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CHAPTER 5

Poincaré's Memoir on the Three Body Problem

5.1. Introduction

Poincaré's attack on the three body problem was remarkable in many ways. His unprecedented qualitative approach to the problem and its intrinsic dynamics is unequivocally more powerful than any previous methodology. By taking a reductionist view and studying the periodic solutions of a system with two degrees of freedom, Poincaré's global qualitative perspective led him to the brilliant discovery of asymptotic solutions. His analysis of the complex nature of the behaviour of these solutions marks a turning point in the history of dynamics. For attendant on this analysis came his discovery of homoclinic points, which embodied the first mathematical description of chaotic motion in a dynamical system. Furthermore, as the comparison of the two versions of the memoir makes clear, Poincaré's first encounter with homoclinic points, as well as being historically important in its own right, has an added significance in the context of the history of the paper itself.

Furthermore, many of the innovative and powerful ideas that Poincaré developed as tools and techniques in order to tackle the three body problem have a more general application not only in the theory of differential equations and celestial mechanics but also in other branches of mathematics. The memoir therefore fulfilled a research role well beyond the confines of the problem it professed to tackle.

For Poincaré himself, the competition had acted as a stimulus to synthesise many of the ideas that he had been developing over the previous decade. The question of the stability of the solar system was one in which he had harboured an interest for several years, and he had for some time been building up a battery of techniques with which to launch an offensive. With the memoir completed, undoubtedly more hurriedly than he would have liked, he continued to work further on these ideas. Three years later saw the publication of the first volume of his three-volume *chef d'œuvre* on celestial mechanics, *Les Méthodes Nouvelles de la Mécanique Céleste*,[153] which was a work founded upon the contents of [P2].

The previous chapter described the circumstances that resulted in Poincaré's making two major changes to the content and structure of the memoir before its publication. The first of these, the addition of the substantial explanatory *Notes*, was made in response to requests for more detail from Mittag-Leffler, and the second, the extensive rewriting, was the result of the discovery of the error. Although Poincaré touched on the subject of these changes in his introduction to the published paper [P2], he did not make clear the extent of the alterations. Unfortunately, it has not been possible to trace the paper Poincaré originally submitted for the

[153] Henceforth referred to as [MN, I-III].

prize, but correspondence at the Institut Mittag-Leffler suggests that, excluding the *Notes*, it assumed a very similar form to the first printed version [P1], copies of which still exist at the Institut. Of especial importance amongst the latter is the one, [P1a], which was personally corrected and extended by Poincaré and to which he attached a note detailing the changes. This copy in its altered form corresponds almost exactly to the published memoir, and it provides a remarkable record of the way the memoir was rewritten. Its existence allows us to follow the metamorphosis of the entire memoir and to provide a complete picture of the exact nature of the error.

This chapter, as well as giving a detailed mathematical analysis of the memoir, also describes how [P2] relates both to the version that actually won the prize and to the *Notes*. The comparison of the versions shows how much of [P1] was retained in [P2], how the *Notes* were integrated (or not) into [P2], and to what extent [P2] was shaped by the detection of the error. Chronicling Poincaré's changes in this way gives a clear picture of why the discovery and correction of the error is all-important with regard to the position the memoir now holds in the history of the mathematical theory of chaos.

Comparing the tables of contents of both printed versions gives a preliminary idea of the overall structure of each version and its relationship to the others.[154] (The differences detailed here are also noted at the appropriate points in the mathematical analysis.) The two introductions are also compared, as this gives an insight into how Poincaré's own perspective on his results changed.

5.2. Tables of contents

Both versions of the memoir are prefaced with an introduction, then separated into two parts: *Generalities* and *The equations of dynamics and the n body problem.* The first part is devoted to developing the theory and the second to applying it. Each part is divided into chapters which are then subdivided into sections. The only difference in the format is that in [P1] the sections are numbered within each chapter, while in [P2] the section numbering runs straight through the memoir, which makes it easier for cross-referencing. [P1] concludes with nine explanatory *Notes*, each topic being labelled with a letter *A* through *I*.

Comparing the tables of contents shows two major changes in the first part of the memoir. Chapter I in [P2] contains four sections as against one in [P1]. In addition to the identical first section on notation and definitions, [P2] includes two sections, §2 and §3, on the method of majorants and a section, §4, on the integration of linear differential equations with periodic coefficients. Most of §2 and all of §3 are taken from *Note E*, and §4 is an exact reproduction of *Note D*. Although the format of Chapter II is identical in both versions, there are significant changes to the content. *Note C* is incorporated at the end of §6, and §8 contains major alterations. Chapter III of Part 1 contains the most important change, with the addition in [P2] of a new concluding section, §14, on the asymptotic solutions of the Hamiltonian equations; the contents of this, apart from including the latter half of

[154]The two tables of contents are reproduced as Appendices 5a and 5b respectively.

Note I, do not appear in [P1]. The other sections in Chapter III, although carrying the same headings in both versions, contain significant changes and additions.

In Part 2 the differences are more marked.

In [P1] the application of the theory to systems with two degrees of freedom is confined to the first chapter, which is divided into five sections. The second chapter is devoted to a general resumé of the results (positive and negative), and the final chapter consists of a single section on Poincaré's endeavours to generalise his results to the n body problem. [P1] concludes with the nine *Notes*.

In [P2] the first section corresponds with that in [P1], but it is the only section in the first chapter. Significantly, the topic of asymptotic surfaces has been revised to merit a chapter in its own right. The new second chapter consists of four sections, none of which retains an exact title from [P1]. One of the sections, §17, contains material from [P1]; one, §18, contains an amended version of *Note F*, together with some additions; and two, §16 and §19, are entirely new. The third chapter, on miscellaneous results, includes as §20 the section on periodic solutions of the second kind taken from the first chapter in Part 2 of [P1]; a section on the divergence of Lindstedt's series, §21, taken both from the negative results in [P1] and *Note A*; and a section on the nonexistence of uniform integrals, §22, which contains the rewritten contents of *Note G*. The last chapter in [P1] on the n body problem is transferred almost intact as the last chapter of [P2]. The section on positive results from [P1] is omitted altogether.

As far as the *Notes* are concerned, with the exception of *Note B*, *New statement of results*, which was entirely deleted, and *Note H*, *Characteristic exponents* (rewritten and expanded to appear as part of §12), these are incorporated, either whole or in part, into the main text of [P2] as indicated above. The exclusion of *Note B*, which was a summary of the main results described in more practical terminology for the benefit of astronomers, is discussed later.[154a]

Whenever a particular piece of the memoir is not specifically ascribed to either [P1] or [P2], it is correct to assume that it appeared in the same form in both versions.

5.3. Poincaré's introductions

In [P1] Poincaré began the introduction by admitting that, although he had written the memoir in response to Question 1 in the competition, he had not been able to provide a full resolution of the problem.[155] He made it clear that he had concentrated on the restricted problem which he specified as follows:

> *I consider three masses, the first very large, the second small but finite, the third infinitely small; I assume that the first two each describe a circle around their common centre of gravity and that the third moves in the plane of these circles. An example would be*

[154a] See Chapter 5, end of Section 5.7.

[155] For details of the Question see Appendix 2.

the case of a small planet perturbed by Jupiter, if the eccentricity of Jupiter and the inclination of the orbits are disregarded.[156]

He then gave an indication of the main mathematical techniques that he had used in the memoir. These included the trigonometric form of the power series solutions derived by Lindstedt and Gyldén (which he had used to avoid the secular terms that arise in the series used by Laplace and Poisson); Cauchy's method of majorants, which he had applied to prove the convergence of the series; his own geometric methods (taken from his earlier memoir on differential equations), which he had used to prove the stability of the solution; and his new idea of invariant integrals, the theory of which he had developed in order to apply his geometric methods to the equations of dynamics.

He emphasised that the central topic of the memoir would be provided by his discussion of periodic solutions and drew attention to the fact that he had been able to develop the theory using Cauchy's methods, since the periodic solutions were untroubled by the problem of small divisors.

[P1] was printed as though it were an exact replica of Poincaré's competition entry and so retained its "anonymous" format. However, even without indicators such as handwriting, a cursory reading of the introduction would have been sufficient to identify the author, since in order to furnish the necessary background to his methods Poincaré needed to reference his own work, and consequently give his own name, no less than five times.

In the case of [P2] Poincaré began by revealing that it was a reworking of his competition entry. He explained that the revision had resulted from incorporating the *Notes*, together with some additional explanations, into the main body of the memoir, a task he considered to be a logical necessity but that he had not had time to do earlier. Although he did mention the error, even acknowledging Phragmén's role in its detection, he adhered to Mittag-Leffler's request and gave no hint of what it might have been.

Nevertheless, he did make it clear that he had included some substantial additions to the opening chapter by the way of reformulation of established theorems. In drawing attention to his work on periodic solutions, he mentioned both asymptotic and doubly asymptotic solutions and in this connection indicated the nature of the restricted three body problem, although this time without giving a complete statement of the problem. He also mentioned his recurrence theorem, but above all he stressed what he called his negative results. These were his proof of the nonexistence of any new single-valued integrals for the restricted problem and his proof of the divergence of Lindstedt's series. To give an indication of the difficulties he had encountered in attempting to generalise his results, he said that he believed a complete solution to the three body problem would require analytic tools quite different and infinitely more complicated than any of those known, which was in contradiction to what Weierstrass had supposed in setting the question.

Finally, in connection with one of the series he had discussed, Poincaré acknowledged an analogy with a paper by Karl Bohlin [1888] which had been published so

[156]Poincaré [P1, *8*].

near to the closing date of the competition that he had not included a reference to it in his memoir.

Comparing the two introductions reveals quite a striking change of emphasis. Poincaré's focus is no longer on mathematical techniques but instead is on so-called negative results. [P1] conveys a sense of optimism about the ultimate resolution of the problem, but in [P2] the tenor is quite different: the future progress of the problem has lost its air of inevitability. The differences between them provide a fair reflection of the far-reaching and totally unexpected mathematical implications of the memoir's essential revision.

5.4. General properties of differential equations

In the first chapter, Poincaré provided the definitions and background for the theory to follow. Although most of the terminology he used would have been familiar to his contemporaries, the fact that the memoir was directed towards an international audience meant there was a special need for precision to avoid ambiguity or misunderstanding.

He considered the system of ordinary differential equations

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, \dots, n),$$

where the X_i are single-valued analytic functions of the n variables $x_1, \dots, x_n$. Given a particular solution $x = \phi(t)$, and a nearby solution $x = \phi + \xi$, he defined the *variational equations*

$$\frac{d\xi_i}{dt} = \frac{dX_i}{dx_1}\xi_1 + \dots + \frac{dX_i}{dx_n}\xi_n.$$

He called the system *canonical* if the variables x could be divided into two series $x_1, \dots, x_p$, and $y_1, \dots, y_p$, where $n = 2p$, and the equations could be written

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \quad (i = 1, \dots, p).$$

For the case when $n = 3$ he made a geometric representation of the system using his idea from [1885] and [1886]. That is, the x_i are considered as the coordinates of a point P in space so that as the time varies P describes a trajectory. By considering the set of trajectories which pass through a given curve in space, he formed a surface which he termed a *surface trajectory*. This representation then led to a definition of stability in which he said the system was stable if all its surface trajectories were closed. In other words the system was stable if any point P stayed within a bounded region of space.

To define a *periodic solution* he said that a particular solution,

$$x_1 = \phi_1(t), \dots, x_n = \phi_n(t),$$

of the differential equations was periodic with period h if when x_i is a linear variable

$$\phi_i(t + h) = \phi_i(t)$$

and when x_i is an angular variable

$$\phi_i(t + h) = \phi_i(t) + 2k\pi \quad (k \text{ an integer}).$$

Having dealt with the definitions, Poincaré moved on to establishing some of the techniques he intended to employ.

In broad terms the three body problem is essentially one of integrating a particular system of differential equations using series methods. While the problem's persistent resistance to attack had made it seem probable that some radical new ideas would be needed if progress was to be made, Weierstrass, in his formulation of the problem for the competition, had indicated that he thought it likely that the answer lay within the bounds of current analytical theory. Poincaré's success, which came not from solving the problem per se but in providing a whole new perspective on it, was the result of a complementary mixture of both old and new ideas and, in particular, the ingenious application of them. He built on a framework of existing results, enhancing and extending their utility in novel and inventive ways.

To launch the theoretical part of the memoir he reviewed some of the available techniques, beginning with the method of majorants.

5.4.1. The method of majorants. The method of majorants originated with Cauchy in [1842] in the search for proofs for the existence of solutions to differential equations. Generally speaking, the method is used to show that a power series in the independent variable (derived by the method of undetermined coefficients) that satisfies the differential equation does have a definite domain of convergence. It had been simplified by Briot and Bouquet [1854], used by Weierstrass in [1842] (although not published until 1894),[157] studied by Fuchs,[158] and, as mentioned in Chapter 3, Poincaré himself had worked on it in his thesis published in 1879.

Poincaré presented Cauchy's basic principle in the following form:

Given a system of differential equations

$$\frac{dy}{dx} = f_1(x, y, z), \quad \frac{dz}{dx} = f_2(x, y, z), \tag{11}$$

where f_1 and f_2 can be expanded in increasing powers of x, y and z, then the equations have a unique solution

$$y = \phi_1(x), \quad z = \phi_2(x),$$

where ϕ_i are Taylor series in x which vanish with x.

To verify that such a solution exists, the series must be shown to be convergent. The two functions f_1 and f_2 are replaced by the majorant function

$$f(x, y, z) = \frac{M}{(1 - \alpha x)(1 - \beta y)(1 - \gamma z)},$$

M, α, β, γ being chosen in such a way that each term of f has a larger coefficient (in absolute value) than the corresponding term in f_1 and f_2. Replacing f_1 and f_2 by f increases the coefficients of ϕ_1 and ϕ_2, and since the series for f is convergent, the two series created by the exchange must be convergent, which in turn implies convergence of the original series for f_1 and f_2.

[157] Weierstrass's work became known to his students and colleagues in the late 1850s. See Cooke [1984, *28*].

[158] See Gray [1984].

In [P1], Poincaré made extensive reference to Cauchy's results, but his own exposition of the method as well as some further developments which he had derived were contained only in *Note E*. In [P2] he extended these developments and added two new sections, §2 and §3, to deal with theory as applied to ordinary differential equations and partial differential equations, respectively. These sections contain almost all the results from *Note E* as well as three new theorems, Theorems III, V and VI.

These new theorems are particularly important within the context of the memoir, since Poincaré's theory of periodic solutions depends fundamentally upon them. By including them in [P2] Poincaré put into place an essential foundation of the theory. In [P1] he used the results frequently but often with little or no reference, which made it extremely difficult to validate his arguments. Although not connected with the error, the addition of these sections represents a significant contribution towards his aim of creating a more logical structure to the memoir.

Most of the theorems contained in §2 are now well known, but they are stated here for completeness.

In the first theorem, Poincaré extended Cauchy's original result by finding an expansion for the solution in terms of a parameter μ as well as in terms of the independent variable t.

THEOREM I. *Suppose that the functions f_1 and f_2 depend not only on x, y and z but also on an arbitrary parameter μ and that they can be expanded as series in x, y, z and μ. Then equations (11) can be written in the form*

$$\frac{dx}{dt} = f(x,y,z,\mu) = 1, \quad \frac{dy}{dt} = f_1(x,y,z,\mu), \quad \frac{dz}{dt} = f_2(x,y,z,\mu), \tag{12}$$

and it is possible to find three series

$$x = \phi(t,\mu,x_0,y_0,z_0) = t + x_0, \quad y = \phi_1(t,\mu,x_0,y_0,z_0), \quad z = \phi_2(t,\mu,x_0,y_0,z_0)$$

that formally satisfy the equations. These series reduce to x_0, y_0 and z_0, respectively, for $t = 0$, and, provided t, μ, x_0, y_0 and z_0 are sufficiently small, they are convergent.

In his proof Poincaré replaced the functions f, f_1 and f_2 by the function

$$f'(x,y,z,\mu) = \frac{M}{(1-\beta\mu)(1-\alpha(x+y+z))}$$

and formed majorant series for x, y and z convergent for sufficiently small values of t, μ, x_0, y_0 and z_0. However, although this gave the desired result—that the series solution is an expansion in ascending powers of the parameter as well as the independent variable—it also contained a severe restriction on the value of t. Since ultimately Poincaré was looking for solutions valid for all t, he wanted to find some way of relaxing this restriction. In the following theorem, which has become a classic in the theory of differential equations depending upon a parameter, he showed that it could be done by proving the existence of a solution which is an expansion in powers of the parameter rather than powers of the independent variable.

THEOREM II. *Excluding one exceptional case, x, y and z can be expanded as powers of μ, x_0, y_0 and z_0 for any value of t, provided μ, x_0, y_0 and z_0 are sufficiently small.*

In his proof Poincaré began by showing that equations (12) have a solution

$$x = \omega_1(t, \mu), \quad y = \omega_2(t, \mu), \quad z = \omega_3(t, \mu),$$

which is such that $x = y = z = 0$ when $t = 0$ and which converges when $0 < t < t_1$. He then replaced x, y, and z in the equations (12) by

$$x = \xi + \omega_1, \quad y = \eta + \omega_2, \quad z = \zeta + \omega_3$$

to get the variational equations

$$\text{(13)} \qquad \frac{d\xi}{dt} = \phi(\xi, \eta, \zeta, \mu), \quad \frac{d\eta}{dt} = \phi_1(\xi, \eta, \zeta, \mu), \quad \frac{d\zeta}{dt} = \phi_2(\xi, \eta, \zeta, \mu),$$

ϕ, ϕ_1 and ϕ_2 vanishing when $\xi = \eta = \zeta = \mu = 0$. Since he had supposed that f, f_1 and f_2 can be expanded in powers of μ, then the same will be true of ϕ, ϕ_1 and ϕ_2, and these expansions can also be shown to be convergent in $0 < t < t_1$. Thus there exists a solution of equations (13) as series in μ, which is such that $\xi = \eta = \zeta = 0$ when $t = 0$ and which converges in $0 < t < t_1$.

The exceptional case occurs when the functions f_1 and f_2 are no longer analytic in the variables x, y and z, i.e., when they become infinite or cease to be single-valued. This is because if the functions are not analytic then they cannot be expanded in series as required. In other words if as t changes the trajectory goes through a singular point, the theorem no longer holds. In the case of the three body problem the functions given by the equations cease to be analytic in the case of a collision. However, since Weierstrass had specifically excluded collisions in the competition question, the theorem was sufficient for Poincaré's purpose.[159]

In his next theorem Poincaré proved explicitly that the solutions depend analytically on the initial conditions. This theorem did not appear in [P1], and it seems likely that he originally believed the result to be self-evident from Theorem II.

THEOREM III. *Let*

$$x = \omega_1(t, \mu, x_0, y_0, z_0), \quad y = \omega_2(t, \mu, x_0, y_0, z_0), \quad z = \omega_3(t, \mu, x_0, y_0, z_0)$$

be the solutions of the differential equations which reduce to x_0, y_0, z_0 for $t = 0$. Then the functions

$$\omega_i(t_1 + \tau, \mu, x_0, y_0, z_0), \quad (i = 1, 2, 3),$$

can be expanded as powers of μ, x_0, y_0, z_0, and τ, provided that these quantities are sufficiently small.

Poincaré attributed Theorem IV, now more familiarly known as the implicit function theorem, to Cauchy and his method of majorants.[160] Although it was not a new result, he had included it in *Note E* because it played such a pivotal role in his own investigations. Theorems V and VI, which only appear in [P2], are direct extensions of it.

[159] Poincaré appears not to have considered the possibility of noncollision singularities. The impossibility of such singularities in the three body problem was proved by Painlevé in [1897]. See Chapter 8.

[160] The history of the implicit function theorem is convoluted and worth further research. It is certainly not clear that Poincaré was right in his attribution.

THEOREM IV. *A system of n equations*

$$f_i(y_1, \ldots, y_n, x_1, \ldots, x_p) = 0, \qquad (i = 1, \ldots, n),$$

where the f are analytic functions of the $n+p$ variables y and x, and vanish with them, can be solved for $y_1, \ldots, y_n$ in increasing powers of $x_1, \ldots, x_p$, if the Jacobian of the functions f with respect to y is not zero when x and y vanish together.

The final two theorems and the accompanying corollaries take account of the case when the Jacobian does vanish. Poincaré did not include them in [P1], although he did make use of the results. In [P2] he did not provide proofs but instead referred to his own thesis and to work by Pusieux.

THEOREM V. *Let y be a function of x defined by the equation*

$$f(y, x) = 0,$$

where f can be expanded in powers of x and y. Suppose that for $x = y = 0$, f and $\dfrac{df}{dy}, \dfrac{d^2 f}{dy^2}, \ldots, \dfrac{d^{m-1} f}{dy^{m-1}}$ vanish, but $\dfrac{d^m f}{dy^m}$ does not vanish. There will exist m series of the following form

$$y = a_1 x^{1/n} + a_2 x^{2/n} + \cdots$$

(n a positive integer, a_1, a_2 ... constant coefficients) which satisfy the original equation.

COROLLARY I. *If the above series satisfy the equation, then so does the series*

$$y = a_1 \alpha x^{1/n} + a_2 \alpha^2 x^{2/n} + \cdots,$$

where α is an nth root of unity.

COROLLARY II. *The number of series of the form given in Theorem V which can be expanded in powers of $x^{1/n}$ (which cannot be expanded in powers of $x^{1/p}, p < n$) is divisible by n.*

COROLLARY III. *If $k_1 n_1$ is the number of the series which can be expanded as powers of x^{1/n_1}, and if $k_p n_p$ is the number of the series which can be expanded as powers of x^{1/n_p}, then*

$$k_1 n_1 + \cdots + k_p n_p = m,$$

and if m is odd, at least one of the numbers $n_1, \ldots, n_p$ is also odd.

THEOREM VI. *Given the p equations:*

$$f_i(y_1, \ldots, y_p, x) = 0 \qquad (i = 1, \ldots, p),$$

where the left-hand sides can be expanded in powers of x and y and vanish with these variables, then, providing the equations are distinct, it is always possible to eliminate $y_2, \ldots, y_p$ and arrive at a unique equation $f(y_1, x) = 0$ of the same form as the equation in Theorem V.

COROLLARY TO THEOREMS V AND VI. *Since Theorem* IV *holds whenever the Jacobian of f is not equal to zero, then whenever the x vanish, $y_1 = \ldots = y_n$ is a simple solution of equations $f_1 = \ldots = f_n = 0$.*

Furthermore, by Theorems V and VI and their Corollaries, Theorem IV will also still hold if the above solution is multiple, provided the order of multiplicity is odd.

In his application of the method of majorants to partial differential equations, Poincaré began with the Cauchy-Kovalevskaya theorem.[161]

In its modern form the simplest case of the theorem can be stated as follows:

Any equation of the form $\dfrac{\partial z}{\partial x} = f(x, y, z, \dfrac{\partial z}{\partial y})$ where the function f is analytic in its arguments for values near the given initial conditions $(x_0, y_0, z_0, \dfrac{\partial z}{\partial y})$ where $\dfrac{\partial z}{\partial y}$ is evaluated at $x = x_0$, $y = y_0$ possesses one and only one solution $z(x, y)$ which is analytic near (x_0, y_0).

The theorem can be generalised to functions of more than two independent variables, to derivatives of higher order and to systems of equations. This is an important result in the theory of partial differential equations which continues to play a major role today.[162] In stating the theorem, Poincaré accorded due credit to Kovalevskaya, and the acknowledgement he gave to her here is often cited, "*Mme Kovalevskaya has considerably simplified Cauchy's proof and has given the theorem its definitive form.*"[163]

As discussed in Chapter 3 above, Poincaré himself had previously extended Kovalevskaya's results in his thesis. He now generalised these results, which concerned the first-order partial differential equation

$$\frac{\partial z}{\partial x_1} X_1 + \cdots + \frac{\partial z}{\partial x_n} X_n = \lambda_1 z,$$

where the X_i are power series in $x_1, \ldots, x_n$, to the equation

$$\frac{\partial z}{\partial t} + \frac{\partial z}{\partial x_1} X_1 + \cdots + \frac{\partial z}{\partial x_n} X_n = \lambda_1 z,$$

and found sufficient conditions for this equation to have an integral which can be expanded in powers of x and which is periodic with respect to t.

He then considered the partial differential equation

$$\frac{\partial z}{\partial t} + \frac{\partial z}{\partial x_1} X_1 + \cdots + \frac{\partial z}{\partial x_n} X_n = 0,$$

and showed that a general integral of this equation is given by

$$z = f(T_1 e^{-\lambda_1 t}, \ldots, T_n e^{-\lambda_n t}),$$

where f is an arbitrary function, and T_i are power series in x and periodic with respect to t. Furthermore since solving this partial differential equation is equivalent to solving a system of ordinary differential equations of the form

$$dt = \frac{dx_1}{X_1} = \ldots = \frac{dx_n}{X_n}, \tag{14}$$

[161] See Kovalevskaya [1875].

[162] A good and thorough study of Kovalevskaya's work, together with some applications of the theorem, are given by Cooke [1984, *22-38*].

[163] Poincaré [P2, *26*].

he observed that a general integral of equations (14) is given by

$$T_1 = K_1 e^{\lambda_1 t}, \dots , T_n = K_n e^{\lambda_n t},$$

where K_i are n constants of integration.[164]

In order to determine the variables $x_1, \dots , x_p$, as functions of $x_{p+1}, \dots , x_n$, he considered

$$\frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial x_{p+1}} X_{p+1} + \cdots + \frac{\partial x_i}{\partial x_n} X_n = X_i \qquad (i = 1, \dots p) \tag{15}$$

and showed that these equations admit a series solution in $x_{p+1}, \dots , x_n$, and sines and cosines of multiples of t, provided the λ satisfy certain conditions. Referring to his earlier work on differential equations [1886, *172*], he was further able to show that providing the initial conditions on λ were changed in a certain way, then the equations (15) have a particular integral of the form

$$x_i = \phi_i(x_{p+1}, \dots , x_n, t), \qquad (i = 1, \dots , p),$$

where the ϕ can be expanded as series in $x_{p+1}, x_{p+2}, \dots , x_n$ and sines and cosines of multiples of t.

In the case where the equations

$$dt = \frac{dx_1}{X_1} = \dots = \frac{dx_n}{X_n}$$

are in the same form as the equations (14) except that the λ no longer satisfy the sufficiency conditions for equations (15) to have an analytic solution, then, using the previous result, Poincaré found, not a general solution, but one containing $n-p$ arbitrary constants.

5.4.2. Trigonometric series. In the final part of Chapter I of [P2], which exactly followed *Note D* in [P1], Poincaré discussed the integration of differential equations using trigonometric series. Since the topic had been the subject of several recent studies (he noted in particular the work of Floquet, Callandreau, Bruns, and Stieltjes) his treatment was not detailed but rather focused on the general case.

Using a result which he had derived in [1886c] concerning the convergence of trigonometric series, he showed that if f is an analytic function and periodic of period 2π, then it can be represented as a trigonometric series of the form

$$f(x) = A_0 + A_1 \cos x + \cdots + A_n \cos nx + \cdots + B_1 \sin x + \cdots + B_n \sin nx + \cdots ,$$

which is absolutely and uniformly convergent.

He then looked for a general solution to the system of linear differential equations

$$\frac{dx_i}{dt} = \phi_{i.1} x_1 + \cdots + \phi_{i.n} x_n \qquad (i = 1, \dots , n), \tag{16}$$

where the n^2 coefficients $\phi_{i.k}$ are periodic functions of t of period 2π.

He began with n linearly independent solutions of equations (16),

$$x_1 = \Psi_{i.1}(t), \dots , x_n = \Psi_{i.n}(t) \qquad (i = 1, \dots , n),$$

[164] This is a standard technique for solving partial differential equations which was introduced by Lagrange and extended by Cauchy. See Kline [1972, *531-535*].

which, since equations (16) are unchanged if t is increased to $t+2\pi$, implied the existence of the solutions

$$x_1 = \Psi_{i.1}(t+2\pi), \dots, x_n = \Psi_{i.n}(t+2\pi).$$

There must be linear combinations of the original solutions, so that

$$\Psi_{i.k}(t+2\pi) = A_{i.1}\Psi_{1.k}(t) + \cdots + A_{i.n}\Psi_{n.k}(t) \qquad (i, k = 1, \dots, n),$$

where the A are constant coefficients. If S_1 is a root of the eigenvalue equation

$$\begin{vmatrix} A_{1.1} - S & A_{1.2} & \dots & A_{1.n} \\ A_{2.1} & A_{2.2} - S & \dots & A_{2.n} \\ \dots & \dots & \dots & \dots \\ A_{n.1} & A_{n.2} & \dots & A_{n.n} - S \end{vmatrix} = 0,$$

then there are constants B_k such that

$$\theta_{1.i}(t+2\pi) = S_1\theta_{1.i}(t),$$

where

$$\theta_{1.i}(t) = \sum_{k=1}^{n} B_k\Psi_{k.i}.$$

If $S_1 = e^{2\alpha_1\pi}$, then $S_1\theta_{1.i}(t)$ will be periodic of period 2π and can be expanded as a trigonometric series $\lambda_{1.i}$. Moreover, providing the functions $\phi_{i.k}$ are analytic, then the series will be absolutely and uniformly convergent.

Hence Poincaré wrote a particular solution to the differential equations as

$$x_i = e^{\alpha_1 t}\lambda_{1.i}(t),$$

which gave a correspondence between each root of the eigenvalue equation and each particular solution of the differential equations. If all the roots of the eigenvalue equation are distinct, then there will be n linearly independent solutions to the differential equations, and Poincaré therefore expressed the general solution as

$$x_i = C_1 e^{\alpha_1 t}\lambda_{1.i}(t) + \cdots + C_n e^{\alpha_n t}\lambda_{n.i}(t),$$

where C and a are constants.

Poincaré also showed that if the eigenvalue equation has a double root, then terms of the form $e^{\alpha_1 t}t\lambda(t)$ will be introduced into the solution of the differential equations. Similarly, a triple root will introduce terms of the form $e^{\alpha_1 t}t^2\lambda(t)$, and so on.

In this analysis Poincaré was augmenting results on the theory of differential equations which had originated with Euler and Johann Bernoulli, been generalised to the complex case by Fuchs, and finally connected to the Jordan canonical form by Hamburger.[165] Poincaré's innovation was to extend the theory to a system of differential equations with periodic coefficients.

[165] See Gray [1984, *1-5*].

5.5. Theory of invariant integrals

Consonant with his qualitative approach to the theory of differential equations, Poincaré's investigations into the three body problem are dominated by his research into the geometry of the problem. As the first stage of this research he made a thorough analysis of the concept of *invariant integrals*, which he had originally introduced in [1886].

Although Poincaré was not the first to recognise the existence and value of invariant integrals—they are earlier encountered in both Liouville [1838] and Boltzmann [1871]—he was the first to formalise a theory centred on the concept. In [1886a] he had used the idea of a particular invariant integral within the context of a problem concerning the stability of the solutions of differential equations. He now considered the whole concept in a broader sense, developing a general theory which revealed that the existence of an invariant integral is a fundamental property of Hamiltonian systems of differential equations. Of particular importance is his use of the theory in connection with the stability of the motion in the restricted three body problem.

The last part of the chapter is devoted to a series of theorems, all of which are characterised by their geometric nature and include one of Poincaré's most celebrated results: the original formulation of his recurrence theorem. These theorems provide Poincaré with the geometrical framework for his later analysis, the qualitative study giving him an insight into the global behaviour of the system. In the introduction to *Note F* in [P1] (which does not appear in [P2]) Poincaré made a remark which gives an insight into how he himself viewed these particular theorems: *"These theorems have been given in a geometric form which has to my eyes the advantage of making clearer the origin of my ideas ..."*.[166]

The chapter is also particularly important with regard to Poincaré's error. For it was at the end of this chapter that he made the first mistake. In essence, he failed to take proper account of the exact geometric nature of a particular curve, and it was in correcting this mistake that he was forced to make dramatic changes in the geometric description of his later results.

5.5.1. Definition of invariant integrals. Poincaré considered the system of differential equations

$$\frac{dx_i}{dt} = X_i, \tag{17}$$

where the X_i are some functions of $x_1 \dots x_n$, and the equations are regarded as defining the motion of a point with coordinates $(x_1, \dots, x_n)$ in an n-dimensional space. Thus, if the initial positions of an infinite number of such points form an arc of a curve C in the n-dimensional space, then at time t they will have formed a displaced arc C', its shape determined by the differential equations.

He defined an invariant integral of the system as an expression of the form $\int \sum Y_i dx_i$ which maintains a constant value at all times t, where the integration is taken over the arc of a curve and the Y_i are some functions of x. He extended

[166] "Ces théorèmes ont été présentés sous une forme géométrique qui avait à mes yeux l'avantage de mieux fair comprendre la genèse de mes idées ...", [P1, *220*].

the definition to encompass double and multiple integrals, where the order of the invariant integral is defined to correspond with the dimension of the region of integration.

To give a dynamical interpretation of the idea, he used the example of the motion of an incompressible fluid, where the motion of the fluid is described by the differential equations

$$\frac{dx}{dt} = X, \qquad \frac{dy}{dt} = Y, \qquad \frac{dz}{dt} = Z,$$

together with the incompressibility condition

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0.$$

Since the fluid is incompressible, the flow is volume preserving, and so the volume, which is given by the triple integral $\int\int\int dxdydz$, is an invariant.

More generally, if the equations (17) have the added relation

$$\sum \frac{\partial X_i}{\partial x_i} = 0,$$

then the "volume" $\int\int \cdots \int dx_1 dx_2 \cdots dx_n$, is always an invariant integral. Thus the equations in Hamiltonian form

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i},$$

where F is a function of the double series of variables $x_1, \dots, x_n, y_1, \dots, y_n$, and the time t, always admit the volume in phase space, $\int \dots \int dx_1 \dots dx_n dy_1 \dots dy_n$, as an invariant integral, since

$$\sum \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial y_i}\right) + \sum \frac{\partial}{\partial y_i}\left(\frac{-\partial F}{\partial x_i}\right) = 0.$$

By considering a particular solution of the variational equations, Poincaré found a second invariant of the Hamiltonian system, namely the double integral $\int\int \sum dx_i dy_i$.

Looking specifically at the n body problem, he found that there existed not only invariant integrals which could be deduced from the ten classical integrals of the problem, but that there was also a further invariant not associated with any integral of the original equations. This additional invariant was given by

$$\int \sum (2x_i dy_i + y_i dx_i) + 3(C_1 - C_0)t,$$

where C_0 and C_1 are the values of the energy constant at the extremities of the arc along which the integral is evaluated (in $6n$ dimensional space).

The proof of these last two results first appeared in *Note C*.

5.5.2. Transformation of invariant integrals. The transformation of variables is one of the most frequently employed methods of solving differential equations in celestial mechanics, and so it was natural for Poincaré to consider the effect of such transformations on the associated invariant integrals.

Considering the system of differential equations (17) with the condition

$$\sum \frac{\partial(MX_i)}{\partial x_i} = 0,$$

such that $J = \int M dx_1 \dots dx_n$ is a positive invariant, he found that the transformation

$$x_i = \Psi_i(z_1, \dots, z_n) \qquad (i = 1, \dots, n)$$

left the invariant J positive, provided that in the domain under consideration the x are single-valued functions of the z and vice versa.[167]

In the case where one of the new variables is chosen to be $z_n = C$, where $F(x_1, \dots, x_n) = C$ is a particular integral of the original equations, he found that the transformed equations

$$\frac{dz_1}{dt} = Z_1, \dots, \frac{dz_{n-1}}{dt} = Z_{n-1},$$

admit a positive invariant integral of order $n-1$.

He further observed that the situation was different if the transformation included the independent variable t. For if equations (17) have an invariant integral of order n, and if t_1 is the new independent variable which is defined by $t_1 = \theta(x_1, \dots, x_n)$, then the new invariant integral is given by

$$\int M \left(\frac{\partial \theta}{\partial x_1} X_1 + \dots + \frac{\partial \theta}{\partial x_n} X_n \right) dx_1 \dots dx_n.$$

From this result Poincaré was led naturally to the consideration of sections transverse to the flow. For in the case where $n = 3$ and the x_i are regarded as the coordinates of a point P in space, then a transverse section S of a surface $\theta(x_1, x_2, x_3) = 0$ is the part of the surface on which all the points satisfy

$$\frac{\partial \theta}{\partial x_1} X_1 + \frac{\partial \theta}{\partial x_2} X_2 + \frac{\partial \theta}{\partial x_3} X_3 \neq 0.$$

In other words, the flow defined by the differential equations goes through the surface S and is nowhere tangent to it.

To investigate the existence of invariant integrals over S, Poincaré used his idea of consequents. In [1882] he had introduced these as point iterations on transverse sections, but he now extended the idea to include curves and areas. He considered a volume V bounded by a surface trajectory, where the surface trajectory was formed from a curve C on S bounding an area A passing to its iterate C' bounding an area A'. He showed that if there is a positive invariant integral which extends to the volume V, then there is another integral which conserves its value over the area A or any of its consequents.

[167]The function M satisfying the linear partial differential equation, called the *last multiplier* of the system of differential equations, was introduced by Jacobi in [1844].

5.5.3. The use of invariant integrals. To look at the role of invariant integrals in relation to the stability of the solutions of the restricted three body problem, Poincaré extended his original definition of stability to include the definition he had used in [1885] and which he now called Poisson stability. In this definition the motion of a point P is said to be stable if it returns infinitely often to positions arbitrarily close to its initial position.

Using the result that today is more familiarly known as his recurrence theorem, Poincaré established that, given certain initial conditions, there are an infinite number of solutions of the restricted problem that are Poisson stable, and that those which are not Poisson stable can be considered exceptional in a sense which he made precise.

THEOREM I (recurrence theorem). *Suppose that the coordinates x_1, x_2, x_3 of a point P in space remain finite, and that the invariant integral $\int\int\int dx_1dx_2dx_3$ exists; then for any region r_0 in space, however small, there will be trajectories which traverse it infinitely often. That is to say, in some future time the system will return arbitrarily close to its initial situation and will do so infinitely often.*

In other words, given a system with three degrees of freedom in which the volume is preserved, there are an infinite number of solutions which are Poisson stable.

Poincaré's proof of the theorem is attractively simple.[168]

Consider a region R with volume V within which the point P remains. Then consider a very small region r_0 of R with volume v which at time t consists of an infinite number of moving points. At time τ these points will have filled out a region r_1, at time 2τ a region r_2, etc., and at time $n\tau$ a region r_n, where r_0 and r_1 have no point in common and r_n is the nth iterate of r_0. Since the volume is preserved, each region $r_0 \dots r_n$ will have the same volume v. Thus if $n > \dfrac{V}{v}$ then at least two of the regions have a part in common. Consideration of the successive iterates of this common region shows that there is a collection of points which belong simultaneously to r_0 and to an infinite number of other regions, and that this collection of points itself forms a region σ. From the definition of the region σ, every trajectory which starts from a point within it goes through the region r_0 infinitely often.

COROLLARY. *It follows from the above that there exist an infinite number of trajectories which pass through the region r_0 infinitely often; but there may exist others which pass through it only a finite number of times, although these latter trajectories may be regarded as exceptional.*

By exceptional Poincaré meant that the probability that a trajectory starting in the region r_0 does not pass through the region more than k times is zero, however

[168]It is sometimes suggested that in order properly to rigorise Poincaré's argument it is necessary to have the concept of the "measure" of a set of points, a concept which was not available until Lebesgue presented his ideas on integration in [1902]. In 1915 Van Vleck [1915, *335*] reformulated the theorem in terms of measure theory, and shortly afterwards Carathéodory [1919] provided a proof. Wintner [1947, *414*] believed Poincaré's proof to be correct, and, according to Brush [1980], this view is endorsed by Clifford Truesdell, who considers Carathéodory's reformulation to be simply "cosmetic".

large k and however small the region r_0. The corollary and its proof were additions to [P2]. In [P1] Poincaré simply stated the claim that the stable trajectories would outnumber the unstable, in direct analogy with the irrational and rational numbers.

Poincaré also pointed out that the theorem holds in a variety of other cases, namely when

i) the volume is not an invariant integral but there exists a positive invariant integral $J = \int\int\int M dx_1 dx_2 dx_3$, which remains finite;

ii) $n > 3$ providing there exists a positive n-dimensional invariant integral and the n coordinates of the point P in the n-dimensional space remain finite;

iii) the positive n-dimensional invariant integral extended over the whole n-dimensional space remains finite, even if the n coordinates are not constrained to remain finite.

He also distinguished between the cases when a known integral of equations (17)

$$F(x_1, \dots \; x_n) = constant,$$

is the equation of a system of closed surfaces in an n-dimensional space, and when the integral is the equation of a system of unbounded surfaces in an n-dimensional space. In the former the conditions of the theorem are satisfied without any further constraints, but in the latter the theorem only holds providing a positive invariant integral exists which has a finite value when extended to all systems of values of x where $C_1 < F < C_2$.

In [P2] Poincaré used this last property to extend a result in Hill's lunar theory. Hill [1878] had proved the existence of an upper bound for the radius vector of the moon. Poincaré was now able to strengthen Hill's result by proving that the moon returned infinitely often to positions as close as desired to its initial position, i.e., that the moon has Poisson stability. In his proof he regarded the variables in Hill's differential equations as representing the coordinates of a point in four-dimensional space so that their accompanying integral represented a system of unbounded surfaces. He then showed that the fourth-order invariant integral of the system extended to all points contained between two of these surfaces was finite. From this it followed that his recurrence theorem held, which in turn implied the existence of trajectories passing infinitely often through any region (however small) of the four-dimensional space.

With regard to the restricted three body problem, Poincaré appealed to Bohlin's [1887] generalisation of Hill's result, in which Bohlin had proved the existence of an upper bound for the radius vector of the planetoid, to show that, providing the Jacobian integral remains within certain limits (which in general it does), the motion of the planetoid also possesses Poisson stability. He also observed that the result could not be extended to the general three body problem since it is then no longer possible to assign limits to the coordinates.

In [P1] Poincaré included very little of the above concerning Hill's theory and made no explicit statement about stability in connection with either the lunar theory or the restricted three body problem. He did not prove the result concerning the fourth-order invariant integral nor did he make its significance more accessible by putting it into the context of a particular problem. It was therefore difficult to understand what he was trying to achieve and why. He partly ameliorated the problem in *Note B*, where he translated his results into the more physical language of the astronomers and gave the relevant references to the work of Hill and Bohlin.[168a] Nevertheless, even *Note B* was not sufficiently clear on the behaviour of the radius vector to alleviate all the confusion, and Mittag-Leffler sought further clarification by asking Poincaré for a summary of his definition of stability.[169] Poincaré's detailed response, in which he carefully spelt out the differences between his results and those of Hill [1878] and Bohlin [1887] (i.e., that he had proved the existence of both lower and upper bounds for the radius vector of the planetoid), formed the basis for *An addition to Note B*, which appears at the end of *Note B* in [P1].[170]

Poincaré's next theorem is a generalisation of the result he had applied in [1886b], when he had used the idea of an invariant integral for the first time. This and the remaining results in the second chapter of the memoir are concerned with the properties of the mapping associated with the flow which takes a transverse section into itself.

THEOREM II. *If x_1, x_2, x_3, represent the coordinates of a point in space, and there exists a positive invariant integral, then there is no* closed *transverse section. For $n > 3$ the theorem can be given analytically.*

After the proof of Theorem II Poincaré took the step of introducing a small parameter into the differential equations, the reason being that many dynamical problems, especially those of celestial mechanics and in particular the restricted three body problem, naturally involve small parameters which are used to form power series expansions of the solutions to the differential equations.

In the restricted problem the natural parameter which arises is that of the mass of the smaller of the two primaries, generally designated by μ. The advantage of using μ as the parameter is that it is possible to change the nature of the problem by changing the value of μ. For if $\mu = 0$, the problem reduces to a pair of two body problems and can therefore be solved. This leads to the idea of starting with a particular solution for which $\mu = 0$, and then seeing if it is possible to find solutions for values of μ which are close to but not equal to zero. This is exactly what Poincaré did.

To apply his theoretical results to the restricted problem, Poincaré considered the differential equations

$$\frac{dx_1}{dt} = X_1, \qquad \frac{dx_2}{dt} = X_2, \qquad \frac{dx_3}{dt} = X_3,$$

as functions of the x_i and μ, the solutions of which can be expanded in terms of the parameter.

LEMMA I. *Consider part of a transverse section* S, *passing through the point a_0, b_0, c_0; if x_0, y_0, z_0 are the coordinates of a point of* S, *and if x_1, y_1, z_1 are the coordinates of its consequent, then x_1, y_1, z_1 can be expanded in powers of $x_0 - a_0, y_0 - b_0, z_0 - c_0$ and μ, providing these quantities are sufficiently small.*

[168a] See Chapter 5, end of Section 5.7.

[169] Mittag-Leffler to Poincaré, 21.12.1888, I M-L.

[170] Poincaré to Mittag-Leffler, 15.1.1889, No. 45a, I M-L.

Lemma I was first included in *Note E*, where its proof referred to what is now Theorem II in §5.4. In [P2] Poincaré adjusted the proof by referring to Theorem III from the same section. It made sense to insert it where it now came because of its role in the proof of the following lemma, which did appear in the main text of [P1].

LEMMA II. *If the distance between two points A_0 and B_0 belonging to part of a transverse section* S *is very small of* n*th order, then it will be the same for their iterates A_1 and B_1.*

In each version of this lemma the expression "small quantity of nth order" had a different meaning. In [P1] Poincaré defined a function of x_1, x_2, x_3 and μ as being a small quantity of nth order if it could be expanded in powers of μ with the first term in the expansion being a term in μ^n. In [P2] he defined a function of μ, which need not have a power series expansion in μ, as a small quantity of nth order if it tended to zero with μ in such a way that the ratio of the function to μ^n tended towards a finite limit. This change was needed to fit in with the subsequent alterations to Theorem III.

Up to this point Poincaré's revisions were more or less cosmetic: theorems added, proofs enhanced but no fundamental alterations to the results in [P1]. Essentially the effect has been to give this early part of the memoir a more coherent structure. The rest of this chapter of the memoir tells a rather different story. The next series of changes that Poincaré made were not simply improvements but were necessitated by the discovery of a mistake in the Corollary to Theorem III [P1]. This resulted in a significant change to the theorem itself and the removal of the Corollary altogether.

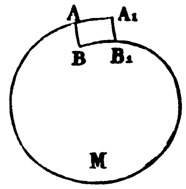
Before discussing these theorems, one definition from [P1] (which does not appear in [P2]) is needed. In Theorem III [P1] Poincaré uses a new term, *quasi-closed*, which he did not use in [P2] but which is important with regard to the error, although unfortunately his definition is not altogether clear. He said that an nth order curve C, by which he meant a curve coincident with its nth iterate, which in general is dependent on μ and is contained on part of a transverse section S, was quasi-closed if there were two points A and B on it which were separated by a finite arc but whose distance apart was very small of pth order.

THEOREM III ([P1]). *If an invariant curve C is quasi-closed such that the distance between the points of closure A and B is very small of* n*th order, and there exists a positive invariant integral, the distance from the point A to its iterate A_1 and that of B to its iterate B_1 are very small of* n*th order.*

In proving the theorem, Poincaré referred to *Figures 5.5.i* and *5.5.ii*[171] and his argument, which hinged on an application of the triangle inequality, showed that the configuration given in *Figure 5.5.ii*, as opposed to that given in *Figure 5.5.i*, was correct.

COROLLARY ([P1]). *If it has been proved that an invariant curve C is quasi-closed so that the distance between the points of closure A and B is very small of* n*th order at least, if moreover it is known that the distance of the point A to its*

[171] Poincaré [P2, *329*]



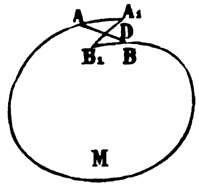


FIGURE 5.5.i FIGURE 5.5.ii

iterate is a finite quantity or a small quantity of $n-1th$ *order* at most*, and finally if there exists a positive invariant integral, then the curve* C *is* closed.

Poincaré gave no proof of the corollary, simply observing that if the curve was only quasi-closed, then the distance AA_1 would have to be of nth order. What he did not explore was the possibility that the curve, rather than being closed, might be self-intersecting. In essence he failed to take into account the full range of possibilities consistent with the constraint of area-preservation imposed by the existence of the invariant integral. Although he knew that the area inside the curve had to remain constant and independent of the iterative process, he focused on a single iteration and appears not to have investigated the possible outcome engendered by the extension of the iterative process. As he later realised and showed in Theorem III [P2], the concept consistent with area preservation was not closure but self-intersection.

Poincaré set up Theorem III [P2] in essentially the same way as the Corollary to Theorem III [P1].

THEOREM III ([P2]). *Let* A_1AMB_1B *be an invariant curve, such that* A_1 *and* B_1 *are the iterates of* A *and* B*. Suppose that the arcs* AA_1 *and* BB_1 *are very small (i.e., they tend to zero with* μ*) but that their curvature is finite. Suppose that the invariant curve and the position of the points* A *and* B *depend upon* μ *according to some rule, and that there exists a positive invariant integral. If the distance* AB *is very small of the* n*th order and the distance* AA_1 *is not very small of the* n*th order, then the arc* AA_1 *intersects the arc* BB_1.

He then discussed the four possible hypotheses:

1. The two arcs AA_1 and BB_1 intersect each other.

2. The curvilinear quadrilateral AA_1B_1B is such that the four arcs which comprise its sides do not have a point in common except for the four corners A, A_1, B, B_1 (as in *Figure 5.5.i*).

3. The two arcs AB and A_1B_1 intersect each other at a point D (as in *Figure 5.5.ii*).

4. One of the arcs AB or A_1B_1 intersects one of the arcs AA_1 or BB_1; but the arcs AA_1 and BB_1 do not intersect each other, neither do the arcs AB and A_1B_1.

He found that he could eliminate the second and the fourth hypotheses because they failed the condition of area preservation; in both cases the area AMB was not

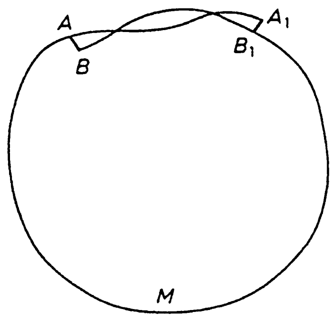


FIGURE 5.5.iii

equal to the area A_1MB_1, although in order to prove the latter case he had to do some additional juggling with the arcs. He also found that the third hypothesis was unacceptable because it implied that the distance AA_1 must be a quantity of nth order (as in Theorem III [P1]) and not a distance of $(n-1)$th order at most as specified. Thus he could conclude that the first hypothesis, that the curve was self-intersecting, was the only one possible.

Poincaré did not include a diagram, but the correct form of the curve is shown in *Figure 5.5.iii*. The reason the curve has to take this slightly complicated shape with two crossing points is to allow for more than one iteration. If after one iteration the curve had only one crossing point, then any subsequent iteration would violate the condition of area preservation.

Why did Poincaré make the mistake in [P1]? Was it simply an oversight? As previously remarked, Poincaré was renowned for paying scant attention to detail, and certainly the deadline for the competition would not have encouraged him to do otherwise. However, perhaps a more convincing argument might be that he had a preconceived idea about how he thought the curve would behave. If he thought he had found what he was expecting, he might not have felt the necessity to scrutinise his results, particularly if he felt pressed for time. As is revealed later, the behaviour of the self-intersecting curve is extremely complex and quite unlike anything Poincaré (or anybody else) had previously encountered. Indeed, when he discovered the mistake and its implication, it came as a complete shock.[172]

In [P1] Poincaré also included two extensions to Theorem III, as well as a fourth theorem. Since none of these appeared in [P2], they have been included as Appendix 6.

5.6. Theory of periodic solutions

Poincaré's discussion of periodic solutions forms the central topic of the memoir. In it he brings together principles and techniques from the previous chapters and from his earlier papers on differential equations and the three body problem. The chapter is dominated by two important ideas connected with the stability of the periodic solutions. The first of these concerns certain constants which appear

[172]Poincaré to Mittag-Leffler, postmarked 1.12.1889, No. 54a, I M-L. The letter is reproduced in full at the end of §5.8.3.

in the solutions and which he originally discussed in [1886]. These are now identified as *characteristic exponents*, and an investigation into their behaviour reveals information about the stability of the solutions. Secondly, there is his remarkable discovery of an entirely new class of solutions which asymptotically approach an unstable periodic solution and which he called *asymptotic solutions*.

In the rewriting Poincaré made several additions and alterations to this chapter; only the section on characteristic exponents survived the transition intact. The most radical change concerned the analytical description of the asymptotic solutions, which underwent a major revision, culminating in the addition, at the end of the chapter, of a completely new section concerned with the asymptotic solutions of the autonomous Hamiltonian equations.

5.6.1. Existence of periodic solutions. Poincaré began with the equations

$$\frac{dx_i}{dt} = X_1 \qquad (i = 1, \dots, n), \tag{18}$$

where the X_i are functions of x, t and the mass parameter μ, and have the additional condition that they are periodic of period 2π with respect to t. Assuming that for $\mu = 0$ there exists a periodic solution $x_i = \phi_i(t)$, where ϕ_i is a periodic function of t with period 2π, then the question Poincaré asked was whether this periodic solution could be analytically continued for small values of the disturbing parameter μ.

Poincaré began by looking for series in powers of μ with periodic coefficients that would satisfy the differential equations. If, having proved the existence of such series, he could also prove their convergence, then he would have proved the existence of the required periodic solutions. However, having got as far as proving the existence of the series, he decided that instead of proving their convergence he would prove the existence of the periodic solutions, which would then imply the convergence of the series. It is not clear why Poincaré changed his approach, especially as he said he thought that the convergence argument could be made directly, although he gave no indication as to how this could be done. Perhaps, as Ian Stewart suggests, he could foresee complications, or maybe he was not absolutely sure how to go about it.[173]

He considered a particular solution close to the original periodic solution

$$x_i(0) = \phi_i(0) + \beta_i, \qquad x_i(2\pi) = \phi_1(0) + \beta_i + \Psi_i,$$

where Ψ_i are analytic functions of μ and β which vanish with these variables, and then sought Ψ_i such that they satisfy the equations

$$\Psi_1 = \dots = \Psi_n = 0. \tag{19}$$

His analysis showed that, providing the Jacobian Δ of Ψ with respect to β was not zero, these equations could be resolved. In other words equations (18) have periodic solutions for small values of μ.

In considering the case when $\Delta = 0$, he used the same method in both [P1] and [P2], but the method's dependence on the method of majorants meant that his improvements to Chapter I in [P2] were particularly beneficial with respect to clarifying his procedure.

[173]Stewart [1989, *67*].

If equations (19) are distinct, then $\beta_1, \dots, \beta_{n-1}$ can be eliminated to give a unique equation $\Phi = 0$. If μ and β_n are then regarded as coordinates of a point in a plane, this equation can be regarded as representing a curve passing through the origin with each of its points corresponding to a periodic solution. By constructing the part of the curve close to the origin, Poincaré was then able to study the behaviour of periodic solutions which correspond to small values of μ and β.

If $\Delta = 0$, then (for $\mu = \beta_n = 0$) $\dfrac{d\Phi}{d\beta_n} = 0$. That is, the curve $\Phi = 0$ is tangent to the line $\mu = 0$ at the origin, and, moreover, when $\mu = 0$, the equation $\Phi = 0$ will be an equation in β_n which admits zero as a multiple root. If the order of multiplicity of the root is m, then, by Theorem V of Chapter I, there exist m series in positive fractions of μ, which vanish with μ and, when substituted for β_n, satisfy $\Phi = 0$. Using these series, Poincaré considered the intersection of the part of $\Phi = 0$ which is close to the origin with the two lines $\mu = \epsilon$, $\mu = -\epsilon$ which are very close to the line $\mu = 0$. If $m_1 (m_2)$ are the number of points of intersection of $\Phi = 0$ with $\mu = \epsilon$ $(\mu = -\epsilon)$ which are real and close to the origin, then Poincaré claimed that m, m_1 and m_2 all have the same parity.[174] Thus if m is odd, then m_1 and m_2 are at least equal to one, and there exist periodic solutions for small values of μ. The result holds for both positive and negative values of μ, although clearly in the context of the restricted three body problem no physical meaning can be attached to the latter.

The above analysis also led Poincaré to the important result that as μ varies the periodic solutions disappear in pairs in the same way as real roots of algebraic equations. For if $m_1 \neq m_2$, then, since they have the same parity, their difference is an even integer, and so as μ increases continuously, the number of periodic solutions which disappear as μ changes sign will be even. In other words, a periodic solution can only disappear when it becomes identical with another periodic solution.

Poincaré looked at the case when for $\mu = 0$ the differential equations admit an infinite number of periodic solutions of the form

$$x_1 = \phi_1(t,h), \dots, x_n = \phi_n(t,h),$$

where h is an arbitrary constant. The equations (19) are no longer distinct for μ $= 0$ and Φ contains μ as a factor, i.e., $\Phi = \mu\Phi_1$. In this case Poincaré showed that the equations still have periodic solutions for small values of μ, but only providing that when $\mu = 0$, the equation $\Phi_1 = 0$ admits $\beta_n = 0$ as a root of odd order.

In [P2] he made the additional point that in the case where the equations admit a single-valued integral $F(x_1, \dots, x_n) = \textit{constant}$, equations (19) will not be distinct unless further conditions are imposed.

Poincaré next considered the existence of periodic solutions when the functions X_i are autonomous and the periodic solutions can be of any period. In other words, if the equations have one periodic solution, they will have an infinite number. For if $x_i = \phi_i(t)$ is a periodic solution, the same will be true of $x_i = \phi_i(t+h)$, whatever the value of the constant h.

[174]The justification for this claim is not immediately obvious, and Poincaré later gave an explanation in [MN I, *70-71*].

If for $\mu = 0$ the equations have a periodic solution $x_i = \phi_i(t)$ of period T, and if for small values of μ

$$x_i(0) = \phi_i(0) + \beta_i, \qquad x_i(T+\tau) = \phi_i(0) + \beta_i + \Psi_i,$$

where Ψ_i are analytic functions of $\mu, \beta_1, \dots, \beta_n, \tau$, then periodic solutions will exist for small values of μ providing it is possible to resolve the n equations

$$\Psi_1 = \Psi_2 = \dots = \Psi_n = 0$$

with respect to the $n+1$ unknowns $\beta_1, \dots, \beta_n, \tau$.

Poincaré showed that having chosen any one of the $\beta_i = 0$, then a sufficient condition for the existence of periodic solutions for small values of μ is that not all the determinants in the matrix

$$\begin{bmatrix} \frac{\partial\Psi_1}{\partial\beta_1} & \frac{\partial\Psi_1}{\partial\beta_2} & \cdots & \frac{\partial\Psi_1}{\partial\beta_n} & \frac{\partial\Psi_1}{\partial\tau} \\ \frac{\partial\Psi_2}{\partial\beta_1} & \frac{\partial\Psi_2}{\partial\beta_2} & \cdots & \frac{\partial\Psi_2}{\partial\beta_n} & \frac{\partial\Psi_2}{\partial\tau} \\ \cdots & \cdots & & \cdots & \cdots \\ \frac{\partial\Psi_n}{\partial\beta_1} & \frac{\partial\Psi_n}{\partial\beta_2} & \cdots & \frac{\partial\Psi_n}{\partial\beta_n} & \frac{\partial\Psi_n}{\partial\tau} \end{bmatrix}$$

are simultaneously zero for $\mu = \beta_i = \tau = 0$, although in this case the periodic solutions have period $T + \tau$ as opposed to period T.

5.6.2. Characteristic exponents. Having established the existence of periodic solutions, Poincaré now turned his attention to the question of their stability. Assuming a periodic solution $\phi(t)$ of equations (18) had been found, he formed the variational equations in order to study the behaviour of nearby solutions.

Since the variational equations are linear differential equations with periodic coefficients, there are n particular solutions (see Chapter 1, [P2]):

$$\xi_{1.k} = e^{\alpha_k t} S_{1k}, \qquad \dots, \qquad \xi_{n.k} = e^{\alpha_k t} S_{nk}, \qquad (k = 1, \dots, n),$$

where the α are constants and the S_{ik} are periodic functions of t with the same period as $\phi(t)$.

The constants α are what Poincaré called the *characteristic exponents* of the periodic solution, and his insight was to realise that they were the key to the stability problem. For if α is purely imaginary then the ξ remain finite and the solution can be said to be stable, and, conversely, if α is not purely imaginary then the solution can be said to be unstable. In other words, investigating the stability of the periodic solutions is equivalent to investigating the properties of their characteristic exponents. As already mentioned, the idea was not entirely new to Poincaré; it had first appeared in [1886], but, as with the case of invariant integrals, he now engaged in a more detailed study.

Drawing further from his results from the first chapter in the memoir, he proceeded to show that if two characteristic exponents are equal then terms of the type $te^{\alpha_k t} S_{jk}$ appear in the solution, and, similarly, if three characteristic exponents are equal then terms which include t^2 outside the exponential and trigonometric functions appear, and so on. He also showed that if the system is autonomous or it has a single-valued integral, then in either case one of the characteristic exponents vanishes.

With regard to Hamiltonian systems, he found that the characteristic exponents can always be arranged in pairs of equal magnitude but opposite sign. Thus if the Hamiltonian system is autonomous, then two of the characteristic exponents are zero. He called the n distinct quantities α^2 the *coefficients of stability* of the periodic solution.

By considering a particular solution of the variational equations in which $\xi_i = \beta_i$ for $t = 0$, and $\xi_i = \beta_i + \Psi_i$ for $t = 2\pi$, he derived the eigenvalue equation

$$\begin{vmatrix} \frac{\partial \Psi_1}{\partial \beta_1} + 1 - e^{2\alpha\pi} & \frac{\partial \Psi_1}{\partial \beta_2} & \cdots & \frac{\partial \Psi_1}{\partial \beta_n} \\ \frac{\partial \Psi_2}{\partial \beta_1} & \frac{\partial \Psi_2}{\partial \beta_2} + 1 - e^{2\alpha\pi} & \cdots & \frac{\partial \Psi_2}{\partial \beta_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \Psi_n}{\partial \beta_1} & \frac{\partial \Psi_n}{\partial \beta_2} & \cdots & \frac{\partial \Psi_n}{\partial \beta_n} + 1 - e^{2\alpha\pi} \end{vmatrix} = 0$$

from which it can be seen that if $\alpha = 0$, then the equation is equivalent to $\Delta = 0$.[175] Conversely, it also implies that if Δ is zero then one of the characteristic exponents must vanish. Consequently, Poincaré could re-express his result concerning the existence of periodic solutions by saying that if equations (18) have a periodic solution for $\mu = 0$ for which none of the characteristic exponents vanish, they will have also have a periodic solution for small values of μ.

5.6.3. Periodic solutions of Hamiltonian systems. Poincaré next considered the existence of periodic solutions in the autonomous Hamiltonian system

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \qquad (i = 1, 2, 3) \tag{20}$$
$$F = F_0 + \mu F_1 + \mu^2 F_2 + \ldots$$

where F_0 is a function of the x only (since in the general problem of dynamics the force function is dependent only on the distance) and $F_1, F_2, \ldots$ are functions of all variables x, y and periodic of period 2π with respect to each y.

Thus when $\mu = 0, x_i$ are constants and $y_i = n_i t + \varpi_i$, where $n_i = -\dfrac{\partial F_0}{\partial x_i}$, and ϖ_i are constants of integration. So for a solution of the differential equations to be periodic when $\mu = 0$, it is necessary and sufficient for the n_i to be commensurable, and, providing the $\dfrac{\partial F_0}{\partial x_i}$ are independent of each other, the x_i can always be chosen so that this condition is fulfilled. The period T will then be the lowest common

[175] If the solution being considered differs only slightly from the periodic solution, so that the squares and higher powers of ξ can be neglected, then the squares and higher powers of β_i can be neglected likewise.

multiple of the $\frac{2\pi}{n_i}$. In other words, when $\mu = 0$ there are an infinite number of choices for the constants x_i which will lead to periodic solutions.

The question then arises of whether these periodic solutions can be analytically continued for small values of μ. Poincaré found that such analytic continuation was possible providing the periodic solutions correspond (in the simplest case) to pairs of Kepler circles with rational frequency ratio and a certain phase relation determined by the critical points of a function Ψ, where Ψ is the mean value of F_1 considered as a periodic function of t.

Although he approached the question of analytic continuation using the same methods in both [P1] and [P2], the two presentations appear rather different. The results are essentially similar, but in [P2] they are expressed in a more logical order with additional explanations. In [P1] Poincaré began by expanding the coordinates as a series in μ and then proving the existence of the periodic solutions, whereas in [P2] he first proved the existence of the periodic solutions before expanding the coordinates and determining the coefficients of the series. Furthermore, [P2] contains a discussion of the application of the theory to the restricted three body problem that is not found in [P1].

Poincaré started in [P2] by supposing that for $\mu \neq 0$ a particular solution at $t = 0$ is given by

$$x_i = a_i + \delta a_i, \qquad y_i = \varpi_i + \delta\varpi_i,$$

and that for $t = T$ the solution has the values

$$x_i = a_i + \delta a_i + \Delta a_i, \qquad y_i = \varpi_i + n_i T + \delta\varpi_i + \Delta\varpi_i.$$

Thus the solution will be periodic if

$$\Delta a_1 = \Delta a_2 = \Delta a_3 = \Delta\varpi_1 = \Delta\varpi_2 = \Delta\varpi_3 = 0.$$

However, since $F =$ *constant* is an integral of equations (20) and F is periodic with respect to y, these equations are not independent. Hence it is only necessary to satisfy five of them. Furthermore, choosing $t = 0$ when $y_1 = 0$, gives $\varpi_1 = \delta\varpi_1 = 0$.

Poincaré showed that the five equations could be satisfied provided both that ϖ_2 and ϖ_3 were chosen in such a way that

$$\frac{\partial\Psi}{\partial\varpi_2} = \frac{\partial\Psi}{\partial\varpi_3} = 0, \tag{21}$$

and that neither the Hessian of Ψ with respect to ϖ_2 and ϖ_3 nor the Hessian of F_0 with respect to x_i^0 were equal to zero.[176]

Since Ψ is finite and periodic in ϖ_2 and ϖ_3, equation (21) is always satisfied, and so, providing μ is sufficiently small and neither of the two Hessians vanish, there exists a periodic solution of period T, where T is determined by the choice of the numbers n_i.

[176]The Hessian of a function is the determinant of the matrix of which the entries are given by the second partial derivatives of the function. It is named after the German geometer Ludwig Otto Hesse (1811-1874).

Furthermore, if $n_i' = n_i(1+\epsilon)$ then, providing ϵ is small, there exists a periodic solution for small values of μ,

$$x_i = \phi_i(t, \mu, \epsilon), \qquad y_i = \phi_i'(t, \mu, \epsilon)$$

with period $T' = \dfrac{T}{1+\epsilon}$ which is nearly equal to T.

In the case of the restricted three body problem, where there are only two degrees of freedom, the function Ψ depends only on ϖ_2 and so the relations (21) reduce to

$$\frac{\partial \Psi}{\partial \varpi_2} = 0 \tag{22}$$

and the Hessian of Ψ reduces to $\dfrac{\partial^2 \Psi}{\partial \varpi_2^2}$. Hence, corresponding to each of the simple roots of equation (22) there is a periodic solution for all sufficiently small values of μ, and, as established earlier, the same is true for each of the roots of odd order.

Returning to the case where the periodic solutions have period T, and having shown that they could be expressed in the form of convergent series in powers of μ,

$$x_i = x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \dots \qquad (i = 1, 2, 3)$$
$$y_i = y_i^0 + \mu y_i^1 + \mu^2 y_i^2 + \dots\,,$$

Poincaré 's next step was to determine the coefficients of the series.

Considering the unperturbed motion gives values for x_i^0 and y_i^0, but calculating the remaining coefficients requires a careful analysis. Poincaré's procedure, although somewhat lengthy, did not, however, impose any further restrictions on the periodic solutions since the only constraint on its validity was that the Hessian of F_0 with respect to x_i^0 did not vanish.

Applying the theory to specific problems, Poincaré began with the system described by the differential equation

$$\frac{d^2\rho}{dt^2} + n^2\rho + m\rho^3 = \mu R(\rho, t).$$

This equation, now generally known as Duffing's equation, is often encountered in celestial mechanics, where it occurs in the theory of libration.[177] It also arises in solid mechanics, where it can be modelled by a pendulum under the action of an imposed periodic force.

To prove the existence of periodic solutions, Poincaré simply applied a series of transformations to put the equation into Hamiltonian form, and it was then straightforward to see that the requisite conditions were fulfilled. Although, as he observed, when the nonlinearity is absent, the Hessian with respect to F_0 is zero, and the theory can no longer be applied.

Turning to the three body problem Poincaré encountered the same difficulty with the vanishing Hessian, although, as he described, in the case of the restricted problem it can be easily overcome. In this particular case the small number of variables means that it is possible to find a function of F which can be used legitimately

[177]Duffing [1918] made an extensive study of this equation in the context of solid mechanics.

to replace F in the Hamiltonian equations and for which the Hessian of F_0 does not vanish. Unfortunately, the same method does not work in the general problem, and an alternative method of establishing the existence of periodic solutions needs to be found.[178]

5.6.4. Calculation of characteristic exponents. To calculate the characteristic exponents of the autonomous Hamiltonian system, Poincaré began by supposing that a periodic solution of the equations was given by $x_i = \phi_i(t)$, $y_i = \Psi_i(t)$ with a nearby solution given by $x_i = \phi_i(t) + \xi_i$, $y_i = \Psi_i(t) + \eta_i$. This leads to the equations of variation

$$\frac{d\xi_i}{dt} = \sum \frac{\partial^2 F}{\partial y_i \partial x_k}\xi_k + \sum \frac{\partial^2 F}{\partial y_i \partial y_k}\eta_k, \qquad (i, k = 1, 2, 3),$$
$$\frac{d\eta_i}{dt} = -\sum \frac{\partial^2 F}{\partial x_i \partial x_k}\xi_k - \sum \frac{\partial^2 F}{\partial x_i \partial y_k}\eta_k,$$

with solutions in the form

$$\xi_i = e^{\alpha t} S_i, \qquad \eta_i = e^{\alpha t} T_i, \tag{23}$$

S_i and T_i being periodic functions of t. Since it is an autonomous Hamiltonian system, two of the characteristic exponents are zero, and so there are only four particular solutions.

When $\mu = 0$, F is reduced to F_0, and the variational equations are reduced to

$$\frac{d\xi_i}{dt} = 0, \qquad \frac{d\eta_i}{dt} = -\sum \frac{\partial^2 F_0}{\partial x_i^0 \partial x_k^0}\xi_k,$$

where the coefficients of the second equation are constants. In this case, the most general solution is given by $\xi_i = 0$ and $\eta_i = \eta_i^0$, where η_i^0 are constants of integration, and so all six characteristic exponents are zero.

To find values for the functions α, S_i and T_i which satisfy equations (23) for small values of μ, Poincaré sought series expansions in powers of the parameter. The difficulty is that, since all the characteristic exponents are zero when $\mu = 0$, α cannot be expanded in integer powers of μ, since the conditions necessary for the implicit function theorem to be valid are no longer fulfilled. This leads to the question of whether α can be expanded in fractional powers of μ.

What Poincaré found was that α, as well as S_i and T_i, could be expanded in powers of $\sqrt{\mu}$, and so could be written[179]

$$\alpha = \alpha_1\sqrt{\mu} + \alpha_2\mu + \dots \qquad (\alpha_0 = 0, \quad \text{since } \mu = 0 \Longrightarrow \alpha = 0)$$
$$S_i = S_i^0 + S_i^1\sqrt{\mu} + S_i^2\mu + \dots$$
$$T_i = T_i^0 + T_i^1\sqrt{\mu} + T_i^2\mu + \dots .$$

To calculate the coefficients in these series, he proceeded by first substituting these series in equations (23), and differentiating with respect to t. Next, having expanded the second derivatives of F as series in integer powers of μ, he made

178 Poincaré resolved this difficulty in [MN I, *133*].

179 Poincaré was not the first to form series in powers of the square root of the parameter. As he acknowledged in his introduction to [P2], series of this type occur in Bohlin [1888], where they are used to overcome the problem of small divisors in planetary perturbation theory. Poincaré later made a careful examination of Bohlin's series in [MN II]. See Chapter 7.

the appropriate substitutions in the variational equations, and then determined the coefficients by equating powers of $\sqrt{\mu}$. By this process he was able to calculate the coefficients as far as α_i, S_i^m and T_i^m.

In [P1] Poincaré went straight into the calculation of the coefficients without first proving that such series do indeed exist, and, moreover, giving no mathematical explanation as to why they should be series in powers of $\sqrt{\mu}$ rather than μ, or indeed rather than any other fractional power of μ. He went some of the way towards rectifying this omission in *Note H*, although his proof for the existence of the series invoked theorems concerning the method of majorants which only appeared in [P2]. [P2] contained a much more detailed existence proof for the series for α, including showing that the expansion for α only contains odd powers of $\sqrt{\mu}$, and it also contained a proof for the existence of the other two series, neither of which had been included in [P1].[180]

In his determination of the coefficients in the series for α, Poincaré found that the sign of α_1^2 depended on the sign of $\dfrac{\partial^2\Psi}{\partial\varpi^2}$; in other words, it depended on the derivative of equation (22), the roots of which correspond to periodic solutions. Since the stability of the periodic solutions depends on the sign of $|\alpha^2|$, if μ is sufficiently small, this translates into the stability being dependent on the sign of α_1^2. Poincaré was therefore interested in the behaviour of equation (22). He considered the general case when the equation only has simple roots, i.e., the roots correspond to maxima and minima of the function Ψ. Since Ψ is a periodic function, there is at least one maximum and one minimum within each period, and exactly the same number of each. Consequently there are precisely as many roots for which the derivative and α_1^2 are positive as roots for which the derivative and α_1^2 are negative. This means that, corresponding to each system of values of n_1 and n_2, there is at least one stable and one unstable periodic solution, and, providing μ is sufficiently small, there are exactly the same number of each.

In [P2] Poincaré also showed how it was possible to continue the calculation of the coefficients for the series for S_i and T_i beyond the terms S_i^m and T_i^m already calculated.

Many of the changes Poincaré made to this section can be directly attributable to intervention from Phragmén. In the introduction to [P2] Poincaré specifically mentions Phragmén's help with regard to the calculation of the coefficients for the series for S_i and T_i. In addition, according to Mittag-Leffler, Poincaré's inclusion of the existence proofs were also prompted by queries from Phragmén.[181]

5.6.5. Asymptotic solutions. Poincaré next turned his attention to the unstable periodic solutions and the behaviour of other solutions in their immediate neighbourhood.

[180] Although Poincaré had shown algebraically why the expansions had to be in powers of $\sqrt{\mu}$, he gave no dynamical explanation of the result. In essence, the square root arises because near a periodic solution a perturbation changes the nature of some of the phase curves, so straightforward perturbation theory cannot be used. However, a local transformation can be found in which the unperturbed Hamiltonian is similar to that of a vertical pendulum for which the separatrix (the special phase curve which separates the phase curves with different properties) width is of the order of $\sqrt{\mu}$, and this automatically introduces a $\sqrt{\mu}$ into the new perturbation.

[181] Mittag-Leffler to Poincaré 16.7.1889, I M-L.

Starting with equations (18), he supposed that

$$x_1 = x_1^0, \dots, x_n = x_n^0$$

was a periodic solution with a neighbouring solution $x_i = x_i^0 = \xi_i$. He then derived a system of equations to determine ξ_i,

$$\frac{d\xi_i}{dt} = \Xi_i, \tag{24}$$

where the Ξ are functions which can be expanded in powers of ξ, are periodic with respect to t, and have no terms independent of ξ. Neglecting powers of ξ, equations (24) reduce to the linear equations of variation with general solution $\xi_i = \sum A_k e^{\alpha_k t}\phi_{ki}$, where A are constants of integration, α characteristic exponents, and ϕ periodic functions of t.

To solve the equations when they include powers of ξ, Poincaré made the linear transformation

$$\xi = \sum \eta_k \phi_{ki}$$

so that (24) become

$$\frac{d\eta_i}{dt} = H_i = H_i^1 + \dots + H_i^n + \dots, \tag{25}$$

where the H_i are functions of t and η of the same form as Ξ, and H_i^p represent the collection of terms of H_i of degree p respect to η. He then looked for general solutions to equations (24) and (25).

By writing

$$\eta_i = \eta_i^1 + \dots + \eta_i^n + \dots,$$

where η_i^p represent the terms of η_i of degree p with respect to A, replacing η_i in H_i^k, and calculating η_i^q by recurrence, he found

$$\frac{d\eta_i^q}{dt} - \alpha_i \eta_i^q = \sum C A^q e^{\Omega t},$$

where $A^q = A_1^{\beta_1} \dots A_n^{\beta_n}$, $\Omega = \gamma\sqrt{-1} + \sum \alpha\beta$, γ is a positive or negative integer, $\sum \alpha\beta = \alpha_1\beta_1 + \dots + \alpha_n\beta_n$, and the β_i are positive integers with sum q.

This equation is satisfied by

$$\eta_i^q = \sum \frac{C A^q e^{\Omega t}}{\Omega - \alpha_i},$$

where C is generally imaginary, excluding the exceptional case when $\Omega - \alpha_i = 0$, when terms in t are introduced.

He proved that the series

$$\eta_i = \sum N \frac{A_1^{\beta_1} A_2^{\beta_2} \dots A_n^{\beta_n}}{\Pi} e^{\Omega t},$$

where Π represents the product of the divisors $\Omega - \alpha_i$, is convergent, providing that $\Omega - \alpha_i$ does not become less than any given quantity ϵ for positive integer values of β and positive or negative integers γ; i.e., if neither of the two convex polygons containing $\alpha \pm \sqrt{-1}$ contain the origin or if the real part of the quantities α are the same sign and not equal to zero. Although he observed that the convergence followed immediately from his results on the method of majorants applied to partial differential equations described earlier, he also provided a direct proof.

With the restricted three body problem in mind, Poincaré considered the particular system represented by the differential equations

$$\frac{dx_1}{dt} = X_1, \qquad \frac{dx_2}{dt} = X_2,$$

with the added condition

$$\frac{\partial X_1}{\partial x_1} = \frac{\partial X_2}{\partial x_2} = 0,$$

which implies that the "volume" is an invariant integral.

As Poincaré pointed out, since a state of the system only depends on the variables x_1, x_2, and t, it can be represented by the position of a point in space with coordinates $e^{x_1}\cos t$, $e^{x_1}\sin t$, x_2. A periodic solution can then be represented by a closed curve, and if the periodic solution is unstable the coefficient of stability α^2 will be real and positive. In this case the η_i can be expanded as a series in $Ae^{\alpha t}$ and $Be^{-\alpha t}$. Thus if $A = 0$, and $t \longrightarrow t + \infty$, then η_1 and $\eta_2 \longrightarrow 0$ and the corresponding solution asymptotically approaches the periodic solution. Similarly, if $B = 0$ and $t \longrightarrow -\infty$, then again η_1 and $\eta_2 \longrightarrow 0$ and the solution again asymptotically approaches the periodic solution. These two series of solutions, the first corresponding to $t = +\infty$ and the second corresponding to $t = -\infty$, are what Poincaré called *asymptotic solutions.* Moreover, since each of these series corresponds to a sequence of curves which asymptotically approaches a closed curve C, Poincaré called the surface formed by the set of these curves an *asymptotic surface.* Thus, there are two asymptotic surfaces, one corresponding to $t = +\infty$ and the other corresponding to $t = -\infty$, and both of these surfaces pass through the closed curve C.

In [P1] Poincaré went through a similar analysis to show that in the case of equations (18) the series for η could be expanded in a convergent series in $A_i e^{\alpha_i t}$. But at the end of the analysis he added the claim that if the differential equations depend on the parameter μ then the series could also be expanded in powers of μ or $\sqrt{\mu}$, according to the circumstances. Nowhere did he prove that such expansions were actually possible. Furthermore, implicit in his claim was that the series in each case was convergent. Particularly significant is the fact that he made no attempt to distinguish between the autonomous and nonautonomous cases. As he later discovered, neglecting to make this distinction was a serious oversight.

In [P2] the ending of the section was quite different. Poincaré proved that the η could be represented by a series in μ, and, moreover, that these series were convergent, subject to three conditions. First, that the differential equations are dependent on the parameter μ and the functions X_i can be expanded in powers of the parameter; second, that for $\mu = 0$, all the characteristic exponents α are distinct and can be expanded in integer powers of μ; and third, it is possible to remove all the constants A which correspond to an α whose real part ≤ 0.

The significant condition to observe is the second one, which concerns the characteristic exponents. For this condition means that if the system under consideration is an autonomous Hamiltonian system, then the series are not convergent, and in particular the series are not convergent in the case of the restricted three body problem. This point is central with regard to Poincaré's error. For in [P1] Poincaré had not appreciated that, in describing the behaviour of asymptotic solutions, there was a critical difference between autonomous and nonautonomous systems, a difference which initially manifests itself in the values of the characteristic exponents. In [P2] the distinction between the two cases is clearly made, the nonautonomous case having been dealt with here and the autonomous case being the subject of the next section.

5.6.6. Asymptotic solutions of Hamiltonian systems. Poincaré had already proved that there were circumstances under which the autonomous Hamiltonian system had periodic solutions, and so to establish the existence of asymptotic solutions he only had to make certain that one of the corresponding characteristic exponents α was real. That being so, it only remained to ascertain the form of the asymptotic solutions.

In the case discussed in the previous section the functions X_i were expanded in powers of μ, and the characteristic exponents were distinct for $\mu = 0$. In the case of the autonomous Hamiltonian equations, the right-hand side of the equations can again be expanded as powers of μ, but now all the characteristic exponents vanish when $\mu = 0$.

This results in several important differences. First, as Poincaré had already described, the expansions for the characteristic exponents are in powers of $\sqrt{\mu}$ rather than μ. Similarly, the expansions of the functions $\phi_{i.k}$ which appear in the general solution to the variational equations and which, in this case, are the expansions of the functions S_i and T_i are also in powers of $\sqrt{\mu}$ rather than μ. Furthermore, this implies that the expansions of the functions H_i are in powers of η, $e^{t\sqrt{-1}}$, $e^{-t\sqrt{-1}}$, and $\sqrt{\mu}$ (and not of μ). Although η_i can be derived as before,

$$\eta_i = \sum N \frac{A_1^{\beta_1} A_2^{\beta_2} \dots A_n^{\beta_n}}{\Pi} e^{\Omega t},$$

the expansions of N and Π are now also in powers of $\sqrt{\mu}$.

These differences led Poincaré to ask the following questions:

1. Since N and Π can be expanded as powers of $\sqrt{\mu}$, can the quotient $\dfrac{N}{\Pi}$ also be expanded in powers of $\sqrt{\mu}$?

2. If the answer to Question 1 is yes, then this implies the existence of series in $\sqrt{\mu}$, $A_i e^{\alpha_i t}$, $e^{t\sqrt{-1}}$, and $e^{-t\sqrt{-1}}$, which *formally* satisfy the equations; are these series convergent?

3. If the series are not convergent, can they be used to approximate the asymptotic solutions?

With regard to Question 1, since both N and Π can be expanded in powers of $\sqrt{\mu}$, Poincaré realised that the only problem that could arise with the expansion of their quotient is the appearance of negative powers of $\sqrt{\mu}$. For if this should occur then the asymptotic solutions would cease to exist for $\mu = 0$. His answer therefore consisted in proving that these negative powers never arise. He had previously recognised the existence of this particular problem and included an earlier version of the proof in *Note I*.

Poincaré had therefore proved the existence of series which formally satisfied the equations, but were these series convergent? Importantly, Poincaré showed that they were not. However, the discovery of their divergence was entirely unexpected, since his analysis in [P1] had led him to believe that they were convergent. Put in the context of the whole memoir, his failure to appreciate the divergence of these series is essentially the analytical analogue of the geometrical mistake that he made at the end of his discussion on invariant integrals in [P1].

In [P2] Poincaré proved that, rather than being convergent, the series belonged to the class of divergent series that he had defined in [1886a] as asymptotic series. [P1a] reveals that he was slightly concerned about the status of this particular proof despite describing it in [P2] as "rigoureuse". In [P1a] the word "rigoureuse" was originally preceded by the word "plus", which was then crossed out and replaced by the word "absolument", which was also crossed out.

He began with the expression $(\Omega - \alpha_i)^{-1}$. If γ is not equal to zero, then this expression can be expanded in powers of $\sqrt{\mu}$, but the radius of convergence of the series will tend to zero as $\dfrac{\gamma}{\sum \beta}$ tends to zero. Thus if the expression $\dfrac{1}{\Pi}$ is expanded in powers of $\sqrt{\mu}$, there will always be an infinite number of such expressions for which the radius of convergence of the expansion is arbitrarily small. If the same is true for $\dfrac{N}{\Pi}$, then this implies that the series are divergent.

Rather than considering the series for η_i, Poincaré began with the simpler series

$$F(w, \mu) = \sum \frac{w^n}{1 + n\mu},$$

where $w = A_i e^{a_i t}$. This series in w is uniformly convergent when $\mu > 0$ and $|w| < w_0 < 1$, and if differentiated the resulting series is also uniformly convergent.

On the other hand, if the function $F(w, \mu)$ is expanded as a series in μ,

$$F(w, \mu) = \sum_{n,p} w^n (-n)^p \mu^p, \tag{26}$$

then, as Poincaré knew from his theory developed in [1886a], the series is not convergent but is an asymptotic expansion.

Poincaré claimed that the series (26) was completely analogous both to the series which represent the functions η_i,

$$\sum \frac{N}{\Pi} w_1^{\beta_1} \dots w_k^{\beta_k} e^{\gamma t \sqrt{-1}} = F(\sqrt{\mu}, w_1, \dots, w_k, t), \qquad (w_i = A_i e^{\alpha_i t}),$$

and to the series

$$\sum w_1^{\beta_1} \dots w_k^{\beta_k} e^{\gamma t \sqrt{-1}} \frac{d^p \left(\frac{N}{\Pi}\right)}{(d\sqrt{\mu})^p} = \frac{d^p F}{(d\sqrt{\mu})^p}.$$

These two series are uniformly convergent when expanded in powers of w provided $|\, w \,| < w_0 < 1$ and $\sqrt{\mu}$ is real, but if $\dfrac{N}{\Pi}$ is expanded in powers of $\sqrt{\mu}$, then they are divergent. Thus if they are analogous to the series (26) they must be asymptotic expansions.

Poincaré first defined $\Phi_p(\sqrt{\mu}, w_1, \dots, w_k, t)$ to be a polynomial of degree p in $\sqrt{\mu}$ which can be expanded in powers of w, and $e^{\pm t\sqrt{-1}}$. The series for $\dfrac{F - \Phi_p}{\sqrt{\mu^p}}$ is then given by

$$\sum \frac{1}{\sqrt{\mu^p}}\left(\frac{N}{\Pi} - H_p\right) w_1^{\beta_1} \dots w_k^{\beta_k} e^{\gamma t\sqrt{-1}},$$

where H_p is the group of terms in the expansion of $\frac{N}{\Pi}$ in which the exponent of $\sqrt{\mu}$ is at most equal to p. To prove that the series for η_i is an asymptotic expansion, this series must be shown to be uniformly convergent with its terms tending to zero as μ tends to zero. This convergence proof turned out to require a long and delicate analysis, and Poincaré's attempt in [P2] included some unproven assertions, which doubtless accounts for his concern about the rigour.[182]

Nevertheless, he correctly concluded that the series for the asymptotic solutions

$$x_i = x_i^0 + \sqrt{\mu x_i^1} + \mu x_i^2 + \dots, \qquad y_i = n_i t + y_i^0 + \sqrt{\mu y_i^1} + \mu y_i^2 + \dots$$

were asymptotic expansions and, in addition, that if they were differentiated they would also give asymptotic expansions.

5.7. Study of the case with two degrees of freedom

In [P1] the opening chapter of the second part of the memoir consisted of five sections and constituted the major part of this half of the memoir. In [P2] Poincaré changed the structure so that only the first of these sections, which concerned the geometric representation of systems of differential equations, was contained in the opening chapter of the second part.

Poincaré's task now was to apply the foregoing theory to the restricted three body problem. He therefore focused on the Hamiltonian system with two degrees of freedom

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \qquad (i = 1, 2), \tag{27}$$

where the x_i are linear variables, y_i are 2π periodic angular variables, and F is an autonomous function of x_i and y_i.

His strategy was to begin by showing how such a system can be given a geometric representation that uniquely identifies its each and every state, the chosen representation dependent on the given constraints of the particular problem under consideration.

First, Poincaré used the property that the four variables are linked by the Jacobian integral

$$F = (x_1, x_2, y_1, y_2) = C,$$

to create a representation in which each state of the system is represented by a point in a three-dimensional space. By adding further conditions he developed representations in which each state of the system was represented either by a point contained between two tori or by an interior point of a torus.

In the case of the restricted three body problem he started by defining the position of the planetoid using the osculating elements, i.e., the variables defined

[182] Later, in the first volume of the *Méthodes Nouvelles*, Poincaré gave a fuller version of the proof in which he supplied the missing details [MN I, *353-382*].

by means of the instantaneous ellipse described by the planetoid round the centre of gravity of the system. He adopted Tisserand's [1887] notation (derived from Delaunay) to write the equations of motion in canonical form

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial R}{\partial l}, \qquad \frac{dl}{dt} = -\frac{\partial R}{\partial L} \\ \frac{dG}{dt} &= \frac{\partial R}{\partial g}, \qquad \frac{dg}{dt} = -\frac{\partial R}{\partial G}, \end{aligned} \tag{28}$$

where l is the mean anomaly of the planetoid, g is the longitude of its perihelion, n is the mean motion of the planetoid, $L = \sqrt{a}$, where a is the semi-major axis of the instantaneous ellipse, and $G = \sqrt{a(1-e^2)}$, where e is the eccentricity of the ellipse. In order to preserve the canonical form, the standard perturbation function is increased by the addition of the term $\frac{1}{2a} = \frac{1}{2L^2}$ to give the Hamiltonian R.

He chose his units so that the masses of the two primaries were 1 - μ and μ, the gravitational constant was equal to one, the mean motion of the smaller of the two primaries was equal to one, and its longitude was equal to t. Under these conditions the angle from which the distance between the two smaller masses is seen from the larger differs from $l + g - t$ by a periodic function of l of period 2π.

Since the distance between the primaries is constant and the distance between the larger of the two primaries and the planetoid is only dependent on L, G and l, the function R is only dependent on L, G, l and $l + g - t$. Moreover, since R is periodic with period 2π with respect to l, and with respect to $l + g - t$,

$$\frac{\partial R}{\partial t} + \frac{\partial R}{\partial g} = 0,$$

and equations (28) admit the integral $R + G = \textit{constant}$.

However, R has an explicit dependence on t, and so equations (28) are not in the required form of equations (27). To remedy this Poincaré made the transformation

$$x_1 = G, \qquad x_2 = L, \qquad y_1 = g - t, \qquad y_2 = l$$
$$F(x_1, x_2, y_1, y_2) = R + G.$$

The function F is dependent on the mass parameter μ, and so can be written

$$F = F_0 + \mu F_1,$$

which if $\mu = 0$ reduces to

$$F = F_0 = \frac{1}{2a} + G = x_1 + \frac{1}{2x_2^2},$$

which is a function of only the linear variables.

By definition, $L^2 \geq G^2$, which implies that $x_2 \geq x_1 \geq -x_2$. If $x_1 = +x_2$, then the eccentricity is zero, and the perturbation function and the state of the system only depend on the difference of the longitude of the two smaller masses, i.e., they only depend on

$$l + g - t = y_1 + y_2.$$

Consequently

$$\frac{\partial F}{\partial y_1} = \frac{\partial F}{\partial y_2},$$

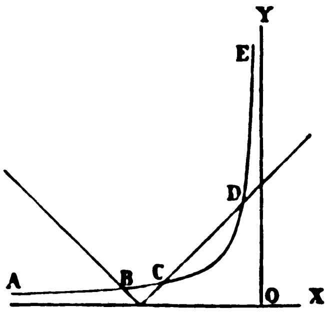


FIGURE 5.7.i

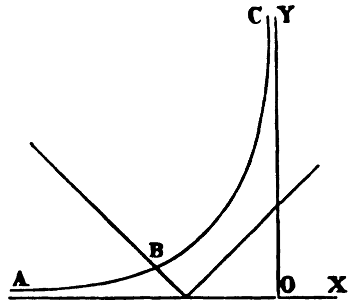


FIGURE 5.7.ii

which implies

$$\frac{d(x_1 - x_2)}{dt} = 0,$$

and since $x_2^2 \geq x_1^2$, the maximum value of $x_1 - x_2$ is 0. (x_1 is not identically equal to x_2 since in equations (27) $x_1 = x_2$ only if there is a singularity.) If $x_2 = -x_1$, again the eccentricity is zero but the motion is then retrograde, which always occurs when x_1 and x_2 are of different sign.

To create a geometric representation for the restricted three body problem, Poincaré had to represent the system using only three variables. He therefore sought to express x_1 and x_2 as single-valued functions of y_1, y_2 and a new variable ξ.

He began with $\mu = 0$ and considered the plane in which the coordinates of a point are defined by

$$X = x_1 - C, \qquad Y = x_2,$$

which, from the definition of F and the constraint on x_1, implies

$$X + \frac{1}{2Y^2} = 0, \qquad Y > X + C > -Y.$$

The construction of the curve $X + \dfrac{1}{2Y^2} = 0$, together with the lines $X + C = \pm Y$, takes two different forms depending on the value of the constant C as shown in *Figures 5.7.i* and *5.7.ii*,[183] the transition point that occurs when the line CD becomes tangent to the curve takes place when $X = \frac{1}{2}$, $Y = 1$, and $C = \frac{3}{2}$. Although the inequalities are satisfied by the curves BC and DE, in *Figure 5.7.i* and BC in *Figure 5.7.ii*, the part of the curve which is of interest with respect to the problem is the part which is bounded, that is, BC in *Figure 5.7.i.*

For $\mu = 0$, choosing $\xi = \dfrac{x_2 - x_1}{x_2 + x_1}$ fulfils the required conditions since along the arc CB, ξ increases constantly from 0 to ∞.

For small values of μ, ξ can be chosen in the same way, but only if $x_2 > 0$, and the Jacobian $\dfrac{\partial(\xi, F)}{\partial(x_1, x_2)}$ is not equal to zero. Providing the value of C is not close

[183]Since the figures are symmetric with respect to the X axis, Poincaré only included diagrams of the top half of the X, Y plane [P2, *402*].

to $\frac{3}{2}$ these conditions are satisfied for small values of μ, and ξ can be taken as the independent variable.

Finally, in order to make a more convenient representation, Poincaré made a further transformation to another set of canonical variables

$$x_1' = x_1 + x_2, \qquad x_2' = x_1 - x_2,$$

$$y_1' = \frac{1}{2}(y_1 + y_2), \qquad y_2' = \frac{1}{2}(y_1 - y_2).$$

In this form y_i' are angular variables which if increased by 2π generate an identical increase in y_i, and so the system remains unchanged. The system also remains unchanged if simultaneously y_i' and y_2' are each increased by π. A state of the system can then be represented by a point in space with rectangular coordinates

$$X = \cos y_1' e^{\xi \cos y_2'} \qquad Y = \sin y_1' e^{\xi \cos y_2'} \qquad Z = \xi \sin y_2'.$$

In this representation each point in space corresponds to a single state of the system, while the two systems of values (x_1', x_2', y_1', y_2') and $(x_1', x_2', y_1' + \pi, y_2' + \pi)$, which correspond to two different points of space, correspond to only one state of the system.

In addition, applying the transformation has the effect of reducing the fourth-order invariant integral of the Hamiltonian equations to a third-order positive invariant.

When $\mu = 0$, equations (28) integrate to give

$$L = constant, \qquad G = constant, \qquad g = constant, \qquad l = nt + constant.$$

These solutions can be represented by trajectories that are closed whenever the mean motion n is a rational number. They lie on the surface trajectories that are defined by the general equation $\xi = constant$ and, consequently, generate closed surfaces of revolution analogous to tori.

In the following chapter in [P2], Poincaré showed the effects on these results when the system no longer remains unperturbed and μ takes on small values.

He concluded the current chapter with the consideration of two more dynamical problems. For the first he returned to the system described by Duffing's equation, and for the second he considered a heavy point mass moving on a frictionless surface in the neighbourhood of a stable equilibrium. In each case he generated a similar representation to the one he had derived for the restricted problem.

By putting this section on geometrical representations into a chapter on its own in [P2], Poincaré has accorded it a higher degree of prominence than its counterpart in [P1] (which was included in a chapter with other sections). This change of emphasis is quite revealing.

It is clear that for Poincaré framing dynamical problems geometrically came naturally (as exemplified by his remarks on the theorems concerning invariant integrals). However, his kind of geometric approach to celestial mechanics represented something quite new in mathematics, and its sheer novelty would probably have been sufficient to make the memoir almost inaccessible to those of a more practical persuasion. This was certainly the view adopted by Mittag-Leffler, who, while studying the original memoir, expressed the concern that Poincaré's resolution of the restricted three body problem was given in a form that would be difficult to understand by anyone except those very familiar with his work. In particular he thought that astronomers would not understand it all.[184] Mittag-Leffler identified as the main source of potential difficulty the fact that Poincaré was working in a three-dimensional *multiplicité* (= manifold) which was not the Euclidean three-dimensional space in which the bodies actually moved.

Poincaré responded to Mittag-Leffler's remarks by translating his most important results into a more traditional format known to be familiar to astronomers, which he added as *Note B*. However, the discovery of the mistake invalidated a large part of the *Note*, and he completely excised all trace of it in the revision.

It could be argued therefore that he chose to use the structure of the memoir to stress the geometrical representation, as opposed to having to revise *Note B*. Given the nature of the new results, the rewriting would have been a delicate undertaking and not one he would have relished in the time he had available. In any event, since he would have been primarily concerned with the response from mathematicians rather than astronomers, it is perhaps not surprising that he chose not to develop this particular side of the problem any further at this stage. However, he did not entirely forget the astronomers, for in the following year he wrote a summary of his results from [P2]:

> ... *for the readers of the Bulletin astronomique who do not have time to read 'in extenso' the original memoir which is very voluminous.*[185]

This summary, although materially similar to *Note B*, was in fact a new paper which had a quite different structure.

5.8. Study of asymptotic surfaces

In Poincaré's geometric representation of the restricted problem, a generating unstable periodic solution and its accompanying family of asymptotic solutions are represented in the three-dimensional solution space by a closed curve and two asymptotic surfaces. In order to understand the behaviour of these asymptotic solutions, he sought the exact equations for these asymptotic surfaces. Although he approached the problem in a similar way in both versions, in [P2] he added an entirely new section simply for the purpose of stating the problem and outlining a strategy for dealing with it—a clear indication of the importance he now attached to the topic.

He first noted that it was possible to move from one surface to the other by changing the sign of the parameter μ in the equations for the surfaces. So by making such a sign change it is possible to generate the second surface from the first. Furthermore, since these two surfaces cut another, they can be considered together as two sides of the same surface. This surface will then have the special

[184] Mittag-Leffler to Poincaré 15.11.88, I M-L.

[185] Poincaré [1891, *480*]

feature of a double curve which identifies the particular series which satisfy the equations for the asymptotic surfaces.

The equations of these surfaces are of the form

$$\frac{x_2}{x_1} = f(y_1, y_2),$$

where x_1 and x_2 are given by the asymptotic series

$$x_1 = s_1(y_1, y_2, \sqrt{\mu}) \qquad x_2 = s_2(y_1, y_2, \sqrt{\mu}) \tag{29}$$

and satisfy

$$\frac{\partial F}{\partial x_1}\frac{\partial x_i}{\partial y_1} + \frac{\partial F}{\partial x_2}\frac{\partial x_i}{\partial y_2} + \frac{\partial F}{\partial y_i} = 0. \tag{30}$$

In order to calculate the equations of the surfaces exactly, Poincaré proceeded in three stages. In the first approximation he calculated the first two coefficients, which, since the series were in powers of $\sqrt{\mu}$, gave an approximation with an error of the order of μ. In the next stage he considered a larger but finite number of coefficients, which give an error of the order μ^p for any fixed p, no matter how large. In the final stage he calculated the exact equations.

5.8.1. First approximation. Most of the section entitled *First approximation* in [P2] came from two consecutive sections in Chapter I [P1]. It began with *The Equation of Asymptotic Surfaces* and ended with the first half of *The Construction of Asymptotic Surfaces* (*first approximation*).

Since Poincaré was concerned with the restricted three body problem, he began with the Hamiltonian equations (27), assuming that F could be expanded in powers of the mass parameter μ,

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots ,$$

with F_0 independent of y. In order to ensure the existence of a generating periodic solution, he supposed that for certain values of x_i, say x_i^0, $\dfrac{\partial F_0}{\partial x_i}(= n_i)$ were commensurable.

This gives

$$x_i = \Phi_i(y_1, y_2), \qquad (i = 1, 2)$$

as the general form of the equation of a surface trajectory, providing the functions Φ_i are chosen such that $F(\Phi_1, \Phi_2, y_1, y_2) = C$, and that they satisfy equations (30).

To integrate equations (30) Poincaré supposed

$$x_i = x_i^0 + \sqrt{\mu}\, x_i^1 + \mu x_i^2 + \dots , \tag{31}$$

where x_i are very close to x_i^0, the latter having been chosen such that the ratio $n_1 : n_2$ is commensurable. It then remained to determine the coefficients x_i^k such that when the series (31) are substituted into the equations (30) the equations are formally satisfied.

To generate a sequence of equations from which he could determine x_i^k, Poincaré substituted the series for x_i in the series for F, then equated powers of μ.

In his determination of the coefficients x_i^k Poincaré first showed that they were periodic functions of y_1 and as such could be expanded as trigonometric series in sines and cosines of multiples of y_1. He then found that

$$x_1^1 = 0, \qquad x_2^1 = \sqrt{\frac{2}{N}([F_1] + C_1)}, \tag{32}$$

where the notation $[F_1]$ represents the average value of the function F_1 considered as a periodic function of y_1, $N = -\dfrac{\partial^2 F_0}{(\partial x_2^0)^2}$ and C_1 is an integration constant. Thus he could write the series to be used in the first approximation as

$$x_1 = x_1^0, \qquad x_2 = x_2^0 + \sqrt{\frac{2\mu}{N}([F_1] + C_1)}.$$

At this point in [P2] Poincaré stopped following the section on the *Equation of Asymptotic Surfaces* from [P1] and continued instead with material taken from the section on *Construction of Asymptotic Surfaces* (*first approximation*).

The remaining pages from the *Equation of Asymptotic Surfaces*, which did not appear in [P2], contained an outline of Poincaré's method for determining the remaining coefficients x_i^k, which was not required for the first approximation and which, for each coefficient, involved the choice of an arbitrary constant of integration C_k. The section concluded with three points raised for discussion:

1. When are the series thus obtained convergent?
2. How should the arbitrary constants $C_1, C_2, \ldots C_{k-1}, \ldots$ be determined?
3. What are the properties of the functions defined by the series?

He addressed all of these points in later sections in his study of asymptotic surfaces.

In [P2] Poincaré considered his results in the context of the geometric representation of the restricted problem. To simplify the notation, he suppressed the primes and called the variables x_i and y_i (not to be confused with the original x_i and y_i, i.e., G, L, $g - t$, and l). The new y_i are linear functions of

$$y_1' = \frac{1}{2}(g - t + l) \qquad y_2' = \frac{1}{2}(g - t - l),$$

and the ratio $\dfrac{x_2}{x_1}$ is a linear function of ξ. Poincaré was now able to define completely the position of a point P in the space so that every relation between y_1, y_2, and the ratio $\dfrac{x_2}{x_1}$ was the equation of a surface, and both y_1 and y_2 could be increased by a multiple of 2π without changing the position of P.

The coefficients from (32) then gave the first approximation for the equation of the surface trajectories

$$\frac{x_2}{x_1} = \frac{x_2^0 + x_2^1\sqrt{\mu}}{x_1^0 + x_1^1\sqrt{\mu}} = \frac{x_2^0}{x_1^0} + \frac{\sqrt{\mu}}{x_1^0}\sqrt{\frac{2}{N}([F_1] + C_1)}. \tag{33}$$

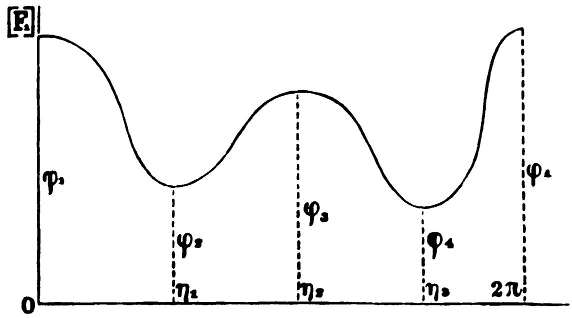


FIGURE 5.8.i

Poincaré's next problem was to identify the particular surfaces which, as he had earlier described, displayed a double curve. This led him to consider the intersections of the surfaces defined by equations (33) with the transverse section S defined by the surface $y_1 = 0$.

The position of a point P on the surface S is defined by the two coordinates $\frac{x_2}{x_1}$ and y_2, which, since they are analogous to polar coordinates, means that the curves $\frac{x_2}{x_1}$ = *constant* are closed concentric curves on the surface S and the position of a point P on S is unchanged when y_2 is increased by 2π. Since $\sqrt{\mu}$ is very small, the intersections of the surfaces defined by equation (33) with the transverse section defined by $y_1 = 0$ differ very little from the curves $\frac{x_2}{x_1}$ = *constant.*

In order to investigate the curves formed by the intersections, Poincaré needed to understand the nature of the function $[F_1]$. He found that it was a finite periodic function of y_2. In other words, it was similar to the function Ψ he had found previously. To look at a general function of this type he supposed that as y_2 varied from 0 to 2π, $[F_1]$ varied as in *Figure 5.8.i*,[186] where $\phi_1 > \phi_3 > \phi_2 > \phi_4$.

He constructed a set of curves defined by

$$y_1 = 0, \qquad \frac{x_2}{x_1} = \frac{x_2^0}{x_1^0} + \frac{\sqrt{\mu}}{x_1^0}\sqrt{\frac{2}{N}([F_1] + C_1)},$$

the shape of each curve depending on the value of the constant C_1 as shown in *Figure 5.8.ii.*[187] Each one of these curves lies in the plane $y_1 = 0$, and so if y_1 is now varied from 0 through to 2π, the curves will each sweep out a surface. More precisely, if through each point on an arbitrary one of these curves is drawn one of the lines defined by the equations y_2 = *constant*, $\frac{x_2}{x_1}$ = *constant*, then the set of all these lines constitutes a closed surface which is exactly one of the surfaces defined by equation (33).

Since each of the roots of the equation $\frac{[dF_1]}{dy_2} = 0$ corresponds to a periodic solution *(cf.* equation (22)), the periodic solutions correspond to the extremum

[186] Poincaré [P2, *418*].
[187] Poincaré [P2, *419*].

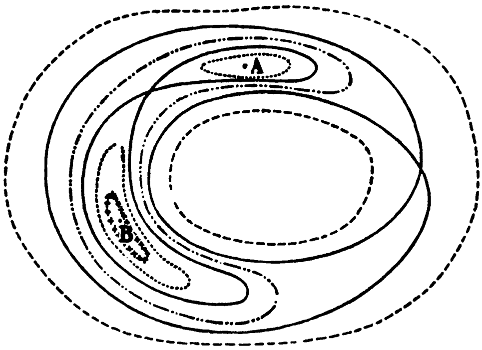


FIGURE 5.8.ii

points of $[F_1]$. In this case (due to the choice of $[F_1]$) there are four extremum points, which represent four periodic solutions, two stable and two unstable. The two stable periodic solutions correspond to the two isolated closed curves of the surfaces $C_1 = -\phi_3$ and $C_1 = -\phi_1$ (points A and B), and the two unstable periodic solutions correspond to the double curves of the surfaces $C_1 = -\phi_2$ and $C_1 = -\phi_4$. By the criteria established earlier, the latter two are the ones in which Poincaré was interested and which represent his first approximation.

In [P1], Poincaré arrived at the same result, but, due to his earlier analysis and his (erroneous) belief that the asymptotic surfaces could be represented by convergent series in $\sqrt{\mu}$, he also included the following conclusions:

1. At the first approximation the asymptotic surfaces are closed surfaces, and this result is confirmed by following approximations.
2. Since every asymptotic surface is a surface trajectory, its intersection with the transverse section S will be an invariant curve C. Consider a curve C':

$$y_1 = 0, \qquad \frac{x_2}{x_1} = \frac{x_2^0}{x_1^0} + \frac{\sqrt{\mu}}{x_1^0}\sqrt{\frac{2}{N}([F_1] - \phi_4)}.$$

This will differ very little from the invariant curve C (up to the order of μ). Its iterate will also differ very little from the iterate of C, i.e., C itself. Thus the curve C' will differ very little from its own iterate (up to the order of μ).
3. The curve C' is a closed curve; the curve C from which it differs only slightly will thus be a quasi-closed curve such that the distance between the points of closure will be of the order of μ. Thus the asymptotic surface cuts the surface $y_1 = 0$ as a quasi-closed curve.

The distance of an arbitrary point P on the surface $C_1 = -\Phi_4$ to its iterate P' will be of order $\sqrt{\mu}$. Likewise the distance of an arbitrary point on the curve C' to its iterate will also be of order $\sqrt{\mu}$.

Later it is shown how Poincaré used these results in [P1], and how they became invalidated by the discovery of the error.

5.8.2. Second approximation. The purpose of the second approximation was to determine some arbitrary number of coefficients of the series (31). Since Poincaré had originally believed the series to be convergent rather than asymptotic, there was no equivalent section in [P1].

However, most of the section is in fact taken from *Note F*, which Poincaré had added to [P1] because he wanted to include an analytic description of the asymptotic surfaces as a complement to his geometric one. *Note F*, therefore, contained what Poincaré then believed to be a description of the entire series.

Poincaré began the second approximation in the same fashion as the first, but he then transformed the problem using Hamilton-Jacobi theory.[188]

Since the system of differential equations is an autonomous Hamiltonian system, the expression $x_1 dy_1 + x_2 dy_2$ is an exact differential and so can be written

$$dS = x_1 dy_1 + x_2 dy_2,$$

where $S(y_1, y_2)$is a solution of the Hamilton-Jacobi partial differential equation

$$F\left(\frac{\partial S}{\partial y_1}, \frac{\partial S}{\partial y_2}, y_1, y_2\right) = C.$$

S can then be expanded as a series in $\sqrt{\mu}$,

$$S = S_0 + S_1\sqrt{\mu} + S_2\mu + S_3\mu\sqrt{\mu} + \dots,$$

with coefficients S_i functions of y_1 and y_2. Moreover, since

$$\frac{\partial S_k}{\partial y_1} = x_1^k, \qquad \frac{\partial S_k}{\partial y_2} = x_2^k,$$

the problem of determining the coefficients of the asymptotic series amounts to determining partial derivatives of the coefficients in the series for S, and hence determining the coefficients in the series for S.

When $C_1 > -\phi_4$ Poincaré proved that $\dfrac{\partial S}{\partial y_1}$ and $\dfrac{\partial S}{\partial y_2}$ could be determined as (divergent) trigonometric series in sines and cosines of multiples of y_1 and y_2. But his main concern was with the case $C_1 = -\phi_4$ when the series represent the asymptotic solutions. In this case, the expression $[F_1] + C_1$ is never negative, and it only reaches zero when $y_2 = \eta_3$.

Choosing η_3 as the origin for y_2, he put the expression into the form of a trigonometric series

$$[F_1] + C_1 = \sum A_m \cos my_2 + \sum B_m \sin my_2.$$

For $y_2 = 0$ this function and its derivative vanish. Since the function is always positive, zero is, therefore, a minimum. As a result, the function

$$\frac{[F_1] + C_1}{sin^2 \frac{y_2}{2}}$$

[188] See Jacobi [1866].

can also be expanded in sines and cosines of multiples of y_2, and since it is a periodic function of y_2 which neither vanishes nor becomes infinite, it is possible to write

$$\frac{\sin\frac{y_2}{2}}{\sqrt{[F_1]+C_1}} = \sum A_m \cos my_2 + \sum B_m \sin my_2.$$

From this expression Poincaré showed that $\frac{\partial S_p}{\partial y_1}$ and $\frac{\partial S_p}{\partial y_2}$ are periodic functions of y_1 and y_2, where the period is 2π with respect to y_1 and 4π with respect to y_2. Furthermore, after a detailed analysis he showed that it was also possible to ensure that the functions remained finite and so could be expanded as sines and cosines of multiples of y_1 and $\frac{y_2}{2}$, where if p is even they will contain only the even multiples of $\frac{y_2}{2}$ and if p is odd they will contain only the odd multiples of $\frac{y_2}{2}$.

He was then able to write the approximate equations for the asymptotic surfaces as the asymptotic series

$$x_1 = \sum_{p=0}^{p=n} \mu^{\frac{p}{2}} \frac{\partial S_p}{\partial y_1}, \qquad x_2 = \sum_{p=0}^{p=n} \mu^{\frac{p}{2}} \frac{\partial S_p}{\partial y_2}.$$

These series, as he had previously proved, are divergent; but since they are asymptotic, if they are stopped at the nth term then the error is very small, providing, of course, μ is very small.

This was a crucially different conclusion from the mistaken one Poincaré had reached in *Note F*, where he had been misled by his false belief in the convergence of the series. There he had said that the equations for the asymptotic surfaces could be represented by the convergent infinite series

$$x_1 = \sum_{p=0}^{p=\infty} \mu^{\frac{p}{2}} \frac{\partial S_p}{\partial y_1}, \qquad x_2 = \sum_{p=0}^{p=\infty} \mu^{\frac{p}{2}} \frac{\partial S_p}{\partial y_2}.$$

5.8.3. Third approximation. In the final refinement Poincaré constructed exactly the asymptotic surfaces or, strictly speaking, their intersection with the transverse section $y_1 = 0$. Here the differences between [P1] and [P2] are quite dramatic.

In [P1] Poincaré's objective was to determine the coefficients of the series defining the asymptotic surfaces. He began by quickly disposing of the two cases where the series were clearly divergent. In the first case, when $C_1 > -\phi_4$, he likened the series to those derived by Lindstedt: divergent but nonetheless useful since the divergence derives from large multipliers rather than small divisors and so is relatively slow. In the second, when $C_1 < -\phi_4$, he gave an analysis which became the introduction to *Periodic solutions of the second class* in [P2] and is discussed later.

He moved on to the case where $C_1 = -\phi_4$, which he then believed gave rise to convergent series defining the asymptotic surfaces. He therefore set about determining the coefficients of the series given the properties he thought he had previously established, namely, that they were periodic with respect to y_1, that they were real and finite, and that they were convergent for sufficiently small values of μ. This

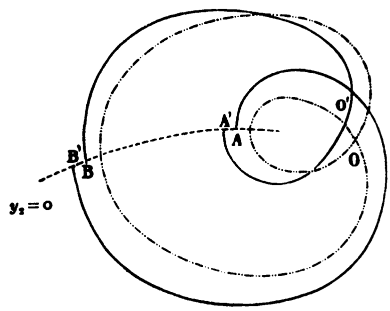


FIGURE 5.8.iii

involved showing it was possible to choose the series of constants derived in the section on the *Equation of Asymptotic Surfaces* so that the series were convergent. That being done, he returned to the geometry in order to give an actual description of the asymptotic surfaces.

To clarify the description, he used *Figure 5.8.iii.*[189] The plain lines which identify the two curves $AO'B'$ and $A'O'B$ represent the two asymptotic surfaces which cut the surface $y_1 = 0$, and the dashed line represents the curve $y_1 = y_2 = 0$. The dotted and dashed line, which is a closed curve with a double point at O, represents the curves with equation

$$y_1 = 0, \qquad \frac{x_2}{x_1} = \frac{x_2^0}{x_1^0} + \frac{\sqrt{\mu}}{x_1^0}\sqrt{\frac{2}{N}([F_1] - \phi_4)}$$

that arise when the surfaces which differ very little from the asymptotic surfaces cut the surface $y_1 = 0$. The generating (unstable) periodic solution is represented by a closed trajectory cutting the surface $y_1 = 0$ at the point O', and the distance OO' is of order μ.

Poincaré used his results from the end of the first approximation to infer, first, that the curve $BO'B'$ is quasi-closed, the distance between the points of closure being infinitely small of order μ, and, second, that the distance of the point B to its iterate is of the order of $\sqrt{\mu}$. Appealing to the (invalid) Corollary to Theorem III he concluded (erroneously) that the curve $BO'B'$ was rigorously closed, i.e., that the points B and B' were coincident, and consequently that the asymptotic surfaces were closed. Furthermore, inherent in this conclusion was the implication of stability.

He ended by noting that a similar argument could be used to establish that the asymptotic surfaces corresponding to the unstable periodic solution $C_1 = -\phi_2$ were also closed.

Thus Poincaré believed he had proved, for sufficiently small values of μ, that relative to a given unstable periodic solution there was a set of asymptotic solutions that could be considered stable in the sense that they remained confined to a given region of space. Moreover, he thought that this set of solutions was well behaved and that their behaviour could be completely understood. His analysis in [P2] led to an entirely different conclusion.

[189]Poincaré [P2, *438*].

In [P2] Poincaré used the same diagram as in [P1] (*Figure 5.8.iii*), with the same labelling, except that in [P2] the dotted and dashed line represents the curves with equation

$$y_1 = 0, \qquad x_1 = s_1^p(0, y_2), \qquad x_2 = s_2^p(0, y_2),$$

where s_i^p are the sums of the first p terms of the series $s_i(y_i, y_2)$ and are periodic functions of period 2π with respect to y_1 and 4π with respect to y_2.

The first question Poincaré asked was whether the curves $AO'B'$ and $A'O'B$ were closed. He could see that they would be if the series s_i were convergent. For in this case the plain curves would differ as little as required from the dotted and dashed curves, since the distance from a point on the former to a point on the latter would tend to zero as p increased indefinitely. But he had already proved that the series were divergent. Nevertheless, the question still remained. Was it possible for the curves $AO'B'$ and $A'O'B$ to be closed even though the series were divergent?

Poincaré tackled the question by looking at the specific example of a simple pendulum weakly coupled to a linear oscillator. In this case the Hamiltonian is given by

$$-F = p + q^2 - 2\mu \sin^2 \frac{y}{2} - \mu\epsilon \cos x\phi(y),$$

where μ and ϵ are two very small parameters, and $\phi(y)$ is a periodic function of y of period 2π. The Hamiltonian equations are

$$\frac{dp}{dt} = \frac{\partial F}{\partial x} = -\epsilon \sin x\phi(y), \qquad \frac{dq}{dt} = \frac{\partial F}{\partial y} = -\mu \sin y + \mu\epsilon \cos x\phi'(y),$$

$$\frac{dx}{dt} = -\frac{\partial F}{\partial p} = 1, \qquad \frac{dy}{dt} = -\frac{\partial F}{\partial q} = 2q,$$

where the variables p, q, x and y correspond to the variables x_1, x_2, y_1 and y_2 respectively in equations (27).

If $\epsilon = 0$, then the equations have a periodic solution

$$x = t, \qquad p = 0, \qquad q = 0, \qquad y = 0,$$

the two nonzero characteristic exponents are equal to $\pm\sqrt{2\mu}$, and the equations of the two asymptotic surfaces are

$$p = \frac{\partial S_0}{\partial x}, \qquad q = \frac{\partial S_0}{\partial y}, \qquad S_0 = \pm 2\sqrt{2\mu}\cos\frac{y}{2};$$

from which

$$p = 0, \qquad q = \pm\sqrt{2\mu}\sin\frac{y}{2},$$

and hence the surfaces enclose a region which has a width of the order of $\sqrt{\mu}$.

Since there are nonzero characteristic exponents for $\epsilon = 0$, there are also periodic solutions for small values of ϵ. The equations of the corresponding asymptotic surfaces are given by

$$p = \frac{\partial S}{\partial x}, \qquad q = \frac{\partial S}{\partial y},$$

where S is a function of x and y, satisfying the equation

$$\frac{\partial S}{\partial x} + \left(\frac{\partial S}{\partial y}\right)^2 = 2\mu \sin^2\frac{y}{2} + \mu\epsilon \cos x\phi(y).$$

Moreover, the existence of nonzero characteristic exponents for $\epsilon = 0$ implies that p and q and therefore S can be expanded as series in ϵ. So S can be put in the form $S = S_0 + \epsilon S_1 + \epsilon^2 S_2 + \dots$. S_0 has already been found, and equating powers of ϵ shows that S_1 must satisfy the equation[190]

$$\frac{\partial S_1}{\partial x} + 2\sqrt{2\mu}\sin\frac{y}{2}\frac{\partial S_1}{\partial y} = \mu\cos x\phi(y).$$

To determine S_1, Poincaré defined a new function $\sum$ to be the function which satisfies

$$\frac{\partial \sum}{\partial x} + 2\sqrt{2\mu}\sin\frac{y}{2}\frac{\partial \sum}{\partial y} = \mu e^{ix}\phi(y), \qquad i = \sqrt{-1},$$

so that S_1 is the real part of $\sum$. This equation can then be satisfied by $\sum = e^{ix}\Psi(y)$, which gives a linear equation in Ψ:

$$i\Psi + 2\sqrt{2\mu}\sin\frac{y}{2}\frac{d\Psi}{dy} = \mu\phi(y).$$

If $\phi = 0$, then[191]

$$\Psi = \left(\tan\frac{y}{4}\right)^{\alpha}, \qquad \alpha = -i\frac{1}{\sqrt{2\mu}},$$

and if $\phi(y)$ is arbitrary, the integral can be written

$$\Psi = \left(\tan\frac{y}{4}\right)^{\alpha}\int\sqrt{\frac{\mu}{8}}\phi(y)\left(\sin\frac{y}{2}\right)^{-1}\left(\tan\frac{y}{4}\right)^{\alpha}dy,$$

where Ψ can be expanded in integer powers of y for small values of y. If $\phi(0) = 0$, then the integral is also equal to zero and hence its limits are 0 and y.

If the curves $AO'B'$ and $A'O'B$ are closed,[192] then the function S and its derivatives will be finite for all values of y as well as being periodic of period 4π with respect to y (*cf.* the functions s_1^p and s_2^p). Since this must be true for any given value of ϵ, it must also be true for S_1, and hence for Ψ.

Thus, for values of y close to 2π, Ψ should be expansible in integer powers of $y - 2\pi$. But since $\left(tan\frac{y}{4}\right)^{\alpha}$ cannot be expanded in this way, the condition can only hold if the integral

$$J = \int_0^{2\pi}\sqrt{\frac{\mu}{8}}\phi(y)\left(\sin\frac{y}{2}\right)^{-1}\left(\tan\frac{y}{4}\right)^{-\alpha}dy$$

is zero. However, evaluating J, using $\phi(y) = \sin y$, gives

$$J = -2\pi i\ \operatorname{sech}\left(\frac{\pi}{2\sqrt{2\mu}}\right),$$

which is clearly not equal to zero, and so the curves $AO'B'$ and $A'O'B$ cannot be closed.

However, the lack of closure still left open the possibility that the extended curves $O'B$ and $O'B'$ could intersect. For if this should occur, any trajectory which passes through the point of intersection would simultaneously belong to both sides

[190] Poincaré omitted the factor 2 in the second term of this equation, although he included it in [P1a] and then crossed it out.

[191] Poincaré put $\alpha = -i\sqrt{\frac{2}{\mu}}$.

[192] Poincaré wrote BO′B′ and AO′A′.

of the asymptotic surface. To distinguish this type of trajectory, Poincaré called them *doubly asymptotic*. He was later to use the term *homoclinic*.[193]

In other words, if C is the closed trajectory which passes through the point O' and represents the periodic solution, then, if the trajectory is doubly asymptotic, it would begin by being very close to C when t is very large and negative. It would then asymptotically move away to deviate greatly from C, before asymptotically reapproaching C when t is very large and positive.

To prove the existence of doubly asymptotic trajectories Poincaré needed to show that the system fulfilled the conditions of Theorem III in Chapter III of Part 1.

To do this he established that none of the curves $O'B$, $O'B'$, $O'A$ and $O'A'$ were self-intersecting, i.e., that none of them have a double point; that the curvature of the curves $O'B$ and $O'B'$ was finite, i.e., that it does not increase indefinitely as μ tends to zero; and that the distances BB', B_1B_1', together with the ratios $\dfrac{BB'}{BB_1}$ and $\dfrac{BB'}{B'B_1'}$, tend to zero as μ tends to zero, where B_1 and B_1' are the iterates of B and B' respectively. Furthermore, since the system is in Hamiltonian form, it also possesses a positive invariant integral, and hence all the conditions of Theorem III are satisfied.

Therefore the arcs BB_1 and $B'B_1'$ intersect each other, i.e., the extended curve $O'B'$ intersects the extended curve $O'B$, and through the point of intersection (today called a *homoclinic* point) passes a doubly asymptotic trajectory.

Poincaré had constructed the figure so that the points B and B' lie on the curve

$$y_1 = y_2 = 0,$$

and since the origin of y_2 is arbitrary he supposed that at the intersection of the curves $O'B$ and $O'B'$, $y_2 = 0$. In this case the points B and B' are coincident and so are their iterates B_1 and B_1'. Thus the two arcs BB_1 and $B'B_1'$ have the same end points. But by Theorem III (in which the area limited by the two arcs is not convex), the two arcs must intersect again at a different point N. Thus there are at least two doubly asymptotic trajectories, one passing through the point B and one passing through the point N.

To show that there are in fact an infinite number of doubly asymptotic trajectories, he supposed that the points B and B' are always coincident, that BMN is the part of the curve $O'B$ between B and N; and BPN is the part of the curve $O'B'$ between the point $B = B'$ and the point N, so that these two arcs limit a certain area α. If the system is one in which the conditions of the recurrence theorem are satisfied, such as the restricted three body problem, then there will be trajectories which cross this area α infinitely often. Hence among the iterates of the area α there will be an infinite number which have a part in common with α.

The closed curve $BMNPB$ which limits the area α has an infinite number of iterates. The arc BMN cannot intersect any of its own iterates; for the arc BMN and its iterates belong to the curve $O'B$ and the curve $O'B$ is not self-intersecting.

[193] Poincaré first used the word 'homocline' in [MN III, *384*].

Similarly, the arc BPN does not intersect any of its own iterates. Therefore either the arc BMN intersects with one of the iterates of BPN, or the arc BPN intersects with one of the iterates of BMN (as in the case under consideration). In either case the curve $O'B$ or its extension will intersect the curve $O'B'$ or its extension.

Thus these two curves intersect each other at an infinite number of points, and an infinite number of these points of intersection will be found either on the arc BMN or on the arc BPN. These points of intersection are all points of intersection of the curve $O'B'$ or its extension with the curve $O'B$ or its extension, and, since through each of these points of intersection passes a doubly asymptotic trajectory, there are an infinite number of doubly asymptotic trajectories. Similarly, the asymptotic surface which cuts the surface $y_1 = 0$ along the curve $O'A$ also contains an infinite number of doubly asymptotic trajectories.

This is arguably the first mathematical description of chaotic motion within a dynamical system. Although Poincaré drew little attention to the complexity of the behaviour he had discovered and made no attempt to draw a diagram, he was profoundly disturbed by his discovery, as he revealed in a letter to Mittag-Leffler:

> *I have written this morning to M. Phragmén to tell him of an error I have made and doubtless he has shown you my letter. But the consequences of this error are more serious than I first thought. It is not true that the asymptotic surfaces are closed, at least in the sense which I originally intended. What is true is that if both sides of this surface are considered (which I still believe are connected to each other) they intersect along an infinite number of asymptotic trajectories (and moreover that their distance becomes infinitely small of order higher than μ^p however great the order of p).*
>
> *I had thought that all these asymptotic curves having moved away from a closed curve representing a periodic solution, would then asymptotically approach the same closed curve. What is true, is that there are an infinity which enjoy this property.*
>
> *I will not conceal from you the distress this discovery has caused me. In the first place, I do not know if you will still think that the results which remain, namely the existence of periodic solutions, the asymptotic solutions, the theory of characteristic exponents, the nonexistence of single-valued integrals, and the divergence of Lindstedt's series, deserve the great reward you have given them.*
>
> *On the other hand, many changes have become necessary and I do not know if you can begin to print the memoir; I have telegraphed Phragmén.*
>
> *In any case, I can do no more than to confess my confusion to a friend as loyal as you. I will write to you at length when I can see things more clearly.*[194]

[194] Poincaré to Mittag-Leffler, postmarked 1.12.1889, No. 54a, I M-L. (The original letter is reproduced on the following pages.)

#: et de plus que leurs distances est un infiniment petit d'ordre plus élevé que μ^p quelque grand que soit p.

[54 a]

Mon cher ami,

J'ai écrit ce matin à M. Phragmén pour lui parler d'une erreur que j'avais commise et il vous a sans doute communiqué ma lettre. Mais les conséquences de cette erreur sont plus graves que je ne l'avais cru d'abord. Il n'est pas vrai que les surfaces asymptotiques soient fermées, au moins dans le sens où je l'entendais d'abord. Ce qui est vrai, c'est que ~~si je~~ considère les deux parties de cette surface (que je croyais liées encore raccordées l'une à l'autre) se coupent suivant une infinité de courbes trajectoires asymptotiques. #

J'avais cru que toutes ces courbes asymptotiques après s'être éloignées d'une courbe fermée représentant une solution périodique, se rapprochaient ensuite asymptotiquement de la même courbe fermée. Ce qui est vrai c'est qu'il y en a une infinité qui jouissent de cette propriété.

Je ne vous dissimulerai pas le chagrin que me cause cette découverte. Je ne sais d'abord si vous jugerez encore que les résultats qui subsistent, à savoir l'existence des solutions périodiques, celle des solutions asymptotiques, la théorie des exposants caractéristiques, la non-existence des intégrales uniformes et la divergence des séries de M Lindstedt méritent encore la haute récompense que vous avez bien voulu leur accorder.

D'autre part, de grands remaniements vont devenir nécessaires et je ne sais si on n'a pas commencé à tirer le mémoire; j'ai télégraphié à M. Phragmén.

En tout cas, je ne puis mieux faire que de confesser mes perplexités à un ami aussi dévoué que vous l'avez toujours été.

Je vous en écrirai plus long quand j'y verrai un peu plus clair dans mes affaires.

Veuillez agréer, mon cher ami, avec mes bien sincères excuses, l'assurance de mon entier dévouement,

Poincaré

Perhaps a further indication of Poincaré's concern and confusion at his discovery of the strange behaviour of these solutions can be detected in the introduction to [P2]. Of all the results in the memoir this was clearly the most extraordinary, and yet it is not amongst those he singled out in his introduction. Possibly this was because he felt unable to do so without ignoring Mittag-Leffler's request not to give details of the error. It would after all have been very difficult to draw attention to the complexity of the doubly asymptotic solutions without mentioning the error. On the other hand perhaps it was simply because he had had so little time in which to assess the implications of his discovery that he felt it wiser not to emphasise it.

5.9. Further results

The penultimate chapter of [P2] is devoted to three separate topics: periodic solutions of the second class, the divergence of Lindstedt's series and the nonexistence of any new integrals for the restricted three body problem.

Most of the chapter is derived from [P1] enhanced by additions. The first section on periodic solutions of the second class contains most of the section with the same name in [P1], although it opens with part of the section entitled *The Exact Construction of Asymptotic Surfaces* and concludes with some new material. The section on the divergence of Lindstedt's series is essentially *Note A*, while the final section on the nonexistence of single-valued integrals is derived from *Note G*.

The last two sections contain the results which Poincaré had emphasised in the introduction to [P2] and which quickly came to be amongst the best known in the memoir. Given the relative importance he now attached to them, it is perhaps surprising that he did not include their complete proofs in [P1]. It may well have been, of course, that they were not in a sufficiently finished form by the closing date of the competition. On the other hand, his emphasis on them in [P2] could possibly represent Weierstrass's opinion, which had been conveyed to him by Mittag-Leffler (see Chapter 6).

5.9.1. Periodic solutions of the second class. In Poincaré's investigation of asymptotic solutions he had begun by showing how the periodic solutions could be represented by curves on the transverse section defined by the surface $y_1 = 0$, the nature of the curves depending on the value of a particular constant C_1, and his discussion had centred on the unstable periodic solution corresponding to $C_1 = \phi_4$. In this section he considers the situation when the value of this constant is less than $-\phi_4$, i.e., when the coefficient $x_2^{\frac{1}{2}} = \sqrt{\frac{2}{N}([F_1] + C_1)}$ in the series for x_2 is not always real.

He found that in this case there were regions of motion but that these regions contained periodic solutions which made more than one revolution around the origin before closing up. These solutions were therefore of a different type from those that he had previously found, and he labelled them periodic solutions of the second class. More formally they can be described by saying that if a system has, for small values of μ, a periodic solution of period T, then they are those periodic solutions which are close to the original periodic solution but whose periods are integral multiples of T.

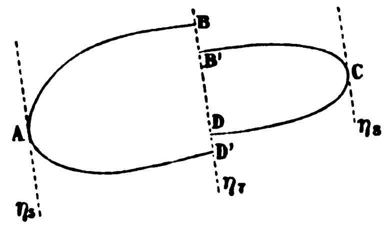


FIGURE 5.9.i

Since $[F_1]$ is a function of y_2, the behaviour of the system for different values of the constant C_1 will depend on y_2. If for a chosen value of C_1, x_2^1 is real as the value of y_2 varies between, say, η_5 and η_6, then since x_i^k are determined by a recurrence relation that is dependent on x_2^1, x_i^k can be determined for all values η_7 of y_2 in this range. The existence of the square root means that x_i^k has two sets of values equal in magnitude but opposite in sign. If $x_{0.i}^k$ are the functions of y_2 when the square root is positive and $x_{1.i}^k$ are the functions of y_2 when it is negative, then the latter will be the analytic continuation of the former.

What Poincaré now wanted to establish was how the behaviour of the system was affected by a change in the constant C_1. Were there regions in which the value of y_2 would remain finite? Since his method of using a transverse section to understand the evolution of the system reduced the dimension of the system by one, it was possible that a small change in the constant could induce some strange behaviour in y_2 which would only manifest itself in this other (unseen) dimension. It was clear that any behaviour of this type would not be captured by a study of the transverse section for individual choices of the constant.

Poincaré therefore looked at the change in values of y_2 for a very small change in the constant C_1. He replaced C_1 by a new constant C_1' very close to C_1, so that $\sqrt{\frac{2}{N}([F_1]+C_1')}$ was real whenever y_2 was between η_7 and a certain value η_8 very close to η_6. Again he could determine the functions x_i^k for all values of y_2 in this range, where $x_{2.i}^k$ are the functions of y_2 when the square root is positive and $x_{3.i}^k$ the functions of y_2 when it is negative.

He then constructed the four branches of the curve:

$$y_1 = 0, \quad x_1 = \phi_{k.1}(y_2), \quad x_2 = \phi_{k.2}(y_2),$$
$$(k = 0, 1 : \eta_5 \le y_2 \le \eta_7; \quad k = 2, 3 : \eta_7 \le y_2 \le \eta_8)$$

where

$$\phi_{p.q}(y_2) = x_{p.q}^0 + x_{p.q}^1\sqrt{\mu} + \ldots + x_{p.q}^k \mu^{k/2},$$

the first and second branches of the curve corresponding to the constant C_1, and meeting at a tangent to the curve $y_2 = \eta_7$, and the third and fourth branches of the curve corresponding to the constant C_1' and meeting at a tangent to the curve $y_2 = \eta_8$ (*Figure 5.9.i*).[195]

Poincaré began by regarding C_1 as given and C_1' as close in value to C_1 but nevertheless arbitrary. He now determined C_1' by imposing the condition that the

[195] Poincaré [P2, *447*].

first and third curves meet, i.e., the points B and B' are coincident. Then appealing to an earlier theorem (invariant integrals, Theorem III), he derived the result that the distance DD' between the second and the fourth curves was infinitely small of the order $\mu^{(k+1)/2}$. Hence he could conclude that for a limited period of time there do exist regions in which the values of y_2 remain finite, and these are those known as the regions of libration. The time constraint was a consequence cf the fact that the series involved were asymptotic rather than convergent.

Poincaré then considered the regions of libration in order to ascertain whether they contained periodic solutions.

The equations

$$x_1 = x_1^0 + \mu x_1^2, \quad x_2 = x_2^0 + \sqrt{\mu}\sqrt{\frac{2}{N}([F_1] + C_1)} + \mu u_2^2 \tag{34}$$

define, up to the order of μ, the surfaces just constructed (see *Figure 5.9.i*) and so approximately satisfy equations (30), where u_2^2 is a finite function of y_1 and y_2 which only differs from x_2^2 by a function of y_2 so that

$$\frac{\partial(u_2^2)}{\partial y_1} = \frac{\partial(x_2^2)}{\partial y_1}.$$

Poincaré then modified the form of F in the Hamiltonian equations so that the equations (34) *exactly* satisfied equations (30). The new form of the equations can then be integrated exactly, and, following the same argument as given previously, shows that there exist an infinite number of closed surface trajectories defined by the equation

$$\frac{x_2}{x_1} = \frac{x_2^0 + \sqrt{\mu}\sqrt{\frac{2}{N}([F_1] + C_1)} + \mu u_2^2}{x_1^0 + \mu x_1^2}, \tag{35}$$

which is of the same form as equation (33).

Thus the same hypotheses can be made about the function $[F_1]$ in equation (35) as about $[F_1]$ in equation (33). The two surfaces of (35) which correspond to the values $-\phi_2$ and $-\phi_4$ of C_1 are therefore closed with a double curve. The space can then be divided into three regions: interior, exterior, and between the two sheets of the surface, the last being the region of libration.

Since the closed surface corresponding to a given value of C_1 (which must be $< -\phi_4$) has the same connectivity as a torus, the existence of periodic solutions depends on the behaviour of the two angular variables defining the surface. By investigating the behaviour of these angular variables Poincaré showed that there were an infinite number of values for C_1 for which periodic solutions exist.

Then, by deriving an equation in which C_1 was defined as a continuous function of μ, he showed that it was possible to make μ so small that the equation no longer had a root. This means that while there exist an infinite number of closed trajectories which represent periodic solutions, these solutions will disappear one after the other as μ decreases. In other words, if along a closed trajectory, μ decreases continuously, then the trajectory deforms continuously and at a certain value of μ will eventually disappear. As a result, when $\mu = 0$ all the periodic solutions in the region of libration will have disappeared. This is in contrast to the behaviour of the periodic solutions of the first class (those which only have one revolution around the primary) which continue to exist for $\mu = 0$.

Since he had proved that in the neighbourhood of a periodic solution (stable or unstable) there exist an infinite number of other periodic solutions, Poincaré considered the possibility that every region of the space, however small, was crossed by an infinite number of periodic solutions. In other words he conjectured that the periodic solutions were everywhere dense. Although he was unable to prove it, Cantor's recent work on set theory, which had shown that it was possible for a set to be perfect without being connected, led him to believe that it was extremely likely.[196]

All the above is contained in both [P1] and [P2]. There is, however, an important addition to the section which only appears in [P2]. In both versions of his proof of the existence of periodic solutions of the second class, Poincaré had shown that periodic solutions exist for small values of a certain parameter ϵ. But in [P1] he had thought that if $\epsilon = \mu\sqrt{\mu}$ it automatically *followed* that periodic solutions also exist for small values of the mass parameter μ. In the revision he realised that this result needed to be established rigorously.

Therefore, in [P2] he returned to the Hamiltonian equations (27) and considered a *stable* periodic solution

$$x_1 = \phi_1(y_1), \qquad y_2 = \phi_2(y_1),$$

of period 2π. In *Figure 5.8.ii* this periodic solution is approximately represented by the isolated closed curve of the surface $C_1 = -\phi_3$. It has two characteristic exponents $\pm\alpha$, the squares of which are real and negative. If

$$x_1 = \phi_1(y_1) + \xi_1, \qquad y_2 = \phi_2(y_1) + \xi_2$$

is a nearby solution and β_1 and β_2 are the initial values of ξ_1 and ξ_2 for $y_1 = 0$, and $\beta_1 + \Psi_1$ and $\beta_2 + \Psi_2$ are the values of ξ_1 and ξ_2 for $y_1 = 2k\pi$ (k an integer), then the solution will be periodic of period 2π if

$$\Psi_1 = \Psi_2 = 0, \tag{36}$$

where Ψ_1 and Ψ_2 can be expanded in powers of β_1 and β_2 which depend on μ.

If β_1, β_2 and μ are regarded as the coordinates of a point in space, then the equations (36) represent a curve, each point of which corresponds to a periodic solution. If $\xi_1 = \xi_2 = 0$, then $\beta_1 = \beta_2 = 0$, which implies $\Psi_1 = \Psi_2 = 0$, and a periodic solution of period 2π is obtained which can also be regarded as being periodic with period $2k\pi$.

Thus the curve (36) consists of the entire μ axis. Poincaré proved that if $k\alpha$ is a multiple of $2i\pi$ when $\mu = \mu_0$, then there exists another branch of the curve (36) which passes through the point

$$\mu = \mu_0, \qquad \beta = 0, \qquad \beta_2 = 0,$$

and so, from the previous theory, for values of μ close to μ_0, there exist periodic solutions other than $\xi_1 = \xi_2 = 0$.

[196]This point is taken up again in [MN I, *82*]. See Chapter 7.

The proof, which depended on the theory of invariant integrals, involved expanding Ψ_1 and Ψ_2 in terms of β_1, β_2 and $(\mu - \mu_0)$, and then showing that β_1 and β_2 could themselves be expanded in positive fractional powers of $(\mu - \mu_0)$. This implies that there exists a series in $(\mu - \mu_0)$ which is not identically zero and which satisfies equations (36). This in turn implies that there exists a system of periodic solutions in which the expressions for the coordinates can be expanded in positive fractional powers of $(\mu - \mu_0)$, the period of which is a multiple of the generating periodic solution, and when $\mu = \mu_0$ the solution is simply the original periodic solution.

5.9.2. Divergence of Lindstedt's series. In [P1] Poincaré had included a section entitled *Negative Results*, in which he had incorporated the result that no analytic single-valued integral of the restricted problem exists apart from the Jacobian integral. He claimed that a consequence of this result was that the series generally used in celestial mechanics, and in particular the series derived by Lindstedt (see Chapter 2), were, contrary to what had been previously thought, divergent. But he gave no evidence of why this assertion should be true. That it was not immediately obvious is plainly expressed by Mittag-Leffler, who told Poincaré that he had spent a month with Weierstrass trying to work it out but without success.[197] Poincaré responded with *Note A*, in which he gave two forms of a proof, but, according to Mittag-Leffler, Weierstrass was still not satisfied. Poincaré had proved that there were circumstances under which Lindstedt's series were not convergent, but he had ignored the wider question of whether convergent trigonometric series solutions could ever be found. Furthermore, since Dirichlet's original remarks had led Weierstrass to believe that such solutions did exist, he was particularly anxious to have this important point clarified.[198] Mittag-Leffler again asked Poincaré for a proof.[199] This time Poincaré replied that he thought that he had covered the point in *Note A*, although he admitted that he could not be sure as he had mislaid his own copy of it.[200] Mittag-Leffler did not pursue the issue, in spite of further requests from Weierstrass to do so, and, significantly, Poincaré left Weierstrass's question unresolved. In [P2] Poincaré gave only the first form of the proof, since the second depended on the proof of the nonexistence of any new integrals which he had placed in the following section.

Poincaré first showed how Lindstedt's method for approximately integrating differential equations of the form

$$\frac{d^2x}{dt^2} + n^2x = \alpha\Phi(x,t)$$

by deriving a formal trigonometric series solution without any secular terms could be adapted to accommodate the system of Hamiltonian equations (27). He considered the system with Hamiltonian

$$F = F_0 + \mu F_1,$$

where F_0 is independent of y_1 and y_2, and F_1 is a trigonometric series of sines and cosines of multiples of y_1 and y_2 with coefficients which are analytic functions of x_1

[197] Mittag-Leffler to Poincaré, 15.11.1888, I M-L.

[198] See Appendix 2, Question 1.

[199] Mittag-Leffler to Poincaré, 23.2.1889, I M-L.

[200] Poincaré to Mittag-Leffler, 1.3.1889, No. 49, I M-L.

and x_2. The x_i and y_i are then regarded as functions of two variables $w_i = \lambda_i t + \varpi_i$ (as opposed to simply functions of the time), where the frequencies λ_i are to be determined, ϖ_i are constants of integration, and

$$
\begin{aligned}
x_i &= x_i^0 + \mu x_i^1 + \mu^2 x_i^2 + \dots \qquad (i = 1, 2) \\
y_i &= w_i + \mu y_i^1 + \mu^2 y_i^2 + \dots \\
\lambda_i &= \lambda_i^0 + \mu \lambda_i^1 + \mu^2 \lambda_i^2 + \dots .
\end{aligned} \tag{37}
$$

The coefficients λ_i^k are constants, and the coefficients x_i^k and y_i^k are trigonometric series in sines and cosines of multiples of w_1 and w_2. Poincaré then sketched a method which, in line with Lindstedt's result, demonstrated that it was possible to determine the $2q+2$ constants $\lambda_i^0, \dots, \lambda_i^q$ so that the $4q$ trigonometric series x_i^0, $\dots$, x_i^q, y_i^0, $\dots$, y_i^q, for arbitrarily large q, satisfy the Hamiltonian equations up to the order of μ^{q+1}.

The frequencies λ_1 and λ_2 can be expanded in powers of μ, ω_1 and ω_2, and the solutions corresponding to the values of ω_1 and ω_2 for which the frequency ratio is commensurable are therefore periodic. Corresponding to each of these periodic solutions are characteristic exponents each of which can be calculated if the general solution of the equations is known. Thus if the series are uniformly convergent and consequently give the general solution of the equations, then the characteristic exponents of the periodic solutions can be calculated.

When Poincaré calculated the characteristic exponents under the assumption that analytic solutions do exist he found that they were all zero. But when he put this result into the eigenvalue equation that determines the characteristic exponents in the restricted problem, he arrived at a contradiction. He therefore concluded that his original assumption—that there is a general solution given by uniformly convergent series—must be false, and hence:

> ... *in the restricted three body problem and consequently in the general case, Lindstedt's series are not uniformly convergent for all the values of the arbitrary constants of integration which they contain.*[201]

But, as Weierstrass had observed, Poincaré's discussion was incomplete. He gave no consideration to the circumstances under which convergence could occur, with the result that he gave no indication of what proportion of the series were divergent. However, Poincaré did not abandon the question. In the second volume of the *Méthodes Nouvelles* he reworked it in greater depth and generality, and his conclusions are described in Chapter 7 below.

5.9.3. Nonexistence of single-valued integrals. The final section in the chapter contained what has become one of the best-known results in the memoir: the proof of the nonexistence of any new transcendental integral for the restricted three body problem. Only two years earlier Heinrich Bruns, a former student of Weierstrass, had proved the nonexistence of any new algebraic integral for the general three body problem [1887],[202] and Poincaré's result was therefore an important complement to that of Bruns'.

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[201] [P2, *470*].

[202] For a clear exposition of Bruns' result see Whittaker [1937, *358*].

Poincaré had given an outline of a proof of this result in the *Negative Results* in [P1], and in response to yet another of Mittag-Leffler's requests for details, he had also produced an extension to the proof which he had intended to appear as *Note G.*[203] In [P2], having reshaped his original thoughts, he completely rewrote the proof.

More specifically, Poincaré proved that if the differential equations (27) possess the solution $F = constant$, where F is a single-valued analytic function of x_i, y_i and μ, which can be expanded in powers of μ and is periodic of period 2π with respect to y_i, then the equations possess no other solution of the same form.

Suppose $\Phi = constant$ is another such solution. If $x_1 = \phi_1$, $x_2 = \phi_2$, $y_1 = \phi_3$, $y_2 = \phi_4$ is a periodic solution of the differential equations such that $x_1 = \phi_1 + \beta_1$, $\ldots, y_2 = \phi_4 + \beta_4$ when $t = 0$ and $x_1 = \phi_1 + \beta_1 + \Psi_1, \ldots, y_2 = \phi_4 + \beta_4 + \Psi_4$, when $t = T$, then Ψ can be expanded as a power series in β, and the eigenvalue equation in S

$$\begin{vmatrix} \frac{\partial \Psi_1}{\partial \beta_1} - S & \frac{\partial \Psi_1}{\partial \beta_2} & \frac{\partial \Psi_1}{\partial \beta_3} & \frac{\partial \Psi_1}{\partial \beta_4} \\ \frac{\partial \Psi_2}{\partial \beta_1} & \frac{\partial \Psi_2}{\partial \beta_2} - S & \frac{\partial \Psi_2}{\partial \beta_3} & \frac{\partial \Psi_2}{\partial \beta_4} \\ \frac{\partial \Psi_3}{\partial \beta_1} & \frac{\partial \Psi_3}{\partial \beta_2} & \frac{\partial \Psi_3}{\partial \beta_3} - S & \frac{\partial \Psi_3}{\partial \beta_4} \\ \frac{\partial \Psi_4}{\partial \beta_1} & \frac{\partial \Psi_4}{\partial \beta_2} & \frac{\partial \Psi_4}{\partial \beta_3} & \frac{\partial \Psi_4}{\partial \beta_4} - S \end{vmatrix} = 0$$

can be formed.

The roots of this equation are $e^{\alpha T} - 1$, α being the characteristic exponents. Since it is the restricted three body problem that is being considered, two of the roots are zero and two are nonzero.

Furthermore,

$$\frac{\partial F}{\partial x_1}\frac{\partial \Psi_1}{\partial \beta_i} + \frac{\partial F}{\partial x_2}\frac{\partial \Psi_2}{\partial \beta_i} + \frac{\partial F}{\partial y_1}\frac{\partial \Psi_3}{\partial \beta_i} + \frac{\partial F}{\partial y_2}\frac{\partial \Psi_4}{\partial \beta_i} = 0 \qquad (i = 1, \ldots, 4)$$

$$\frac{\partial \Phi}{\partial x_1}\frac{\partial \Psi_1}{\partial \beta_i} + \frac{\partial \Phi}{\partial x_2}\frac{\partial \Psi_2}{\partial \beta_i} + \frac{\partial \Phi}{\partial y_1}\frac{\partial \Psi_3}{\partial \beta_i} + \frac{\partial \Phi}{\partial y_2}\frac{\partial \Psi_4}{\partial \beta_i} = 0,$$

where, in the derivatives of F and Φ, x_i and y_i are replaced by $\phi_i(T)(i = 1, \ldots, 4)$. Hence either

$$\frac{\partial F}{\partial x_1} \Big/ \frac{\partial \Phi}{\partial x_1} = \frac{\partial F}{\partial x_2} \Big/ \frac{\partial \Phi}{\partial x_2} = \frac{\partial f}{\partial y_1} \Big/ \frac{\partial \Phi}{\partial y_1} = \frac{\partial F}{\partial y_2} \Big/ \frac{\partial \Phi}{\partial y_2} \tag{38}$$

or the Jacobian of Ψ with respect to β is zero, together with all the minors of first order.

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[203]Mittag-Leffler to Poincaré 21.12.1888, I M-L.

On the other hand, if $\phi'(t)$ is the derivative of $\phi(t)$, then

$$\sum \frac{\partial \Psi_i}{\partial \beta_j} \phi_j'(0) = 0 \qquad (i = 1, \dots, 4)$$

$$\frac{\partial F}{\partial x_1}\phi_1'(0) + \frac{\partial F}{\partial x_2}\phi_2'(0) + \frac{\partial F}{\partial y_1}\phi_3'(0) + \frac{\partial F}{\partial y_2}\phi_4'(0) = 0$$
$$\frac{\partial \Phi}{\partial x_1}\phi_1'(0) + \frac{\partial \Phi}{\partial x_2}\phi_2'(0) + \frac{\partial \Phi}{\partial y_1}\phi_3'(0) + \frac{\partial \Phi}{\partial y_2}\phi_4'(0) = 0,$$

and if it can be shown that if equations (38) are not satisfied, then either

$$\phi_1'(0) = \phi_2'(0) = \phi_3'(0) = \phi_4'(0) = 0 \tag{39}$$

or the equation in S has three zero roots. But since S only has two zero roots and equations (39) are only satisfied for certain particular periodic solutions where the planetoid has a circular orbit,[204] the equations (38) must be satisfied for $x_1 = \phi_1(T)$, .., $y_2 = \phi_4(T)$. Furthermore, since the origin of the time is arbitrary, they must also be satisfied for $x_1 = \phi_1(t)$, .., $y_2 = \phi_4(t)$, that is for all points of the periodic solutions.

For the final part of the proof Poincaré showed that equations (38) were in fact satisfied identically, which proves that Φ is a function of F. Hence the two solutions F and Φ cannot be distinct. Thus the equations (37) do not admit any new single-valued transcendental integral, providing the value of μ is sufficiently small.

5.9.4. Positive and negative results. In [P1] Poincaré did not have a chapter dealing specifically with any of the three topics described above, that is, periodic solutions of the second class, the divergence of Lindstedt's series, and the nonexistence of integrals; although, as has been described, much of the material did appear in [P1] in a less organised fashion. However, he did include a chapter in [P1] in which he gave a general resumé of his results. The chapter was brief, amounting to only five pages, and was divided into two parts, positive and negative results, of which very little was reproduced in [P2].

With regard to the positive results Poincaré mainly concluded that the trajectories in the restricted three body problem could be classified into three types: closed trajectories corresponding to periodic solutions; asymptotic trajectories; and the general trajectories that did not fit into either of the above categories. He commented that the difficult and unexpected result which (he believed) he had established was that the trajectories which asymptotically approach an unstable closed trajectory were the same trajectories as those which asymptotically move away from the same unstable closed trajectory, and that the set of these asymptotic trajectories formed a closed asymptotic surface. It was of course the latter part of this result which was erroneous and which caused him so much trouble.

Of the results he described as negative, although he mentioned the divergence of Lindstedt's series, the one he considered the most important was the one concerning the nonexistence of any new integrals. He gave only a brief outline of the proof of this result in which he related it to the restricted problem, although, as mentioned, he did add a more detailed proof in *Note G*.

[204]See Laplace [1789-1825, **X**, Chapter VI].

5.10. Attempts at generalisation

5.10.1. The n body problem. In the last chapter of the memoir Poincaré returned to the competition question: the n body problem. Initially he had hoped in some way to generalise his earlier results, even allowing for the fact that there would be complications arising from the increased number of variables and the consequent impossibility of creating a realisable geometric representation. Unfortunately, this did not turn out to be the case, although the difficulties only really became apparent when he attempted to generalise the second half of the memoir.

The first part of the memoir presented him with little problem, since it was only a question of extending the number of dimensions of the representation space. This is quite straightforward, since in a system with two degrees of freedom a state of the system is represented by the position of a point in a space of 3 dimensions, while in a system with p degrees of freedom a state is represented by the position of a point in a space of $2p-1$ dimensions. Thus many of the conclusions concerning periodic and asymptotic solutions of the restricted problem can be generalised with relative ease. For example, the theory extends in a natural way to show that in the n body problem there are an infinite number of periodic solutions, stable and unstable, as well as an infinite number of asymptotic solutions.

It was in his attempt to generalise the second part of the memoir that Poincaré found himself beset with difficulties. He cited, for example, the case where the autonomous Hamiltonian equations have three degrees of freedom, and the problem is then to find three functions, $x_i = \Phi_i(y_1, y_2, y_3)$, satisfying

$$\frac{\partial x_i}{\partial y_1}\frac{\partial F}{\partial x_1} + \frac{\partial x_i}{\partial y_2}\frac{\partial F}{\partial x_2} + \frac{\partial x_i}{\partial y_3}\frac{\partial F}{\partial x_3} + \frac{\partial F}{\partial y_i} = 0 \qquad (i = 1, 2, 3).$$

He found that even this relatively simple case led to the consideration of three different situations, two of which led to the problem of small divisors, while the third led to inscrutable integrals.

A second difficulty he faced concerned the motion of the perihelions. In the unperturbed case when the system is in a state of Keplerian motion, since the Hessian of F_0 with respect to the linear variables x_i is zero, as well as the Hessian of any arbitrary function of F_0, the perihelions remain fixed. This difficulty does not arise in the restricted problem because it is not necessary to use the longitude of the perihelion, g, as a variable, since the variable $(g-t)$ can be chosen instead.

Poincaré also drew attention to the fact that he had not made a full investigation of the periodic solutions of the unperturbed motion in the three body problem, that is, when the orbits of the two smaller bodies or planets reduce to Keplerian ellipses. In his analysis he had only considered the obvious case of periodic solutions that arise when the two mean motions are commensurable, and he had not considered the possibility of any others. Thinking about this particular question led him to another idea: that of periodic motion resulting from two planets passing infinitely close to each other without actually colliding. It occurred to him that if the planets did move in such a way, then this would give rise to a change in their orbits which would give the appearance of a collision. He thought then that it might be possible to choose the initial conditions in such a way that these "collisions" occurred periodically. If this were the case, then discontinuous solutions would be obtained which would be proper periodic solutions of the Keplerian motion. He did not have time at this stage to pursue the idea further, although he did discuss it some ten years later [MN III, Chapter XXXI], when he called these solutions *periodic solutions of the second species.*[205]

Although it was evident that some of the difficulties Poincaré had encountered in trying to generalise his results would be overcome in the fulness of time, there were others that appeared to be beyond the scope of available techniques. In any event, Poincaré had made it clear that the n body problem was still far from being solved.

With characteristic modesty, Poincaré concluded by saying that he regarded his work as only a preliminary survey from which he hoped future progress would result.

[205]These solutions are discussed further in Chapter 7.

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CHAPTER 6

Reception of Poincaré's Memoir

6.1. Introduction

The discovery of the error in Poincaré's original memoir and the accompanying delay in the publication meant that almost two and a half years had elapsed between the submission of the manuscript to the competition and the publication of the corrected memoir in *Acta*. As a result it is possible to distinguish between the responses to each version. The response to [P1] consisted of the opinions of the prize commission, together with those whose knowledge of it was derived from Mittag-Leffler's report. [P2] had a rather wider audience.

At the time the prize was awarded the only publicly available information about the mathematical content of the prize-winning memoir was Mittag-Leffler's brief report. The memoir had not been made publicly available, and so the rest of the press reports covering Poincaré's triumph contained no mathematical commentaries. Furthermore, the prepublication copies of [P1] which made a brief appearance at the end of 1889 were in circulation for such a short period that none of the recipients would have had time to master the contents in order to make a fully informed judgement. Thus the only people who had had the opportunity to scrutinize [P1] were the members of the prize commission. Nevertheless, despite the paucity of information available in Mittag-Leffler's report, it was sufficient to provoke an adverse reaction from Hugo Gyldén.

When [P2] finally appeared, it drew widespread praise, although it is clear that certain aspects of Poincaré's mathematics were well beyond the grasp of his fellow mathematicians. With regard to the error, Mittag-Leffler's campaign of secrecy plus the delay caused by the backlog of publishing at *Acta* meant that when [P2] was published, those members of the mathematical community who had heard any rumours about the error had had plenty of time to forget the details. The result was that by the time [P2] finally appeared, any remaining concerns about the error, if indeed they existed, seemed to have vanished, despite Poincaré's brief allusion to it in the introduction.

6.2. The views of the prize commission

As described in Chapter 4 above, the three members of the prize commission were quick to come to a unanimous decision concerning the overall merit of Poincaré's entry. However, in their correspondence during the adjudication period, the only one who ventured anything more than a general opinion on the mathematics was Weierstrass. Mittag-Leffler, although he openly indicated to Poincaré the points in the memoir which he felt needed further elaboration, fought shy of discussing the relative merits of any of the results with either Poincaré or the members of the commission. His prime concern appears to have been in fulfilling his role as a mediator between Hermite and Weierstrass, communicating information from one to the other.

Of a more public nature was Mittag-Leffler's responsibility for the commission's general report. But since the report was intended to present the opinion of the whole commission on the results of the competition and was written with help from Hermite and Weierstrass, it gives no insight into Mittag-Leffler's personal views. In any case, since the task of providing a mathematical analysis of Poincaré's memoir had been left to Weierstrass, the report contained no details of the winning entry beyond giving a general indication of the nature of the results and emphasising the power of analytic methods in treating questions of celestial mechanics.[206]

In addition to writing the general report, Mittag-Leffler had been asked by the King to give a resumé of Poincaré's results at the February meeting of the Swedish Academy of Sciences. As it happened, Mittag-Leffler's talk was not given until March, and then in rather less than favourable circumstances, the background to which is described later in the chapter.

Hermite, by virtue of being in Paris, was in the unique position of being able to speak directly to Poincaré about the memoir. He was unequivocal in his opinion of it, but, as he told Mittag-Leffler, he too had sought help from Poincaré over the details:

> *Poincaré's memoir is of such rare depth and power of invention, it will certainly open a new scientific era from the point of view of analysis and its consequences for astronomy. But greatly extended explanations will be necessary and at the moment I am asking the distinguished author to enlighten me on several important points.*[207]

It is not clear exactly which parts of [P1] Hermite felt needed explaining, nor did he give any indication of the results he considered the most important. The implication that Hermite felt uneasy about his ability fully to comprehend the mathematics is confirmed by his reaction to the suggestion that he might have to write the official report on the memoir. Although it was more or less understood that Weierstrass as proposer of the question ought to be the author of the report, Mittag-Leffler, as a result of Weierstrass's declining health, had expressed concern to Hermite about Weierstrass's fitness for the undertaking.[208] Unfortunately, Hermite's response was not what Mittag-Leffler was looking for:

> *The task of writing the report falls by right to Weierstrass who proposed the question, who can with authority express reservations which would put me into an indescribably difficult position should I have to make them. Indeed what would be my position vis-à-vis Poincaré to whom I would have to appeal for explanations in order to understand the most important points of the memoir; I would no longer be in the*

[206]See Appendix 4.

[207]Hermite to Mittag-Leffler, 17.10.1888, *Cahiers* **6** (1985), *146*.

[208]Mittag-Leffler to Hermite, 17.10.1888, I M-L.

> *role of judge, and I must tell you in all frankness, if I have to make the report, it would be the echo of what I had heard from listening to the author, with the intention of justifying my admiration for his genius. ... Besides, to satisfy the demand of public opinion, and taking into account the importance and seriousness of the announcement of the prize, do you think that it is advisable that the prize awarded to a Frenchman should rest on the report of a Frenchman who is his colleague and friend?*[209]

Mittag-Leffler needed no further convincing, and the responsibility for the report remained with Weierstrass, who had also displayed doubts about Hermite's ability to deal with the mathematics unaided.

At the 1889 public meeting of the Paris Academy of Sciences, Hermite, in his official capacity as Vice President, prompted by Mittag-Leffler's letter to the Secretary, used the occasion to comment on the results of the competition and commend the contents of Poincaré's memoir. In particular he drew attention to Poincaré's discovery of the asymptotic character of the series used in celestial mechanics and ironically chose to describe this result in terms of Poincaré having discovered an error: "*The error having been recognised, it opens a new avenue in the study of the three body problem, and this is where Poincaré's talent is displayed with brilliance.*"[210] He could have had no idea how prescient those words would turn out to be.

Weierstrass's opinion of [P1] is mostly revealed in three letters to Mittag-Leffler, parts of which were published in *Acta* in Mittag-Leffler's [1912] biography of Weierstrass. The most significant of these is the first, in which he gave his judgement of the competition entries. Although he commented on five of them, he wrote more than four times more on Poincaré's work than on the other four put together.[211]

Weierstrass considered the most important results to be what Poincaré had described as *negative results*, that is, the divergence of Lindstedt's series and the theorem on the nonexistence of single-valued integrals. Although he thought that these showed that an entirely new approach would be needed for the problem to be solved, he was still convinced that a solution existed. With regard to the positive results, he singled out Poincaré's discoveries in stability, invariant integrals, periodic solutions and asymptotic solutions as being especially notable, and was very enthusiastic about the treatment of the analytic solutions of algebraic differential equations. But it was not unadulterated praise. Weierstrass had also found the memoir extremely difficult to read, and he was concerned about its general lack of rigour.

In the second letter written some seven weeks later, although he was even more enthusiastic about the memoir than before, Weierstrass confessed that despite working hard on it he still had not been able to master it completely.[212] He now believed that the results on periodic solutions and the discovery of asymptotic motion were achievements of the highest importance, describing them as no less

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[209] Hermite to Mittag-Leffler, 22.10.1888, *Cahiers* **6** (1985), *147-148*.

[210] Hermite [1889].

[211] Weierstrass to Mittag-Leffler, 15.11.1888, I M-L. Mittag-Leffler [1912, *50-52*].

[212] Weierstrass to Mittag-Leffler, 8.1.1889, I M-L. Mittag-Leffler [1912, *53-55*].

than epoch-making. On a critical note he was concerned about Poincaré's treatment of the stability question in the restricted three body problem. He queried the physical validity of Poincaré's definition of stability, which appeared to put an upper bound on the distance between two points without considering what would happen if two points became infinitely close. As he pointed out, if this should occur, it would inevitably affect the form of motion, and thus, he argued, a distinction ought to be drawn that would take this into account.

The manuscript of this letter also reveals a careful piece of editing by Mittag-Leffler. Tactfully omitted from the published version is Weierstrass's remark in which he confided to Mittag-Leffler that he thought Hermite must have had somebody to explain the memoir to him.

In the third letter Weierstrass explained why, contrary to what he had said in his previous letter, he was now satisfied that Poincaré's analysis did ensure that the planetoid could not come infinitely close to the other two bodies.[213] He had simply overlooked Poincaré's incorporation of the condition that the value of the constant C in the equation

$$\frac{1}{2a} + G + \mu F_1 = C$$

had to be essentially greater than $\frac{3}{2}$. This was also the letter in which, as mentioned in the previous chapter, he questioned Poincaré's claim that the nonexistence of any new single-valued solution necessarily implied the nonexistence of convergent trigonometric solutions.

Finally, he told Mittag-Leffler that, although he would definitely have the report finished by the end of the week, he was having difficulty with its introduction. This was because he believed it should begin with a justification of the question in order to counter the adverse criticism which it had been lodged against it. The criticism was on two fronts: there were those who claimed that the question as it stood was completely insoluble, while others censured the limitation induced by the assumption that a collision between two points can never take place. Weierstrass indicated to Mittag-Leffler his intended response to these accusations, but his real concern was how to condense into a few lines something which he felt warranted a long discussion. Clearly much of the criticism was due to Kronecker [1888], and, given the brittle nature of their relationship, Weierstrass was keen for his defence to be carefully drafted.

Despite his intentions, Weierstrass never finished the report, although he did complete the introduction, sending a copy to Mittag-Leffler in March 1889. The introduction threw no light on his judgement of the memoir, but it did explain why he had chosen to formulate the n body problem in the way that he had, as well as giving the criteria he had used in judging the entries which had attempted to solve it.[214]

In the question Weierstrass had asked for an expansion of the coordinates as infinite series of known functions of time which were uniformly convergent for *unbegrenzter Dauer* (= unlimited time), the implication being that these series did actually exist, and it was the phrase *unbegrenzter Dauer* that was the main cause

[213] Weierstrass to Mittag-Leffler, 2.2.1889, I M-L. Mittag-Leffler [1912, *55-58*].
[214] Mittag-Leffler [1912, *63-65*].

of the misunderstanding. It had been interpreted as meaning that Weierstrass required series which were uniformly convergent for infinite time, i.e., for $0 \leq t \leq \infty$, rather than, as he had intended, series which were uniformly convergent for a fixed value of time, however large, i.e., for $0 \leq t \leq a$ ($a < \infty$). The distinction was critical because Weierstrass believed he had proved that such series did exist in the latter case—the proof being dependent on its ability to show that the distance between any two points can never become either infinitely small or infinitely large as time approaches a finite limit—but he had no proof for the former.

Showing that the distance between two points can never become infinitely small amounts to dealing with the possibility of collisions, both binary and multiple. Weierstrass's difficulty was that he thought that when he had originally set the question he had constructed a proof that overcame the problem of collisions, but he could not remember it. In his letter to Mittag-Leffler he gave an outline of a proof that dealt with binary collisions and stated a conjecture about triple collisions, but gave no consideration to collisions of greater multiplicity.

To deal with a binary collision he assumed that the time t was sufficiently close to the moment of collision t_0 so that the coordinates of all the points could be expanded in positive powers of $(t_0 - t)^{\frac{1}{3}}$, and the expressions would contain not $6n$ but $6n - 2$ arbitrary constants.[215]

With regard to triple collisions, Weierstrass claimed that it was easy to show that all three points can only collide when the three constants of angular momentum are simultaneously zero. With regard to the three body problem this was clearly an important result, but unfortunately Weierstrass did not give a proof, and Mittag-Leffler did not press him for one. It was not until the beginning of the next century that Sundman, unaware of Weierstrass's conjecture, provided a proof of this result; this is discussed in Chapter 8.

Rather curiously Weierstrass does not appear to have considered the possibility of the mutual distances becoming infinitely large, and it was not until 1895 that

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[215]Saari [1990] gives a clear account of the derivation of this expansion. Briefly, the equations of motion for two colliding points in the collinear central force problem are given by $\frac{d^2x}{dt^2} = -\frac{1}{x^2}$ with solution $x(t) \sim A(t - t_0)^{\frac{2}{3}}$ as $t \to t_0$, where A is a positive constant and t_0 is the time of collision, and it can be shown for the n body problem that the same rate of approach holds for collisions of any kind taking place at $t = t_0$. Assuming that $t_0 = 0$ and substituting $X(t)t^{\frac{2}{3}} = x(t)$ into the equations of motion gives

$$t^2\frac{d^2X}{dt^2} + \frac{4}{3}t\frac{dX}{dt} - \frac{2}{9}X = -\frac{1}{X^2}.$$

Making the change of variable $s = t^{\frac{1}{3}}$ leads to

$$s^2\frac{d^2X}{ds^2} + 2s\frac{dX}{ds} - 2X = -9\frac{1}{X^2},$$

which has an analytic solution in s. Then for arbitrary initial conditions, the probability of a collision between any two of the points would be infinitely small and so could properly be ignored. Weierstrass did, however, admit that he was concerned by the fact that this method did leave open the possibility that after an infinitely long period of time two points could approach each other infinitely closely without actually colliding.

Paul Painlevé formally proved that in the three body problem such a situation cannot arise.[216]

Weierstrass ended his letter to Mittag-Leffler by saying that he felt that he had provided sufficient results to validate the claim that, in general, the coordinates of the points in the three body problem could be developed in series of the form he had specified in the question. However, since he had neither provided a proof of the impossibility of triple collision nor eliminated the possibility that the mutual distances cannot become infinitely large, his claim was somewhat tenuous.

Nevertheless, since Weierstrass did consider it legitimate to suppose that, given an unlimited time interval, the coordinates in the n body problem were single-valued continuous functions of time and as such could be represented by a series as specified in the question, he did believe that a solution was possible, and so his question was then whether such a solution was actually feasible. That was why he had asked for the description of a method which would calculate successive terms of the series rather than asking for a complete expansion. In other words, he believed it was possible to give an approximate expression for the functions such that the difference between the expression and the function did not exceed a specified arbitrarily small limit within a time interval of arbitrary length. If this could be done then the function would be represented by an absolutely and uniformly convergent series, and the problem would be solved as required.

In setting the question Weierstrass had hoped to achieve a better understanding of the true nature of the motion of celestial bodies as well as obtaining a reliable result concerning the stability of the solar system. He had little doubt that the latter could be achieved, even without a solution which was valid for infinite time.

He explained that earlier attempts to obtain a solution had resulted in the coordinates of the planets or variable orbital elements being represented by series of the form

$$\sum_{\nu_1,\nu_2,\ldots} \{C_{\nu_1\nu_2}\ldots \sin(c_0+\nu_1 c_1+\nu_2 c_2+\ldots)t\}\,,$$

where ν_i are integers, t is the time and $(C_{\nu_1\nu_2\ldots}, c_1, c_2, \ldots)$ are independent of t. Since it had been shown that, under certain assumptions, such series do formally satisfy the differential equations, what remained to be resolved was whether such series were convergent and thus true expressions of the quantities to be represented. Because this problem was clearly one of the fundamental issues raised by the competition question, its treatment provided Weierstrass with a criteria on which to base his judgement.

It was unfortunate that Weierstrass never completed his report, but it seems very probable that his analysis of [P1] would have been largely based on the letters described above. It is clear from the later correspondence that his delay in producing a report was due to his continuing difficulties over parts of the memoir that he felt it necessary to master and that he considered insufficiently explained.

[216] Painlevé's contribution and the question of noncollision singularities in the n body problem are discussed in Chapter 8.

He did however make one further comment concerning [P1]; that was to criticise Mittag-Leffler for the letter he had sent to the Secretary of the Paris Academy announcing the results of the competition.[217] It seems that the total French triumph had proved rather hard for the German mathematicians to bear and, in particular, exception had been taken to Mittag-Leffler's description of Poincaré's memoir as being one of "*... the most important pieces of mathematics of the century ...*".

While the Germans may well have been justified in detecting an element of self-interest in Mittag-Leffler's remark, it is fair to argue that history has proved him right.

6.3. Gyldén

One of the first people outside the commission to hear about Poincaré's memoir was Gyldén. As a member of the editorial board of *Acta*, as well as being a lecturer in astronomy at the Stockholm Högskola, he was in close touch with Mittag-Leffler and well placed to hear about the result of the competition. However, unfortunately for Mittag-Leffler, when Gyldén learnt of the results in Poincaré's memoir, his reaction mirrored that of Kronecker more than three years earlier. Gyldén, having seen the general report with its remarks about the discovery of asymptotic solutions, believed that he had already discovered similar results which he had had published in *Acta*.[218]

Mittag-Leffler appears to have had some idea of Gyldén's views almost immediately, because only a matter of days after the report was published he wrote to Poincaré to ask him for his opinion on a result in Gyldén's paper. The result appeared to conflict with something that Poincaré had written. The particular point at issue concerned the convergence of certain power series: Gyldén claimed that the series were definitely convergent while Poincaré had stated that the evidence for convergence was inconclusive. Poincaré responded swiftly to Mittag-Leffler but he admitted that he had found Gyldén's paper extremely hard to read.[219] In order to give a definitive answer he would have to make a much more detailed study of it, a task he was reluctant to undertake. He was therefore unable to say whether Gyldén's method led to a proof of either convergence or divergence, although he believed divergence more likely. He was also unhappy with the fact that Gyldén's method did not allow successive terms in the expansion to be deduced recurrently but involved making choices at each stage of the calculation, a feature which incorporated an element of chance into the process.

Meanwhile, Mittag-Leffler, at the King's request, had been due to give a review of Poincaré's paper at the February meeting of the Swedish Academy of Sciences. In the end illness prevented him from attending, although he had expressed reluctance to talk publicly about Poincaré's work without the support of Weierstrass's report. Gyldén, on the other hand, did attend the meeting and, moreover, did talk

[217] *Comptes Rendus* **108** (8) (25 February 1889), *387*.

[218] Gyldén [1887].

[219] Mittag-Leffler published Poincaré's side of the correspondence as part of the *Acta* volume dedicated to him.

Poincaré to Mittag-Leffler, 5.2.1889, No. 48, I M-L. *Acta* **38**, *163-164*.

Hugo Gyldén

about Poincaré's memoir. He declared his own position on Poincaré's results and effectively claimed priority.[220]

Once again Mittag-Leffler was placed in an awkward position. The King made it plain that he expected him to reply to Gyldén at the meeting the following month. Mittag-Leffler knew he could not rely on having Weierstrass's report in time, and so it became a matter of urgency to have detailed comments on Gyldén's paper from Poincaré.

On hearing from Mittag-Leffler about Gyldén's position Poincaré responded again and at length.[221] He made the point that the dispute brought into sharp focus the difference between mathematicians and astronomers with regard to their interpretation of convergence. He reasoned in detail against the rigour of Gyldén's method, reiterating that he believed Gyldén's method to rely heavily on questions of judgement, and, in his final letter on the subject, showed clearly why he believed that Gyldén's argument actually led to divergent series.[222]

Briefly, Poincaré began with the equation

$$\frac{d^2V}{dt^2} + n^2 sA \sin V \cos V = n^2(X),$$

where

$$(X) = \sum s_1 A_1 \sin(\lambda_1 nt + mV + h),$$

h is a constant and λ_1 and m are integers, and following Gyldén he put

$$V = V_0 + V_1, \qquad V_0 = -2\arctan e^{-\xi} + \frac{\pi}{2}, \qquad \xi = \alpha nt + c.$$

Gyldén's method then involved integrating by successive approximations and at each stage of the approximation choosing suitable values for the two constants of integration and the coefficient α.

Poincaré's argument hinged on the fact that he did not consider it legitimate for these choices to be arbitrary. With regard to the constants he believed that Gyldén's method meant that there was in fact only one particular value of the constants out of an infinite number of choices which would lead to a convergent series and hence to a proof of the existence of asymptotic solutions. Moreover, from what he could see, Gyldén's method gave no way of recognising which of the series was convergent. As far as α was concerned, he emphasised that its value was completely determined and could not, as Gyldén proposed, be changed with each new approximation, adding that αn was equivalent to what in his own memoir he had called the characteristic exponent.

Nevertheless, despite the critical appearance of his side of the correspondence, Poincaré did in fact maintain a high regard for Gyldén's work, appreciating the flexibility and practical advantages of his methods. He had not intended to demolish Gyldén but rather he had wanted to show how words such as *proof* and *convergence* take on different meanings depending on whether the user is a mathematician or an astronomer. Moreover, he was sensitive to the fact that Gyldén's approach was coloured by a practical interest in the problem which he himself did not share.

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220 Mittag-Leffler to Weierstrass, 22.2.1889, I M-L.

221 Poincaré to Mittag-Leffler, 1.3.1889, No. 49, I M-L. *Acta* **38**, *164-169*.

222 Poincaré to Mittag-Leffler, 5.3.1889, No. 50, I M-L. *Acta* **38**, *169-173*.

Hermite and Weierstrass were also drawn into the polemic. Hermite, who had first heard about the dispute from Kovalevskaya, thought Gyldén's series, like Lindstedt's, were asymptotic but carefully avoided drawing a direct comparison between the two memoirs. He had himself received a letter from Gyldén, but since it was written in Swedish he had been unable to read it, although he had deduced that it concerned the convergence question.

Meanwhile Mittag-Leffler gave his talk at the Academy and wrote in jubilation to Weierstrass and Poincaré, certain that those who had heard him had been convinced that Poincaré deserved the prize.[223] Although Gyldén had raised objections, insisting that his series were convergent for all time, he had admitted that in the neighbourhood of any set of the constants $c_1, \ldots, c_n$ there were other values for which the series did not converge.

However Mittag-Leffler's feeling of triumph was short lived. The academic community in Stockholm decided to weigh in on the side of Gyldén and, despite the fact that Poincaré's memoir was not in the public domain, adopted the view that Gyldén had indeed published proofs of everything Poincaré had done.[224] The consensus was that Mittag-Leffler's denial of Gyldén's results had been motivated by jealousy; this idea was reinforced by the mathematician Bäcklund, who drew attention to the fact that Gyldén's memoir had recently been awarded the St. Petersburg prize.

Meanwhile Gyldén himself steadfastly maintained that the values of the constants $c_1, \ldots, c_n$ for which his series diverged formed only a countable set, and so it was infinitely unlikely that the series was actually divergent. Mittag-Leffler continued to argue against him since, with Poincaré, he believed that the series were divergent not just for a countable set but for a perfect set in the neighbourhood of these constants. Moreover, he told Weierstrass that he thought Gyldén not enough of a mathematician to understand.[225]

With the publication of the memoir not scheduled for several months, the controversy gradually died down. Nevertheless when the memoir finally appeared, Gyldén did attempt to reopen the debate by writing directly to Hermite.[226] Possibly he thought he could count on Hermite's support, since Hermite was known to share his interest in the applications of elliptic function theory in celestial mechanics.[227] But Hermite was not to be drawn. He stood by the judgement of the commission, declaring his loyalty to Mittag-Leffler and Weierstrass. Shortly afterwards Gyldén sent Hermite part of his [1891] *Acta* paper for comment. This time Hermite avoided the issue completely by replying with the claim that the paper was outside his

[223] Mittag-Leffler to Weierstrass, 24.3.1889, and Mittag-Leffler to Poincaré, 28.3.1889, I M-L.

[224] Mittag-Leffler to Weierstrass, 15.4.1889, I M-L.

[225] Although not directly relevant to the disputes over Poincaré's memoir, it is of interest to record that in May that year Gyldén met with Kronecker in Berlin, a meeting which, within the context of competition, Mittag-Leffler would surely have viewed with some misgivings. In any case, the occasion prompted Mittag-Leffler to remark to Weierstrass that, although he had been led to believe that his two adversaries had understood each other perfectly, he suspected that Gyldén really understood as little of Kronecker as Kronecker understood of Gyldén. Mittag-Leffler to Weierstrass, 12.5.1889, I M-L.

[226] Hermite to Mittag-Leffler, 10.1.1891. *Cahiers* **6** (1985), *188-189* .

[227] Picard [1902] specifically mentions Hermite's sympathy for the work of Gyldén in this respect. See Hermite [1877].

own mathematical domain.[228] As he indicated later to Mittag-Leffler, he was not impressed by Gyldén's grasp of analysis, describing Gyldén as a ghost from a bygone age, who had been left behind as the world of analysis transformed about him.[229]

6.4. Minkowski

One of the earliest documented comments about [P2] came from the young Hermann Minkowski, a lecturer at the University of Bonn. In a letter dated 22nd December 1890 to David Hilbert, he revealed that he had studied the first third of the memoir, and what he had seen had reminded him of Dirichlet.[230]

Minkowski was also the author of the 1890 *Jahrbuch über die Fortschritte der Mathematik* report,[231] which appeared in 1893, by which time Minkowski had been promoted to an associate professor. This report, which seems to be the first mathematical commentary on [P2], was of a remarkable length. Most reports in the *Jahrbuch* merited at most a single page; Minkowski's report on [P2] ran to seven.

Since the function of the *Jahrbuch* was to provide information about the current state of mathematical research, Minkowski's priority would have been to provide a factual rather than a critical account of the memoir. Nevertheless, it is clear from the report that he had a good grasp of Poincaré's ideas. He skilfully picked out the salient features, emphasised their relative importance, and presented them in an accessible way.

There are various aspects of Minkowski's report which invite special comment: his clear and concise description of the theory of invariant integrals, in which he drew attention to the recurrence theorem; his discussion of Poincaré's use of the method of analytic continuation in the theory of periodic solutions; and the clarity with which he distinguished between Poincaré's use of the parameter μ and his use of its square root. Especially notable is the fact that he freely acknowledged the difficulties associated with Poincaré's doubly asymptotic solutions. Paradoxically, this probably indicates that Minkowski had a better understanding of the concept than most of his contemporaries, who abstained from passing comment on these solutions.

6.5. Hill

The first person to openly question some of the results in Poincaré's memoir and to do so in an entirely formal setting was Hill. On December 27, 1895, Hill

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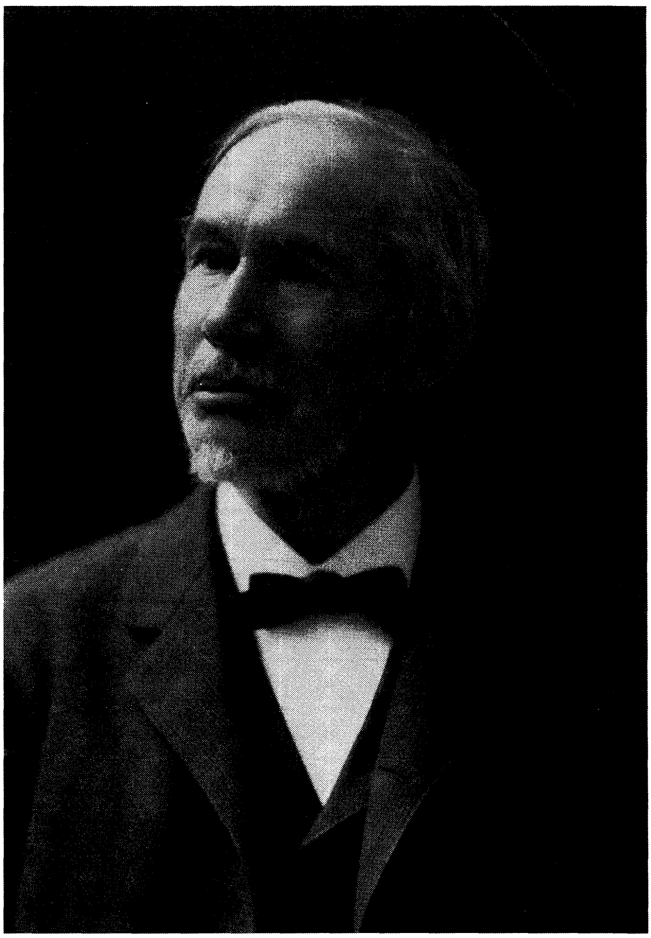
228 Hermite to Mittag-Leffler, 1.3.1891. *Cahiers* **6** (1985), *193*.

229 Hermite to Mittag-Leffler, 17.3.1891. *Cahiers* **6** (1985), *195-196*.

230 L. Rüdenberg and H. Zassenhaus, *Hermann Minkowski Briefe an David Hilbert*, Springer-Verlag, 1973, *40*. It seems likely that Hilbert had asked Minkowski for his opinion on the memoir, since he had already told Klein that he had arranged for a report on the memoir to be made at his Königsberg seminar. See G. Frei, *Der Briefwechsel David Hilbert-Felix Klein (1886-1918)*, Vandenhoek & Ruprecht, Göttingen 1985, *72*.

231 *Jahrbuch über die Fortschritte der Mathematik* **22**, *907-914*.

The *Jahrbuch* report was certainly not the first report of some of the ideas in [P2], although it does appear to be the first full report of the memoir. The first volume of Poincaré's *Mécanique Céleste*, which was derived from parts of [P2], was published in 1892, and a review appeared in the *Bulletin of the New York Mathematical Society* later the same year. See Chapter 7.



George William Hill

delivered the presidential address to the American Mathematical Society.[232] His speech was on the progress of celestial mechanics during the past fifty years, and, although meant for a general mathematical audience, the threads of his arguments were difficult to unravel, and in general it was hard to follow.

He began with a description of Delaunay's contribution and continued, "*Perhaps the most conspicuous labours in our subject, during the period of time we consider, are those of Professor Gyldén and M. Poincaré.*"[233] With regard to Poincaré's work, he cited both [P2] and the first two volumes of the *Méthodes Nouvelles*, which contained many of the results from [P2] reworked in an extended and clearer form. He centred his discussion primarily on the *Méthodes Nouvelles*, but most of his comments apply equally well to both works.

The fact that Hill took the opportunity presented by the occasion to query some of Poincaré's results can be partly explained by his belief that his own concerns were shared by other astronomers whom he thought would feel reassured by his criticism, especially those with less mathematical insight than himself. He may also have thought that a straightforward presentation of the results in a form accessible to astronomers would be somewhat superfluous, as Poincaré himself had already published a simplified account.[234] Another one might have been considered at best repetitious or at worst confusing.

Hill was particularly concerned by Poincaré's proof of the divergence of Lindstedt's series, a result which was of great practical importance to astronomers. Prior to the speech he had written an article in direct response to Poincaré's proof which had focused on the case where the mean motions are incommensurable.[235] In the article Hill demonstrated the existence of a class of cases where convergence can be shown, although he made no attempt to disprove Poincaré's argument. In the speech he aimed at reinforcing the article, and although in both cases he quoted results from [MN II] rather than [P2], it was the essential principle of the divergence of the series which was at issue. He questioned Poincaré's assertion that the convergence of Lindstedt's series would imply the nonexistence of asymptotic solutions, arguing that this was an irrelevant observation since the domains of the two things were quite distinct, i.e., where Lindstedt's series were applicable there were no asymptotic solutions and vice versa.

Since these objections concerned what Poincaré considered to be one of his most important results, and since their author was someone whose academic integrity Poincaré respected, he responded immediately.

In [1896], Poincaré's reply to Hill's first article appeared in the *Comptes Rendus* for March 2. He made it clear that there was no contradiction between their results—they had both, and in a similar way, proved the existence of cases where the series converge—but he did emphasise that it was possible for the convergence not to be uniform.

In [1896a] Poincaré countered most of the claims made in Hill's speech. Hill had believed that the series converged provided the variables remained within a

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[232] Hill [1896a].

[233] Hill's view of Gyldén's work is described in Chapter 2.

[234] Poincaré [1891].

[235] Hill [1896] and Poincaré [MN II, *277-280*].

certain domain. Poincaré showed that the series could not converge in any part of a domain which contained a periodic solution and that every domain, however small, contained a periodic solution. Thus if the series were convergent, they could only be convergent for certain discrete values of the variables and could not be convergent for values between given limits, no matter how small the limits.

Hill had also drawn attention to the asymptotic solutions and the role of the associated characteristic exponents. His objection concerned the actual use of asymptotic solutions. He reasoned that since most of practical astronomy is concerned with systems which describe almost circular motion, a first approximation can be given by a periodic solution. By assuming this was the case, he was led to the situation where all the characteristic exponents were imaginary, and thus the coefficients of stability were real and negative, which was a situation of no interest to the working astronomer.

Poincaré pointed out that Hill's premise was based on the mistaken idea that asymptotic solutions could only exist when the variables satisfied certain inequalities. He made it clear that he had actually proved the existence of asymptotic solutions for the restricted three body problem in any domain, however small, for sufficiently small values of the perturbing mass. He attributed Hill's error to the fact that he had only considered periodic solutions of the first kind.

6.6. Whittaker

In 1898 Edmund Whittaker, then a fellow at Trinity College, Cambridge, was asked by the British Association for the Advancement of Science to draw up a report on the current state of planetary theory.[236] He responded with a substantial review of recent work on the three body problem [1899]. Whittaker's report, which was essentially an exhaustive account of the development of dynamical astronomy from 1868 to 1898 (the dates being chosen to coincide with the publication of the last volume of Delaunay's *Lunar Theory* and the third and last volume of Poincaré's *Méthodes Nouvelles*), naturally included a detailed account of [P2], which was the first commentary on the memoir to be published in English.

[236] Whittaker became one of the most influential British mathematicians of his generation. He was appointed Astronomer Royal for Ireland in 1906, and was elected to the chair of mathematics at Edinburgh in 1912, a post which he held until his retirement in 1946. His great treatise *Modern Analysis* [1927], which was first published in 1902, was followed only two years later by the first edition of his comprehensive *Analytical Dynamics* [1937]. The latter, which remains a standard work on the subject, includes a thorough introduction to the three body problem and contains much of his own research related to topics in dynamical astronomy.

Several of Whittaker's early papers are of interest in relation to the work of Poincaré. Of note are [1901], in which he gave a new method for expressing the solution of a dynamical problem in terms of trigonometric series; [1902] and [1902a], in which he established a new criterion for finding periodic solutions of the differential equations of dynamics and for the restricted three body problem; and in particular [1917], which concerns his discovery of the adelphic ("brotherly") integral of a dynamical system. The adelphic integral is associated with infinitesimal contact transformations between adjacent periodic solutions, and Whittaker showed that when such an integral can be constructed, the equations can be integrated and the convergence difficulties indicated by Poincaré [P2, *470*] overcome. Whittaker also gained international recognition through an article on perturbation theory and orbits which he contributed to Klein's *Encyklopädie* [1912]. For a sensitive and informative biography of Whittaker, see McCrea [1957].

As befitted the nature of the report in which it appeared, Whittaker's review of [P2] was an objective summary rather than a subjective discussion. Nevertheless, echoing Minkowski's treatment in the *Jahrbuch*, Whittaker afforded the memoir greater attention than any of the other works included in his review. In contrast to Hill, his treatment of [P2] was both complimentary and easy to follow. He began: "*A new impetus was given to Dynamical Astronomy in 1890 by the publication of a memoir by Poincaré.*"[237] He then gave a clear and concise description of many of the ideas discussed in the memoir: invariant integrals, stability, periodic solutions, characteristic exponents, asymptotic solutions, doubly asymptotic solutions, and periodic solutions of the second class. He explained Poincaré's terminology and emphasised important results, such as the recurrence theorem and the theorem concerning the nonexistence of any new single-valued integrals. His concluding remark about the final section of the memoir was, however, somewhat ambiguous in that he did not make it clear that Poincaré was raising questions concerning the general n body problem rather than solving them.

Curiously, given the extent of his report, Whittaker made no attempt to relate [P2] to Poincaré's earlier papers on differential equations beyond a single reference to his result concerning the conditions for stability. Nor did he attempt to describe Poincaré's geometric representation and the innovative technique of using a transverse section to make the problem more tractable. This may have been because he felt that the conceptual difficulty of the ideas would distract from the actual results, although that had not earlier prevented him from revealing some of the complicated details of Gyldén's method.

With regard to the doubly asymptotic solutions, he simply described them as being "*...approximately periodic when $t = -\infty$ and $t = +\infty$, but not periodic in the meantime.*"[238] While this is certainly true, it hardly gave an indication of the complex nature of the behaviour of these solutions. Admittedly Poincaré himself had not stressed this point in [P2], but he certainly did in [MN III]. But Whittaker did not mention the complexity aspect in his review of [MN III] either. It is possible that he thought it inappropriate to emphasise these solutions, since the probability of their appearance in reality was negligible. This seems unlikely, however, since the same is also true of all Poincaré's periodic solutions. Nevertheless, since he again passed over the point in his treatise on analytical dynamics, perhaps it was because he felt his own understanding was not sufficiently adequate to provide a discussion.

6.7. Other commentators

The continuing interest in the three body problem in the decade following Whittaker's report was described by Edgar Lovett [1912], who charted mathematical developments relating to the n body problem between 1898 and 1908. Apart from the burgeoning literature on the problem by way of journal articles, the period was especially notable for the publication of Hill's *Collected Works*, Moulton's

237 Whittaker [1899, *144*].
238 Whittaker [1899, *149*].

Introduction to Celestial Mechanics and Whittaker's *Analytical Dynamics*. Furthermore, by the end of the decade the publication of Poincaré's lectures on celestial mechanics [LMC] was almost complete, and the publication of the *Collected Papers* of the applied mathematician George Darwin, the pioneer of the quantitative study of periodic orbits, was about to begin.

Significantly, part of the structure of Lovett's essay leads straight back to Poincaré. Of the five headings he used, he included one on the qualitative resolution of the problem and one on periodic solutions. More specifically, he made several references to Poincaré's methods, identifying certain areas in which the influence of Poincaré's methods had been clearly felt. In particular he noted how Poincaré's preference for the canonical form of the differential equations had led to the adoption of this formulation in other investigations. Lovett also referred to developments in the theory of invariant integrals, as well as the importance of Poincaré's theorem on the nonexistence of any new single-valued integrals for the problem.

With regard to particular results in [P2], some interesting observations were made by the mathematical physicist Lord Kelvin [1891], who, having had the memoir brought to his attention by Arthur Cayley, was especially struck by the relationship between some of Poincaré's results and some conclusions of his own which he had published the previous year. In particular, he drew attention to the similarity between Poincaré's conjecture concerning the denseness of the periodic solutions [P2, *454*] and a proposition of Maxwell's concerning the distribution of energy. Maxwell had proposed "*that the system if left to itself in its actual state of motion, will, sooner or later, pass through every phase which is consistent with the equation of energy*",[239] which, as Kelvin pointed out, was essentially equivalent to saying that every region of space would be traversed in every direction by every trajectory. If this proposition were true, which Kelvin believed to be highly likely, then he concluded it was a necessary consequence that every motion would be infinitely close to a periodic motion. In addition, he also commented on the agreement between Poincaré's results and his own results on the instability of periodic motion, observing that "*Poincaré's investigation and mine are as different as two investigations of the same subject could well be, and it is very satisfactory to find perfect agreement in conclusions.*"[240]

As Brush [1966] and Gray [1992] have described, one of the first of Poincaré's ideas from [P2] to emerge in a different context was that of his recurrence theorem. This was because the theorem appeared to demonstrate the futility of contemporary efforts to deduce the second law of thermodynamics from classical mechanics. In 1896 a debate took place in *Annalen der Physik* between Ernst Zermelo, who believed that Poincaré's theorem disproved the absolute validity of the second law of thermodynamics, and Ludwig Boltzmann, who believed in the correctness of Poincaré's theorem but disputed Zermelo's application of it.[241] According to Zermelo, Poincaré's theorem implied that there were no "irreversible" processes at work, and hence the concept of a system with continuously increasing entropy was invalid. Boltzmann's defence was that the theorem was evidence of sudden brief

[239] Quoted in Thomson [1891, *512*].
[240] Thomson [1891, *512*].
[241] Translations of the papers by Zermelo and Boltzmann are contained in Brush [1966].

moments of decreasing entropy but that the statistical nature of his kinetic theory predicted that these moments would be so far apart that they would never actually be observed, and so entropy would in general increase. Although Zermelo and Boltzmann's personal debate came to an end within a year, the controversy continued to arouse interest and eventually became one of the sources for the foundation of modern ergodic theory.[242]

Further attention was drawn to Poincaré's work on the three body problem by his compatriot and predecessor in the chair of celestial mechanics at the Sorbonne, Félix Tisserand. In the fourth and final volume of his acclaimed *Mécanique Céleste* [1896], which was published in the year of his death, Tisserand included a chapter which consisted of Poincaré's own summary [1891] of [P2], together with some further explanations about Poincaré's periodic solutions.

Various aspects of [P2] and its underlying role in the *Méthodes Nouvelles* were naturally mentioned in Poincaré's numerous obituaries.[243] In addition, Volume 38 of *Acta*, which was dedicated to Poincaré, included two long articles describing his work: one on his mathematics by Hadamard and the other on his celestial mechanics and astronomy by von Zeipel, both articles placing a firm emphasis on the significance of the memoir.[244] Hadamard [1921] concentrated on the relationship of [P2] to Poincaré's earlier memoirs on differential equations, while von Zeipel [1921] considered its results in conjunction with the *Méthodes Nouvelles*. Of particular note is the fact that both authors quoted the passage from [MN III] where Poincaré described the complexity of the doubly asymptotic solutions. There is no doubt that the importance of these solutions had by this date been recognised, even if little further had been discovered about them. The fact that [P2] featured so strongly in these two extensive appreciations of Poincaré's career is a fitting compliment to the breadth of vision it embraced.

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[242]See Chapter 9.

[243]See for example Baker [1914] and Darboux [1914].

[244]Mittag-Leffler had begun preparing *Acta* **38** soon after Poincaré's death, but the outbreak of the First World War meant that publication was delayed until 1921.

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CHAPTER 7

Poincaré's Related Work after 1889

7.1. Introduction

After the revision of the memoir, Poincaré channelled much of his energy into amplifying the results it contained. Within two years of its publication the first volume of his celebrated *Les Méthodes Nouvelles de la Mécanique Céleste* was published. Its appearance heralded the start of an enterprise that occupied him in part for almost twenty years. The second volume was published in 1893, and the third (and final) volume was completed in 1899. The *Méthodes* was followed by its didactical counterpart, the *Leçons de Mécanique Céleste*. The *Leçons* were based on lectures Poincaré gave at the Sorbonne in his role as professor of mathematical astronomy and celestial mechanics, the position to which he had succeeded on the death of Tisserand in 1896. They contained a treatment of perturbation theory, the lunar theory, and the theory of tides, and were published in three volumes between 1905 and 1910, although according to Sarton [1913, *10*] the project was unfinished.

As well as producing these major works, Poincaré also published several more papers on different topics in celestial mechanics, some of which were connected to ideas which had appeared in [P2] and the *Méthodes*, and some of which were in response to the work of other mathematicians. There were, for example, several papers on the expansion of the perturbation function, two notes connecting the principle of least action with the theory of periodic solutions, and papers on the form of the equations in the three body problem. Other related papers included a correction to Bruns' theorem on the integrals of the three body problem, discussions of Gyldén's *horistic* methods, and some general articles.

There were also two important papers in which Poincaré continued his research into the periodic solutions of the three body problem, but outside the specific context of celestial mechanics. The first of these, which he originally presented to the American Mathematical Society at the St Louis Congress in 1904, was an investigation into the geodesics on a convex surface. This paper centred on the closed geodesics, since they enjoy an analogous role to the periodic solutions in the three body problem. The second was the paper in which he announced what is today known as his *Last Geometric Theorem*. This paper came to prominence not only because of the importance of the theorem it contained, but also because, despite strenuous efforts, Poincaré had been unable to provide a general proof.

7.2. "Les Méthodes Nouvelles de la Mécanique Céleste"

On the occasion of presenting the medal of the Royal Astronomical Society to Poincaré in 1900, George Darwin, in describing *Les Méthodes Nouvelles de la*