A Discussion of Homoclinic Orbits in the Circular Restricted Three-Body Problem

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1 Introduction

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2 The Competition

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3 Circular Restricted Three Body Problem

Before studying periodic orbits, their manifolds, and the process of locating homoclinic orbits, it is necessary to provide the reader a solid foundation of the Circular Restricted Three Body Problem (CR3BP) - the dynamical system in which the aforementioned trajectories are created and analyzed. In the CR3BP (Fig. 1), the origin of the system is set at the barycenter of the two main bodies in the system (e.g., the Earth & Moon), and the frame rotates so these bodies remain stationary on the x-axis. The bodies are assumed to move in perfectly circular orbits and act as point masses from a gravitational perspective. The restricted problem is then to ascertain the motion of the third body whose mass is considered negligible. The system is typically normalized so that the masses of the two primary bodies sum to 1 (i.e., $m_1 = \mu$ and $m_2 = 1 - \mu$, where $\mu = m_2/(m_1 + m_2)$ is known as the three-body parameter), the distance between the primaries is 1, the orbital period period of the primaries is 2π, and the gravitational constant G is equal to 1. Under these conditions, the equations of motion for the CR3BP are shown in Equations 1-3:

$$\ddot{x} = 2\dot{y} + x - (1-\mu)\left(\frac{x+\mu}{r_1^3}\right) - \mu\left(\frac{x-1+\mu}{r_2^3}\right) \tag{1}$$

$$\ddot{y} = -2\dot{x} + y\left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 1\right) \tag{2}$$

$$\ddot{z} = z\left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right), \tag{3}$$

where

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2} \tag{4}$$

$$r_2 = \sqrt{(x+-1+\mu)^2 + y^2 + z^2}. \tag{5}$$

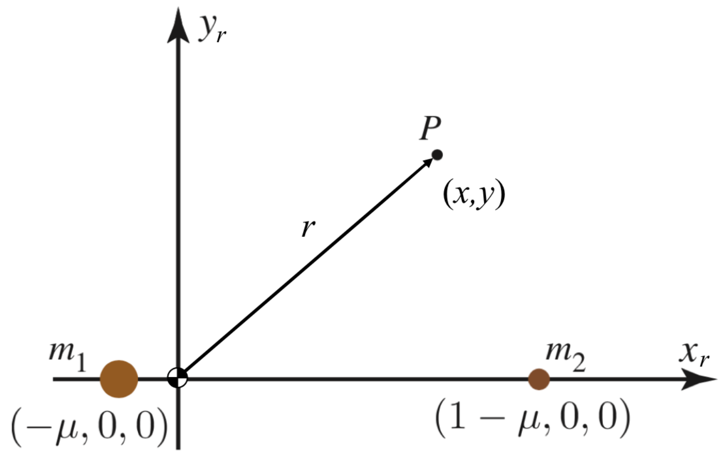

Figure 1: This figure shows the layout of the Circular Restricted Three-Body Problem [?]

In most simplified astrodynamic systems (e.g., Keplerian motion, CR3BP, n-body porblem, etc...), there are important parameters, known as integrals of motion, that are constant throughout the motion of a system that can be used to define the system. For example, the orbital elements of a trajectory in a Keplerian system are the system's integrals of motion. The integral of motion for the CR3PB is called the Jacobi constant and is given by

$$J = -\dot{x}^2 - \dot{y}^2 - \dot{z}^2 + x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2}. \tag{6}$$

For a given Jacobi constant, the motion of a particle is limited to certain regions of space due to the constraint that the velocity cannot have imaginary components. These restrictive regions, known as zero-velocity curves, are computed by setting the velocity in Equation 6 to zero and mapping the resultant surfaces in the CR3PB. The motion of an object with a specific Jacobi constant in bound within its respective zero-velocity curve and can only cross the boundaries under some non-conservative force.

3.1 Equilibrium Point Locations

Complex dynamical system such as the CR3PB often times have equilibrium points that result in a constant solution to the system's differential equations. In the CR3BP, these points, known as Lagrange points, mark positions where the combined gravitational pull of the two large masses provides precisely the centripetal force required to orbit with them (i.e., the Lagrange points are stationary within the rotating system of the CR3BP). There are five such points labeled L_1 - L_5 located in the plane of the two primary masses. The first three Lagrange points lie on the line connecting the two primary bodies, and the last two points, L_4 and L_5, are located at the vertex of an equilateral triangle formed with the two primary bodies [?].

In order to find the location of the five equilibrium points in the CR3BP, the velocity and acceleration must be set to zero in the system's equations of motion. This results in the following equations:

$$0 = x - (1-\mu)\left(\frac{x+\mu}{r_1^3}\right) - \mu\left(\frac{x-1+\mu}{r_2^3}\right) \tag{7}$$

$$0 = y\left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 1\right) \tag{8}$$

$$0 = z. \tag{9}$$

If y is set to zero, a quintic equation in x emerges. Solving this equation to first order results in the following location for the first three Lagrange points:

$$L_1 = \left(1 - \left(\frac{\mu}{3}\right)^{1/3}, 0, 0\right) \tag{10}$$

$$L_2 = \left(1 + \left(\frac{\mu}{3}\right)^{1/3}, 0, 0\right) \tag{11}$$

$$L_3 = -\left(1 + \left(\frac{5\mu}{12}\right), 0, 0\right). \tag{12}$$

Using Equations 10-12 as initial conditions, a Newton-Raphson iteration can then be used to numerically find the exact location of the first three Lagrange points to machine precision. Equations 7-9 can also be used to find the two triangular equilibrium points, L_4 and L_5. Since the equilibrium points form an equilateral triangle with the primary bodies, $r_1 = r_2 = 1$ is substituted into Equation 7 and Equation 8. These equations are subsequently solved to provide the exact locations for the remaining equilibrium points:

$$L_4 = \left(\frac{1}{2} - \mu, \frac{\sqrt{3}}{2}, 0\right) \tag{13}$$

$$L_5 = \left(\frac{1}{2} - \mu, -\frac{\sqrt{3}}{2}, 0\right). \tag{14}$$

Using the Earth-Moon CR3BP three-body parameter ($\mu = 0.01214$), Table 1 shows the non-dimensional locations of the five Lagrange points to six significant figures. Additionally, Figure 2 graphically shows these locations along with a contour plot of the CR3BP's zero-velocity curves for varying Jacobi constants. As Figure 2 shows, L_1, L_2, and L_3 lie at saddle points, while L_4 and L_5 lie at extrema values.

Table 1: Summary of the Dimensionless Earth-Moon CR3BP Equilibrium Points

Lagrange Point	**x**	**y**	**z**
L_1	0.836880	0	0
L_2	1.15571	0	0
L_3	-1.00507	0	0
L_4	0.487842	0.866025	0
L_5	0.487842	-0.866025	0

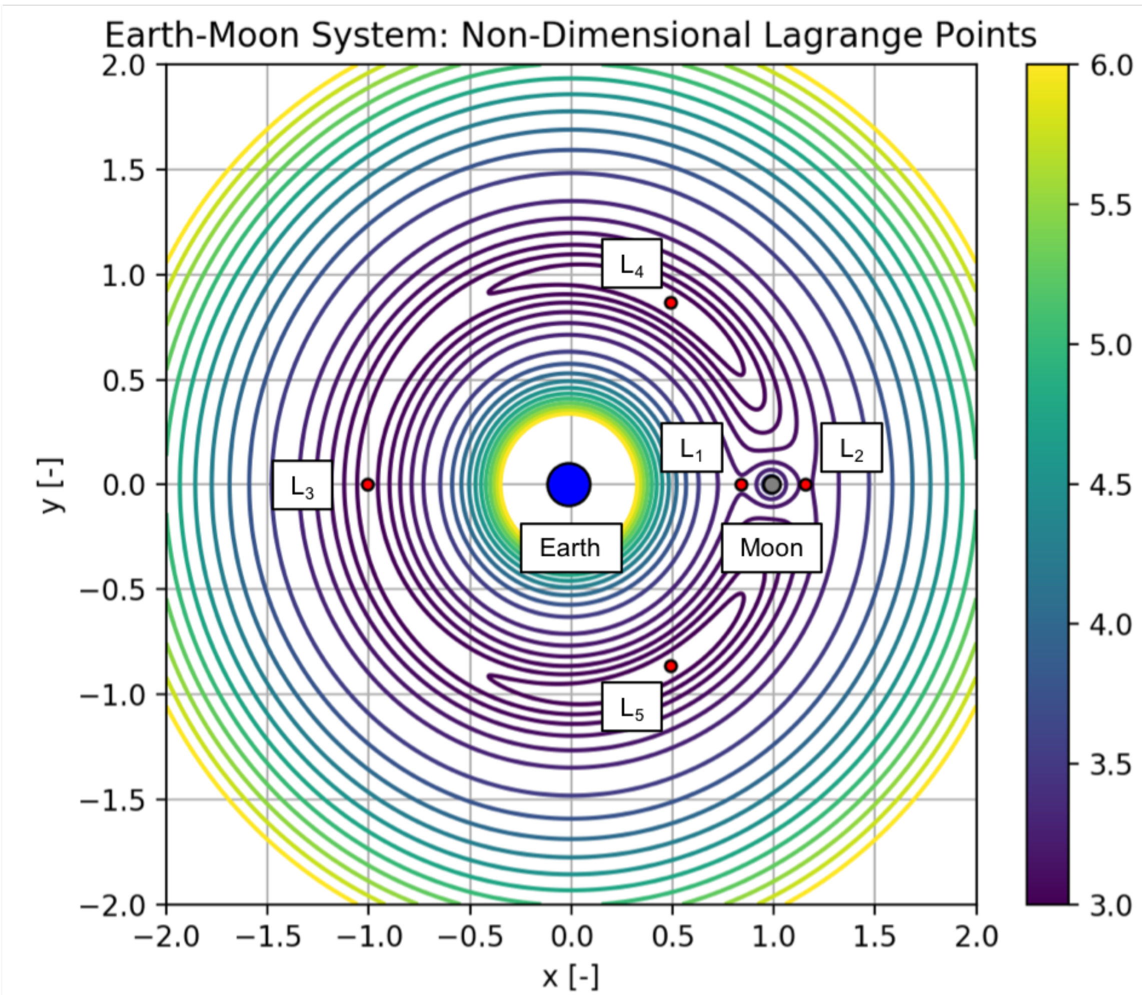

Figure 2: This figure depicts the the 5 equilibrium points and the zero-velocity curves in the non-dimensional Earth-Moon CR3BP system.

3.2 Equilibrium Point Stability

In order to study the planar stability of the equilibrium points in the Earth-Moon CR3BP, a linear stability analysis about each Lagrange point must be performed. This entails linearizing the rotational equations of motion about each equilibrium solution and solving for small departures from equilibrium.

3.2.1 L_1 and L_2 Stability

3.2.2 L_3 Stability

3.2.3 L_4 and L_5 Stability

3.3 Periodic Orbits

3.3.1 Predictor-Corrector Method

3.3.2 Orbit Families

3.4 Periodic Orbit Manifolds

3.5 Poincaré Sections and Homoclinic Orbits

4 Conclusion

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