')$ is a Cauchy net in F. Hence we deduce from (1.4.13) and the completeness of F that $g' \circ T \in F$ for each $g' \in G'$.

Now suppose that $g'_\alpha \xrightarrow{w^*} g'$. Then

$$g'_\alpha \circ T(f') = \langle g'_\alpha, Tf' \rangle_G \to \langle g', Tf' \rangle_G \ , \qquad f' \in F' \ .$$

This shows that ${}^tT \in \mathcal{L}(G'_{w^*}, F_w)$, where ${}^tT$ is the algebraic dual of T. Let q be a continuous seminorm on G. Then $\mathbb{B}_q^\circ$ is w^*-compact by the Alaoglu-Bourbaki theorem. Hence $K := {}^tT(\mathbb{B}_q^\circ)$ is weakly compact as well, thus weakly bounded. Consequently, K is bounded in F. Hence $V' := K^\circ$ is a neighborhood of zero in F'. Observe that $f' \in V'$ iœ $|\langle f', {}^tTg' \rangle_F| = |\langle g', Tf' \rangle_G| \le 1$ for $g' \in \mathbb{B}_q^\circ$, that is, iœ $Tf' \in {}^\circ(\mathbb{B}_q^\circ)$, where, given $C' \subset G'$,

$${}^\circ C' := \left\{ g \in G \ ; \ |\langle g', g \rangle| \le 1 \text{ for } g' \in C' \right\}$$

is the polar of C' in G. By the bipolar theorem ${}^\circ(\mathbb{B}_q^\circ) = \bar{\mathbb{B}}_q$. Hence

$$\sup\left\{ q(Tf') \ ; \ f' \in V' \right\} \le 1 \ . \tag{1.4.14}$$

Since V' is the polar of K, it is absolutely convex and w^*-closed. Thus it is weakly closed by the reÆexivity of F. Hence it is closed. Denoting by p the Minkowski functional of V', it follows that $f' \in V'$ iœ $p(f') \le 1$. From this and (1.4.14) we infer that

$$q(Tf') \le p(f') \ , \qquad f' \in F' \ ,$$

which shows that $T \in \mathcal{L}(F', G)$, thanks to the fact that p is a continuous seminorm on F'. ÷

The Abstract Kernel Theorem

After these preparations we can prove the abstract KERNEL THEOREM in a form that is most suitable for our purposes.

1.4.8 Theorem Let F and G be complete LCSs such that F is reÆexive, nuclear, and conuclear. Then τ is (that is, extends to) a toplinear isomorphism

$$\tau : \ F \mathbin{\widetilde{\otimes}} G \to \mathcal{L}(F', G) \ , \tag{1.4.15}$$

the canonical isomorphism.

Proof It follows from Lemma 1.4.7 that $\mathcal{L}(F', G)$ is complete. Since F is conuclear, we infer from Lemma 1.4.6 that the linear subspace $\mathcal{R}$ of maps of ønite rank is dense in $\mathcal{L}(F', G)$. Since F is also nuclear, Lemma 1.4.4 guarantees that τ is a toplinear isomorphism from $F \otimes_\pi G$ onto $\mathcal{R}$. Now the assertion is an easy consequence of the density of $F \otimes_\pi G$ in $F \mathbin{\widetilde{\otimes}} G$ and a well-known result about continuous extensions of continuous linear maps (e.g., [Jar81, Theorem 3.4.2]). ÷

Tensor Product Characterizations of Some Distribution Spaces

As a ørst application of this general theorem we can prove the following characterizations for spaces of vector-valued distributions. They are the basis for deøning bilinear maps of vector-valued distributions in the next subsection.

1.4.9 Theorem Let E be a Banach space. Then

$$\mathbf{F}(X) \mathbin{\widetilde{\otimes}} E = \mathbf{F}(X, E) \cong \mathcal{L}\bigl(\mathbf{F}'(X), E\bigr) \ , \qquad \mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\} \ ,$$

and

$$\mathbf{F}'(X) \mathbin{\widetilde{\otimes}} E = \mathbf{F}'(X, E) = \mathcal{L}\bigl(\mathbf{F}(X), E\bigr) \ , \qquad \mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\} \ ,$$

where $\cong$ denotes the canonical toplinear isomorphism.

Proof From Theorem 1.1.2 we know that $\mathbf{F}(X)$ and $\mathbf{F}'(X)$ are complete and reÆexive. Furthermore, $\mathbf{F}(X)$ and $\mathbf{F}'(X)$ are both nuclear and conuclear, thanks to Theorem 1.4.3. Hence it follows from Theorem 1.4.8 that

$$\tau : \ \mathbf{F}(X) \mathbin{\widetilde{\otimes}} E \to \mathcal{L}\bigl(\mathbf{F}'(X), E\bigr)$$

for $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\}$, and

$$\tau : \ \mathbf{F}'(X) \mathbin{\widetilde{\otimes}} E \to \mathcal{L}\bigl(\mathbf{F}(X), E\bigr) = \mathbf{F}'(X, E) \ .$$

for $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$ are toplinear isomorphisms. This proves the second assertion and part of the ørst one. It remains to show that $\mathbf{F}(X) \mathbin{\widetilde{\otimes}} E = \mathbf{F}(X, E)$. Since $\mathbf{F}(X) \otimes E$ is dense in $\mathbf{F}(X, E)$ by Theorem 1.3.6, we have to verify that $\mathbf{F}(X, E)$ induces on $\mathbf{F}(X) \otimes E$ the projective topology.

Suppose that $\mathbf{F} \in \{\mathcal{E}, \mathcal{S}, \mathcal{O}_M\}$ and let p be one of the seminorms (1.1.3), (1.1.7), or (1.1.9), respectively. Then

$$p(\varphi \otimes e) = p(\varphi)\, |e| \ , \qquad \varphi \in \mathbf{F}(X) \ , \quad e \in E \ ,$$

where on the left side p is the seminorm on $\mathbf{F}(X, E)$, and on the right side p denotes the corresponding seminorm on $\mathbf{F}(X)$. From this we easily deduce that $\mathbf{F}(X, E)$ induces on $\mathbf{F}(X) \otimes E$ a topology that is weaker than the projective tensor product topology.

Conversely, let $\sum \varphi_j \otimes e_j$ be a representation of $z \in \mathbf{F}(X) \otimes E$. Then, given $\varphi' \in \mathbf{F}'(X)$ and $e' \in E'$,

$$\sum \langle \varphi', \varphi_j \rangle \langle e', e_j \rangle = \left\langle e', \sum \langle \varphi', \varphi_j \rangle e_j \right\rangle = \left\langle e', \left\langle \varphi', \sum \varphi_j \otimes e_j \right\rangle \right\rangle .$$

Consequently (cf. (1.4.7)),

$$p \otimes_\varepsilon |\cdot|_E(z) = \sup_{\varphi' \in \mathcal{B}_p^0} |\langle \varphi', z \rangle|_E \le p(z)$$

by the bipolar theorem. This shows that the topology induced by $\mathbf{F}(X, E)$ on $\mathbf{F}(X) \otimes E$ is stronger than the injective tensor product topology. Since $\mathbf{F}(X)$ is nuclear, the latter coincides with the projective topology.

Finally, given $K \subset\subset X$, the last string of arguments shows that $\mathcal{D}_K(X, E)$ induces on $\mathcal{D}_K(X) \otimes E$ the projective topology. Since $\mathcal{D}(X, E)$ induces on $\mathcal{D}_K(X, E)$ its original topology, it follows that $\mathcal{D}(X, E)$ induces on $\mathcal{D}_K(X) \otimes E$ the projective topology. From

$$\mathcal{D}(X) \otimes E = \varinjlim_{K \subset\subset X} \mathcal{D}_K(X) \otimes E$$

(e.g., [Jar81, Corollary 4 to Theorem 15.5.3]) it follows that $\mathcal{D}(X, E)$ induces on $\mathcal{D}(X) \otimes E$ the projective tensor product topology. Thus, since $\mathcal{D}(X) \otimes E$ is dense in $\mathcal{D}(X, E)$ by Theorem 1.3.6, it follows that $\mathcal{D}(X) \mathbin{\widetilde{\otimes}} E = \mathcal{D}(X, E)$. ÷

Suppose that F and G are Frchet spaces such that F is nuclear. Then

$$(F \mathbin{\widetilde{\otimes}} G)' = F' \mathbin{\widetilde{\otimes}} G' \tag{1.4.16}$$

by means of the duality pairing induced by

$$\langle f' \otimes g', f \otimes g \rangle := \langle f', f \rangle \langle g', g \rangle , \qquad (f, g) \in F \times G , \quad (f', g') \in F' \times G' , \tag{1.4.17}$$

(e.g., [Sch71, Theorem IV.9.9]). By means of this fact we can identify the dual of $\mathbf{F}(X, E)$ for $\mathbf{F} \in \{\mathcal{E}, \mathcal{S}\}$.

1.4.10 Corollary Let E be a Banach space and suppose that $\mathbf{F} \in \{\mathcal{E}, \mathcal{E}, \mathcal{S}, \mathcal{S}'\}$. Then

$$\mathbf{F}(X, E)' = \big(\mathbf{F}(X) \mathbin{\widetilde{\otimes}} E\big)' = \mathbf{F}'(X) \mathbin{\widetilde{\otimes}} E' = \mathbf{F}'(X, E')$$

by means of the duality pairing induced by (1.4.17). If E is reÆexive then $\mathbf{F}(X, E)$ is reÆexive as well.

There are various versions of the abstract kernel theorem in the textbook literature (e.g., [Jar81, Theorem 21.5.9], [Tre67, Section 50]). However, it is always assumed that F and G are Frchet spaces. Since we need to apply Theorem 1.4.8 in the special case that $F = \mathcal{O}_M(\mathbb{R}^n)$ and since $\mathcal{O}_M(\mathbb{R}^n)$ is not metrizable, this assumption does not øt our purposes. For this reason we have included a complete proof of Theorem 1.4.8.

1.5 Extending Bilinear Maps

In this subsection we prove a general extension theorem for bilinear maps on tensor products. For this we need a technical lemma guaranteeing a suitable kind of uniformity for equicontinuous sets of nuclear maps.

1.5.1 Lemma Let F be a nuclear LCS and E a Banach space. Suppose that $\mathcal{T}$ is an equicontinuous subset of $\mathcal{L}(F,E)$. Then there exist a summable sequence (λ_j) in $\mathbb{K}$, an equicontinuous sequence (f_j') in F', and a bounded map

$$\mathcal{T} \to B(\mathbb{N},E) \ , \quad T \mapsto \big(e_j(T)\big)_{j\in\mathcal{N}}$$

such that

$$Tf = \sum_j \lambda_j \langle f_j', f\rangle e_j(T) \ , \qquad f \in F \ , \quad T \in \mathcal{T} \ .$$

Proof By the equicontinuity of $\mathcal{T}$, the set $\bigcap\big\{\, T^{-1}(\bar{\mathbb{B}}_E) \ ; \ T \in \mathcal{T} \,\big\}$ is a closed absolutely convex neighborhood of zero. Let p be its Minkowski functional. Then p is a continuous seminorm on F and

$$|Tf|_E \le p(f) \ , \qquad f \in F \ , \quad T \in \mathcal{T} \ . \tag{1.5.1}$$

The nuclearity of F implies, thanks to Lemma 1.4.5, the existence of a continuous seminorm $q \ge p$ on F such that the canonical map $S : \ F_q \to \widetilde{F}_p$ is nuclear. Hence there exist a summable sequence (λ_j) in $\mathbb{K}$ and bounded sequences $\big(\widehat{f}_j'\big)$ and $(\widehat{g}_j)$ in $(F_q)'$ and $\widetilde{F}_p$, respectively, such that

$$S\widehat{f} = \sum_j \lambda_j \big\langle \widehat{f}_j', \widehat{f}\big\rangle \widehat{g}_j \ , \qquad \widehat{f} \in F_q \ . \tag{1.5.2}$$

It is easy to verify that for each $T \in \mathcal{T}$ there exists $\widetilde{T} \in \mathcal{L}\big(\widetilde{F}_p, E\big)$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\pi_q"] \arrow[d, "T"'] & F_q \arrow[d, "S"] \\ E & \widetilde{F}_p \arrow[l, "\widetilde{T}"'] \end{tikzcd}$$

is commutative, where π_q is the quotient map. Hence, letting $f_j' := (\pi_q)'\widehat{f}_j'$ and $e_j(T) := \widetilde{T}\widehat{g}_j$, the assertion follows since (1.5.1) and (1.4.8) imply $\big\|\widetilde{T}\big\| \le 1$. ÷

Now we are in a position to prove the following general extension theorem for bilinear maps. Throughout the remainder of this subsection we assume that E_j,

$j = 0, 1, 2$, are Banach spaces and

$$E_1 \times E_2 \to E_0 \ , \quad (e_1, e_2) \mapsto e_1 \bullet e_2 \tag{1.5.3}$$

is a multiplication. Furthermore, given an LCS F, we put $F(E) := \mathcal{L}(F', E)$ and we remind the reader of the deønition of point-wise multiplication in Subsection II.1.1 of [Ama95].

1.5.2 Proposition Let F_j, $j = 0, 1, 2$, be LCSs such that

(i) F_1 is reÆexive, complete, nuclear, and conuclear;

(ii) $F_0(E_0)$ is complete.

Suppose that there is given a hypocontinuous bilinear map

$$F_1 \times F_2(E_2) \to F_0(E_2) \ , \quad (f_1, v) \mapsto f_1 - v \ . \tag{1.5.4}$$

Deøne a bilinear map

$$-_\bullet : \ (F_1 \otimes E_1) \times F_2(E_2) \to F_0(E_0)$$

by

$$(f_1 \otimes e_1, v) \mapsto e_1 \bullet (f_1 - v) \ , \qquad (e_1, f_1) \in E_1 \times F_1 \ , \tag{1.5.5}$$

and by bilinear extension otherwise. Then $-_\bullet$ possesses a unique hypocontinuous bilinear extension

$$-_\bullet : \ (F_1 \widetilde{\otimes} E_1) \times F_2(E_2) \to F_0(E_0) \ , \quad (u, v) \mapsto u -_\bullet v \ .$$

Proof Uniqueness follows immediately from the density of $F_1 \otimes E_1$ in $F_1 \widetilde{\otimes} E_1$.

Given $u := \sum f_j \otimes e_j \in F_1 \otimes E_1$ and $v \in F_2(E_2)$,

$$u -_\bullet v = \sum e_j \bullet (f_j - v) \in F_0(E_0) \ . \tag{1.5.6}$$

Let B_0' be a bounded subset of F_0'. Then

$$p(w) := \sup_{f' \in B_0'} |wf'|_{E_0} \ , \qquad w \in F_0(E_0) \ ,$$

and

$$p_2(v) := \sup_{f' \in B_0'} |vf'|_{E_2} \ , \qquad v \in F_0(E_2) \ ,$$

deøne continuous seminorms on $F_0(E_0)$ and $F_0(E_2)$, respectively. Hence

$$p(u -_\bullet v) \le \sum_j p_2(f_j - v)\, |e_j|_{E_1} \ . \tag{1.5.7}$$

Let B_2 be a bounded subset of $F_2(E_2)$. Then (1.5.7) and the hypocontinuity of (1.5.4) imply the existence of a continuous seminorm p_1 on F_1 such that

$$p(u \multimap_\bullet v) \le \sum_j p_1(f_j)\, |e_j|_{E_1} \ , \qquad v \in B_2 \ .$$

Since p_1 is independent of the particular representation of u, it follows that

$$p(u \multimap_\bullet v) \le (p_1 \otimes_\pi |\cdot|)(u) \ , \qquad u \in F_1 \otimes E_1 \ , \quad v \in B_2 \ .$$

This shows that

$$(u \mapsto u \multimap_\bullet v) \in \mathcal{L}\bigl(F_1 \otimes_\pi E_1, F_0(E_0)\bigr) \ , \tag{1.5.8}$$

uniformly with respect to v in bounded subsets of $F_2(E_2)$.

Given $v \in F_2(E_2)$, there exists a unique continuous extension

$$U_v \in \mathcal{L}\bigl(F_1 \mathbin{\widetilde{\otimes}} E_1, F_0(E_0)\bigr) \tag{1.5.9}$$

of (1.5.8), thanks to assumption (ii). Then

$$\bigl[(u,v) \mapsto u \multimap_\bullet v := U_v(u)\bigr] : \ (F_1 \mathbin{\widetilde{\otimes}} E_1) \times F_2(E_2) \to F_0(E_0) \tag{1.5.10}$$

is an extension of (1.5.5) that is trivially linear in the first variable. For $u \in F_1 \mathbin{\widetilde{\otimes}} E_1$ choose a net (u_α) in $F_1 \otimes E_1$ such that $u_\alpha \to u$ in $F_1 \mathbin{\widetilde{\otimes}} E_1$. Then, given $\lambda_j \in \mathbb{K}$ and $v_j \in F_2(E_2)$ for $j = 1, 2$,

$$\begin{aligned} u \multimap_\bullet (\lambda_1 v_1 + \lambda_2 v_2) &= \lim_\alpha \bigl[u_\alpha \multimap_\bullet (\lambda_1 v_1 + \lambda_2 v_2)\bigr] \\ &= \lambda_1 \lim_\alpha (u_\alpha \multimap_\bullet v_1) + \lambda_2 \lim_\alpha (u_\alpha \multimap_\bullet v_2) \\ &= \lambda_1 (u \multimap_\bullet v_1) + \lambda_2 (u \multimap_\bullet v_2) \ . \end{aligned}$$

Thus (1.5.10) is bilinear and it remains to show that it is hypocontinuous.

The uniformity assertion contained in (1.5.8) implies that the family of linear operators (1.5.8) is equicontinuous if v stays in bounded subsets of $F_2(E_2)$. Thus, given a closed neighborhood V_0 of zero in $F_0(E_0)$ and a bounded subset B_2 of $F_2(E_2)$, there exists a neighborhood V_1 of zero in $F_1 \otimes_\pi E_1$ such that $u \multimap_\bullet v \in V_0$ for $(u,v) \in V_1 \times B_2$. Consequently,

$$U_v(\overline{V}_1) \subset V_0 \ , \qquad v \in B_2 \ ,$$

where $\overline{V}_1$ is the closure of V_1 in $F_1 \mathbin{\widetilde{\otimes}} E_1$. This shows that $\{\, U_v \ ; \ v \in B_2 \,\}$ is equicontinuous in $\mathcal{L}\bigl(F_1 \mathbin{\widetilde{\otimes}} E_1, F_0(E_0)\bigr)$. Hence $u \multimap_\bullet v \to 0$ in $F_0(E_0)$ as $u \to 0$ in $F_1 \mathbin{\widetilde{\otimes}} E_1$, uniformly with respect to v in bounded subsets of $F_2(E_2)$.

Now let B be a bounded subset of $F_1 \mathbin{\widetilde{\otimes}} E_1$. Thanks to assumption (i) and Theorem 1.4.8,

$$F_1 \mathbin{\widetilde{\otimes}} E_1 \cong \mathcal{L}(F_1', E_1) = F_1(E_1) \ . \tag{1.5.11}$$

Put $B_1 := \tau(B)$, where τ is the isomorphism of (1.5.11). Then B_1 is bounded in $F_1(E_1)$. Since F_1' is barreled, being the dual of a reÆexive space, hence reÆexive, the uniform boundedness principle shows that B_1 is equicontinuous. Thus Lemma 1.5.1 guarantees the existence of a summable sequence (λ_j) in $\mathbb{K}$, a bounded sequence (f_j) in F_1, and a bounded set C in E_1 such that each $u \in B_1$ has a representation $u = \sum_j \lambda_j f_j \otimes e_j$, where $e_j \in C$ and the series converges in $F_1 \mathbin{\widetilde{\otimes}} E_1$. Hence, given $v \in F_2(E_2)$, it follows from (1.5.9), (1.5.10), and (1.5.6) that

$$u \mathbin{-_{\bullet}} v = \sum_j \lambda_j e_j \bullet (f_j - v) \ .$$

Consequently,

$$p(u \mathbin{-_{\bullet}} v) \le \sum_j |\lambda_j|\, p_2(f_j - v)\, |e_j|_{E_1} \ . \tag{1.5.12}$$

From the hypocontinuity of (1.5.4) and the boundedness of the sequence (f_j) in F_1 we deduce the existence of a continuous seminorm q on $F_2(E_2)$ such that

$$p_2(f_j - v) \le q(v) \ , \qquad v \in F_2(E_2) \ , \quad j \in \mathbb{N} \ . \tag{1.5.13}$$

Since $e_j \in C$ and C is bounded in E_1, we infer from (1.5.12) and (1.5.13) that

$$p(u \mathbin{-_{\bullet}} v) \le c q(v) \ , \qquad v \in F_2(E_2) \ , \quad u \in B_1 \ .$$

This proves that

$$(v \mapsto u \mathbin{-_{\bullet}} v) \in \mathcal{L}\big(F_2(E_2), F_0(E_0)\big) \ ,$$

uniformly with respect to u in bounded subsets of $F_1 \mathbin{\widetilde{\otimes}} E_1$. Thus the map (1.5.10) is hypocontinuous. ÷

By specializing we deduce from the preceding proposition the following fundamental extension result:

1.5.3 Theorem Let F_j, $j = 0, 1, 2$, be reÆexive, complete, nuclear, and conuclear LCSs. Suppose that there are a bilinear map

$$F_1 \times F_2 \to F_0 \ , \quad (f_1, f_2) \mapsto f_1 \odot f_2 \tag{1.5.14}$$

and a hypocontinuos bilinear map

$$F_1 \times (F_2 \mathbin{\widetilde{\otimes}} E_2) \to F_0 \mathbin{\widetilde{\otimes}} E_2 \ , \quad (f_1, v) \mapsto f_1 \diamondsuit v \tag{1.5.15}$$

such that

$$f_1 \diamondsuit (f_2 \otimes e_2) = (f_1 \odot f_2) \otimes e_2 \ , \qquad f_j \in F_j \ , \quad j = 1, 2 \ , \quad e_2 \in E_2 \ . \qquad (1.5.16)$$

Then there exists a unique hypocontiuous bilinear map

$$\odot_\bullet : (F_1 \widetilde{\otimes} E_1) \times (F_2 \widetilde{\otimes} E_2) \to F_0 \widetilde{\otimes} E_0 \ , \quad (u, v) \mapsto u \odot_\bullet v \qquad (1.5.17)$$

satisfying

$$(f_1 \otimes e_1) \odot_\bullet (f_2 \otimes e_2) = (f_1 \odot f_2) \otimes (e_1 \bullet e_2) \qquad (1.5.18)$$

for $f_j \in F_j$ and $e_j \in E_j$, $j = 1, 2$.

Proof Since $E_j \otimes F_j$ is dense in $E_j \widetilde{\otimes} F_j$ for $j = 1, 2$, there exists at most one separately continuous bilinear map $\odot_\bullet$ from $(F_1 \widetilde{\otimes} E_1) \times (F_2 \widetilde{\otimes} E_2)$ to $F_0 \widetilde{\otimes} E_0$ satisfying (1.5.18).

Thanks to Theorem 1.4.8 we can identify $F_j \widetilde{\otimes} E_j$ with $F_j(E_j)$ by means of the respective canonical isomorphisms. Then it follows from (1.5.16) that, given $f_0' \in F_0'$ and $v := f_2 \otimes e_2 \in F_2 \otimes E_2$,

$$\begin{aligned} \big[e_1 \bullet (f_1 \diamondsuit v)\big](f_0') &= e_1 \bullet \big[(f_1 \odot f_2) \otimes e_2\big](f_0') = e_1 \bullet \langle f_0', f_1 \odot f_2 \rangle e_2 \\ &= \langle f_0', f_1 \odot f_2 \rangle (e_1 \bullet e_2) = \big[(f_1 \odot f_2) \otimes (e_1 \bullet e_2)\big](f_0') \end{aligned}$$

for $f_1 \in F_1$ and $e_1 \in E_1$. Hence

$$e_1 \bullet (f_1 \diamondsuit v) = (f_1 \odot f_2) \otimes (e_1 \bullet e_2) \qquad (1.5.19)$$

for $f_1 \otimes e_1 \in F_1 \otimes E_1$ and $v := f_2 \otimes e_2 \in F_2 \otimes E_2$. By Proposition 1.5.2 there exists a unique hypocontinuous bilinear extension (1.5.17) of (1.5.15) satisfying

$$(f_1 \otimes e_1) \odot_\bullet v = e_1 \bullet (f_1 \diamondsuit v) \ , \qquad f_1 \otimes e_1 \in F_1 \otimes E_1 \ , \quad v \in F_2 \widetilde{\otimes} E_2 \ .$$

Thus (1.5.19) implies (1.5.18). ÷

The results of this subsection are due to L. Schwartz. In fact, they are special cases of much more general theorems given in [Sch57b, chap. II] (also see [Sch57a]).

General Hypothesis

Throughout the remainder of this section we suppose that

E, F, and E_j, $j = 0, 1, 2$, are Banach spaces and
$E_1 \times E_2 \to E_0$, $(e_1, e_2) \mapsto e_1 \bullet e_2$ (1.5.20)
is a multiplication. Moreover, X is a nonempty subset of $\mathbb{R}^n$
and convention (1.1.10) is eœective.

1.6 Point-Wise Multiplication

As a ørst application of the general extension result of Theorem 1.5.3 we deøne point-wise multiplication of a vector-valued smooth function with a vector-valued distribution and study some of its properties.

It is an immediate consequence of Leibniz' rule that for each $m \in \bar{\mathbb{N}}$ point-wise multiplication induced by (1.5.20),

$$C^m(X, E_1) \times C^m(X, E_2) \to C^m(X, E_0) \ , \quad (a_1, a_2) \mapsto a_1 \bullet a_2 \ , \tag{1.6.1}$$

is well-deøned, bilinear and continuous.

Given $a \in \mathcal{E}(X)$ and $v \in \mathcal{D}'(X, E)$, we recall that $av \in \mathcal{D}'(X, E)$ is deøned by

$$av(\varphi) := v(a\varphi) \ , \qquad \varphi \in \mathcal{D}(X) \ . \tag{1.6.2}$$

It is easily veriøed that

$$(\varphi \mapsto a\varphi) \in \mathcal{L}\big(\mathcal{D}(X)\big) \ , \qquad a \in \mathcal{E}(X) \ .$$

A Characterization of $\mathcal{O}_M$

First we prove a characterization of $\mathcal{O}_M(\mathbb{R}^n, E)$ that extends [Ama95, (III.4.1.9)]. In addition, it explains the name 'space of multipliers' for $\mathcal{O}_M(\mathbb{R}^n, E)$.

1.6.1 Proposition Suppose that $a \in \mathcal{E}(\mathbb{R}^n, E)$. Then

$$a \in \mathcal{O}_M(\mathbb{R}^n, E) \quad \text{iœ} \quad (\varphi \mapsto \varphi a) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n, E)\big) \ .$$

Proof Given $a \in \mathcal{O}_M(\mathbb{R}^n, E)$ and $k, m \in \mathbb{N}$, Leibniz' rule and (1.1.8) imply

$$q_{k,m}(\varphi a) \le c \max_{|\alpha| \le m} \sum_{\beta \le \alpha} \sup_{x \in \mathcal{R}^n} (1 + |x|^2)^k \, |\partial^\beta \varphi(x)| \, |\partial^{\alpha - \beta} a(x)| \le c q_{\ell,m}(\varphi) \tag{1.6.3}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where $\ell := k + \max\{\, m_{\alpha-\beta} \ ; \ \beta \le \alpha, \ |\alpha| \le m \,\}$ and where $m_{\alpha-\beta} \in \mathbb{N}$ are such that there exists $c_{\alpha-\beta}$ satisfying $|\partial^{\alpha-\beta} a(x)| \le c_{\alpha-\beta}(1 + |x|^2)^{m_{\alpha-\beta}}$ for $x \in \mathbb{R}^n$. This shows that

$$(\varphi \mapsto \varphi a) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n, E)\big) \ . \tag{1.6.4}$$

Conversely, let (1.6.4) be true. Then, given $\alpha \in \mathbb{N}$, there exist $k, m \in \mathbb{N}$ and a positive constant c such that

$$\|\partial^\alpha (\varphi a)\|_\infty \le q_{0,|\alpha|}(\varphi a) \le c q_{k,m}(\varphi) \ , \qquad \varphi \in \mathcal{S}(\mathbb{R}^n) \ .$$

Fix $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that ψ equals 1 near zero. Since $\tau_x\psi \in \mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ for $x \in \mathbb{R}^n$, it follows that

$$\begin{aligned} |\partial^\alpha a(x)| &= \bigl|\partial^\alpha\bigl((\tau_x\psi)a\bigr)(x)\bigr| \le c q_{k,m}(\tau_x\psi) \\ &\le c \sup_{\substack{|\alpha|\le m \\ y\in\mathbb{R}^n}} (1+|x-y|^2)^k\,|\partial^\alpha\psi(y)| \le c(1+|x|^2)^k \end{aligned} \tag{1.6.5}$$

for $x \in \mathbb{R}^n$, thanks to the trivial inequality $1+|x-y|^2 \le 2(1+|x|^2)(1+|y|^2)$. Hence $a \in \mathcal{O}_M(\mathbb{R}^n, E)$. ÷

1.6.2 Corollary Suppose that $a \in \mathcal{E}(\mathbb{R}^n)$. Then

$$a \in \mathcal{O}_M(\mathbb{R}^n) \quad \text{iœ} \quad (u \mapsto au) \in \mathcal{L}\bigl(\mathcal{S}'(\mathbb{R}^n)\bigr)\ .$$

Proof For simplicity, we put $\mathcal{O}_M = \mathcal{O}_M(\mathbb{R}^n)$ etc. If $a \in \mathcal{O}_M$, it follows from Proposition 1.6.1 that $(\varphi \mapsto \varphi a) \in \mathcal{L}(\mathcal{S})$. Hence the dual of this linear map, which is given by $u \mapsto au$ for $u \in \mathcal{S}'$, satisøes $(u \mapsto au) \in \mathcal{L}(\mathcal{S}')$. Conversely, $(u \mapsto au) \in \mathcal{L}(\mathcal{S}')$ implies $(u \mapsto au)' \in \mathcal{L}(\mathcal{S})$ by the reÆexivity of $\mathcal{S}$. Since the latter dual is the map $\varphi \mapsto \varphi a$ for $\varphi \in \mathcal{S}$, the assertion follows. ÷

The General Theorem

Now it is easy to prove the following technical lemma that will be used in the proof of the next theorem, one of the main results of this subsection.

1.6.3 Lemma Suppose that $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0)$ is one of the triplets $(\mathcal{E}, \mathcal{D}'; \mathcal{D}')$, $(\mathcal{E}, \mathcal{E}'; \mathcal{E}')$, $(\mathcal{S}, \mathcal{O}_M; \mathcal{S})$, $(\mathcal{O}_M, \mathcal{O}_M; \mathcal{O}_M)$, $(\mathcal{O}_M, \mathcal{S}'; \mathcal{S}')$, $(\mathcal{E}, \mathcal{E}; \mathcal{E})$, or $(\mathcal{E}, \mathcal{D}; \mathcal{D})$. Then 'point-wise multiplication'

$$\mathbf{F}_1(X) \times \mathbf{F}_2(X, E) \to \mathbf{F}_0(X, E)\ , \quad (a, u) \mapsto au \tag{1.6.6}$$

is a well-deøned hypocontinuous bilinear map.

Proof It follows from (1.1.8) and (1.1.9) that B is a bounded subset of $\mathcal{O}_M(\mathbb{R}^n, E)$ iœ, given $\alpha \in \mathbb{N}^n$, there exist $m_\alpha \in \mathbb{N}$ and $c_\alpha > 0$ such that

$$|\partial^\alpha u(x)| \le c_\alpha (1+|x|^2)^{m_\alpha}\ , \qquad x \in \mathbb{R}^n\ , \quad u \in B\ . \tag{1.6.7}$$

Hence (1.6.3) shows that

$$(a \mapsto au) \in \mathcal{L}\bigl(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n, E)\bigr)\ ,$$

uniformly with respect to u in bounded subsets of $\mathcal{O}_M(\mathbb{R}^n, E)$. Since the function deøned by $x \mapsto (1+|x|^2)^k \partial^\beta\varphi(x)$ belongs to $\mathcal{S}(\mathbb{R}^n)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $k \in \mathbb{N}$, and

$\beta \in \mathbb{N}^n$, it follows from (1.6.3) that, given $k, m \in \mathbb{N}$ and $a \in \mathcal{S}(\mathbb{R}^n)$, there exists $a_{k,\alpha} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$q_{k,m}(au) \leq c \max_{|\alpha| \leq k} \|a_{k,\alpha} \partial^\alpha u\|_\infty \ .$$

Hence

$$(u \mapsto au) \in \mathcal{L}\big(\mathcal{O}_M(\mathbb{R}^n, E), \mathcal{S}(\mathbb{R}^n, E)\big) \ , \qquad a \in \mathcal{S}(\mathbb{R}^n) \ . \tag{1.6.8}$$

Thus, letting $M_u := (a \mapsto au) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n, E)\big)$, we see from (1.6.8) that

$$(u \mapsto M_u) \in \mathcal{L}\Big(\mathcal{O}_M(\mathbb{R}^n, E), \mathcal{L}_s\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n, E)\big)\Big) \ .$$

Consequently, $\mathcal{S}(\mathbb{R}^n)$ being a Montel space, the Banach-Steinhaus theorem guarantees that

$$(u \mapsto M_u) \in \mathcal{L}\Big(\mathcal{O}_M(\mathbb{R}^n, E), \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n, E)\big)\Big) \ ,$$

that is, the map $u \mapsto au$ is continuous from $\mathcal{O}_M(\mathbb{R}^n, E)$ into $\mathcal{S}(\mathbb{R}^n, E)$, uniformly with respect to a in bounded subsets of $\mathcal{S}(\mathbb{R}^n)$. This proves the assertion for the case $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0) = (\mathcal{S}, \mathcal{O}_M; \mathcal{S})$.

Given $a \in \mathcal{O}_M(\mathbb{R}^n)$ and $u \in \mathcal{O}_M(\mathbb{R}^n, E)$, it follows from Leibniz' rule that

$$\|\varphi \partial^\alpha (au)\|_\infty \leq c \Big\| \varphi \sum_{\beta \leq \alpha} \partial^\beta a \partial^{\alpha - \beta} u \Big\|_\infty \ , \qquad \varphi \in \mathcal{S}(\mathbb{R}^n) \ .$$

Thus (1.6.7) implies the existence of $m \in \mathbb{N}$ such that

$$\|\varphi \partial^\alpha (au)\|_\infty \leq c \sum_{\beta \leq \alpha} \|(1 + |x|^2)^m \varphi \partial^\beta a\|_\infty \ , \qquad u \in B \ .$$

This shows that $a \mapsto au$ maps $\mathcal{O}_M(\mathbb{R}^n)$ continuously into $\mathcal{O}_M(\mathbb{R}^n, E)$, uniformly with respect to u in bounded subsets of $\mathcal{O}_M(\mathbb{R}^n, E)$. Similarly, we see that $u \mapsto au$ maps $\mathcal{O}_M(\mathbb{R}^n, E)$ continuously into itself, uniformly with respect to a in bounded subsets of $\mathcal{O}_M(\mathbb{R}^n)$. This proves the assertion if $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0) = (\mathcal{O}_M, \mathcal{O}_M; \mathcal{O}_M)$.

Since $\mathrm{supp}(au) \subset \mathrm{supp}(a) \cap \mathrm{supp}(u)$ for $a \in \mathcal{E}(X)$ and $u \in \mathcal{D}(X, E)$, Leibniz' rule and the properties of LF-spaces easily imply that point-wise multiplication is separately continuous from $\mathcal{E}(X) \times \mathcal{D}(X, E)$ into $\mathcal{D}(X, E)$. Now hypocontinuity in the case $(\mathcal{E}, \mathcal{D}; \mathcal{D})$ follows from the fact that $\mathcal{E}(X)$ and $\mathcal{D}(X, E)$ are barreled.

If $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0) = (\mathcal{E}, \mathcal{E}; \mathcal{E})$, the assertion follows from (1.6.1). In all other cases Proposition 1.6.1 and Corollary 1.6.2 guarantee that the map (1.6.6) is well-deøned, if this is not obvious anyhow. Moreover, in these cases

$$M_a := (\varphi \mapsto a\varphi) \in \mathcal{L}\big(\mathbf{F}_0'(X), \mathbf{F}_2'(X)\big) \ , \qquad a \in \mathbf{F}_1(X) \ ,$$

and $\mathcal{L}\big(\mathbf{F}_j'(X), E\big) = \mathbf{F}_j(X, E)$, $j = 0, 2$, by reÆexivity. Thus (1.6.6) is the linear map

$$\mathcal{L}\big(\mathbf{F}_0'(X), \mathbf{F}_2'(X)\big) \times \mathcal{L}\big(\mathbf{F}_2'(X), E\big) \to \mathcal{L}\big(\mathbf{F}_0'(X), E\big) \ , \quad (a, u) \mapsto u \circ M_a \ ,$$

and the assertion follows from Lemma 1.1.1. ÷

Now we can extend point-wise multiplication to the case that both factors are vector-valued.

1.6.4 Theorem There exists a unique hypocontinuous bilinear map

$$\mathcal{E}(X, E_1) \times \mathcal{D}'(X, E_2) \to \mathcal{D}'(X, E_0) , \quad (a, u) \mapsto a \bullet u , \tag{1.6.9}$$

called point-wise multiplication induced by (1.5.20), such that

$$(\varphi \otimes e_1) \bullet (\psi \otimes e_2) = \varphi\psi \otimes (e_1 \bullet e_2) \tag{1.6.10}$$

for $a := \varphi \otimes e_1 \in \mathcal{D}(X) \otimes E_1$ and $u := \psi \otimes e_2 \in \mathcal{D}(X) \otimes E_2$. It restricts to a hypocontinuous bilinear map

$$\mathbf{F}_1(X, E_1) \times \mathbf{F}_2(X, E_2) \to \mathbf{F}_0(X, E_0) ,$$

where $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0)$ is any one of the triplets $(\mathcal{E}, \mathcal{E}'; \mathcal{E}')$, $(\mathcal{S}, \mathcal{O}_M; \mathcal{S})$, $(\mathcal{O}_M, \mathcal{O}_M; \mathcal{O}_M)$, $(\mathcal{O}_M, \mathcal{S}'; \mathcal{S}')$, $(\mathcal{E}, \mathcal{E}; \mathcal{E})$, or $(\mathcal{E}, \mathcal{D}; \mathcal{D})$.

Proof Let $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0)$ be as stated or $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0) = (\mathcal{E}, \mathcal{D}'; \mathcal{D}')$, and write F_j for $\mathbf{F}_j(X)$, $j = 0, 1, 2$. Then the spaces F_j are reÆexive, complete, nuclear, and conuclear by Theorems 1.1.2 and 1.4.3. Moreover, $F_j \mathbin{\widetilde{\otimes}} E_j = \mathbf{F}_j(X, E_j)$, $j = 0, 1, 2$, thanks to Theorem 1.4.9. Deøne the bilinear maps (1.5.14) and (1.5.15) by point-wise multiplication. It is easily veriøed that (1.5.16) is true and it follows from Lemma 1.6.3 that the map (1.5.15) is hypocontinuous. Hence Theorem 1.5.3 guarantees the existence of a unique hypocontinuous bilinear map

$$\mathbf{F}_1(X, E_1) \times \mathbf{F}_2(X, E_2) \to \mathbf{F}_0(X, E_0) , \quad (a, u) \mapsto a \bullet u \tag{1.6.11}$$

satisfying (1.6.10) for $a := \varphi \otimes e_1 \in \mathbf{F}_1(X) \otimes E_1$ and $u := v \otimes e_2 \in \mathbf{F}_2(X) \otimes E_2$. The assertion is now an obvious consequence of the almost trivial fact that the separately continuous bilinear map (1.6.11) is uniquely determined by its restriction to the subspace $\bigl(\mathcal{D}(X) \otimes E_1\bigr) \times \bigl(\mathcal{D}(X) \otimes E_2\bigr)$ that is dense by Theorem 1.3.6. ÷

Basic Properties of Multiplications

1.6.5 Remarks (a) Since $E_2 \times E_1 \to E_0$, $(e_2, e_1) \mapsto e_1 \bullet e_2$ is a multiplication as well, the assertions of Theorem 1.6.4 are 'symmetric' with respect to E_1 and E_2, that is, the rles of E_1 and E_2 can be interchanged. This fact will often be employed in the following, usually without further mention.

(b) Point-wise multiplication induced by (1.5.20), as deøned in Theorem 1.6.4, coincides on regular distributions with point-wise multiplication in the usual sense. In other words, if $a \in \mathcal{E}(X, E_1)$ and $u \in L_{1,\mathrm{loc}}(X, E_2)$ then $a \bullet u \in L_{1,\mathrm{loc}}(X, E_0)$ and

$$a \bullet u(\cdot) = a(\cdot) \bullet u(\cdot) . \tag{1.6.12}$$

Observe that this justiøes the use of the name 'point-wise multiplication' for the bilinear map of Theorem 1.6.4.

Proof Since $L_{1,\mathrm{loc}}(X,E_2) \hookrightarrow \mathcal{D}'(X,E_2)$, it follows from Theorem 1.6.4 and an obvious estimate that both sides of (1.6.12) are separately continuous bilinear maps

$$\mathcal{E}(X,E_1) \times L_{1,\mathrm{loc}}(X,E_2) \to \mathcal{D}'(X,E_0) \ .$$

Theorem 1.3.6(viii) immediately implies

$$\mathcal{D}(X) \otimes E_2 \overset{d}{\hookrightarrow} L_{1,\mathrm{loc}}(X,E_2) \ . \tag{1.6.13}$$

Thus we deduce from (1.6.10) that both sides of (1.6.12) coincide on the dense subspace $\big(\mathcal{D}(X) \otimes E_1\big) \times \big(\mathcal{D}(X) \otimes E_2\big)$ of $\mathcal{E}(X,E_1) \times L_{1,\mathrm{loc}}(X,E_2)$. Hence the assertion follows. ÷

(c) Leibniz' rule is valid in the general case as well. More precisely, if p is a polynomial in n indeterminates then

$$p(\partial)(a \bullet u) = \sum_\beta \frac{1}{\beta!}(\partial^\beta a) \bullet p^{(\beta)}(\partial)u \tag{1.6.14}$$

for $a \in \mathcal{E}(X,E_1)$ and $u \in \mathcal{D}'(X,E_2)$, where $p^{(\beta)} := \partial^\beta p$. In particular:

$$\partial^\alpha(a \bullet u) = \sum_{\beta \le \alpha} \tbinom{\alpha}{\beta}(\partial^\beta a) \bullet \partial^{\alpha-\beta} u \ , \qquad a \in \mathcal{E}(X,E_1) \ , \quad u \in \mathcal{D}'(X,E_2) \ . \tag{1.6.15}$$

Proof It follows from $\partial^\alpha \in \mathcal{L}\big(\mathcal{E}(X,E_1)\big)$ and $p^{(\beta)}(\partial) \in \mathcal{L}\big(\mathcal{D}'(X,E_2)\big)$ that both sides of (1.6.14) deøne separately continuous bilinear maps:

$$\mathcal{E}(X,E_1) \times \mathcal{D}'(X,E_2) \to \mathcal{D}'(X,E_0) \ .$$

Thanks to Lemma 1.3.2 and to property (1.6.10) they coincide on the linear subspace $\big(\mathcal{D}(X) \otimes E_1\big) \times \big(\mathcal{D}(X) \otimes E_2\big)$. Since, by Theorem 1.3.6(ii) and (v), this subspace is dense in $\mathcal{E}(X,E_1) \times \mathcal{D}'(X,E_2)$, they coincide everywhere. ÷

(d) If $(a,u) \in \mathcal{E}(X,E_1) \times \mathcal{D}'(X,E_2)$ then

$$\mathrm{supp}(a \bullet u) \subset \mathrm{supp}(a) \cap \mathrm{supp}(u) \ .$$

Proof Let $K := \mathrm{supp}(a)$ and denote by $K_\varepsilon := \big\{\, x \in X \ ; \ \mathrm{dist}(x,K) < \varepsilon \,\big\}$ the open ε-neighborhood of K in X for $\varepsilon > 0$. Choose $\psi_\varepsilon \in \mathcal{E}(X)$ with $\mathrm{supp}(\psi_\varepsilon) \subset K_\varepsilon$ and $\psi \,|\, K = 1$. Since $\mathcal{E}(X) \otimes E_1$ is dense in $\mathcal{E}(X,E_1)$, there exists a sequence (a_j) in $\mathcal{E}(X) \otimes E_1$ converging in $\mathcal{E}(X,E_1)$ towards a. Then $b_j := \psi_\varepsilon a_j \in \mathcal{E}(X) \otimes E_1$ and $\mathrm{supp}(b_j) \subset K_\varepsilon$. It is easily veriøed that $\mathrm{supp}(b_j \bullet u) \subset \overline{K}_\varepsilon \cap \mathrm{supp}(u) =: M_\varepsilon$ and that

$$\mathcal{D}'_{M_\varepsilon}(X,E_0) := \big\{\, v \in \mathcal{D}'(X,E_0) \ ; \ \mathrm{supp}(v) \subset M_\varepsilon \,\big\}$$

is a closed linear subspace of $\mathcal{D}'(X,E_0)$. Now we deduce from Theorem 1.6.4 that $b_j \bullet u \to a \bullet u$ in $\mathcal{D}'(X,E_0)$, hence in $\mathcal{D}'_{M_\varepsilon}(X,E_0)$. Consequently, $\mathrm{supp}(a \bullet u) \subset M_\varepsilon$ for each $\varepsilon > 0$, which implies the assertion. ÷

(e) Let E_3, E_4, and E_5 be further Banach spaces and suppose that there are multiplications

$$\begin{tikzcd}[column sep=small]
E_1 \times E_2 \arrow[d] & & E_2 \times E_3 \arrow[d] \\
E_0 \times E_3 \arrow[dr] & & E_1 \times E_4 \arrow[dl] \\
& E_5 &
\end{tikzcd}$$

all denoted by $\bullet$, that are associative, that is,

$$(e_1 \bullet e_2) \bullet e_3 = e_1 \bullet (e_2 \bullet e_3) \ , \qquad e_j \in E_j \ , \quad j = 1,2,3 \ . \tag{1.6.16}$$

Then point-wise multiplication is associative as well, when deøned, that is,

$$(u_1 \bullet u_2) \bullet u_3 = u_1 \bullet (u_2 \bullet u_3) \ , \qquad u_j \in \mathbf{F}_j(X, E_j) \ , \quad j = 1,2,3 \ , \tag{1.6.17}$$

where $(\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3) = \big\{(\mathcal{E}, \mathcal{E}, \mathcal{D}'), (\mathcal{O}_M, \mathcal{O}_M, \mathcal{S}')\big\}$.

Proof It follows from Theorem 1.6.4 that both sides of (1.6.17) deøne separately continuous trilinear maps

$$\prod_{j=1}^{3} \mathbf{F}_j(X, E_j) \to \mathbf{F}_3(X, E_5)$$

that coincide on the dense linear subspace $\prod_{j=1}^{3} \mathcal{D}(X) \otimes E_j$, hence everywhere. ÷

(f) Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M, \mathcal{D}', \mathcal{E}', \mathcal{S}'\}$. Then the map

$$E_1 \times \mathbf{F}(X, E_2) \to \mathbf{F}(X, E_0) \ , \quad (e_1, u) \mapsto e_1 \bullet u := (\geqslant \otimes e_1) \bullet u \tag{1.6.18}$$

is bilinear and continuous, and

$$\partial^\alpha (e_1 \bullet u) = e_1 \bullet \partial^\alpha u \ , \qquad \alpha \in \mathbb{N}^n \ , \quad e_1 \in E_1 \ , \quad u \in \mathbf{F}(X, E_2) \ . \tag{1.6.19}$$

Proof Since $\geqslant \otimes e_1 \in \mathcal{O}_M(\mathbb{R}^n, E_1) \hookrightarrow \mathcal{E}(\mathbb{R}^n, E_1)$, it follows from Theorem 1.6.4 and Remark (a) that (1.6.18) is a well-deøned separately continuous bilinear map. Moreover, given $u = v \otimes e_2 \in \mathbf{F}(X) \otimes E_2$ and $\varphi \in \mathbf{F}'(X)$,

$$\begin{aligned}(e_1 \bullet u)(\varphi) &= \big[(\geqslant \otimes e_1) \bullet (v \otimes e_2)\big](\varphi) = \langle v, \varphi \rangle_{F'} e_1 \bullet e_2 = e_1 \bullet \big(\langle v, \varphi \rangle_{F'} e_2\big) \\ &= e_1 \bullet u(\varphi) \ .\end{aligned}$$

Thus the density of $\mathbf{F}(X) \otimes E_2$ in $\mathbf{F}(X, E_2)$ implies

$$(e_1 \bullet u)(\varphi) = e_1 \bullet u(\varphi) \ , \qquad e_1 \in E_1 \ , \quad u \in \mathbf{F}(X, E_2) \ , \quad \varphi \in \mathbf{F}'(X) \ . \tag{1.6.20}$$

Consequently, given a bounded subset B of $\mathbf{F}'(X)$,

$$\sup_{\varphi \in B} |(e_1 \bullet u)(\varphi)|_{E_0} \le |e_1|_{E_1} \sup_{\varphi \in B} |u(\varphi)|_{E_2} \ , \qquad e_1 \in E_1 \ , \quad u \in \mathbf{F}(X, E_2) \ .$$

This proves the continuity of (1.6.18), thanks to Theorem 1.4.9. Lastly, (1.6.19) is a consequence of (c). ÷

(g) Let $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M, \mathcal{D}', \mathcal{E}', \mathcal{S}'\}$ and consider the multiplication

$$\mathcal{L}(E,F) \times E \to F \ , \quad (T,e) \mapsto T \bullet e := Te \ . \tag{1.6.21}$$

Then it follows from (f) that (1.6.21) induces a continuous bilinear map

$$\mathcal{L}(E,F) \times \mathbf{F}(X,E) \to \mathbf{F}(X,F) \ , \quad (T,u) \mapsto T \bullet u := Tu \ .$$

In particular, each $T \in \mathcal{L}(E,F)$ induces a continuous linear map via (1.6.21),

$$(u \mapsto T \bullet u) \in \mathcal{L}\big(\mathbf{F}(X,E), \mathbf{F}(X,F)\big) \ . \tag{1.6.22}$$

We denote it by T as well and call it the linear map induced by $T \in \mathcal{L}(E,F)$ via point-wise multiplication. Moreover, (1.6.19) implies $\partial^\alpha \circ T = T \circ \partial^\alpha$ for $\alpha \in \mathbb{N}^n$, that is, the diagram

$$\begin{tikzcd}
\mathbf{F}(X,E) \arrow[r, "T"] \arrow[d, "\partial^\alpha"'] & \mathbf{F}(X,F) \arrow[d, "\partial^\alpha"] \\
\mathbf{F}(X,E) \arrow[r, "T"] & \mathbf{F}(X,F)
\end{tikzcd}$$

is commutative for $\alpha \in \mathbb{N}^n$. If $T \in \mathcal{L}(E,F)$ is injective, then (1.6.22) is injective as well. Consequently,

$$i : \ E \hookrightarrow F \quad \text{implies} \quad i : \ \mathbf{F}(X,E) \hookrightarrow \mathbf{F}(X,F) \ .$$

Proof We have to prove only the injectivity assertion. Thus suppose that $Tu = 0$ for some $u \in \mathbf{F}(X,E)$ and that $T \in \mathcal{L}(E,F)$ is injective. Then we obtain from (1.6.20) that $T\big(u(\varphi)\big) = 0$ for each $\varphi \in \mathbf{F}'(X)$. Hence $u(\varphi) = 0$ for each $\varphi \in \mathbf{F}'(X)$, that is, $u = 0$. ÷

(h) Suppose that $E_1 = E_2 =: E$ and multiplication (1.5.20) is symmetric, that is, $e_1 \bullet e_2 = e_2 \bullet e_1$ for $e_j \in E$. Then $a \bullet u = u \bullet a$ for $a \in \mathcal{E}(X,E)$ and $u \in \mathcal{D}'(X,E)$.

Proof This is an immediate consequence of (b), the density of $\mathcal{D}(X,E) \times \mathcal{D}(X,E)$ in $\mathcal{E}(X,E) \times \mathcal{D}'(X,E)$, and the separate continuity of point-wise multiplication. ÷

1.6.6 Corollary Suppose that $(E, \bullet)$ is a [commutative] Banach algebra and that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{O}_M\}$. Then $\mathbf{F}(X,E)$ is a locally convex [commutative] algebra with respect to point-wise multiplication, a multiplication algebra. Multiplication is continuous if $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$, and hypocontinuous if $\mathbf{F} = \mathcal{O}_M$. If E possesses a unit, e_0, then $\mathcal{E}(X,E)$ and $\mathcal{O}_M(\mathbb{R}^n, E)$ possess a unit as well, namely $\mathbb{1} \otimes e_0$.

Proof This follows from Theorem 1.6.4, Remarks 1.6.5(e) and (h), appropriate continuous injections given in Theorem 1.3.6, and the fact that hypocontinuous maps are continuous on barreled spaces. ÷

1.7 Scalar Products and Duality Pairings

In this subsection we show, in particular, that, given $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$, there exists a unique hypocontinuous bilinear map

$$\mathbf{F}'(X, E') \times \mathbf{F}(X, E) \to \mathbb{K} \tag{1.7.1}$$

that extends in a natural way the duality pairing $\mathbf{F}'(X) \times \mathbf{F}(X) \to \mathbb{K}$. In fact, we consider more general situations that will be needed in the remaining subsections.

1.7.1 Lemma Let $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{D}', \mathcal{E}', \mathcal{S}'\}$. Then the bilinear map

$$\mathbf{F}(X, E) \times \mathbf{F}'(X) \to E \ , \quad (u, \varphi) \mapsto u(\varphi) \tag{1.7.2}$$

is hypocontinuous.

Proof Since $\mathbf{F}'' = \mathbf{F}$, it follows from Theorem 1.4.9 that $\mathbf{F}(X, E) = \mathcal{L}\big(\mathbf{F}'(X), E\big)$. Hence the map (1.7.2) is well-deøned and continuous in u, uniformly on bounded subsets of $\mathbf{F}'(X)$. Thanks to the fact that $\mathbf{F}'(X)$ is barreled, bounded subsets of $\mathcal{L}\big(\mathbf{F}'(X), E\big)$ are equicontinuous by the uniform boundedness principle. Thus (1.7.2) is continuous in φ, uniformly with respect to u varying in bounded subsets of $\mathbf{F}(X, E)$. ÷

It is easy to prove the following general existence theorem that will imply, in particular, the desired extension (1.7.1).

1.7.2 Theorem Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{D}', \mathcal{E}', \mathcal{S}'\}$. Then there exists a unique hypocontinuous bilinear map

$$\mathbf{F}'(X, E_1) \times \mathbf{F}(X, E_2) \to E_0 \ , \quad (u', u) \mapsto \langle u' \bullet u \rangle_F \ , \tag{1.7.3}$$

the scalar product induced by multiplication (1.5.20), such that

$$\big\langle (\varphi \otimes e_1) \bullet (\psi \otimes e_2) \big\rangle_F = \langle \varphi, \psi \rangle_{\mathcal{D}} (e_1 \bullet e_2) \tag{1.7.4}$$

for $\varphi, \psi \in \mathcal{D}(X)$ and $e_j \in E_j, \ \ j = 1, 2$.

Proof Put $F_0 := \mathbb{K}$, $F_1 := \mathbf{F}'(X)$, and $F_2 := \mathbf{F}(X)$. Then the F_j are reÆexive, complete, nuclear, and conuclear LCSs. Consequently, Theorem 1.4.9 guarantees that $F_2 \mathbin{\widetilde{\otimes}} E_2 = \mathbf{F}(X, E_2)$, and it is trivially true that $F_0 \otimes E_0 = F_0 \mathbin{\widetilde{\otimes}} E_0 = E_0$. Deøne the bilinear maps (1.5.14) and (1.5.15) by $f_1 \odot f_2 := \langle f_1, f_2 \rangle_{F_2}$ and $f_1 \diamondsuit v := v(f_1)$, respectively. Then Lemma 1.7.1 shows that (1.5.15) is hypocontinuous. Since it is obvious that condition (1.5.16) is satisøed, Theorem 1.5.3 gives the assertion. ÷

1.7.3 Remark Suppose that $u \in L_{1,\mathrm{loc}}(X, E_1)$ and $v \in \mathcal{D}(X, E_2)$. Then

$$\langle u \bullet v \rangle_{\mathcal{D}} = \int_X u(x) \bullet v(x) \, dx \ . \tag{1.7.5}$$

Proof Since $L_{1,\mathrm{loc}}(X,E_1) \overset{d}{\hookrightarrow} \mathcal{D}'(X,E_1)$, Theorem 1.7.2 implies that the map

$$L_{1,\mathrm{loc}}(X,E_1) \times \mathcal{D}(X,E_2) \to E_0 \ , \quad (u,v) \mapsto \langle u \bullet v\rangle_{\mathcal{D}} \tag{1.7.6}$$

is bilinear and separately continuous. From (1.6.13) and Theorem 1.3.6 we infer that

$$\big(\mathcal{D}(X) \otimes E_1\big) \times \big(\mathcal{D}(X) \otimes E_2\big) \overset{d}{\subset} L_{1,\mathrm{loc}}(X,E_1) \times \mathcal{D}(X,E_2) \ . \tag{1.7.7}$$

It is obvious that (1.7.5) is true for $u \in \mathcal{D}(X) \otimes E_1$ and $v \in \mathcal{D}(X) \otimes E_2$. Hence the assertion follows from (1.7.7) and the separate continuity of the map (1.7.6). ÷

Parseval's Formula

As a ørst application of Theorem 1.7.2 we obtain a natural extension of Parseval's formula for the Fourier transform.

1.7.4 Proposition Parseval's formula is valid for vector-valued distributions, that is,

$$\langle u \bullet \varphi\rangle_{\mathcal{S}} = (2\pi)^{-n}\langle \widehat{u} \bullet \check{\widehat{\varphi}}\rangle_{\mathcal{S}} \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E_1) \ , \quad \varphi \in \mathcal{S}(\mathbb{R}^n, E_2) \ . \tag{1.7.8}$$

Proof Thanks to Theorem 1.7.2 and the fact that the Fourier transform and reÆection are toplinear automorphisms, it follows that both sides of (1.7.8) are separately continuous bilinear maps from $\mathcal{S}'(\mathbb{R}^n, E_1) \times \mathcal{S}(\mathbb{R}^n, E_2)$ into E_0. By (1.7.4) and the well-known Parseval formula for scalar distributions the two expressions on either side of (1.7.8) coincide on the dense linear subspace

$$\big(\mathcal{S}'(\mathbb{R}^n) \otimes E_1\big) \times \big(\mathcal{S}(\mathbb{R}^n) \otimes E_2\big) \ .$$

Hence they are equal. ÷

Duality Pairings

By specializing the above results to the case where (1.5.20) is a duality pairing we obtain the desired map (1.7.1).

1.7.5 Theorem Let E be a Banach space and let $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}\}$. Then there exists a unique hypocontinuous bilinear map

$$\mathbf{F}'(X,E') \times \mathbf{F}(X,E) \to \mathbb{K} \ , \quad (u', u) \mapsto \langle u', u\rangle_{F(X,E)} \ ,$$

the duality pairing between $\mathbf{F}'(X,E')$ and $\mathbf{F}(X,E)$, such that

$$\langle u', u\rangle_{F(X,E)} = \int_X \big\langle u'(x), u(x)\big\rangle_E \, dx \tag{1.7.9}$$

for $u' \in \mathcal{D}(X,E')$ and $u \in \mathcal{D}(X,E)$.

Proof Let $E_1 := E'$, $E_2 := E$, and $E_0 := \mathbb{K}$, and put $e_1 \bullet e_2 := \langle e_1, e_2\rangle_E$. Then the assertion follows from Theorem 1.7.2 and Remark 1.7.3. ÷

1.7.6 Remarks (a) It is also true that

$$\langle u', u\rangle_{\mathcal{D}(X,E)} = \int_X \bigl\langle u'(x), u(x)\bigr\rangle_E \, dx \;, \qquad (u', u) \in L_{1,\mathrm{loc}}(X, E') \times \mathcal{D}(X, E) \;,$$

and

$$\langle u', u\rangle_{\mathcal{S}(\mathbb{R}^n,E)} = \int_{\mathbb{R}^n} \bigl\langle u'(x), u(x)\bigr\rangle_E \, dx \;, \qquad (u', u) \in L_p(\mathbb{R}^n, E') \times \mathcal{S}(\mathbb{R}^n, E) \;,$$

where $1 \le p < \infty$.

Proof The first part of the assertion is a consequence of Remark 1.7.3. The second one follows from

$$\mathcal{D}(\mathbb{R}^n, E') \overset{d}{\hookrightarrow} L_p(\mathbb{R}^n, E') \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^n, E') \;, \qquad 1 \le p < \infty \;,$$

from $\mathcal{D}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^n, E)$, and from the separate continuity of the bilinear map

$$(u', u) \mapsto \int_{\mathbb{R}^n} \bigl\langle u'(x), u(x)\bigr\rangle_E \, dx$$

on $L_p(\mathbb{R}^n, E') \times \mathcal{S}(\mathbb{R}^n, E)$. ÷

(b) Theorem 1.7.5 contains a more precise description of the duality pairing of Corollary 1.4.10. ÷

1.8 Tensor Products of Distributions and Kernel Theorems

It is the purpose of this subsection to define tensor products of arbitrary vector-valued distributions. These results are of interest for their own sake. In addition, they will be the basis for an important extension of convolutions of vector-valued distributions given in the next section.

Approximation by Tensor Products

1.8.1 Theorem Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{D}', \mathcal{E}', \mathcal{S}'\}$ and $X_j \subset \mathbb{R}^{n_j}$, $j = 1, 2$, are open. Then $\mathcal{D}(X_1) \otimes \mathcal{D}(X_2, E)$ is sequentially dense in $\mathbf{F}(X_1 \times X_2, E)$.

Proof Let $\varphi \in \mathcal{D}(X_1 \times X_2, E)$ be given and put $K_j := \mathrm{pr}_j\bigl(\mathrm{supp}(\varphi)\bigr)$, where

$$\mathrm{pr}_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_j} \;, \qquad j = 1, 2 \;,$$

are the canonical projections. Then $K_j \subset\subset X_j$ and there exist $\psi_j \in \mathcal{D}(X_j)$ with $\psi_j \,|\, K_j = 1$. It follows that $\psi_1 \otimes \psi_2 \in \mathcal{D}(X_1 \times X_2)$ and $\psi_1 \otimes \psi_2 \,|\, \mathrm{supp}(\varphi) = 1$.

Thanks to Lemma 1.3.7 there exists a sequence (p_k) of polynomials on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with coeŒcients in E, converging in $\mathcal{E}(X_1 \times X_2, E)$ towards φ. Then the sequence $\big((\psi_1 \otimes \psi_2)p_k\big)_{k\in\mathcal{N}}$ lies in $\mathcal{D}(X_1) \otimes \mathcal{D}(X_2, E)$ and converges in $\mathcal{E}(X_1 \times X_2, E)$ towards φ. Since the supports of $(\psi_1 \otimes \psi_2)p_k$ are contained in a øxed compact set, the convergence takes place in $\mathcal{D}(X_1 \times X_2, E)$. It is clear that

$$\mathcal{D}(X_1) \otimes \mathcal{D}(X_2, E) \subset \mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E) \subset \mathbf{F}(X_1 \times X_2, E) \ . \tag{1.8.1}$$

Since $\mathcal{D}(X_1 \times X_2, E)$ is sequentially dense in $\mathbf{F}(X_1 \times X_2, E)$ by [Ama95, Corollary V.2.4.2] and Remark 1.3.4, the assertion follows from (1.8.1) and what has already been proven, provided $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{F}\}$. If $\mathbf{F} \in \{\mathcal{D}', \mathcal{E}', \mathcal{F}'\}$, we obtain from the already proven facts and Proposition 1.3.3 that

$$\mathcal{D}(X_1) \otimes \mathcal{D}(X_2, E) \overset{d}{\subset} \mathcal{D}(X_1 \times X_2, E) \overset{d}{\subset} \mathbf{F}(X_1 \times X_2, E)$$

with sequential density. ÷

1.8.2 Corollary If $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}, \mathcal{S}, \mathcal{D}', \mathcal{E}', \mathcal{S}'\}$ then $\mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$ is sequentially dense in $\mathbf{F}(X_1 \times X_2, E)$.

Proof This is now obvious from $\mathcal{D}(X_1) \otimes \mathcal{D}(X_2, E) \subset \mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$ and from observation (1.3.16). ÷

Tensor Products of Distributions

As a preparation for deøning tensor products of distributions we establish the following technical result.

1.8.3 Lemma Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}\}$. If $\varphi \in \mathbf{F}(X_1 \times X_2)$ and $u_1 \in \mathbf{F}'(X_1, E)$ then

$$\big[x_2 \mapsto u_1\big(\varphi(\cdot, x_2)\big)\big] \in \mathbf{F}(X_2, E)$$

and

$$\partial^\alpha u_1\big(\varphi(\cdot, x_2)\big) = u_1\big(\partial_2^\alpha \varphi(\cdot, x_2)\big) \ , \qquad \alpha \in \mathbb{N}^{n_2} \ .$$

Proof If $\varphi \in \mathcal{D}(X_1 \times X_2)$, let $K_j := \mathrm{pr}_j\big(\mathrm{supp}(\varphi)\big)$. Then $\mathrm{supp}(\varphi) \subset K_1 \times K_2$ and $K_j \subset\subset X_j$. From this it follows easily that

$$\big[x_2 \mapsto \varphi(\cdot, x_2)\big] \in \mathbf{F}\big(X_2, \mathbf{F}(X_1)\big) \ .$$

Now, using $u_1 \in \mathbf{F}'(X_1, E) = \mathcal{L}\big(\mathbf{F}(X_1), E\big)$, it is not diŒcult to derive the asserted properties. ÷

Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}\}$ and let $\varphi \in \mathbf{F}(X_1 \times X_2)$ and $u_j \in \mathbf{F}'(X_j, E_j)$ be given. Then we put

$$u_1\big(\varphi(x_1, x_2)\big) := \big[x_2 \mapsto u_1\big(\varphi(\cdot, x_2)\big)\big] \tag{1.8.2}$$

and

$$u_2\big(\varphi(x_1, x_2)\big) := \big[x_1 \mapsto u_2\big(\varphi(x_1, \cdot)\big)\big] \ . \tag{1.8.3}$$

In other words, the symbolic notation $u_j\big(\varphi(x_1, x_2)\big)$ means that the distribution u_j acts on the function following it as a function of the variable x_j, the other variable being a free parameter.

It follows from Lemma 1.8.3 and Theorem 1.7.2 that, given $u_j \in \mathbf{F}'(X_j, E_j)$, $j = 1, 2$, and $\varphi \in \mathbf{F}(X_1 \times X_2)$,

$$\big\langle u_1 \bullet u_2\big(\varphi(x_1, x_2)\big)\big\rangle_F \ , \quad \big\langle u_2 \bullet u_1\big(\varphi(x_1, x_2)\big)\big\rangle_F \tag{1.8.4}$$

are well-deøned elements of E_0.

Now we are prepared for the proof of the main result of this subsection.

1.8.4 Theorem Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}\}$. Given $u_j \in \mathbf{F}'(X_j, E_j)$, $j = 1, 2$, there exists a unique distribution $u_1 \otimes_\bullet u_2 \in \mathbf{F}'(X_1 \times X_2, E_0)$, the tensor product of u_1 and u_2 with respect to the multiplication (1.5.20), satisfying

$$u_1 \otimes_\bullet u_2(\varphi_1 \otimes \varphi_2) = u_1(\varphi_1) \bullet u_2(\varphi_2) \ , \qquad \varphi_j \in \mathbf{F}(X_j) \ , \tag{1.8.5}$$

such that the map

$$\mathbf{F}'(X_1, E_1) \times \mathbf{F}'(X_2, E_2) \to \mathbf{F}'(X_1 \times X_2, E_0) \ , \quad (u_1, u_2) \mapsto u_1 \otimes_\bullet u_2 \tag{1.8.6}$$

is bilinear and hypocontinuous. It can be evaluated by Fubini's rule:

$$u_1 \otimes_\bullet u_2(\varphi) = \big\langle u_1 \bullet u_2\big(\varphi(x_1, x_2)\big)\big\rangle_F = \big\langle u_1\big(\varphi(x_1, x_2)\big) \bullet u_2\big\rangle_{F'} \tag{1.8.7}$$

for $\varphi \in \mathbf{F}(X_1 \times X_2)$.

Proof Put $X_0 := X_1 \times X_2$ and $F_j := \mathbf{F}'(X_j)$, $j = 0, 1, 2$. Then the F_j satisfy the hypotheses of Theorem 1.5.3 and $F_j \mathbin{\widetilde{\otimes}} E_j = \mathbf{F}'(X_j, E_j)$, $j = 0, 1, 2$, by Theorem 1.4.9. Deøne the maps (1.5.14) and (1.5.15) by

$$\varphi \mapsto f_1 \odot f_2(\varphi) := \big\langle f_1, f_2\big(\varphi(x_1, x_2)\big)\big\rangle_F$$

and

$$\varphi \mapsto f_1 \diamondsuit v(\varphi) \ := v\big(f_1\big(\varphi(x_1, x_2)\big)\big)$$

for $\varphi \in \mathbf{F}(X_0)$, respectively. It follows from Lemma 1.8.3 that they are well-deøned. From that lemma we also infer that

$$T_{f_1} := \big[\varphi \mapsto f_1\big(\varphi(x_1, \cdot)\big)\big] \in \mathcal{L}\big(\mathbf{F}(X_0), \mathbf{F}(X_2)\big) \ . \tag{1.8.8}$$

Thus, since $f_1 \diamondsuit v = v \circ T_{f_1}$, we obtain from Lemma 1.1.1 that the map (1.5.15) is hypocontinuous. Given $f_j \in F_j,\ \varphi_j \in \mathbf{F}(X_j),\ j = 1, 2$, and $e_2 \in E_2$, it is clear that

$$\bigl[f_1 \diamondsuit (f_2 \otimes e_2)\bigr](\varphi_1 \otimes \varphi_2) = \langle f_1, \varphi_1\rangle_F \langle f_2, \varphi_2\rangle_F e_2 = \bigl[(f_1 \odot f_2) \otimes e_2\bigr](\varphi_1 \otimes \varphi_2) \ .$$

Since $\mathbf{F}(X_1) \otimes \mathbf{F}(X_2)$ is dense in $\mathbf{F}(X_0)$ by Corollary 1.8.2, we infer that condition (1.5.16) is satisfied. Now the first part of the assertion follows from Theorem 1.5.3.

From Lemma 1.8.3 we deduce that

$$\bigl[\varphi \mapsto u_j\bigl(\varphi(x_1, x_2)\bigr)\bigr] \in \mathcal{L}\bigl(\mathbf{F}(X_0), \mathbf{F}(X_{3-j}, E_{3-j})\bigr) \ , \qquad u_j \in \mathbf{F}'(X_j, E_j) \ , \quad j = 1, 2 \ .$$

Hence, given $\varphi \in \mathbf{F}(X_0)$, Theorem 1.7.2 guarantees the existence of hypocontinuous scalar products $\bigl\langle u_1 \bullet u_2\bigl(\varphi(x_1, x_2)\bigr)\bigr\rangle_F$ and $\bigl\langle u_1\bigl(\varphi(x_1, x_2)\bigr) \bullet u_2\bigr\rangle_{F'}$ satisfying

$$\bigl\langle (f_1 \otimes e_1) \bullet (f_2 \otimes e_2)\bigl(\varphi(x_1, x_2)\bigr)\bigr\rangle_F = \bigl\langle f_1, f_2\bigl(\varphi(x_1, x_2)\bigr)\bigr\rangle_F (e_1 \bullet e_2)$$

and

$$\bigl\langle (f_1 \otimes e_1)\bigl(\varphi(x_1, x_2)\bigr) \bullet (f_2 \otimes e_2)\bigr\rangle_{F'} = \bigl\langle f_1\bigl(\varphi(x_1, x_2)\bigr), f_2\bigr\rangle_{F'} (e_1 \bullet e_2)$$

for $u_j := f_j \otimes e_j \in \mathbf{F}'(X_j) \otimes E_j,\ \ j = 1, 2$, respectively. Choosing $\varphi := \varphi_1 \otimes \varphi_2$ in $\mathbf{F}(X_1) \otimes \mathbf{F}(X_2)$, we see that

$$\begin{aligned} \bigl\langle u_1 \bullet u_2\bigl(\varphi(x_1, x_2)\bigr)\bigr\rangle_F &= \bigl\langle u_1\bigl(\varphi(x_1, x_2)\bigr) \bullet u_2\bigr\rangle_{F'} \\ &= \langle f_1, \varphi_1\rangle_F \langle f_2, \varphi_2\rangle_F (e_1 \bullet e_2) \\ &= \bigl(\langle f_1, \varphi_1\rangle_F e_1\bigr) \bullet \bigl(\langle f_2, \varphi_2\rangle_F e_2\bigr) \\ &= u_1(\varphi_1) \bullet u_2(\varphi_2) = u_1 \otimes_\bullet u_2(\varphi_1 \otimes \varphi_2) \ . \end{aligned} \tag{1.8.9}$$

From $u_1 \otimes_\bullet u_2 \in \mathbf{F}(X_0, E_0)$, from (1.8.8), and from Theorem 1.7.2 we infer that each one of the trilinear maps

$$\mathbf{F}(X_0) \times \mathbf{F}'(X_1, E_1) \times \mathbf{F}'(X_2, E_2) \to E_0$$

sending (φ, u_1, u_2)

$$\text{to } u_1 \otimes_\bullet u_2(\varphi) \ , \quad \text{to } \bigl\langle u_1 \bullet u_2\bigl(\varphi(x_1, x_2)\bigr)\bigr\rangle_F \ , \quad \text{or to } \bigl\langle u_1\bigl(\varphi(x_1, x_2)\bigr) \bullet u_2\bigr\rangle_{F'} \ ,$$

respectively, is separately continuous. By (1.8.9) they are all equal on the linear subspace

$$\bigl(\mathbf{F}(X_1) \otimes \mathbf{F}(X_2)\bigr) \times \bigl(\mathbf{F}'(X_1) \otimes E_2\bigr) \times \bigl(\mathbf{F}'(X_2) \otimes E_2\bigr) \ ,$$

which is dense by Theorem 1.3.6 and Corollary 1.8.2. Hence they are all equal, that is, Fubini's rule is valid. ÷

Basic Properties

1.8.5 Remarks (a) If $u_j \in \mathcal{D}'(X_j, E_j)$, $j = 1, 2$, then

$$\operatorname{supp}(u_1 \otimes_\bullet u_2) \subset \operatorname{supp}(u_1) \times \operatorname{supp}(u_2)$$

with equality, provided $e_1 \bullet e_2 \neq 0$ whenever $e_j \in \ddot{E}_j$.

Proof Let $A_j := \operatorname{supp}(u_j)$. If $\operatorname{supp}(\varphi) \subset \mathbb{R}^n \times A_2^c$ then

$$u_2\bigl(\varphi(x_1, x_2)\bigr) = 0 \ , \qquad x_1 \in X_1 \ ,$$

hence $u_1 \otimes_\bullet u_2(\varphi) = 0$ by Fubini's rule. Similarly, we see that $u_1 \otimes_\bullet u_2(\varphi) = 0$ if $\operatorname{supp}(\varphi) \subset A_1^c \times \mathbb{R}^{n_2}$. Consequently, $\operatorname{supp}(u_1 \otimes_\bullet u_2) \subset A_1 \times A_2$. On the other hand, given $(x_1, x_2) \in A_1 \times A_2$ and any product neighborhood $U_1 \times U_2$ of (x_1, x_2) in $X_1 \times X_2$, there exist $\varphi_j \in \mathcal{D}(U_j)$ such that $u_j(\varphi_j) \neq 0$. Hence

$$u_1 \otimes_\bullet u_2(\varphi_1 \otimes \varphi_2) = u_1(\varphi_1) \bullet u_2(\varphi_2) \neq 0$$

if $e_1 \bullet e_2 \neq 0$ for $e_j \in \ddot{E}_j$. ÷

(b) Given $u_j \in \mathcal{D}'(X_j, E_j)$,

$$\partial_1^{\alpha_1} \partial_2^{\alpha_2} (u_1 \otimes_\bullet u_2) = (\partial_1^{\alpha_1} u_1) \otimes_\bullet (\partial_2^{\alpha_2} u_2) \ , \qquad \alpha_j \in \mathbb{N}^{n_j} \ . \tag{1.8.10}$$

Proof It follows from Theorem 1.8.4 and the continuity properties of the derivative that both sides of (1.8.10) deøne separately continuous bilinear maps from $\mathcal{D}'(X_1, E_1) \times \mathcal{D}'(X_2, E_2)$ into $\mathcal{D}'(X_1 \times X_2, E_0)$. From (1.8.5) it is obvious that they coincide on the dense linear subspace $\bigl(\mathcal{D}'(X_1) \otimes E_1\bigr) \times \bigl(\mathcal{D}'(X_2) \otimes E_2\bigr)$, hence everywhere. ÷

(c) Let the associativity hypotheses of Remark 1.6.5(e) be satisøed and let X_3 be a nonempty open subset of $\mathbb{R}^{n_3}$. Then

$$(u_1 \otimes_\bullet u_2) \otimes_\bullet u_3 = u_1 \otimes_\bullet (u_2 \otimes_\bullet u_3) \ , \qquad u_j \in \mathcal{D}'(X_j, E_j) \ , \quad j = 1, 2, 3 \ . \tag{1.8.11}$$

Proof By Theorem 1.8.4 both sides of (1.8.11) deøne separately continuous trilinear maps

$$\prod_{j=1}^{3} \mathcal{D}'(X_j, E_j) \to \mathcal{D}'(X_1 \times X_2 \times X_3, E_5) \ .$$

By (1.8.5), Remark 1.4.1(c), and Example 1.4.2(c) they coincide on the dense linear subspace $\prod_{j=1}^{3} \mathcal{D}'(X_j) \otimes E_j$, hence everywhere. ÷

(d) If $u_j \in L_{1,\mathrm{loc}}(X_j, E_j)$ then $u_1 \otimes_\bullet u_2 \in L_{1,\mathrm{loc}}(X_1 \times X_2, E_0)$ and

$$u_1 \otimes_\bullet u_2(x_1, x_2) = u_1(x_1) \otimes_\bullet u_2(x_2) \qquad \text{for a.a. } (x_1, x_2) \in X_1 \times X_2 \ .$$

Proof This follows easily from (1.8.7) and the classical Fubini theorem. ÷

(e) Distributivity Suppose that

$$\begin{tikzcd}[column sep=small]
E_1 \times E_2 \arrow[d] & E_3 \times E_4 \arrow[d] & & E_1 \times E_3 \arrow[d] & E_2 \times E_4 \arrow[d] \\
E_0 \arrow[r, phantom, "\times"] & E_5 \arrow[dr] & & E_6 \arrow[r, phantom, "\times"] & E_7 \\
& & E_8 \arrow[from=2-4, phantom] \arrow[from=2-4] & &
\end{tikzcd}$$

are multiplications between Banach spaces, all denoted by $\bullet$. Also suppose that

$$(e_1 \bullet e_2) \bullet (e_3 \bullet e_4) = (e_1 \bullet e_3) \bullet (e_2 \bullet e_4) \ , \qquad e_j \in E_j \ , \quad j = 1, \ldots, 4 \ ,$$

and $a_j \in \mathcal{E}(X_j, E_j)$ and $u_j \in \mathcal{D}'(X_j, E_{j+2})$, $j = 1, 2$. Then

$$(a_1 \otimes_\bullet a_2) \bullet (u_1 \otimes_\bullet u_2) = (a_1 \bullet u_1) \otimes_\bullet (a_2 \bullet u_2) \ . \tag{1.8.12}$$

Proof First we note that, thanks to (d), $a_1 \otimes_\bullet a_2 \in \mathcal{E}(X_1 \times X_2, E_0)$. Hence Theorems 1.8.4 and 1.6.4 guarantee that both sides of (1.8.12) are well-deøned quadrilinear separately continuous maps from

$$\mathcal{E}(X_1, E_1) \times \mathcal{E}(X_2, E_2) \times \mathcal{D}'(X_1, E_3) \times \mathcal{D}'(X_2, E_4)$$

into $\mathcal{D}'(X_1 \times X_2, E_8)$. It follows from (1.8.5) and (d) that they coincide on the dense linear subspace

$$\big(\mathcal{E}(X_1) \otimes E_1\big) \times \big(\mathcal{E}(X_2) \otimes E_2\big) \times \big(\mathcal{D}'(X_1) \otimes E_3\big) \times \big(\mathcal{D}'(X_2) \otimes E_4\big) \ ,$$

hence everywhere. ÷

(f) Let F_j, $j = 0, 1, 2$, be Banach spaces and suppose that

$$F_1 \times F_2 \to F_0 \ , \quad (f_1, f_2) \mapsto f_1 \bullet f_2$$

is a multiplication. Also suppose that $T_j \in \mathcal{L}(E_j, F_j)$, $j = 0, 1, 2$, satisfy

$$(T_1 e_1) \bullet (T_2 e_2) = T_0(e_1 \bullet e_2) \ , \qquad e_j \in E_j \ , \quad j = 1, 2 \ ,$$

that is, the diagram

$$\begin{tikzcd}
E_1 \times E_2 \arrow[r, "\bullet"] \arrow[d, "T_1 \times T_2"'] & E_0 \arrow[d, "T_0"] \\
F_1 \times F_2 \arrow[r, "\bullet"] & F_0
\end{tikzcd}$$

is commutative. Then, given $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}\}$, the diagram

$$\begin{tikzcd} \mathbf{F}'(X_1, E_1) \times \mathbf{F}'(X_2, E_2) \arrow[r, "\otimes_\bullet"] \arrow[d, "T_1 \times T_2"] & \mathbf{F}'(X_1 \times X_2, E_0) \arrow[d, "T_0"] \\ \mathbf{F}'(X_1, F_1) \times \mathbf{F}'(X_2, F_2) \arrow[r, "\otimes_\bullet"] & \mathbf{F}'(X_1 \times X_2, F_0) \end{tikzcd}$$

is commutative as well.

Proof We have to show that

$$(T_1 u_1) \otimes_\bullet (T_2 u_2) = T_0(u_1 \otimes_\bullet u_2) \ , \qquad u_j \in \mathbf{F}'(X_j, E_j) \ , \quad j = 1, 2 \ .$$

From Remark 1.6.5(g) and Theorem 1.8.4 we infer that both sides of this equality define separately continuous bilinear maps that coincide on the dense linear subspace $\bigl(\mathbf{F}'(X_1) \otimes E_1\bigr) \times \bigl(\mathbf{F}'(X_2) \otimes E_2\bigr)$. Thus they are equal. $\square$

Examples

Finally, we include a few simple examples demonstrating the usefulness of tensor products.

1.8.6 Examples (a) The Dirac distribution $\delta \in \mathcal{E}'(\mathbb{R}^n)$ on $\mathbb{R}^n$ is the n-fold tensor product of the 'one-dimensional' Dirac distribution $\delta \in \mathcal{E}'(\mathbb{R})$. Thus, in symbolic notation,

$$\delta(x) = \delta(x_1) \otimes \delta(x_2) \otimes \cdots \otimes \delta(x_n) \ , \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ ,$$

where $\delta(y) := \delta \in \mathcal{E}'(\mathbb{R}^m)$ for $y \in \mathbb{R}^m$ and $\otimes$ is the tensor product induced by the standard multiplication in $\mathbb{K}$.

(b) Suppose that $n = n_1 + n_2$ with $0 \le n_1 \le n - 1$. Let X_1 be open in $\mathbb{R}^{n_1}$ and let X_2 be a nonempty open cube in $\mathbb{R}^{n_2}$. Also suppose that $u \in \mathcal{D}'(X_1 \times X_2, E)$ satisfies $\partial_j u = 0$, $n_1 + 1 \le j \le n$. Then $u = u_1 \otimes \mathbf{1}_{X_2}$ for some $u_1 \in \mathcal{D}'(X_1, E)$, where $\mathcal{D}'\bigl(\{0\}, E\bigr) := E$. Lastly, $u \in \mathcal{D}'(X, E)$ is constant if $\partial_j u = 0$ for $1 \le j \le n$ and X is connected.

Proof (i) First we consider the case $n_1 = 0$ and $n = 1$. Thus X is a nonempty open interval in $\mathbb{R}$ and $u \in \mathcal{D}'(X, E)$ satisfies $\partial u = 0$. Fix any $\varphi_0 \in \mathcal{D}(X)$ with $\langle \mathbf{1}, \varphi_0 \rangle = 1$. Then, given $\varphi \in \mathcal{D}(X)$, put $\psi := \varphi - \langle \mathbf{1}, \varphi \rangle \varphi_0 \in \mathcal{D}(X)$. Let

$$\chi(x) := \int_{-\infty}^{x} \psi(t)\, dt \ , \qquad x \in \mathbb{R} \ ,$$

and observe that $\chi \in \mathcal{D}(X)$. Consequently,

$$0 = (\partial u)(\chi) = -u(\partial \chi) = -u(\psi) = \langle \mathbf{1}, \varphi \rangle u(\varphi_0) - u(\varphi) \ ,$$

which shows that $u = u(\varphi_0) \mathbf{1}_X$, that is, u is constant.

(ii) If Y_j are open subsets of $\mathbb{R}^{m_j}$ for $j = 0, 1, 2$ and $Y_0 := Y_1 \times Y_2$, then $\geqslant_{Y_0} = \geqslant_{Y_1} \otimes \geqslant_{Y_2}$. Using this fact and induction, it suffices to consider the case $n_2 = 1$.

Given $\varphi_1 \in \mathcal{D}(X_1)$, it is easily verified that

$$u_{\varphi_1} := \big(\varphi_2 \mapsto u(\varphi_1 \otimes \varphi_2)\big) \in \mathcal{D}'(X_2, E) \ . \tag{1.8.13}$$

By assumption, $\partial u_{\varphi_1} = 0$. Hence (i) implies the existence of $e(\varphi_1) \in E$ such that $u_{\varphi_1} = e(\varphi_1)\geqslant_{X_2}$. Consequently,

$$u(\varphi_1 \otimes \varphi_2) = e(\varphi_1)\langle \geqslant_{X_2}, \varphi_2 \rangle \ , \qquad \varphi_2 \in \mathcal{D}(X_2) \ . \tag{1.8.14}$$

Now fix $\varphi_2 \in \mathcal{D}(X_2)$ with $\langle \geqslant_{X_2}, \varphi_2 \rangle = 1$ and note that

$$u_1 := \big(\varphi_1 \mapsto u(\varphi_1 \otimes \varphi_2)\big) \in \mathcal{D}'(X_1, E) \ .$$

By (1.8.14) we see that $u_1(\varphi_1) = e(\varphi_1)$ so that

$$u(\varphi_1 \otimes \varphi_2) = u_1(\varphi_1)\langle \geqslant_{X_2}, \varphi_2 \rangle \ , \qquad \varphi_j \in \mathcal{D}(X_j) \ .$$

Now the assertion follows from Theorem 1.8.4.

(iii) If $u \in \mathcal{D}'(X, E)$ satisfies $\partial_j u = 0$ for $1 \le j \le n$, we infer from (ii) that each point $x \in X$ has a neighborhood U_x in X such that $u \,|\, U_x = e_x \geqslant_{U_x}$ for some $e_x \in E$. If $U_x \cap U_y \neq \emptyset$ then it follows from

$$e_x\langle \geqslant, \varphi \rangle = u(\varphi) = e_y\langle \geqslant, \varphi \rangle \ , \qquad \varphi \in \mathcal{D}(U_x \cap U_y) \ ,$$

that $e_x = e_y$. Let x and y be any two points in X. Since X is connected there is a continuous path $\gamma : [0,1] \to X$ connecting x and y. Since $\gamma\big([0,1]\big) \subset \bigcup_{x \in X} U_x$ and $\gamma\big([0,1]\big)$ is compact and connected, we can find a finite chain $U_0, U_1, \ldots, U_m$ of these neighborhoods covering $\gamma\big([0,1]\big)$ and satisfying $U_{j-1} \cap U_j = \emptyset$. Then the above considerations imply that $e_x = e_y$ for $x, y \in X$, so that $u = e \otimes \geqslant_X$ for some $e \in E$. ÷

(c) Let $\Theta := \chi_{\mathcal{R}^+} \in L_{1,\mathrm{loc}}(\mathbb{R})$ be the Heaviside function. Then

$$\chi_P := \underbrace{\chi_{\mathcal{R}^+} \otimes \cdots \otimes \chi_{\mathcal{R}^+}}_{n}$$

is the characteristic function of the natural positive cone $P := (\mathbb{R}^+)^n$ of $\mathbb{R}^n$. Since $\partial\Theta = \delta$, we deduce from Remark 1.8.5(b) and from (a) that

$$\partial_1 \cdots \partial_n \chi_P = \partial\chi_{\mathcal{R}^+} \otimes \cdots \otimes \partial\chi_{\mathcal{R}^+} = \delta(x_1) \otimes \cdots \otimes \delta(x_n) = \delta(x) \ .$$

(d) Suppose that $n = n_1 + n_2$ with $0 \le n_1 \le n - 1$, and that $u \in \mathcal{E}'(\mathbb{R}^n, E)$ has its support in $\mathbb{R}^{n_1} \times \{0\}$. Then there exist $m \in \mathbb{N}$ and $u_\alpha \in \mathcal{E}'(\mathbb{R}^{n_1}, E)$ for $\alpha \in \mathbb{N}^{n_2}$ with $|\alpha| \le m$ such that

$$u = \sum_{|\alpha| \le m} u_\alpha \otimes \partial^\alpha \delta(x_2) \ , \qquad x_2 \in \mathbb{R}^{n_2} \ .$$

This representation is unique.

Proof Since $u \in \mathcal{E}'(\mathbb{R}^n, E)$, there exist $m \in \mathbb{N}$ and $K \subset\subset \mathbb{R}^n$ such that

$$|u(\varphi)| \le c p_{m,K}(\varphi) \ , \qquad \varphi \in \mathcal{E}(\mathbb{R}^n) \ . \tag{1.8.15}$$

Let $\{ \varphi_\varepsilon \ ; \ \varepsilon > 0 \}$ be a molliøer on $\mathbb{R}^{n_2}$. Then, putting $\ell := n_2$ and

$$\psi_\varepsilon := \varphi_{\varepsilon/4} * \chi_{(\varepsilon/2)\mathbb{B}^\ell} \ , \qquad \varepsilon > 0 \ ,$$

it is easily veriøed that $\psi_\varepsilon \in \mathcal{D}(\mathbb{R}^{n_2})$ satisøes $\psi_\varepsilon(0) = 1$, $\operatorname{supp}(\psi_\varepsilon) \subset \varepsilon\mathbb{B}^{n_2}$, and

$$\|\partial^\alpha \psi_\varepsilon\|_\infty \le c(\alpha, n_2)\varepsilon^{-|\alpha|} \ , \qquad \alpha \in \mathbb{N}^{n_2} \ , \quad \varepsilon > 0 \ . \tag{1.8.16}$$

Thus, given $\varphi_j \in \mathcal{E}(\mathbb{R}^{n_j})$, $j = 1, 2$, we see that $\varphi_1 \otimes (1 - \psi_\varepsilon)\varphi_2 \in \mathcal{E}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ has its support in $\mathbb{R}^{n_1} \times (\mathbb{R}^{n_2})^{\bullet}$. Consequently,

$$u(\varphi_1 \otimes \varphi_2) = u(\varphi_1 \otimes \psi_\varepsilon\varphi_2) + u\big(\varphi_1 \otimes (1 - \psi_\varepsilon)\varphi_2\big) = u(\varphi_1 \otimes \psi_\varepsilon\varphi_2) \ , \qquad \varepsilon > 0 \ ,$$

thanks to $\operatorname{supp}(u) \subset \mathbb{R}^{n_1} \times \{0\}$. Hence we infer from (1.8.15) and (1.8.16) that

$$\begin{aligned} |u(\varphi_1 \otimes \varphi_2)| &\le c p_{m,K}(\varphi_1 \otimes \psi_\varepsilon\varphi_2) \\ &\le c \max_{|\alpha_1|+|\alpha_2| \le m} p_{K_1}(\partial^{\alpha_1}\varphi_1) p_{\varepsilon\mathbb{B}^\ell}\big(\partial^{\alpha_2}(\psi_\varepsilon\varphi_2)\big) \\ &\le c \max_{|\alpha_1| \le m} p_{K_1}(\partial^{\alpha_1}\varphi_1) \sum_{|\beta| \le m-|\alpha_1|} \varepsilon^{|\beta|-m+|\alpha_1|} p_{\varepsilon\mathbb{B}^\ell}(\partial^\beta\varphi_2) \ , \end{aligned} \tag{1.8.17}$$

where K_1 is the canonical projection of K into $\mathbb{R}^{n_1}$.

Now suppose that $\partial^\alpha\varphi_2(0) = 0$ for $|\alpha| \le m$. Then Taylor's theorem implies

$$|\partial^\beta\varphi_2(x_2)| \le \varepsilon^{m-|\alpha_1|-|\beta|} \sum_{|\gamma|=m-|\alpha_1|} p_{\varepsilon\mathbb{B}^\ell}(\partial^\gamma\varphi_2) \ , \qquad \varepsilon > 0 \ , \quad x_2 \in \varepsilon\mathbb{B}^\ell \ ,$$

for $|\beta| \le m - |\alpha_1|$ and $|\alpha_1| \le m$. Thus we deduce from (1.8.17) that

$$|u(\varphi_1 \otimes \varphi_2)| \le c p_{m,K_1}(\varphi_1) p_{m,\varepsilon\mathbb{B}^\ell}(\varphi_2) \ , \qquad \varepsilon > 0 \ ,$$

for $\varphi_j \in \mathcal{E}(\mathbb{R}^{n_j})$, $j = 1, 2$, with $\partial^\alpha\varphi_2(0) = 0$ for $|\alpha| \le m$. Observe that the right-hand side of the last estimate tends to zero as $\varepsilon \to 0$ thanks to $\partial^\alpha\varphi_2(0) = 0$ for $|\alpha| \le m$. Consequently,

$$u(\varphi_1 \otimes \varphi_2) = 0 \quad \text{for} \quad \varphi_j \in \mathcal{E}(\mathbb{R}^{n_j}) \qquad \text{with} \quad \partial^\alpha\varphi_2(0) = 0 \ , \quad |\alpha| \le m \ . \tag{1.8.18}$$

Given any $\varphi_2 \in \mathcal{E}(\mathbb{R}^{n_2})$, Taylor's theorem gives

$$\varphi_2(x_2) = \sum_{|\alpha| \le m} \frac{1}{\alpha!}\partial^\alpha\varphi_2(0)x_2^\alpha + \psi(x_2) \ , \qquad x_2 \in \mathbb{R}^{n_2} \ ,$$

where $\psi \in \mathcal{E}(\mathbb{R}^{n_2})$ satisfies $\partial^\alpha \psi(0) = 0$ for $|\alpha| \le m$. Thus (1.8.18) implies

$$u(\varphi_1 \otimes \varphi_2) = u\Big(\varphi_1 \otimes \sum_{|\alpha| \le m} \frac{1}{\alpha!} \partial^\alpha \varphi_2(0) x_2^\alpha\Big) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} u_\alpha(\varphi_1) \partial^\alpha \varphi_2(0)$$

for $\varphi_j \in \mathcal{E}(\mathbb{R}^{n_j})$, where

$$u_\alpha(\varphi_1) := \frac{(-1)^{|\alpha|}}{\alpha!} u(\varphi_1 \otimes x_2^\alpha) \ , \qquad |\alpha| \le m \ .$$

Now Theorem 1.8.4 entails

$$u = \sum_{|\alpha| \le m} u_\alpha \otimes \partial^\alpha \delta(x_2)$$

with $u_\alpha \in \mathcal{E}'(\mathbb{R}^{n_1}, E)$.

Finally, suppose that there are $\widetilde{m} \in \mathbb{N}$ and $\widetilde{u}_\alpha \in \mathcal{E}'(\mathbb{R}^{n_1}, E)$ such that

$$u = \sum_{|\alpha| \le \widetilde{m}} \widetilde{u}_\alpha \otimes \partial^\alpha \delta(x_2) \ .$$

By replacing m and $\widetilde{m}$ by $\max\{m, \widetilde{m}\}$, we can assume that $m = \widetilde{m}$. Then

$$0 = \sum_{|\alpha| \le m} (u_\alpha - \widetilde{u}_\alpha) \otimes \partial^\alpha \delta(x_2)(\varphi_1 \otimes x^\beta) = (-1)^{|\beta|} \beta! \, (u_\beta - \widetilde{u}_\beta)(\varphi_1)$$

for $|\beta| \le m$ and $\varphi_1 \in \mathcal{E}(\mathbb{R}^{n_1})$. This proves the asserted uniqueness. ÷

Topological Tensor Products of Distributions

Now we consider the special case of tensor products of a scalar and a vector-valued distribution.

1.8.7 Theorem Suppose that $\mathbf{F} \in \{\mathcal{E}, \mathcal{S}\}$. Then

$$\mathbf{F}(X_1) \mathbin{\widetilde{\otimes}} \mathbf{F}(X_2, E) = \mathbf{F}(X_1 \times X_2, E) \ .$$

Proof Thanks to Corollary 1.8.2 it suffices to show that $\mathbf{F}(X_1 \times X_2, E)$ induces on $\mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$ the projective topology.

Given a continuous seminorm p on $\mathbf{F}(X_1 \times X_2, E)$ belonging to the families (1.1.3) and (1.1.7), respectively, it is easily verified that there exist continuous seminorms r on $\mathbf{F}(X_1)$ and s on $\mathbf{F}(X_2, E)$, respectively, such that

$$p(\varphi \otimes \psi) \le r(\varphi) s(\psi) \ , \qquad \varphi \in \mathbf{F}(X_1) \ , \quad \psi \in \mathbf{F}(X_2, E) \ .$$

Hence, given a representation $\sum \varphi_j \otimes \psi_j$ of $z \in \mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$, it follows that

$$p(z) \leq \sum p(\varphi_j \otimes \psi_j) \leq \sum r(\varphi_j) s(\psi_j) \ .$$

This implies that $p(z) \leq r \otimes_\pi s(z)$ for $z \in \mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$. Thus $\mathbf{F}(X_1 \times X_2, E)$ induces on $\mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$ a topology weaker than the projective topology.

Conversely, let r and s be continuous seminorms on $\mathbf{F}(X_1)$ and $\mathbf{F}(X_2, E)$, respectively, belonging to the respective families (1.1.3) and (1.1.7), and let $\sum \varphi_j \otimes \psi_j$ be a representation of $z \in \mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$. Observe that

$$\sum \langle \varphi', \varphi_j \rangle \langle \psi', \psi_j \rangle = \Big\langle \varphi', \Big\langle \psi', \sum \varphi_j \otimes \psi_j \Big\rangle \Big\rangle \ , \qquad \varphi' \in \mathbf{F}'(X_1) \ , \quad \psi' \in \mathbf{F}'(X_2, E) \ .$$

Hence, cf. (1.4.7),

$$r \otimes_\varepsilon s(z) = \sup_{\substack{\varphi' \in \mathbb{B}_r^0 \\ \psi' \in \mathbb{B}_s^0}} \Big| \sum \langle \varphi', \varphi_j \rangle \langle \psi', \psi_j \rangle \Big| = r\Big(s\Big(\sum \varphi_j \otimes \psi_j \Big) \Big)$$

by the bipolar theorem, where $\mathbb{B}_r$ is the open r-unit-ball in $\mathbf{F}(X_1)$ and $\mathbb{B}_s$ is the open s-unit-ball in $\mathbf{F}(X_2, E)$. From this we infer the existence of a continuous seminorm p on $\mathbf{F}(X_1 \times X_2, E)$ such that $r \otimes_\varepsilon s(z) \leq p(z)$ for $z \in \mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$. Thus $\mathbf{F}(X_1 \times X_2, E)$ induces on $\mathbf{F}(X_1) \otimes \mathbf{F}(X_2, E)$ a topology that is stronger than the injective tensor product topology. Since $\mathbf{F}(X_1)$ is nuclear, the latter coincides with the projective topology (e.g. [Tre67, Theorem 50.1(f)]). ÷

1.8.8 Corollary If $\mathbf{F} \in \{\mathcal{E}, \mathcal{S}\}$ then

$$\mathbf{F}'(X_1) \mathbin{\widetilde{\otimes}} \mathbf{F}'(X_2, E') = \mathbf{F}(X_1 \times X_2, E)' \ .$$

Proof From Corollary 1.4.10 and Theorem 1.8.7 we know that

$$\mathbf{F}'(X_1 \times X_2, E') = \mathbf{F}(X_1 \times X_2, E)' = \big(\mathbf{F}_1(X_1) \mathbin{\widetilde{\otimes}} \mathbf{F}(X_2, E) \big)' \ .$$

Hence the assertion follows from (1.4.16) and by applying Corollary 1.4.10 once more. ÷

It should be remarked that

$$\mathcal{D}(X_1) \mathbin{\widetilde{\otimes}} \mathcal{D}(X_2) \neq \mathcal{D}(X_1 \times X_2)$$

as topological vector spaces, although these spaces are equal as vector spaces. Moreover,

$$\mathcal{D}'(X_1) \mathbin{\widetilde{\otimes}} \mathcal{D}'(X_2) \neq \big(\mathcal{D}(X_1) \mathbin{\widetilde{\otimes}} \mathcal{D}(X_2) \big)'$$

(cf. [Sch57b, chap. I, p. 95]).

Kernel Theorems

After these preparations we can prove some ‘vector-valued kernel theorems’:

1.8.9 Theorem Let E be a Banach space and let X_j be open in $\mathbb{R}^{n_j}$, $j = 1, 2$. Then

$$\begin{aligned} \mathcal{E}(X_1 \times X_2, E) &\cong \mathcal{L}\big(\mathcal{E}'(X_1), \mathcal{E}(X_2, E)\big) \ , \\ \mathcal{E}'(X_1 \times X_2, E') &\cong \mathcal{L}\big(\mathcal{E}(X_1), \mathcal{E}'(X_2, E')\big) \ , \\ \mathcal{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, E) &\cong \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^{n_1}), \mathcal{S}(\mathbb{R}^{n_2}, E)\big) \ , \\ \mathcal{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, E') &\cong \mathcal{L}\big(\mathcal{S}(\mathbb{R}^{n_1}), \mathcal{S}'(\mathbb{R}^{n_2}, E')\big) \ , \end{aligned}$$

where $\cong$ denotes the canonical toplinear isomorphism.

Proof These assertions are immediate consequences of Theorems 1.4.8 and 1.8.7, and of Corollary 1.8.8. ÷

For completeness we also include a proof of the ‘classical’ Schwartz kernel theorem for scalar distributions. To do this we rely on the following functional-analytical result:

1.8.10 Lemma Suppose that $\mathbf{F} \in \{\mathcal{D}, \mathcal{E}\}$. Let (u_α) be a net in $\mathbf{F}'(X_1) \otimes_\pi \mathbf{F}'(X_2)$. Then (u_α) converges to zero iœ $\langle u_\alpha, \varphi_1 \otimes \varphi_2 \rangle \to 0$, uniformly with respect to φ_j in bounded subsets of $\mathbf{F}(X_j)$, $j = 1, 2$.

Proof Let F and G be complete LCSs such that F is nuclear. Then it follows from [Sch71, Corollaries 1 and 2 to Theorem IV.9.4], for example, that a net (v_α) converges in $F \mathbin{\widetilde{\otimes}} G$ towards zero iœ $v_\alpha(f' \otimes g') \to 0$, uniformly with respect to f' and g' in equicontinuous subsets of F'_{w^*} and G'_{w^*}, respectively. Put $F := \mathbf{F}'(X_1)$ and $G := \mathbf{F}'(X_2)$. Let B_j be a bounded subset of $\mathbf{F}(X_j)$. Then B_j is weakly bounded, hence bounded in $\mathbf{F}(X_j)_{w^*} = \mathcal{L}_s\big(\mathbf{F}'(X_j), \mathbb{C}\big)$, by the reÆexivity of $\mathbf{F}(X_j)$. Thus the uniform boundedness principle implies that B_j is equicontinuous in $\mathbf{F}(X_j)_{w^*}$. Thus if (u_α) is a net in $\mathbf{F}'(X_1) \mathbin{\widetilde{\otimes}} \mathbf{F}'(X_2)$ converging towards zero, it follows from $\big[\mathbf{F}'(X_j)\big]'_{w^*} = \mathbf{F}(X_j)_{w^*}$ that $\langle u_\alpha, \varphi_1 \otimes \varphi_2 \rangle \to 0$, uniformly with respect to φ_j in bounded subsets of $\mathbf{F}(X_j)$, $j = 1, 2$, as asserted. ÷

1.8.11 Theorem $\mathcal{D}'(X_1 \times X_2) \cong \mathcal{L}\big(\mathcal{D}(X_1), \mathcal{D}'(X_2)\big)$ by means of the canonical toplinear isomorphism.

Proof It is easily veriøed that the bilinear map

$$\mathcal{D}(X_1) \times \mathcal{D}(X_2) \to \mathcal{D}(X_1 \times X_2) \ , \quad (\varphi_1, \varphi_2) \mapsto \varphi_1 \otimes \varphi_2$$

is continuous, hence hypocontinuous. Given $u \in \mathcal{D}'(X_1 \times X_2)$ and $\varphi_1 \in \mathcal{D}(X_1)$, put

$$\varphi_1 \cdot u(\varphi_2) := u(\varphi_1 \otimes \varphi_2) \ , \qquad \varphi_2 \in \mathcal{D}(X_2) \ ,$$

(using obvious identiøcations). It follows that $\varphi_1 \cdot u \in \mathcal{D}'(X_2)$ and

$$T_u := (\varphi_1 \mapsto \varphi_1 \cdot u) \in \mathcal{L}\big(\mathcal{D}(X_1), \mathcal{D}'(X_2)\big) .$$

Let (u_α) be a net in $\mathcal{D}'(X_1 \times X_2)$ converging towards zero. Then $\langle u_\alpha, \varphi\rangle \to 0$, uniformly with respect to φ in bounded subsets of $\mathcal{D}(X_1 \times X_2)$. Consequently, $T_{u_\alpha}\varphi_1 = \varphi_1 \cdot u_\alpha \to 0$ in $\mathcal{D}'(X_2)$, uniformly with respect to φ_1 in bounded subsets of $\mathcal{D}(X_1)$. This shows that

$$T := (u \mapsto T_u) \in \mathcal{L}\Big(\mathcal{D}'(X_1 \times X_2), \mathcal{L}\big(\mathcal{D}(X_1), \mathcal{D}'(X_2)\big)\Big) . \tag{1.8.19}$$

By Theorem 1.4.8 we know that

$$\mathcal{D}'(X_1) \mathbin{\widetilde{\otimes}} \mathcal{D}'(X_2) \cong \mathcal{L}\big(\mathcal{D}(X_1), \mathcal{D}'(X_2)\big) . \tag{1.8.20}$$

If $u = u_1 \otimes u_2 \in \mathcal{D}'(X_1) \otimes \mathcal{D}'(X_2)$, we see that $T_u\varphi_1 = \langle u_1, \varphi_1\rangle u_2$ for $\varphi_1 \in \mathcal{D}(X_1)$. This shows that $T\,|\,\mathcal{D}'(X_1) \otimes \mathcal{D}'(X_2)$ equals the canonical injection (1.4.5) of the tensor product $\mathcal{D}'(X_1) \otimes \mathcal{D}'(X_2)$ in $\mathcal{L}\big(\mathcal{D}(X_1), \mathcal{D}'(X_2)\big)$. Thus we infer from (1.8.19) that $\mathcal{D}'(X_1 \times X_2)$ induces on $\mathcal{D}'(X_1) \otimes \mathcal{D}'(X_2)$ a topology that is stronger than the projective topology.

Conversely, let (u_α) be a net in $\mathcal{D}'(X_1) \otimes_\pi \mathcal{D}'(X_2)$ converging towards zero. Then $\langle u_\alpha, \varphi_1 \otimes \varphi_2\rangle \to 0$, uniformly with respect to φ_j in bounded subsets of $\mathcal{D}(X_j)$, $j = 1, 2$, thanks to Lemma 1.8.10. Hence, given any $\psi_j \in \mathcal{D}(X_j)$, $j = 1, 2$, it follows that

$$\big\langle (\psi_1 \otimes \psi_2)u_\alpha, \varphi_1 \otimes \varphi_2 \big\rangle_{\mathcal{E}} = \langle u_\alpha, \psi_1\varphi_1 \otimes \psi_2\varphi_2\rangle_{\mathcal{D}} \to 0 ,$$

uniformly with respect to φ_j in bounded subsets of $\mathcal{E}(X_j)$, $j = 1, 2$. Now we infer from Lemma 1.8.10 that $(\psi_1 \otimes \psi_2)u_\alpha \to 0$ in $\mathcal{E}'(X_1) \otimes \mathcal{E}'(X_2)$. Theorem 1.8.9 implies $(\psi_1 \otimes \psi_2)u_\alpha \to 0$ in $\mathcal{E}'(X_1 \times X_2)$ for each pair $(\psi_1, \psi_2) \in \mathcal{D}(X_1) \times \mathcal{D}(X_2)$. This entails $u_\alpha\,|\,\mathcal{D}_K(X_1 \times X_2) \to 0$ for $K \subset\subset X_1 \times X_2$. Consequently, $u_\alpha \to 0$ in $\mathcal{D}'(X_1 \times X_2)$, which shows that the projective topology is stronger than the topology induced by $\mathcal{D}'(X_1 \times X_2)$ on $\mathcal{D}'(X_1) \otimes \mathcal{D}'(X_2)$.

We know from Corollary 1.8.2 that $\mathcal{D}'(X_1) \otimes \mathcal{D}'(X_2)$ is dense in $\mathcal{D}'(X_1 \times X_2)$. Hence it follows that

$$\mathcal{D}'(X_1) \mathbin{\widetilde{\otimes}} \mathcal{D}'(X_2) = \mathcal{D}'(X_1 \times X_2) , \tag{1.8.21}$$

which, together with (1.8.20), proves the assertion. ÷

1.8.12 Remarks (a) It may be useful to reinterprete the Kernel Theorems 1.8.9 and 1.8.11. Suppose, for example, that $k \in \mathcal{E}'(X_1 \times X_2, E)$ and deøne K by the relation

$$(K\varphi_1)(\varphi_2) = k(\varphi_1 \otimes \varphi_2) , \qquad \varphi_j \in \mathcal{E}(X_j) , \quad j = 1, 2 . \tag{1.8.22}$$

Then Theorem 1.8.9 guarantees that

$$K \in \mathcal{L}\big(\mathcal{E}(X_1), \mathcal{E}'(X_2, E)\big) . \tag{1.8.23}$$

Conversely, for any K satisfying (1.8.23) there exists a unique $k \in \mathcal{E}'(X_1 \times X_2, E)$ such that (1.8.22) is true. Moreover, the map

$$\mathcal{E}'(X_1 \times X_2, E) \to \mathcal{L}\big(\mathcal{E}(X_1), \mathcal{E}'(X_2, E)\big) \ , \quad k \mapsto K \tag{1.8.24}$$

is a toplinear isomorphism. The distribution k is said to be the kernel of the map K, and K is associated to the kernel k. In symbolic notation the isomorphism (1.8.24) is written as

$$K\varphi_1 = \int_{X_1} k(x_1, \cdot)\varphi_1(x_1)\, dx_1 \ , \qquad \varphi_1 \in \mathcal{E}(X_1) \ .$$

In this case (1.8.22) takes the suggestive form

$$\int_{X_2} (K\varphi_1)(x_2)\varphi_2(x_2)\, dx_2 = \int_{X_1 \times X_2} k(x_1, x_2)\varphi_1(x_1)\varphi_2(x_2)\, d(x_1, x_2) \ .$$

The kernel $k \in \mathcal{E}'(X_1 \times X_2, E)$ is said to be regularizing if its associated map K can be extended from $\mathcal{D}(X_1)$ to a continuous linear map of $\mathcal{E}'(X_1)$ into $\mathcal{E}(X_2, E)$. Thanks to Theorem 1.8.9 this is the case iœ $k \in \mathcal{E}(X_1 \times X_2, E)$. Similar deønitions and conventions apply to tempered distributions and to the classical case of the Schwartz Kernel Theorem (1.8.11).

(b) It is easily veriøed that the ørst part of the proof of Theorem 1.8.11 remains valid if we replace $\mathcal{D}'(X_2)$ and $\mathcal{D}'(X_1 \times X_2)$ by $\mathcal{D}'(X_2, E)$ and $\mathcal{D}'(X_1 \times X_2, E)$, respectively. From this we infer that

$$\mathcal{D}'(X_1 \times X_2, E) \hookrightarrow \mathcal{L}\big(\mathcal{D}(X_1), \mathcal{D}'(X_2, E)\big) \tag{1.8.25}$$

by means of obvious identiøcations. This signiøes that each kernel distribution $k \in \mathcal{D}'(X_1 \times X_2, E)$ induces a unique continuous linear operator K from $\mathcal{D}(X_1)$ to $\mathcal{D}'(X_2, E)$, and that the map $k \mapsto K$ is continuous and injective. But we do not know whether (1.8.25) is surjective. However,

$$\mathcal{L}\big(\mathcal{D}(X_1), \mathcal{D}'(X_2, E)\big) \cong \mathcal{D}'(X_1) \,\widetilde{\otimes}\, \mathcal{D}'(X_2) \,\widetilde{\otimes}\, E \ ,$$

where $\cong$ denotes the canonical toplinear isomorphism.

Proof From Lemma 1.4.7 we infer that $\mathcal{D}'(X_2, E)$ is complete. Hence Theorem 1.4.8 guarantees that

$$\mathcal{L}\big(\mathcal{D}(X_1), \mathcal{D}'(X_2, E)\big) \cong \mathcal{D}'(X_1) \,\widetilde{\otimes}\, \mathcal{D}'(X_2, E) \ .$$

Now the assertion follows from Theorem 1.4.9 and the natural associativity of topological tensor products (cf. [Gro55, I.1.4]). ÷

1.9 Convolutions of Vector-Valued Distributions

In Subsection 1.2 we have already defined convolutions of a vector-valued and a scalar distribution. In this subsection we extend these definitions to the case of two vector-valued distributions. In addition, we weaken the support restrictions.

The Basic Theorem

1.9.1 Theorem Suppose that either $u_1 \in \mathcal{D}'(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{E}'(\mathbb{R}^n, E_2)$, or $u_1 \in \mathcal{S}(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{S}'(\mathbb{R}^n, E_2)$. Then there exists a unique distribution in $\mathcal{D}'(\mathbb{R}^n, E_0)$ or $\mathcal{O}_M(\mathbb{R}^n, E_0)$, respectively, the convolution of u_1 and u_2 with respect to multiplication (1.5.20), denoted by $u_1 *_\bullet u_2$, such that

$$(v_1 \otimes e_1) *_\bullet (v_2 \otimes e_2) = (v_1 * v_2) \otimes (e_1 \bullet e_2) \tag{1.9.1}$$

for $v_1, v_2 \in \mathcal{D}(\mathbb{R}^n)$, $e_j \in E_j$, and $j = 1, 2$, and such that the 'convolution maps' $(u_1, u_2) \mapsto u_1 *_\bullet u_2$ are bilinear and hypocontinuous:

$$\mathcal{D}'(\mathbb{R}^n, E_1) \times \mathcal{E}'(\mathbb{R}^n, E_2) \to \mathcal{D}'(\mathbb{R}^n, E_0) \tag{1.9.2}$$

and

$$\mathcal{S}(\mathbb{R}^n, E_1) \times \mathcal{S}'(\mathbb{R}^n, E_2) \to \mathcal{O}_M(\mathbb{R}^n, E_0) , \tag{1.9.3}$$

respectively. In addition, the convolution maps restrict to hypocontinuous bilinear maps:

$$\mathcal{E}'(\mathbb{R}^n, E_1) \times \mathcal{E}'(\mathbb{R}^n, E_2) \to \mathcal{E}'(\mathbb{R}^n, E_0) , \tag{1.9.4}$$

$$\mathcal{E}(\mathbb{R}^n, E_1) \times \mathcal{E}'(\mathbb{R}^n, E_2) \to \mathcal{E}(\mathbb{R}^n, E_0) , \tag{1.9.5}$$

$$\mathcal{D}(\mathbb{R}^n, E_1) \times \mathcal{D}'(\mathbb{R}^n, E_2) \to \mathcal{E}(\mathbb{R}^n, E_0) , \tag{1.9.6}$$

$$\mathcal{D}(\mathbb{R}^n, E_1) \times \mathcal{E}'(\mathbb{R}^n, E_2) \to \mathcal{D}(\mathbb{R}^n, E_0) , \tag{1.9.7}$$

$$\mathcal{S}(\mathbb{R}^n, E_1) \times \mathcal{S}(\mathbb{R}^n, E_2) \to \mathcal{S}(\mathbb{R}^n, E_0) . \tag{1.9.8}$$

Proof Let $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0)$ be either of the triplets $(\mathcal{D}', \mathcal{E}'; \mathcal{D}')$, $(\mathcal{S}, \mathcal{S}'; \mathcal{O}_M)$, $(\mathcal{E}', \mathcal{E}'; \mathcal{E}')$, $(\mathcal{E}, \mathcal{E}'; \mathcal{E})$, $(\mathcal{D}, \mathcal{D}'; \mathcal{E})$, $(\mathcal{D}, \mathcal{E}'; \mathcal{D})$, or $(\mathcal{S}, \mathcal{S}; \mathcal{S})$, and put $F_j := \mathbf{F}_j(\mathbb{R}^n)$, $j = 0, 1, 2$. Then the F_j are reflexive, complete, nuclear, and conuclear by Theorems 1.1.2 and 1.4.3. Moreover, $F_j \widetilde{\otimes} E_j = \mathbf{F}_j(\mathbb{R}^n, E_j)$ by Theorem 1.4.9. Define the maps (1.5.14) and (1.5.15) by $f_1 \odot f_2 := f_1 * f_2$ and $f_1 \diamondsuit v := f_1 * v$, respectively. From Propositions 1.2.1, 1.2.3, and 1.2.7 we know that they are well-defined and that (1.5.15) is hypocontinuous. Definition (1.2.1) of convolution immediately implies that condition (1.5.16) is satisfied. Hence Theorem 1.5.3 guarantees the existence of a unique hypocontinuous bilinear map

$$\mathbf{F}_1(X, E_1) \times \mathbf{F}_2(X, E_2) \to \mathbf{F}_0(X, E_0) , \quad (u_1, u_2) \mapsto u_1 *_\bullet u_2 \tag{1.9.9}$$

satisfying (1.9.1) for $u_j \in \mathbf{F}_j(X)$ and $e_j \in E_j$, $j = 1, 2$. By Theorem 1.3.6 the linear subspace $\big(\mathcal{D}(X) \otimes E_1\big) \times \big(\mathcal{D}(X) \otimes E_2\big)$ is dense in $\mathbf{F}_1(X, E_1) \times \mathbf{F}_2(X, E_2)$.

Hence the map (1.9.9) is determined by its restriction to this subspace. This proves the theorem. ÷

1.9.2 Remarks (a) It should be observed that Theorem 1.9.1 is 'symmetric in E_1 and E_2', that is, the rles of $E\ _1$ and E_2 can be interchanged. This fact will be used throughout, usually without further mention.

(b) Suppose that either $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$, $j = 1, 2$, and that u_1 or u_2 has compact support, or $u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{S}(\mathbb{R}^n, E_2)$. Then

$$u_1 *_\bullet u_2(\varphi) = u_1 *_\bullet (u_2 * \check{\varphi})(0) = \big\langle u_1 \bullet (\check{u}_2 * \varphi)\big\rangle_{\mathcal{D}} \tag{1.9.10}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

Proof We can assume that $u_2 \in \mathcal{E}'(\mathbb{R}^n, E_2)$ in the ørst case. Then by Theorems 1.7.2 and 1.9.1 each one of the three expressions in (1.9.10) deønes a separately continuous trilinear map

$$\mathcal{D}'(\mathbb{R}^n, E_1) \times \mathcal{E}'(\mathbb{R}^n, E_2) \times \mathcal{D}(\mathbb{R}^n) \to E_0$$

in the ørst case, and

$$\mathcal{S}'(\mathbb{R}^n, E_1) \times \mathcal{S}(\mathbb{R}^n, E_2) \times \mathcal{D}(\mathbb{R}^n) \to E_0$$

in the second one. Suppose that $u_j = v_j \otimes e_j \in \mathcal{D}(\mathbb{R}^n) \times E_j$, $j = 1, 2$. Then we deduce from (1.9.1), (1.2.11), and (1.2.13) that

$$\begin{aligned} u_1 *_\bullet u_2(\varphi) &= (v_1 \otimes e_1) *_\bullet (v_2 \otimes e_2)(\varphi) = v_1 * v_2(\varphi)(e_1 \bullet e_2) \\ &= v_1 * (v_2 * \check{\varphi})(0)(e_1 \bullet e_2) = \big[v_1 * (v_2 * \check{\varphi})\big] \otimes (e_1 \bullet e_2)(0) \\ &= u_1 *_\bullet (u_2 * \check{\varphi})(0)\ . \end{aligned}$$

Moreover, by these calculations, (1.2.11), Remark 1.2.4(c), and (1.7.4),

$$\begin{aligned} u_1 *_\bullet u_2(\varphi) &= v_1 * (v_2 * \check{\varphi})(0)(e_1 \bullet e_2) = \big\langle v_1, (v_2 * \check{\varphi})\check{}\,\big\rangle_{\mathcal{D}}(e_1 \bullet e_2) \\ &= \langle v_1, \check{v}_2 * \varphi\rangle_{\mathcal{D}}(e_1 \bullet e_2) = \big\langle (v_1 \otimes e_1) \bullet \big[(\check{v}_2 * \varphi) \otimes e_2\big]\big\rangle_{\mathcal{D}} \\ &= \big\langle u_1 \bullet (\check{u}_2 * \varphi)\big\rangle_{\mathcal{D}}\ . \end{aligned}$$

Thus all three terms in (1.9.10) agree on the dense linear subspace

$$\big(\mathcal{D}(\mathbb{R}^n) \otimes E_1\big) \times \big(\mathcal{D}(\mathbb{R}^n) \otimes E_2\big) \times \mathcal{D}(\mathbb{R}^n)\ ,$$

and so everywhere. ÷

L_p-Functions with Compact Supports

Given $K \subset\subset X$, we put

$$L_{p,K}(X, E) := \big\{\, u \in L_p(X, E)\ ;\ \mathrm{supp}(u) \subset \overline{K}\,\big\}\ , \qquad 1 \le p \le \infty\ .$$

Then $L_{p,K}(X,E)$ is a closed linear subspace of the Banach space $L_p(X,E)$, hence a Banach space as well. Moreover,

$$\bigcup_{K \subset\subset X} L_{p,K}(X,E) = \{\, u \in L_p(X,E) \;;\; \mathrm{supp}(u) \subset\subset X \,\} \ .$$

Thus, recalling (1.1.5) and (1.1.6),

$$L_{p,c}(X,E) := \varinjlim_{K \subset\subset X} L_{p,K}(X,E) \ , \qquad 1 \le p \le \infty \ , \tag{1.9.11}$$

is a well-deøned LF-space, the space of E-valued L_p-functions with compact supports. It is obvious that

$$L_{p,c}(X,E) \hookrightarrow L_p(X,E) \hookrightarrow L_{1,\mathrm{loc}}(X,E) \hookrightarrow \mathcal{D}'(X,E) \tag{1.9.12}$$

for $1 \le p \le \infty$, and it is not diŒcult to see that

$$\mathcal{D}(X,E) \overset{d}{\hookrightarrow} L_{p,c}(X,E) \ , \qquad 1 \le p < \infty \ , \tag{1.9.13}$$

by using Theorem 1.3.6(viii) with $k = 0$.

Convolutions of Regular Distributions

Our next proposition shows that the convolution product of regular distributions is again a regular distribution, if deøned.

1.9.3 Proposition Suppose that $u_1 \in L_{1,\mathrm{loc}}(\mathbb{R}^n, E_1)$ and $u_2 \in L_{1,c}(\mathbb{R}^n, E_2)$. Then $u_1 *_\bullet u_2 \in L_{1,\mathrm{loc}}(\mathbb{R}^n, E_0)$ and

$$u_1 *_\bullet u_2(x) = \int_{\mathbb{R}^n} u_1(x-y) \bullet u_2(y)\, dy = \int_{\mathbb{R}^n} u_1(y) \bullet u_2(x-y)\, dy \tag{1.9.14}$$

for a.a. $x \in \mathbb{R}^n$. Moreover, the convolution map $(u_1, u_2) \mapsto u_1 *_\bullet u_2$ is bilinear and continuous:

$$L_{1,\mathrm{loc}}(\mathbb{R}^n, E_1) \times L_{1,c}(\mathbb{R}^n, E_2) \to L_{1,\mathrm{loc}}(\mathbb{R}^n, E_0) \ . \tag{1.9.15}$$

Proof We deduce from Theorem 1.9.1 and (1.9.12) that the convolution map is separately continuous from $L_{1,\mathrm{loc}}(\mathbb{R}^n, E_1) \times L_{1,c}(\mathbb{R}^n, E_2)$ into $\mathcal{D}'(\mathbb{R}^n, E_0)$.

Suppose that $K, K_2 \subset\subset \mathbb{R}^n$. Then, letting $K_1 := K - K_2 \subset\subset \mathbb{R}^n$, Tonelli's theorem implies

$$\begin{aligned} \int_K \Big| \int_{\mathbb{R}^n} u_1(x-y) \bullet u_2(y)\, dy \Big|\, dx &\le \int_K \int_{K_2} |u_1(x-y)|\, |u_2(y)|\, dy\, dx \\ &\le \|u_1\|_{1,K_1} \|u_2\|_{1,K_2} \end{aligned}$$

for $u_j \in L_{1,\mathrm{loc}}(\mathbb{R}^n, E_j)$ with $\mathrm{supp}(u_2) \subset K_2$. From this we deduce that

$$(u_1, u_2) \mapsto \int_{\mathcal{R}^n} u_1(\cdot - y) \bullet u_2(y)\, dy$$

is a bilinear and continuous map

$$L_{1,\mathrm{loc}}(\mathbb{R}^n, E_1) \times L_{1,c}(\mathbb{R}^n, E_2) \to L_{1,\mathrm{loc}}(\mathbb{R}^n, E_0) \hookrightarrow \mathcal{D}'(\mathbb{R}^n, E_0)\ .$$

By [Ama95, (III.4.2.17)] and (1.9.1) we infer that both sides of (1.9.14) coincide on the linear subspace $L_{1,\mathrm{loc}}(\mathbb{R}^n, E_1) \times \big(\mathcal{D}(\mathbb{R}^n) \otimes E_2\big)$, which is dense by (1.9.13). Thus the assertion follows. ÷

Tensor Products and Convolutions

Given the hypotheses of Proposition 1.9.3, we deduce from (1.9.14) by Fubini's theorem, a change of variables, and Remark 1.8.5(d) that

$$\begin{aligned} u_1 *_\bullet u_2(\varphi) &= \int_{\mathcal{R}^n}\int_{\mathcal{R}^n} \varphi(x) u_1(x-y) \bullet u_2(y)\, dy\, dx \\ &= \int_{\mathcal{R}^n \times \mathcal{R}^n} \varphi(x+y) u_1(x) \bullet u_2(y)\, d(x,y) = u_1 \otimes_\bullet u_2\big(\varphi(x_1+x_2)\big) \end{aligned}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$, where in the last term we use our symbolic notation, meaning that u_j acts on the function following it as a variable of $x_j \in \mathbb{R}^n$. The next theorem shows that this formula holds in general.

1.9.4 Theorem Suppose that one of the distributions $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$, $j = 1, 2$, has compact support. Then

$$u_1 *_\bullet u_2(\varphi) = u_1 \otimes_\bullet u_2\big(\varphi(x_1+x_2)\big)\ , \qquad \varphi \in \mathcal{D}(\mathbb{R}^n)\ .$$

Proof By Remark 1.9.2(b) and Fubini's rule for tensor products

$$\begin{aligned} u_1 *_\bullet u_2(\varphi) &= \big\langle u_1 \bullet (\check{u}_2 * \varphi)\big\rangle_{\mathcal{D}} = \big\langle u_1 \bullet \check{u}_2(\tau_{x_1}\check{\varphi})\big\rangle_{\mathcal{D}} = \big\langle u_1 \bullet u_2\big(\tau_{x_1}\check{\varphi}(-x_2)\big)\big\rangle_{\mathcal{D}} \\ &= \big\langle u_1 \bullet u_2\big(\varphi(x_1+x_2)\big)\big\rangle = u_1 \otimes_\bullet u_2\big(\varphi(x_1+x_2)\big) \end{aligned}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$. ÷

Theorem 1.9.4 enables us to deøne convolutions in certain cases where neither u_1 nor u_2 has compact support. For this we recall that a mapping between metric spaces is proper, if preimages of compact sets are compact. Given nonempty closed subsets $A_1, \ldots, A_m$ of $\mathbb{R}^n$, we say $(A_1, \ldots, A_m)$ satisøes condition (Σ) if the map

$$A_1 \times \cdots \times A_m \to \mathbb{R}^n\ , \quad (x_1, \ldots, x_m) \mapsto x_1 + \cdots + x_m$$

is proper.

1.9.5 Remarks (a) A proper continuous mapping is closed, that is, it maps closed sets onto closed sets. Hence if $(A_1,\dots,A_m)$ satisfies condition (Σ) then the set $A_1+\cdots+A_m$ is closed in $\mathbb{R}^n$.

(b) If $A_1,\dots,A_m$ are closed in $\mathbb{R}^n$ and all but one are compact, then $(A_1,\dots,A_m)$ satisfies condition (Σ).

(c) Let Γ be a closed convex proper cone in $\mathbb{R}^n$. (A cone Γ is proper if it does not contain a full line, that is, if $\Gamma\cap(-\Gamma)=\{0\}$.) Then the m-tuple $(\Gamma,\dots,\Gamma)$ satisfies condition (Σ). If $\mathsf{H} := \{\, y\in\mathbb{R}^n \ ;\ \langle\eta,y\rangle \ge 0 \,\}$ is a closed half-space with interior normal η lying in the interior of the dual cone $\Gamma' := \{\, \xi\in\mathbb{R}^n \ ;\ \langle\xi,x\rangle\ge 0,\ x\in\Gamma \,\}$, then the m-tuple $(\Gamma,\dots,\Gamma,\mathsf{H})$ satisfies condition (Σ). Note that $\mathrm{int}(\Gamma')\ne\emptyset$.

Proof First we assume that $m=2$. Thus let $\big((x_j,y_j)\big)$ be a sequence in $\Gamma\times\mathsf{B}$, where $\mathsf{B}\in\{\Gamma,\mathsf{H}\}$, such that $x_j+y_j\to z$ in $\mathbb{R}^m$. Then, by the local compactness and the closedness of Γ and B, it suffices to prove that (x_j) and (y_j) are bounded sequences. If this is not the case, clearly both sequences have to be unbounded. Thus, by passing to a suitable subsequence we can assume that $|x_j|\to\infty$ and $x_j/|x_j|\to x\in\dot{\Gamma}$. Hence $y_j/|x_j|\to -x\in\dot{\mathsf{B}}$. If $\mathsf{B}=\Gamma$, this is impossible since Γ is proper. If $\mathsf{B}=\mathsf{H}$, it follows that $0<\langle\eta,x\rangle$ since $\eta\in\mathrm{int}(\Gamma')$ and $x\in\dot{\Gamma}$, and $\langle\eta,x\rangle\le 0$ since $-x\in\mathsf{H}$, which is again a contradiction. The general case is derived from an obvious induction argument. ÷

In the following, the distributions $u_1,\dots,u_m$ are said to have convolutive supports or to satisfy condition (Σ) if their supports satisfy condition (Σ).

Let $u_j\in\mathcal{D}'(\mathbb{R}^n,E_j)$, $j=1,2$, satisfy condition (Σ), and let X be a bounded open subset of $\mathbb{R}^n$. Then there exists $\rho:=\rho_X>0$ such that

$$\big(x_j\in\mathrm{supp}(u_j)\ ,\ x_1+x_2\in X\big)\quad\Longrightarrow\quad\big(x_j\in\rho\mathbb{B}^n\ ,\ j=1,2\big)\ . \tag{1.9.16}$$

Thus, given

$$\psi_j\in\mathcal{D}(\mathbb{R}^n)\quad\text{with}\quad\psi_j\,|\,\rho\mathbb{B}^n=1\ ,\qquad j=1,2\ , \tag{1.9.17}$$

convolution $(\psi_1u_1)*_\bullet(\psi_2u_2)$ is well-defined, since $\psi_ju_j\in\mathcal{E}'(\mathbb{R}^n,E_j)$. Moreover, for $\chi_j\in\mathcal{D}(\mathbb{R}^n)$ with $\chi_j\,|\,\rho\mathbb{B}^n=1$ it follows from Remarks 1.8.5(a) and (e), the fact that $\psi_j-\chi_j$ vanish on $\rho\mathbb{B}^n$, and from (1.9.16) that, given $\varphi\in\mathcal{D}$,

$$\begin{aligned}
&\big[(\psi_1u_1)*_\bullet(\psi_2u_2)-(\chi_1u_1)*_\bullet(\chi_2u_2)\big](\varphi)\\
&\quad=\big[(\psi_1u_1)\otimes_\bullet(\psi_2u_2)-(\chi_1u_1)\otimes_\bullet(\chi_2u_2)\big]\big(\varphi(x_1+x_2)\big)\\
&\quad=\big[(\psi_1-\chi_1)u_1\otimes_\bullet(\psi_2u_2)+(\chi_1u_1)\otimes_\bullet(\psi_2-\chi_2)u_2\big]\big(\varphi(x_1+x_2)\big)\\
&\quad=\Big\{\big[(\psi_1-\chi_1)\otimes\psi_2\big](u_1\otimes_\bullet u_2)+\big[\chi_1\otimes(\psi_2-\chi_2)\big](u_1\otimes_\bullet u_2)\Big\}\big(\varphi(x_1+x_2)\big)\\
&\quad=(u_1\otimes_\bullet u_2)\Big\{\big[(\psi_1-\chi_1)(x_1)\psi_2(x_2)+\chi_1(x_1)(\psi_2-\chi_2)(x_2)\big]\varphi(x_1+x_2)\Big\}=0\ .
\end{aligned}$$

This shows that $(\psi_1 u_1) *_\bullet (\psi_2 u_2)$ deønes a distribution in $\mathcal{D}'(X, E_0)$, independently of the choice of the ψ_j satisfying (1.9.17). From this and the fact that a distribution is determined by its values on the open subsets of $\mathbb{R}^n$ (e.g., [Hr83, Theorem 2.2.4] and note that the proof carries over to the E-valued case) we see that

$$\left.\begin{array}{l}\text{given } u_j \in \mathcal{D}'(\mathbb{R}^n, E_j),\quad j = 1,2, \text{ having convolutive supports,}\\ \text{there exists a unique distribution } u_1 *_\bullet u_2 \in \mathcal{D}'(\mathbb{R}^n, E_0), \text{ the con-}\\ \text{volution of } u_1 \text{ and } u_2 \text{ with respect to multiplication (1.5.20),}\\ \text{satisfying}\\ \qquad u_1 *_\bullet u_2 \,|\, X = (\psi_1 u_1) *_\bullet (\psi_2 u_2) \,|\, X \ , \qquad X = \mathring{X} \subset\subset \mathbb{R}^n \ ,\\ \text{where } \psi_j \text{ satisøes (1.9.17) for } j = 1,2.\end{array}\right\} \tag{1.9.18}$$

Basic Properties

In the following remarks we collect some of the most important properties of convolutions.

1.9.6 Remarks Unless explicit restrictions are given, we suppose that either $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$ for $j = 1,2$, with convolutive supports, or that $u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{S}(\mathbb{R}^n, E_2)$.

(a) It is an obvious consequence of (1.9.18) and Remark 1.9.2(b) that

$$u_1 *_\bullet u_2(\varphi) = u_1 * (u_2 * \check{\varphi})(0) = \big\langle u_1 \bullet (\check{u}_2 * \varphi)\big\rangle_{\mathcal{D}}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(b) If $u_j \in L_{1,\mathrm{loc}}(\mathbb{R}^n, E_j)$ satisfy condition (Σ) then $u_1 *_\bullet u_2 \in L_{1,\mathrm{loc}}(\mathbb{R}^n, E_0)$ and

$$u_1 *_\bullet u_2(x) = \int_{\mathcal{R}^n} u_1(x-y) \bullet u_2(y)\,dy = \int_{\mathcal{R}^n} u_1(y) \bullet u_2(x-y)\,dy\ , \qquad \text{a.a. } x \in \mathbb{R}^n\ .$$

Proof Thanks to (1.9.18) we can assume that u_j has compact support for $j = 1,2$. Then the assertion follows from Proposition 1.9.3. ÷

(c) *Associativity* Let the associativity hypotheses of Remark 1.6.5(e) be satisøed and suppose that either $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$, $j = 1,2,3$, have convolutive supports, or $u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u_k \in \mathcal{S}(\mathbb{R}^n, E_k)$, $k = 2,3$. Then

$$u_1 *_\bullet (u_2 *_\bullet u_3) = (u_1 *_\bullet u_2) *_\bullet u_3\ . \tag{1.9.19}$$

Proof If condition (Σ) is satisøed,[2] it suŒces, by (1.9.18), to prove the assertion for $u_j \in \mathcal{E}'(X, E_j)$, $j = 1,2,3$. Then Theorem 1.9.1 implies that both sides of

[2] We leave it to the reader to verify that the convolutions on either side of (1.9.19) are well-deøned if (Σ) is satisøed (also cf. (f)).

(1.9.19) deøne separately continuous trilinear maps

$$\mathcal{E}'(\mathbb{R}^n, E_1) \times \mathcal{E}'(\mathbb{R}^n, E_2) \times \mathcal{E}'(\mathbb{R}^n, E_3) \to \mathcal{E}'(\mathbb{R}^n, E_0)$$

if condition (Σ) is satisøed, and

$$\mathcal{S}'(\mathbb{R}^n, E_1) \times \mathcal{S}(\mathbb{R}^n, E_2) \times \mathcal{S}(\mathbb{R}^n, E_3) \to \mathcal{O}_M(\mathbb{R}^n, E_0)$$

otherwise. Thanks to (1.9.1) and (1.2.13) they coincide on the dense linear subspace $\prod_{j=1}^{3} \mathcal{D}(X) \otimes E_j$. Thus they are equal. ÷

(d) Commutativity Suppose that $E_1 = E_2 =: E$ and multiplication (1.5.20) is symmetric. Then $u_1 *_\bullet u_2 = u_2 *_\bullet u_1$.

Proof By (1.9.18) we can assume that $u_j \in \mathcal{E}'(\mathbb{R}^n, E)$ if condition (Σ) is satisøed. Thus, by Theorem 1.9.1 and the density of $\mathcal{D}(\mathbb{R}^n, E)$ in $\mathcal{E}'(\mathbb{R}^n, E)$ and in $\mathcal{S}'(\mathbb{R}^n, E)$ and $\mathcal{S}(\mathbb{R}^n, E)$, respectively, we can assume that $u_j \in \mathcal{D}(\mathbb{R}^n, E)$. Now the assertion follows from (b). ÷

(e) Distributivity Let the distributivity hypotheses of Remark 1.8.5(e) be satisøed and suppose that $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$ as well as $v_j \in \mathcal{D}'(\mathbb{R}^n, E_{j+2})$, $j = 1, 2$, have convolutive supports. Then

$$(u_1 *_\bullet u_2) \otimes_\bullet (v_1 *_\bullet v_2) = (u_1 \otimes_\bullet v_1) *_\bullet (u_2 \otimes_\bullet v_2) \ . \tag{1.9.20}$$

Proof First we show that $u_j \otimes_\bullet v_j \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n, E_{j+5})$ satisfy condition (Σ). By Remark 1.8.5(a)

$$\operatorname{supp}(u_j \otimes_\bullet v_j) \subset \operatorname{supp}(u_j) \times \operatorname{supp}(v_j) \ , \qquad j = 1, 2 \ . \tag{1.9.21}$$

Let $X \subset \mathbb{R}^n$ be bounded and open. Then there exists $\rho > 0$ such that $x_j \in \operatorname{supp}(u_j)$ and $y_j \in \operatorname{supp}(v_j)$ with $x_1 + x_2$ and $y_1 + y_2$ belonging to X imply $x_j, y_j \in \rho\mathbb{B}^n$. Thus, if $(x_1, y_1) + (x_2, y_2) \in X \times X$, it follows from (1.9.21) that

$$(x_j, y_j) \in \rho\mathbb{B}^n \times \rho\mathbb{B}^n \subset \rho_1 \mathbb{B}^{2n} \ , \qquad j = 1, 2 \ ,$$

for some $\rho_1 > 0$. Since every bounded open subset of $\mathbb{R}^{2n}$ is contained in a product set $X \times X$, we see that $u_j \otimes v_j$ satisfy condition (Σ). Hence both sides of (1.9.20) are well-deøned.

Given $\psi_j, \psi_{j+2} \in \mathcal{D}(\mathbb{R}^n)$, it follows from Remark 1.8.5(e) that

$$(\psi_j u_j) \otimes_\bullet v_j(\psi_{j+2} v_j) = (\psi_j \otimes \psi_{j+2})(u_j \otimes_\bullet v_j) \ , \qquad j = 1, 2 \ .$$

Using this and (1.9.18) it is easily seen that it suŒces to prove the assertion for $u_j \in \mathcal{E}'(\mathbb{R}^n, E_j)$ and $v_j \in \mathcal{E}'(\mathbb{R}^n, E_{j+2})$, $j = 1, 2$. Now Theorems 1.8.4 and 1.9.1 together with a density argument show that we can assume that $u_j \in \mathcal{D}(\mathbb{R}^n, E_j)$

and $v_j \in \mathcal{D}(\mathbb{R}^n, E_{j+2})$. Thus, by (b) and Remark 1.8.5(d), the left-hand side of (1.9.20) equals

$$\int_{\mathcal{R}^n} u_1(x_1 - y_1) \bullet u_2(y_1)\, dy_1 \bullet \int_{\mathcal{R}^n} v_1(x_2 - y_2) \bullet v_2(y_2)\, dy_2 \ , \quad (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n \ .$$

By Fubini's theorem and (1.8.12) this product takes the form

$$\int_{\mathcal{R}^{2n}} \big(u_1(x_1 - y_1) \bullet v_1(x_2 - y_2)\big) \bullet \big(u_2(y_1) \bullet v_2(y_2)\big)\, d(y_1, y_2) \ ,$$

which is the right-hand side of (1.9.20). ÷

(f) Support Theorem Suppose that $u_j \in \mathcal{D}'(\mathbb{R}^n, E_j)$, $j = 1, 2$, have convolutive supports. Then $\mathrm{supp}(u_1 *_\bullet u_2) \subset \mathrm{supp}(u_1) + \mathrm{supp}(u_2)$.

Proof From Remark 1.9.5(a) we know that $C := \mathrm{supp}(u_1) + \mathrm{supp}(u_2)$ is closed in $\mathbb{R}^n$. Let $\varphi \in \mathcal{D}(C^c)$. Then the support of the function $(x_1, x_2) \mapsto \varphi(x_1 + x_2)$ does not meet $\mathrm{supp}(u_1) \times \mathrm{supp}(u_2)$, hence not $\mathrm{supp}(u_1 \otimes_\bullet u_2)$ by Remark 1.8.5(a). Thus $(\psi_1 u_1) *_\bullet (\psi_2 u_2)(\varphi) = (\psi_1 u_1) \otimes_\bullet (\psi_2 u_2)\big(\varphi(x_1 + x_2)\big) = 0$ for all $\psi_j \in \mathcal{D}(X)$, $j = 1, 2$. Consequently, (1.9.18) implies $u_1 *_\bullet u_2(\varphi) = 0$, that is, the assertion. ÷

(g) $\partial^{\alpha+\beta}(u_1 *_\bullet u_2) = \partial^\alpha u_1 *_\bullet \partial^\beta u_2$, $\alpha, \beta \in \mathbb{N}^n$.

Proof First suppose that u_1 and u_2 satisfy condition (Σ). Since

$$\mathrm{supp}(\partial^\gamma u) \subset \mathrm{supp}(u) \ , \qquad u \in \mathcal{D}'(\mathbb{R}^n, E) \ , \quad \gamma \in \mathbb{N}^n \ ,$$

it follows that $\partial^\alpha u_1$ and $\partial^\beta u_2$ satisfy condition (Σ). Hence $\partial^\alpha u_1 *_\bullet \partial^\beta u_2$ is well-deøned. Given $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\partial^{\alpha+\beta}(u_1 *_\bullet u_2)(\varphi) = (-1)^{|\alpha|+|\beta|}(u_1 *_\bullet u_2)(\partial^{\alpha+\beta}\varphi) \ .$$

Let X be a bounded open subset of $\mathbb{R}^n$ containing $\mathrm{supp}(\varphi)$ and choose $\psi_j \in \mathcal{D}(\mathbb{R}^n)$ with $\psi_j \,|\, \rho_X \mathbb{B}^n = 1$. Then

$$\begin{aligned}
(-1)^{|\alpha|+|\beta|}(u_1 *_\bullet u_2)(\partial^{\alpha+\beta}\varphi) &= (-1)^{|\alpha|+|\beta|}(\psi_1 u_1) *_\bullet (\psi_2 u_2)(\partial^{\alpha+\beta}\varphi) \\
&= (-1)^{|\alpha|+|\beta|}(\psi_1 u_1) \otimes_\bullet (\psi_2 u_2)\big(\partial^{\alpha+\beta}\varphi(x_1 + x_2)\big) \\
&= \partial^\alpha(\psi_1 u_1) \otimes_\bullet \partial^\beta(\psi_2 u_2)\big(\varphi(x_1 + x_2)\big) \\
&= (\psi_1 \partial^\alpha u_1) \otimes_\bullet (\psi_2 \partial^\beta u_2)\big(\varphi(x_1 + x_2)\big) \\
&= (\psi_1 \partial^\alpha u_1) *_\bullet (\psi_2 \partial^\beta u_2)(\varphi) \\
&= \partial^\alpha u_1 *_\bullet \partial^\beta u_2(\varphi) \ ,
\end{aligned}$$

thanks to Remark 1.8.5(b), Leibniz' rule, and the properties of ψ_j.

If $u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{S}(\mathbb{R}^n, E_2)$, we can restrict ourselves as usual to the case $u_j \in \mathcal{D}(\mathbb{R}^n) \otimes E_j$. Then the assertion follows from Remark 1.2.4(b). ÷

(h) $\tau_a(u_1 *_\bullet u_2) = (\tau_a u_1) *_\bullet u_2 = u_1 *_\bullet \tau_a u_2$, $a \in \mathbb{R}^n$, and

$$(u_1 *_\bullet u_2)\check{} = \check{u}_1 *_\bullet \check{u}_2 \ .$$

Proof It is easily veriøed that these convolutions are all well-deøned. If u_1 and u_2 satisfy condition (Σ), we deduce from (1.9.18) that we can assume that u_1 and u_2 have compact supports. Thus in either case it suŒces by Theorem 1.9.1 to consider the case where $u_j \in \mathcal{D}(\mathbb{R}^n) \otimes E_j$. Now the assertion follows from Remark 1.2.4(c). ÷

(i) Suppose that $a \in E_1$ and $u \in \mathcal{D}'(\mathbb{R}^n, E_2)$. Then

$$\partial^\alpha\big[(\delta \otimes a) *_\bullet u\big] = (\partial^\alpha \delta \otimes a) *_\bullet u = a \bullet \partial^\alpha u = \partial^\alpha (a \bullet u)$$

for $\alpha \in \mathbb{N}^n$.

Proof Since $\partial^\beta \delta \otimes a \in \mathcal{E}'(\mathbb{R}^n, E_1)$, we see that the above convolutions are well-deøned. Thanks to (g) and Remark 1.6.5(f) we can assume that $\alpha = 0$. By Theorems 1.6.4 and 1.9.1 it suŒces to consider $u \in \mathcal{D}(\mathbb{R}^n) \times E_2$. Then the assertion follows from (1.9.1) and Remark 1.2.4(a). ÷

Convolution Algebras

Given a nonempty subset K of $\mathbb{R}^n$, we put

$$\mathcal{D}'_K(\mathbb{R}^n, E) := \big\{\, u \in \mathcal{D}'(\mathbb{R}^n, E) \ ; \ \operatorname{supp}(u) \subset \overline{K} \,\big\} \qquad (1.9.22)$$

and, as usual, $\mathcal{D}'_K(\mathbb{R}^n) := \mathcal{D}'_K(\mathbb{R}^n, \mathbb{K})$, if no confusion seems likely. We also set

$$\mathcal{D}'_+(E) := \mathcal{D}'_{\mathbb{R}^+}(\mathbb{R}, E) \ . \qquad (1.9.23)$$

Note that $\mathcal{D}'_K(\mathbb{R}^n, E)$ is a closed linear subspace of $\mathcal{D}'(\mathbb{R}^n, E)$.

In the following theorem we collect some of the properties of convolutions in a particularly important setting.

1.9.7 Theorem Let Γ be a proper closed convex cone in $\mathbb{R}^n$ and let

$$\mathbf{F} \in \{\mathcal{D}, \mathcal{S}, \mathcal{E}', \mathcal{D}'_\Gamma\} \ .$$

Then convolution is a well-deøned hypocontinuous bilinear map

$$\mathbf{F}(\mathbb{R}^n, E_1) \times \mathbf{F}(\mathbb{R}^n, E_2) \to \mathbf{F}(\mathbb{R}^n, E_0) \ , \quad (u_1, u_2) \mapsto u_1 *_\bullet u_2$$

which possesses the associativity and commutativity properties of Remarks 1.9.6(c) and (d), respectively. In particular, if $(E, \bullet)$ is a [commutative] Banach algebra then $\big(\mathbf{F}(\mathbb{R}^n, E), *_\bullet\big)$ is a [commutative] algebra, a convolution algebra. If $(E, \bullet)$ has a unit e_0 and $\mathbf{F} \in \{\mathcal{E}', \mathcal{D}'_\Gamma\}$, then $\big(\mathbf{F}(\mathbb{R}^n, E), *_\bullet\big)$ has a unit as well, namely $\delta \otimes e_0$.

Proof Everything, except the hypocontinuity of the convolution map if $\mathbf{F} = \mathcal{D}'_\Gamma$, follows easily from Theorem 1.9.1, the injection $\mathcal{D}(\mathbb{R}^n, E_2) \hookrightarrow \mathcal{E}'(\mathbb{R}^n, E_2)$, and Remarks 1.9.5(c), 1.9.6(f), and 1.2.4(a). Recall that $\mathcal{D}'_\Gamma(\mathbb{R}^n, E)$ is a closed linear subspace of $\mathcal{D}'(\mathbb{R}^n, E)$. Thus, if (u_α) is a net in $\mathcal{D}'_\Gamma(\mathbb{R}^n, E_1)$ converging to zero, it converges to zero in $\mathcal{D}'(\mathbb{R}^n, E_1)$. Let V be a bounded subset of $\mathcal{D}'_\Gamma(\mathbb{R}^n, E_2)$, hence of $\mathcal{D}(\mathbb{R}^n, E_2)$. Then, given any bounded subset B of $\mathcal{D}(\mathbb{R}^n)$, there exists $K := K_B \subset\subset \mathbb{R}^n$ such that $\mathrm{supp}(\varphi) \subset \overline{K}$ for $\varphi \in B$. Hence condition (Σ) guarantees the existence of $\rho > 0$ such that $x, y \in \Gamma$ with $x + y \subset \overline{K}$ implies $x, y \in \rho\mathbb{B}^n$. Thus, letting $\psi_j \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\psi \,|\, \rho\mathbb{B}^n = 1$, it follows from (1.9.18) that

$$u_\alpha *_\bullet v(\varphi) = (\psi_1 u_\alpha) *_\bullet (\psi_2 v)(\varphi) \ , \qquad \varphi \in B \ , \quad v \in V \ . \tag{1.9.24}$$

Note that $\psi_2 v(\chi) = v(\psi_2 \chi)$ and $\psi_2 \chi \in \mathcal{D}_{\mathrm{supp}(\psi_2)}(\mathbb{R}^n)$ for $\chi \in \mathcal{E}(\mathbb{R}^n)$ imply that $\{\, \psi_2 v \ ; \ v \in V \,\}$ is bounded in $\mathcal{E}'(\mathbb{R}^n, E_2)$. Now the hypocontinuity of the map (1.9.2) and (1.9.24) entail $u_\alpha *_\bullet v(\varphi) \to 0$ in E_0, uniformly with respect to $v \in V$ and $\varphi \in B$. Consequently, $u_\alpha *_\bullet v \to 0$ in $\mathcal{D}'(\mathbb{R}^n, E_0)$, hence in $\mathcal{D}'_\Gamma(\mathbb{R}^n, E_0)$, uniformly with respect to v in bounded subsets of $\mathcal{D}'_\Gamma(\mathbb{R}^n, E_0)$. This proves, by symmetry, the asserted hypocontinuity. ÷

1.9.8 Remarks (a) Let Γ be a proper, closed, and convex cone in $\mathbb{R}^n$. Then $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n)$, and $\mathcal{D}'_\Gamma(\mathbb{R}^n)$ are commutative convolution algebras. Moreover, $\mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{D}'_\Gamma(\mathbb{R}^n)$ possess the unit δ.

(b) Since $\mathcal{D}(\mathbb{R}^n, E)$ and $\mathcal{S}(\mathbb{R}^n, E)$ are barreled, the convolution product is, in fact, continuous on these spaces.

(c) Suppose that $u_j \in \mathcal{D}'_+(E_j)$ are regular distributions. Then $u_1 *_\bullet u_2 \in \mathcal{D}'_+(E_0)$ is a regular distribution as well, and

$$u_1 *_\bullet u_2(t) = \int_0^t u_1(t-s) \bullet u_2(s)\, ds \ , \qquad \text{a.a. } t \in \mathbb{R}^+ \ .$$

Proof This is an easy consequence of Remarks 1.9.6(b) and (f). ÷

Convolutions of Bounded and Integrable Functions

For the reader's convenience and for later use we now collect the properties of convolution on certain Banach spaces of vector-valued convolutions, thus extending, in particular, assertions [Ama95, (III.4.2.19)˘(III.4.2.22)] (note that in that book (III.4.2.18) is a special case of (III.4.2.21)).

1.9.9 Theorem Suppose that $(\mathbf{F}_1, \mathbf{F}_2; \mathbf{F}_0)$ is any one of the triplets

$$(BUC, L_1; BUC), (C_0, L_1; C_0), (L_p, L_1; L_p), (L_\infty, L_1; BUC), (L_q, L_{q'}; C_0) \ ,$$

where $1 \le p < \infty$ and $1 < q < \infty$. Then convolution with respect to multiplication (1.5.20) extends from $\mathbf{F}_1(\mathbb{R}^n, E_1) \times \mathcal{D}(\mathbb{R}^n, E_2)$ to a multiplication

$$\mathbf{F}_1(\mathbb{R}^n, E_1) \times \mathbf{F}_2(\mathbb{R}^n, E_2) \to \mathbf{F}_0(\mathbb{R}^n, E_0) \ .$$

It is given by

$$u_1 *_\bullet u_2(x) = \int_{\mathcal{R}^n} u_1(x-y) \bullet u_2(y)\, dy = \int_{\mathcal{R}^n} u_1(y) \bullet u_2(x-y)\, dy \ , \qquad \text{a.a. } x \in \mathbb{R}^n \ .$$

Proof Since $\mathbf{F}_1(\mathbb{R}^n, E_1) \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^n, E_1) \cap \mathcal{S}'(\mathbb{R}^n, E_1)$, we see from (1.9.3) that

$$u_1 *_\bullet u_2 \in \mathcal{O}_M(\mathbb{R}^n, E_0) \ , \qquad u_1 \in \mathbf{F}_1(\mathbb{R}^n, E_1) \ , \quad u_2 \in \mathcal{D}(\mathbb{R}^n, E_2) \ ,$$

and Proposition 1.9.3 guarantees that $u_1 *_\bullet u_2$ is given by the above integral in this case. Since $\mathcal{D}(\mathbb{R}^n, E_2)$ is dense in $\mathbf{F}_2(\mathbb{R}^n, E_2)$, it suŒces to show that $u_1 *_\bullet u_1 \in \mathbf{F}_0(\mathbb{R}^n, E_0)$ and that the estimate

$$\|u_1 *_\bullet u_2\|_{F_0(\mathcal{R}^n,E_0)} \le \|u_1\|_{F_1(\mathcal{R}^n,E_1)}\, \|u_2\|_{F_2(\mathcal{R}^n,E_2)} \tag{1.9.25}$$

is valid for $u_1 \in \mathbf{F}_1(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{D}(\mathbb{R}^n, E_2)$. Then the assertion follows by continuous extension.

If $u_1 \in L_r(\mathbb{R}^n, E_1)$, $1 \le r \le \infty$, and $u_2 \in \mathcal{D}(\mathbb{R}^n, E_2)$, it follows from

$$|u_1 * u_2(x)| \le \int_{\mathcal{R}^n} |u_1(x-y)|\, |u_2(y)|\, dy \ , \qquad x \in \mathbb{R}^n \ ,$$

and the classical scalar Young inequality that

$$\|u_1 *_\bullet u_2\|_{L_r(\mathcal{R}^n,E_0)} \le \|u_1\|_{L_r(\mathcal{R}^n,E_1)}\, \|u_2\|_{L_1(\mathcal{R}^n,E_2)} \tag{1.9.26}$$

and

$$\|u_1 *_\bullet u_2\|_{L_\infty(\mathcal{R}^n,E_0)} \le \|u_1\|_{L_r(\mathcal{R}^n,E_1)}\, \|u_2\|_{L_{r'}(\mathcal{R}^n,E_2)} \tag{1.9.27}$$

(cf. [Ama95, (III.4.2.20)˘(III.4.2.22)]). If $u_j \in \mathcal{D}(\mathbb{R}^n, E_j)$ for $j = 1, 2$, we know from (1.9.7) that $u_1 *_\bullet u_2 \in \mathcal{D}(\mathbb{R}^n, E_0)$. Since $\mathcal{D}(\mathbb{R}^n, E_1)$ is dense in $\mathbf{F}_1(\mathbb{R}^n, E_1)$ if $\mathbf{F}_1$ belongs to $\{C_0, L_p, L_q\}$ and since $C_0(\mathbb{R}^n, E_0)$ is a closed linear subspace of $L_\infty(\mathbb{R}^n, E_0)$ containing $\mathcal{D}(\mathbb{R}^n, E_0)$, we infer from (1.9.26) and (1.9.27) that the assertion is true for $\mathbf{F}_1 \in \{C_0, L_p, L_q\}$.

If $u_1 \in BUC(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{D}(\mathbb{R}^n, E_2)$ then, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|u_1(x) - u_1(y)| \le \varepsilon$ for $x, y \in \mathbb{R}^n$ with $|x - y| \le \delta$. Hence

$$|u_1 *_\bullet u_2(x) - u_1 *_\bullet u_2(y)| \le \int_{\mathcal{R}^n} |u_1(x-z) - u_1(y-z)|\, |u_2(z)|\, dz \le \varepsilon\, \|u_2\|_1$$

for $x, y \in \mathbb{R}^n$ with $|x - y| \le \delta$. This shows that $u_1 *_\bullet u_2 \in BUC(\mathbb{R}^n, E_0)$ in this case. Hence the assertion follows for $\mathbf{F}_1 = BUC$ from (1.9.26) and the fact that $BUC(\mathbb{R}^n, E_0)$ is a closed linear subspace of $L_\infty(\mathbb{R}^n, E_0)$.

Finally, suppose that $u_1 \in L_\infty(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{D}(\mathbb{R}^n, E_2)$. Then we infer from Remark 1.9.6(h) and (1.9.26) that

$$\|\tau_a(u_1 *_\bullet u_2) - u_1 *_\bullet u_2\|_\infty = \|u_1 *_\bullet (\tau_a u_2 - u_2)\|_\infty \le \|u_1\|_\infty \, \|\tau_a u_2 - u_2\|_1$$

for $a \in \mathbb{R}^n$. Since the translation group is strongly continuous on $L_1(\mathbb{R}^n, E_2)$, it follows that $u_1 *_\bullet u_2 \in BUC(\mathbb{R}^n, E_0)$ in this case. Thus the assertion holds for $\mathbf{F}_1 = L_\infty$ as well. ÷

Given the hypotheses of Theorem 1.9.9, inequality (1.9.25), that holds for all $u_j \in \mathbf{F}_j(\mathbb{R}^n, E_j)$, $j = 1, 2$, is again called Young's inequality.

The Convolution Theorem

Lastly, we extend the CONVOLUTION THEOREM to vector-valued distributions.

1.9.10 Theorem If $u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u_2 \in \mathcal{S}(\mathbb{R}^n, E_2)$ then $(u_1 *_\bullet u_2)\widehat{\ } = \widehat{u}_1 \bullet \widehat{u}_2$.

Proof Since $\mathcal{F}$ is a toplinear automorphism of $\mathcal{S}(\mathbb{R}^n, E)$ and of $\mathcal{S}'(\mathbb{R}^n, E)$, it follows from Theorems 1.6.4 and 1.9.1, together with

$$\mathcal{S}(\mathbb{R}^n, E) \hookrightarrow \mathcal{O}_M(\mathbb{R}^n, E) \hookrightarrow \mathcal{S}'(\mathbb{R}^n, E) ,$$

that the bilinear maps $(u_1, u_2) \mapsto (u_1 *_\bullet u_2)\widehat{\ }$ and $(u_1, u_2) \mapsto \widehat{u}_1 \bullet \widehat{u}_2$ are well-deøned and hypocontinuous from $\mathcal{S}'(\mathbb{R}^n, E_1) \times \mathcal{S}(\mathbb{R}^n, E_2)$ into $\mathcal{S}'(\mathbb{R}^n, E_0)$. Thanks to

$$(u \otimes e)\widehat{\ } = \widehat{u} \otimes e , \qquad u \otimes e \in \mathcal{S}'(\mathbb{R}^n) \otimes E ,$$

we see from (1.6.10), (1.9.1), and the scalar convolution theorem that they coincide on the dense linear subspace $\big(\mathcal{S}'(\mathbb{R}^n) \otimes E_1\big) \times \big(\mathcal{S}(\mathbb{R}^n) \otimes E_2\big)$, hence everywhere. ÷

1.9.11 Remarks (a) Let

$$\mathcal{O}_C'(\mathbb{R}^n, E) := \mathcal{F}^{-1}\mathcal{O}_M(\mathbb{R}^n, E) := \big\{\, u \in \mathcal{S}'(\mathbb{R}^n, E) \ ; \ \mathcal{F}u \in \mathcal{O}_M(\mathbb{R}^n, E) \,\big\} ,$$

endowed with the unique locally convex topology such that

$$\mathcal{F} \in \mathcal{L}\mathrm{is}\big(\mathcal{O}_C'(\mathbb{R}^n, E), \mathcal{O}_M(\mathbb{R}^n, E)\big) .$$

Then it follows from Theorem 1.9.10, the continuity properties of $\mathcal{F}$, and Theorem 1.6.4 that we can deøne a hypocontinuous bilinear map

$$\mathcal{S}'(\mathbb{R}^n, E_1) \times \mathcal{O}_C'(\mathbb{R}^n, E_2) \to \mathcal{S}'(\mathbb{R}^n, E_0) , \quad (u_1, u_2) \mapsto u_1 *_\bullet u_2 , \tag{1.9.28}$$

the convolution with respect to multiplication (1.5.20), by putting

$$u_1 *_\bullet u_2 := \mathcal{F}^{-1}(\widehat{u}_1 \bullet \widehat{u}_2) , \qquad u_1 \in \mathcal{S}'(\mathbb{R}^n, E_1) , \quad u_2 \in \mathcal{O}_C'(\mathbb{R}^n, E_2) . \tag{1.9.29}$$

We leave it to the interested reader to carry over the properties of Remarks 1.9.6 to this more general case.

(b) Given $a \in \mathcal{O}_M(\mathbb{R}^n, E_1)$, we put

$$a(D)u := \mathcal{F}^{-1} a_\bullet \mathcal{F} u := \mathcal{F}^{-1}(a \bullet \widehat{u}) \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E_2) \ .$$

Then it follows from the isomorphism properties of the Fourier transform and from Theorem 1.6.4 that

$$a(D) \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E_2), \mathcal{S}'(\mathbb{R}^n, E_0)\big) \ .$$

Moreover,

$$a(D)u := \mathcal{F}^{-1}(a) *_\bullet u \ , \qquad u \in \mathcal{S}(\mathbb{R}^n, E_2) \ ,$$

thanks to the Convolution Theorem 1.9.10. Motivated by these facts we put

$$a(D)u := \mathcal{F}^{-1}(a_\bullet \mathcal{F})u := \mathcal{F}^{-1}(a) *_\bullet u \tag{1.9.30}$$

whenever $a \in \mathcal{S}'(\mathbb{R}^n, E_1)$ and $u \in \mathcal{S}'(\mathbb{R}^n, E_2)$ are such that the convolution product on the right-hand side is well-deøned. In these general situations $a(D)$ is said to be a translation-invariant (pseudodiœerential) operator with symbol a (related to multiplication (1.5.20)). Of course, this is motivated by Theorem 1.2.2. Note that

$$a(D) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E_2), \mathcal{S}(\mathbb{R}^n, E_0)\big) \cap \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E_2), \mathcal{S}'(\mathbb{R}^n, E_0)\big) \tag{1.9.31}$$

if $a \in \mathcal{O}_M(\mathbb{R}^n, E_1)$. ÷

Convention In the remainder of this work the multiplications which occur belong most of the time to one of the classes given in Examples 1.3.1(a)ˇ(e). If this is the case, we usually omit the symbol $\bullet$ everywhere, e.g., we simply write $*$ for $*_\bullet$, etc. Note, for instance, that, if $\bullet$ is the multiplication of Example 1.3.1(b), then (1.9.31) reduces to

$$a(D) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E)\big) \cap \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E)\big) \ ,$$

an assertion that has already been given in Chapter III. Similarly, using this particular multiplication, many of the results of the present subsection reduce to assertions given, and having been used already, in [Ama95]. Nevertheless, even in this simpler situation we shall refer to the theorems and results formulated and proven in the present subsection since this is the place where all essential properties have been collected.

1.10 Laplace Transforms

In this subsection we restrict ourselves to the consideration of distributions in one variable, that is, on $\mathbb{R}$ only, although most of the results below have extensions to distributions on $\mathbb{R}^n$. However, the one-dimensional case is the most important one and the only one we shall make use of.

Deønition of Laplace Transforms

For abbreviation we put

$$e_z := (x \mapsto e^{-zx}) \in \mathcal{E}(\mathbb{R}) \ , \qquad z \in \mathbb{C} \ .$$

Then, given $u \in \mathcal{D}'(\mathbb{R}, E)$, we set

$$I_u := \left\{ \xi \in \mathbb{R} \ ; \ e_\xi u \in \mathcal{S}'(\mathbb{R}, E) \right\} \ .$$

Of course, I_u may be empty, as is the case for $u \in L_{1,\mathrm{loc}}(\mathbb{R}, E)$ with $u(x) = e^{x^2} u_0$ for some nonzero $u_0 \in E$. Moreover, $0 \in I_u$ iœ $u \in \mathcal{S}'(\mathbb{R}, E)$. The following lemma shows that I_u is convex.

1.10.1 Lemma I_u is an interval.

Proof For $\xi_0, \xi_1 \in I_u$ and $t \in [0,1]$ put $\xi := (1-t)\xi_0 + t\xi_1$. Since

$$e_\xi = (e_{\xi_0})^{1-t} (e_{\xi_1})^t \le e_{\xi_0} + e_{\xi_1}$$

it is easily veriøed that $\chi := e_\xi \big/ (e_{\xi_0} + e_{\xi_1}) \in BC^\infty(\mathbb{R}) \hookrightarrow \mathcal{O}_M(\mathbb{R})$.Consequently, $e_\xi u = \chi e_{\xi_0} u + \chi e_{\xi_1} u \in \mathcal{S}'(\mathbb{R}, E)$, which proves the assertion. ÷

The next lemma gives more detailed information on I_u if the support of u is limited to the left.

1.10.2 Lemma Suppose that $\inf \mathrm{supp}(u) > -\infty$. Then $\sup I_u = \infty$ if $I_u \neq \emptyset$.

Proof Let $\mathrm{supp}(u) \subset [\alpha, \infty)$ for some $\alpha \in \mathbb{R}$. Fix $\xi_0 \in I_u$ and $\psi \in \mathcal{E}(\mathbb{R})$ with $\psi \,|\, [\alpha - 1, \infty) = 1$ and $\psi \,|\, (-\infty, \alpha - 2] = 0$. Then, given $\xi > \xi_0$,

$$a(\xi) := e_{\xi - \xi_0} \psi \in BC^\infty(\mathbb{R}) \hookrightarrow \mathcal{O}_M(\mathbb{R})$$

and, consequently, $e_\xi u = a(\xi) e_{\xi_0} u \in \mathcal{S}'(\mathbb{R}, E)$. Hence $\xi \in I_u$ for $\xi > \xi_0$. ÷

1.10.3 Remarks (a) If $u \in \mathcal{D}'_+(E)$ and $\xi_0 \in I_u$ then $[\xi_0, \infty) \subset I_u$. In particular, $\mathbb{R}^+ \subset I_u$ if $u \in \mathcal{S}'(\mathbb{R}, E) \cap \mathcal{D}'_+(E)$.

Proof This is an obvious consequence of Lemma 1.10.2. ÷

(b) If $u \in \mathcal{E}'(\mathbb{R}, E)$ then $I_u = \mathbb{R}$.

Proof Since $e_\xi \in \mathcal{E}(\mathbb{R})$ for $\xi \in \mathbb{R}$ we know from Lemma 1.6.3 that $e_\xi u \in \mathcal{E}'(\mathbb{R}, E)$ for $u \in \mathcal{E}'(\mathbb{R}, E)$. Hence the assertion follows from $\mathcal{E}' \hookrightarrow \mathcal{S}'$. ÷

Next we prove the basic technical lemma that will allow us to deøne Laplace transformations for certain distributions.

1.10.4 Lemma Suppose that $u \in \mathcal{D}'(\mathbb{R}, E)$. Then $\widehat{e_\xi u} \in \mathcal{O}_M(\mathbb{R}, E)$ for $\xi \in \mathring{I}_u$, and the map

$$\mathring{I}_u \to \mathcal{O}_M(\mathbb{R}, E) \ , \quad \xi \mapsto \widehat{e_\xi u}$$

is smooth.

Proof Fix $\xi_0, \xi_1 \in \mathring{I}_u$ with $\xi_0 < \xi_1$, and $\varphi_0 \in \mathcal{E}(\mathbb{R})$ with $0 \le \varphi_0 \le 1$ such that $\varphi_0 \,|\, \mathbb{R}^+ = 1$ and $\varphi_0(x) = 0$ for $x \le -1$. Also put $\varphi_1 := 1 - \varphi_0$. Then

$$e_\xi u = e_{\xi-\xi_0}\varphi_0 e_{\xi_0} u + e_{\xi-\xi_1}\varphi_1 e_{\xi_1} u \ , \qquad \xi \in \mathbb{R} \ . \tag{1.10.1}$$

It is easily verified that

$$a_j := (\xi \mapsto e_{\xi-\xi_j}\varphi_j) \in C^\infty\big((\xi_0, \xi_1), \mathcal{S}(\mathbb{R})\big) \ , \qquad j = 0, 1 \ .$$

Hence

$$\widehat{a}_j \in C^\infty\big((\xi_0, \xi_1), \mathcal{S}(\mathbb{R})\big) \ , \qquad j = 0, 1 \ , \tag{1.10.2}$$

since $\mathcal{F} \in \mathcal{L}\mathrm{aut}\big(\mathcal{S}(\mathbb{R})\big)$. From (1.10.1) and the convolution theorem we infer that

$$\widehat{e_\xi u} = \widehat{a_0 e_{\xi_0} u} + \widehat{a_1 e_{\xi_1} u} = (2\pi)^n \big(\widehat{a}_0 * \widehat{e_{\xi_0} u} + \widehat{a}_1 * \widehat{e_{\xi_1} u}\big) \tag{1.10.3}$$

for $\xi \in (\xi_0, \xi_1)$, thanks to $\widehat{\widehat{u}} = (2\pi)^n \check{u}$ and Remark 1.9.6(h). Since $\mathcal{F}$ is a continuous automorphism of $\mathcal{S}$ and of $\mathcal{S}'$ we infer from Proposition 1.2.7 that

$$(\varphi \mapsto \widehat{\varphi} * \widehat{v}) \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}), \mathcal{O}_M(\mathbb{R}, E)\big) \ , \qquad v \in \mathcal{S}'(\mathbb{R}, E) \ .$$

Now the assertion is a consequence of (1.10.2) and (1.10.3). ÷

Suppose that $u \in \mathcal{D}'(\mathbb{R}, E)$ and $\mathring{I}_u \ne \emptyset$. Then u is said to be Laplace transformable and the E-valued function

$$\widetilde{u} : \ \mathring{I}_u + i\mathbb{R} \to E \ , \quad \zeta := \xi + i\eta \mapsto \widetilde{u}(\zeta) := \widehat{e_\xi u}(\eta)$$

is the Laplace transform (or transformation) of u. Note that $\widetilde{u}$ is well-defined by Lemma 1.10.4.

Weakly Holomorphic Functions

Now we prove some facts about E-valued holomorphic functions that we need to derive further properties of Laplace transforms.

Recall that a function $f : X \to E$, where X is an open subset of $\mathbb{C}$ (and E is a complex Banach space) is holomorphic iff it is (complex) differentiable. It is weakly holomorphic if $e' \circ f : X \to \mathbb{C}$ is holomorphic for each $e' \in E'$.

1.10.5 Proposition Let X be open in $\mathbb{C}$. Then $f \in E^X$ is holomorphic iff it is weakly holomorphic.

Proof Suppose that f is weakly holomorphic. Let $z_0 \in X$ be given and choose $r > 0$ such that $\bar{\mathbb{B}}_{\mathbb{C}}(z_0, 2r) \subset X$. Given $e' \in E'$, Cauchy's integral formula implies

$$\begin{aligned}(\alpha-\beta)^{-1}\Big[\frac{e'\circ f(z_0+\alpha)-e'\circ f(z_0)}{\alpha}-\frac{e'\circ f(z_0+\beta)-e'\circ f(z_0)}{\beta}\Big]\\ =\frac{1}{2\pi i}\int_{|z-z_0|=2r}\frac{e'\circ f(z)\,dz}{(z-z_0-\alpha)(z-z_0-\beta)(z-z_0)}\end{aligned}$$

for $\alpha, \beta \in r\mathring{\mathbb{B}}_{\mathbb{C}}$ with $\alpha \neq \beta$. Since the integral is uniformly bounded for these α and β, the uniform boundedness principle implies the existence of a constant c such that

$$\Big|\frac{f(z_0+\alpha)-f(z_0)}{\alpha}-\frac{f(z_0+\beta)-f(z_0)}{\beta}\Big| \le c\,|\alpha-\beta| \ , \qquad \alpha,\beta \in r\mathring{\mathbb{B}}_{\mathbb{C}} \ .$$

Now we deduce from the completeness of E that f is differentiable in z_0. Hence f is holomorphic. The converse is obvious. □

As usual, we identify $\mathbb{C}$ and $\mathbb{R}^2$ by means of the canonical identification

$$\mathbb{C} \ni z = x + iy \quad \longleftrightarrow \quad (x, y) \in \mathbb{R}^2 \ .$$

Then $f \in E^X$ can be interpreted as a function of the complex variable $z \in X \subset \mathbb{C}$ or as a function of the real variables $(x, y) \in X \subset \mathbb{R}^2$. We write ∂ for the complex derivative ∂_z and say that f is real differentiable if f is differentiable with respect to the real variables $(x, y) \in X \subset \mathbb{R}^2$.

1.10.6 Corollary Let X be open in $\mathbb{C}$. Then $f \in E^X$ is holomorphic iff f is real differentiable and $\overline{\partial} f = 0$, where

$$\overline{\partial} := \partial_{\overline{z}} := \frac{1}{2}(\partial_x + i\partial_y) \ , \qquad z = x + iy \ .$$

If this is the case then $\partial f = \partial_z f = \partial_x f$.

Proof Given $e' \in E'$, it follows that $\overline{\partial}(e' \circ f) = e' \circ (\overline{\partial} f)$. Hence $\overline{\partial} f = 0$ iff $e' \circ f$ satisfies the Cauchy-Riemann differential equations for each $e' \in E'$, that is, iff f is weakly holomorphic, hence holomorphic by Proposition 1.10.5. If this is true then

$$e' \circ \partial f = \partial(e' \circ f) = \partial_x\big(\mathrm{Re}(e' \circ f)\big) + i\,\partial_x\big(\mathrm{Im}(e' \circ f)\big) = \partial_x(e' \circ f) = e' \circ \partial_x f$$

for each $e' \in E'$. Hence $\partial f = \partial_x f$. □

Holomorphy of Laplace Transforms

After these preparations we can show that the Laplace transform of a distribution is a holomorphic function. At the end of this subsection we shall derive more precise information about its behavior at infinity.

1.10.7 Theorem If $u \in \mathcal{D}'(\mathbb{R}, E)$ is Laplace transformable then $\widetilde{u}$ is holomorphic.

Proof From Lemma 1.10.4 we know already that $\widetilde{u} \in C^\infty(\mathring{I}_u \times i\mathbb{R}, E)$. Writing $\zeta = \xi + i\eta$ it is easily seen that

$$\partial_\xi \widetilde{u}(\zeta) = -(X e_\xi u)\widehat{\ }(\eta) , \tag{1.10.4}$$

where X denotes multiplication with the independent variable. On the other hand,

$$\begin{aligned}(\partial_\eta \widetilde{u})(\varphi) &= -\widetilde{u}(\partial\varphi) = -\widehat{e_\xi u}(\partial\varphi) = -u\big(e_\xi \widehat{\partial\varphi}\big) \\ &= -i\,u\big(e_\xi X \widehat{\varphi}\big) = -i\,(X e_\xi u)\widehat{\ }(\varphi)\end{aligned}$$

for $\varphi \in \mathcal{D}(\mathbb{R})$. Hence $\overline{\partial}\widetilde{u} = 0$ and the assertion follows from Corollary 1.10.6. ÷

1.10.8 Corollary If $u \in \mathcal{E}'(\mathbb{R}, E)$ then $\widehat{u}$ extends to an E-valued entire analytic function on $\mathbb{C}$, the Laplace transform $\widetilde{u}$ of u.

Properties of Laplace Transforms

Thanks to the close relationship between the Fourier and the Laplace transforms the latter enjoy many properties that are similar to some of those we know to hold for the former.

1.10.9 Proposition Suppose that $u \in \mathcal{D}'(\mathbb{R}, E)$ is Laplace transformable. Then

(i) $\partial^k \widetilde{u} = (-1)^k (X^k u)\widetilde{\ }$, $k \in \mathbb{N}$, where X^k denotes multiplication with the function $x \mapsto x^k$.

(ii) $I_{\partial^k u} \supset I_u$ and

$$(\partial^k u)\widetilde{\ }(\zeta) = \zeta^k\, \widetilde{u}(\zeta) , \qquad \zeta \in \mathring{I}_u + i\mathbb{R} ,$$

for $k \in \mathbb{N}$.

(iii) $\tau_\alpha u$ is Laplace transformable for $\alpha \in \mathbb{R}$ and

$$(\tau_\alpha u)\widetilde{\ }(\zeta) = e^{-\alpha\zeta}\, \widetilde{u}(\zeta) , \qquad \zeta \in \mathring{I}_u + i\mathbb{R} .$$

Proof (i) follows from (1.10.4), Theorem 1.10.7, and Corollary 1.10.6.

(ii) Note that $e_\xi \partial u = \partial(e_\xi u) + \xi e_\xi u$ for $\xi \in \mathbb{R}$. Thus $e_\xi \partial u \in \mathcal{S}'(\mathbb{R}, E)$ if $\xi \in I_u$. Consequently,

$$(\partial u)\widetilde{\ }(\zeta) = (e_\xi \partial u)\widehat{\ }(\eta) = \big(\partial(e_\xi u)\big)\widehat{\ } + \xi\,\widetilde{u} = i\eta\,\widehat{e_\xi u} + \xi\,\widetilde{u} = \zeta\,\widetilde{u}$$

for $\zeta \in \mathring{I}_u + i\mathbb{R}$. Now the assertion follows by induction.

(iii) Observe that $e_\xi \tau_\alpha u = e^{-\alpha\xi}\tau_\alpha(e_\xi u)$ for $\xi \in \mathbb{R}$. Consequently,

$$\begin{aligned}(\tau_\alpha u)^\sim(\zeta) &= (e_\xi \tau_\alpha u)^\wedge(\eta) = e^{-\alpha\xi}\big(\tau_\alpha(e_\xi u)\big)^\wedge(\eta) \\ &= e^{-(\alpha\xi+i\alpha\eta)}\widehat{e_\xi u}(\eta) = e^{-\alpha\zeta}\,\widetilde{u}(\zeta)\end{aligned}$$

for $\zeta \in \mathring{I}_u + i\mathbb{R}$, thanks to [Ama95, (III.4.2.6)]. ÷

1.10.10 Example $(\partial^k \delta_\alpha)^\sim(\zeta) = \zeta^k e^{-\alpha\zeta}$, $\zeta \in \mathbb{C}$, $\alpha \in \mathbb{R}$.

Next we prove an important and useful result, namely the CONVOLUTION THEOREM for Laplace transforms.

1.10.11 Theorem Let $E_1 \times E_2 \to E_0$, $(e_1, e_2) \mapsto e_1 \bullet e_2$ be a multiplication and suppose that $u_j \in \mathcal{D}'(\mathbb{R}, E_j)$, $j = 1, 2$, are Laplace transformable, have convolutive supports, and satisfy $\mathring{I}_{u_1} \cap \mathring{I}_{u_2} \neq \emptyset$. Then $u_1 *_\bullet u_2 \in \mathcal{D}'(\mathbb{R}, E_0)$ is Laplace transformable as well, and

$$(u_1 *_\bullet u_2)^\sim(\zeta) = \widetilde{u}_1(\zeta) \bullet \widetilde{u}_2(\zeta)\ , \qquad \zeta \in \big(\mathring{I}_{u_1} \cap \mathring{I}_{u_2}\big) + i\mathbb{R}\ .$$

Proof From (1.9.18) we know that $u_1 *_\bullet u_2$ is a well-deøned element of $\mathcal{D}'(\mathbb{R}, E_0)$. Given $\xi \in \mathring{I}_{u_1} \cap \mathring{I}_{u_2}$, it follows from $e_\xi u_j \in \mathcal{S}'(\mathbb{R}, E_j)$, Remark 1.9.11(a), and Lemma 1.10.4 that

$$\big((e_\xi u_1) *_\bullet (e_\xi u_2)\big)^\wedge = \widehat{e_\xi u_1} \bullet \widehat{e_\xi u_2}\ . \tag{1.10.5}$$

Let X be a bounded interval in $\mathbb{R}$. Since u_1 and u_2 satisfy condition (Σ) there exists $\rho > 0$ such that $x_j \in \mathrm{supp}(u_j)$ and $x_1 + x_2 \in X$ imply $|x_j| < \rho$ for $j = 1, 2$. Let $\psi_j \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\psi_j \,|\, [-\rho, \rho] = 1$. Then, thanks to the deønition of $(e_\xi u_1) * (e_\xi u_2)$, given $\varphi \in \mathcal{D}(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subset X$, it follows that

$$\begin{aligned}(e_\xi u_1) *_\bullet (e_\xi u_2)(\varphi) &= e_\xi \psi_1 u_1 \otimes_\bullet e_\xi \psi_2 u_2\big(\varphi(x_1 + x_2)\big) \\ &= \psi_1 u_1 \otimes_\bullet \psi_2 u_2\big((e_\xi \varphi)(x_1 + x_2)\big) = u_1 *_\bullet u_2(e_\xi \varphi) \\ &= e_\xi(u_1 *_\bullet u_2)(\varphi)\ .\end{aligned}$$

Thus we infer from (1.10.5) that

$$\big(e_\xi(u_1 *_\bullet u_2)\big)^\wedge = \widehat{e_\xi u_1} \bullet \widehat{e_\xi u_2}\ , \qquad \xi \in \mathring{I}_1 \cap \mathring{I}_2\ ,$$

which, thanks to the deønition of the Laplace transform, proves the theorem. ÷

Laplace Transformable Distributions in $\mathcal{D}'_+(E)$

In the remainder of this subsection we restrict ourselves to the most important case, that is, to the case of Laplace transformable distributions whose supports are bounded below. Thanks to Proposition 1.10.9(iii) and the fact that

$$\operatorname{supp}(u) \subset [\alpha, \infty) \iff \operatorname{supp}(\tau_\alpha u) \subset \mathbb{R}^+ \tag{1.10.6}$$

for $\alpha \in \mathbb{R}$ and $u \in \mathcal{D}'(\mathbb{R}, E)$, it is no loss of generality to consider only Laplace transformable distributions belonging to $\mathcal{D}'_+(E)$. Thus we put

$$\mathcal{L}\mathrm{ap}_+(E) := \{\, u \in \mathcal{D}'_+(E) \ ; \ I_u \neq \emptyset \,\} \ .$$

From Lemma 1.10.2 we know that $\sup I_u = \infty$ for $u \in \mathcal{L}\mathrm{ap}_+(E)$, and we introduce $\omega_0(u)$, the abscissa of holomorphy of $\widetilde{u}$, by

$$\omega_0(u) := \inf I_u \ .$$

Then $\widetilde{u}$ is deøned and holomorphic on $\bigl[\operatorname{Re} z > \omega_0(u)\bigr]$ with values in E.

Suppose that $u \in \mathcal{L}\mathrm{ap}_+(E)$. Fix $\psi_j \in \mathcal{E}(\mathbb{R})$, satisfying $\operatorname{supp}(\psi_j) \subset [-2, \infty)$ and $\psi_j \,|\, [-1, \infty) = 1$ for $j = 0, 1$. Also let $\xi_1 > \xi_0 > \omega_0(u)$. Then $\psi_j e_{\zeta - \xi_j} \in \mathcal{S}(\mathbb{R})$ for $\operatorname{Re}\zeta > \xi_1$ and $e_{\xi_j} u \in \mathcal{S}'(\mathbb{R}, E)$ for $j = 0, 1$. Hence $e_{\xi_j} u(\psi_j e_{\zeta - \xi_j})$ is well-deøned for $j = 0, 1$ and $\operatorname{Re}\zeta > \xi_1$. Note that

$$\begin{aligned} e_{\xi_0} u(\psi_0 e_{\zeta - \xi_0}) &= e_{\xi_0} u(\psi_0 \psi_1 e_{\xi_1 - \xi_0} e_{\zeta - \xi_1}) = \psi_0 e_{\xi_1 - \xi_0} e_{\xi_0} u(\psi_1 e_{\zeta - \xi_1}) \\ &= e_{\xi_1} u(\psi_1 e_{\zeta - \xi_1}) \end{aligned} \tag{1.10.7}$$

for $\operatorname{Re}\zeta > \xi_1$, thanks to $\psi_0 e_{\xi_1 - \xi_0} \in \mathcal{S}(\mathbb{R})$ and $\psi_j = 1$ in a neighborhood of $\operatorname{supp}(u)$. This shows that, given any $u \in \mathcal{L}\mathrm{ap}_+(E)$ and $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > \omega_0(u)$,

$$u(e_\zeta) := e_{\xi_0} u(\psi_0 e_{\zeta - \xi_0}) \tag{1.10.8}$$

is well-deøned, independently of $\xi_0 \in \bigl(\omega_0(u), \operatorname{Re}\zeta\bigr)$ and $\psi_0 \in \mathcal{E}(\mathbb{R})$, provided $\psi_0 = 1$ in a neighborhood of $\operatorname{supp}(u)$ and ψ_0 vanishes near $-\infty$.

1.10.12 Theorem If $u \in \mathcal{L}\mathrm{ap}_+(E)$ then $\widetilde{u}(\zeta) = u(e_\zeta)$ for $\operatorname{Re}\zeta > \omega_0(u)$.

Proof Write $\xi + i\eta = \zeta$ and øx $\xi_0 \in \bigl(\omega_0(u), \xi\bigr)$ and $\psi_0 \in \mathcal{E}(\mathbb{R})$ vanishing near $-\infty$ and being equal to 1 in $[-1, \infty)$. Then, given $\varphi \in \mathcal{D}(\mathbb{R})$, the arguments of (1.10.7) show that

$$\widetilde{u}(\xi + i\ \cdot\ )(\varphi) = \widehat{e_\xi u}(\varphi) = e_\xi u(\widehat{\varphi}) = e_{\xi_0} u(e_{\xi - \xi_0} \psi_0 \widehat{\varphi}) \ .$$

Note that

$$(e_{\xi - \xi_0} \psi_0 \widehat{\varphi})(\eta) = \int_{\mathcal{R}} e^{-(\xi - \xi_0)\eta} \psi_0(\eta) e^{-i\eta x} \varphi(x)\, dx$$

and $e_{\xi_0} u \in \mathcal{S}'_+(E)$ imply

$$\begin{aligned} e_{\xi_0} u(e_{\xi - \xi_0} \psi_0 \widehat{\varphi}) &= \int_{\mathcal{R}} e_{\xi_0} u(e_{\xi - \xi_0 + ix} \psi_0) \varphi(x)\, dx = \int_{\mathcal{R}} u(e_{\xi + ix}) \varphi(x)\, dx \\ &= u(e_{\xi + i\cdot})(\varphi) \ , \end{aligned}$$

as is easily veriøed. This proves the assertion. ÷

1.10.13 Corollary Suppose that $u \in L_{1,\mathrm{loc}}(\mathbb{R}, E)$ has its support in $\mathbb{R}^+$ and satisøes

$$\omega(u) := \inf\bigl\{ \omega \in \mathbb{R} \ ; \ e_\omega u \in L_1(\mathbb{R}, E) \bigr\} < \infty \ . \tag{1.10.9}$$

Then $u \in \mathcal{L}\mathrm{ap}_+(E)$ with $\omega_0(u) \le \omega(u)$, and

$$\widetilde{u}(\zeta) = \int_0^\infty e^{-\zeta x} u(x)\, dx \ , \qquad \mathrm{Re}\, \zeta > \omega(u) \ . \tag{1.10.10}$$

Proof It suŒces to recall that $L_1(\mathbb{R}, E) \hookrightarrow \mathcal{S}'(\mathbb{R}, E)$. ÷

It should be remarked that $\omega(u)$ is called abscissa of absolute convergence of the Laplace integral of u, which is the integral appearing in (1.10.10).

1.10.14 Examples Suppose that $A \in \mathcal{G}(E, M, \sigma)$ for some $M \ge 1$ and $\sigma \in \mathbb{R}$.

(a) Put $e_A(t) := e^{-tA}$ for $t \ge 0$, and $e_A(t) := 0$ for $t < 0$. Then $e_A \in \mathcal{L}\mathrm{ap}_+\bigl(\mathcal{L}(E)\bigr)$ and $\widetilde{e_A}(\zeta) = (\zeta + A)^{-1}$ for $\mathrm{Re}\, \zeta > \sigma$.

Proof It is easily veriøed that $\omega(e_A) \le \sigma$. From semigroup theory it is known that

$$(\zeta + A)^{-1} = \int_0^\infty e^{-\zeta t} e^{-tA}\, dt \ , \qquad \mathrm{Re}\, \zeta > \sigma \ .$$

Hence Corollary 1.10.13 implies the assertion. ÷

(b) Given $z \in \mathbb{C}$, put $u_z(t) := t^{z-1} e^{-tA}$ for $t > 0$, and $u_z(t) := 0$ for $t < 0$. Then $u_z \in \mathcal{L}\mathrm{ap}_+\bigl(\mathcal{L}(E)\bigr)$ for $\mathrm{Re}\, z > 0$, and $\widetilde{u}_z(\zeta) = \Gamma(z)(\zeta + A)^{-z}$ for $\mathrm{Re}\, \zeta > \sigma$.

Proof Again $\sigma(u_z) \le \sigma$ for $\mathrm{Re}\, z > 0$, and

$$\widetilde{u}(\zeta) = \int_0^\infty t^{z-1} e^{-(\zeta + A)t}\, dt \ , \qquad \mathrm{Re}\, \zeta > \sigma \ ,$$

by Corollary 1.10.13. Now the assertion follows from [Ama95, Theorem III.4.6.6]. ÷

An Inversion Formula

Using the fact that $\mathcal{F}$ is an automorphism of $\mathcal{S}'(\mathbb{R}, E)$, it is possible to recover the distribution $u \in \mathcal{L}\mathrm{ap}_+(E)$ from its Laplace transform.

1.10.15 Theorem Suppose that $u \in \mathcal{L}\mathrm{ap}_+(E)$. Then

$$u = e_{-\xi} \mathcal{F}^{-1} \widetilde{u}(\xi + i\,\cdot\,) \ , \qquad \xi > \omega_0(u) \ . \tag{1.10.11}$$

If $\widetilde{u}(\xi + i\,\cdot\,) \in L_1(\mathbb{R}, E)$ for some $\xi > \omega_0(u)$ then $e_\xi u \in C_0(\mathbb{R}, E)$ and

$$u(x) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} e^{xz}\, \widetilde{u}(z)\, dz \ , \qquad x \in \mathbb{R} \ . \tag{1.10.12}$$

Proof The first assertion follows directly from the definition of $\widetilde{u}$. If $\widetilde{u}(\xi + i\,\cdot\,)$ is integrable over $\mathbb{R}$ for some $\xi > \omega_0(u)$ then the Riemann-Lebesgue theorem guarantees that $e_\xi u = \mathcal{F}^{-1}\widetilde{u}(\xi + i\,\cdot\,) \in C_0(\mathbb{R}, E)$. Moreover,

$$u(x) = \frac{e^{\xi x}}{2\pi} \int_{-\infty}^{\infty} e^{ix\eta}\,\widetilde{u}(\xi + i\eta)\,d\eta\ , \qquad x \in \mathbb{R}\ ,$$

so that (1.10.12) is obvious. ÷

1.10.16 Remarks (a) From (1.10.11) we obtain immediately a uniqueness theorem for Laplace transforms: If $u, v \in \mathcal{L}\mathrm{ap}_+(E)$ and there is $\xi > \max\bigl(\omega_0(u), \omega_0(v)\bigr)$ with $\widetilde{u}\,|\,(\xi + i\mathbb{R}) = \widetilde{v}\,|\,(\xi + i\mathbb{R})$, then $u = v$. ÷

(b) It should be remarked that $\mathcal{L}\mathrm{ap}_+(E)$ is a vector subspace of $\mathcal{D}'_+(E)$ and that the Laplace transform satisfies

$$(u + \alpha v)\widetilde{\ }(\zeta) = \widetilde{u}(\zeta) + \alpha\widetilde{v}(\zeta)\ , \qquad \operatorname{Re}\zeta > \omega_0(w) \vee \omega_0(v)\ ,$$

for $u, v \in \mathcal{L}\mathrm{ap}_+(E)$ and $\alpha \in \mathbb{R}$. ÷

Characterization of Laplace Transforms

Let X be open in $\mathbb{K}^n$. Then $f \in E^X$ is said to be polynomially bounded in X if there exist $c \geq 0$ and $k \in \mathbb{N}$ such that

$$|f(x)| \leq c(1 + |x|)^k\ , \qquad x \in X\ .$$

Using this definition we now characterize those E-valued functions that are Laplace transforms of distributions in $\mathcal{D}'_+(E)$.

1.10.17 Theorem Suppose that $u \in \mathcal{L}\mathrm{ap}_+(E)$. Then $\widetilde{u}$ is holomorphic in the half-plane $\bigl[\operatorname{Re} z > \omega_0(u)\bigr]$ and polynomially bounded on $[\operatorname{Re} z \geq \omega]$ for each $\omega > \omega_0(u)$.

Conversely, suppose that $\omega \geq 0$ and $v : [\operatorname{Re} z > \omega] \to E$ is holomorphic and polynomially bounded on $[\operatorname{Re} z \geq \omega']$ for each $\omega' > \omega$. Then v is the Laplace transform of some $u \in \mathcal{D}'_+(E)$, and $\omega_0(u) \leq \omega$.

Proof (i) Suppose that $u \in \mathcal{L}\mathrm{ap}_+(E)$. Then $\widetilde{u}$ is holomorphic in $\bigl[\operatorname{Re} z > \omega_0(u)\bigr]$ by Lemma 1.10.2 and Theorem 1.10.7. Fix $\omega > \omega_0(u)$ and put $v := e_\omega u \in \mathcal{S}'_+(E)$. Then there exist $k, m \in \mathbb{N}$ such that

$$|v(\varphi)| \leq c q_{k,m}(\varphi)\ , \qquad \varphi \in \mathcal{S}(\mathbb{R})\ . \tag{1.10.13}$$

Choose $\psi \in \mathcal{E}(\mathbb{R})$ such that $\operatorname{supp}(\psi) \subset (-2, \infty)$ and $\psi\,|\,(-1, \infty) = 1$. Then, thanks to (1.10.8) and Proposition 1.10.12,

$$\widetilde{u}(\zeta) = v(\psi e_{\xi - \omega})\ , \qquad \operatorname{Re}\zeta \geq \omega > \omega_0\ . \tag{1.10.14}$$

From this and (1.10.13) we infer that $\widetilde{u}$ is polynomially bounded on $[\operatorname{Re} z \geq \omega]$ for each $\omega > \omega_0(u)$.

(ii) First we consider the special case that $f : [\operatorname{Re} z > \omega] \to E$ is holomorphic for some $\omega \in \mathbb{R}$, and that

$$|f(\zeta)| \leq c(1+|\zeta|)^{-2}\ , \qquad \operatorname{Re}\zeta > \omega\ . \tag{1.10.15}$$

Then $f(\xi + i\,\cdot\,) \in L_1(\mathbb{R}, E)$ for $\xi > \omega$. Motivated by Theorem 1.10.15 we put

$$u(x) := \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} e^{x\zeta} f(\zeta)\,d\zeta\ , \qquad x \in \mathbb{R}\ . \tag{1.10.16}$$

Since (1.10.15) implies

$$|e^{x\zeta} f(\zeta)| \leq \frac{ce^{x\xi}}{1+\eta^2}\ , \qquad \zeta = \xi + i\eta\ , \quad \xi > \omega\ ,$$

a standard application of Cauchy's theorem shows that the integral in (1.10.16) is independent of $\xi > \omega$. Rewriting (1.10.16) in the form

$$(e_\xi u)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\eta} f(\xi + i\eta)\,d\eta = \mathcal{F}^{-1}\big(f(\xi + i\,\cdot\,)\big)(x)\ , \qquad x \in \mathbb{R}\ , \tag{1.10.17}$$

we deduce from the Riemann-Lebesgue lemma that $e_\xi u \in C_0(\mathbb{R}, E)$ and that

$$f(\xi + i\eta) = \widehat{e_\xi u}(\eta)\ , \qquad \xi > \omega\ , \quad \eta \in \mathbb{R}\ .$$

From (1.10.15) and (1.10.17) we infer that

$$|u(x)| \leq c\frac{e^{\xi x}}{2\pi}\int_{-\infty}^{\infty}\frac{d\eta}{1+\eta^2} \leq ce^{\xi x}\ , \qquad \xi > \omega\ , \quad x \in \mathbb{R}\ . \tag{1.10.18}$$

Since $u \in C(\mathbb{R}, E)$ we see from this estimate that $u(x) = 0$ for $x < 0$. Hence $\operatorname{supp}(u)$ is contained in $\mathbb{R}^+$. This shows that $u \in \mathcal{L}\mathrm{ap}_+(E)$ and $\widetilde{u}(\zeta) = f(\zeta)$ for $\operatorname{Re}\zeta > \omega$.

Now suppose that $\omega \geq 0$ and $g : [\operatorname{Re} z > \omega] \to E$ is holomorphic and, given $\omega' > \omega$, there exist $c > 0$ and $m \in \mathbb{N}$ such that

$$|g(\zeta)| \leq c(1+|\zeta|)^m\ , \qquad \operatorname{Re}\zeta > \omega'\ . \tag{1.10.19}$$

Put $f(\zeta) := \zeta^{-m-2} g(\zeta)$ for $\operatorname{Re}\zeta > \omega$. Then f is holomorphic in $[\operatorname{Re} z > \omega]$ and satisøes an estimate of the form (1.10.15) on $[\operatorname{Re} z \geq \omega']$. Hence, by what has already been shown and since (1.10.19) holds for each $\omega' > \omega$, there exists $u \in \mathcal{L}\mathrm{ap}_+(E)$ with $\widetilde{u}(\zeta) = f(\zeta)$ for $\operatorname{Re}\zeta > \omega$. Then $v := \partial^{m+2} u$ belongs to $\mathcal{L}\mathrm{ap}_+(E)$ and $\widetilde{v}(\zeta)$ equals $g(\zeta)$ for $\operatorname{Re}\zeta > \omega$, thanks to Proposition 1.10.9(ii). ÷

1.10.18 Remarks (a) The above proof shows that estimate (1.10.15) guarantees that the holomorphic function f is the Laplace transformation of a continuous function with support in $\mathbb{R}^+$. A characterization of Laplace transforms of functions in $L_\infty(\mathbb{R}, E)$ with support in $\mathbb{R}^+$ has been given by Arendt in [Are87] (also see [Pr93, Theorem 0.2]). The standard reference for vector-valued Laplace transforms of functions and measures is [HP57, Chapter VI].

(b) For simplicity we restricted ourselves to distributions on $\mathbb{R}$. However, it is not diŒcult to verify that the proofs given in [Sch66] or [Hr83] for scalar Laplace transforms in n variables carry over to the case of vector-valued distributions. This is also true for vector-valued versions of the Paley-Wiener theorems characterizing Laplace transforms of test functions and distributions with compact supports. ÷

The above proofs are adaptions of and straight-forward extensions from the scalar to the vector-valued case of the proofs given in [Sch66], [Sch65], and [Hr83] (also see [DL90, Chapter XVI]). A convolution theorem more general than Theorem 1.10.11 is given, of course, in [Sch57b].

2 Dyadic Decompositions and Multiplier Theorems

This section is to a large extent of technical nature. The heart of the expounded material are the results on dyadic decompositions and Fourier multiplier theorems.

The technique of dyadic decompositions is well-known in harmonic analysis and belongs to the fundament on which one can build large parts of the theory of function spaces. Multiplier theorems are the principal tool for proving resolvent estimates for elliptic diœerential and pseudodiœerential operators. As we know from [Ama95], resolvent estimates belong to the cornerstones on which the theory of parabolic evolution equations rests.

2.1 Lebesgue Spaces

In this subsection we collect some more or less well-known results on L_p-spaces of vector-valued functions. In particular, we consider sequence spaces where the elements of the sequences lie in Banach spaces.

Radon Measures

Let X be a σ-compact metrizable space, that is, X is a locally compact metrizable space that can be written as a countable union of compact sets. This is the case iœ X is a locally compact separable metric space. By a positive Radon measure μ on X we mean a complete $\bar{\mathbb{R}}^+$-valued measure on a σ-algebra $\mathcal{A}_\mu$ containing the Borel σ-algebra of X, which is ønite on compact sets. Then μ is regular, that is,

$$\mu(A) = \inf\{\, \mu(O)\ ;\ O \supset A,\ O \text{ open} \,\} = \sup\{\, \mu(K)\ ;\ K \subset A,\ K \text{ compact} \,\}$$

for $A \in \mathcal{A}_\mu$ (e.g., [Rud70, Theorem 2.18]). Clearly, if X is an open or a closed subset of $\mathbb{R}^n$ and μ is the (restriction of the) n-dimensional Lebesgue measure on X then X is σ-compact and μ is a positive Radon measure on X.

Now let X be a σ-compact metrizable space, μ a positive Radon measure on X, and $E := (E, |\cdot|)$ a Banach space. In the following, we make free use (as we already did in [Ama95]) of the Bochner integral and refer the reader to [DS57], [HP57], or [Yos65] for details. In particular, we recall that PETTIS' THEOREM guarantees that a function $u \in E^X$ is μ-measurable iœ it is weakly μ-measurable and μ-almost separably valued. In particular, every $u \in C(X, E)$ is μ-measurable.

For $u \in \mathcal{L}_1(X, \mu, E)$ and $A \in \mathcal{A}_\mu$ with $0 < \mu(A) < \infty$ we deøne the average of u over A by

$$⨍_A u\, d\mu := \frac{1}{\mu(A)} \int_A u\, d\mu\ .$$

2.1.1 Proposition Let $u \in \mathcal{L}_1(X, \mu, E)$ and let S be a closed subset of E. If $⨍_A u\, d\mu \in S$ for every $A \in \mathcal{A}_\mu$ with $0 < \mu(A) < \infty$ then $u(x) \in S$ for μ-a.e. $x \in X$.

Proof We can assume that $S \neq E$. By changing u on a μ-null set we can assume that $u(X)$ is separable, hence that E is separable. Let $e \in S^c$ be given and let $r > 0$ be so small that $\bar{\mathbb{B}}(e,r) \cap S = \emptyset$. Let $K = \overline{K} \subset\subset X$ and suppose that $A := \big[u(x) \in \mathbb{B}(e,r)\big] \cap K$ has positive measure. Then

$$\Big| \fint_A u\,d\mu - e \Big| \leq \fint_A |u-e|\,d\mu \leq r$$

which contradicts $\fint_A u\,d\mu \in S$. Hence we infer from the σ-compactness of X that $\mu\big(\big[u(x) \in \mathbb{B}(e,r)\big]\big) = 0$. Thanks to the separability of E there exists a countable dense subset M of S^c. Consequently, S^c can be written as a countable union of open balls with centers in M and rational radii. Now the assertion is obvious. ÷

2.1.2 Corollary If $u \in \mathcal{L}_1(X,\mu,E)$ and $\int_A u\,d\mu = 0$ for each $A \in \mathcal{A}_\mu$ with ønite measure then $u = 0$ μ-a.e.

Dual Spaces

For $1 \leq p \leq \infty$, $u \in L_p(X,\mu,E)$, and $v \in L_{p'}(X,\mu,E')$ with $p' := p/(p-1)$, it follows from Hlder's inequality that

$$\Big| \int_X \langle v,u\rangle_E\,d\mu \Big| \leq \int_X |\langle v,u\rangle_E|\,d\mu \leq \int_X |v|_{E'}\,|u|_E\,d\mu \leq \|v\|_{p'}\,\|u\|_p\ . \tag{2.1.1}$$

Thus, given $v \in L_{p'}(X,\mu,E')$ and putting

$$T_v u := \int_X \langle v,u\rangle_E\,d\mu\ , \qquad u \in L_p(X,\mu,E)\ ,$$

we see that

$$T_v \in L_p(X,\mu,E)'\ , \quad \|T_v\| \leq \|v\|_{p'}\ , \tag{2.1.2}$$

thanks to (2.1.1). Consequently,

$$(v \mapsto T_v) \in \mathcal{L}\big(L_{p'}(X,\mu,E'), L_p(X,\mu,E)'\big)\ . \tag{2.1.3}$$

In fact, more is true:

2.1.3 Proposition Given $p \in [1,\infty]$, the map $v \mapsto T_v$ is a linear isometry from $L_{p'}(X,\mu,E')$ into $L_p(X,\mu,E)'$.

Proof Let $p > 1$ and let $v := \sum e'_j \chi_{A_j}$ be a μ-simple E'-valued function so that the A_j are pair-wise disjoint. Let $\varepsilon > 0$ be given. Then there exist $e_j \in E$ such that $|e_j| = 1$ and $\langle e'_j, e_j\rangle \geq (1-\varepsilon)\,\big|e'_j\big|$, where $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_E$. Put

$$u := \sum \big|e'_j\big|^{p'-1} e_j \chi_{A_j}$$

and observe that

$$\langle v,u\rangle = \sum \left|e_j'\right|^{p'-1} \langle e_j', e_j\rangle \chi_{A_j} \geq (1-\varepsilon) \sum \left|e_j'\right|^{p'} \chi_{A_j} \ .$$

Hence

$$\int_X \langle v,u\rangle \, d\mu \geq (1-\varepsilon) \sum \left|e_j'\right|^{p'} \mu(A_j) = (1-\varepsilon)\, \|v\|_{p'}^{p'}$$

and

$$\|u\|_p = \Big(\sum \left|e_j'\right|^{(p'-1)p} \mu(A_j)\Big)^{1/p} = \|v\|_{p'}^{p'/p} = \|v\|_{p'}^{p'-1}$$

if $p<\infty$. If $p=\infty$ and $v\neq 0$ then $\|u\|_\infty = 1$. Thus

$$T_v u = \int_X \langle v,u\rangle \, d\mu \geq (1-\varepsilon)\, \|v\|_{p'} \, \|u\|_p \ .$$

Consequently, $\|T_v\| \geq (1-\varepsilon)\, \|v\|_{p'}$ and, letting $\varepsilon\to 0$ and observing (2.1.2), we see that

$$\|T_v\| = \|v\|_{p'} \tag{2.1.4}$$

for each μ-simple E'-valued function v on X. As the set of these functions is dense in $L_{p'}(X,\mu,E')$, it follows from (2.1.3) that (2.1.4) is true for each $v\in L_{p'}(X,\mu,E')$, provided $1<p\leq\infty$.

Let $p=1$ and $v\in L_\infty(X,\mu,E')$. Denote by $\widetilde{v}$ an arbitrary representative of v. By changing $\widetilde{v}$ on a μ-null set we may assume that $\widetilde{v}(X)$ is separable. Hence $F := \overline{\operatorname{span}}\, \widetilde{v}(X)$ is a separable closed subspace of E'. Choose $e'\in F$ such that $|e'| > \|T_v\|$, let $0<r<|e'|-\|T_v\|$, and put $B := \big[\, |\widetilde{v}(x)-e'|<r \big]$. Fix a positive ε strictly less than $(|e'|-\|T_v\|-r)/|e'|$ and choose $e\in E$ with $|e|=1$ and $\langle e',e\rangle > (1-\varepsilon)\,|e'|$. Then

$$\begin{aligned}\big|\langle \widetilde{v}(x),e\rangle - |e'|\big| &\leq \big|\langle \widetilde{v}(x),e\rangle - \langle e',e\rangle\big| + \big|\langle e',e\rangle - |e'|\big| \leq |\widetilde{v}(x)-e'| + \varepsilon\, |e'| \\ &< |e'| - \|T_v\|\end{aligned} \tag{2.1.5}$$

for $x\in B$. Observe that

$$\Big|\int_X \langle \widetilde{v}, e\chi_A\rangle \, d\mu\Big| \leq \|T_v\|\, \mu(A) \ , \qquad A\in\mathcal{A}_\mu \ ,$$

so that

$$⨍_A \langle \widetilde{v}, e\rangle \, d\mu \in \bar{\mathbb{B}}\,(0,\|T_v\|)$$

for each $A\in\mathcal{A}_\mu$ with $0<\mu(A)<\infty$. Hence $|\langle\widetilde{v}(x),e\rangle| \leq \|T_v\|$ for μ-a.e. $x\in X$, thanks to Proposition 2.1.1. Thus it follows from (2.1.5) that $\mu(B)=0$. Now the separability of F implies ¯ similarly as in the proof of Proposition 2.1.1 ¯ that $|\widetilde{v}(x)| \leq \|T_v\|$ for μ-a.e. $x\in X$. Consequently, $\|v\|_\infty \leq \|T_v\|$ and (2.1.2) implies the assertion in this case as well. ÷

Convention By means of this isometry we identify $L_{p'}(X, \mu, E')$ for $1 \leq p \leq \infty$ with a closed linear subspace of $L_p(X, \mu, E)'$ so that

$$\langle v, u \rangle_{L_p} = \int_X \bigl\langle v(x), u(x) \bigr\rangle_E \, d\mu(x) \ , \qquad v \in L_{p'}(X, \mu, E') \ , \quad u \in L_p(X, \mu, E) \ .$$

Note that this identiøcation has already been used in [Ama95, (III.4.3.18)].

The following theorem gives suŒcient conditions for equality.

2.1.4 Theorem Suppose that $1 \leq p < \infty$ and E is reÆexive or E' is separable. Then

$$L_p'(X, \mu, E) := L_p(X, \mu, E)' = L_{p'}(X, \mu, E') \ .$$

Proof This follows from [DU77, Theorem 1 on p. 98, Theorem 1 on p. 79, Corollary 4 on p. 82], and from the usual procedure for extending results from ønite measure spaces to the σ-ønite case (e.g., [HS65, proof of Theorem 20.19]). Cf. also [KJF77, Section 2.2] and [GGZ74, Satz IV.1.14] for special cases. ÷

2.1.5 Corollary If $\mu \neq 0$ and $1 < p < \infty$ then $L_p(X, \mu, E)$ is reÆexive iœ E is reÆexive.

Proof If E is reÆexive, the assertion follows from Theorem 2.1.4. The converse assertion is a consequence of the isometry of E to the closed linear subspace $\varphi \otimes E$ of $L_p(X, \mu, E)$, where $\varphi \in L_p(X, \mu)$ satisøes $\|\varphi\|_p = 1$, and the fact that a closed linear subspace of a reÆexive Banach space is reÆexive. ÷

In fact, it is known that the assertion of Theorem 2.1.4 is valid iœ E' has the Radon-Nikodym property. The latter is true, in particular, if E' is reÆexive (which is the case iœ E is reÆexive) or E' is separable (cf. [DU77]).

In general, we put

$$\langle v, u \rangle := \langle v, u \rangle_X := \langle v, u \rangle_{X,\mu} := \int_X \bigl\langle v(x), u(x) \bigr\rangle_E \, d\mu(x)$$

whenever $u : X \to E$ and $v : X \to E'$ are (equivalence classes of) μ-measurable functions such that $\bigl\langle v(\cdot), u(\cdot) \bigr\rangle_E \in L_1(X, \mu)$, and we call $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_X = \langle \cdot, \cdot \rangle_{X,\mu}$ the natural duality pairing or the L_p-duality pairing. Note that this is consistent with the deønition of $\langle \cdot, \cdot \rangle$ in Subsection III.1.1 of [Ama95] and, in particular, with Theorem 1.7.5.

Sequence Spaces

In the remainder of this subsection we derive some more or less known results on spaces of vector-valued sequences. These results are of great importance for the precise foundation of Besov spaces.

We endow $\mathbb{Z}$ with the discrete topology and denote by $\mathcal{H}_0$ the counting measure on $\mathbb{Z}$ (the zero-dimensional Hausdorœ measure). Thus, if $A \subset \mathbb{Z}$ is ønite then $\mathcal{H}_0(A) := \operatorname{card}(A)$, and $\mathcal{H}_0(A) := \infty$ otherwise. Then $\mathcal{H}_0$ is a positive Radon measure on the σ-compact metrizable space $\mathbb{Z}$. Moreover, $C(\mathbb{Z}, E) = E^{\mathbb{Z}}$.

We put

$$\ell_p(E) := L_p(\mathbb{Z}, \mathcal{H}_0, E) \ , \qquad 1 \le p \le \infty \ ,$$

and

$$c_0(E) := C_0(\mathbb{Z}, E) \ .$$

As usual,

$$\ell_p := \ell_p(\mathbb{K}) \ , \quad c_0 := c_0(\mathbb{K}) \ ,$$

if no confusion seems possible.

More generally, let $\boldsymbol{E} := (E_j)$ be a doubly inønite sequence of $\mathbb{K}$-Banach spaces. Then we put

$$\ell_p(\boldsymbol{E}) := \Bigl(\bigl\{ \boldsymbol{u} := (u_j) : \mathbb{Z} \to \textstyle\bigcup_j E_j \ ; \ u_j \in E_j, \ \|\boldsymbol{u}\|_{\ell_p} < \infty \bigr\}, \ \|\cdot\|_{\ell_p} \Bigr)$$

for $1 \le p \le \infty$, where

$$\|\cdot\|_{\ell_p} := \begin{cases} \bigl(\sum_j \|u_j\|_{E_j}^p \bigr)^{1/p} \ , & 1 \le p < \infty \ , \\ \sup_j \|u_j\|_{E_j} \ , & p = \infty \ . \end{cases}$$

We also put

$$c_0(\boldsymbol{E}) := \bigl\{ \boldsymbol{u} \in \ell_\infty(\boldsymbol{E}) \ ; \ \|u_j\|_{E_j} \to 0 \text{ as } j \to \pm\infty \bigr\} \ .$$

Clearly, if $E_j = E$ for $j \in \mathbb{Z}$, then

$$\ell_p(\boldsymbol{E}) = \ell_p(E) \ , \quad 1 \le p \le \infty \ , \qquad c_0(\boldsymbol{E}) = c_0(E) \ . \tag{2.1.6}$$

It is an easy exercise to prove that $\ell_p(\boldsymbol{E})$ is a Banach space for $1 \le p \le \infty$, that $c_0(\boldsymbol{E})$ is a closed linear subspace of $\ell_\infty(\boldsymbol{E})$, and that

$$\|\boldsymbol{u}\|_{\ell_{p_0}} \le \|\boldsymbol{u}\|_{\ell_{p_1}} \ , \qquad 1 \le p_1 \le p_0 \le \infty \ , \tag{2.1.7}$$

so that

$$\ell_{p_1}(\boldsymbol{E}) \hookrightarrow \ell_{p_0}(\boldsymbol{E}) \ , \qquad 1 \le p_1 \le p_0 \le \infty \ . \tag{2.1.8}$$

Moreover,

$$\ell_{p_1}(\boldsymbol{E}) \overset{d}{\hookrightarrow} \ell_{p_0}(\boldsymbol{E}) \overset{d}{\hookrightarrow} c_0(\boldsymbol{E}) \hookrightarrow \ell_\infty(\boldsymbol{E}) \ , \qquad 1 \le p_1 \le p_0 < \infty \ . \tag{2.1.9}$$

Let $\boldsymbol{E}' := (E_j')$. Then Hlder's inequality implies that

$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle_{\ell_p} := \sum_j \langle v_j, u_j \rangle_{E_j} \ , \qquad \boldsymbol{v} \in \ell_{p'}(\boldsymbol{E}') \ , \quad \boldsymbol{u} \in \ell_p(\boldsymbol{E}) \ , \tag{2.1.10}$$

is well-defined and satisfies

$$|\langle \boldsymbol{v}, \boldsymbol{u} \rangle_{\ell_p}| \leq \|\boldsymbol{v}\|_{\ell_{p'}(\boldsymbol{E}')} \, \|\boldsymbol{u}\|_{\ell_p(\boldsymbol{E})} \ . \tag{2.1.11}$$

The following theorem shows that more is true:

2.1.6 Theorem With respect to the ℓ_p-duality pairing (2.1.10)

(i) $$\big[\ell_p(\boldsymbol{E})\big]' = \ell_{p'}(\boldsymbol{E}') \ , \qquad 1 \leq p < \infty \ ,$$

and

(ii) $$\big[c_0(\boldsymbol{E})\big]' = \ell_1(\boldsymbol{E}') \ .$$

Proof (i) Define $T_{\boldsymbol{v}}$ for $\boldsymbol{v} \in \ell_{p'}(\boldsymbol{E}')$ by

$$T_{\boldsymbol{v}} \boldsymbol{u} := \langle \boldsymbol{v}, \boldsymbol{u} \rangle_{\ell_p} \ , \qquad \boldsymbol{u} \in \ell_p(\boldsymbol{E}) \ .$$

Then an obvious modification of the first part of the proof of Proposition 2.1.3 shows that $\boldsymbol{v} \mapsto T_{\boldsymbol{v}}$ is a linear isometry from $\ell_{p'}(\boldsymbol{E}')$ into $\big[\ell_p(\boldsymbol{E})\big]'$ if $1 < p \leq \infty$.

If $p = 1$ then

$$T_{\boldsymbol{v}} \in \big[\ell_1(\boldsymbol{E})\big]' \quad \text{and} \quad \|T_{\boldsymbol{v}}\| \leq \|\boldsymbol{v}\|_{\ell_\infty} \ , \qquad \boldsymbol{v} \in \ell_\infty(\boldsymbol{E}') \ .$$

Given $\boldsymbol{v} \in \ell_\infty(\boldsymbol{E}')$ and $\varepsilon > 0$, there exists $k \in \mathbb{Z}$ with $|v_k|_{E'_k} \geq (1-\varepsilon) \, \|\boldsymbol{v}\|_{\ell_\infty}$. If $E_k \neq \{0\}$ then we can find $u_k \in E_k$ with $|u_k|_{E_k} = 1$ and

$$\langle v_k, u_k \rangle_{E_k} \geq (1-\varepsilon) \, |v_k|_{E'_k} \geq (1-\varepsilon)^2 \, \|\boldsymbol{v}\|_{\ell_\infty} \ .$$

Thus, letting $\boldsymbol{u} := (u_k \delta_{kj})_{j \in \mathbb{Z}}$, where δ_{kj} is the Kronecker symbol, we see that $\boldsymbol{u} \in \ell_1(\boldsymbol{E})$ with $\|\boldsymbol{u}\|_{\ell_1} = 1$ and

$$|T_{\boldsymbol{v}} \boldsymbol{u}| = \langle v_k, u_k \rangle_{E_k} \geq (1-\varepsilon)^2 \, \|\boldsymbol{v}\|_{\ell_\infty} \ .$$

This implies $\|T_{\boldsymbol{v}}\| \geq \|\boldsymbol{v}\|_{\ell_\infty}$, so that $\boldsymbol{v} \mapsto T_{\boldsymbol{v}}$ is a linear isometry from $\ell_\infty(\boldsymbol{E}')$ into $\big[\ell_1(\boldsymbol{E})\big]'$ as well. Thus it remains to show that $\boldsymbol{v} \mapsto T_{\boldsymbol{v}}$ is surjective.

Let $f \in \big[\ell_p(\boldsymbol{E})]'$ be given. For each $k \in \mathbb{Z}$ define $e'_k \in E'_k$ by

$$\langle e'_k, e_k \rangle := f\big((e_k \delta_{kj})_{j \in \mathbb{Z}}\big) \ , \qquad e_k \in E_k \ .$$

Let $\boldsymbol{v} := (e'_j)$ and put $\boldsymbol{v}^N := \boldsymbol{v} \chi_{[\,|j| \leq N]}$ for $N \in \mathbb{N}$. Then $\boldsymbol{v}^N \in \ell_{p'}(\boldsymbol{E}')$ and

$$\langle \boldsymbol{v}^N, \boldsymbol{u} \rangle_{\ell_p} = \sum_{j=-N}^{N} \langle e'_j, u_j \rangle_{E_j} = f(\boldsymbol{u}^N) \ , \qquad \boldsymbol{u} \in \ell_p(\boldsymbol{E}) \ , \tag{2.1.12}$$

where $\boldsymbol{u}^N := \boldsymbol{u}\chi_{[\,|j|\leq N]}$. This means that

$$T_{\boldsymbol{v}^N}\boldsymbol{u} = f(\boldsymbol{u}^N) , \qquad \boldsymbol{u} \in \ell_p(\boldsymbol{E}) .$$

From this and the first part of the proof we deduce that

$$\|\boldsymbol{v}^N\|_{\ell_{p'}} = \|T_{\boldsymbol{v}^N}\|_{[\ell_p(\boldsymbol{E})]'} \leq \|f\|_{[\ell_p(\boldsymbol{E})]'} , \qquad N \in \mathbb{N} .$$

It is an easy consequence of this estimate that $\boldsymbol{v} \in \ell_{p'}(\boldsymbol{E})$. Since $\boldsymbol{u}^N \to \boldsymbol{u}$ in $\ell_p(\boldsymbol{E})$ as $N \to \infty$, thanks to $1 \leq p < \infty$, we now infer from (2.1.12) that $\langle \boldsymbol{v}, \boldsymbol{u}\rangle_{\ell_p} = f(\boldsymbol{u})$ for $\boldsymbol{u} \in \ell_p(\boldsymbol{E})$. This proves the surjectivity of the map $\boldsymbol{v} \mapsto T_{\boldsymbol{v}}$.

(ii) The above proof applies to $f \in \bigl[c_0(\boldsymbol{E})\bigr]'$ as well. ÷

The proof of Theorem 2.1.6 is a trivial extension of the corresponding result for the case where $E_j = E$ for all $j \in \mathbb{Z}$, which can be found in Lions and Peetre [LP64], although it was probably already known to Grothendieck much earlier. It should be noted that there are no restrictions on the Banach spaces E_j.

Now suppose that

$$E_j := \{0\} , \qquad j \notin \{1, \ldots, n\} . \tag{2.1.13}$$

Then

$$\|\boldsymbol{u}\|_\infty \leq \|\boldsymbol{u}\|_{\ell_p} \leq \sqrt[p]{n}\, \|u\|_\infty , \qquad 1 \leq p < \infty , \tag{2.1.14}$$

so that

$$\ell_p(\boldsymbol{E}) \doteq c_0(\boldsymbol{E}) \doteq \ell_2(\boldsymbol{E}) =: E_1 \times \cdots \times E_n , \qquad 1 \leq p \leq \infty . \tag{2.1.15}$$

Note that the last definition means that we equip the product Banach space $E_1 \times \cdots \times E_n$ with the 'Euclidean norm', unless otherwise stated. If, in addition, $E_1 = \cdots = E_n = E$, then in abuse of notation

$$\ell_p^{(n)}(E) := (E^n, |\cdot|_{\ell_p}) \tag{2.1.16}$$

and $\ell_p^{(n)} := \ell_p^{(n)}(\mathbb{K})$.

Weighted Sequence Spaces

In connection with dyadic resolutions of the identity we shall encounter the sequence spaces

$$\ell_p^s(E) := L_p(\mathbb{Z}, \mu_s; E) , \qquad 1 \leq p \leq \infty , \quad s \in \mathbb{R} , \tag{2.1.17}$$

where μ_s is the Radon measure on $\mathbb{Z}$ that assigns the value 2^{ks} to the singleton $\{k\}$. Letting

$$E_k^s := 2^{ks}E := (E, 2^{ks}\,|\cdot|_E) \doteq E , \qquad k \in \mathbb{Z} , \quad s \in \mathbb{R} , \tag{2.1.18}$$

and $\boldsymbol{E}^s := (E_k^s)$, it follows that

$$\ell_p^s(E) = \ell_p(\boldsymbol{E}^s) , \qquad 1 \le p \le \infty , \quad s \in \mathbb{R} . \tag{2.1.19}$$

Observe that

$$\ell_p(E) = \ell_p^0(E) , \qquad 1 \le p \le \infty . \tag{2.1.20}$$

We also put

$$c_0^s(E) := c_0(\boldsymbol{E}^s) ,$$

so that $c_0^0(E) = c_0(E)$.

The following theorem is an almost immediate consequence of the above duality theorem for ℓ_p spaces.

2.1.7 Theorem With respect to the ℓ_p-duality pairing:

$$\bigl[\ell_p^s(E)\bigr]' = \ell_{p'}^{-s}(E') , \qquad 1 \le p < \infty ,$$

and

$$\bigl[c_0^s(E)\bigr]' = \ell_1^{-s}(E')$$

for $s \in \mathbb{R}$.

Proof Since $(E_k^s)' = (E')_k^{-s}$ with respect to the E'-E duality pairing, Theorem 2.1.6 implies the assertion. ÷

For later use we include a simple embedding theorem for ℓ_p^s-spaces.

2.1.8 Proposition Suppose that $s, t \in \mathbb{R}$ and $p, q \in [1, \infty]$. Then

$$\ell_p^s(E) \overset{d}{\hookrightarrow} \ell_q^s(E) \overset{d}{\hookrightarrow} c_0^s(E) \hookrightarrow \ell_\infty^s(E) , \qquad p \le q < \infty , \tag{2.1.21}$$

and, if $E_k = \{0\}$ for $k < 0$, then

$$\ell_p^s(E) \overset{d}{\hookrightarrow} \ell_q^t(E) , \qquad s > t . \tag{2.1.22}$$

If E_0 and E_1 are Banach spaces with $E_1 \hookrightarrow E_0$ then

$$\ell_p^s(E_1) \hookrightarrow \ell_p^s(E_0) \quad \text{and} \quad c_0^s(E_1) \hookrightarrow c_0^s(E_0) . \tag{2.1.23}$$

If (E_0, E_1) is a densely injected Banach couple then

$$\ell_p^s(E_1) \overset{d}{\hookrightarrow} \ell_p^s(E_0) \text{ for } p < \infty \quad \text{and} \quad c_0^s(E_1) \overset{d}{\hookrightarrow} c_0^s(E_0) . \tag{2.1.24}$$

Proof It follows from (2.1.8), (2.1.9), and (2.1.19) that the injections in (2.1.21) are well-deøned and continuous. Put

$$C_c(\mathbb{Z}, E) := \bigl\{\, u : \mathbb{Z} \to E \ ; \ \operatorname{supp}(u) \text{ is ønite} \,\bigr\}$$

and note that

$$C_c(\mathbb{Z},E) \overset{d}{\subset} c_0^s(E) \ , \qquad s \in \mathbb{R} \ . \tag{2.1.25}$$

Since it is also easily verified that

$$C_c(\mathbb{Z},E) \overset{d}{\subset} \ell_p^s(E) \ , \qquad s \in \mathbb{R} \ , \quad 1 \le p < \infty \ , \tag{2.1.26}$$

assertion (2.1.21) follows. If $E_k = \{0\}$ for $k < 0$ then

$$\begin{aligned} \|\boldsymbol{u}\|_{\ell_q^t} = \left\|(2^{kt}u_k)_{k\in\mathcal{N}}\right\|_{\ell_q} &\le \left\|(2^{k(t-s)})_{k\in\mathcal{N}}\right\|_{\ell_q}\left\|(2^{ks}u_k)_{k\in\mathcal{N}}\right\|_{\ell_\infty} \\ &= c\,\|\boldsymbol{u}\|_{\ell_\infty^s} \le c\,\|\boldsymbol{u}\|_{\ell_p^s} \end{aligned} \tag{2.1.27}$$

for $\boldsymbol{u} := (u_k) \in \ell_p^s(E)$, where the last inequality follows from (2.1.21). Hence we see that $\ell_p^s(E) \hookrightarrow \ell_q^t(E)$ for $s > t$. The density assertion is now a consequence of (2.1.25) and (2.1.26).

Assertion (2.1.23) is obvious. If E_1 is also dense in E_0 then (2.1.24) is an easy consequence of (2.1.25) and (2.1.26), ÷

Of course, if $E_k = \{0\}$ for $k < 0$ then $\ell_p(E) = L_p(\mathbb{N},\mathcal{H}_0,E)$, $1 \le p \le \infty$, and $c_0(E) = C_0(\mathbb{N},E)$, etc.

2.2 Vector Measures and the Riesz Representation Theorem

In connection with duality theory and Besov spaces we need some explicit information on $C_0(\mathbb{R}^n,E)'$. For this reason we present in this subsection some elements of the theory of vector measures and prove a generalization of the well-known Riesz representation theorem.

Measures of Bounded Variation

Throughout this subsection $E := (E,|\cdot|)$ is a Banach space, X is a σ-compact metrizable space, and $\mathcal{B}_X$ denotes the Borel σ-algebra of X. By an E-valued vector measure μ on X we mean a σ-additive map $\mu:\ \mathcal{B}_X \to E$ satisfying $\mu(\emptyset) = 0$. For such a vector measure μ we define the variation $|\mu|:\ \mathcal{B}_X \to \mathbb{R}^+ \cup \{\infty\}$ by

$$|\mu|\,(B) := \sup_{\pi(B)} \sum_{A\in\pi(B)} |\mu(A)| \ , \qquad B \in \mathcal{B}_X \ ,$$

where the supremum is taken over all partitions $\pi(B)$ of B into a finite number of pair-wise disjoint Borel subsets. Then μ is said to be of bounded variation if

$$\|\mu\|_{BV} := |\mu|\,(X) < \infty \ .$$

The set of all E-valued vector measures on X of bounded variation is denoted by

$$\mathcal{M}_{BV}(X,E) := \big(\mathcal{M}_{BV}(X,E),\ \|\cdot\|_{BV}\big)\ .$$

It is clear that $\mathcal{M}_{BV}(X,E)$ is a normed vector space with the obvious linear structure (as a vector subspace of $E^{\mathcal{B}_X}$).

Given $\mu \in \mathcal{M}_{BV}(X,E)$, it follows from $|\mu(B)| \le |\mu|(B)$ for $B \in \mathcal{B}_X$ that $|\mu|$ is a positive Borel measure on X. We denote its completion, deøned on the σ-algebra $\mathcal{A}_{|\mu|}$, consisting of all $A \subset X$ of the form $A = B \cup N$ where $B \in \mathcal{B}_X$ and N is a subset of a $|\mu|$-null set, again by $|\mu|$. Then $|\mu|$ is a positive ønite Radon measure on X.

2.2.1 Remark Denote by dx a positive Radon measure on X. Then, given any function $u \in L_1(X,dx,E)$,

$$(u\,dx)(B) := \int_B u\,dx\ , \qquad B \in \mathcal{B}_X\ ,$$

deønes an E-valued measure of bounded variation and $\|u\,dx\|_{BV} = \|u\|_1$ (cf. [DU77, Theorem 4(iv) on p. 46] or [Lan69, Theorem XI.9]). Hence

$$(u \mapsto u\,dx) :\ L_1(X,dx,E) \to \mathcal{M}_{BV}(X,E)$$

is a linear isometry. Occasionally, it will be convenient to identify u with $u\,dx$, that is, to consider $L_1(X,dx,E)$ as a closed linear subspace of $\mathcal{M}_{BV}(X,E)$. ÷

Integrals with Respect to Vector Measures

Let E_0, E_1, and E_2 be Banach spaces and suppose that

$$E_1 \times E_2 \to E_0\ , \quad (e_1,e_2) \mapsto e_1 \bullet e_2 \tag{2.2.1}$$

is a multiplication. Let $\mathcal{B}(X,E_1)$ be the closure in $B(X,E_1)$ of the linear subspace $S(X,E_1)$ of all simple functions

$$u = \sum_{B\in\pi} \chi_B \otimes e_B\ , \qquad e_B \in E_1\ , \quad \pi := \pi(X)\ . \tag{2.2.2}$$

If $u \in S(X,E_1)$ is given by (2.2.2) and $\mu \in \mathcal{M}_{BV}(X,E_2)$ then we put

$$\int u\,d\mu := \int_X u\,d\mu := \sum_{B\in\pi} e_B \bullet \mu(B) \in E_0\ .$$

It follows that

$$S(X,E_1) \times \mathcal{M}_{BV}(X,E_2) \to E_0\ , \quad (u,\mu) \mapsto \int u\,d\mu$$

is a well-deøned bilinear map satisfying

$$\Big|\int u\,d\mu\Big|_{E_0} \le \sum_{B\in\pi} |e_B|\,|\mu(B)| \le \sum_{B\in\pi} |e_B|\,|\mu|\,(B) \le \|u\|_\infty\,\|\mu\|_{BV}\ . \tag{2.2.3}$$

Hence it possesses a unique continuous bilinear extension of norm at most one over $\mathcal{B}(X,E_1)\times\mathcal{M}_{BV}(X,E_2)$. Note that $C_0(X,E_1)$ is a closed linear subspace of $\mathcal{B}(X,E_1)$. Thus, by restriction, we obtain a well-deøned multiplication

$$C_0(X,E_1)\times\mathcal{M}_{BV}(X,E_2)\to E_0\ ,\quad (u,\mu)\mapsto\int u\,d\mu := \int_X u\,d\mu \tag{2.2.4}$$

and $\int_X u\,d\mu$ is said to be the integral of u over X with respect to μ (and the multiplication (2.2.1)). In fact, from (2.2.3) we deduce that

$$\Big|\int u\,d\mu\Big|_{E_0} \le \int |u|\,d\,|\mu| \le \|u\|_\infty\,\|\mu\|_{BV} \tag{2.2.5}$$

for $u\in C_0(X,E_1)$ and $\mu\in\mathcal{M}_{BV}(X,E_2)$.

Now suppose that there are further Banach spaces E_3, E_4, and E_5, and multiplications satisfying the associativity assumption of Remark 1.6.5(e). Then it follows from the deønition of $\int u\,d\mu$ that, for each u of the form (2.2.2),

$$\int u\,d\mu\bullet e = \sum_{B\in\pi} e_B\bullet\big(\mu(B)\bullet e\big)\in E_5\ ,\qquad e\in E_3\ . \tag{2.2.6}$$

Note that, given $e\in E_3$,

$$\mu\bullet e := \mu(\cdot)\bullet e:\ \mathcal{B}_X\to E_4\ ,\quad B\mapsto\mu(B)\bullet e$$

is σ-additive. Moreover,

$$|\mu\bullet e(B)|\le|\mu(B)|\,|e|\le|\mu|\,(B)\,|e|\ ,\qquad B\in\mathcal{B}_X\ ,$$

plainly implies that $\mu\bullet e$ is an E_4-valued vector measure on X of bounded variation, and that

$$\mathcal{M}_{BV}(X,E_2)\times E_3\to\mathcal{M}_{BV}(X,E_4)\ ,\quad(\mu,e)\mapsto\mu\bullet e \tag{2.2.7}$$

is a multiplication. From this and (2.2.6) we infer that

$$\Big(\int u\,d\mu\Big)\bullet e = \int u\,d(\mu\bullet e) \tag{2.2.8}$$

and that

$$C_0(X,E_1)\times\mathcal{M}_{BV}(X,E_2)\times E_3\to E_5\ ,\quad(u,\mu,e)\mapsto\int u\,d(\mu\bullet e) \tag{2.2.9}$$

is a trilinear multiplication.

Similarly as above, we see that (2.2.1) induces a multiplication

$$E_1 \times \mathcal{M}_{BV}(X, E_2) \to \mathcal{M}_{BV}(X, E_0)\ , \quad (e, \mu) \mapsto e \bullet \mu\ . \tag{2.2.10}$$

Thus, given

$$\varphi = \sum_{B\in\pi} \alpha_B \chi_B \in S(X, \mathbb{K})\ ,$$

it follows from

$$\varphi \otimes e = \sum_{B\in\pi} \alpha_B \chi_B \otimes e = \sum_{B\in\pi} \chi_B \otimes (\alpha_B e) \in S(X, E_1)$$

for $e \in E_1$ that

$$\int \varphi \otimes e\, d\mu = \sum_{B\in\pi} \alpha_B e \bullet \mu(B) = \int \varphi\, d(e \bullet \mu)\ .$$

From this we deduce that

$$\int \varphi \otimes e\, d\mu = \int \varphi\, d(e \bullet \mu)\ , \qquad \varphi \otimes e \in C_0(X) \otimes E_1\ , \quad \mu \in \mathcal{M}_{BV}(X, E_2)\ . \tag{2.2.11}$$

In the following, we leave it to the reader to identify the choices for the spaces $E_0, \ldots, E_5$ and the multiplications that we are using in concrete situations.

Vector Measures as Distributions

Now suppose that X is open in $\mathbb{R}^n$. Since $\mathcal{D}(X)$ is dense in $C_0(X)$, it follows that the multiplication

$$C_0(X) \times \mathcal{M}_{BV}(X, E) \to E\ , \quad (\varphi, \mu) \mapsto \int \varphi\, d\mu$$

is completely determined by its restriction to $\mathcal{D}(X) \times \mathcal{M}_{BV}(X, E)$. Put

$$T_\mu \varphi := \int \varphi\, d\mu\ , \qquad (\varphi, \mu) \in \mathcal{D}(X) \times \mathcal{M}_{BV}(X, E)\ .$$

Then $\mathcal{D}(X) \hookrightarrow C_0(X)$ implies

$$(\mu \mapsto T_\mu) \in \mathcal{L}\big(\mathcal{M}_{BV}(X, E), \mathcal{D}'(X, E)\big)\ . \tag{2.2.12}$$

Suppose that $T_\mu = 0$ in $\mathcal{D}'(X, E)$ for some $\mu \in \mathcal{M}_{BV}(X, E)$. Then we infer from (2.2.8) that

$$\langle e', T_\mu \varphi \rangle_E = \int \varphi\, d\langle e', \mu \rangle_E = 0\ , \qquad \varphi \in \mathcal{D}(X)\ , \quad e' \in E'\ .$$

Since $\langle e', \mu \rangle_E \in \mathcal{M}_{BV}(X) := \mathcal{M}_{BV}(X, \mathbb{K})$, this implies $\langle e', \mu \rangle_E = 0$ for each $e' \in E'$. Thus $\big\langle e', \mu(B) \big\rangle_E = 0$ for $e' \in E'$ and $B \in \mathcal{B}_X$, which gives $\mu = 0$. This shows that (2.2.12) is an injection. Thus the following convention is justiøed.

Convention Let X be a nonempty open subset of $\mathbb{R}^n$. Then we identify $\mu \in \mathcal{M}_{BV}(X, E)$ with $T_\mu \in \mathcal{D}'(X, E)$ so that

$$\mathcal{M}_{BV}(X, E) \hookrightarrow \mathcal{D}'(X, E) \tag{2.2.13}$$

and

$$\mu(\varphi) = \int_X \varphi \, d\mu \ , \qquad \mu \in \mathcal{M}_{BV}(X, E) \ , \quad \varphi \in \mathcal{D}(X) \ . \tag{2.2.14}$$

Convolutions Involving Vector Measures

Observe that $\mathcal{S}(\mathbb{R}^n) \overset{d}{\hookrightarrow} C_0(\mathbb{R}^n)$ implies

$$\mathcal{M}_{BV}(\mathbb{R}^n, E) \hookrightarrow \mathcal{S}'(\mathbb{R}^n, E) \ . \tag{2.2.15}$$

Hence we know from Proposition 1.2.7 that the convolution product $\varphi * \mu$ is a well-deøned element of $\mathcal{O}_M(\mathbb{R}^n, E)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\mu \in \mathcal{M}_{BV}(\mathbb{R}^n, E)$. Moreover,

$$(\varphi * \mu)(x) = \mu(\tau_x \check{\varphi}) = \int_{\mathcal{R}^n} \varphi(x - y) \, d\mu(y) \ , \qquad x \in \mathbb{R}^n \ , \tag{2.2.16}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\mu \in \mathcal{M}_{BV}(\mathbb{R}^n, E)$. Hence, by (2.2.5),

$$|\varphi * \mu(x)|_E \le \int_{\mathcal{R}^n} |\varphi(x - y)| \, d\,|\mu|\,(y) = |\varphi| * |\mu|\,(x) \ , \qquad x \in \mathbb{R}^n \ ,$$

so that, by Fubini's theorem,

$$\|\varphi * \mu\|_1 \le \|\varphi\|_1 \, \|\mu\|_{BV} \ , \qquad \varphi \in \mathcal{S}(\mathbb{R}^n) \ , \quad \mu \in \mathcal{M}_{BV}(\mathbb{R}^n, E) \ . \tag{2.2.17}$$

2.2.2 Proposition Convolution is a well-deøned multiplication:

$$L_1(\mathbb{R}^n) \times \mathcal{M}_{BV}(\mathbb{R}^n, E) \to L_1(\mathbb{R}^n, E) \ .$$

Proof This follows from (2.2.17) by continuous extension. ÷

If $E = \mathbb{K}$ then, thanks to Remark 2.2.1, Proposition 2.2.2 reduces to the well-known fact that $L_1(\mathbb{R}^n)$ is an ideal in the convolution algebra of all $\mathbb{K}$-valued Borel measures (e.g., [Pet83, Theorem I.4.5]).

2.2.3 Lemma Suppose $\varphi \in L_1(\mathbb{R}^n)$, $u \in C_0(\mathbb{R}^n, E)$, and $\mu \in \mathcal{M}_{BV}(\mathbb{R}^n, E')$. Then

$$\int_{\mathcal{R}^n} \varphi * u \, d\mu = \int_{\mathcal{R}^n} u \, d(\check{\varphi} * \mu) = \int_{\mathcal{R}^n} \langle \check{\varphi} * \mu, u \rangle_E \, dx = \langle \check{\varphi} * \mu, u \rangle \ . \tag{2.2.18}$$

Proof From Theorem 1.9.9, Remark 2.2.1, Proposition 2.2.2, and (2.2.4) we infer that each one of the maps that send

$$(\varphi, u, \mu) \in L_1(\mathbb{R}^n) \times C_0(\mathbb{R}^n, E) \times \mathcal{M}_{BV}(\mathbb{R}^n, E')$$

into one of the integrals in (2.2.18) is a trilinear $\mathbb{K}$-valued multiplication. Thus, since $C_0(\mathbb{R}^n) \otimes E$ is dense in $C_0(\mathbb{R}^n, E)$ by Theorem 1.3.6(x), it suŒces to consider the case where $u = \psi \otimes e \in C_0(\mathbb{R}^n) \otimes E$. Since $\varphi * (\psi \otimes e) = (\varphi * \psi) \otimes e$ we deduce from (2.2.11), (2.2.16), and Fubini's theorem that

$$\begin{aligned} \int \varphi * u \, d\mu = \int \varphi * \psi \, d\langle \mu, e\rangle_E &= \iint \varphi(x-y) \, d\langle \mu, e\rangle_E(x) \psi(y) \, dy \\ &= \int \bigl(\check{\varphi} * \langle \mu, e\rangle_E\bigr) \psi \, dx \ . \end{aligned} \tag{2.2.19}$$

Moreover, thanks to (2.2.8),

$$\begin{aligned} \check{\varphi} * \langle \mu, e\rangle_E(x) &= \int \check{\varphi}(x-y) \, d\langle \mu, e\rangle_E(y) = \Bigl\langle \int \check{\varphi}(x-y) \, d\mu(y), e \Bigr\rangle_E \\ &= \bigl\langle \check{\varphi} * \mu(x), e \bigr\rangle_E \ , \end{aligned}$$

so that the last integral in (2.2.19) equals

$$\int \bigl\langle \check{\varphi} * \mu(x), \psi(x) \otimes e \bigr\rangle_E \, dx = \int \langle \check{\varphi} * \mu, u\rangle_E \, dx = \langle \check{\varphi} * \mu, u\rangle \ .$$

This proves the lemma. ÷

The Riesz Representation Theorem

Finally, we prove a generalization of the classical (scalar) Riesz Representation Theorem.

2.2.4 Theorem Let X be a σ-compact metrizable space and let E be a Banach space. Then

$$C_0(X, E)' = \mathcal{M}_{BV}(X, E')$$

with respect to the duality pairing

$$\langle \mu, u\rangle_{C_0} := \int_X u \, d\mu \ , \qquad \mu \in \mathcal{M}_{BV}(X, E') \ , \quad u \in C_0(X, E) \ .$$

Proof Suppose that $\mu \in \mathcal{M}_{BV}(X, E')$. Then it follows from (2.2.4) that

$$\Bigl(u \mapsto \int u \, d\mu\Bigr) \in C_0(X, E)'$$

and

$$\Big|\int u\,d\mu\Big| \le \|\mu\|_{BV}\,\|u\|_\infty\ , \qquad u\in C_0(X,E)\ . \tag{2.2.20}$$

Conversely, let $w\in C_0(X,E)'$ be øxed. Then, given any $e\in E$, the scalar Riesz representation theorem (e.g., [Rud70, Theorems 2.14, 6.2, and 6.19]) implies the existence of a unique regular $\mathbb{K}$-valued Radon measure μ_e on X satisfying

$$w(\varphi\otimes e)=\int\varphi\,d\mu_e\ , \qquad \varphi\in C_0(X)\ , \tag{2.2.21}$$

and

$$\|\mu_e\|_{BV}=\sup_{\|\varphi\|_\infty\le 1}\Big|\int\varphi\,d\mu_e\Big| = \sup_{\|\varphi\|_\infty\le 1}|w(\varphi\otimes e)| \le \|w\|_{C_0'}\,|e|_E\ .$$

Since, by uniqueness, the map $E\to(\mathcal{B}_X\to\mathbb{K})$, $e\mapsto\mu_e$ is linear, it follows that

$$\big\langle\mu(B),e\big\rangle := \mu_e(B)\ , \qquad B\in\mathcal{B}_X\ , \tag{2.2.22}$$

deønes a map $\mu:\ \mathcal{B}_X\to E'$ that is easily seen to be ønitely additive.

In order to show that μ has bounded variation we ørst consider a family $\{\mathcal{O}_1,\ldots,\mathcal{O}_m\}$ of pair-wise disjoint open subsets of X. For each $j\in\{1,\ldots,m\}$ we choose $\varphi_j\in C_0(X)$ with $\operatorname{supp}(\varphi_j)\subset\mathcal{O}_j$ and $\|\varphi_j\|_\infty\le 1$, as well as $e_j\in\mathbb{B}_E$. Then, putting

$$\varepsilon_j := \operatorname{sign} w(\varphi_j\otimes e_j)\in\mathbb{K}$$

and using (2.2.21),

$$\begin{aligned}\sum_{j=1}^m\Big|\int_{\mathcal{O}_j}\varphi_j\,d\mu_{e_j}\Big| &= \sum_{j=1}^m|w(\varphi_j\otimes e_j)| = \sum_{j=1}^m\varepsilon_j w(\varphi_j\otimes e_j)\\ &= w\Big(\sum_{j=1}^m\varphi_j\otimes\varepsilon_j e_j\Big)\le\|w\|_{C_0'}\ .\end{aligned}$$

Consequently,

$$\sum_{j=1}^m\big|\mu_{e_j}\big|(\mathcal{O}_j)\le\|w\|_{C_0'}\ , \qquad e_j\in\mathbb{B}_E\ , \quad 1\le j\le m\ . \tag{2.2.23}$$

Now let $\pi:=\{B_1,\ldots,B_m\}$ be an arbitrary partition of X into Borel sets. Then, given $\varepsilon>0$, we ønd for each $j\in\{1,\ldots,m\}$ an element $e_j\in\mathbb{B}_E$ such that

$$|\mu(B_j)|_{E'}\le\big|\big\langle\mu(B_j),e_j\big\rangle_E\big|+\varepsilon/m\ . \tag{2.2.24}$$

The regularity of μ_{e_j} implies the existence of a compact subset K_j of B_j with

$$\big|\mu_{e_j}\big|(B_j)\le\big|\mu_{e_j}\big|(K_j)+\varepsilon/m\ . \tag{2.2.25}$$

Since $K_j \cap K_k = \emptyset$ for $j \neq k$ we ønd pair-wise disjoint open sets $\mathcal{O}_1, \ldots, \mathcal{O}_m$ in X such that $K_j \subset \mathcal{O}_j$ for $1 \leq j \leq m$. Now we infer from (2.2.23)˘(2.2.25) that

$$\begin{aligned}\sum_{j=1}^{m} |\mu(B_j)|_{E'} &\leq \sum_{j=1}^{m} \bigl|\bigl\langle \mu(B_j), e_j \bigr\rangle_E\bigr| + \varepsilon \leq \sum_{j=1}^{m} \bigl|\mu_{e_j}\bigr| (B_j) + \varepsilon \\ &\leq \sum_{j=1}^{m} \bigl|\mu_{e_j}\bigr| (K_j) + 2\varepsilon \leq \sum_{j=1}^{m} \bigl|\mu_{e_j}\bigr| (\mathcal{O}_j) + 2\varepsilon \leq \|w\|_{C_0'} + 2\varepsilon \ .\end{aligned}$$

Hence μ is of bounded variation.

Let now (B_j) be a sequence in $\mathcal{B}_X$ such that $B_j \cap B_k = \emptyset$ for $j \neq k$, and put $B := \bigcup B_j$. Then it is clear that

$$\sum_j |\mu(B_j)| \leq \|\mu\|_{BV} < \infty \ .$$

Thus $\bigl\langle \sum_j \mu(B_j), e \bigr\rangle_E$ is well-deøned for $e \in E$ and

$$\Bigl\langle \sum_j \mu(B_j), e \Bigr\rangle_E = \sum_j \bigl\langle \mu(B_j), e \bigr\rangle = \sum_j \mu_e(B_j) = \mu_e(B) = \bigl\langle \mu(B), e \bigr\rangle \qquad (2.2.26)$$

by the σ-additivity of μ_e. Since (2.2.26) is true for every $e \in E$, and E separates the points of E', it follows that $\mu(B) = \sum_j \mu(B_j)$. Now (2.2.11), (2.2.21), (2.2.22), and the density of $C_0(X) \otimes E$ in $C_0(X, E)$ imply

$$\langle w, u \rangle_{C_0} = \int u \, d\mu \ , \qquad u \in C_0(X, E) \ .$$

From this and (2.2.20) the assertion follows. ÷

The above proof of the generalized Riesz representation theorem is due to W. Arendt (personal communication).

2.3 Dyadic Decompositions

Dyadic decompositions are locally ønite partitions of unity on $\mathbb{R}^n$ enjoying particularly 'nice' properties. They are used in the Fourier image of tempered distributions and give a convenient and easy way to control decay properties of Fourier transforms, hence regularity properties of the original distributions. These properties are the basis for introducing the important scales of Besov spaces and for their study.

Elementary Properties of the Fourier Transform

For the reader's convenience, and for later use, we begin by recalling and extending some of the basic properties of the Fourier transform.

Let E be a Banach space. Recall that

$$\mathcal{F} \in \mathcal{L}\mathrm{aut}\big(\mathcal{S}(\mathbb{R}^n, E)\big) \cap \mathcal{L}\mathrm{aut}\big(\mathcal{S}'(\mathbb{R}^n, E)\big) \tag{2.3.1}$$

and that

$$\mathcal{F}^{-1}u = (2\pi)^{-n}\, \check{\widehat{u}} = (2\pi)^{-n}\, \widehat{\check{u}}\ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E)\ . \tag{2.3.2}$$

Moreover,

$$\widehat{D^\alpha u} = \xi^\alpha \widehat{u}\ , \qquad \alpha \in \mathbb{N}^n\ , \quad u \in \mathcal{S}'(\mathbb{R}^n, E)\ . \tag{2.3.3}$$

By replacing u by $\mathcal{F}^{-1}u$ in (2.3.3) we see that $D^\alpha(\mathcal{F}^{-1}u) = \mathcal{F}^{-1}(\xi^\alpha u)$. Thus (2.3.2) implies $D^\alpha\, \check{\widehat{u}} = \big(\widehat{\xi^\alpha u}\big)\check{}$. Since reÆection is its own inverse and

$$D^\alpha \check{u} = (-1)^{|\alpha|}(D^\alpha u)\check{}\ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E)\ , \quad \alpha \in \mathbb{N}^n\ , \tag{2.3.4}$$

it follows that

$$\widehat{\xi^\alpha u} = (-1)^{|\alpha|} D^\alpha \widehat{u}\ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E)\ , \quad \alpha \in \mathbb{N}^n\ . \tag{2.3.5}$$

We deøne the dilation $\sigma_t u$ by $t > 0$ of $u \in E^{(\mathcal{R}^n)}$ by[3]

$$\sigma_t u(x) := u(x/t)\ , \qquad x \in \mathbb{R}^n\ , \tag{2.3.6}$$

and by

$$\sigma_t u(\varphi) := t^n u(\sigma_{t^{-1}}\varphi)\ , \qquad u \in \mathcal{D}'(\mathbb{R}^n, E)\ , \quad \varphi \in \mathcal{D}(\mathbb{R}^n)\ . \tag{2.3.7}$$

Then $\{\, \sigma_t\ ;\ t > 0 \,\}$, the group of dilations, is strongly continuous on $\mathbf{F}(\mathbb{R}^n, E)$, where $\mathbf{F} \in \{\mathcal{D}, \mathcal{D}', \mathcal{E}, \mathcal{E}', \mathcal{S}, \mathcal{S}'\}$, and

$$\sigma_t^{-1} = \sigma_{1/t} = \sigma_{t^{-1}}\ , \qquad t > 0\ . \tag{2.3.8}$$

It is easily veriøed that $\mathcal{F}\sigma_t u = t^n \sigma_{t^{-1}} \mathcal{F}u$ for $u \in \mathcal{S}(\mathbb{R}^n, E)$ and, thanks to (2.3.7), that this is true for $u \in \mathcal{S}'(\mathbb{R}^n, E)$ as well, that is,

$$\mathcal{F}\sigma_t = t^n \sigma_{t^{-1}}\mathcal{F}\ , \qquad t > 0\ . \tag{2.3.9}$$

From this, (2.3.1), and (2.3.8) we infer that

$$\mathcal{F}^{-1}\sigma_t = t^n \sigma_{t^{-1}}\mathcal{F}^{-1}\ , \qquad t > 0\ . \tag{2.3.10}$$

---

[3]Now we prefer to use deønitions (2.3.6) and (2.3.7) instead of [Ama95, (III.4.2.7) and (III.4.2.8)], respectively. This is, of course, simply a matter of convenience.

The chain rule implies $\partial^\alpha(\sigma_t u) = t^{-|\alpha|}\sigma_t(\partial^\alpha u)$ for $u \in \mathcal{D}(\mathbb{R}^n)$. Now we infer from (2.3.7) that this formula remains valid for $u \in \mathcal{D}'(\mathbb{R}^n, E)$, that is,

$$\partial^\alpha \sigma_t = t^{-|\alpha|}\sigma_t \partial^\alpha \ , \qquad t > 0 \ , \quad \alpha \in \mathbb{N}^n \ . \tag{2.3.11}$$

Lastly, it is easily veriøed that $L_p(\mathbb{R}^n, E)$ is invariant under $\{\, \sigma_t \ ; \ t > 0 \,\}$ for $1 \le p \le \infty$ and that

$$\|\sigma_t u\|_p = t^{n/p}\, \|u\|_p \ , \qquad u \in L_p(\mathbb{R}^n, E) \ , \quad 1 \le p \le \infty \ , \quad t > 0 \ . \tag{2.3.12}$$

Dyadic Resolutions of the Identity

Now we introduce the concept of dyadic decompositions. For this we øx a function satisfying the following conditions:

$$\psi \in \mathcal{D}(\mathbb{R}^n) \ , \quad \psi \,|\, \mathbb{B}^n = 1 \ , \quad \operatorname{supp}(\psi) \subset 2\mathbb{B}^n =: \Omega_0 \ . \tag{2.3.13}$$

Then we put $\widetilde{\psi} := \psi - \sigma_{2^{-1}}\psi = \psi - \psi(2\,\cdot\,)$ and

$$\psi_0 := \psi \ , \quad \psi_k := \sigma_{2^k}\widetilde{\psi} = \psi(2^{-k}\,\cdot\,) - \psi(2^{-k+1}\,\cdot\,) \ , \qquad k \in \dot{\mathbb{N}} \ , \tag{2.3.14}$$

and

$$\eta_j := \sigma_{2^j}\widetilde{\psi} = \psi(2^{-j}\,\cdot\,) - \psi(2^{-j+1}\,\cdot\,) \ , \qquad j \in \mathbb{Z} \ . \tag{2.3.15}$$

We also let

$$\Omega_k := [2^{k-1} \le |\xi| \le 2^{k+1}] \ , \qquad k \in \dot{\mathbb{N}} \ , \tag{2.3.16}$$

and

$$\Sigma_j := [2^{j-1} \le |\xi| \le 2^{j+1}] \ , \qquad j \in \mathbb{Z} \ . \tag{2.3.17}$$

Then it follows that

$$\psi_k, \eta_j \in \mathcal{D}(\mathbb{R}^n) \ , \quad \operatorname{supp}(\psi_k) \subset \Omega_k \ , \quad \operatorname{supp}(\eta_j) \subset \Sigma_j \tag{2.3.18}$$

for $k \in \mathbb{N}$ and $j \in \mathbb{Z}$. Moreover,

$$\sum_{k=0}^{m} \psi_k(\xi) = \psi(2^{-m}\xi) \ , \qquad \xi \in \mathbb{R}^n \ , \quad m \in \mathbb{N} \ , \tag{2.3.19}$$

and

$$\sum_{j=-m'}^{m} \eta_j(\xi) = \psi(2^{-m}\xi) - \psi(2^{m'+1}\xi) \ , \qquad \xi \in \mathbb{R}^n \ , \quad m, m' \in \mathbb{N} \ . \tag{2.3.20}$$

Consequently,

$$\sum_{k=0}^{\infty} \psi_k(\xi) = 1 \ , \qquad \xi \in \mathbb{R}^n \ , \tag{2.3.21}$$

and

$$\sum_{j=-\infty}^{\infty} \eta_j(\xi) = 1 \ , \qquad \xi \in (\mathbb{R}^n)^{\cdot} \ , \tag{2.3.22}$$

where for each $\xi \in (\mathbb{R}^n)^{\cdot}$ at most two terms in the above series are diœerent from zero. Thus $(\psi_k) := (\psi_k)_{k\in\mathcal{N}}$ and $(\eta_j) := (\eta_j)_{j\in\mathcal{Z}}$ are resolutions of the identity on $\mathbb{R}^n$ and on $(\mathbb{R}^n)^{\cdot}$, respectively, the dyadic resolutions of the identity on $\mathbb{R}^n$ and on $(\mathbb{R}^n)^{\cdot}$, respectively, induced by ψ.

Basic Properties

Letting $\psi_{-1} := 0$, put

$$\chi_k := \psi_{k-1}+\psi_k+\psi_{k+1} \ , \quad \lambda_j := \eta_{j-1}+\eta_j+\eta_{j+1} \ , \qquad k \in \mathbb{N} \ , \quad j \in \mathbb{Z} \, . \tag{2.3.23}$$

Then it is an easy consequence of (2.3.11) and (2.3.13)˘(2.3.18) that

$$\psi_k = \psi_k\chi_k \ , \quad \eta_j = \eta_j\lambda_j \ , \qquad k \in \mathbb{N} \ , \quad j \in \mathbb{Z} \, , \tag{2.3.24}$$

and

$$2^{k|\alpha|}\,|\partial^\alpha \psi_k| \le c(\alpha,\psi)\chi_{\Omega_k} \ , \qquad k \in \mathbb{N} \ , \quad \alpha \in \mathbb{N}^n \ , \tag{2.3.25}$$

as well as

$$2^{j|\alpha|}\,|\partial^\alpha \eta_j| \le c(\alpha,\psi)\chi_{\Sigma_j} \ , \qquad j \in \mathbb{Z} \, , \quad \alpha \in \mathbb{N}^n \ . \tag{2.3.26}$$

Since $\psi_k, \eta_j \in \mathcal{D}(\mathbb{R}^n)$ it follows from the Convolution Theorem 1.9.10 and from (1.9.3) that

$$\psi_k(D), \eta_j(D) \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n,E), \mathcal{O}_M(\mathbb{R}^n,E)\big) \ , \qquad k \in \mathbb{N} \ , \quad j \in \mathbb{Z} \, . \tag{2.3.27}$$

We also infer from (2.3.10) that

$$\mathcal{F}^{-1}\psi_k = 2^{kn}\sigma_{2^{-k}}\mathcal{F}^{-1}\widetilde{\psi} \ , \quad \mathcal{F}^{-1}\eta_j = 2^{jn}\sigma_{2^{-j}}\mathcal{F}^{-1}\widetilde{\psi} \ , \qquad k \in \dot{\mathbb{N}} \ , \quad j \in \mathbb{Z} \, . \tag{2.3.28}$$

Similarly, given $u \in \mathcal{S}'(\mathbb{R}^n,E)$ and $k \in \dot{\mathbb{N}}$,

$$\psi_k(D)u = \mathcal{F}^{-1}\psi_k\mathcal{F}u = \mathcal{F}^{-1}\sigma_{2^k}(\widetilde{\psi}\sigma_{2^{-k}}\mathcal{F}u) = \sigma_{2^{-k}}\mathcal{F}^{-1}\widetilde{\psi}\mathcal{F}\sigma_{2^k}u \ .$$

Consequently,

$$\psi_k(D) = \sigma_{2^{-k}}\widetilde{\psi}(D)\sigma_{2^k} \ , \quad \eta_j(D) = \sigma_{2^{-j}}\widetilde{\psi}(D)\sigma_{2^j} \ , \qquad k \in \dot{\mathbb{N}} \ , \quad j \in \mathbb{Z} \, . \tag{2.3.29}$$

Since $\int u\,dx = \widehat{u}(0)$ for $u \in L_1(\mathbb{R}^n,E)$, it follows that

$$\int_{\mathcal{R}^n} \mathcal{F}^{-1}\psi\,dx = 1 \ , \quad \int_{\mathcal{R}^n} \mathcal{F}^{-1}\psi_k\,dx = 0 \ , \quad \int_{\mathcal{R}^n} \mathcal{F}^{-1}\eta_j\,dx = 0 \tag{2.3.30}$$

for $k \in \dot{\mathbb{N}}$ and $j \in \mathbb{Z}$. Thus, letting $\varphi_1 := \mathcal{F}^{-1}\psi$ and

$$\varphi_\varepsilon := \mathcal{F}^{-1}\sigma_{\varepsilon^{-1}}\psi = \varepsilon^{-n}\sigma_\varepsilon\varphi_1 = \varepsilon^{-n}\varphi_1(\varepsilon^{-1}\,\cdot\,) \ , \qquad \varepsilon > 0 \ , \tag{2.3.31}$$

we see that $\{\,\varphi_\varepsilon \ ; \ \varepsilon > 0\,\}$ is an approximate identity.

Given any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$,

$$\|\varphi \partial^\alpha (\sigma_{2^m} \psi - \mathbb{1})\|_\infty \to 0 \qquad \text{as } m \to \infty \ . \tag{2.3.32}$$

Consequently, $\sigma_{2^m}\psi \to \mathbb{1}$ in $\mathcal{O}_M(\mathbb{R}^n)$ as $m \to \infty$, so that (2.3.19) implies

$$\sum_{k=0}^{\infty} \psi_k = \mathbb{1} \qquad \text{in } \mathcal{O}_M(\mathbb{R}^n) \ . \tag{2.3.33}$$

Thus, given $u \in \mathcal{S}'(\mathbb{R}^n, E)$, it follows from Theorem 1.6.4 that $\sum_{k=0}^{\infty} \psi_k \widehat{u} = \widehat{u}$ in $\mathcal{S}'(\mathbb{R}^n, E)$. Hence we obtain from (2.3.1) that

$$\sum_{k=0}^{\infty} \psi_k(D) u = u \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E) \ , \tag{2.3.34}$$

that is,

$$\sum_{k=0}^{\infty} \psi_k(D) = 1_{\mathcal{S}'(\mathcal{R}^n, E)} \qquad \text{in } \mathcal{L}_s\big(\mathcal{S}'(\mathbb{R}^n, E)\big) \ . \tag{2.3.35}$$

This shows that the dyadic resolution of the identity on $\mathbb{R}^n$ induced by ψ gives a dyadic decomposition

$$u = \sum_{k=0}^{\infty} u_k \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E) \ , \tag{2.3.36}$$

into smooth functions

$$u_k := \psi_k(D) u \in \mathcal{O}_M(\mathbb{R}^n, E) \ , \qquad k \in \mathbb{N} \ . \tag{2.3.37}$$

Although (2.3.22) is true, it is not true that $\sum_{j=-\infty}^{\infty} \eta_j = \mathbb{1}$ in $\mathcal{O}_M(\mathbb{R}^n)$. This follows from the fact that $\sigma_{2^{-m}}\psi$ does not converge to zero in $\mathcal{O}_M(\mathbb{R}^n)$ as $m \to \infty$. Consequently, a relation analogous to (2.3.34), namely

$$\sum_{j=-\infty}^{\infty} \eta_j(D) u = u \qquad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \ , \tag{2.3.38}$$

does not hold for all $u \in \mathcal{S}(\mathbb{R}^n, E)$. On the other hand, for each $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it is obvious that $\langle \sigma_{2^{-m}}\psi, \varphi \rangle \to 0$ as $m \to \infty$. Thus the Banach-Steinhaus theorem and the fact that $\mathcal{S}(\mathbb{R}^n)$ is a Montel space imply that $\sigma_{2^{-m}}\psi \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $m \to \infty$. Together with $\sigma_{2^m}\psi \to \mathbb{1}$ in $\mathcal{O}_M(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ as $m \to \infty$, this entails, thanks to (2.3.20),

$$\sum_{j=-\infty}^{\infty} \eta_j = \mathbb{1} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n) \ . \tag{2.3.39}$$

Now we deduce from Theorem 1.6.4 that

$$\sum_{j=-\infty}^{\infty} \eta_j u = u \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) , \qquad u \in \mathcal{O}_M(\mathbb{R}^n, E) . \tag{2.3.40}$$

Since $\mathcal{S} \hookrightarrow \mathcal{O}_M$, we infer from (2.3.40) that

$$\sum_{j=-\infty}^{\infty} \eta_j(D) u = u \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) , \qquad u \in \mathcal{S}(\mathbb{R}^n, E) , \tag{2.3.41}$$

thanks to (2.3.1).

Dyadic decompositions are a standard tool in harmonic analysis and the theory of function spaces (see [Ste70], [Ste93], [Tri78], [Tri83], [Tri92], for example, and the references in these books).

2.4 Multiplier Spaces

In this subsection we prove some simple technical results that will be useful for verifying that certain maps deøne Fourier multipliers. Since positively homogeneous functions play a particularly important rle in this connection, we also introduce spaces of homogeneous maps and collect some examples. For later purposes we allow suitable parameter dependencies.

Computing Derivatives

Let X be a nonempty open subset of $\mathbb{R}^n$, let $E_0, E_1, \ldots, E_\ell$ be Banach spaces, and let

$$E_1 \times \cdots \times E_\ell \to E_0 , \quad (x_1, \ldots, x_\ell) \mapsto x_1 \bullet \cdots \bullet x_\ell$$

be a multiplication. Given $a_j \in C^k(X, E_j)$, it follows from the product rule (1.3.2) by induction that $a_1 \bullet \cdots \bullet a_\ell \in C^k(\mathbb{R}^n, E_0)$ and

$$\partial^\alpha (a_1 \bullet \cdots \bullet a_\ell) = \sum_{\substack{\alpha_1, \ldots, \alpha_\ell \in \mathbb{N}^n \\ \alpha_1 + \cdots + \alpha_\ell = \alpha}} \partial^{\alpha_1} a_1 \bullet \cdots \bullet \partial^{\alpha_\ell} a_\ell \tag{2.4.1}$$

for $|\alpha| \le k$.

Let X be a topological space, E a Banach space, and $\mathbf{F}(X, E)$ a LCS satisfying $\mathbf{F}(X, E) \hookrightarrow C(X, E)$. Then, given any nonempty subset Y of E,

$$\mathbf{F}(X, Y) := \big\{ u \in \mathbf{F}(X, E) \ ; \ u(X) \subset Y \big\} ,$$

endowed with the topology induced by $\mathbf{F}(X, E)$.

Let F be a second Banach space. Recall that $\mathcal{L}\mathrm{is}(E,F)$ is open in $\mathcal{L}(E,F)$ and that the inversion map

$$\mathrm{inv}:\ \mathcal{L}\mathrm{is}(E,F)\to\mathcal{L}\mathrm{is}(F,E)\ ,\quad A\mapsto A^{-1} \tag{2.4.2}$$

is smooth with

$$\big[\partial\,\mathrm{inv}(A)\big]B=-A^{-1}BA^{-1}\ ,\qquad A\in\mathcal{L}\mathrm{is}(E,F)\ ,\quad B\in\mathcal{L}(E,F), \tag{2.4.3}$$

(e.g., [Die69, Theorem (8.3.2) and Example (8.12.11)]).

2.4.1 Lemma   Let X be open in $\mathbb{R}^n$ and suppose that $a\in C^k\big(X,\mathcal{L}\mathrm{is}(E,F)\big)$ for some $k\in\dot{\mathbb{N}}$. Then $a^{-1}\in C^k\big(X,\mathcal{L}\mathrm{is}(F,E)\big)$ and

$$\partial^\alpha a^{-1}=\varepsilon_\alpha\sum_{j=1}^{|\alpha|}\ \sum_{\substack{\alpha_1,\ldots,\alpha_j\in(\mathbb{N}^n)^{\cdot}\\ \alpha_1+\cdots+\alpha_j=\alpha}} a^{-1}(\partial^{\alpha_1}a)a^{-1}\cdots(\partial^{\alpha_j}a)a^{-1}\ ,\qquad |\alpha|\le k\ ,$$

where $\varepsilon_\alpha\in\{-1,1\}$.

Proof   From (2.4.3) and the chain rule we infer

$$\partial_j a^{-1}=-a^{-1}(\partial_j a)a^{-1}\ ,\qquad 1\le j\le n\ . \tag{2.4.4}$$

Now (2.4.1) (with the composition of bounded linear operators as a multiplication) implies the assertion by induction. ÷

Multilinear Symmetric Maps

An m-linear map $T:\ E^m\to F$ is symmetric if

$$T(e_1,\ldots,e_m)=T(e_{\sigma(1)},\ldots,e_{\sigma(m)})\ ,\qquad (e_1,\ldots,e_m)\in E^m\ ,\quad \sigma\in\mathsf{S}_m\ ,$$

where S_m is the group of all permutations of m elements. It is clear that

$$\mathcal{L}^m_{\mathrm{sym}}(E,F):=\big\{\,T\in\mathcal{L}^m(E,F)\ ;\ T\text{ is symmetric}\,\big\}$$

is a closed linear subspace of $\mathcal{L}^m(E,F)$, hence a Banach space. Recall that, given an open subset U of E,

$$\partial^m f\in C\big(U,\mathcal{L}^m_{\mathrm{sym}}(E,F)\big)\ ,\qquad f\in C^m(U,F)\ . \tag{2.4.5}$$

The Chain Rule

Now suppose that X is open in $\mathbb{R}^n$ and $\varphi\in C^m(X,U)$. Then $\varphi^*f:=f\circ\varphi$, the pull-back of f by φ, belongs to $C^m(X,F)$ and

$$\partial_j(\varphi^*f)=(\varphi^*\partial f)\partial_j\varphi\ ,\qquad 1\le j\le n\ , \tag{2.4.6}$$

by the chain rule. In the following lemma we describe the qualitative form of the higher derivatives of φ^*f, where we write $[e]^m$ for the m-tuple $(e,e,\ldots,e)$.

2.4.2 Lemma Suppose that X is open in $\mathbb{R}^n$, that U is open in E, and that $\varphi \in C^m(X,U)$ and $f \in C^m(U,F)$. Then, given $\alpha \in \mathbb{N}^n$ with $|\alpha| = m > 0$,

$$\partial^\alpha(\varphi^* f) = \sum c_{\beta,\gamma}\varphi^* \partial^{|\beta|} f\big([\partial^{\gamma_1}\varphi]^{\beta_1}, \ldots, [\partial^{\gamma_m}\varphi]^{\beta_m}\big) \ ,$$

where the summation extends over all $(\beta,\gamma) \in \mathbb{N}^m \times (\mathbb{N}^n)^m$ satisfying $|\gamma_i| > 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m \beta_i\,|\gamma_i| = m$. The constants $c_{\beta,\gamma}$ depend on β, γ, n, and m only.

Proof This follows by induction from the chain rule. ÷

For $z \in \mathbb{C}$ we put

$$\Lambda^z(\xi,\rho) := \Lambda^z_\rho(\xi) := (|\xi|^2 + \rho^2)^{z/2} \ , \qquad (\xi,\rho) \in \mathbb{R}^n \times \mathbb{R} \ .$$

Thus $\Lambda := \Lambda^1$ is the Euclidean norm on $\mathbb{R}^{n+1}$. (We do not indicate n, the dimension of $\mathbb{R}^n$, in this notation, since its value will always be clear from the context.)

Spaces of Multipliers

For $m \in \mathbb{R}$ and $\rho \in \mathbb{R}^+$ we denote by

$$S^m_\rho(\mathbb{R}^n, E) := \big(S^m_\rho(\mathbb{R}^n, E), \|\cdot\|_{S^m_\rho}\big)$$

the set of all $a \in C^{n+1}\big((\mathbb{R}^n)^{\bullet}, E\big)$ such that

$$\|a\|_{S^m_\rho} := \max_{|\alpha| \le n+1} \|\Lambda^{|\alpha|-m}_\rho \partial^\alpha a\|_\infty < \infty \ .$$

It is easily verified that $S^m_\rho(\mathbb{R}^n, E)$ is a Banach space such that

$$S^m_\rho(\mathbb{R}^n, E) \hookrightarrow BC^{n+1}\big((\mathbb{R}^n)^{\bullet}, E\big) \hookrightarrow L_\infty(\mathbb{R}^n, E) \ , \qquad m \le 0 \ , \quad \rho > 0 \ ,$$

and

$$S^m_\rho(\mathbb{R}^n, E) \hookrightarrow W^{n+1}_{1,\mathrm{loc}}(\mathbb{R}^n, E) \ , \qquad m > 0 \ , \quad \rho > 0 \ .$$

Each $a \in S^m_\rho(\mathbb{R}^n, E)$ is a regular distribution on $\mathbb{R}^n$ whose distributional derivatives of order $\le n+1$ are represented on $(\mathbb{R}^n)^{\bullet}$ by continuous functions. Hence it is obvious that

$$S^0_\rho(\mathbb{R}^n, E) \hookrightarrow S^0_0(\mathbb{R}^n, E) \hookrightarrow \mathcal{M}_M(\mathbb{R}^n, E) \ , \qquad \rho \ge 0 \ , \tag{2.4.7}$$

where the norms of these injections are bounded by 1 and where $\mathcal{M}_M(\mathbb{R}^n, E)$ is the (E-valued analogue) of the multiplier space introduced in [Ama95, (III.4.4.6)]. As usual,

$$S^m_\rho(\mathbb{R}^n) := S^m_\rho(\mathbb{R}^n, \mathbb{K}) \quad \text{and} \quad S^m(\mathbb{R}^n, E) := S^m_1(\mathbb{R}^n, E) \ .$$

The reason for considering $n+1$ derivatives in the deønition of S^m_ρ will become apparent by looking at the multiplier theorems of Subsection 2.5.

In the following,

$$E_1 \times E_2 \to E_0 \ , \quad (e_1, e_2) \mapsto e_1 \bullet e_2 \tag{2.4.8}$$

is a multiplication.

2.4.3 Remarks (a) For $m_1, m_2 \in \mathbb{R}$ point-wise multiplication induced by (2.4.8),

$$S_\rho^{m_1}(\mathbb{R}^n, E_1) \times S_\rho^{m_2}(\mathbb{R}^n, E_2) \to S_\rho^{m_1+m_2}(\mathbb{R}^n, E_0) \ , \quad (a_1, a_2) \mapsto a_1 \bullet a_2 \ ,$$

is well-deøned, bilinear, and continuous.

Proof Leibniz' rule. ÷

(b) If $\rho, \sigma > 0$ then $S_\rho^m(\mathbb{R}^n, E) \doteq S_\sigma^m(\mathbb{R}^n, E)$ and

$$\big(1 \wedge (\sigma/\rho)^{n+1-m}\big) \, \|\cdot\|_{S_\rho^m} \le \|\cdot\|_{S_\sigma^m} \le \big(1 \vee (\sigma/\rho)^{n+1-m}\big) \, \|\cdot\|_{S_\rho^m} \ .$$

Proof This is implied by the estimate

$$1 \wedge (\sigma/\rho) \le \Lambda_\sigma \Lambda_\rho^{-1} \le 1 \vee (\sigma/\rho) \ , \qquad \rho, \sigma > 0 \ , \tag{2.4.9}$$

which is obvious. ÷

(c) Suppose that $a \in S_\rho^0\big(\mathbb{R}^n, \mathcal{L}\mathrm{is}(E, F)\big)$ satisøes $a^{-1} \in B\big((\mathbb{R}^n)^{\text{"}}, \mathcal{L}(F, E)\big)$. Then $a^{-1} \in S_\rho^0\big(\mathbb{R}^n, \mathcal{L}(F, E)\big)$ and

$$\|a^{-1}\|_{S_\rho^0} \le c\big(\|a\|_{S_\rho^0}, \|a^{-1}\|_\infty\big) \ ,$$

where $c \in C\big((\mathbb{R}^+)^2\big)$ is independent of a.

Proof This follows from Lemma 2.4.1. ÷

(d) Let U be open in E and suppose that $f \in C^{n+1}(U, F)$ satisøes

$$\|f\|_{n+1,\infty} := \max_{0 \le k \le n+1} \|\partial^k f\|_\infty < \infty \ .$$

Then $f(a) \in S_\rho^0(\mathbb{R}^n, F)$ for $a \in S_\rho^0(\mathbb{R}^n, U)$, and

$$\|f(a)\|_{S_\rho^0} \le c\big(\|a\|_{S_\rho^0}\big) \, \|f\|_{n+1,\infty} \ ,$$

where $c \in C(\mathbb{R}^+)$ and is independent of a and f.

Proof The assertion is easily deduced from Lemma 2.4.2. ÷

(e) If $m_1 \ge m_0$ then $S_\rho^{m_0}(\mathbb{R}^n, E) \hookrightarrow S_\rho^{m_1}(\mathbb{R}^n, E)$ for $\rho > 0$, and the norm of this injection is bounded by $\rho^{m_0 - m_1}$. ÷

Homogeneous Functions

The problem of verifying that a given $a \in C^{n+1}(\mathbb{R}^n, E)$ belongs to $S^m_\rho(\mathbb{R}^n, E)$ and of estimating the corresponding norm can often easily be solved if a is suitably positively homogeneous. For this reason we now collect some facts about maps of this type.

Let V and W be vector spaces and let P be a cone in V. Then $a : \mathring{P} \to W$ is positively homogeneous of degree $z \in \mathbb{C}$ if

$$a(tv) = t^z a(v) \ , \qquad t > 0 \ , \quad v \in \mathring{P} \ .$$

If this is the case and $\operatorname{Re} z > 0$, then we always extend a to a map from P to W, again denoted by a, by putting $a(0) := 0$.

2.4.4 Remarks (a) The set of all $a : \mathring{P} \to W$ that are positively homogeneous of degree z for some øxed $z \in \mathbb{C}$ is a vector subspace of $W^{\mathring{P}}$.

(b) If $a_j : \mathring{P} \to E_j$ is positively homogeneous of degree z_j for $j = 1, 2$, then the map $a_1 \bullet a_2 : \mathring{P} \to E_0$ is positively homogeneous of degree $z_1 + z_2$.

(c) Suppose that W_1 and W_2 are vector spaces and $a : \mathring{P} \to \operatorname{Hom}(W_1, W_2)$ is positively homogeneous of degree z. If $a(v)$ is bijective for each $v \in \mathring{P}$, then

$$a^{-1} : P \to \operatorname{Hom}(W_2, W_1)$$

is positively homogeneous of degree $-z$.

Proof From

$$1_{W_1} = a^{-1}(tv)a(tv) = t^z a^{-1}(tv)a(v)$$

and

$$1_{W_2} = a(tv)a^{-1}(tv) = a(v)t^z a^{-1}(tv)$$

for $v \in \mathring{P}$ and $t > 0$ we see that $t^z a^{-1}(tv) = a^{-1}(v)$, thanks to the uniqueness of the inverse. ÷

(d) Suppose that $u \in \mathcal{D}'(\mathbb{R}^n, E)$. Then u is positively homogeneous of degree $z \in \mathbb{C}$ if $\sigma_{t^{-1}} u = t^z u$, $t > 0$. This agrees apparently with the above deønition if u is a regular distribution. If $u \in \mathcal{D}'(\mathbb{R}^n, E)$ is positively homogeneous of degree z for some $z \in \mathbb{C}$, then $u \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\widehat{u}$ and $\mathcal{F}^{-1}u$ are positively homogeneous of degree $-z-n$. If, in addition, $u \in C^k\big((\mathbb{R}^n)^{\circ}, E\big)$ for some $k > \operatorname{Re} z + n$, then $\partial^\alpha \widehat{u} \in C\big((\mathbb{R}^n)^{\circ}, E\big)$ for $|\alpha| < k - \operatorname{Re} z - n$.

Proof The proof of [Hr83, Theorem 7.1.18] carries over to the E-valued case to show that $u \in \mathcal{S}'(\mathbb{R}^n, E)$ and that the last assertion is true. Hence we can apply (2.3.9) to obtain

$$t^z \mathcal{F}u = \mathcal{F}(t^z u) = \mathcal{F}(\sigma_{t^{-1}} u) = t^n \sigma_t \mathcal{F}u \ .$$

Consequently, $\sigma_{t^{-1}}\widehat{u} = t^{-z-n}\,\widehat{u}$. Thus, by replacing $\mathcal{F}$ by $\mathcal{F}^{-1}$ in this argument, we obtain the second assertion. ■

Now we suppose that H is a cone in some Banach space $\mathbf{H} := (\mathbf{H}, |\cdot|)$. Then

$$\mathbb{R}^n \times \mathbf{H} \to \mathbb{R}\ , \quad (\xi,\eta) \mapsto \Lambda(\xi, |\eta|)$$

is the 'Euclidean' norm on $\mathbb{R}^n \times \mathbf{H}$ and

$$(\mathbb{R}^n \times \mathbf{H})^{\bullet} \to \big[\Lambda(\xi,|\eta|) = 1\big]\ , \quad (\xi,\eta) \mapsto (\xi^*,\eta^*) := (\xi,\eta)\Lambda^{-1}(\xi,|\eta|)$$

is the 'Euclidean' radial retraction onto the unit-sphere of $\mathbb{R}^n \times \mathbf{H}$.

For $a:\ (\mathbb{R}^n)^{\bullet} \times \mathsf{H} \to E$ we often set

$$a_\eta := a(\cdot,\eta):\ (\mathbb{R}^n)^{\bullet} \to E\ .$$

Of course, if $\mathsf{H} = \{0\}$, then we identify $(\mathbb{R}^n)^{\bullet} \times \mathsf{H}$ and $(\mathbb{R}^n)^{\bullet}$ and omit any reference to $\eta = 0$.

2.4.5 Lemma Suppose that $a:\ (\mathbb{R}^n)^{\bullet} \times \mathsf{H} \to E$ is positively homogeneous of degree z for some $z \in \mathbb{C}$ and that $a_\eta \in C^k\big((\mathbb{R}^n)^{\bullet}, E\big)$ for some $k \in \dot{\mathbb{N}}$ and each $\eta \in \mathsf{H}$. Then $\partial_\xi^\alpha a:\ (\mathbb{R}^n)^{\bullet} \times \mathsf{H} \to E$ is positively homogeneous of degree $z - |\alpha|$ and

$$\partial^\alpha a_\eta(\xi) = \Lambda^{z-|\alpha|}(\xi,|\eta|)\partial_\xi^\alpha a(\xi^*,\eta^*)\ , \qquad (\xi,\eta) \in (\mathbb{R}^n)^{\bullet} \times \mathsf{H}\ ,$$

for $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$.

Proof By differentiating the identity $a(t\xi, t\eta) = t^z a(\xi,\eta)$ with respect to ξ it follows that

$$(\partial_\xi^\alpha a)(t\xi, t\eta) = t^{z-|\alpha|}\partial_\xi^\alpha a(\xi,\eta)\ , \qquad t > 0\ , \quad (\xi,\eta) \in (\mathbb{R}^n)^{\bullet} \times \mathsf{H}\ ,$$

proving the first assertion. The second one follows now if we replace (ξ,η) by (ξ^*,η^*) and t by $\Lambda(\xi,|\eta|)$, respectively. ■

Spaces of Homogeneous Functions and Multipliers

For $k \in \mathbb{N}$ and $z \in \mathbb{C}$ we denote by

$$\mathcal{H}_z^k(\mathbb{R}^n \times \mathsf{H}, E) := \big(\mathcal{H}_z^k(\mathbb{R}^n \times \mathsf{H}, E), \|\cdot\|_{\mathcal{H}_z^k}\big)$$

the normed vector space of all $a:\ (\mathbb{R}^n)^{\bullet} \times \mathsf{H} \to E$ that are positively homogeneous of degree z and satisfy $a_\eta \in C^k\big((\mathbb{R}^n)^{\bullet}, E\big)$ for $\eta \in \mathsf{H}$ as well as

$$\|a\|_{\mathcal{H}_z^k} := \max_{|\alpha| \le k}\ \sup_{(\xi,\eta)\in(\mathbb{R}^n)^{\bullet}\times\mathsf{H}} |\partial_\xi^\alpha a(\xi^*,\eta^*)| < \infty\ .$$

Moreover,

$$\mathcal{H}_z^\infty(\mathbb{R}^n \times \mathsf{H}, E) := \bigcap_{k=0}^{\infty} \mathcal{H}_z^k(\mathbb{R}^n \times \mathsf{H}, E) \ ,$$

equipped with the obvious projective limit topology. As usual,

$$\mathcal{H}_z^k(\mathbb{R}^n \times \mathsf{H}) := \mathcal{H}_z^k(\mathbb{R}^n \times \mathsf{H}, \mathbb{K}) \ , \qquad k \in \bar{\mathbb{N}} \ ,$$

and

$$\mathcal{H}_z(\mathbb{R}^n \times \mathsf{H}, E) := \mathcal{H}_z^{n+1}(\mathbb{R}^n \times \mathsf{H}, E) \ , \qquad z \in \mathbb{C} \ .$$

2.4.6 Remarks (a) $(a \mapsto a_\eta) \in \mathcal{L}\big(\mathcal{H}_z(\mathbb{R}^n \times \mathsf{H}, E), S_{|\eta|}^{\operatorname{Re} z}(\mathbb{R}^n, E)\big)$ with the norm bounded by 1.

Proof This is a consequence of Lemma 2.4.5. ÷

(b) Point-wise multiplication induced by (2.4.8)

$$\mathcal{H}_{z_1}^k(\mathbb{R}^n \times \mathsf{H}, E_1) \times \mathcal{H}_{z_2}^k(\mathbb{R}^n \times \mathsf{H}, E_2) \to \mathcal{H}_{z_1+z_2}^k(\mathbb{R}^n \times \mathsf{H}, E_0) \ , \quad (a_1, a_2) \mapsto a_1 \bullet a_2$$

is well-deøned, bilinear, and continuous for $k \in \bar{\mathbb{N}}$ and $z_1, z_2 \in \mathbb{C}$.

Proof Leibniz' rule and Remark 2.4.4(b). ÷

(c) Suppose that $a \in \mathcal{H}_z^k\big(\mathbb{R}^n \times \mathsf{H}, \mathcal{L}(E, F)\big)$ satisøes $a(\xi^*, \eta^*) \in \mathcal{L}\mathrm{is}(E, F)$ and

$$\|a^{-1}\|_{\infty,[\Lambda=1]} = \sup_{(\xi,\eta)\in(\mathbb{R}^n)^{\cdot}\times \mathsf{H}} |a^{-1}(\xi^*, \eta^*)|_{\mathcal{L}(F,E)} < \infty \ .$$

Then $a^{-1} \in \mathcal{H}_{-z}^k\big(\mathbb{R}^n \times \mathsf{H}, \mathcal{L}(F, E)\big)$ and

$$\|a^{-1}\|_{\mathcal{H}_{-z}^k} \le c\big(\|a\|_{\mathcal{H}_z^k}, \|a^{-1}\|_{\infty,[\Lambda=1]}\big) \ ,$$

where $c \in C\big((\mathbb{R}^+)^2\big)$.

Proof It follows from Remark 2.4.4(c) that a^{-1} is well-deøned as a map from $(\mathbb{R}^n)^{\cdot} \times \mathsf{H}$ into $\mathcal{L}(F, E)$ and is positively homogeneous. Now the assertion is a consequence of Lemma 2.4.1. ÷

To conclude, we consider three simple but most important examples. They will be the basis for many computations in connection with Fourier multipliers.

2.4.7 Examples (a) $\big[(\xi, \eta) \mapsto \Lambda^z(\xi, |\eta|)\big] \in \mathcal{H}_z^\infty(\mathbb{R}^n \times \mathsf{H})$ and

$$\|\Lambda^z\|_{\mathcal{H}_z^k} \le c(k)(1 + |z|^k) \ , \qquad z \in \mathbb{C} \ , \quad k \in \mathbb{N} \ .$$

Proof First we note that $\Lambda_{|\eta|} \in C^\infty\big((\mathbb{R}^n)^{\cdot}\big)$ for $\eta \in \mathsf{H}$ and that Λ is positively homogeneous of degree 1. Moreover,

$$\partial_j \Lambda_{|\eta|}(\xi) = \xi_j \Lambda_{|\eta|}^{-1}(\xi) \ , \qquad \xi = (\xi_1, \ldots, \xi_n) \in (\mathbb{R}^n)^{\cdot}, \quad \eta \in \mathsf{H} \ , \quad 1 \le j \le n \ .$$

Now we infer from (2.4.1) and Lemma 2.4.1 that

$$|\partial_\xi^\alpha \Lambda(\xi^*, |\eta^*|)| \le c(\alpha) , \qquad \alpha \in \mathbb{N}^n .$$

Consequently, $\Lambda \in \mathcal{H}_1^k(\mathbb{R}^n \times \mathsf{H})$ and $\|\Lambda\|_{\mathcal{H}_1^k} \le c(k)$. Next we put $f(t) := t^z$ for $z \in \mathbb{C}$. Then $f \in C^\infty(\dot{\mathbb{R}}^+)$ and

$$\partial^j f(t) = z(z-1)\cdots(z-j+1)t^{z-j} , \qquad t > 0 , \quad j \in \mathbb{N} .$$

Hence $|\partial^j f(1)| \le c(j)(1+|z|^j)$, and the assertion follows from $\Lambda^z = f(\Lambda)$ and Lemma 2.4.2. ÷

(b) If $\eta \in \dot{\mathsf{H}}$ then $\Lambda_{|\eta|}^z \in \mathcal{O}_M(\mathbb{R}^n)$.

Proof We obtain the assertion from (a) and Lemma 2.4.5 thanks to the fact that $\Lambda_\rho^z \in C^\infty(\mathbb{R}^n)$ and $\Lambda_\rho \ge \rho$. ÷

(c) Let $a_\alpha : \mathsf{H} \to E$ be positively homogeneous of degree $m - |\alpha|$ for $|\alpha| \le m$ and put

$$a(\xi, \eta) := \sum_{|\alpha| \le m} a_\alpha(\eta)\xi^\alpha , \qquad (\xi, \eta) \in \mathbb{R}^n \times \mathsf{H} .$$

Then $a \in \mathcal{H}_m^\infty(\mathbb{R}^n \times \mathsf{H}, E)$ and

$$\|a\|_{\mathcal{H}_m^k} \le c \max_{|\alpha| \le m} \sup_{|\eta| \le 1} |a_\alpha(\eta)| , \qquad k \in \mathbb{N} ,$$

since $\partial^\alpha a_\eta = 0$ for $|\alpha| > m$. ÷

Below we often identify Λ^z with $\Lambda^z 1_E$. In other words, we consider Λ^z as an element of $\mathcal{H}_z^\infty\big(\mathbb{R}^n \times \mathsf{H}, \mathcal{L}(E)\big)$. Then the estimates of Example 2.4.7(a) are valid in this case as well.

2.5 Fourier Multipliers

In this subsection we derive some most important multiplier theorems for operator-valued symbols, of which Lemma 2.5.3 has to be especially mentioned. It entails, in particular, Lemma 2.5.6 that develops its full power in the framework of Besov spaces.

First we recall the multinomial theorem:

$$(x_1 + \cdots + x_n)^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha , \qquad m \in \mathbb{N} , \quad n \in \dot{\mathbb{N}} , \quad x_j \in \mathbb{K} . \tag{2.5.1}$$

Moreover, putting

$$\binom{m}{\alpha} := \frac{m!}{\alpha!\,(m-|\alpha|)!}\ , \qquad m \in \mathbb{N}\ , \quad \alpha \in \mathbb{N}^n\ ,$$

it follows from (2.5.1) that

$$(1 + x_1 + \cdots + x_n)^m = \sum_{|\alpha| \le m} \tbinom{m}{\alpha} x^\alpha\ , \qquad m \in \mathbb{N}\ , \quad n \in \dot{\mathbb{N}}\ , \quad x_j \in \mathbb{K}\ . \quad (2.5.2)$$

Observe that these formulas remain valid if $\mathbb{K}$ is replaced by an algebra with unit, and $x_1, \ldots, x_m$ by commutative elements thereof.

The Multiplier Space $\mathcal{F}L_1$

Now let $E := (E, |\cdot|)$ be a Banach space. We put

$$\mathcal{F}L_1(\mathbb{R}^n, E) := \Big(\big\{\, u \in \mathcal{S}'(\mathbb{R}^n, E)\ ;\ \mathcal{F}^{-1}u \in L_1(\mathbb{R}^n, E) \,\big\},\ \|\cdot\|_{\mathcal{F}L_1} \Big)\ ,$$

where

$$\|u\|_{\mathcal{F}L_1} := \|\mathcal{F}^{-1}u\|_1\ . \qquad (2.5.3)$$

As usual,

$$\mathcal{F}L_1(\mathbb{R}^n) := \mathcal{F}L_1(\mathbb{R}^n, \mathbb{C})\ .$$

2.5.1 Proposition

(i) $\mathcal{F}L_1(\mathbb{R}^n, E)$ is a Banach space and

$$\mathcal{S}(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{F}L_1(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} C_0(\mathbb{R}^n, E)\ . \qquad (2.5.4)$$

(ii) If E is a Hilbert space and $k > n/2$, then

$$W_2^k(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} \mathcal{F}L_1(\mathbb{R}^n, E)\ .$$

Moreover,

$$\|u\|_{\mathcal{F}L_1} \le c\,\|u\|_2^{1-\theta} \Big(\max_{|\alpha|=k} \|\partial^\alpha u\|_2 \Big)^\theta \le c\,\|u\|_{2,k}\ , \qquad u \in W_2^k(\mathbb{R}^n, E)\ ,$$

where $\theta := n/(2k)$.

Proof (i) It is obvious that $\mathcal{F}L_1$ is a Banach space. Thanks to the Riemann-Lebesgue lemma,

$$\|u\|_{C_0} = \big\|\widehat{\mathcal{F}^{-1}u}\big\|_{C_0} \le c\,\|\mathcal{F}^{-1}u\|_1 = c\,\|u\|_{\mathcal{F}L_1}\ , \qquad u \in \mathcal{F}L_1(\mathbb{R}^n, E)\ ,$$

hence $\mathcal{F}L_1 \hookrightarrow C_0$. Since $\mathcal{S}$ is dense in L_1 and $\mathcal{F} \in \mathcal{L}\mathrm{aut}(\mathcal{S}) \cap \mathcal{L}\mathrm{is}(L_1, \mathcal{F}L_1)$, we infer that $\mathcal{S} \overset{d}{\hookrightarrow} \mathcal{F}L_1$. Consequently, (2.5.4) follows from $\mathcal{S} \overset{d}{\hookrightarrow} C_0$.

(ii) Let $t > 0$. By the Cauchy-Schwarz inequality and since $\mathcal{F}^{-1}u = (2\pi)^{-n}\,\check{\widehat{u}}$,

$$\begin{aligned}\int_{|x|>t} |\mathcal{F}^{-1}u|\,dx &= (2\pi)^{-n}\int_{|\xi|>t} |\xi|^{-k}\,|\xi|^{k}\,|\widehat{u}(\xi)|\,d\xi \\ &\le (2\pi)^{-n}\Big(\int_{|\xi|>t} |\xi|^{-2k}\,d\xi\Big)^{1/2}\Big(\int_{|\xi|>t} |\xi|^{2k}\,|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2} \\ &\le ct^{n/2-k}\max_{|\alpha|=k}\|D^\alpha u\|_2\ ,\end{aligned}$$

where we used (2.5.1) and Plancherel's theorem in the last step. Similarly, we ﬁnd

$$\int_{|x|\le t} |\mathcal{F}^{-1}u|\,dx \le (2\pi)^{-n}\Big(\int_{|\xi|\le t} 1\,d\xi\Big)^{1/2}\Big(\int_{|\xi|\le t} |\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2} \le ct^{n/2}\,\|u\|_2\ .$$

Thus

$$\|\mathcal{F}^{-1}u\|_1 \le ct^{n/2}\Big(\|u\|_2 + t^{-k}\max_{|\alpha|=k}\|\partial^\alpha u\|_2\Big)\ ,\qquad t>0\ ,$$

and the assertion follows by estimating the minimum of the right-hand side with respect to $t > 0$. ÷

Multiplier Theorems

Now suppose that $E_j,\ \ j = 0, 1, 2$, are Banach spaces and

$$E_1\times E_2\to E_0\ ,\quad (e_1,e_2)\mapsto e_1\bullet e_2 \tag{2.5.5}$$

is a multiplication. The following simple multiplier result is the reason for introducing the space $\mathcal{F}L_1$.

2.5.2 Theorem Suppose that $\mathbf{F} \in \{\, BUC, C_0, L_p\ ;\ 1 \le p < \infty\,\}$. Then

$$\big[a\mapsto a(D)\big]\in\mathcal{L}\Big(\mathcal{F}L_1(\mathbb{R}^n,E_1),\mathcal{L}\big(\mathbf{F}(\mathbb{R}^n,E_2),\mathbf{F}(\mathbb{R}^n,E_0)\big)\Big)\ .$$

Moreover,

$$\big[a\mapsto a(D)\big]\in\mathcal{L}\Big(\mathcal{F}L_1(\mathbb{R}^n,E_1),\mathcal{L}\big(L_\infty(\mathbb{R}^n,E_2),BUC(\mathbb{R}^n,E_0)\big)\Big)\ .$$

The norms of these linear maps are bounded by 1.

Proof This follows immediately from Theorem 1.9.9 and Remark 1.9.11(b). ÷

Now we prove some criteria for a distribution to belong to $\mathcal{F}L_1$. They will play an important rle in the sequel. For this we ﬁx $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfying (2.3.13) and denote by (ψ_k) and (η_j) the dyadic resolutions on $\mathbb{R}^n$ and on $(\mathbb{R}^n)^{\dot{}}$, respectively, induced by ψ. We also recall that $\Omega_0 := 2\mathbb{B}^n$ and $\Sigma_j := 2^{j+1}\bar{\mathbb{B}}^n\setminus(2^{j-1}\mathbb{B}^n)$ for $j \in \mathbb{Z}$.

2.5.3 Lemma Suppose that $a \in \mathcal{D}\big((\mathbb{R}^n)'', E\big)$ and all its derivatives of order less than or equal to $n+1$ are regular distributions.

(i) Given $j \in \mathbb{Z}$, suppose that

$$\mu_j := \max_{|\alpha| \le n+1} \|\Lambda_0^{|\alpha|} \partial^\alpha a\|_{\infty, \Sigma_j} < \infty \ .$$

Then $\eta_j a \in \mathcal{F}L_1(\mathbb{R}^n, E)$ and

$$\|\eta_j a\|_{\mathcal{F}L_1} \le c\mu_j \ ,$$

where $c = c(n, \psi)$ is independent of a and j.

(ii) If $a \in W_\infty^{n+1}(\Omega_0, E)$, then $\psi a \in \mathcal{F}L_1(\mathbb{R}^n, E)$ and

$$\|\psi a\|_{\mathcal{F}L_1} \le c \, \|a\|_{n+1, \infty, \Omega_0} \ ,$$

where c is independent of a.

Proof (i) Since $\eta_j a = (\sigma_{2^j} \widetilde{\psi}) a = \sigma_{2^j}(\widetilde{\psi} \sigma_{2^{-j}} a)$, it follows from (2.3.10) that

$$\mathcal{F}^{-1}(\eta_j a) = 2^{jn} \sigma_{2^{-j}} \mathcal{F}^{-1}(\widetilde{\psi} \sigma_{2^{-j}} a) \ .$$

Now we obtain from (2.3.12) and (2.5.3) that

$$\|\eta_j a\|_{\mathcal{F}L_1} = \|\widetilde{\psi} \sigma_{2^{-j}} a\|_{\mathcal{F}L_1} \ . \tag{2.5.6}$$

Leibniz' rule and (2.3.11) imply

$$\partial^\alpha(\widetilde{\psi} \sigma_{2^{-j}} a) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} 2^{j|\beta|} (\sigma_{2^{-j}} \partial^\beta a) \partial^{\alpha - \beta} \widetilde{\psi} \ .$$

Thus, since $\mathrm{supp}(\partial^{\alpha-\beta} \widetilde{\psi}) \subset \Sigma_0$, it follows that

$$|\partial^\alpha(\widetilde{\psi} \sigma_{2^{-j}} a)(\xi)| \le c \sum_{\beta \le \alpha} 2^{j|\beta|} \, |\partial^\beta a(2^j \xi)| \, \chi_{\Sigma_0}(\xi) \le c\mu_j \chi_{\Sigma_0}(\xi)$$

for $|\alpha| \le n+1$ and a.a. $\xi \in \mathbb{R}^n$. Hence $\partial^\alpha(\widetilde{\psi} \sigma_{2^{-j}} a) \in L_1(\mathbb{R}^n, E)$ and

$$\|\partial^\alpha(\widetilde{\psi} \sigma_{2^{-j}} a)\|_1 \le c\mu_j \ , \qquad |\alpha| \le n+1 \ . \tag{2.5.7}$$

Consequently, by (2.3.2), (2.3.3), and the Riemann-Lebesgue lemma,

$$x^\alpha \mathcal{F}^{-1}(\widetilde{\psi} \sigma_{2^{-j}} a) = \mathcal{F}^{-1}\big(D^\alpha(\widetilde{\psi} \sigma_{2^{-j}} a)\big) \in C_0(\mathbb{R}^n, E)$$

and

$$|x^\alpha \mathcal{F}^{-1}(\widetilde{\psi} \sigma_{2^{-j}} a)(x)| \le c\mu_j \ , \qquad x \in \mathbb{R}^n \ , \quad |\alpha| \le n+1 \ . \tag{2.5.8}$$

Since, by (2.5.1),

$$|x|^k = \big((x_1^2+\cdots+x_n^2)^k\big)^{1/2} \le c(k,n)\sum_{|\alpha|=k}|x^\alpha|\ , \qquad k\in\mathbb{N}\ , \tag{2.5.9}$$

it follows from (2.5.8) that

$$|\mathcal{F}^{-1}(\widetilde{\psi}\sigma_{2^{-j}}a)(x)| \le c\mu_j\,|x|^{-n-1}\ , \qquad x\in(\mathbb{R}^n)^{\text{"}}\ . \tag{2.5.10}$$

Hence, thanks to (2.5.8) and (2.5.10),

$$\|\widetilde{\psi}\sigma_{2^{-j}}a\|_{\mathcal{F}L_1} = \|\mathcal{F}^{-1}(\widetilde{\psi}\sigma_{2^{-j}}a)\|_1 \le c\mu_j\Big[\int_{|x|\le 1}1\,dx + \int_{|x|\ge 1}|x|^{-n-1}\,dx\Big] = c\mu_j\ .$$

Now the assertion is a consequence of (2.5.6).

(ii) From Leibniz' rule we infer that

$$|\partial^\alpha(\psi a)| \le c\sum_{\beta\le\alpha}|\partial^\beta a|\,\chi_{\Omega_0} \le c\,\|a\|_{n+1,\infty,\Omega_0}\,\chi_{\Omega_0}$$

for $|\alpha|\le n+1$. Now the arguments of (i) apply if we replace $\widetilde{\psi}$ by ψ and set $j:=0$ and $\mu_0 := \|a\|_{n+1,\infty,\Omega_0}$. ÷

2.5.4 Theorem Suppose that $a\in L_\infty(\mathbb{R}^n,E)$ and all derivatives of order $\le n+1$ are regular distributions on $(\mathbb{R}^n)^{\text{"}}$. Put

$$\mu_j := \max_{|\alpha|\le n+1}\|\Lambda_0^{|\alpha|}\partial^\alpha a\|_{\infty,\Sigma_j}\ , \qquad j\in\mathbb{Z}\ ,$$

and

$$\lambda_0 := \max_{|\alpha|\le n+1}\|\partial^\alpha a\|_{\infty,\Omega_0}\ .$$

Also suppose that

$$\kappa := \Big(\sum_{j=-\infty}^{\infty}\mu_j\Big)\wedge\Big(\lambda_0+\sum_{j=1}^{\infty}\mu_j\Big) < \infty\ .$$

Then $a\in\mathcal{F}L_1(\mathbb{R}^n,E)$ and $\|a\|_{\mathcal{F}L_1}\le c\kappa$, where c is independent of κ and a.

Proof Given $\varphi\in\mathcal{S}(\mathbb{R}^n)$, it follows from (2.3.20) and Lebesgue's theorem that

$$\Big(\sum_{j=-k'}^{k}\eta_j a\Big)(\varphi) = \int_{\mathcal{R}^n} a\big(\psi(2^{-k}\cdot)-\psi(2^{k'+1}\cdot)\big)\varphi\,dx \to a(\varphi)$$

as $k,k'\to\infty$. Thus the Banach-Steinhaus theorem and the fact that $\mathcal{S}(\mathbb{R}^n)$ is a Montel space imply

$$\sum_{j=-\infty}^{\infty}\eta_j a = a \qquad \text{in } \mathcal{S}'(\mathbb{R}^n,E)\ . \tag{2.5.11}$$

Similarly,

$$\sum_{k=0}^{\infty} \psi_k a = a \qquad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \ . \tag{2.5.12}$$

Lemma 2.5.3 gives

$$\sum_{j=-k'}^{k} \|\eta_j a\|_{\mathcal{F}L_1} \leq c \sum_{j=-\infty}^{\infty} \mu_j$$

and, thanks to $\psi_j = \eta_j$ for $j \in \mathbb{N}$,

$$\sum_{j=0}^{k} \|\psi_k a\|_{\mathcal{F}L_1} \leq \lambda_0 + c \sum_{j=0}^{\infty} \mu_j$$

for $k, k' \in \mathbb{N}$. Hence the series on the left-hand sides of (2.5.11) and (2.5.12) converge in the Banach space $\mathcal{F}L_1(\mathbb{R}^n, E)$ if the right-hand sides in the corresponding estimates are ønite. Now the assertion follows from the continuous injection of $\mathcal{F}L_1(\mathbb{R}^n, E)$ in $\mathcal{S}'(\mathbb{R}^n, E)$ and from (2.5.11) and (2.5.12), respectively. ÷

2.5.5 Corollary Suppose that $a \in W_\infty^{n+1}(\mathbb{R}^n, E)$ and there exists $\varepsilon > 0$ such that

$$\|a\|_{n+1,\infty} + \max_{|\alpha| \leq n+1} \|\Lambda_0^{|\alpha|+\varepsilon} \partial^\alpha a\|_\infty \leq \mu < \infty \ .$$

Then $a \in \mathcal{F}L_1(\mathbb{R}^n, E)$ and the estimate $\|a\|_{\mathcal{F}L_1} \leq c\mu$ holds, where $c := c(\varepsilon, n)$ is independent of μ and a.

Proof Note that $\lambda_0 \leq \|a\|_{n+1,\infty} \leq \mu$ and

$$\max_{|\alpha| \leq n+1} |\xi|^\alpha \, |\partial^\alpha a(\xi)| \leq \mu \, |\xi|^{-\varepsilon} \ , \qquad \text{a.a. } \xi \in (\mathbb{B}^n)^c \ .$$

Hence $\mu_j \leq \mu 2^{-(j-1)\varepsilon}$ for $j \in \dot{\mathbb{N}}$ so that $\kappa \leq \mu\big[1 + (1 - 2^{-\varepsilon})^{-1}\big]$. ÷

It should be noted that there is no restriction on the Banach space E. This is in contrast to Proposition 2.5.1(ii) where E has to be a Hilbert space since its proof is based on Plancherel's theorem.

2.5.6 Theorem Suppose that $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\psi \,|\, \mathbb{B}^n = 1$ and $\operatorname{supp}(\psi) \subset 2\mathbb{B}^n$, and let (ψ_k) be the dyadic resolution of the identity on $\mathbb{R}$ induced by ψ. Also suppose that

$$\mathbf{F} \in \{\, BUC, C_0, L_p \ ; \ 1 \leq p < \infty \,\} \ .$$

Then

$$(\psi_k a)(D) \in \mathcal{L}\big(\mathbf{F}(\mathbb{R}^n, E_2), \mathbf{F}(\mathbb{R}^n, E_0)\big)$$

and

$$(\psi_k a)(D) \in \mathcal{L}\big(L_\infty(\mathbb{R}^n, E_2), BUC(\mathbb{R}^n, E_0)\big)$$

for $a \in S^0(\mathbb{R}^n, E_1)$, and

$$\sup_{k\in\mathcal{N}} \|(\psi_k a)(D)\| \le c\,\|a\|_{S^0}\ , \qquad a \in S^0(\mathbb{R}^n, E_1)\ ,$$

where c is independent of a.

Proof If $a \in S^0(\mathbb{R}^n, E)$ then $\|a\|_{n+1,\infty} \le \|a\|_{S^0}$ and $\mu_j \le \|a\|_{S^0}$ for $j \in \mathbb{N}$. Hence the assertion follows from Lemma 2.5.3 and Theorem 2.5.2. ÷

2.5.7 Corollary Given the hypotheses of Theorem 2.5.6,

$$\psi_k(D) \in \mathcal{L}\big(\mathbf{F}(\mathbb{R}^n, E)\big) \cap \mathcal{L}\big(L_\infty(\mathbb{R}^n, E), BUC(\mathbb{R}^n, E)\big)\ , \qquad k \in \mathbb{N}\ ,$$

and $\sup_{k\in\mathcal{N}} \|\psi_k(D)\| < \infty$.

Proof Choose $a := 1_E$. ÷

Multiplier theorems of the type of Theorem 2.5.4 and its corollary are well-known in the scalar case (e.g., [BL76, Exercise 6.8.3] or [Hie91, Lemma 3.3]). In this case the proof is usually based on Proposition 2.5.1(ii) so that the order of diœer-entiation can be reduced from $n+1$ to $[n/2]+1$. This requires, however, the use of Plancherel's theorem that is valid only if E is a Hilbert space. (Thus Lemma 6.1.5 in [BL76] does not hold for $E = \mathcal{L}(H_0, H_1)$, in general, since $\mathcal{L}(H_0, H_1)$ is not a Hilbert space if H_0 and H_1 are inønite-dimensional Hilbert spaces.) For our purpose it will be important that in Corollary 2.5.5 and Theorem 2.5.6 we do not have to impose restrictions on the Banach space E. These results are due to the author [Ama97].

A Limiting Case

We close this subsection by proving that Corollary 2.5.5 is close to being optimal in the sense that we cannot expect to obtain a similar result with $\varepsilon = 0$.

2.5.8 Proposition Suppose $a \in L_\infty(\mathbb{R}^n) \cap C^{n+1}\big((\mathbb{R}^n)^{\cdot}\big)$ is a Fourier multiplier for $\mathbf{F}(\mathbb{R}^n)$, where $\mathbf{F} \in \{BUC, C_0, L_1, L_\infty\}$. If a is positively homogeneous of degree 0 then a is constant.

Proof By assumption $a(D) \in \mathcal{L}\big(\mathbf{F}(\mathbb{R}^n)\big)$ and

$$\|\mathcal{F}^{-1}a * u\|_F \le c\,\|u\|_F\ , \qquad u \in \mathbf{F}(\mathbb{R}^n)\ . \tag{2.5.13}$$

First we suppose that $\mathbf{F} \ne L_1$. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then we know that $\mathcal{F}^{-1}a * \varphi \in \mathcal{E}(\mathbb{R}^n)$ and that $\mathcal{F}^{-1}a * \varphi(0) = \langle \mathcal{F}^{-1}a, \check{\varphi}\rangle$. Hence it follows from (2.5.13) that

$$|\langle \mathcal{F}^{-1}a, \varphi\rangle| \le c\,\|\varphi\|_\infty\ , \qquad \varphi \in \mathcal{D}(\mathbb{R}^n)\ . \tag{2.5.14}$$

By restricting ourselves to $\varphi \in \mathcal{D}\big((\mathbb{R}^n)^{\cdot}\big)$ and by using the density of $\mathcal{D}\big((\mathbb{R}^n)^{\cdot}\big)$ in $C_0\big((\mathbb{R}^n)^{\cdot}\big)$ we deduce from (2.5.14) that $f := \mathcal{F}^{-1}a\,|\,(\mathbb{R}^n)^{\cdot}$ belongs to $C_0'\big((\mathbb{R}^n)^{\cdot}\big)$.

Thus f is a bounded measure on $(\mathbb{R}^n)^{''}$ by the Riesz representation theorem. From Remark 2.4.4(d) we know that $f \in C\big((\mathbb{R}^n)^{''}\big)$ and that $\mathcal{F}^{-1}a$ is positively homogeneous of degree $-n$. This implies that f belongs to $L_1\big((\mathbb{R}^n)^{''}\big)$ and that it is positively homogeneous of degree $-n$. Hence, if $f \neq 0$,

$$\begin{aligned} \infty > \int_{(\mathcal{R}^n)^{''}} |f|\,dx &= \int_0^\infty r^{n-1} \int_{S^{n-1}} |f(r\omega)|\,d\omega\,dr \\ &= \int_0^\infty r^{-1}\,dr \int_{S^{n-1}} |f(\omega)|\,d\omega = \infty\ , \end{aligned}$$

which is impossible. Thus $f = 0$ so that $\operatorname{supp}(\mathcal{F}^{-1}a) \subset \{0\}$. Now we deduce from Example 1.8.7(d) that

$$\mathcal{F}^{-1}a = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha \delta$$

for suitable $k \in \mathbb{N}$ and $a_\alpha \in \mathbb{C}$. Hence we infer from (2.3.3) that a is a polynomial. Since it has to be homogeneous of degree 0, we see that $a = \text{const}$. Lastly, it is well-known that a is a Fourier multiplier for $L_1(\mathbb{R}^n)$ iœ it is one for $L_\infty(\mathbb{R}^n)$ (e.g., [Ste70, IV, 3.1]). This covers the case $\mathbf{F} = L_1$. ÷

The above proof follows the arguments in [Gui93, Lemma 1.10]. Note that a similar result fails to be true for $L_p(\mathbb{R}^n)$-multipliers if $1 < p < \infty$, as Mikhlin's theorem shows.

3 The Heat Equation and Related Problems

The theory of linear and nonlinear parabolic evolution equations is built on the concept of strongly continuous analytic semigroups. So far we have not seen any concrete semigroup of this type but remained exclusively within an abstract framework. Now we turn to a ørst concrete situation: we study in detail the heat semigroup.

But ørst, in Subsection 3.1, we give a simple application of the theory of convolutions of Banach-space-valued distributions to linear evolution equations. This subsection is included mostly for curiosity since it throws new light on the possible concepts of generalized solutions to abstract evolution equations.

In Subsection 3.2 we investigate the fundamental Gau-Weierstra or heat semigroup and identify its generator on the 'standard' function spaces. Since this semigroup is generated by the Laplace operator, it is the prototype of an 'elliptic convolution semigroup'. This class of semigroups is at the basis of the general theory of elliptic and parabolic systems of diœerential and pseudodiœerential operators. The information that we obtain from the study of the Gau-Weierstra semigroup is deepened in Subsection 3.3. There we derive additional properties of its generator.

3.1 Distributional Solutions of Evolution Equations

As a ørst application of the general theory of vector-valued distributions we give a short discussion of linear evolution equations in the framework of distribution theory. For this we assume that

$$A \in \mathcal{C}(E) \ .$$

Then, given $f \in \mathcal{D}'_+(E)$, we consider the linear evolution equation

$$(\partial + A)u = f \qquad \text{in } \mathcal{D}'_+(E) \ . \tag{3.1.1}$$

By a distributional solution of (3.1.1) we mean an element $u \in \mathcal{D}'_+\big(D(A)\big)$ satisfying $\partial u + Au = f$ in $\mathcal{D}'_+(E)$. To be more precise, recall from Example 1.3.1(d) that

$$\mathcal{L}\big(D(A), E\big) \times D(A) \to E \ , \quad (B, a) \mapsto Ba \tag{3.1.2}$$

is a multiplication. Moreover, letting $i : \ D(A) \hookrightarrow E$, it follows that i and A belong to $\mathcal{L}\big(D(A), E\big)$. Hence we obtain from Remark 1.6.5(c) and (d) that, recalling Remark 1.6.5(g),

$$i, A \in \mathcal{L}\big(\mathcal{D}'_+(D(A)), \mathcal{D}'_+(E)\big) \tag{3.1.3}$$

and that $\partial \circ i = i \circ \partial$. Thus, employing the natural identiøcation $\partial := i \circ \partial$, we see that

$$\partial + A \in \mathcal{L}\big(\mathcal{D}'_+(D(A)), \mathcal{D}'_+(E)\big) \ .$$

This shows that the above deønition of a distributional solution is meaningful.

Fundamental Solutions

Denoting the convolution with respect to multiplication (3.1.2) simply by $*$, it follows from Remark 1.9.6(i) that

$$(\partial + A)u = (\partial\delta \otimes i + \delta \otimes A) * u \ , \qquad u \in \mathcal{D}'_+\big(D(A)\big) \ . \tag{3.1.4}$$

From this representation we easily deduce the following solvability criterion for equation (3.1.1).

3.1.1 Theorem Suppose that there exists

$$G \in \mathcal{D}'_+\big(\mathcal{L}(E)\big) \cap \mathcal{D}'_+\big(\mathcal{L}(E, D(A))\big) \ , \tag{3.1.5}$$

a fundamental solution for $\partial + A$, satisfying

$$(\partial\delta \otimes i + \delta \otimes A) * G = \delta \otimes 1_E \tag{3.1.6}$$

and

$$G * (\partial\delta \otimes i + \delta \otimes A) = \delta \otimes i \ . \tag{3.1.7}$$

Then equation (3.1.1) possesses for each $f \in \mathcal{D}'_+(E)$ a unique distributional solution $u \in \mathcal{D}'\big(D(A)\big)$. It is given by $u := G * f$.

Proof Put $E_0 := \mathcal{L}(E)$, $E_1 := \mathcal{L}\big(D(A), E\big)$, $E_2 := \mathcal{L}\big(E, D(A)\big)$, $E_3 := E_5 := E$, and $E_4 := D(A)$. Then the associativity assumption of Remark 1.6.5(e) is satisøed with respect to the obvious multiplications. Hence the associativity of the associated convolution products, guaranteed by Remark 1.9.6(c), implies

$$(\partial + A)(G * f) = (\partial\delta \otimes i + \delta \otimes A) * G * f = (\delta \otimes 1_E) * f = f$$

in $\mathcal{D}'_+(E)$, thanks to (3.1.5), (3.1.6), and Remark 1.9.6(i). Thus $G * f$ is a distributional solution of (3.1.1).

Conversely, let $E_0 := E_2 := \mathcal{L}\big(D(A), E\big)$, $E_1 := \mathcal{L}(E)$, $E_3 := D(A)$, and $E_4 := E_5 := E$. Then the hypotheses of Remark 1.6.5(e) are again satisøed with respect to the obvious multiplications. Hence we deduce once more from (3.1.5) and Remark 1.9.6(c) that, given a distributional solution u of (3.1.1),

$$G * f = G * (\partial + A)u = G * (\partial\delta \otimes i + \delta \otimes A) * u = (\delta \otimes i) * u = u \ ,$$

thanks to (3.1.7) and Remark 1.9.6(i). This gives uniqueness. ÷

3.1.2 Remark Given the hypotheses of Theorem 3.1.1, it follows from Theorem 1.9.7 that

$$(f \mapsto u := G * f) \in \mathcal{L}\big(\mathcal{D}'_+(E), \mathcal{D}'_+(D(A))\big) \ ,$$

that is, the unique solution u of (3.1.1) depends continuously on f (in the topology of $\mathcal{D}'_+$). This is often expressed by saying that problem (3.1.1) is well-posed (in the sense of Hadamard). ÷

Fundamental Solutions and Laplace Transforms

Theorem 3.1.1 guarantees that we can solve the evolution equation $(\partial + A)u = f$ in $\mathcal{D}'_+(E)$ provided we can guarantee the existence of a fundamental solution for $\partial + A$. This can be achieved by means of Laplace transforms, provided we impose the following assumption:

$$\begin{aligned} &\text{there exists } \omega \geq 0 \text{ such that } [\operatorname{Re} z > \omega] \subset \rho(-A) \\ &\text{and the map } [\operatorname{Re} z > \omega'] \to \mathcal{L}(E),\ \lambda \mapsto (\lambda + A)^{-1} \\ &\text{is polynomially bounded for each } \omega' > \omega. \end{aligned} \tag{3.1.8}$$

Since the resolvent $\lambda \mapsto (\lambda + A)^{-1}$ is a holomorphic $\mathcal{L}(E)$-valued function it follows from Theorem 1.10.17 that there exists $G \in \mathcal{D}'_+\big(\mathcal{L}(E)\big)$ such that $(\lambda + A)^{-1}$ is the Laplace transform of G at $\lambda \in [\operatorname{Re} z > \omega]$. The following theorem shows that G is a fundamental solution of $\partial + A$, and that it is the only one.

3.1.3 Theorem Let assumption (3.1.8) be satisfied. Then there exists a unique fundamental solution G for $\partial + A$.

Proof Suppose that there exists a fundamental solution $G \in \mathcal{D}'_+\big(\mathcal{L}(E, D(A))\big)$ which is Laplace transformable. Then we obtain from (3.1.6) and (3.1.7), thanks to Example 1.10.10, Theorem 1.10.11, and Remark 1.10.16(b),

$$(\lambda i + A)\widetilde{G}(\lambda) = 1_E\ , \quad \widetilde{G}(\lambda)(\lambda i + A) = i\ , \quad \operatorname{Re}\lambda > \omega_0(G)\ . \tag{3.1.9}$$

Hence $\widetilde{G}(\lambda) = (\lambda + A)^{-1}$ for $\operatorname{Re}\lambda > \omega \vee \omega_0(G)$. Thus the identity theorem for holomorphic functions implies that $\omega_0(G) \leq \omega$, and the uniqueness of G is a consequence of Remark 1.10.16.

By the observations preceding this theorem we know that there exists a unique $G \in \mathcal{L}\mathrm{ap}_+\big(\mathcal{L}(E)\big)$ with $\widetilde{G}(\lambda) = (\lambda + A)^{-1}$ for $\operatorname{Re}\lambda > \omega$. Therefore $\widetilde{G}(\lambda)$ belongs to $\mathcal{L}\big(E, D(A)\big)$ so that $G \in \mathcal{D}'_+\big(\mathcal{L}(E, D(A))\big)$. Hence $\widetilde{G}$ satisfies (3.1.9) for $\operatorname{Re}\lambda > \omega$ and the convolution theorem implies that (3.1.6) and (3.1.7) are valid. ÷

3.1.4 Remark Given $A \in \mathcal{C}(E)$, one can ask for necessary and sufficient conditions for $\partial + A$ to be an isomorphism from $\mathcal{D}'_+\big(D(A)\big)$ onto $\mathcal{D}'_+(E)$. This problem has been solved by J.L. Lions by means of the theory of distribution semigroups in [Lio60]. In that paper it is also shown that condition (3.1.8) is necessary and sufficient for a distribution semigroup to be Laplace transformable. ÷

Fundamental Solutions and Semigroups

Lastly, we consider the particularly important case that $-A$ is the infinitesimal generator of a strongly continuous semigroup. In this case it can be expected that the fundamental solution is related to the semigroup. The next theorem shows that this, indeed, is the case.

3.1.5 Theorem Suppose that $A \in \mathcal{G}(E)$. Then there exists a fundamental solution for $\partial + A$. It is given by the regular distribution e_A, where

$$e_A(t) := \begin{cases} e^{-tA} , & t \geq 0 , \\ 0 , & t < 0 . \end{cases}$$

Thus problem (3.1.1) is well-posed.

Proof The Hille-Yosida theorem implies, in particular, that $[\operatorname{Re} z > \omega] \subset \rho(-A)$ for some $\omega \in \mathbb{R}$ and that

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(E)} \leq M(\operatorname{Re}\lambda - \omega)^{-1} , \qquad \operatorname{Re}\lambda > \omega ,$$

for some $M \geq 1$. Hence condition (3.1.8) is trivially satisøed. Therefore Theorem 3.1.3 guarantees the existence of a unique fundamental solution G for $\partial + A$, and the proof of that theorem shows that $(\lambda + A)^{-1} = \widetilde{G}(\lambda)$ for $\operatorname{Re}\lambda > \omega$. Now we infer from Example 1.10.14(a) and the uniqueness assertion of Remark 1.10.16(a) that $G = e_A$. ÷

Given $(x, g) \in E \times L_{1,\mathrm{loc}}(\mathbb{R}^+, E)$, denote by $g^* \in L_{1,\mathrm{loc}}(\mathbb{R}, E)$ the trivial extension of g over $\mathbb{R}$ and put $f := \delta \otimes x + g^*$. Then $g^* \in \mathcal{D}'_+(E)$, and it follows from Remarks 1.9.6(i) and 1.9.8(c) that $e_A * f$ is the regular distribution given by

$$e_A * f(t) = e^{-tA}x + \int_0^t e^{-(t-\tau)A} g(\tau)\, d\tau , \qquad t \geq 0 .$$

Thus the following result is an immediate consequence of Theorem 3.1.5.

3.1.6 Corollary Suppose that $A \in \mathcal{G}(E)$ and $(x, g) \in E \times L_{1,\mathrm{loc}}(\mathbb{R}^+, E)$. Then the dicerential equation

$$(\partial + A)u = \delta \otimes x + g^* \qquad \text{in } \mathcal{D}'_+(E)$$

is well-posed and its solution $u \in \mathcal{D}'_+\big(D(A)\big)$ is the mild solution of the linear Cauchy problem

$$\dot{u} + Au = g(t) , \quad t > 0 , \qquad u(0) = x .$$

It should be remarked that Laplace transform techniques can also be applied to evolution equations of the form $\partial^k u + Au = f$ for $k \geq 2$, that is, to higher order equations (e.g., [DL90, Chapter XVI]).

3.2 The Gau-Weierstra Semigroup

In this subsection we give a detailed investigation of the Gau-Weierstra semigroup on various spaces of functions and distributions. This semigroup is perhaps the best-known semigroup (of convolutions) and has numerous applications in mathematics. It has the advantage of possessing an explicit analytic representation (through the heat kernel). Moreover, its generator is the Laplace operator, perhaps the most-studied elliptic diœerential operator. The results on the heat semigroup and its generator that we derive in this subsection are fundamental for the study of Bessel potential and Besov spaces.

The Heat Kernel

We deøne the Gau-Weierstra (or heat) kernel $\{\, w_t \,;\, t > 0 \,\}$ on $\mathbb{R}^n$ by

$$w_t(x) := (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} \ , \qquad x \in \mathbb{R}^n \ , \quad t > 0 \ . \tag{3.2.1}$$

Observe that $w_t \in \mathcal{S}(\mathbb{R}^n)$ and $w_t(x) > 0$ for $x \in \mathbb{R}^n$ and $t > 0$.

3.2.1 Lemma $w_t \in L_1(\mathbb{R}^n)$ and $\int w_t\, dx = 1$. Moreover, $w_t = \mathcal{F}^{-1} e^{-t\,|\cdot|^2}$ for $t > 0$.

Proof The ørst part of the assertion has already been obtained in the proof of Lemma 1.3.7.

Thanks to $e^{-t|\cdot|^2} = \sigma_{\sqrt{t}}^{-1} e^{-|\cdot|^2}$ we obtain from (2.3.2) and (2.3.10) that

$$\mathcal{F}^{-1} e^{-t|\cdot|^2} = t^{-n/2} \sigma_{\sqrt{t}} \mathcal{F}^{-1} e^{-|\cdot|^2} = (2\pi)^{-n} t^{-n/2} \sigma_{\sqrt{t}} \mathcal{F} e^{-|\cdot|^2} \ , \tag{3.2.2}$$

since $e^{-|\cdot|^2}$ is even.

Put $f_n := e^{-|\cdot|^2} \in \mathcal{S}(\mathbb{R}^n)$ and note that $f_n = f_1 \otimes \cdots \otimes f_1$ with n factors. Hence, by Fubini's theorem, $\widehat{f_n} = \widehat{f_1} \otimes \cdots \otimes \widehat{f_1}$, and also $\partial f_1 = -2x_1 f_1$. Thus (2.3.3) and (2.3.5) imply $i\xi_1 \widehat{f_1} = \widehat{\partial f_1} = -2\widehat{x_1 f_1} = -2i\partial \widehat{f_1}$. Hence $\widehat{f_1}$ solves the initial value problem

$$\dot{\psi} = -(t/2)\psi \ , \quad \psi(0) = \widehat{f_1}(0) = \int f_1\, dx = \sqrt{\pi} \ .$$

Consequently, $\widehat{f_1}(t) = \sqrt{\pi} e^{-t^2/4}$ for $t \in \mathbb{R}$, so that $\widehat{f_n} = \pi^{n/2} e^{-|\cdot|^2/4}$. Now the last assertion follows from (3.2.2). ÷

The Heat Semigroup

Let $E := (E, |\cdot|)$ be a Banach space. We put $w_0 := \delta$, where δ is the Dirac distribution, and

$$W(t)u := w_t * u \ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E) \ , \qquad t \geq 0 \ .$$

Note that $W(0)u = u$. Then $\{\, W(t) \,;\, t \geq 0 \,\}$ is said to be the Gau-Weierstra (or heat) semigroup. The following theorem justiøes this name. As usual, we denote by

$$\Delta := \Delta_n := \partial_1^2 + \cdots + \partial_n^2 = -\Lambda_0^2(D) \in \mathcal{L}\big(\mathcal{D}'(\mathbb{R}^n, E)\big)$$

the (n-dimensional) Laplace operator.

A map $U : \mathbb{R}^+ \to \mathcal{L}(F)$, where $F := (F, \mathcal{P})$ is a LCS, is locally uniformly bounded if for each $p \in \mathcal{P}$ and $T > 0$ there exist $q \in \mathcal{P}$ and $c > 0$ such that

$$p\big(U(t)f\big) \le cq(f)\ , \qquad f \in F\ , \quad t \in [0, T]\ .$$

3.2.2 Theorem (i) Suppose that $\mathbf{F} \in \{\mathcal{S}, \mathcal{S}'\}$. Then the Gau-Weierstra semigroup is a locally uniformly bounded semigroup on $\mathbf{F}(\mathbb{R}^n, E)$ with inønitesimal generator $\Delta \in \mathcal{L}\big(\mathbf{F}(\mathbb{R}^n, E)\big)$, and

$$W(\cdot)u \in C^\infty\big(\mathbb{R}^+, \mathbf{F}(\mathbb{R}^n, E)\big)\ , \qquad u \in \mathbf{F}(\mathbb{R}^n, E)\ .$$

(ii) Suppose that $\mathbf{F} \in \{\, BUC, C_0, L_p\ ;\ 1 \le p < \infty \,\}$. Then the Gau-Weierstra semigroup restricts from $\mathcal{S}'(\mathbb{R}^n, E)$ to a strongly continuous analytic contraction semigroup on the Banach space $\mathbf{F}(\mathbb{R}^n, E)$, whose inønitesimal generator is the $\mathbf{F}(\mathbb{R}^n, E)$-realization Δ_F of $\Delta \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E)\big)$.

Proof (i) Since $e^{-t\,|\cdot|^2} \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{O}_M(\mathbb{R}^n)$ for $t > 0$ and since $\geqslant \in \mathcal{O}_M(\mathbb{R}^n)$, it follows from (1.9.31) and Lemma 3.2.1 that

$$W(t) = w_t * \; = \mathcal{F}^{-1} e^{-t\,|\cdot|^2} \mathcal{F} \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E)\big)\ , \qquad t \ge 0\ . \tag{3.2.3}$$

Note that $\{\, e^{-t|\xi|^2}\ ;\ t \ge 0 \,\}$ is for each $\xi \in \mathbb{R}^n$ a subsemigroup of the multiplicative group $(\dot{\mathbb{R}}^+, \cdot)$. Hence $\big\{\, W(t)\ ;\ t \ge 0 \,\big\}$ is a semigroup on $\mathcal{S}(\mathbb{R}^n, E)$.

Let

$$a_j(t) := \partial_t^j e^{-t|\cdot|^2} = (-1)^j\, |\cdot|^{2j}\, e^{-t|\cdot|^2}\ , \qquad j \in \mathbb{N}\ , \quad t \ge 0\ .$$

Then, by Taylor's theorem,

$$a_j(t+h) - a_j(t) - a_{j+1}(t)h = h^2 b_j(t,h) \tag{3.2.4}$$

point-wise with respect to $\xi \in \mathbb{R}^n$, where

$$b_j(t,h) := \int_0^1 (1-s)a_{j+2}(t+sh)\,ds$$

for $t \in \mathbb{R}^+$ and $h \in \dot{\mathbb{R}}$ with $t + h \ge 0$. Observe that

$$|\partial_\xi^\alpha a_j(t)(\xi)| \le c(m,T)(1+|\xi|^2)^{j+m} \tag{3.2.5}$$

for $\xi \in \mathbb{R}^n$, $0 \le t \le T < \infty$, and $|\alpha| \le m$. Hence we infer from (3.2.4) that, given $\varphi \in \mathcal{S}(\mathbb{R}^n, E)$ and $j, k, m \in \mathbb{N}$,

$$q_{k,m}\Bigl(h^{-1}\bigl[a_j(t+h) - a_j(t) - a_{j+1}(t)h\bigr]\varphi\Bigr) \le |h|\, c(m,T) q_{\ell,m}(\varphi)$$

for $t \in \mathbb{R}^+$ and $h \in \mathbb{R}$ with $t + h \ge 0$, where $\ell := j + 2 + k + m$. This shows that for $\varphi \in \mathcal{S}(\mathbb{R}^n, E)$, $t \in \mathbb{R}^+$, and $j \in \mathbb{N}$,

$$h^{-1}\bigl[a_j(t+h) - a_j(t)\bigr]\varphi \to a_{j+1}(t)\varphi \qquad \text{in } \mathcal{S}(\mathbb{R}^n, E) \tag{3.2.6}$$

as $h \to 0$. Thanks to $a_j(t) \in \mathcal{O}_M(\mathbb{R}^n)$,

$$a_j(t)(D) = \mathcal{F}^{-1} a_j(t) \mathcal{F} \in \mathcal{L}\bigl(\mathcal{S}(\mathbb{R}^n, E)\bigr) , \qquad t \ge 0 , \quad j \in \mathbb{N} .$$

Consequently, (3.2.6) and (2.3.1) imply that

$$\begin{aligned} \frac{a_j(t+h)(D)\varphi - a_j(t)(D)\varphi}{h} &= \mathcal{F}^{-1}\Bigl(\frac{a_j(t+h) - a_j(t)}{h}\widehat{\varphi}\Bigr) \\ &\to a_{j+1}(t)(D)\varphi \end{aligned} \tag{3.2.7}$$

in $\mathcal{S}(\mathbb{R}^n, E)$ as $h \to 0$ for each $\varphi \in \mathcal{S}(\mathbb{R}^n, E)$, $t \ge 0$, and $j \in \mathbb{N}$, and it follows that

$$a_j(t)(D) = (-1)^j(-\Delta)^j W(t) = \Delta^j W(t) , \qquad t \ge 0 , \quad j \in \mathbb{N} . \tag{3.2.8}$$

From (3.2.7) and (3.2.8) we deduce that

$$W(\cdot)\varphi \in C^\infty\bigl(\mathbb{R}^+, \mathcal{S}(\mathbb{R}^n, E)\bigr) , \qquad \varphi \in \mathcal{S}(\mathbb{R}^n, E) ,$$

and that

$$\partial_t^j W(t)\varphi = \Delta^j W(t)\varphi , \qquad t > 0 , \quad j \in \mathbb{N} , \quad \varphi \in \mathcal{S}(\mathbb{R}^n, E) . \tag{3.2.9}$$

Given $\varphi \in \mathcal{S}(\mathbb{R}^n, E)$ and $k, m \in \mathbb{N}$, (3.2.5) entails

$$q_{k,m}\bigl(a_0(t)\varphi\bigr) \le c(m,T) q_{k+m,m}(\varphi) , \qquad 0 \le t \le T < \infty .$$

Thanks to $W(t)\varphi = \mathcal{F}^{-1}\bigl(a_0(t)\widehat{\varphi}\bigr)$, this implies, together with (2.3.1), that the Gau-Weierstra semigroup is locally uniformly bounded on $\mathcal{S}(\mathbb{R}^n, E)$.

Since $a_j(t) \in \mathcal{O}_M(\mathbb{R}^n)$, it follows from (1.9.31) that $a_j(t)(D) \in \mathcal{L}\bigl(\mathcal{S}'(\mathbb{R}^n, E)\bigr)$. Moreover,

$$a_j(t)(D)u(\varphi) = \mathcal{F}^{-1} a_j(t) \mathcal{F} u(\varphi) = u\bigl(\mathcal{F} a_j(t) \mathcal{F}^{-1}\varphi\bigr) \tag{3.2.10}$$

for $u \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. It is obvious that the estimates leading to (3.2.7) also imply that, given $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}\Bigl(\frac{a_j(t+h) - a_j(t)}{h}\mathcal{F}^{-1}\varphi\Bigr) \to \mathcal{F} a_{j+1}(t) \mathcal{F}^{-1}\varphi$$

as $h \to 0$ in $\mathcal{S}(\mathbb{R}^n)$. Hence it follows from (3.2.10) that

$$\frac{a_j(t+h)(D)u - a_j(t)(D)u}{h}(\varphi) \to a_{j+1}(D)u(\varphi) \qquad \text{as } h \to 0 \tag{3.2.11}$$

for $u \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is a Montel space, the Banach-Steinhaus theorem guarantees that the convergence in (3.2.11) holds uniformly for φ in bounded subsets of $\mathcal{S}(\mathbb{R}^n)$, that is,

$$h^{-1}\big[a_j(t+h)(D)u - a_j(t)(D)u\big] \to a_{j+1}(t)(D)u \qquad \text{in } \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), E\big) = \mathcal{S}'(\mathbb{R}^n, E)$$

as $h \to 0$. This shows that

$$W(\cdot)u \in C^\infty\big(\mathbb{R}^+, \mathcal{S}'(\mathbb{R}^n, E)\big)\ , \qquad u \in \mathcal{S}'(\mathbb{R}^n, E)\ ,$$

and that (3.2.9) is true for $\varphi \in \mathcal{S}'(\mathbb{R}^n, E)$. A similar argument guarantees that the map $t \mapsto W(t)$ from $\mathbb{R}^+$ to $\mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E)\big)$ is locally uniformly bounded. Since $\big\{\, W(t)\ ;\ t \geq 0 \,\big\}$ is a semigroup on $\mathcal{S}'(\mathbb{R}^n, E)$ as well, assertion (i) has been completely proven.

(ii) It follows from Lemma 3.2.1 and Theorem 1.9.9 that $W(t) \in \mathcal{L}(\mathbf{F})$ with $\|W(t)\| \leq 1$, where $\mathbf{F} := \mathbf{F}(\mathbb{R}^n, E)$. Hence we deduce from (i) that $\big\{\, W(t)\ ;\ t \geq 0 \,\big\}$ restricts to a contraction semigroup on $\mathbf{F}$. Next observe that

$$w_\varepsilon(x) = \big(\sqrt{\varepsilon}\big)^{-n} w_1\big(x\big/\sqrt{\varepsilon}\big)\ , \qquad \varepsilon > 0\ , \quad x \in \mathbb{R}^n\ .$$

Since $w_1 \in L_1(\mathbb{R}^n)$ and $\int w_1\, dx = 1$ by Lemma 3.2.1, we see that $\{\, w_{\varepsilon^2}\ ;\ \varepsilon > 0 \,\}$ is an approximate identity. Hence [Ama95, (III.4.2.23)$\check{}$(III.4.2.25)] and the closedness of C_0 in BUC imply that $W(t)u \to u$ in $\mathbf{F}$ as $t \to 0$ for each $u \in \mathbf{F}$. Thus $\big\{\, W(t)\ ;\ t \geq 0 \,\big\}$ is a strongly continuous semigroup on $\mathbf{F}$.

Let A be the inøniteesimal generator of $\big\{\, W(t)\ ;\ t \geq 0 \,\big\}$ on $\mathbf{F}$. Then, given $u \in \operatorname{dom}(A)$, it follows that

$$t^{-1}\big(W(t)u - u\big) \to Au \tag{3.2.12}$$

in $\mathbf{F}$ as $t \to 0$. Since $\mathbf{F} \hookrightarrow \mathcal{S}'(\mathbb{R}^n, E)$ and the left-hand side of (3.2.12) converges in $\mathcal{S}'(\mathbb{R}^n, E)$ towards Δu by (i), we see that $A \subset \Delta_F$.

On the other hand, given $u \in \operatorname{dom}(\Delta_F)$, the strong continuity of W implies $W(\cdot)\Delta u \in C(\mathbb{R}^+, \mathbf{F})$ and, consequently,

$$\int_0^t W(\tau)\Delta u\, d\tau \in \mathbf{F}\ , \qquad t > 0\ .$$

Thus we infer from Fubini's theorem that, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\int_0^t W(\tau)\Delta u\,d\tau(\varphi) &= \int_{\mathcal{R}^n}\int_0^t \bigl(W(\tau)\Delta u\bigr)(x)\varphi(x)\,d\tau\,dx\\ &= \int_0^t W(\tau)\Delta u(\varphi)\,d\tau = \int_0^t u\bigl(\Delta W(\tau)\varphi\bigr)\,d\tau\\ &= \int_0^t u\bigl(\partial_\tau W(\tau)\varphi\bigr)\,d\tau = \int_0^t \partial_\tau\bigl[u\bigl(W(\tau)\varphi\bigr)\bigr]\,d\tau\\ &= u\bigl((W(t)-1)\varphi\bigr) = \bigl(W(t)-1\bigr)u(\varphi)\ ,\end{aligned}$$

where we used the fact that

$$\mathcal{F}^{-1}e^{-t|\cdot|^2}\mathcal{F} = \mathcal{F}e^{-t|\cdot|^2}\mathcal{F}^{-1}$$

by the evenness of $\xi \mapsto e^{-t|\xi|^2}$, and (3.2.9). Consequently,

$$\int_0^t W(\tau)\Delta u\,d\tau = W(t)u - u$$

so that

$$t^{-1}\bigl(W(t)u-u\bigr) = \frac{1}{t}\int_0^t W(\tau)\Delta u\,d\tau \to \Delta u$$

in $\mathbf{F}$ as $t \to 0$. This shows that $u \in \mathrm{dom}(A)$ so that $A = \Delta_F$.

Clearly,

$$\partial_t w_t(x) = \frac{1}{t}\Bigl(-\frac{n}{2}+\frac{|x|^2}{4t}\Bigr)w_t(x) =: \dot w_t(x)\ ,\qquad t>0\ ,\quad x\in\mathbb{R}^n\ .$$

Hence $\|\dot w_t\|_1 \le c(\varepsilon)$ for $t \ge \varepsilon > 0$ and, thanks to Lebesgue's theorem,

$$(t \mapsto \dot w_t) \in C\bigl((0,\infty), L_1(\mathbb{R}^n)\bigr)\ .$$

Thus we infer from Theorem 1.9.9 and Remark 1.9.6(g) that

$$(t \mapsto w_t * u) \in C^1\bigl((0,\infty),\mathbf{F}\bigr)$$

with

$$\partial_t\bigl(W(t)u\bigr) = \partial_t(w_t*u) = \dot w_t * u\ ,\qquad u\in\mathbf{F}\ ,\quad t>0\ . \tag{3.2.13}$$

Consequently,

$$\mathrm{im}\bigl(W(t)\bigr) \subset \mathrm{dom}(A)\ ,\qquad t>0\ . \tag{3.2.14}$$

Note that

$$t\dot w_t = -\frac{n}{2}w_t + v_t \qquad \text{with}\quad v_t(x) := \frac{|x|^2}{4t}w_t(x)\ . \tag{3.2.15}$$

Since

$$v_t = \bigl(\sqrt{4t}\bigr)^{-n}\sigma_{\sqrt{4t}}h \qquad \text{with}\quad h(x) := \pi^{-n/2}\,|x|^2\,e^{-|x|^2}\ ,$$

we see from (2.3.12) that

$$\begin{aligned}\|v_t\|_1 = \|h\|_1 &= \pi^{-n/2}\operatorname{vol}(S^{n-1})\int_0^\infty r^{n+2}e^{-r^2}\,dr/r\\ &= 2^{-1}\pi^{-n/2}\operatorname{vol}(S^{n-1})\Gamma\Big(\frac{n}{2}+1\Big) = n/2\ ,\end{aligned}\tag{3.2.16}$$

since $\operatorname{vol}(S^{n-1}) = n\omega_n$, where

$$\omega_n := \operatorname{vol}(\mathbb{B}^n) = \pi^{n/2}\Big/\Gamma\Big(\frac{n}{2}+1\Big)\ .\tag{3.2.17}$$

Thus we obtain from Lemma 3.2.1 and (3.2.13)ˇ(3.2.16) that

$$\big\|t\partial_t\big(W(t)u\big)\big\| = \|tAW(t)u\| \le n\,\|u\|\ ,\qquad t>0\ .$$

Now Yosida's characterization of analytic semigroups (e.g., [Dav80, Theorem 2.3.9]) implies that $\big\{\,W(t)\ ;\ t\ge 0\,\big\}$ is an analytic semigroup on $\mathbf{F}$. ÷

3.2.3 Remarks (a) Since $e^{-t|\cdot|^2}$ is an even function,

$$W(t)u(\varphi) = u\big(W(t)\varphi\big)\ ,\qquad u\in\mathcal{S}'(\mathbb{R}^n,E)\ ,\quad \varphi\in\mathcal{S}(\mathbb{R}^n)\ ,$$

for $t>0$, as noticed in the above proof.

(b) In the following we often put

$$e^{t\Delta} := W(t)\ ,\qquad t\ge 0\ .$$

Thanks to (3.2.3) this somewhat imprecise notation is consistent with the general notation $m(D) := \mathcal{F}^{-1}m\mathcal{F}$. Due to Theorem 3.2.2(ii) it is also consistent with our general notation $\{\,e^{-tA}\ ;\ t\ge 0\,\}$ for the semigroup generated by $-A$.

(c) Suppose that $a\in\mathcal{O}_M(\mathbb{R}^n)$ with $\inf\operatorname{Re} a\ge\omega>-\infty$. Then $e^{-ta}\in\mathcal{O}_M(\mathbb{R}^n)$ for $t\ge 0$ and

$$e^{-ta(D)} := \mathcal{F}^{-1}e^{-ta}\mathcal{F}\ ,\qquad t\ge 0\ ,$$

is well-deøned. An obvious modiøcation of the proof of Theorem 3.2.2(i) shows that $\{\,e^{-ta(D)}\ ;\ t\ge 0\,\}$ is a locally uniformly bounded semigroup on $\mathcal{S}(\mathbb{R}^n,E)$ and on $\mathcal{S}'(\mathbb{R}^n,E)$ with inønitesimal generator $-a(D)$. Moreover, given $u\in\mathcal{S}(\mathbb{R}^n,E)$ or $u\in\mathcal{S}'(\mathbb{R}^n,E)$, the map $t\mapsto e^{-ta(D)}u$ is smooth from $\mathbb{R}^+$ into $\mathcal{S}(\mathbb{R}^n,E)$ or $\mathcal{S}'(\mathbb{R}^n,E)$, respectively.

The above assumptions are satisøed, for example, if $a(\xi) := i\,\langle h,\xi\rangle$ for $\xi\in\mathbb{R}^n$ and a øxed vector $h := (h^1,\dots,h^n)\in\mathbb{R}^n$, or if $a(\xi) = z\,|\xi|^2$ for $\xi\in\mathbb{R}^n$ and some øxed $z\in[\operatorname{Re} z\ge 0]$. In the ørst case $a(D)=\partial_h$, the directional derivative in the direction h, that is,

$$\partial_h := \sum_{j=1}^n h^j\partial_j\ ,$$

and $e^{-ta(D)} = e^{-t\partial_h} = \tau_{th}$, where τ_{th} denotes right translation by the vector th (cf. [Ama95, (III.4.2.6))]. In the second case, letting in particular $z := i$, we obtain the Schrdinger semigroup.

(d) Let $F := (F, \mathcal{P})$ be an LCS and suppose that $\{ U(t) \ ; \ t \geq 0 \}$ is a locally uniformly bounded semigroup on F such that $U(\cdot)f \in C^1(\mathbb{R}^+, F)$ for each $f \in F$. Then, deøning A by $Af = \partial_t\big(U(t)f\big)\big|_{t=0}$ for $f \in F$, it follows that $A \in \mathcal{L}(F)$ and that the Cauchy problem

$$\dot{u} = Au \ , \quad t > 0 \ , \qquad u(0) = f$$

has for each $f \in F$ a unique solution $u \in C^1(\mathbb{R}^+, F)$, namely $u(\cdot) = U(\cdot)f$.

This follows by mimicking the corresponding proof for the case that F is a Banach space, $-A \in \mathcal{G}(F)$, and $f \in \mathrm{dom}(A)$.

(e) From Theorem 3.2.2 and (d) we infer that, given $u^0 \in \mathcal{S}'(\mathbb{R}^n, E)$, the Cauchy problem

$$\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times \dot{\mathbb{R}}^+ \ , \qquad u(\cdot, 0) = u^0 \tag{3.2.18}$$

has a unique solution $u(\cdot, t, u^0)$ with

$$\big[t \mapsto u(\cdot, t, u^0)\big] \in C^\infty\big(\mathbb{R}^+, \mathcal{S}'(\mathbb{R}^n, E)\big) \ ,$$

and that it is given by $u(\cdot, t, u^0) = w_t * u^0$ for $t > 0$. Since $\partial_t^j w_t \in \mathcal{S}(\mathbb{R}^n)$ for $t > 0$ and $j \in \mathbb{N}$, Proposition 1.2.7 and Remark 1.9.6(g) show that

$$\partial_t^j u(\cdot, t, u^0) \in \mathcal{O}_M(\mathbb{R}^n, E) \ .$$

From this one easily derives

$$\big[(x, t) \mapsto u(x, t, u^0)\big] \in C^\infty(\mathbb{R}^n \times \dot{\mathbb{R}}^+, E) \tag{3.2.19}$$

for $u^0 \in \mathcal{S}'(\mathbb{R}^n, E)$. This shows that the Gau-Weierstra semigroup possesses a rather strong 'smoothing property'.

Recall that $\mathcal{E}'(\mathbb{R}^n, E) \hookrightarrow \mathcal{S}'(\mathbb{R}^n, E)$. Thus the Cauchy problem (3.2.18) has for each initial value u^0 that is a distribution with compact support the unique solution $u(\cdot, t, u^0) \in \mathcal{S}'(\mathbb{R}^n, E)$, and it satisøes (3.2.19). In particular, we can choose $u^0 = \delta$ and ønd that the heat kernel is the unique temperate solution of the heat equation

$$\partial_t u = \Delta u \quad \text{on } \mathbb{R}^n \ , \qquad t > 0 \ ,$$

satisfying $u(0) = \delta$. In other words, w_t is for $t > 0$ the temperature distribution evolving according to the heat equation from a unit-strength point-source at $(x, t) = (0, 0)$. The fact that w_t is smooth (indeed: analytic) for $t > 0$ reÆects the property of inønite speed of propagation of the heat equation.

(f) The elements of $\mathcal{S}'(\mathbb{R}^n)$ are slowly increasing distributions. More precisely, $u \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $\mathcal{S}'(\mathbb{R}^n)$ iœ there exist $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $f \in BC(\mathbb{R}^n)$

such that $u = \partial^\alpha(\Lambda_1^k f)$ (cf. [Sch66, Chap. VII, Thorme VI]). Thus, given any continuous function u^0 on $\mathbb{R}^n$ growing at most polynomially, the initial value problem (3.2.18) has a unique solution $u(\cdot, u^0)$ that is smooth and such that $u(t, u^0)$ grows at most polynomially in x for each $t > 0$. If we drop the requirement that $u(t, u^0) \in \mathcal{S}'(\mathbb{R}^n)$ for $t \geq 0$, uniqueness can no longer be guaranteed. In fact, Tychonov constructed a classical solution of the heat equation vanishing for $t = 0$ and growing faster than exponentially in x (cf. [Hel60, I.4.3] or [DiB95, V.4.1]). The same construction also shows that the backward Cauchy problem

$$\partial_t u = \Delta u \ , \quad t < 0 \ , \qquad u(\cdot, 0) = u^0$$

is not well-posed. ÷

3.3 Bessel Potentials

Since $-\Delta$ generates the Gau-Weierstra semigroup on the classical function spaces we can deøne the fractional powers $(\rho^2 - \Delta)^z$ of $\rho^2 - \Delta$ by means of the general theory developed in [Ama95, Subsection III.6]. On the other hand, we can also deøne $(\rho^2 - \Delta)^z$ naturally by means of Fourier transforms. It is reasonable to expect that these two deønitions give the same result. That this is indeed the case, at least if $\operatorname{Re} z \neq 0$, is proven in Proposition 3.3.2. If we assume that E is a UMD space then we can show that this equality prevails for every $z \in \mathbb{C}$. More importantly, we show in Theorem 3.3.3 that $\rho^2 - \Delta$ has bounded imaginary powers. Lastly, we complement these results by some classical representation theorems.

Bessel Kernels

Let $E := (E, |\cdot|)$ be a Banach space. In the following we put

$$J_\rho^z := \Lambda_\rho^z(D) \ , \quad J^z := J_1^z \ , \qquad z \in \mathbb{C} \ , \quad \rho > 0 \ . \tag{3.3.1}$$

Since $\Lambda_\rho^z \in \mathcal{O}_M(\mathbb{R}^n)$ by Example 2.4.7(b), it follows that

$$J_\rho^z \in \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E)\big) \cap \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E)\big) \ . \tag{3.3.2}$$

It is also obvious that

$$J_\rho^{z_1+z_2} = J_\rho^{z_1} J_\rho^{z_2} \ , \quad J_\rho^0 = \mathrm{id} \ , \qquad z_1, z_2 \in \mathbb{C} \ , \quad \rho > 0 \ , \tag{3.3.3}$$

so that $\{\, J_\rho^z \ ; \ z \in \mathbb{C} \,\}$ is a group of continuous automorphisms of $\mathcal{S}(\mathbb{R}^n, E)$ and of $\mathcal{S}'(\mathbb{R}^n, E)$, respectively.

Noting that $\Lambda_\rho^z = \rho^z \sigma_\rho \Lambda_1^z$, it follows from (2.3.9) and (2.3.10) that

$$J_\rho^z = \rho^z \sigma_\rho^{-1} J^z \sigma_\rho \ , \qquad z \in \mathbb{C} \ , \quad \rho > 0 \ . \tag{3.3.4}$$

Observe that $J_\rho^2 = \rho^2 - \Delta$. Hence it is natural to put

$$(\rho^2 - \Delta)^{z/2} := J_\rho^z \ , \qquad z \in \mathbb{C} \ , \quad \rho > 0 \ . \tag{3.3.5}$$

Then (3.3.3) implies that $(\rho^2 - \Delta)^k$ are the ordinary powers of

$$\rho^2 - \Delta \in \mathcal{L}\bigl(\mathcal{S}(\mathbb{R}^n, E)\bigr) \cap \mathcal{L}\bigl(\mathcal{S}'(\mathbb{R}^n, E)\bigr)$$

for $k \in \mathbb{Z}$. Thus (3.3.5) is a natural way of deøning the powers of $\rho^2 - \Delta$ for every $z \in \mathbb{C}$.

Now let

$$\mathbf{F} \in \{\, BUC, C_0, L_p \ ; \ 1 \le p < \infty \,\} \tag{3.3.6}$$

and let Δ_F be the $\mathbf{F}(\mathbb{R}^n, E)$-realization of $\Delta \in \mathcal{L}\bigl(\mathcal{S}'(\mathbb{R}^n, E)\bigr)$. Then, Δ_F being the inønitesimal generator of the Gau-Weierstra semigroup on $\mathbf{F}(\mathbb{R}^n, E)$, it follows that $\rho^2 - \Delta_F$ is of positive type for $\rho > 0$. Hence the fractional powers $(\rho^2 - \Delta_F)^z$ are well-deøned by [Ama95, Theorem III.4.6.5]. In the following we show that $(\rho^2 - \Delta_F)^z$ equals the $\mathbf{F}(\mathbb{R}^n, E)$-realization $\bigl[(\rho^2 - \Delta)^z\bigr]_F$ of

$$(\rho^2 - \Delta)^z = J_\rho^{2z} \in \mathcal{L}\bigl(\mathcal{S}'(\mathbb{R}^n, E)\bigr)$$

for $\rho > 0$ and $\operatorname{Re} z \ne 0$. This shows that (3.3.5) is also consistent with the theory of fractional powers developed in [Ama95, Subsection III.4.6], provided $\operatorname{Re} z \ne 0$. To prove this we need some preparation.

We deøne the Bessel potentials or Bessel kernels $\{\, G_\rho^z \ ; \ \operatorname{Re} z > 0 \,\}$ for $\rho > 0$ by

$$G_\rho^z(x) := \frac{1}{\Gamma(z/2)} \int_0^\infty t^{z/2} e^{-t\rho^2} w_t(x)\, dt/t \ , \qquad x \in \mathbb{R}^n \ . \tag{3.3.7}$$

Note that $G_\rho^z(x) > 0$ for $z > 0$.

3.3.1 Lemma $G_\rho^z = \mathcal{F}^{-1}\Lambda_\rho^{-z} \in L_1(\mathbb{R}^n)$ with $\|G_\rho^z\|_1 \le \rho^{-\operatorname{Re} z} \bigl|\Gamma(\operatorname{Re} z/2)/\Gamma(z/2)\bigr|$. If $z > 0$ then $\|G_\rho^z\|_1 = \int G_\rho^z \, dx = \rho^{-z}$.

Proof Observe that, trivially, $\Lambda_\rho^2(\xi) \in \mathcal{G}(\mathbb{C}\,; 1, -\rho^2)$ for $(\xi, \rho) \in \mathbb{R}^n \times \mathbb{R}$. Thus Theorem III.4.6.6 of [Ama95] implies

$$\Lambda_\rho^{-2z}(\xi) = \frac{1}{\Gamma(z)} \int_0^\infty t^z e^{-t\rho^2} e^{-t|\xi|^2}\, dt/t \ , \qquad (\xi, \rho) \in \mathbb{R}^n \times \ddot{\mathbb{R}} \ ,$$

for $\operatorname{Re} z > 0$. Since $\Lambda_\rho^{-2z} \in L_\infty(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ for $\rho > 0$ and $\operatorname{Re} z > 0$, Fubini's theorem guarantees that, given $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \Lambda_\rho^{-2z}, \varphi \rangle = \frac{1}{\Gamma(z)} \int_0^\infty t^z e^{-t\rho^2} \bigl\langle e^{-t\,|\cdot|^2}, \varphi \bigr\rangle\, dt/t \ .$$

By replacing φ by $\mathcal{F}^{-1}\varphi$ and observing Lemma 3.2.1 we infer that

$$\begin{aligned}\left\langle \mathcal{F}^{-1}\Lambda_\rho^{-2z}, \varphi\right\rangle = \left\langle \Lambda_\rho^{-2z}, \mathcal{F}^{-1}\varphi\right\rangle &= \frac{1}{\Gamma(z)}\int_0^\infty t^z e^{-t\rho^2}\left\langle e^{-t\,|\cdot|^2}, \mathcal{F}^{-1}\varphi\right\rangle dt/t \\ &= \frac{1}{\Gamma(z)}\int_0^\infty t^z e^{-t\rho^2}\langle w_t, \varphi\rangle\, dt/t = \langle G_\rho^{2z}, \varphi\rangle\end{aligned}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n)$, where in the last step we used Fubini's theorem once more. This proves that $\mathcal{F}^{-1}\Lambda_\rho^{-2z} = G_\rho^{2z}$ for $\operatorname{Re} z > 0$ and $\rho > 0$.

Since $w_t \in \mathcal{S}(\mathbb{R}^n)$ for $t > 0$ and $\int w_t\, dx = 1$ by Lemma 3.2.1, Fubini's theorem implies

$$\|G_\rho^{2z}\|_1 \le \frac{1}{\Gamma(z)}\int_0^\infty t^{\operatorname{Re} z} e^{-t\rho^2}\, dt/t = \rho^{-2\operatorname{Re} z}$$

for $\operatorname{Re} z > 0$ and $\rho > 0$, and $\|G_\rho^z\|_1 = \rho^{-z}$ if $z > 0$. ÷

Powers of $1 - \Delta$

3.3.2 Proposition Suppose that $\rho > 0$. Then $\left[(\rho^2 - \Delta)^z\right]_F \supset (\rho^2 - \Delta_F)^z$ and $\left[(\rho^2 - \Delta)^z\right]_F = (\rho^2 - \Delta_F)^z$ for $\operatorname{Re} z \neq 0$.

Proof If $\operatorname{Re} z > 0$ then the fact that $\Lambda_\rho^{-2z} \in \mathcal{O}_M(\mathbb{R}^n)$, Lemma 3.3.1, and Remark 1.9.11(b) imply

$$J_\rho^{-2z} u = G_\rho^{2z} * u\ , \qquad u \in \mathbf{F} := \mathbf{F}(\mathbb{R}^n, E)\ . \tag{3.3.8}$$

Thus we infer from $G_\rho^{2z} \in L_1(\mathbb{R}^n)$ and Theorem 1.9.9 that

$$(J_\rho^{-2z})_F = \left[(\rho^2 - \Delta)^{-z}\right]_F = J_\rho^{-2z}\,\big|\,\mathbf{F} \in \mathcal{L}(\mathbf{F})\ , \qquad \operatorname{Re} z > 0\ . \tag{3.3.9}$$

By means of Fubini's theorem we obtain from (3.3.7) and (3.3.8) that

$$J_\rho^{-2z} u = \frac{1}{\Gamma(z)}\int_0^\infty t^z e^{-t\rho^2} w_t * u\, dt/t\ , \qquad \operatorname{Re} z > 0\ .$$

Thus, thanks to Theorems 3.2.2 and [Ama95, III.4.6.6],

$$J_\rho^{-2z} u = \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1} e^{-t(\rho^2 - \Delta_F)} u\, dt = (\rho^2 - \Delta_F)^{-z} u\ , \qquad \operatorname{Re} z > 0\ , \tag{3.3.10}$$

for $u \in \mathbf{F}$. Now it follows from (3.3.9) and (3.3.10) that the assertion is true for $\operatorname{Re} z < 0$.

Suppose that $\operatorname{Re} z > 0$ and $u \in \operatorname{dom}\big((J_\rho^{2z})_F\big)$. Then $u \in \mathbf{F}$ and $v := J_\rho^{2z} u \in \mathbf{F}$. Hence

$$(\rho^2 - \Delta_F)^{-z} v = (J_\rho^{-2z})_F v = J_\rho^{-2z} J_\rho^{2z} u = u\ ,$$

that is, $u \in \mathrm{dom}\big((\rho^2-\Delta_F)^z\big)$ and $(\rho^2-\Delta_F)^z u = v = \big[(\rho^2-\Delta)^z\big]_F u$. Thus

$$(\rho^2-\Delta_F)^z \supset \big[(\rho^2-\Delta)^z\big]_F\ , \qquad \mathrm{Re}\, z > 0\ .$$

Since $(\rho^2-\Delta_F)^z$ is injective and $(J_\rho^{2z})_F$ is surjective, [Ama95, Lemma V.1.4.2] implies that the assertion is true for $\mathrm{Re}\, z > 0$ as well.

Lastly, suppose that $\mathrm{Re}\, z = 0$. Then [Ama95, Theorem III.4.6.5(ii)] and what has already been proven show that

$$\begin{aligned}(\rho^2-\Delta_F)^z u &= \frac{\sin \pi z}{\pi z}\int_0^\infty s^z\big(s+(\rho^2-\Delta)_F\big)^{-2}(\rho^2-\Delta)_F u\, ds\\ &= \frac{\sin \pi z}{\pi z}\int_0^\infty s^z (s+J_\rho^2)^{-2} J_\rho^2 u\, ds\end{aligned} \tag{3.3.11}$$

for $u \in \mathrm{dom}(\rho^2-\Delta_F)$. Since Λ_ρ^2 and $(s+\Lambda_\rho^2)^{-2}$ belong to $\mathcal{O}_M(\mathbb{R}^n)$, we see that

$$(s+J_\rho^2)^{-2}J_\rho^2 = \mathcal{F}^{-1}(s+\Lambda_\rho^2)^{-2}\Lambda_\rho^2\mathcal{F}\ .$$

Given any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}^n$, it is easily veriøed that

$$\big\|\varphi\partial^\alpha\big[(s+\Lambda_\rho^2)^{-2})\Lambda_\rho^2\big]\big\|_\infty \le c(\alpha,\varphi,\rho)(s+\rho^2)^{-2}\ , \qquad s > 0\ .$$

This implies that the integral

$$\int_0^\infty s^z(s+\Lambda_\rho^2)^{-2}\Lambda_\rho^2\, ds \tag{3.3.12}$$

converges absolutely in $\mathcal{O}_M(\mathbb{R}^n)$. Using these facts and the continuity of $\mathcal{F}^{-1}$ in $\mathcal{S}'(\mathbb{R}^n, E)$, it follows that

$$\begin{aligned}\int_0^\infty s^z(s+J_\rho^2)^{-2}J_\rho^2 u\, ds &= \int_0^\infty \mathcal{F}^{-1}\big(s^z(s+\Lambda_\rho^2)^{-2}\Lambda_\rho^2\widehat{u}\big)\, ds\\ &= \mathcal{F}^{-1}\int_0^\infty s^z(s+\Lambda_\rho^2)^{-2}\Lambda_\rho^2\, ds\, \mathcal{F}u = \frac{\pi z}{\sin \pi z}\mathcal{F}^{-1}\Lambda_\rho^{2z}\mathcal{F}u\end{aligned}$$

for $u \in \mathbf{F} \hookrightarrow \mathcal{S}'(\mathbb{R}^n, E)$, where the last equality follows from Theorem III.4.6.5(ii) of [Ama95] as well. Hence we obtain from (3.3.11) that

$$(\rho^2-\Delta_F)^z u = \big[(\rho^2-\Delta)^z\big]_F u\ , \qquad u \in \mathrm{dom}(\rho^2-\Delta_F) = \mathrm{dom}(\Delta_F)\ . \tag{3.3.13}$$

From the deønition of $(\rho^2-\Delta_F)^z$ in [Ama95, (III.4.6.21)] we know that $\mathrm{dom}(\Delta_F)$ is a core for $(\rho^2-\Delta_F)^z$. Thus, since $\big[(\rho^2-\Delta)^z\big]_F$ is closed in $\mathbf{F}$, it follows from (3.3.13) that $\big[(\rho^2-\Delta)^z\big]_F \supset (\rho^2-\Delta_F)^z$ for $\mathrm{Re}\, z = 0$. ÷

After these preparations we can prove the main result of this subsection.

3.3.3 Theorem Let E be a UMD space and let Δ_p be the $L_p(\mathbb{R}^n, E)$-realization of $\Delta \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E)\big)$. Then $\rho^2 - \Delta_p \in \mathcal{BIP}\big(L_p(\mathbb{R}^n, E)\big)$ for $1 < p < \infty$ and $\rho > 0$, and

$$\|(\rho^2 - \Delta_p)^{it}\| \le c(n)(1 + |t|^{n+1}) , \qquad t \in \mathbb{R} , \quad \rho > 0 .$$

Proof We infer from Example 2.4.7(a) and Remark 2.4.6(a) that

$$\Lambda_\rho^{it} \in S_\rho^0(\mathbb{R}^n) \qquad \text{with} \quad \|\Lambda_\rho^{it}\|_{S_\rho^0} \le c(n)(1 + |t|^{n+1}) , \qquad t \in \mathbb{R} .$$

Hence (2.4.7) and [Ama95, Theorem III.4.4.3] imply

$$J_\rho^{it} = \big[(\rho^2 - \Delta)^{it}\big]_{L_p(\mathcal{R}^n, E)} \in \mathcal{L}\big(L_p(\mathbb{R}^n, E)\big) , \qquad t \in \mathbb{R} , \quad \rho > 0 ,$$

with the norm bounded by $c(n)(1 + |t|^{n+1})$. Consequently, by Proposition 3.3.2,

$$\|(\rho^2 - \Delta_p)^{it} u\|_p = \|J_\rho^{it} u\|_p \le c(n)(1 + |t|^{n+1}) \, \|u\|_p , \qquad \rho > 0 ,$$

for $u \in \mathrm{dom}(\Delta_p)$ and $t \in \mathbb{R}$. Since $\mathrm{dom}(\Delta_p)$ is dense in $L_p(\mathbb{R}^n, E)$, the assertion follows. ÷

Fourier Transforms of Radial Functions

We close this subsection by giving another representation of the Bessel potentials, at least for $\mathrm{Re}\, z > n$, explaining the name. For this we ørst prove a general theorem on Fourier transforms.

Recall that the Bessel function J_ν (of the ørst kind) of index $\nu \in \mathbb{C} \setminus (-\ddot{\mathbb{N}})$ is deøned by

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{n=0}^\infty \frac{(-1)^n}{n!\, \Gamma(n + \nu + 1)} \Big(\frac{z}{2}\Big)^{2n} , \qquad z \in \mathbb{C} , \tag{3.3.14}$$

where we use the principal value of the power z^ν. If $\mathrm{Re}\, \nu > -1/2$, the integral representation

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\pi (\sin\vartheta)^{2\nu} e^{-iz\cos\vartheta} \, d\vartheta , \qquad z \in \mathbb{C} , \tag{3.3.15}$$

is valid, and from (3.3.14) it follows easily that

$$J_{-1/2}(z) = \frac{\sqrt{2}\cos z}{\sqrt{\pi z}} , \qquad z \in \mathbb{C} , \tag{3.3.16}$$

(cf. [Wat44, Section 3.3] or [Sch65, Chap. IX]).

Now we compute the Fourier transform of a radial function. That is, we assume that

$$f(x) = F(|x|) \ , \qquad x \in \mathbb{R}^n \ ,$$

for some $F : \mathbb{R}^+ \to E$. Fubini's theorem implies that

$$f \in L_1(\mathbb{R}^n, E) \quad \Longleftrightarrow \quad F \in L_1(\mathbb{R}^+, r^{n-1}\,dr; E) \ . \tag{3.3.17}$$

First suppose that $n = 1$. Then, by Euler's formula for the exponential function and since sin is odd,

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} F(|x|)\,dx = 2\int_0^{\infty} \cos(|\xi|\,r) F(r)\,dr \ , \qquad \xi \in \mathbb{R} \ . \tag{3.3.18}$$

If $n \geq 2$,

$$\widehat{f}(\xi) = \int_0^{\infty} \int_{|x|=r} F(r) e^{-i\langle \xi, x\rangle}\,d\sigma(x)\,dr \ , \qquad \xi \in \mathbb{R}^n \ , \tag{3.3.19}$$

where $d\sigma$ is the volume measure of rS^{n-1}. If $\xi \neq 0$, we introduce polar coordinates in the inner integral such that $\xi/|\xi|$ is the north pole and ϑ is the angle between x and ξ. Then

$$\int_{|x|=r} e^{-i\langle \xi, x\rangle}\,d\sigma(x) = r\int_0^{\pi} e^{-i|\xi| r\cos\vartheta}\,\mathrm{vol}(r\sin\vartheta S^{n-2})\,d\vartheta \ .$$

Since

$$\begin{aligned}\mathrm{vol}(r\sin\vartheta S^{n-2}) &= r^{n-2}(\sin\vartheta)^{n-2}(n-1)\omega_{n-1} \\ &= 2r^{n-2}(\sin\vartheta)^{n-2}\pi^{(n-1)/2}\Big/\Gamma\Big(\frac{n-1}{2}\Big) \ ,\end{aligned}$$

thanks to (3.2.17), and since

$$\Gamma\Big(\frac{n+1}{2}\Big) = \frac{n-1}{2}\Gamma\Big(\frac{n-1}{2}\Big) \ ,$$

it follows that

$$\int_{|x|=r} e^{-i\langle \xi, x\rangle}\,d\sigma(x) = \frac{2r^{n-1}\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\int_0^{\pi} (\sin\vartheta)^{n-2} e^{-i|\xi| r\cos\vartheta}\,d\vartheta \ . \tag{3.3.20}$$

Now the next theorem is almost obvious.

3.3.4 Theorem Suppose that $f \in L_1(\mathbb{R}^n, E)$ with $f(x) = F(|x|)$ for $x \in \mathbb{R}^n$. Then

$$\widehat{f}(\xi) = \frac{(2\pi)^{n/2}}{|\xi|^{(n-2)/2}} \int_0^{\infty} F(r) r^{n/2} J_{(n-2)/2}(|\xi|\,r)\,dr \ , \qquad \xi \in \mathbb{R}^n \ .$$

Proof This follows from (3.3.15)$^\vee$(3.3.20). ÷

3.3.5 Corollary Suppose that $\rho > 0$ and $\operatorname{Re} z > n$. Then

$$G_\rho^z(x) = (2\pi)^{-n/2} \, |x|^{(2-n)/2} \int_0^\infty (\rho^2 + r^2)^{-z/2} r^{n/2} J_{(n-2)/2}(|x| \, r) \, dr \ .$$

Proof Since $\Lambda_\rho^{-z} \in L_1(\mathbb{R}^n)$ for $\operatorname{Re} z > n$, the assertion follows from Lemma 3.3.1, Theorem 3.3.4, and (2.3.2). ÷

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Index