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Some Homoclinic Phenomena in the Three-Body Problem

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We prove the existence of transversal homoclinic points in the collinear three-body problem, restricted and general, and in the planar circular restricted three-body problem. As a consequence the shift of Bernoulli is proved to be included as a subsystem of a suitable section of the flow for the three cases studied. Then the existence of all the possible types of final evolution follows.

1. INTRODUCTION AND MAIN RESULTS

In 1960 Sitnikov [9] gave a proof of the existence of oscillatory solutions in the three-body problem. For such solutions $\underline{\lim}(\sup(r_{ij}))$ is bounded but $\overline{\lim}(\sup(r_{ij}))$ is not. Oscillatory motions were predicted by Chazy as early as 1929 [2]. The results of Sitnikov were made rigourous and widely extended by Alekseev in 1967 [1]. The machinery is quite impressive but Moser [7] was able to do things in a more tractable way.

What is actually proved in the Sitnikov problem (two equal masses describing a Keplerian bounded orbit in a plane and a third infinitesimal one in the line orthogonal to the plane through the center of the two finite masses) is that near the parabolic orbits of the third body the Bernoulli shift based on countable many symbols can be included as a subsystem of a Poincaré map associated to the flow. This implies not only the existence of oscillatory solutions but that of an infinity of periodic, capture and escape orbits. A basic fact used to prove the embedding of the shift is the existence of transversal homoclinic points.

The purpose of this paper is to give different examples of transversal homoclinic phenomena in the three-body problem. First we consider a restricted collinear problem. Let $m_1 = m$, $m_2 = 1 - m$, $-x_1$, x_2 be the masses and positions of the primaries and x the position of the infinitesimal body. We

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suppose that the motion of the primaries is rectilinear elliptic with semiaxis of length one. The infinitesimal body collides with m_2 and goes near infinity. Let k be the (integer part of the) number of collisions of m_1 and m_2 between two successive collisions of m_2 and m_3. Our main result is

THEOREM A. *Let $0 < m$ be sufficiently small. Then there exists an integer $b = b(m)$ such that given any sequence $\{b_n\}$ with $b_n \geqslant b$, $n \in \mathbb{Z}$, the successive values taken by k equal the prescribed b_n, $n \in \mathbb{Z}$.*

Eventually the sequence $\{b_n\}$ can end (on one side or both) if $b_n = +\infty$. Take b_n bounded for all n. We get a Lagrange stable (or bounded) solution. If $\overline{\lim}\, b_n = +\infty$, $\underline{\lim}\, b_n < +\infty$, we obtain oscillatory solutions. Capture orbits are associated to sequences $(\infty, b_j, b_{j+1}, \ldots)$ with b_i bounded for all $i > j$. The reverse gives escape orbits, and $(\infty, b_j, \ldots, b_i, \infty)$ produces capture, interplay and escape.

As in the Sitnikov problem a first step consists in introducing a Poincaré map f. We obtain a region D_0 in which f is defined. The boundary ∂D_0 is related to the parabolic orbits. Symmetric regions D_1 and ∂D_1 are obtained for f^{-1}.

Following McGehee [5] we study the neighbourhood of infinity. The parabolic orbits are on invariant manifolds (stable and unstable) and they cut the Poincaré "surface de section" in ∂D_0, ∂D_1, respectively. We are interested in a homoclinic point p belonging to $\partial D_0 \cap \partial D_1$. Then we prove that ∂D_0 and ∂D_1 are transversal at p. This is the more technical part.

We point out a significant difference with respect to the case of Sitnikov: one needs to deal with collisions of m_1, m_2 and of m_2, m_3. This produces some difficulties in choosing regular variables that are simple enough to perform the computation of transversality.

After the restricted collinear problem we turn to the general one. We remark that the motion of m_2 was regular at the m_2, m_3 collisions in the restricted case, but now it is not. However we can extend the ideas provided m_3 is small enough. Defining k as in the restricted problem we obtain the following result:

THEOREM B. *Let $m_1 = m(1 - \epsilon)$, $m_2 = 1 - m$, $m_3 = m\epsilon$. There exist m_0, ϵ_0 such that given any sequence $\{b_n\}$, $n \in \mathbb{Z}$, $b_n \geqslant b$, the successive values taken by k equal the prescribed b_n, $n \in \mathbb{Z}$, where $b = b(m, \epsilon)$ for any $m \in (0, m_0)$ and $\epsilon \in (0, \epsilon_0)$.*

The remarks made after Theorem A are obviously applicable.

The last case we study is the restricted circular problem in the plane. We suppose that the Jacobi constant is large and that the infinitesimal body is far from the primaries. This fact is preserved forever due to the existence of zero velocity curves. Let us confine our study to orbits of the third body with the excentricity bounded from below. Using the node elimination we can define a

Poincaré map. Now k is the number of revolutions of m_1, m_2 between two successive minima of the distance of m_3 to the origin. We state the result:

THEOREM C. *Let* $m_1 = m$, $m_2 = 1 - m$. *There exist* m_0, C_0 *such that if* $m \in (0, m_0)$ *and the Jacobi constant* C *is greater than* C_0, *then there is some* $b = b(m, C)$ *such that given any sequence* $\{b_n\}$, $n \in \mathbb{Z}$, $b_n \geqslant b$, *the successive values taken by* k *equal the prescribed* b_n, $n \in \mathbb{Z}$.

The proof of transversality in the planar case will appear in a forthcoming paper [3], where a similar result is established without using the node reduction.

We remark that what is proved in the three cases is the embedding of the Bernoulli shift as a subsystem of the Poincaré map near a transversal homoclinic point.

2. THE EQUATIONS IN THE RESTRICTED COLLINEAR CASE

We consider the motion of two punctual bodies of masses m_1, m_2 on a line in a degenerate elliptic orbit with regularized collisions. Let us select units of length, time and mass such that the length of the major axis equals 2, the period is 2π and $m_1 = m$, $m_2 = 1 - m$ for some $m \in (0, 1)$. We take the center of masses at the origin. Let $-x_1$, x_2 ($x_i \geqslant 0$) be the coordinates of m_1, m_2, respectively. Then we have the expressions

$$x_1 = (1 - m)(1 - \cos E), \quad x_2 = m(1 - \cos E) \quad \text{with } t = E - \sin E. \tag{2.1}$$

The parameter E is the so-called excentric anomaly and the origin of time is taken at a collision between m_1 and m_2. We remind that t, E are defined modulus 2π and from now on this will be understood without explicit mention each time that t or E appears.

Let $x \geqslant 0$ be the coordinate of the third body of mass $m_3 = 0$. We suppose $x \geqslant x_2$ (see Fig. 1). The differential equation for the motion of that body is

$$\ddot{x} = -(1 - m)/(x - x_2)^2 - m/(x + x_1)^2, \tag{2.2}$$

with x_1, x_2 given by (2.1). In what follows m_i, $i = 1, 2, 3$ will be used to designate the ith body as well as its mass.

We have a singularity in (2.2) when $x = x_2$. If $x_2 > 0$ we encounter a binary collision between m_2, m_3. A change of variables will regularize such collisions

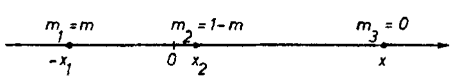

FIGURE 1

as usual. If $x_2 = 0$ then $x_1 = 0$ too and we obtain a triple collision. We do not need to study near triple collisions. In fact the properties that we shall use for our results deal only with collisions $x = x_2$ in the vicinity of $E = \pi$.

Let $h_{123} = \dot{x}^2/2 - (1-m)/(x - x_2) - m/(x + x_1)$ be the energy per unit mass of the third body, and $h_{23} = (\dot{x} - \dot{x}_2)^2/2 - (1-m)/(x - x_2)$ the energy associated to the binary $m_2 - m_3$. We intend to relate the motion close to collisions of $m_2 - m_3$ with the motion of the third body near infinity. As h_{123} does not have a finite limit when $x - x_2 \to 0$ and h_{23} is not suitable when $x \to \infty$, we define

$$h = (x^r - x_2^r)\, h_{123}/x^r + x_2^r h_{23}/x^r \qquad \text{with} \quad r \geqslant 1.$$

Using the behaviour of bodies near collision (see Siegel and Moser [8, p. 30]) we see that h is well defined along solutions of (2.2) (if and only if $r \geqslant 1$) and that $h \to h_{23}$ if $x - x_2 \to 0$, $h \to h_{123}$ if $x \to \infty$.

We scale the time in order to regularize collisions $m_1 - m_2$ and $m_2 - m_3$ introducing the s variable (see, for instance, Stiefel and Scheifele [10, p. 20])

$$dt = (x - x_2)(1 - \cos E)\, ds.$$

Then (2.2) can be written

$$\begin{aligned}
\frac{dx}{ds} &= \left\{\frac{x_2^r \dot{x}_2}{x^r} + \left[\left(\frac{x_2^r \dot{x}_2}{x^r}\right)^2 + 2\left(\frac{x^r - x_2^r}{x^r}\,\frac{m}{x + x_1} + \frac{1-m}{x - x_2} + h - \frac{x_2^r \dot{x}_2^2}{2x^r}\right)\right]^{1/2}\right\}\frac{dt}{ds} \\
\frac{dh}{ds} &= \frac{x^r - x_2^r}{x^r}\,\frac{dh_{123}}{ds} + \frac{x_2^r}{x^r}\,\frac{dh_{23}}{ds} + (h_{23} - h_{123})\,\frac{d}{ds}\left(\frac{x_2^r}{x^r}\right) \qquad (2.3) \\
\frac{dt}{ds} &= (x - x_2)(1 - \cos E),
\end{aligned}$$

where

$$\begin{aligned}
dh_{123}/ds &= (m\dot{x}_1/(x + x_1)^2 - (1-m)\,\dot{x}_2/(x - x_2)^2) \cdot dt/ds, \\
dh_{23}/ds &= (\dot{x} - \dot{x}_2)(-m/(x + x_1)^2 - \ddot{x}_2) \cdot dt/ds, \\
d/ds(x_2/x)^r &= r x_2^{r-1}(\dot{x}_2 x - x_2 \dot{x})/x^{r+1} \cdot dt/ds, \\
h_{23} - h_{123} &= -\dot{x}\dot{x}_2 + \dot{x}_2^2/2 + m/(x + x_1), \\
\dot{x}_1 &= (1-m)\sin E/(1 - \cos E), \\
\dot{x}_2 &= m \sin E/(1 - \cos E), \\
\ddot{x}_1 &= -(1-m)/(1 - \cos E)^2, \\
\ddot{x}_2 &= -m/(1 - \cos E)^2.
\end{aligned}$$

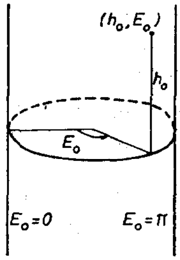

FIGURE 2

We remark that $r \geqslant 1$ is enough to make dx/ds and $(x^r - x_2{}^r)\, x^{-r} dh_{123}/ds$ regular, but values $r \geqslant 2$ and $r \geqslant 3$ are necessary for $x_2{}^r x^{-r} dh_{23}/ds$ and $(h_{23} - h_{123}) d/ds(x_2/x)^r$, respectively. From now on we take $r = 3$.

PROPOSITION 2.1. *Orbits of the third body excluding triple collision are determined by points of a cylinder excluding one generatrix.*

Proof. From (2.3) we see that the motion of m_3 is well defined through the initial conditions

$$s = s_0\,, \qquad (x - x_2)(s_0) = 0, \qquad h(s_0) = h_0\,, \qquad t(s_0) = t_0$$

(the orbit of m_3 has at least one collision $m_2 - m_3$).

Due to the autonomous character of the equations the value of s_0 is irrelevant. Let us take $s_0 = 0$. Then $h(0) = h_0$, $t(0) = t_0$, or, equivalently $E(0) = E_0$, are enough to determine the motion if we suppose that bodies m_2, m_3 are at collision. Under these hypotheses one orbit of the third body can be given through a point of a cylinder C with angle E_0 and height h_0 (see fig. 2). ∎

More than one point can give the same orbit. The generatrix $E_0 = 0$ which is related to triple collision is excluded from now on, but we keep calling C the cylinder without that generatrix.

3. THE POINCARÉ MAP

For $m = 0$ our problem becomes two independent two-body problems with $m_2 = 1$ at the origin and $m_2 = m_3 = 0$.

PROPOSITION 3.1. *For $m = 0$ the cylinder C is divided by the circle $h_0 = 0$ (parabolic orbits) in two regions: elliptic orbits ($h_0 < 0$) and hyperbolic ones ($h_0 > 0$).*

The proof is elementary using the fact that now $h = h_{123} = h_{23}$ is an integral of (2.3). We shall extend the proposition above to the case of positive m.

We define a mapping f in a subset D_0 of C, f: $D_0 \to C$ in the following way: Given a point $(h_0, E_0) \in C$ and $s_0 = 0$ let us consider the orbit defined by (h_0, E_0) as stated in Proposition 2.1. We have $(x - x_2)(s_0) = 0$. Let s_1 be the next zero of $x - x_2$ for increasing values of s, if it exists. Then, if $h(s_1) = h_1$, $E(s_1) = E_1$, we define $f(h_0, E_0) = (h_1, E_1)$.

We call f the Poincaré map of the collinear restricted three-body problem associated to C.

The set D_0 where f is defined is associated to the orbits of m_3 which collide again with m_2. We call loosely D_0 the set of elliptic orbits for increasing time. It is clear that D_0 is open.

Let us study the complement of D_0. If f is not defined on the point (h_0, E_0), as $\ddot{x}(t) < 0$ for all $t > t_0$, the function $\dot{x}(t) > 0$ is monotonically decreasing. There exist $\lim_{t\to+\infty} \dot{x}(t) \equiv \dot{x}(+\infty) \geqslant 0$ and $\lim_{t\to+\infty} x(t) = +\infty$. We say (h_0, E_0) is hyperbolic (parabolic) for $t \to +\infty$ if $\dot{x}(+\infty) > 0$ ($\dot{x}(+\infty) = 0$). The set of hyperbolic (parabolic) initial conditions for $t \to +\infty$ in C will be called $H_0(P_0)$.[1] The set H_0 is open.

PROPOSITION 3.2. *If $m > 0$ is sufficiently small the set P_0 of parabolic initial conditions for $t \to +\infty$ in C is a simple closed curve (if we add the generatrix $E_0 = 0$ to C) which divides C in two components: D_0 and H_0.*

Before giving the proof of Proposition 3.2 we need some study of the flow of (2.3) for m_3 in the neighborhood of $+\infty$. Following McGehee [6] we can introduce the variables

$$x = 2/q^2, \qquad \dot{x} = -p, \qquad dt = 4/q^3 \cdot dw \qquad \text{with} \quad 0 < q < +\infty.$$

Then (2.2) becomes

$$\begin{aligned} dq/dw &= p, \\ dp/dw &= [(1 - m)/(1 - q^2x_2/2)^2 + m/(1 + q^2x_1/2)^2]q, \\ dt/dw &= 4/q^3, \end{aligned} \tag{3.1}$$

with x_1, x_2 given by (2.1).

In [6, Sect. 8] it is proved that parabolic orbits form two real analytic submanifolds: stable (parabolic orbits for $t \to +\infty$) and unstable (for $t \to -\infty$). Due to the invariance of (3.1) with respect to the symmetry $(q, p, t, w) \to (q, -p, -t, -w)$, if one manifold has the expression $q = F(p, t)$ the other is given by $q = F(-p, -t)$.

[1] We recall that H_0, P_0 are sets leading to escape when f is not defined. Other escapes can occur, f being defined a finite number of times. For details in the case of the Sitnikov problem see Llibre and Simó [4].

Note. $F(p, t)$ is 2π-periodic in t and is not analytic in t but in E. It is not difficult to verify that $F(p, t) = \sum_{n\geq 0} a_n(t)\, p^n$, with

$$a_0 = a_2 = a_3 = a_4 = a_6 = 0, \qquad a_1 = 1,$$
$$a_5 = -5m(1-m)/16, \qquad a_7 = 35m(1-m)(1-2m)/128,$$
$$a_8 = -3m(1-m)\left(-5t/2 + \int_0^t (1-\cos E)^3\, dE\right), \ldots .$$

The computation is done through Fourier expansion of the coefficients $a_n(t)$, derivation of $q = F(p, t)$ with respect to t and insertion of $\dot{q}$, $\dot{p}$ as power series in p given by (3.1). We get a system $\dot{a}_n(t) = f_n(t)$ that is solved recursively.

We can introduce the variables $u = (q - F(-p, -t))/4$, $v = (q - F(p, t))/4$ (see McGehee [5]) in order to transform the invariant manifolds into coordinate planes. Then

$$du/dw = u + O_5\,, \quad dv/dw = -v + O_5\,, \quad dt/dw = 1/(2(u+v)^3 + O_7) \tag{3.2}$$

in $q > 0$, where O_n is a function $f(q, p, t)$ of class C^∞ in (p, q), 2π-periodic in t, such that $f(kq, kp, t)/k^n$ is uniformly bounded in t when $k \to 0^+$.

Suppressing the O_n terms in (3.2) and with the elimination of w we have that $(0, 0)$ is a degenerated hyperbolic point for the flow without linear part. The $q > 0$ condition restricts the flow to $u + v > 0$. The pattern is the same if we add the suppressed terms. The u (v) axis is associated to parabolic orbits for $t \to -\infty$ ($t \to +\infty$), the first quadrant to elliptic orbits. Points in the allowed part of the second (fourth) quadrant are associated to hyperbolic orbits for $t \to +\infty$ ($t \to -\infty$). See Fig. 19 in Moser [7, p. 159].

Proof of Proposition 3.2. We know that parabolic orbits for $t \to +\infty$ are given in a neighbourhood of infinity by the set $\{(u, v, E), u = 0, v > 0\}$. As E is an angular variable, the intersection of this set with $v = v_0 > 0$, v_0 sufficiently small, is a simple closed curve γ. The intersection is transversal, because if this were not the case, we would have $\dot{v} = 0$ in $v = v_0$ and the parabolic orbit would not reach infinity. We consider the mapping $(u = 0, v = v_0, E) \to (x - x_2 = 0, h_0, E_0)$ induced by the flow for decreasing time and the first intersection with C (in finite time). We know that such a mapping is a diffeomorphism. Then P_0 is a simple closed curve in $C \cup \{E_0 = 0\}$. Some point in γ will be in the triple collision orbit (see Fig. 3). ∎

More than the result given in Proposition 3.2 will be needed. We claim that using regularized variables, the invariant manifolds of the origin ($p = q = 0$) for Eqs. (3.1) are analytic in m and, if m_3 is not zero, are also analytic in m_3. This will be used in Section 6. The proof of our claim is easy. We need only trace back the analytical dependence on parameters in the proof of the stable

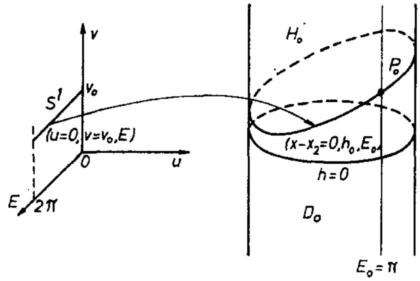

FIGURE 3

manifold theorem for degenerate fixed points in [6]. One gets some uniformly bounded sequence of functions in a product of open sets of the complex plane and then the result follows.

PROPOSITION 3.3. *The curve P_0 near $E_0 = \pi$ is analytic with respect to m and E. If the system* (3.1) *is analytically perturbed and the perturbation depends on some parameter μ then P_0 is also analytic with respect to μ.*

Proof. The curve γ used on the proof of Proposition 3.2 depends analytically on m (and μ). The curve is carried by the flow. However the collision $m_2 - m_3$ is a singularity. We consider the (transversal) intersection of the negative orbit of γ with the manifold $x - x_2 = \delta$ for the first time (δ is chosen small, fixed; for instance, $\delta = 0.1$). We only take into account a small arc on the intersection around the point which will be mapped on P_0 at the point $E_0 = \pi$. As there h is a regular function, and as $h = 0$ if $m = 0$ we can use E as a parameter of the arc. We have $h = h(E, m, \mu)$ as the equation of the arc, which is analytic in all the variables. Now we change the equations of motion using Levi Civita regularization. Let τ be the regularized time. Then the time interval $\Delta\tau$ and the variations in E, h from $x - x_2 = \delta$ to $x - x_2 = 0$ are analytic functions in E, h, m, μ, i.e., in E, m, μ since $h = h(E, m, \mu)$. Let h_c, E_c the points obtained in C (we recall that using Levi Civita variables the flow reaches the manifold of collision transversally). Then

$$\begin{aligned} h_c &= h(E, m, \mu) + \Delta h(E, m, \mu), \\ E_c &= E + \Delta E(E, m, \mu). \end{aligned} \tag{3.3}$$

Since δ is small, ΔE is small too. Then we can invert the second of the relations (3.3) and we have $E = E(E_c, m, \mu)$. Inserting in the first relation we have the analytical expression $h_c = h_c(E_c, m, \mu)$ we wanted. ∎

COROLLARY 3.1. *If $0 < m$ is sufficiently small there exists $\partial g/\partial E_0 |_{E_0=\pi}$, where $h_0 = g(E_0, m)$ is the expression of the curve P_0 near $E_0 = \pi$.*

Now we consider the inverse Poincaré map. Given $(h_0, E_0) \in C$ for $s_0 = 0$, let s_{-1} be the next zero of $x - x_2$ along the orbit defined through (h_0, E_0) for decreasing values of s, if it exists. Then, if $h_{-1} = h(s_{-1})$, $E_{-1} = E(s_{-1})$, it is clear that $f^{-1}(h_0, E_0) = (h_{-1}, E_{-1})$. If f^{-1} is defined in a subset D_1, we have $D_1 = f(D_0)$. D_1 is called the set of elliptic orbits for decreasing time. Of course D_1 is the complement of the orbits such that $\lim_{t\to-\infty} \dot{x}(t) \geqslant 0$. We define $H_1(P_1)$ as hyperbolic (parabolic) orbits for $t \to -\infty$ in a natural way. An assertion symmetrical to Proposition 3.2 is:

PROPOSITION 3.4. *The set P_1 of parabolic initial conditions for $t \to -\infty$ in C is a simple closed curve (if we add the generatrix $E_0 = 0$ to C) provided m is sufficiently small. P_1 divides C in two components: D_1 and H_1.*

We remark that Eq. (2.2) is invariant with respect to the symmetry $(x, \dot{x}, t) \to (x, -\dot{x}, -t)$. Then $x(-t, -x_0, -t_0) = x(t, x_0, t_0)$.

If S is the symmetry in C given by $(h_0, E_0) \to (h_0, 2\pi - E_0)$, we have $f^{-1} = S^{-1} \circ f \circ S$ and hence $D_1 = S(D_0)$, $P_1 = S(P_0)$, $H_1 = S(H_0)$.

4. TRANSVERSALITY IF $m > 0$

For $m = 0$ the Poincaré map is given by

$$f(h_0, E_0) = (h_1, E_1), \qquad h_1 = h_0, \qquad E_1 = E_0 + E(2\pi/(-2h_0)^{3/2}),$$

where $E(t)$ is the solution of $t = E - \sin E$. Of course $D_1 = f(D_0) = D_0 = \{(h, E) \in C \mid h < 0\}$.

A fundamental fact used to prove Theorem A is contained in the following result:

THEOREM 4.1. *If $m > 0$ is sufficiently small the intersection of P_0 and P_1 on $\{E_0 = \pi\}$ is transversal.*

Proof. Let P_0 be given by $h_0 = g(E_0, m)$. Due to the symmetry $P_1 = S(P_0)$ we obtain for P_1 the expression $h_0 = g(2\pi - E_0, m)$. Therefore $P_0 \cap P_1$ has a point on $\{E_0 = \pi\}$. Using Proposition 3.3 we have that $P_0 \cap P_1$ has exactly one point on $\{E_0 = \pi\}$. Again by symmetry it is enough to show $\partial g/\partial E_0 |_{E_0=\pi} \neq 0$, if m is small enough. This is true if

$$\left.\frac{\partial^2 g}{\partial E_0\, \partial m}\right|_{\substack{m=0\\E_0=\pi}} \neq 0.$$

We shall prove it is positive.

Consider the integral of the function $(dh/ds)(s; s_0, E_0, m)$ along the parabolic orbit $x(s; s_0, E_0, m)$ for $s \to +\infty$ given by the initial conditions

$(x - x_2)(s_0) = 0$, $h(s_0) = g(E_0, m)$, $s_0 = 0$. From (2.3) it is immediate that the integral $\int_0^\infty (dh/ds)(s; E_0, m)\, ds$ is convergent; its value is

$$h(+\infty) - h(0) = -g(E_0, m). \tag{4.1}$$

Let us introduce the following functions:

$$\begin{aligned}
J_1 &= m(1-m)\sin E[(x-x_2)^2/(x+x_1)^2 - 1]/x,\\
J_2 &= m^2(1-m)\sin E(1-\cos E)[(x-x_2)^2/(x+x_1)^2 - 1]/x^2,\\
J_3 &= m^3(1-m)\sin E(1-\cos E)^2[(x-x_2)^2/(x+x_1)^2 - 1]/x^3,\\
J_4 &= m^4(1-\cos E)^2(x-x_2)(\dot x-\dot x_2)[1-(1-\cos E)^2/(x+x_1)^2]/x^3,\\
J_5 &= -3m^4\sin^2 E(1-\cos E)(x-x_2)^2(\dot x-\dot x_2)/x^4,\\
J_6 &= -3m^5\sin^3 E(x-x_2)^2/2x^4,\\
J_7 &= 3m^4\sin E(1-\cos E)^2(x-x_2)^2/x^4(x+x_1),\\
J_8 &= 3m^4\sin E(1-\cos E)^3(x-x_2)(\dot x-\dot x_2)^2/x^4,\\
J_9 &= 3m^5\sin^2 E(1-\cos E)^2(x-x_2)(\dot x-\dot x_2)^2/2x^4,\\
J_{10} &= -3m^4(1-\cos E)^4(x+x_2)(\dot x-\dot x_2)/x^4(x+x_1).
\end{aligned}$$

Again from (2.3) we have $(dh/ds)(s; E_0, m) = \sum_{i=1}^{10} J_i$. We define

$$I_i(E_0, m) = \int_0^\infty J_i(s; E_0, m)\, ds, \qquad i = 1 \div 10.$$

Write $I_i = I_i^0 + I_i^\infty$, where $I_i^0 = \int_0^{\bar s} J_i\, ds$ is the contribution of I_i near the collision $m_2 - m_3$, and $I_i^\infty = \int_{\bar s}^\infty J_i\, ds$ the contribution in the unbounded region. Of course

$$\int_0^\infty \frac{dh}{ds}(s; E_0, m)\, ds = \sum_{i=1}^{10} I_i. \tag{4.2}$$

We say that a function $F(x, m)$ is of order α in m, $O(F) = \alpha$ by definition, if there exist bounded functions $G(x)$, $H(x, m)$ such that $F(x, m) = G(x)\, m^\alpha + H(x, m)\, m^\beta$ with $\alpha < \beta$ for m in an interval $(0, m_0)$.

We state three technical lemmas that will be used in the proof of Theorem 4.1. The proofs of the lemmas are given at the end of this section.

Lemma 4.1. *If $|E_0 - \pi|$ is small, in a neighbourhood of $s = 0$ the following relations hold:*

(i) $x - x_2 = 2s^2 + O(h_{23}s^4) + O(ms^2)$,

(ii) $\dot x - \dot x_2 = s^{-1} + O(h_{23}s) + O(ms^{-1})$,

(iii) $E = E_0 + 2s^3/3 + O(h_{23}s^5) + O(ms^3)$,

(iv) $\sin E = \sin E_0 + 2\cos E_0 \cdot s^3/3 + O(s^6) + O(h_{23}s^5) + O(ms^3)$,

(v) $1 - \cos E = 1 - \cos E_0 + 2\sin E_0 \cdot s^3/3 + O(s^6) + O(h_{23}s^5) + O(ms^3)$,

(vi) $(x - x_2)^2/(x + x_1)^2 - 1 = -1 + s^4 + O(h_{23}s^6) + O(ms^4)$,

(vii) $1 - (1 - \cos E)^2/(x + x_1)^2 = (1 + \cos E_0) - (1 + \cos E_0)^2/4 + O(m) + O(s^2)$.

LEMMA 4.2. *If m and $|E_0 - \pi|$ are sufficiently small and $\bar{s} = O(m^{1/4})$, then:*

(i) $O(I_1^0) = O(I_2^0) = O(I_3^0) = O(I_8^0) = 1/2$, $O(I_{10}^0) = 1$,

$O(I_4^0) = O(I_5^0) = O(I_9^0) = 2$, $O(I_7^0) = 5/2$, $O(I_6^0) = 7/2$.

(ii) $\int_0^\infty (A_1 + A_2 + A_3 + A_8)\, ds = 0$, *where A_i is the dominant term in J_i in the interval $(0, \bar{s})$, i.e., the term of smaller order in m.*

(iii) $O(I_i^\infty) \geqslant O(I_i^0)$, $i = 1 \div 10$.

(iv) $O\left(I_1^0 - \int_0^{\bar{s}} A_1\, ds\right) > 1$; $O\left(I_i^0 - \int_0^{\bar{s}} A_i\, ds\right) = 2$, $i = 2, 3, 8$.

(v) $O\left(I_1^\infty - \int_{\bar{s}}^\infty A_1\, ds\right) = 1$; $O\left(I_i^\infty - \int_{\bar{s}}^\infty A_i\, ds\right) > 1$, $i = 2, 3, 8$.

(vi) $O\left(\sum_{i=1}^{10} I_i\right) = O\left(I_1^\infty - \int_{\bar{s}}^\infty A_i\, ds + I_{10}^0\right) = 1$.

LEMMA 4.3. *Let*

$$C_1 = m \int_{\bar{E}}^{E'} \frac{\sin E\, dE}{[(9/2)^{1/3}(E - \sin E - E_0 + \sin E_0)^{2/3} + 1 - \cos E]^2},$$

$$C_2 = m \int_{\bar{E}}^{E'} \frac{\sin E_0 - \sin E}{(9/2)^{2/3}(E - \sin E - E_0 + \sin E_0)^{4/3}}\, dE,$$

where $\bar{E} = E_0 + m^{3/4}$ and E' is sufficiently large. Define

$$\bar{C}_i = -\left.\frac{\partial^2 C_i}{\partial E_0\, \partial m}\right|_{\substack{m=0\\ E_0=\pi}}, \qquad i = 1, 2.$$

Then $\bar{C}_i > 0$, $i = 1, 2$.

We return to the proof of Theorem 4.1. From (4.1), and Lemma 4.2 we have

$$g(E_0, m) = -\left(I_1^\infty - \int_{\bar{s}}^\infty A_1\, ds + I_{10}^0\right) + o(m).$$

As $I_{10}^0 = (1 - \cos E_0)^4 O(m) + O(m^2)$, we obtain

$$\frac{\partial^2 g}{\partial E_0 \, \partial m}\bigg|_{\substack{m=0\\E_0=\pi}} = -\frac{\partial}{\partial E_0}\frac{\partial}{\partial m}\left(I_1^\infty - \int_{\bar{s}}^\infty A_1 \, ds\right)\bigg|_{\substack{m=0\\E_0=\pi}}. \tag{4.3}$$

In I_1^∞ we introduce the variable E through $(x - x_2)\, ds = dE$ and fix $\bar{s} = (3/2)^{1/3} m^{1/4}$. Using Lemma 4.1(iii) we have

$$I_1^\infty = \int_{\bar{E}}^\infty m \sin E(x - x_2)[(x + x_1)^{-2} - (x - x_2)^{-2}]\, x^{-1}\, dE + O(m^2),$$

with $\bar{E}$ as in Lemma 4.3.

For $m = 0$ the parabolic orbit of (2.2) is $x(E; E_0, 0) = (9/2)^{1/3}[E - \sin E - E_0 + \sin E_0]^{2/3}$. If $m > 0$ is sufficiently small

$$x(E; E_0, m) = x(E; E_0, 0) + O(h_{23}) + O(m).$$

But $h_{23} = O(m)$ on any compact $[E_0, E']$ with $E' > E_0$, provided that m is small enough. Then

$$x(E; E_0, m) = (9/2)^{1/3}[E - \sin E - E_0 + \sin E_0]^{2/3} + O(m). \tag{4.4}$$

Therefore in (4.3) we can split the interval of integration into $[E_0 + m^{3/4}, E']$ and $[E', \infty]$ such that the integrals involved in $[E', \infty]$ are $O(m^\alpha)$, $\alpha > 1$.

We conclude that $I_1^\infty = \int_{\bar{E}}^{E'} m \sin E(x - x_2)[(x + x_1)^{-2} - (x - x_2)^{-2}]\, x^{-1} dE + O(m^\alpha)$ and $\int_{\bar{s}}^\infty A_1\, ds = \int_{\bar{E}}^{E'} -(m/2) \sin E_0 (x - x_2)^{-1}(m + s^2)^{-1}\, dE + O(m^\alpha)$ (see the proof of Lemma 4.2(i)).

Inserting (4.4) in the last two formulas, performing the expansion in powers of m and taking into account that $x_2 = O(m)$ and $x \geqslant O(m^{1/2})$, we have

$$I_1^\infty - \int_{\bar{s}}^\infty A_1\, ds = C_1 + C_2 + O(m^\beta)$$

with $\beta > 1$ and where C_1, C_2 are the integrals given in Lemma 4.3. From (4.3) and Lemma 4.3 we get

$$\frac{\partial^2 g}{\partial E_0 \, \partial m}\bigg|_{\substack{m=0\\E_0=\pi}} = \bar{C}_1 + \bar{C}_2 > 0. \quad \blacksquare$$

Now we give the proofs of the lemmas.

Proof of Lemma 4.1. (i) We introduce variables u, v through $u^2 = x - x_2$, $dt = u^2\, dv = u^2(1 - \cos E)\, ds$. Then the equation of motion (2.2) can be written

$$u'' - (h_{23}/2)u = mu^3\{-[u^2 + (1 - \cos E)]^{-2} + (1 - \cos E)^{-2}\}/2, \tag{4.5),}$$

where $h_{23} = [2u'^2 - (1-m)]/u^2$ if $u \neq 0$ and $h_{23} = \lim_{u\to 0}[2u'^2 - (1-m)]/u^2$ if $u = 0$, and $' = d/dv$.

The integration of (4.5) by the Euler method with initial conditions $u(0) = 0$, $u'(0) = ((1-m)/2)^{1/2}$ (for $m = 0$ the orbit with these initial conditions would be parabolic) produces $u(v) = 2^{-1/2}v + O(h_{23}v^3) + O(mv)$. The approximation $1 - \cos E_0 = 2$ ($|E_0 - \pi|$ is small) gives (i).

Part (ii) is obtained from (i) and from the definition of s since $t = 4s^3/3 + \cdots$

From $\int_0^s (x - x_2)\, d\sigma = \int_{E_0}^{E} d\epsilon$ and (i) we deduce (iii). Parts (iv) and (v) are obtained elementarily from (iii). Part (vi) is obtained from (i) and the expressions of x_1, x_2 in terms of E. Part (vii) results from (i), (v) and (vi). ▮

Proof of Lemma 4.2. (i) If in I_i^0 we consider only the dominant terms in m and introduce $s = m^{1/2}t$, $\bar{t} = m^{-1/2}\bar{s}$, from Lemma 4.1 we have

$$I_1^0 = -\frac{m^{1/2}\sin E_0}{2}\int_0^{\bar{t}} \frac{dt}{1+t^2} = O(m^{1/2}),$$

$$I_2^0 = -\frac{m^{1/2}\sin E_0}{2}\int_0^{\bar{t}} \frac{dt}{(1+t^2)^2} = O(m^{1/2})$$

$$I_3^0 = -\frac{m^{1/2}\sin E_0}{2}\int_0^{\bar{t}} \frac{dt}{(1+t^2)^3} = O(m^{1/2}),$$

$$I_4^0 = m^2 K(E_0)\int_0^{\bar{t}} \frac{t\,dt}{(1+t^2)^3} = O(m^2),$$

$$I_5^0 = -\frac{3m^2\sin^2 E_0}{2}\int_0^{\bar{t}} \frac{t^3\,dt}{(1+t^2)^4} = O(m^2),$$

$$I_6^0 = -\frac{3m^{7/2}\sin^3 E_0}{8}\int_0^{\bar{t}} \frac{t^4\,dt}{(1+t^2)^4} = O(m^{7/2}).$$

$$I_7^0 = \frac{3m^{5/2}\sin E_0}{2}\int_0^{\bar{t}} \frac{t^4\,dt}{(1+t^2)^4} = O(m^{5/2}),$$

$$I_8^0 = 3m^{1/2}\sin E_0\int_0^{\bar{t}} \frac{dt}{(1+t^2)^4} = O(m^{1/2}),$$

$$I_9^0 = \frac{3m^2\sin^2 E_0}{4}\int_0^{\bar{t}} \frac{t\,dt}{(1+t^2)^4} = O(m^2),$$

$$I_{10}^0 = -3m\int_0^{\bar{t}} \frac{t\,dt}{(1+t^2)^2} = O(m).$$

where $K(E_0) = (1 + \cos E_0) - (1 + \cos E_0)^2/4$.

(ii) The integral $\int_0^\infty (A_1 + A_2 + A_3 + A_4)\, ds$ equals

$$\frac{m^{1/2} \sin E_0}{2} \int_0^\infty \left(-\frac{1}{1+t^2} - \frac{1}{(1+t^2)^2} - \frac{1}{(1+t^2)^3} + \frac{6}{(1+t^2)^4} \right) dt = 0,$$

as was expected due to the analyticity with respect to m.

Part (iii) follows easily from the definition of J_i and (i).

(iv) Let $B_i = I_i^0 - \int_0^{\bar{s}} A_i\, ds$, $i = 1, 2, 3, 8$. From Lemma 4.1 we obtain

$$O(B_1) = O\left(-\frac{m \cos E_0}{3} \int_0^{\bar{s}} \frac{s^3\, ds}{m + s^2} \right) = O\left(m^2 \int_0^{\bar{t}} \frac{t^3\, dt}{1 + t^2} \right) > 1.$$

The other cases are obtained in an analogous way.

(v) We have

$$\begin{aligned} I_1^\infty - \int_{\bar{s}}^\infty A_1\, ds &= \int_{\bar{s}}^\infty \left[m(1-m) \sin E \left(\frac{(x - x_2)^2}{(x + x_1)^2} - 1 \right) \frac{1}{x} + \frac{m \sin E_0}{2(m + s^2)} \right] ds \\ &= II_1 + II_2 + II_3 + II_4, \end{aligned}$$

where

$$II_1 = m \int_{\bar{s}}^1 \left(\frac{\sin E_0}{2(m + s^2)} - \frac{\sin E}{x} \right) ds,$$

$$II_2 = m \int_{\bar{s}}^1 \sin E \frac{(x - x_2)^2}{(x + x_1)^2 x}\, ds,$$

$$II_3 = m \int_1^\infty \frac{\sin E_0}{2(m + s^2)} + \frac{\sin E}{x} \left(\frac{(x - x_2)^2}{(x + x_1)^2} - 1 \right) ds,$$

$$II_4 = -m^2 \int_{\bar{s}}^\infty \sin E \left(\frac{(x - x_2)^2}{(x + x_1)^2} - 1 \right) \frac{1}{x}\, ds.$$

It is clear that $O(II_3) = 1$, $O(II_4) \geqslant 3/2$ and from Lemma 4.1 follows $O(II_1) = O(II_2) = 1$.

Part (vi) follows from the previous cases. ∎

Proof of Lemma 4.3. We have

$$\begin{aligned} \bar{C}_1 = &-(32/3)^{1/2}(1 - \cos E_0) \\ &\times \int_E^{E'} \frac{\sin E (E - \sin E - E_0 + \sin E_0)^{-1/3}\, dE}{[(9/2)^{1/3}(E - \sin E - E_0 + \sin E_0)^{2/3} + 1 - \cos E]^3} \Bigg|_{E_0 = \pi} \end{aligned}$$

The proof that $\bar{C}_1 > 0$ merely requires that

$$\int_\pi^\infty \frac{\sin E\, dE}{[(9/2)^{1/3}(E - \sin E - \pi)^{2/3} + 1 - \cos E]^3 (E - \sin E - \pi)^{1/3}} < 0.$$

Let $z = E - \pi$ and

$$F(z) = \frac{\sin z}{[(9/2)^{1/3}(z + \sin z)^{2/3} + 1 + \cos z]^3 (z + \sin z)^{1/3}}.$$

Then $\bar{C}_1 = \int_0^\infty F(z)\, dz = \sum_{k \geqslant 0} \alpha_k$, where $\alpha_k = \int_{k\pi}^{(k+1)\pi} F(z)\, dz$, $k \geqslant 0$.

We have $\alpha_{2k} > 0$, $\alpha_{2k+1} < 0$ and $\alpha_{2k} > | \alpha_{2k+1} |$ because if $z_1 \in [k\pi, (k+1)\pi)$ and $z_1 + z_2 = 2(k+1)\pi$, then $|F(z_1)| > |F(z_2)|$ since $\sin z_1 = -\sin z_2$, $\cos z_1 = \cos z_2$ and $z_1 + \sin z_1 < z_2 + \sin z_2$. We conclude that $\bar{C}_1 > 0$.

Computing $\bar{C}_2$ we obtain

$$\bar{C}_2 = -(2/9)^{2/3} \frac{\partial}{\partial E_0} \int_{E_0 + m^{3/4}}^\infty \frac{\sin E_0 - \sin E}{(E - \sin E - E_0 + \sin E_0)^{4/3}}\, dE \,\Big|_{E_0 = \pi}$$

$$= -(2/9)^{2/3} \left\{ \int_{z_0}^\infty \frac{(5/3)\sin z - z}{(z + \sin z)^{7/3}}\, dz - \frac{\sin z_0}{(z_0 + \sin z_0)^{4/3}} \right\},$$

where $z = E - \pi$ and $z_0 = m^{3/4}$.

Let

$$\delta_1 = \int_{5/3}^\infty \frac{(5/3)\sin z - z}{(z + \sin z)^{7/3}}\, dz,$$

$$\delta_2 = \int_{z_0}^{5/3} \frac{(5/3)\sin z - z}{(z + \sin z)^{7/3}}\, dz - \frac{\sin z_0}{(z_0 + \sin z_0)^{4/3}}.$$

As $(5/3)\sin z - z < 0$ if $z > 5/3$ we have $\delta_1 < 0$. On the other hand

$$\int_{z_0}^{5/3} \frac{z - (5/3)\sin z}{(z + \sin z)^{7/3}}\, dz > \int_{z_0}^{5/3} \frac{-2z/3}{(2z)^{7/3}}\, dz = 2^{-4/3}[(3/5)^3 - z_0^{-1/3}].$$

Therefore $\delta_2 < 2^{-4/3} z_0^{-1/3} - 2^{-4/3}(3/5)^3 - (z_0 - z_0^3/6)(z_0 + \sin z_0)^{-4/3}$. As for $m \to 0$ the right-hand side tends to $-2^{-4/3}(3/5)^3$ we conclude $\delta_2 < 0$ if m is sufficiently small. ∎

5. The Embedding of the Shift and Proof of Theorem A

First we recall results of Moser [7] that will be used later.

To every orbit of m_3 in the restricted collinear problem we associate a sequence of integers. Let $(x - x_2)(t_0) = 0$. Consider the successive zeros $\{t_n\}$ of $(x - x_2)(t)$. Four cases are possible:

(a) t_n exists for every $n \in \mathbb{Z}$.

(b) t_n exists for $n > 0$ but there is a $k < 0$ such that t_k does not exists. Then we take $t_k = -\infty$.

(c) t_n exists for $n < 0$ but there is an $l > 0$ such that t_l does not. We take $t_l = +\infty$.

(d) t_n does not exist for a $k < 0$ and for an $l > 0$. Then we take $t_k = -\infty$ and $t_l = +\infty$.

The integers

$$a_n = \left[\frac{t_n - t_{n-1}}{2\pi}\right] \tag{5.1}$$

([ ] = integer part) are the number of collisions of the bodies $m_1 - m_2$ between two successive collisions $m_2 - m_3$. This calls for some results of symbolic dynamics.

Let A be the set $\mathbb{N} \cup \{\infty\}$, where we have the usual order extended by $a < \infty$, for all $a \in \mathbb{N}$. Let S be the set of sequences of elements belonging to A of the types:

(a) $(..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...)$ with $a_n \neq \infty$ for all $n \in \mathbb{Z}$.

(b) $(a_k, a_{k+1}, a_{k+2}, ...)$ with $a_k = \infty$, $k \leqslant 0$ and $a_n \neq \infty$ for all $n > k$.

(c) $(..., a_{l-2}, a_{l-1}, a_l)$ with $a_l = \infty$, $l \geqslant 1$ and $a_n \neq \infty$ for all $n < l$.

(d) $(a_k, a_{k+1}, ..., a_{l-1}, a_l)$ with $a_k = a_l = \infty$, $k \leqslant 0$, $l \geqslant 1$, $a_n \neq \infty$ for all n such that $k < n < l$.

We introduce in S a topology through the basis $\{U_j(a), a \in S, j \in \mathbb{N}\}$ where for a of each one the types (a), (b), (c), (d), $U_j(a)$ is defined by

$$\begin{aligned}
U_j(a) &= \{a' \in S \mid a'_n = a_n \quad \text{if} \quad |n| < j\},\\
U_j(a) &= \{a' \in S \mid a'_n = a_n \quad \text{if} \quad k < n \leqslant j, a'_k \geqslant j\},\\
U_j(a) &= \{a' \in S \mid a'_n = a_n \quad \text{if} \quad -j \leqslant n < l, a'_l \geqslant j\},\\
U_j(a) &= \{a' \in S \mid a'_n = a_n \quad \text{if} \quad k < n < l, a'_k, a'_l \geqslant j\},
\end{aligned}$$

respectively. Using the standard technique of continued fractions we have

PROPOSITION 5.1. *With the given topology, S is compact.*

Let σ: $S \to S$ be the Bernoulli shift defined by $(\sigma(a))_n = a_{n+1}$. σ is defined in $D(\sigma) = \{a \in S \mid a_0 \neq \infty\}$. We want to know that for $f: Q = [0, 1] \times [0, 1] \to \mathbb{R}^2$ a suitable mapping, there exists a homeomorphism h: $S \to h(S) \subset Q$ such that in $h(D(\sigma))$, f is conjugated to σ, i.e., $\sigma = h^{-1} \circ f \circ h$.

We need some definitions:

Let $p \in (0, 1)$. A curve $y = h(x)$ is a horizontal curve in Q if: (i) $h([0, 1]) \subset [0, 1]$; (ii) h is p-lipschitz in $[0, 1]$. Given two horizontal curves defined by h_1, h_2, and if $h_1(x) < h_2(x)$ for all $x \in [0, 1]$, the set $H = \{(x, y) \in Q \mid h_1(x) \leqslant y \leqslant h_2(x)\}$ is called an horizontal strip. Vertical notions are similar.

Let $f: Q \to \mathbb{R}^2$ be a C^1 mapping such that the following two conditions are satisfied:

(I) There exist a family of mutually disjoint horizontal strips $\{H_n\}$ and a family of mutually disjoint vertical strips $\{V_n\}$ such that $f(V_n) = H_n$ for all $n \in \mathbb{N}$, and horizontal (vertical) components of ∂V_n are mapped onto horizontal (vertical) components of ∂H_n preserving the order (left part onto left part, upper part onto upper part, etc.). Furthermore we order the strips V_n in such a way that if V_n is defined through the vertical curves $v_{2n-1} < v_{2n}$, then $v_{2n} < v_{2n+1}$. Let $V_\infty = \{(x, y) \in Q \mid x = 1\}$. Then $V_n \to V_\infty$ when $n \to \infty$. Symmetric conditions hold for $\{H_n\}$.

(II) There exists $p \in (0, 1)$ such that the sector bundle S^+ defined on $\bigcup_1^\infty V_n$ by $|v| \leqslant p\,|u|$ (u, v components of the vectors in the tangent space to Q) is mapped into itself by Df: $(Df)(S^+) \subset S^+$. Furthermore if $(u_0, v_0)_a \in S^+$ and $(Df(a))(u_0, v_0)_a = (u_1, v_1)_{f(a)}$, then $|u_1| \geqslant p^{-1}\,|u_0|$. If S^- is the sector bundle on $\bigcup_1^\infty H_n$ defined by $|u| \leqslant p\,|v|$, then $(Df^{-1})(S^-) \subset S^-$. Furthermore if $(u_1, v_1)_{f(a)} \in S^-$ and $(Df(a))(u_0, v_0)_a = (u_1, v_1)_{f(a)}$, then $|v_0| \geqslant p^{-1}\,|v_1|$.

Then the following result holds.

THEOREM 5.1 (Moser [7]). *If $f: Q \to \mathbb{R}^2$ is of class C^1 and satisfies conditions* I, II *with $0 < p < 1/2$ then there is a homeomorphism h: $S \to h(S) \subset Q$ such that on the subset $h(D(\sigma))$ of Q, f is conjugated to the shift defined on $D(\sigma)$. For $q \in h(D(\sigma))$, $f^{-n}(q) \in V_{a_n}$ for all a_n, if $a = \{a_n\}$ is the sequence associated to q.*

We can say that the shift has been embedded as a subsystem of the mapping f.

In a first approximation the collinear elliptic restricted three-body problem is like the Sitnikov problem in the neighbourhood of the infinity. Then two lemmas identical to Lemmas 4, 5 of Moser [7, pp. 167–181] follow. Using both and Theorem 5.1 we prove

THEOREM 5.2. *The Bernoulli shift σ defined on $D(\sigma)$ is conjugated to the Poincaré map f, associated to the collinear elliptic restricted three-body problem on a subset of D_0, provided the mass $m > 0$ of m_1 is sufficiently small.*

Theorem A now follows from Theorem 5.2.

Proof of Theorem A. For $m = 0$ the image under the Poincaré map f of an arc in D_0 ending at P_0 is a curve which spirals towards P_1. This is also true for $m > 0$.

Let $r > 0$ be sufficiently small. We define $D_0(r) = \{q \in D_0 \mid d(q, P_0) < r\}$ where the distance d is measured on C. $D_1(r)$ is the symmetric region: $D_1(r) = S(D_0(r))$.

Let R be the connected component of $D_0(r) \cap D_1(r)$ which contains the point $p \in P_0 \cap P_1$ of Theorem 4.1 (see Fig. 4). For small r the boundary ∂R consists

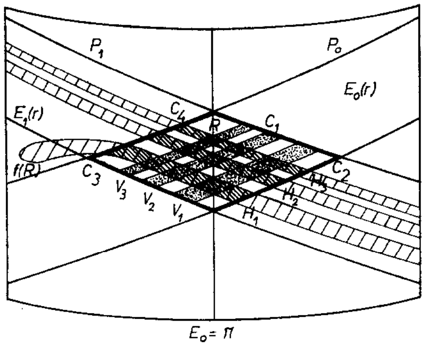

FIGURE 4

of four C^1 arcs: C_1, C_2, C_3, C_4. Here R plays the role of Q in Theorem 5.1. The arcs C_1, C_3 are transversal to P_0 due to Theorem 4.1. Then $f(R)$ is a narrow strip which spirals towards P_1. $f(R) \cap R$ is the disjoint union of an infinity of "horizontal" strips numbered in a suitable way to satisfy condition I, i.e., C_1 plays the role of H_∞. We define "vertical" strips $V_n = S(H_n)$ and one has $f(V_n) = H_n$.

From the sequence $\{b_n\}$ of Theorem A we construct a new sequence $\{a_n\}$ such that $a_n = b_n - b$. The points in V_k are initial conditions for the motion of the third body such that the time up to the next $m_2 - m_3$ collision is $2\pi(k + b + \theta)$ where b is related to the integer number of turns given by $f(R)$ around C before reaching C_4, and $\theta \in [0, 1)$. It is clear that b depends on m because the angle between P_0, P_1 at p is a function of m. ∎

The sequence $\{a_n\}$ belongs to $D(\sigma)$ and by Theorem 5.2 there is a unique point $q \in R$ associated to it. By Theorem 5.1 we know that $f^{-n}(q) \in V_{a_n}$ for all a_n of the sequence $\{a_n\}$. Therefore for the orbit defined by q the integers b_n measure the number of collisions m_1, m_2 as in the statement of Theorem A.

From Theorem A we can derive some consequences about the final evolutions in our problem. Bodies m_1, m_2 are always in an elliptic (degenerated) orbit. We recall that the third body is in hyperbolic (parabolic) motion when $t \to +\infty$ if $h \to h(+\infty) > 0$ ($h(+\infty) = 0$). For $t \to +\infty$, let HE^+, PE^+, OS^+, L^+ denote final evolutions of hyperbolic elliptic, parabolic elliptic, oscillatory and Lagrange stable (bounded) types. Replace $+$ by $-$ for $t \to -\infty$. From Theorem 4.1 we know that in a neighbourhood of the homoclinic point p there are points in C such that they determine orbits with final evolution of types $HE^- \cap HE^+$, $PE^- \cap HE^+$, $HE^- \cap PE^+$ (the orbit associated to p has final evolution of type $PE^- \cap PE^+$).

Using Theorem A we have that in $h(D(\sigma))$ there are points with associated orbits of other types of final evolution. Table I gives the types of final evolutions that we encounter related to the four types of sequences (a), (b), (c), (d). Further-

TABLE I

Type of sequence	Related possible final evolutions			
(a)	$L^- \cap L^+$,	$L^- \cap OS^+$,	$OS^- \cap L^+$,	$OS^- \cap OS^+$
(b)	$HE^- \cap L^+$,	$HE^- \cap OS^+$,	$PE^- \cap L^+$,	$PE^- \cap OS^+$
(c)	$L^- \cap HE^+$,	$L^- \cap PE^+$,	$OS^- \cap HE^+$,	$OS^- \cap PE^+$
(d)	$HE^- \cap HE^+$,	$HE^- \cap PE^+$,	$PE^- \cap HE^+$,	$PE^- \cap PE^+$

more we can establish the existence of an infinity of periodic orbits. Let $\{b_n\}$ be an m-periodic sequence with $b_n \geqslant b$. Then Theorem A assures the existence of a point $q = h(\{b_n\})$ in R such that the associated orbit of the third body is periodic (of period $2\pi m$) because

$$f^m(q) = h\sigma^m h^{-1}(q) = h\sigma^m(\{b_n\}) = h(\{b_n\}) = q.$$

6. The General Collinear Three-Body Problem

Let us suppose that the third body has a finite mass $m_3 > 0$. We normalize the mass unit and introduce parameters m, ϵ such that $m_1 = m(1 - \epsilon)$, $m_2 = 1 - m$, $m_3 = m\epsilon$. The center of masses is fixed at the origin. With the names of the positions as in Section 2 we have the equations of motion

$$\ddot{x} = -\frac{m_2}{(x - x_2)^2} - \frac{m_1}{(x + x_1)^2}; \qquad \ddot{x}_2 = \frac{m_3}{(x - x_2)^2} - \frac{m_1}{(x + x_2)^2}. \tag{6.1}$$

For $x = x_2$, $x_2 = -x_1$, Eqs. (6.1) are singular. Both binary collisions can be regularized.

Equations (6.1) have the energy integral

$$\begin{aligned} H &= \tfrac{1}{2}m_1 \dot{x}_1{}^2 + \tfrac{1}{2}m_2\dot{x}_2{}^2 - \frac{m_1 m_2}{x_1 + x_2} + m_3 h_{123}, \\ h_{123} &= \tfrac{1}{2}\dot{x}^2 - \frac{m_2}{x - x_2} - \frac{m_1}{x + x_1}. \end{aligned} \tag{6.2}$$

In the restricted case the fact that the second equation of (6.1) is integrable reduces the dimension of the phase space from 4 to 3. Now it is (6.2) which allows the same reduction.

Equations (6.1) are a perturbation of (2.2). Then, using Proposition 3.3 where the parameter μ is now ϵ, we have

PROPOSITION 6.1. *The invariant manifolds of the point at infinity associated to the general collinear three-body problem (manifolds of parabolic orbits) are transversal along a homoclinic orbit provided m, ϵ are sufficiently small.*

We need m small and ϵ smaller than m in order to keep the transversality. Due to the analyticity of the flow (in regularized variables) it is unnecessary to specify the transversal section where the Poincaré map is defined. However we can define a cylinder as in the restricted case which allows all the geometrical interpretations made before. Define an angle $\tilde{E} = 2 \operatorname{arctg}(-\dot{x}_1/m_1)$. We see that $\tilde{E}$ becomes the excentric anomaly if $m_3 = 0$. The averaged energy h is defined as in the restricted case. Then the variables $\tilde{E}$, h define a cylinder $\tilde{C}$ whose points, when $x = x_2$, determine an orbit of Eqs. (6.1) (regularized).

On $\tilde{C}$ we define curves $\tilde{P}_0$, $\tilde{P}_1$ as in Section 3. They are perturbations of P_0, P_1. The symmetry with respect to $\tilde{E} = \pi$ when we change the sign of time is preserved (the system is reversible). Then the homoclinic point p whose orbit is referred to in Proposition 6.1 is on $\tilde{E} = \pi$.

Using the transversality and Theorem 5.1 as in Section 5 we prove:

THEOREM 6.1. *The Bernoulli shift is a subsystem of the Poincaré map associated to the collinear three-body problem in $\tilde{C}$ provided m, ϵ are sufficiently small.*

Theorem B follows from Theorem 6.1. The same consequences are obtained.

7. EMBEDDING OF THE SHIFT IN THE RESTRICTED CIRCULAR PLANAR PROBLEM OF THREE BODIES

Let m_1, m_2 be the masses of the two bodies (primaries) describing circular orbits around the origin. We suppose $m_1 = 1 - m$, $m_2 = m$, $m \in (0, 1)$, and the distance between m_1 and m_2 and mean motion normalized to 1.

The motion of the third body in synodic coordinates (rotating axes) is given by

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x}, \qquad \ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y}, \tag{7.1}$$

where $\Omega(x, y) = (x^2 + y^2)/2 + (1 - m)/r_1 + m/r_2 + m(1 - m)/2$, $r_1{}^2 = (x - m)^2 + y^2$, $r_2{}^2 = (x + 1 - m)^2 + y^2$. The system (7.1) has a first integral (Jacobi): $C = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2) = 2M - 2h$, with $M(h)$ the momentum (energy) of the third body in sidereal (fixed axis) coordinates. We refer to Szebehely [11] for the derivation and geometry.

Working with C sufficiently large we now define a Poincaré map. We know that then the zero velocity curves have three components. We suppose from now on that motion takes place on the unbounded region.

First we suppose that $m = 0$. Then m_3 is in a synodic two-body motion.

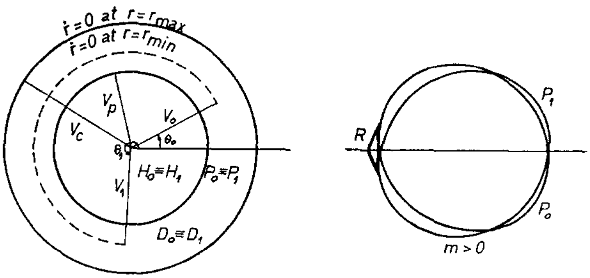

FIGURE 5

Define a Poincaré map in the following way: Let (θ_0, r_0) be the polar synodic coordinates when r_0 is a minimum, i.e., $\dot{r}_0 = 0$. Let (θ_1, r_1) be the corresponding values for the next minimum. As $m = 0$ we have $r_1 = r_0$, $\theta_1 = \theta_0 - 2\pi(-1/2h)^{3/2}$, where h, the energy of m_3 with respect to m_1 is supposed to be negative. Then the Poincaré map f associates to (θ_0, v_0) the value (θ_1, v_1) where v_i denotes the synodic velocity in (θ_i, r_i), $i = 0, 1$.

Some remarks: (1) The map f is not defined if m_3 is in a circular orbit. As we are interested in what occurs near parabolic orbits we can restrict f to orbits with the excentricity bounded from below.

(2) Near parabolic orbits the synodic velocity is $v = (C^2/8 - 4/C) + h(8/C^2 - C^4/64) + O(h^2)$ at the minimum distance point. Parabolic orbits are represented by points $v_p = C^2/8 - 4/C$. This is a circle whose inner part is the hyperbolic orbits and outer part the elliptic orbits. Circular orbits are associated to points with a given $v_c(C) > v_p(C)$. The outer part of the circle of radius v_c is related to synodical velocities not at the point of minimum distance but at the point of maximum distance (see Fig. 5).

(3) The circle $v = v_p$ is the intersection of the manifold of zero radial velocity with the stable $\equiv$ unstable invariant manifold of infinity (see McGehee [6]).

(4) The full picture is slightly analytically perturbed if m is sufficiently small. Now we have that the stable and unstable manifolds are different and we get two different curves P_0 and P_1: orbits going (coming) parabolically to (from) infinity.

The main fact is contained in the following result whose proof is found in Llibre and Simó [3]:

THEOREM 7.1. *If m is sufficiently small and C is large enough the curves D_0 and D_1 have a transversal intersection.*

Then, observing that the expression of f for $m = 0$ assures the suitable spiraling near D_0 and that we have symmetry due to the reversibility, the usual techniques provide us the proof of the embedding of the shift and, as a consequence, we obtain Theorem C.

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