Poincaré's Discovery of Homoclinic Points

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The modern theory of dynamical systems begins with HENRI POINCARÉ. He introduced global, topological methods into the study of non-linear differential equations and he invented general procedures for finding periodic solutions. For Hamiltonian systems he obtained results about non-existence of first integrals and proved the well-known recurrence theorem. However, the most radical break with prevailing conceptions was his discovery of homoclinic points, which nowadays figure so prominently in studies of "chaotic" motions. The presence of a (transverse) homoclinic point in a dynamical system complicates the orbit structure considerably and implies the existence of trajectories with quite unpredictable long-time behaviour. POINCARÉ first encountered homoclinic points in 1889 in connection with his work on the three-body problem of celestial mechanics. The discovery was made under rather dramatic circumstances just as the printing was completed of the famous memoir for which he had been awarded a prize by the Swedish king OSCAR II. This incident is briefly alluded to in the introduction to POINCARÉ's paper [1890], but for various reasons the immediate circumstances of the affair were carefully hushed up. At the Mittag-Leffler Institute outside Stockholm are kept documents from which one can obtain a clear picture of what actually happened. The purpose of this article is to give publicity to these documents and thereby throw some light on this interesting episode. In particular it will became clear what POINCARÉ originally was aiming at in his attack on the three-body problem.

1. The Prize of Oscar II

In 1882 a new mathematical journal was founded in Sweden, Acta Mathematica. It was an ambitious enterprise directed by GÖSTA MITTAG-LEFFLER, professor at Stockholms Högskola. Patron of the project was King OSCAR II. The journal was a success from the beginning. In no small measure this was due to POINCARÉ, who contributed a number of important papers to the first volumes. In the spring of 1884 MITTAG-LEFFLER began to cultivate a new plan, designed to draw attention to Acta Mathematica. A prize intended to encourage discoveries in higher mathematical analysis was to be established and the king's name used to give prestige to the prize. The king gave his consent and negotiations started about the detailed arrangement. In particular MITTAG-LEFFLER relied on his old teacher WEIERSTRASS for advice. At first WEIERSTRASS was opposed to the idea of set prize questions and wanted to reward discoveries made in the past. However, in the end it was agreed that a committee consisting of HERMITE, WEIERSTRASS and MITTAG-LEFFLER should formulate the questions and also evaluate the memoirs competing for the prize. On March 28, 1885 MITTAG-LEFFLER could write to WEIERSTRASS:[1]

> At last everything is set about the prize. The King will receive me in audience on April 7 and has ordered me to present him with the proposal of the committee for him to sign. I send you now a draft of this proposal, with which I hope that you will be satisfied. I have chosen two of your questions and two of Mr. Hermite's. . . .

The first question, which is the one that is relevant in the present context, was suggested by WEIERSTRASS and concerns a system of point masses moving according to NEWTON's law of gravitation. Under the assumption that collisions do not occur, it was asked to represent the motion in terms of uniformly convergent series involving only known functions of time. If this question was not settled before the end of the contest, the prize could be given to a work containing the solution of another important problem of mechanics.

The prize was first announced in a circular distributed in July 1885 and then in volume 7 of Acta Mathematica. It was to be awarded on the 21st of January 1889, the sixtieth birthday of the king. Papers entering the contest should be sent to the editor of Acta Mathematica before June 1, 1888. On July 13, 1887 MITTAG-LEFFLER writes to POINCARÉ:

> . . . I take the liberty to remind you that the memoirs designed for the prize of king Oscar shall be sent to me before June 1, 1888. I dare hope that you will be kind enough to send something. As you know any memoir on the theory of functions may compete. If you will send something it is hardly probable that anybody will surpass you. . . .

POINCARÉ's reply, dated July 16, is printed in volume 38 of Acta Mathematica. He writes that he has not forgotten the prize. On the contrary, he has been exclusively occupied with it for the last couple of months. He is working on the first question and reports that for the restricted body problem he has found a rigorous proof of stability, giving precise bounds for the elements of the three bodies. He adds that he has come to the conclusion that it is hopeless to try to find the solutions of the problem in terms of known functions. In Acta Mathematica this letter is accompanied by a note relating the statement about stability to what is now known as POINCARÉ's recurrence theorem. As will be seen below, this note is very misleading.

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[1] All letters quoted without a reference are from the archives of the Mittag-Leffler Institute. They have been translated into English by me.

On December 25, POINCARÉ writes again to MITTAG-LEFFLER:

> . . . I am constantly working on the first question, but in this problem one meets with very great difficulties. This work and the preparation of my course on optics takes all my time. . . .

Finally on May 17, 1888 he sends in his memoir for the competition.

In the middle of November MITTAG-LEFFLER confidentially informs POINCARÉ that he will get the prize. The committee is very impressed with his work, but WEIERSTRASS wants him to clarify a few points. On January 21, 1889 it was officially announced that the prize was given to POINCARÉ. At the same time a gold medal and an honorable mention was awarded to PAUL APPELL for a memoir entitled: *Sur les intégrales de fonctions à multiplicateurs et leur application au développement des fonctions abéliennes en séries trigonometrique.* On March 23, Count LEWENHAUPT, Swedish ambassador to France, presented POINCARÉ with the prize consisting of 2500 Swedish crowns and a gold medal bearing the portrait of the king.

2. Poincaré's paper. The first printed version

The printing of POINCARÉ's paper was carried out between July 3 and November 13, 1889. To the original manuscript was added a large number of explanatory notes. As will be seen in the next section, these notes were composed by POINCARÉ in response to various questions concerning his manuscript. In the present section I shall briefly describe the contents of the paper printed in November 1889.

In the introduction is mentioned the classical work of LAPLACE and POISSON concerning the stability of the solar system and it is pointed out that their results are valid only in an approximate sense. Then various formal solutions by means of trigonometric series are considered and POINCARÉ claims that he can prove that most of these series are divergent. As a contrast CAUCHY's method of majorants is put forward as a tool for proving convergence of power series solutions. POINCARÉ then refers to the geometrical methods that he has developed in four big papers on differential equations in Journal de Mathématiques [1881–1886]. He claims that using these methods he can rigorously prove stability for the restricted three-body problem and other problems of mechanics having only two degrees of freedom. Finally he mentions that he has a general theory for periodic solutions and that he has invented a new concept, *integral invariants*, which plays a crucial rôle in connection with applications of the geometrical method. It is worth noticing that the theorem about non-existence of first integrals is mentioned nowhere in this introduction.

The body of the paper consists of two parts. In the first part certain general principles are developed which are then applied in the second part mainly to systems with two degrees of freedom. The first part has three chapters. It starts with a brief chapter containing some basic definitions. The second chapter is devoted to integral invariants. These are differential forms whose integrals over suitable manifolds preserve their value when the manifolds are transported by the flow. Important examples discussed by POINCARÉ are the symplectic two-form and the volume in phase space. That the volume is an integral invariant for Hamiltonian systems had been noted before by LIOUVILLE and BOLTZMANN, but POINCARÉ makes use of this fact in new and ingenious ways. As a first application he proves the recurrence theorem, which is stated roughly as follows. Suppose that the volume is an integral invariant and that R is a domain with finite volume which is made up of complete trajectories. Then, for every subdomain r_0 of R, there is a trajectory which passes r_0 an infinite number of times. This result is followed by a number of other applications, which are mainly of interest in connection with the problems discussed in the second part of the paper. We shall return to integral invariants in that context.

The third chapter deals with periodic solutions and solutions that are asymptotic to periodic solutions. Here POINCARÉ describes his continuation method for obtaining periodic solutions of systems of differential equations depending on a parameter μ. Given a periodic solution for $\mu = 0$ he proves, using the implicit function theorem, that there is a periodic solution also for small $\mu \neq 0$, provided that some appropriate functional determinant does not vanish. To compute such functional determinants one must study the so called equation of variation corresponding to the periodic solution for $\mu = 0$. After having recalled some general facts about equations of variation and characteristic exponents, POINCARÉ specializes to Hamiltonian systems which he writes

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \qquad i = 1, \ldots, p .$$

He assumes that F depends on a small parameter μ and can be expanded in a convergent series

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \cdots$$

where the coefficients are analytic functions such that F_0 is independent of $y_1, \ldots, y_n$ and $F_1, F_2, \ldots$ are 2π-periodic in each of the y-variables. The y:s are angular variables which are only determined up to integer multiples of 2π. This is the standard form for a perturbation of an integrable system. For simplicity we consider only $p = 2$, which is the case discussed in the second part of the paper. For $\mu = 0$, the solutions then have the form

$$x_i = x_i^0, \qquad y_i = n_i t + y_i^0, \quad \text{where} \quad n_i = -\frac{dF_0}{dx_i}(x_1^0, x_2^0), \quad i = 1, 2 .$$

The corresponding trajectories describe windings on a torus given by $x_1 = x_1^0$, $x_2 = x_2^0$. They close up and represent periodic motions if and only if n_1 and n_2 are commensurable. In this situation one may always, by means of a simple change of variables, obtain that $n_2 = 0$. Suppose thus that x_1^0 and x_2^0 are given so that $n_2 = 0$. On the corresponding torus only y_1 varies along the trajectories. Define $\psi(y_2)$ as the mean value of F_1 along the trajectory corresponding to y_2, *i.e.*

$$\psi(y_2) = \frac{1}{2\pi} \int_0^{2\pi} F_1(x_1^0, x_2^0, y_1, y_2)\, dy_1 \; .$$

Then POINCARÉ proves that, in general, local maxima and minima of ψ identify periodic solutions which persist for $\mu > 0$. More precisely, if the Hessian of F_0 at (x_1^0, x_2^0) is nondegenerate and if $\psi'(y_2^0) = 0$, $\psi''(y_2^0) \neq 0$, then there is for small μ a periodic solution $(x_\mu(t), y_\mu(t))$ which can be expanded in a power series in μ and which reduces to the one given by $x_1 = x_1^0$, $x_2 = x_2^0$, $y_2 = y_2^0$ for $\mu = 0$. Moreover generally $(x_\mu(t), y_\mu(t))$ can be chosen so that F is equal to $F_0(x_1^0, x_2^0)$ along the corresponding orbit.

POINCARÉ goes on to consider the equation of variation associated with the periodic solution $(x_\mu(t), y_\mu(t))$. Here two of the characteristic exponents always vanish, because the original system is Hamiltonian, but it is shown that in general the remaining two exponents do not vanish when $\mu > 0$ and can be expanded in convergent power series in $\sqrt{\mu}$. That they must be expanded in terms of $\sqrt{\mu}$ instead of μ is related to the fact that all four characteristic exponents vanish for $\mu = 0$ since the full system is then integrable. Depending on the sign of $\psi''(y_2^0)$ the non-vanishing exponents are purely imaginary (the elliptic case) or real with opposite signs (the hyperbolic case). In the hyperbolic case it is shown that there are two surfaces intersecting along the orbit of the periodic solution $(x_\mu(t), y_\mu(t))$ and which are generated by solutions that approach the periodic solution as t tends to $+\infty$ and $-\infty$ respectively. These surfaces correspond to what are nowadays called the stable and unstable curves of the POINCARÉ return map associated with the periodic solution $(x_\mu(t), y_\mu(t))$. It is claimed that these asymptotic surfaces can be parametrized by functions which can be expanded in convergent power series in $\sqrt{\mu}$.

The second part of the paper contains a long chapter, mainly devoted to the question of stability of systems with two degrees of freedom, followed by two very short chapters where the results are summed up and possible generalizations are considered. The first chapter begins with a discussion of various ways to represent trajectories geometrically. In particular the restricted three-body problem is examined. Here two point masses 1 and μ are moving in a plane in circular orbits around their common centre of gravity and the problem is to describe the motion of a third point of infinitesimal mass moving in the plane under the influence of the two given masses. It is shown that, under natural assumptions, this problem fits the general scheme of the paper.

In order to prove stability POINCARÉ returns to the asymptotic surfaces belonging to the periodic solutions $(x_\mu(t), y_\mu(t))$ discussed in the first part of the paper. His idea is to prove that the two asymptotic surfaces actually are branches of one closed surface which, within a given energy level, bounds a three-dimensional space from which solutions cannot escape. That trajectories cannot pass through an asymptotic surface follows from the fact that these surfaces are invariant under the flow. For small values of μ the asymptotic surfaces are supposed to lie close to the torus $x_1 = x_1^0, x_2 = x_2^0$. Thus POINCARÉ attempts to represent them by expressing x_1 and x_2 as functions of the angular variables y_1 and y_2 of the form

$$x_1 = x_1^0 + x_1^1\sqrt{\mu} + x_1^2\mu + x_1^3\mu\sqrt{\mu} + \cdots,$$

$$x_2 = x_2^0 + x_2^1\sqrt{\mu} + x_2^2\mu + x_2^3\mu\sqrt{\mu} + \cdots,$$

where the coefficients x_1^k and x_2^k for $k \geqq 1$ depend on y_1 and y_2 and are 2π-periodic with respect to y_1. Suppose that $n_2 = 0$ and that $\psi(y_2)$ is the mean value of F_1 defined above. Using only that the asymptotic surfaces are invariant under the flow and contained in the set where F is equal to $F_0(x_1^0, x_2^0)$, POINCARÉ shows that $x_1^1 = 0$ and that

$$(*) \qquad (x_2^1)^2 = \frac{2}{N}(\psi(y_2) + C) \ .$$

Here N is the second derivative of $-F_0$ with respect to x_2 at (x_1^0, x_2^0) and C is a constant to be determined. Assume that $N > 0$. Then the minima of ψ correspond to periodic solutions of hyperbolic type for $\mu > 0$. POINCARÉ illustrates his method for the case when ψ has two minima at $y_2 = \alpha$ and $y_2 = \beta$ and $\psi(\alpha) < \psi(\beta)$. Near the torus $x_1 = x_1^0, x_2 = x_2^0$ it is possible to parametrize the energy level where F is equal to $F_0(x_1^0, x_2^0)$ by means of x_2 and the two angular variables y_1 and y_2. It is no restriction to assume that $x_2^0 > 0$. The intersection of the asymptotic surfaces with $y_1 = 0$ can thus be represented by curves in a plane with polar coordinates x_2 and y_2. In first approximation these curves are given by $x_2 = x_2^0 + x_2^1\sqrt{\mu}$, where x_2^1 satisfies $(*)$. They are plotted in Figure 1 for a suitable function ψ and some representative values of the constant C. The continuous lines correspond to the values $-\psi(\alpha)$ and $-\psi(\beta)$ of C. POINCARÉ concludes that in first approximation these two curves represent the asymptotic surfaces belonging to the hyperbolic periodic solutions related to the two minima α and β of the function ψ. Consequently the asymptotic surfaces are closed in first approximation. To prove that also the exact asymptotic

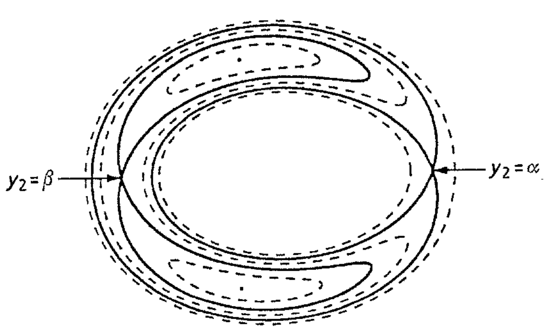

Fig. 1.

surfaces are closed, POINCARÉ makes use of integral invariants. Consider the periodic solution of the perturbed system corresponding to the minimum point α of ψ. In Figure 2 this solution is represented by the point P which is in general at a distance of order μ from $x_2 = x_2^0, y_2 = \alpha$. The curves PA and PB describe the intersection with $y_1 = 0$ of parts of the exact asymptotic surfaces through P corresponding to the outer loop of the curves through $x_2 = x_2^0, y_2 = \alpha$ in Figure 1. We assume that the curve from P to A represent solutions approaching the periodic solution at P as t tends to $-\infty$. If a trajectory starts on this curve close to P, then after one turn following the angular variable y_1 it returns to the curve further away from P. Correspondingly points on the curve from B to P return closer to P after one turn. Let A' and B' be the points obtained from A and B by following a trajectory from $y_1 = 0$ to $y_1 = 2\pi$. Suppose now that the outward distance from B to A, corresponding to the angle y_2, is non-vanishing and can be expanded in a power series with respect to $\sqrt{\mu}$. Denote by $c_k(y_2)$ the coefficient of $\mu^{k/2}$ in that series and let $c_m(y_2)$ be the first coefficient which does not vanish identically. Since the asymptotic curves are closed in first approximation, we know that $m \geqq 2$ and it can be assumed that $c_m(y_2) > 0$ in a neighbourhood of some fixed angle as in Figure 2. Consider the return map R defined by following the flow from $y_1 = 0$ to $y_1 = 2\pi$. We are mainly interested in R close to the curves representing the asymptotic surfaces. Therefore the Hamiltonian F may be modified and put equal to F_0 when (x_1, x_2) is at some distance from (x_1^0, x_2^0). Then there will be a circle Γ which is invariant under R and which lies inside the asymptotic curves in Figure 2. Now POINCARÉ notes that, since the flow preserves volume, the return map R is area-preserving in a generalized sense. From this he obtains a contradiction by observing that the distance from A to A' and from B to B' is of the order $\sqrt{\mu}$ and thus the domain bounded by the line segment AB, the curve $BB'PA$ and the circle Γ is mapped onto a domain with larger area.[2] He

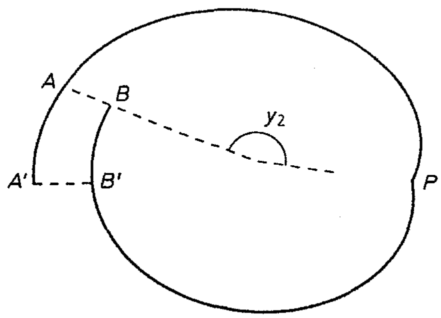

Fig. 2.

[2] POINCARÉ's argument has been slightly modified at this point. The circle Γ is not used by him.

concludes that all coefficients $c_k(y_2)$ vanish identically and that the asymptotic surfaces match exactly. In the same way it is proved that the asymptotic surfaces corresponding to the inner loop in Figure 1 close up. POINCARÉ finally remarks that it can also be proved that the asymptotic surfaces corresponding to the other minimum point β of ψ form a closed surface. The chapter ends with a discussion of a special type of periodic solutions.

The next chapter contains a general summary of the results obtained. Here POINCARÉ remarks that in general an analytic Hamiltonian system has no global analytic integrals independent of the Hamiltonian. To support this statement he simply refers to the abundance of hyperbolic periodic solutions and asymptotic surfaces that he has found for perturbations of integrable systems. He also claims that the Lindstedt series and various other formal solutions of the three-body problem must be divergent, since their convergence would imply integrability of the three-body problem.

In the last chapter the possibility of generalization to systems with more than two degrees of freedom is discussed. POINCARÉ points out that, even if many results remain valid, there are new substantial difficulties in the general case. In particular his proof of stability does not work since a closed manifold of dimension p does not define a bounded region in $(2p-1)$-dimensional space when $p > 2$.

The notes following the main text take up almost one hundred pages, which is more than one third of the paper. Note A deals with the divergence of the Lindstedt series and in note B the results about the three-body problem are presented in terms of the coordinates that are usually employed in astronomy. In the notes C, D, E are treated integral invariants, linear differential equations with periodic coefficients and CAUCHY's method of majorants. Note F is about the construction of asymptotic surfaces. Here POINCARÉ makes use of the formalism of JACOBI's *Vorlesungen über Dynamik*. The equations of the asymptotic surfaces are written in the form

$$x_1 = dS/dy_1, \qquad x_2 = dS/dy_2 \ ,$$

where

$$S = S_0 + S_1\sqrt{\mu} + S_2\mu + S_3\mu\sqrt{\mu} + \cdots$$

and the coefficients S_k, which are functions of the angular variables y_1 and y_2, are to be determined. The object is to prove that the asymptotic surfaces are closed, using only analytical methods, without resorting to geometric arguments. The proof is carried out in the same situation as before, *i.e.* for the asymptotic surfaces corresponding to the minimum point α of the function ψ. In this case it is shown that all coefficients S_k must be periodic functions of y_1 and y_2 with periods 2π and 4π respectively.

Note G concerns the non-existence of first integrals and the last two notes H and I are about characteristic exponents and asymptotic solutions. We shall have more to say about the notes H and I in the next section.

3. Questions and replies

It was mainly WEIERSTRASS who had to take the responsibility for evaluating POINCARÉ's paper. The letters from WEIERSTRASS to MITTAG-LEFFLER concerning this matter are published in volume 35 of Acta Mathematica. In a letter of November 15, 1888 WEIERSTRASS states his reasons for nominating POINCARÉ for the prize and suggests a statement to be presented to the king. However, he complains that the paper is very difficult to read and expresses his hopes that POINCARÉ will revise the memoir before publication. On the same date, November 15, MITTAG-LEFFLER writes to POINCARÉ:

> ... I will not conceal from you that the study of your work has offered us very great difficulties ... Mr. Weierstrass has asked me ... to suggest ... that you append to your memoir before it is published some developments of those essential points which have so far been treated very briefly. ... Among the points which I think need a more thorough treatment is first of all your statement that expansions of Lindstedt type are divergent. I have spent a full month this summer together with Mr. Weierstrass exclusively occupied with the study of your memoir. When I left we had still not understood how you prove that proposition. ... Another point to which I want to draw your attention is the following: In the case of the three-body problem where you have succeeded in finding a complete solution you express this solution in a form which is difficult to understand ... the multiplicity of three dimensions which you consider is not ordinary space where the bodies are found. Couldn't you translate your results in such a manner that your propositions are directly valid in ordinary space? ...

POINCARÉ immediately replies that he will write two notes as requested. He also asks MITTAG-LEFFLER to inform him of other points where the presentation needs to be clarified. MITTAG-LEFFLER recommends him to speak to HERMITE to get his opinion about which parts he finds most obscure. Now also the young assistant editor of Acta Mathematica LARS EDVARD PHRAGMÉN[3] is becoming involved in the reading of POINCARÉ's manuscript. On December 16, 1888 PHRAGMÉN writes to MITTAG-LEFFLER:

> ... I now return P:s memoir with the notes. There are certainly many things therein which I still do not fully understand, but the main reason for this is probably that I have not had the time to penetrate the details ... For example I have not completely understood what he means by stability. ...

[3] L. E. PHRAGMÉN (1863–1937) succeeded SONJA KOVALEVSKI in 1892 as professor of Higher Mathematical Analysis at Stockholms Högskola. In 1903 he became head of the Swedish Insurance Inspection and from 1908 he was managing director of a private insurance company. In pure mathematics he is mainly known for the Phragmén-Lindelöf principle.

PHRAGMÉN also complains that he cannot follow POINCARÉ's proof for the non-existence of first integrals. MITTAG-LEFFLER communicates PHRAGMÉN's remarks to POINCARÉ, who writes back on December 25:

> ... I have begun to compose various explanatory notes. The notes C and D which are already written concern integral invariants and linear differential equations with periodic coefficients. Note E which is almost finished, deals with the "Calcul des Limites". As you know it is by this expression, however badly justified, that Cauchy denotes the collection of procedures by which one may prove the convergence of series solutions of differential equations. I will also add one note F on the asymptotic surfaces and one note G on the non-existence of analytic integrals. On this last point I can at once give you some explanations: ...

Here follow some explanations pertaining to PHRAGMÉN's questions.

On February 2, 1889 WEIERSTRASS writes to MITTAG-LEFFLER (see MITTAG-LEFFLER [1912]) that he still cannot understand why the Lindstedt series must diverge. MITTAG-LEFFLER informs POINCARÉ who replies that he believes that he has already answered this question in note A.

At last this exchange of questions and replies appears to have come to an end and preparations are being made to begin the printing of POINCARÉ's memoir. Then, more than three months later, on June 11 in the middle of a letter to MITTAG-LEFFLER about other things PHRAGMÉN remarks:

> ... Unfortunately I have got stuck again at a detail – important I'm afraid – from POINCARÉ, but I hope that it will come out all right. ...

On June 26 PHRAGMÉN writes again to MITTAG-LEFFLER:

> ... I hereby return Weierstrass' letter. I concluded, probably correctly, that W. never obtained a copy of note A which deals with exactly these things, and therefore I sent as fast as possible a copy of it. I suppose that this note completely clarifies the point in question (at least Poincaré's attitude to it).
>
> The point where I got stuck was not this one, but the question about the way in which μ enters into the equations for the asymptotic surface. In fact, in the asymptotic solutions appear certain divisors
>
> $$\sum_k \alpha_k \beta_k + \gamma\sqrt{-1} - \alpha_i$$
>
> and here α_k is of the form $\sqrt{\mu}\,\alpha'_k + \dots$. For $\gamma = 0$ one thus obtains $\sqrt{\mu}$ in the denominator and this fact has caused me some problems, which I have not yet overcome. ...

It was decided to write to POINCARÉ, and PHRAGMÉN draws up a letter which is signed and sent by MITTAG-LEFFLER on July 16. First POINCARÉ is asked to clear up some minor points. In particular he is requested to prove that the characteristic exponents of the perturbed problem can be expanded in convergent power series with respect to $\sqrt{\mu}$. Then comes the main question, which is phrased as follows:

> ... "We suppose that x_1 and x_2 are expanded with respect to powers of $\sqrt{\mu}$ and we write
>
> $$x_1 = x_1^0 + x_1^1\sqrt{\mu} + x_1^2\mu + \cdots$$
>
> $$x_2 = x_2^0 + x_2^1\sqrt{\mu} + x_2^2\mu + \cdots \text{"}$$
>
> How does one know that these expansions are possible? In fact, it seems to me that in the asymptotic solutions ... $\sqrt{\mu}$ appears in the denominator for those divisors $\gamma\sqrt{-1} + \sum\alpha\beta - \alpha_i$ where γ vanishes and it is not very easy to see how one can get rid of this problem.
>
> I shall be very grateful, my dear friend, if you will be kind enough to explain to me how this last difficulty can be evaded. ...

POINCARÉ's response is to write two more notes H and I. In H he indicates how one can prove that the series expansions of the characteristic exponents are convergent. The second note I concerns certain formulas given near the end of the first part of the paper for the asymptotic solutions which generate the asymptotic surfaces used in the proof of stability. PHRAGMÉN's question refers to these formulas and in note I POINCARÉ proves that, due to cancellations, $\sqrt{\mu}$ does not appear as a factor in the denominators.

For information about what happened next, we have to rely on some notes which are kept at the Mittag-Leffler Institute and which were written by MITTAG-LEFFLER more than twenty years later. The relevant letters are missing. According to MITTAG-LEFFLER there was an interchange of letters between PHRAGMÉN and POINCARÉ ending with a letter from PHRAGMÉN sent on September 16 (in one place MITTAG-LEFFLER writes November 16, but the earlier date seems more plausible). We can only guess what was in that letter, but it seems highly probable that PHRAGMÉN inquired about the *convergence* of the series representing the asymptotic solutions. MITTAG-LEFFLER writes that there was no answer from POINCARÉ for a long time. Then on November 30 arrives a telegram from POINCARÉ asking that printing be stopped awaiting a letter explaining the situation. On December 1 POINCARÉ writes to MITTAG-LEFFLER:

> My dear friend,
>
> I have written this morning to Mr. Phragmén to tell him about an error which I have committed and he has undoubtedly informed you of my letter. But the consequences of this error are more serious than I first thought. It is not true that the asymptotic surfaces are closed, at least not in the sense that I meant before. What is true, is that if one considers the two parts of

that surface (which I yesterday still believed *coincided* with each other) they intersect along infinitely many asymptotic trajectories and furthermore their distance is an infinitesimal of higher order than μ^p however big p is.

I did believe that *all* asymptotic curves, after having left a closed curve representing a periodic solution, approached again asymptotically the *same* closed curve. What is true, is that there are infinitely many [asymptotic curves] having this property.

I don't conceal from you the trouble this discovery gives me. First of all I don't know whether you think that the results which subsist, i.e. the existence of periodic solutions, of asymptotic solutions; the theory of characteristic exponents, the non-existence of analytic integrals and the divergence of the series of Mr. Lindstedt, still merit the high award that you have granted them.

On the other hand, extensive revisions will be necessary and I don't know if one has begun to print the memoir; I have sent a telegram to Mr. Phragmén about this. . . .

For MITTAG-LEFFLER this was certainly bad news. In Sweden the astronomer GYLDÉN had openly voiced his disappointment that the prize had been given to POINCARÉ and in Germany KRONECKER was very dissatisfied with the way the whole prize matter had been handled (see MITTAG-LEFFLER [1912]). The reputation of Acta Mathematica was involved and also the king. MITTAG-LEFFLER acted rapidly. He stopped all distribution of the journal and wrote letters to a number of mathematicians asking them to return the copies that they had received. He wrote to POINCARÉ on December 5 and suggested that the memoir should be revised and provided with a new introduction saying that it was a revised and extended version of the crowned memoir in which an error had been corrected. MITTAG-LEFFLER explained that he was accountable to the parliaments of the Scandinavian countries, which were subsidizing Acta Mathematica, and he asked if POINCARÉ was willing to carry the cost for the suppressed memoir. POINCARÉ agreed and started immediately to rewrite the paper.[4] The whole operation was successful. Of the original memoir there remain today only a few copies. The one on which the account in section 2 was based is kept at the Mittag-Leffler Institute and bears the inscription (in Swedish): "The whole edition was destroyed. M.L." On January 5, 1890 POINCARÉ sent in the new version of his paper.

4. The revised memoir

In its final form the prize memoir appeared in volume 13 of Acta Mathematica in December 1890. In the introduction PHRAGMÉN is thanked for his

[4] The final costs for the suppressed memoir amounted to 3585 kronor and 63 öre in Swedish currency. This sum was paid by POINCARÉ.

contribution. There is no longer any reference to the question of stability, except that the recurrence theorem is mentioned. Instead emphasis is laid on the results about existence of periodic solutions, asymptotic solutions and doubly asymptotic solutions. Also the negative results about non-existence of analytic integrals and divergence of the Lindstedt series are mentioned.

The first part of the paper has grown considerably, since the notes have been worked into the main text. An interesting addition is that the statement of the recurrence theorem has been made more precise. It is proved in an informal way that with probability one a randomly chosen trajectory is recurrent in the sense of that theorem. This may be the first occurrence of a measure theoretic statement of this kind in connection with dynamical systems. Asymptotic solutions are treated at length and note I from the first version is included and extended. However, it is stressed that the expansions of the solutions with respect to the parameter $\sqrt{\mu}$ are only formal.

In the second part of the paper the discussion of the asymptotic surfaces still takes up the largest part. Note F of the old version is included and followed by an example for which a proof is indicated that the series constructed in that note are divergent. In the same notation as before, this example concerns the Hamiltonian

$$F = -x_1 - x_2^2 + 2\mu \sin^2(y_2/2) + \mu\varepsilon\varphi(y_2)\cos y_1 \ .$$

For $\varepsilon = 0$, the corresponding dynamical system has the periodic solution $x_1 = x_2 = y_2 = 0$, $y_1 = t$. The non-vanishing characteristic exponents belonging to this solution are $\pm\sqrt{2\mu}$ and the associated asymptotic surfaces are given by

$$x_1 = 0, \qquad x_2 = \pm\sqrt{2\mu}\sin(y_2/2) \ .$$

They form a single closed surface

$$x_1 = \frac{dS_0}{dy_1}, \qquad x_2 = \frac{dS_0}{dy_2}, \quad \text{where} \quad S_0 = 2\sqrt{2\mu}\cos(y_2/2) \ .$$

Since two characteristic exponents are non-vanishing for $\varepsilon = 0$, there are, locally for small $\varepsilon > 0$, asymptotic surfaces given by similar equations with S_0 replaced by a convergent series

$$S = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots \ .$$

Poincaré proceeds to write down the equation for S_1 and by an explicit calculation, in the case when $\varphi(y) = \sin y$, he arrives at the conclusion that the asymptotic surfaces do not form a closed surface for $\varepsilon > 0$. From this it follows that the expansions of note F must be divergent in this case, since their convergence would imply that the asymptotic surfaces were closed.

Even if the asymptotic surfaces do not match completely, Poincaré claims that they must intersect. To show this he employs essentially the same geometric argument as before, involving integral invariants. A point of intersection of the two asymptotic surfaces corresponds to a doubly asymptotic solution, *i.e.* a solution which approaches the same periodic solution asymptotically both when t tends to $-\infty$ and to $+\infty$. By appeal to the recurrence theorem he proves that if there is one doubly asymptotic solution, there must be infinitely many.

Although POINCARÉ's argument is not completely rigorous, it was sufficient to convince him around December 1, 1889 that in general the asymptotic surfaces do not close up but that they intersect along infinitely many doubly asymptotic trajectories. To a reader of his paper, who was not familiar with its history, this result must have come as an anticlimax after the long and intricate constructions from note F which preceeded it.

5. Epilogue

POINCARÉ returned to celestial mechanics in his monumental work *Les méthodes nouvelles de la mécanique céleste* [1892–1899]. To a large extent this is just an expanded version of his Acta paper. The last chapter of the third volume is devoted to doubly asymptotic solutions. Here a non-periodic solution which tends towards the same periodic solution both when t tends to $-\infty$ and to $+\infty$ is called a *homoclinic* solution. For a periodic solution of hyperbolic type of a Hamiltonian system with two degrees of freedom, such a solution corresponds to a point of intersection (a *homoclinic* point) of the stable and unstable curves associated with the corresponding return map. POINCARÉ examines the consequences of the existence of a single homoclinic point for the geometrical behaviour of these stable and unstable curves and he sums up his observations in the following often quoted words:

> When one tries to imagine the figure formed by these two curves and their infinitely many intersections each corresponding to a doubly asymptotic solution, these intersections form a kind of lattice, web or network with infinitely tight loops; neither of the two curves must ever intersect itself, but it must bend in such a complex fashion that it intersects all the loops of the network infinitely many times.
>
> One is struck by the complexity of this figure which I am not even attempting to draw. Nothing can give us a better idea of the complexity of the three-body problem and of all problems of dynamics where there is no analytic integral and Bohlin's series diverge.[5]

This complex picture arises when the stable and unstable curves intersect transversally. It should be compared with the simple picture that would have resulted if they had matched exactly as POINCARÉ originally believed.

The behaviour of flows near homoclinic trajectories was analysed more in detail by G.D. BIRKHOFF [1935]. He related this behaviour to symbolic

[5] Series of the type occuring in the note F were called Bohlin series by POINCARÉ, after the Swedish astronomer KARL BOHLIN (1860–1939).

dynamics and proved that the existence of a transverse homoclinic point implies that a Bernoulli shift can be embedded into the dynamics of the return map. Thus there are solutions whose trajectories depend in a very sensitive way on the initial conditions. BIRKHOFF's work was generalized and made more precise by S. SMALE [1965] who connected homoclinic points with his famous horseshoe. Today the study of homoclinic intersections and tangencies is at the center of many investigations of non-linear dynamics. However, somewhat ironically, it still remains to be proved that homoclinic orbits exist in the planar restricted three-body problem.[6] ... what?

Concerning the question of stability it turned out that one should not start from the toruses corresponding to commensurable frequencies n_1 and n_2 as POINCARÉ did, but rather from those with incommensurable frequencies. According to the so called KAM theorem of KOLMOGOROV, ARNOLD and MOSER most toruses with incommensurable frequencies survive small pertubations and are only slightly deformed. From this result follows the stability, in the sense of POINCARÉ, of the restricted three-body problem and other problems with only two degrees of freedom.

References

BIRKHOFF, G. D. 1935. *Nouvelles recherches sur les systèmes dynamiques*, Mem. Pont. Acad. Novi Lyncaei **1**, 85–216.

MITTAG-LEFFLER, G. 1912. *Zur Biographie von Weierstrass*, Acta Math. **35**, 29–65.

POINCARÉ, H. 1881–1886. *Mémoire sur les courbes définies par une équation différentielle I-IV*, Journal de Mathématiques pures et appliquées, 3[e] série, t.**VII**, 375–422, t.**VIII**, 251–296; 4[e] série, t.**I**, 167–244, t.**II**, 151–217.

POINCARÉ, H. 1890. *Sur le problème des trois corps et les équations de la dynamique*, Acta Math. **13**, 1–271.

POINCARÉ, H. 1892–1899. *Les méthodes nouvelles de la mécanique céleste I-III*, Gauthiers-Villars.

POINCARÉ, H. 1921. *Lettres à M. Mittag-Leffler concernant le mémoire couronné du prix de S.M. le roi Oscar II*, Acta Math. **38**, 161–173.

SMALE, S. 1965. *Diffeomorphisms with many periodic points*, in Differential and Combinatorial Topology, Princeton University Press, 63–80.

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[6] I owe this remark to J. PALIS.