Automatic Flight Control System Design: Part 1
AE4301

Based on Lectures from Erik-Jan van Kampen

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1 Introduction to Automatic Flight Control

1.1 Evolution of Flight Control Systems

1.1.1 Mechanical Flight Control Systems

Early and current light aircraft use mechanically actuated Flight Control Systems (FCS).

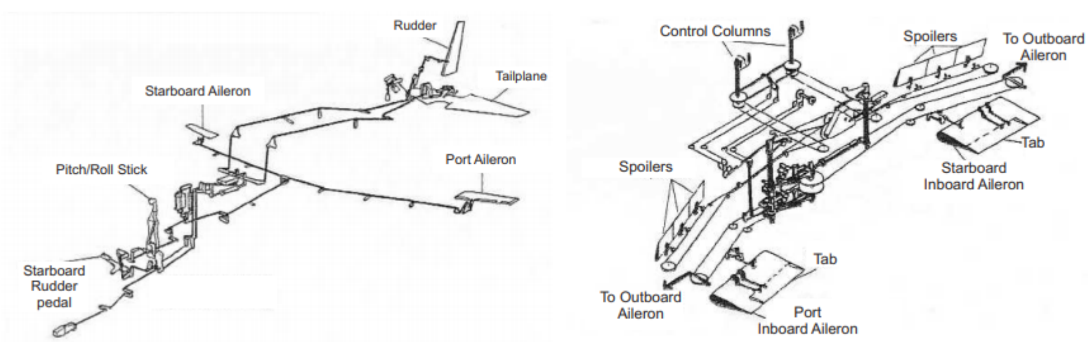


Figure 1.1: Mechanical Flight Control System, **left**: roll, pitch & yaw, **right**: roll

Early automation of FCS used gyroscopic heading and attitude indicator linked to elevator and rudder. Hands-Free demo in Paris 1914.

1.1.2 Fly-By-Wire Flight Control Systems

Civil airliners, military transport and jets use fly-by-wire FCS.

Pilot and sensor inputs are feed into the flight control computer (FCC). The FCC processes these inputs and then sends the output to control the control surface servos and actuator.

The flight control computer varies in size depending on the aircraft size. It is typically located in the electronics and equipment bay on commercial aircraft below the cockpit. Aircraft are also typically equipped with a Flight Management and Guidance Computer (FMGC) and a Flight Augmentation Computer (FAC).

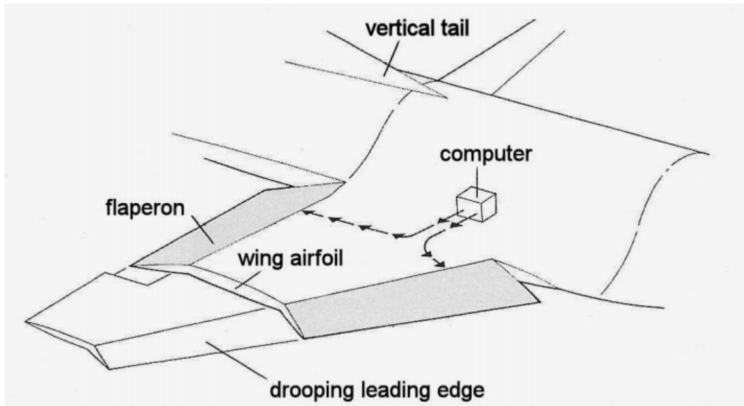


Figure 1.2: Fly-by-Wire FCS

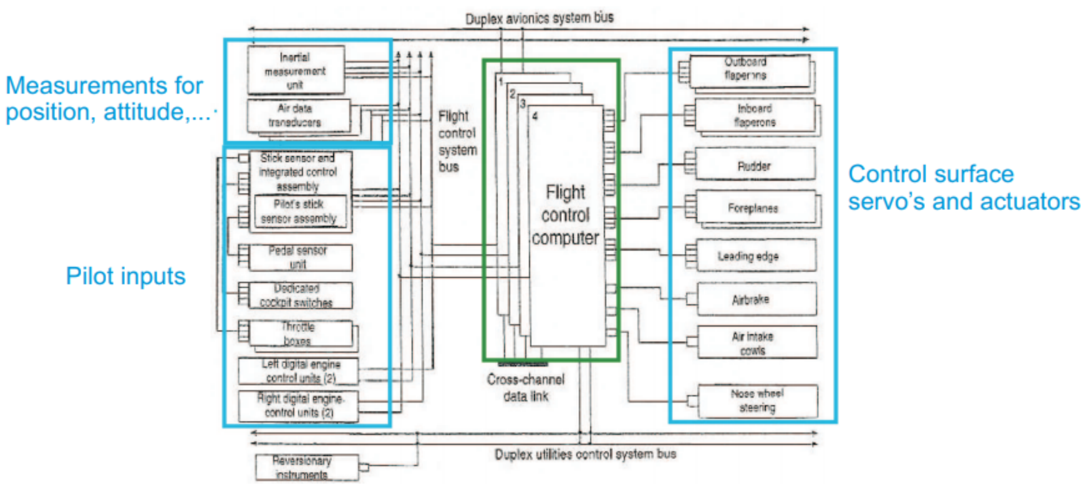


Figure 1.3: Fly-By-Wire FCS diagram of Euro-fighter Tycoon

Acronyms

- **FCS** - Flight Control System
- **FCC** - Flight Control Computer
- **FMGC** - Flight Management and Guidance Computer
- **FAC** - Flight Augmentation Computer

1.2 Reference Frames

Earth-Fixed Frame F_E	**Body-Fixed Frame F_B**	**Stability Frame F_S**	**Aircraft Frame F_R**
-	-	Describes attitude w.r.t. reference flight condition	Describes location of control surfaces and other aircraft parts
Right-Handed Orthogonal	Right-Handed Orthogonal	Right-Handed Orthogonal	\textbf{Left}-Handed Orthogonal
Origin in Aircraft CoG at beginning of considered motion	Origin in Aircraft CoG	Origin in Aircraft CoG	Origin is arbitrary, but fixed and in-variate
Frame is fixed to the Earth	Frame is fixed to the Aircraft	Frame is fixed to the Aircraft	Frame is fixed to the Aircraft
X_E-Axis points towards the North	X_B-Axis points forwards	X_S-Axis points in the plane of symmetry parallel to the velocity vector of the CoG in steady flight preceding the disturbed motion \textbf{(i.e. X_S points in the direction of motion of the plane)}	X_R-Axis points to the rear
Y_E-Axis points towards the East	Y_B-Axis points towards the right (of the plane)	Y_S-Axis points towards the right (of the plane)	Y_R-Axis points to the left
Z_E-Axis points downwards to the Earth‘s Center	Z_B-Axis points downwards in normal flight	Z_S-Axis completes right-handed system	Z_R-Axis points upwards in normal flight

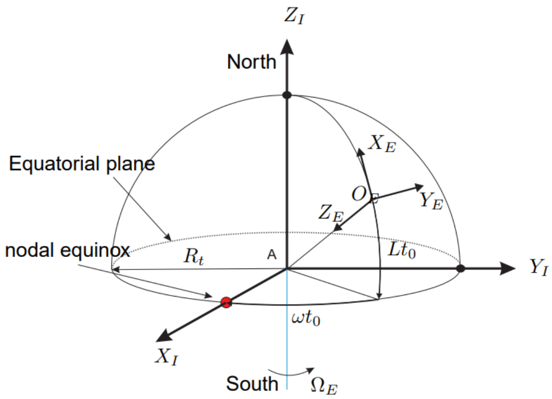


Figure 1.4: Earth-Fixed F_E Frame of Reference

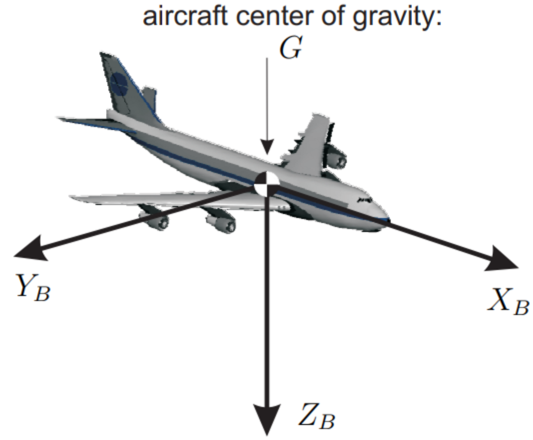


Figure 1.5: Body-Fixed F_B Frame of Reference

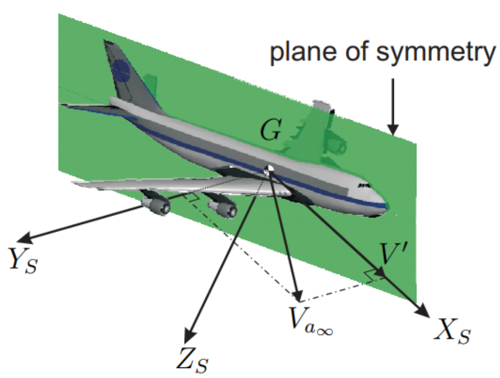


Figure 1.6: Stability F_S Frame of Reference

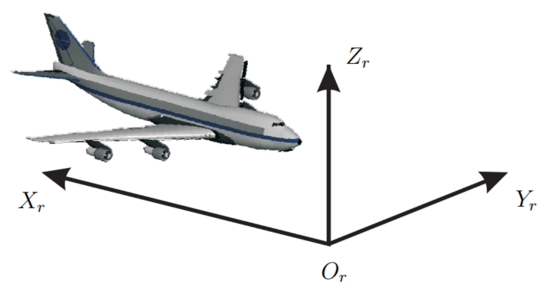


Figure 1.7: Aircraft F_R Frame of Reference

Relationship between rotation in Earth-Frame to Body-Frame and vice versa.

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\phi \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{1.1}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{1.2}$$

1.3 Rotating Reference Frames

Euler‘s Second Law: **The change in angular momentum of a body is equal to the applied external torque.**

Angular Momentum: $B = \mathbf{r} \cdot m\mathbf{v}$

For a mass element i:

$$\left.\frac{d\mathbf{B}_i}{dt}\right|_I = \left.\frac{d\left(\mathbf{r}_i \times m_i\dot{\mathbf{R}}_{P,i}\right)}{dt}\right|_I = \left.\frac{d\mathbf{r}_i}{dt}\right|_I \times m_i\dot{\mathbf{R}}_{P,i} + \mathbf{r}_i \times \left.\frac{d\left(m_i\dot{\mathbf{R}}_{P,i}\right)}{dt}\right|_I \tag{1.3}$$

$$\left.\frac{d\mathbf{B}_i}{dt}\right|_I = \mathbf{r}_i \times \left.\frac{d\left(m_i\dot{\mathbf{R}}_{P,i}\right)}{dt}\right|_I \tag{1.4}$$

Summed over all mass elements i, the first cross product is zero if the origin is the CoG proving Euler‘s Second Law.

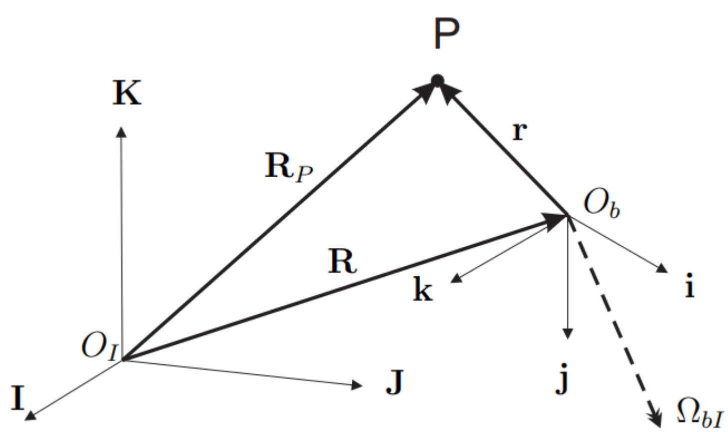


Figure 1.8: Notation of Coordinate System

The point R_P expressed in the inertial frame is given by:

$$\mathbf{R}_P = \mathbf{R} + \mathbf{r} = R_I\mathbf{I} + R_J\mathbf{J} + R_K\mathbf{K} + \mathbb{T}_{Ib}\left(r_i\mathbf{i} + r_j\mathbf{j} + r_k\mathbf{k}\right) \tag{1.5}$$

The velocity of point R_P expressed in the inertial frame is given by:

$$\left.\frac{d\mathbf{R}_P}{dt}\right|_I = \left.\frac{d\mathbf{R}}{dt}\right|_I + \left.\frac{d\mathbf{r}}{dt}\right|_I = \left.\frac{d\mathbf{R}}{dt}\right|_I + \left.\frac{d\mathbf{r}}{dt}\right|_b + \mathbf{\Omega}_{bI} \times \mathbf{r} \tag{1.6}$$

Substituting Equation 1.5 into the first term of Equation 1.3 gives:

$$\sum_i \left.\frac{d\mathbf{r}_i}{dt}\right|_I \times m_i \dot{\mathbf{R}}_{P,i} = \sum_i \left(\left.\frac{d\mathbf{r}_i}{dt}\right|_b + \mathbf{\Omega}_{bI} \times \mathbf{r}_i \right) \times m_i \left(\left.\frac{d\mathbf{R}}{dt}\right|_I + \left.\frac{d\mathbf{r}_i}{dt}\right|_b + \mathbf{\Omega}_{bI} \times \mathbf{r}_i \right) \tag{1.7}$$

We assume that $\left.\frac{d\mathbf{r}_i}{dt}\right|_b$ is zero due to the rigid body assumption and define the velocity of the origin $\left.\frac{d\mathbf{R}_i}{dt}\right|_I$ as v_0. Furthermore, the cross product of two equal vectors is zero.

$$\begin{aligned}
\sum_i \left.\frac{d\mathbf{r}_i}{dt}\right|_I \times m_i \dot{\mathbf{R}}_{P,i} &= \sum_i (\mathbf{\Omega}_{bI} \times \mathbf{r}_i) \times m_i (v_0 + \mathbf{\Omega}_{bI} \times \mathbf{r}_i) && (1.8)\\
&= \sum_i m_i (\mathbf{\Omega}_{bI} \times \mathbf{r}_i) \times v_0 + \sum_i m_i (\mathbf{\Omega}_{bI} \times \mathbf{r}_i) \times (\mathbf{\Omega}_{bI} \times \mathbf{r}_i) && (1.9)\\
&= \mathbf{\Omega}_{bI} \times \left(\sum_i m_i \mathbf{r}_i \right) \times v_0 \qquad = 0 && (1.10)
\end{aligned}$$

The sum of all mass elements times their distance to the origin is zero if the origin is at the CoG ($(\sum_i m_i \mathbf{r}_i) = 0$). Thus it has been proven that the first term of Equation 1.3 is indeed 0 when the origin is the CoG.

1.4 Nonlinear Equations of Motion

Assumptions

- Constant Aircraft Mass
- Aircraft is a Rigid Body
- Symmetric Mass Distribution along $X_B, Z_B - Plane$
- Earth Rotation in Space Neglected
- Earth Surface Curvature Neglected

From Newton‘s Second Law:

$$\mathbf{A}_G^b = \left.\frac{d\mathbf{V}_G}{dt}\right|_E^b = \left.\frac{d\mathbf{V}_G}{dt}\right|_b^b + \mathbf{\Omega}_{bE}^b \times \mathbf{V}_G^b = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{1.11}$$

With force equations:

$$
\begin{aligned}
F_x &= -W\sin\theta + X &&= m(\dot{u} + qw - rv)\\
F_y &= W\cos\theta\sin\phi + Y &&= m(\dot{v} + ru - pw)\\
F_z &= W\cos\theta\cos\phi + Z &&= m(\dot{w} + pv - qu)
\end{aligned}
\tag{1.12}
$$

From Euler's Equation:

$$
\mathbf{M}_{ext} = \left.\frac{d\mathbf{B}_G}{dt}\right|_E^b = \left.\frac{d\mathbf{B}_G}{dt}\right|_b^b + \mathbf{\Omega}_{bO}^b \times \mathbf{B}_G^b = I_G^b \left.\frac{d\mathbf{\Omega}_{bE}}{dt}\right|_b^b + \mathbf{\Omega}_{bE}^b \times \left(I_G^b \mathbf{\Omega}_{bE}^b\right) \tag{1.13}
$$

$$
\mathbf{M}_{ext} = \begin{bmatrix} I_{xx} & 0 & -J_{xz} \\ 0 & I_{yy} & 0 \\ -J_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \left(\begin{bmatrix} I_{xx} & 0 & -J_{xz} \\ 0 & I_{yy} & 0 \\ -J_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right) \tag{1.14}
$$

With moment equations:

$$
\begin{aligned}
M_x &= L &&= I_{xx}\dot{p} + (I_{zz} - I_{yy})\, qr - J_{xz}(\dot{r} + pq)\\
M_y &= M &&= I_{yy}\dot{q} + (I_{xx} - I_{zz})\, rp + J_{xz}\left(p^2 - r^2\right)\\
M_z &= N &&= I_{zz}\dot{r} + (I_{yy} - I_{xx})\, pq - J_{xz}(\dot{p} - rq)
\end{aligned}
\tag{1.15}
$$

1.5 Linearization of Equations of Motion

Linearization at point x_0

$$
f(x) \approx f(x_0) + f'(x_0)(x - x_0) = f(x_0 + \Delta x) \tag{1.16}
$$

- Replace quantity X by $X_0 + \Delta X$
  - X_0 is the nominal value from steady flight conditions
  - ΔX is the small variation from the nominal flight condition
- Eliminate equilibrium equation
- Apply small angle assumptions ($cos(\Delta\alpha) \approx 1$,$sin(\Delta\alpha) \approx 0$)
- Apply small production approximation ($\Delta X \Delta Y \approx 0$,$\Delta X \Delta X \approx 0$)
- Develop forces and moments as first order Taylor series

$$
\Delta X = X_u \Delta u + X_w \Delta w + X_q \Delta q + X_{\delta_e} \Delta\delta_e + X_{\delta_t} \Delta\delta_t \tag{1.17}
$$

$$
X_u = \frac{\partial X}{\partial u}, X_w = \frac{\partial X}{\partial w}, X_q = \frac{\partial X}{\partial q}, X_{\delta_e} = \frac{\partial X}{\partial \delta_e}, X_{\delta_t} = \frac{\partial X}{\partial \delta_t} \tag{1.18}
$$

For example, linearizing the forces in x-direction from Equation 1.12.

$$F_x = -W\sin\theta + X = m(\dot{u} + qw - rv) \tag{1.19}$$

Performing the Taylor expansion $(f(x) \approx f(x_0) + f'(x_0)(\Delta x))$

$$\begin{aligned} &-W\sin\theta_0 + X_0 + \tfrac{\partial F_x}{\partial \theta}\Delta\theta + \tfrac{\partial F_x}{\partial u}\Delta u + \tfrac{\partial F_x}{\partial w}\Delta w + \tfrac{\partial F_x}{\partial q}\Delta q + \tfrac{\partial F_x}{\partial \delta_e}\Delta\delta_e + \tfrac{\partial F_x}{\partial \delta_t}\Delta\delta_t = \\ &= m(\dot{u}_0 + q_0 w_0 - r_0 v_0) + m\left(\tfrac{\partial F_x}{\partial \dot{u}}\Delta\dot{u} + \tfrac{\partial F_x}{\partial q}\Delta q + \tfrac{\partial F_x}{\partial w}\Delta w + \tfrac{\partial F_x}{\partial r}\Delta r + \tfrac{\partial F_x}{\partial v}\Delta v\right) \end{aligned} \tag{1.20}$$

Removing the linearization point:

$$\begin{aligned} &\tfrac{\partial F_x}{\partial \theta}\Delta\theta + \tfrac{\partial F_x}{\partial u}\Delta u + \tfrac{\partial F_x}{\partial w}\Delta w + \tfrac{\partial F_x}{\partial q}\Delta q + \tfrac{\partial F_x}{\partial \delta_e}\Delta\delta_e + \tfrac{\partial F_x}{\partial \delta_t}\Delta\delta_t = \\ &= m\left(\tfrac{\partial F_x}{\partial \dot{u}}\Delta\dot{u} + \tfrac{\partial F_x}{\partial q}\Delta q + \tfrac{\partial F_x}{\partial w}\Delta w + \tfrac{\partial F_x}{\partial r}\Delta r + \tfrac{\partial F_x}{\partial v}\Delta v\right) \end{aligned} \tag{1.21}$$

Working out the derivatives in the linearization point:

$$\begin{aligned} &-W\cos\theta_0\Delta\theta + X_u\Delta u + X_w\Delta w + X_q\Delta q + X_{\delta_e}\Delta\delta_e + X_{\delta_t}\Delta\delta_t = \\ &= m(1\Delta\dot{u} + w_0\Delta q + q_0\Delta w - v_0\Delta r - r_0\Delta v) \end{aligned} \tag{1.22}$$

Applying the assumptions of straight, symmetric flight ($q_0 = v_0 = r_0 = 0$):

$$\begin{aligned} &-W\cos\theta_0\Delta\theta + X_u\Delta u + X_w\Delta w + X_q\Delta q + X_{\delta_e}\Delta\delta_e + X_{\delta_t}\Delta\delta_t = \\ &= m(\Delta\dot{u} + w_0\Delta q) \end{aligned} \tag{1.23}$$

Body frame to stability frame, by definition $w_0 = 0$ and simply notation $\Delta x \Leftrightarrow x$. Note that x is now the deviation from nominal flight conditions:

$$-W\cos\theta_0\theta + X_u u + X_w w + X_q q + X_{\delta_e}\delta_e + X_{\delta_t}\delta_t = m\dot{u} \tag{1.24}$$

Assumptions:

- Aerodynamically symmetric aircraft w.r.t. $X_B, Z_B - Plane$
- Motions about steady, straight, symmetric flight
- Disturbances and deviations small enough to permit linearization of the equations of motion

Symmetric Motion

$$
\begin{aligned}
-W\cos\theta_0\theta + X_u u + X_w w + X_q q + X_{\delta_e}\delta_e + X_{\delta_t}\delta_t &= m\dot{u} \\
-W\sin\theta_0\theta + Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q + Z_{\delta_e}\delta_e + Z_{\delta_t}\delta_t &= m(\dot{w}) - qV \\
M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q + M_{\delta_e}\delta_e + M_{\delta_t}\delta_t &= I_y\dot{q} \\
\dot{\theta} &= q
\end{aligned}
\tag{1.25}
$$

Dimensionless Symmetric Motion

- D_C is the derivative operator
- μ_C is the dimensionless mass
- C_{Z_0} the dimensionless form of $-W cos(\theta_0)$
- $\alpha = \omega/V$

$$
\begin{aligned}
\left(C_{X_u} - 2\mu_c D_c\right)\hat{u} + C_{X_\alpha}\alpha + C_{Z_0}\theta + C_{X_q}\frac{q\bar{c}}{V_{\bar{c}}} + C_{X_{\delta_e}}\delta_e &= 0 \\
C_{Z_u}\hat{u} + \left[C_{Z_\alpha} + \left(C_{Z_{\dot{\alpha}}} + 2\mu_c\right)D_c\right]\alpha - C_{X_0}\theta + \left(C_{Z_q} + 2\mu_c\right)\frac{q\bar{c}}{V} + C_{Z_{\delta_e}}\delta_e &= 0 \\
-D_c\theta + \frac{q\bar{c}}{V} &= 0 \\
C_{m_u}\hat{u} + \left(C_{m_\alpha} + C_{m_{\dot{\alpha}}}c\right)\alpha + \left(C_{m_q} - 2\mu_c K_Y^2 D_c\right)\frac{q\bar{c}}{\overline{V}} + C_{m_{\delta_e}}\delta_e &= 0
\end{aligned}
\tag{1.26}
$$

Asymmetric Motion

$$
\begin{aligned}
W\cos\theta_0\phi + Y_v v + Y_{\dot{v}}\dot{v} + Y_p p + Y_r r + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r &= m(\dot{v} + rV) \\
L_v v + L_p p + L_r r + L_{\delta_a}\delta_a + L_{\delta_r}\delta_r &= I_x\dot{p} - J_{xz}\dot{r} \\
N_v v + N_{\dot{v}}\dot{v} + N_p p + N_r r + N_{\delta_a}\delta_a + N_{\delta_r}\delta_r &= I_x\dot{r} - J_{xz}\dot{p} \\
\dot{\psi} &= \frac{r}{\cos\theta_0} \\
\dot{\phi} &= p + r\tan\theta_0
\end{aligned}
\tag{1.27}
$$

Dimensionless Asymmetric Motion

$$
\begin{aligned}
\left[C_{Y_\beta} + \left(C_{Y_{\dot{\beta}}} - 2\mu_b\right)D_b\right]\beta + C_L\phi + C_{Y_p}\frac{pb}{2V} + \left(C_{Y_r} - 4\mu_b\right)\frac{rb}{2V} + C_{Y_{\delta_\alpha}}\delta_\alpha + C_{Y_{\delta_r}}\delta_r &= 0 \\
-\frac{1}{2}D_b\phi + \frac{pb}{2V} &= 0 \\
C_{l_\beta}\beta + \left(C_{l_p} - 4\mu_b K_X^2 D_b\right)\frac{pb}{2V} + \left(C_{l_r} + 4\mu_b K_{xz} D_b\right)\frac{rb}{2V} + C_{l_{\delta_g}}\delta_\alpha + C_{l_{\delta_r}}\delta_r &= 0 \\
\left(C_{n_\beta} + C_{n_{\dot{\beta}}}D_b\right)\beta + \left(C_{n_p} + 4\mu_b K_{xz} D_b\right)\frac{pb}{2V} + \left(C_{n_r} - 4\mu_b K_Z^2 D_b\right)\frac{rb}{2V} + C_{n_{\delta_\alpha}}\delta_\alpha + C_{n_{\delta_r}}\delta_r &= 0
\end{aligned}
\tag{1.28}
$$

1.6 Linearity Dynamics

- Rewrite dimensionless equations in matrix form
- Group all terms without and with differential operator D_c or D_b in different matrices
- Resulting equation: $P\dot{\mathbf{x}} = Q\mathbf{x} + R\mathbf{u}$
- Rewritten in state-space form:

$$\begin{aligned} &\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad A = P^{-1}Q, \quad B = P^{-1}R \\ &\mathbf{y} = C\mathbf{x} + D\mathbf{u} \end{aligned}$$

Symmetric State-Space Form

$$\begin{bmatrix} \dot{\hat{u}} \\ \dot{\alpha} \\ \dot{\theta} \\ \frac{\dot{q}\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \delta_e \tag{1.29}$$

Asymmetric State-Space Form

$$\begin{bmatrix} \dot{\beta} \\ \dot{\phi} \\ \frac{\dot{p}b}{2V} \\ \frac{\dot{r}b}{2V} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ \hat{A}_{31} & \hat{A}_{32} & \hat{A}_{33} & \hat{A}_{34} \\ \hat{A}_{41} & \hat{A}_{42} & \hat{A}_{43} & \hat{A}_{44} \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} + \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \\ \hat{B}_{31} & \hat{B}_{32} \\ \hat{B}_{41} & \hat{B}_{42} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{1.30}$$

The **trim point** or equilibrium condition is the set of controls and states for which all accelerations on the aircraft are zero.

Trimming is performed by minimizing a cost functions that contains a weighted measure of accelerations.

$$J = \sum_{i=1}^{n} W_i \dot{x}_i^2, \quad x = [u, v, w, p, q, r] \tag{1.31}$$

2 Laplace Transform

To analyze real-life systems and design controllers for them, real-life systems should be modeled mathematically. Usually, the dynamics of real-life systems evolves in time. Therefore, the corresponding mathematical model may appear as a set of differential equations. Laplace transform allows us to look at a system within the frequency domain (complex variable s) instead of the time domain (time variable t). With Laplace transform an ordinary differential equation is transformed into an algebraic equation. Laplace transform simplifies a convolution integral as a multiplication.

Laplace transform $F(S)$ of function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \tag{2.1}$$

The Laplace transform of a function $f(t)$ exists if the Laplace integral converges.

Unit step function $1(t)$	$\frac{1}{s}$
Unit impulse (Dirac delta) function $\delta(t)$	1
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Figure 2.1: Laplace Transform Table

2.0.1 Special Cases

- Multiplication of $f(t)$ by e^{-at}:

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a) \tag{2.2}$$

- Change of time-scale by a:

$$\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} = aF(as) \tag{2.3}$$

2.1 Differentiation Theorem

Partial Integration: $\int_{v_0}^{v_f} u\mathrm{d}v = u_f v_f - u_0 v_0 - \int_{u_0}^{u_f} v\mathrm{d}u$

For a function $f(t)$ of exponential order, the Laplace transform can be determined by using the partial integration theorem with $u = e^{-st}$ and $v = f(t)$.

$$\begin{aligned}\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} &= \int_0^\infty \frac{df(t)}{dt}e^{-st}dt \\ &= e^{-s\infty}f(\infty) - e^{-s0}f(0) + s\int_0^\infty f(t)e^{-st}dt \\ &= sF(s) - f(0)\end{aligned} \tag{2.4}$$

For a second order derivative the Laplace transform function is given as:

$$\mathcal{L}\left\{\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\right\} = s^2F(s) - sf(0) - \frac{\mathrm{d}f}{\mathrm{d}t}(0) \tag{2.5}$$

Laplace transform for integration

$$\mathcal{L}\left\{\int_{-\infty}^{t} f(t)\mathrm{d}t\right\} = \frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} f(t)\mathrm{d}t \tag{2.6}$$

Inverse Laplace transform

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\mathrm{d}s \tag{2.7}$$

2.2 Convolution Integral

A **linear** time-invariant system means that if we think of the input as the sum of different input functions, the output will be the sum of the two associated output functions. Any input to this linear system can be thought of as an infinite sum of impulse functions (Dirac Comb), the resulting output will also be an infinite sum of impulse functions.

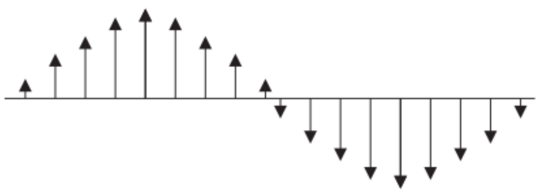


Figure 2.2: Dirac Comb

The unit impulse is a function with a width that approaches zero and an area equal to one.

$$\delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{2.8}$$

$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1 \tag{2.9}$$

If the unit impulse response of the linear system is a known function $h(t)$, the response of the linear system to the input function $u(t)$ is given as:

$$y(t) \approx \sum_{n=0}^{n=t/(\Delta t)} u(n\Delta t)h(t - n\Delta t)\Delta t \tag{2.10}$$

For the limit $\Delta t \to 0$ this turns into the convolution integral:

$$y(t) = \int_0^t u(\tau)h(t-\tau)\mathrm{d}\tau \tag{2.11}$$

Where $h(t)$ is the output of the system to an impulse function and $u(t)$ is equal to the impulse function, $y(t)$ is equal to the impulse response. The convolution operation slides $u(t)$ and $h(t)$ across each other to compute $y(t)$.

The Laplace transform of the convolution integral is

$$\mathcal{L}\left\{\int_0^t u(\tau)h(t-\tau)\mathrm{d}\tau\right\} = U(s)H(s) \tag{2.12}$$

3 | Transfer Functions

Transfer function is the ratio of Laplace transform of system's output, $Y(s)$ and Laplace transform of input, $U(s)$, assuming all initial conditions are zero. The transfer function is the Laplace transform of the unit impulse response of the system $\mathcal{L}\{h(t)\} = H(s)$.

$$H(S) = \frac{Y(S)}{U(S)} \tag{3.1}$$

The transfer function characterizes a system if:

- The system is linear w.r.t inputs and outputs
- The system is time-invariant

Once an input signal $U(S)$ is chosen (you need its Laplace representation from Figure 2.1), you can calculate the output signal $Y(S) = H(S)U(S)$.

Example, converting the differential equation $\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_1\dot{u}(t) + b_0 u(t)$ into Laplace domain. Assuming zero initial conditions $(y(0) = 0, \dot{y}(0) = 0, u(0) = 0)$.

Converting into Laplace domain gives:

$$\begin{aligned} s^2Y(s) + a_1 sY(s) + a_0 Y(s) &= b_1 sU(s) + b_0 U(s) \\ Y(s)\left(s^2 + a_1 s + a_0\right) &= U(s)\left(b_1 s + b_0\right) \end{aligned} \tag{3.2}$$

Re-writing into a transfer function $H(S)$:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \tag{3.3}$$

Open-Loop Control/Feed-Forward Control

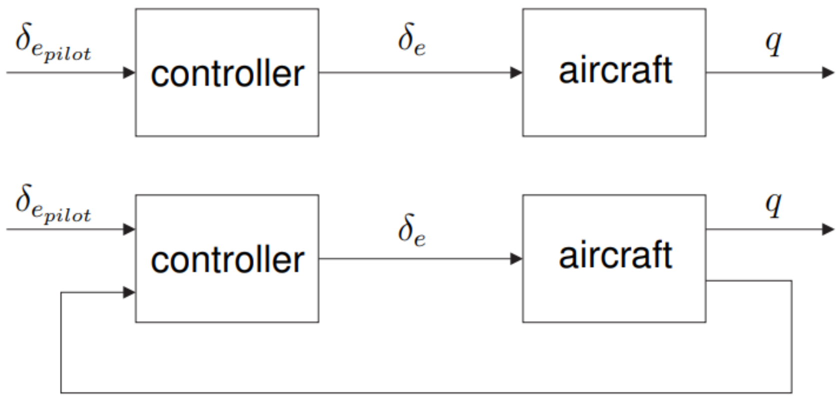


Figure 3.1: Open-Loop Control (top) vs. Close-Loop Control

- Modifies the "reference" (in this case pilot input), to produce a control signal
- Will not compensate for additional disturbances (turbulence etc.)
- Cannot stabilize an unstable system

Close-Loop Control/Feed-Back Control

- Uses both the reference and the output from the system to produce a control signal
- Can stabilize an unstable system

3.1 Equivalent Transfer Function for Closed Loop

The equivalent transfer function for a close-loop system shown in Figure 3.2 is given by:

$$\left.\begin{array}{l} U(s) - Y(s)G_2(s) = E(s) \\ E(s) = \frac{Y(s)}{G_1(s)} \end{array}\right\} \Rightarrow U(s) = Y(s)\left(G_2(s) + \frac{1}{G_1(s)}\right) \tag{3.4}$$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)} \tag{3.5}$$

3.2 Manipulation of Block Diagrams

- Combine blocks in series into blocks with the product of the transfer functions
- Combine blocks in parallel into blocks with the sum of the transfer function

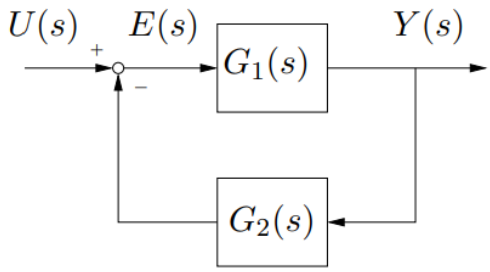


Figure 3.2: Close-Loop Controller

- Move summation and branch points (avoid moving summation over branch points or vice versa) to simplify the diagram

Moving Branches

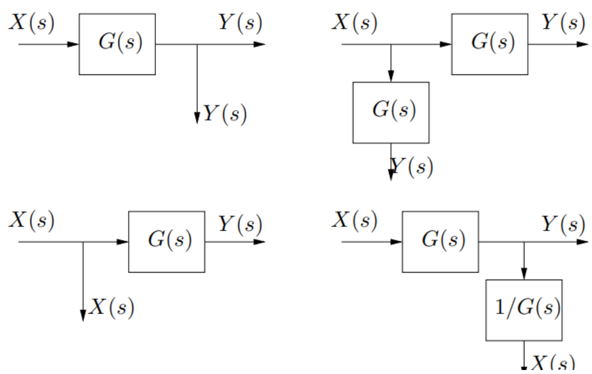


Figure 3.3: Moving Branches in Block Diagrams

Moving Summation Points

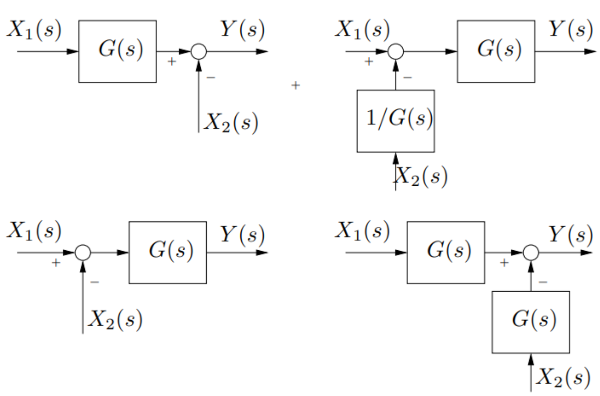


Figure 3.4: Moving Sums in Block Diagrams

4 First Order Systems

Zeros: Roots of Numerator
Poles: Roots of Denominator

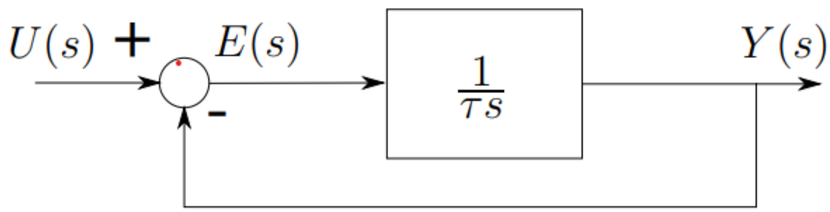


Figure 4.1: Block Diagram of First Order System

$$H_c(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{\tau s}}{1 + \frac{1}{\tau S}} = \frac{1}{\tau s + 1} \tag{4.1}$$

With a closed-loop pole of $s = -1/\tau$

4.1 Unit-Step Response

The response of the first-order dynamical system to a unit-step function $f(t) = 1$ for $t \geq 1$ is analyzed. Following the convolution integral theory we can multiply the transfer function by the Laplace transform of unit-step function is $U(s) = 1/s$.

$$Y(s) = \frac{1}{\tau s + 1} \cdot \frac{1}{s} \tag{4.2}$$

Performing partial fraction expansion

$$Y(s) = \frac{1}{s} - \frac{\tau}{\tau s + 1} = \frac{1}{s} - \frac{1}{s + 1/\tau} \Rightarrow y(t) = 1 - e^{-t/\tau} \tag{4.3}$$

Initially at $t = 0$ the output is 0. Finally, as $t \Rightarrow \infty$ output becomes 1 for $\tau > 0$ and output goes to $-\infty$ for $\tau < 0$.

Characteristics:

- At $t = \tau$ the response of the system is independent of τ and is $y(t) = 1 - e^{-1} = 0.632$.
- At $t = 0$ the slope is $1/\tau$ $(\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \frac{1}{\tau}e^{-t/\tau})$
- The smaller the time constant τ, the faster the response of the 1st-order dynamical system.

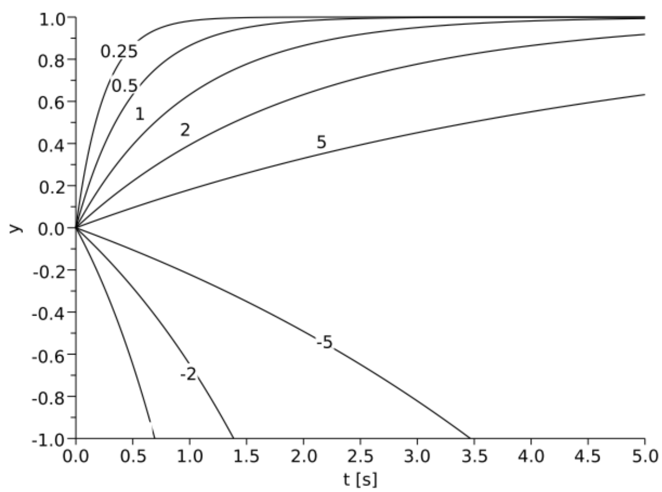


Figure 4.2: Response of First Order System to Unit-Step Function

- A negative pole close to 0 (corresponding to large positive τ) results in slow response.
- A negative pole far from 0 (corresponding to small positive τ) results in fast response.
- A positive pole (corresponding to negative τ) results in unstable behavior

4.2 Unit-Ramp Response

The Laplace transform of the unit-ramp function is $F(s) = 1/s^2$.

$$Y(s) = \frac{1}{\tau s + 1} \cdot \frac{1}{s^2} \tag{4.4}$$

$$Y(s) = \frac{1}{s^2} - \frac{\tau}{s} - \frac{\tau^2}{\tau s + 1} \Rightarrow y(t) = t - \tau + \tau e^{-t/\tau} \tag{4.5}$$

4.3 Unit-Impulse Response

The Laplace transform of the unit-impulse function is $F(s) = 1$.

$$Y(s) = \frac{1}{\tau s + 1} = \frac{1}{\tau} \cdot \frac{1}{s + 1/\tau} \Rightarrow y(t) = \frac{1}{\tau} e^{-t/\tau} \tag{4.6}$$

Recap:

Unit-ramp response: $y(t) = t - \tau + \tau e^{-t/\tau}$
Unit-step response: $y(t) = 1 - e^{-t/\tau}$
Unit-impulse response: $y(t) = \frac{1}{\tau} e^{-t/\tau}$

For linear system the **response of the system to the derivative of an input function is the same as the derivative of the response of the system to the input function itself.**

5 | Second Order Systems

Standard Form of Second Order Systems

$$H_c(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.1}$$

With poles (roots of denominator):
$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad p_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

Undamped Natural Frequency: $\omega_n = \sqrt{k/m}$
Damping Ratio: $\zeta = \frac{c}{2\sqrt{km}}$
Damped Natural Frequency: $\omega_d = \omega_n\sqrt{1 - \zeta^2} > 0$

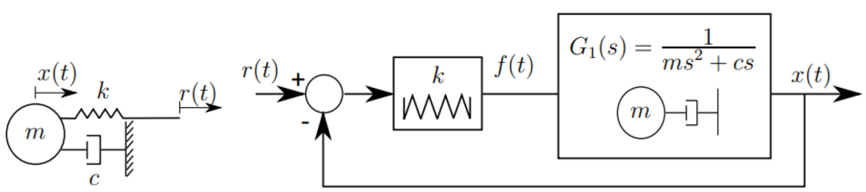


Figure 5.1: Block Diagram of Second Order System

With corresponding close loop transfer function

$$H_c(s) = \frac{\frac{k}{ms^2+cs}}{1 + \frac{k}{ms^2+cs}} = \frac{k}{ms^2 + cs + k} \tag{5.2}$$

5.1 Damping

- Under-damped $0 < \zeta < 1$: closed-loop poles are complex conjugates in left-hand s-plane
- Critically damped $\zeta = 1$: closed-loop poles lie on each other in left-hand s-plane
- Over-damped $\zeta > 1$: closed-loop poles are real negative values in the s-plane

5.2 Unit-Step Response

5.2.1 Under-Damped $(0 < \zeta < 1)$

Multiply transfer function Equation 5.2 by unit-step transfer function $U(s) = 1/s$ to obtain $Y(s)$.

$$Y(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} \tag{5.3}$$

Using partial fraction expansion

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2} \end{aligned} \tag{5.4}$$

Finding the inverse Laplace transform of $Y(s)$

$$\begin{aligned} \mathcal{L}^{-1}\{Y(s)\} = y(t) &= 1 - e^{-\zeta\omega_n t}\left(\cos\left(\omega_d t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t\right)\right) \\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \arctan\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \end{aligned} \tag{5.5}$$

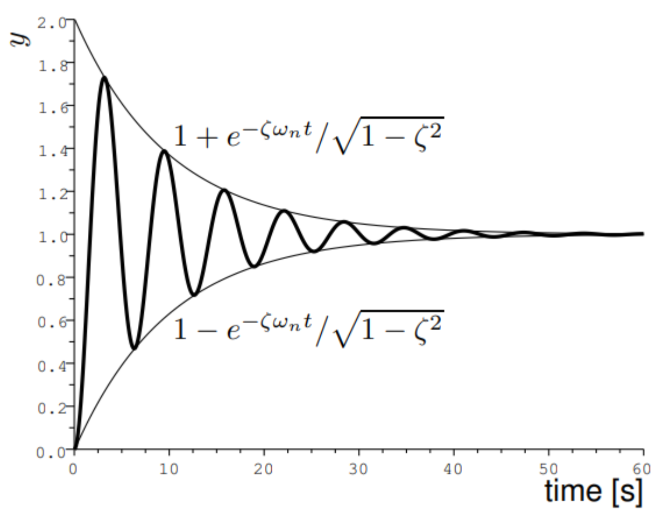


Figure 5.2: Under-Damped Transient Response

5.2.2 Critically Damped $\zeta = 1$

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \tag{5.6}$$

5.2.3 Over-Damped $\zeta > 1$

$$y(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \tag{5.7}$$

$$\text{With } s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n, \quad s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$$

5.3 Effect of ζ on Unit-Step Response

Shape of the response function is determined by ζ.

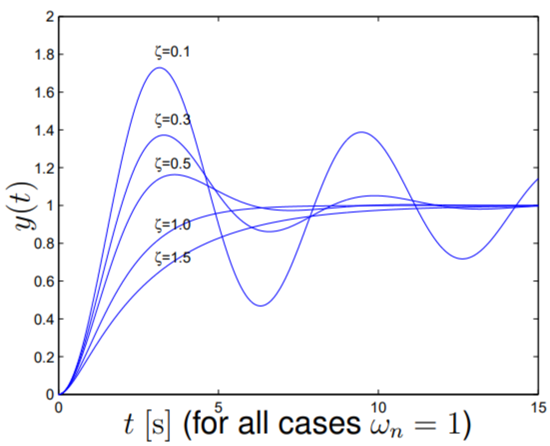


Figure 5.3: Effect of ζ on Unit-Step Response

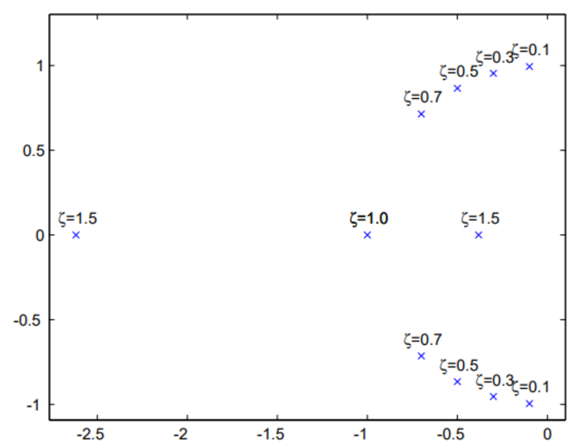


Figure 5.4: Pole Locations for Different Damping Coefficients

5.4 Effect of ω_n on Unit-Step Response

Speed of the response function is determined by ω_n ($\zeta = 0.7$).

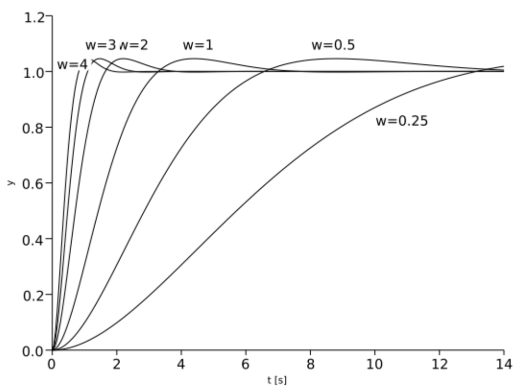


Figure 5.5: Effect of ω_n on Unit-Step Response

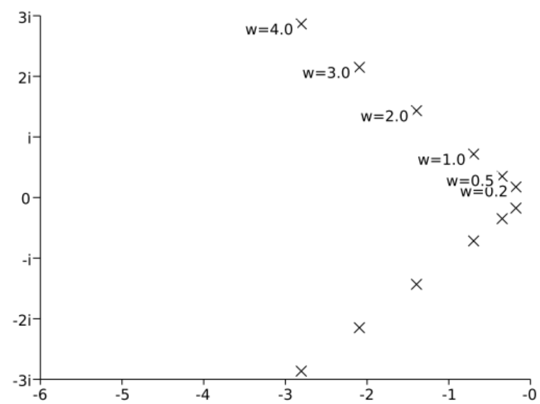


Figure 5.6: Pole Locations for Different Natural Frequencies

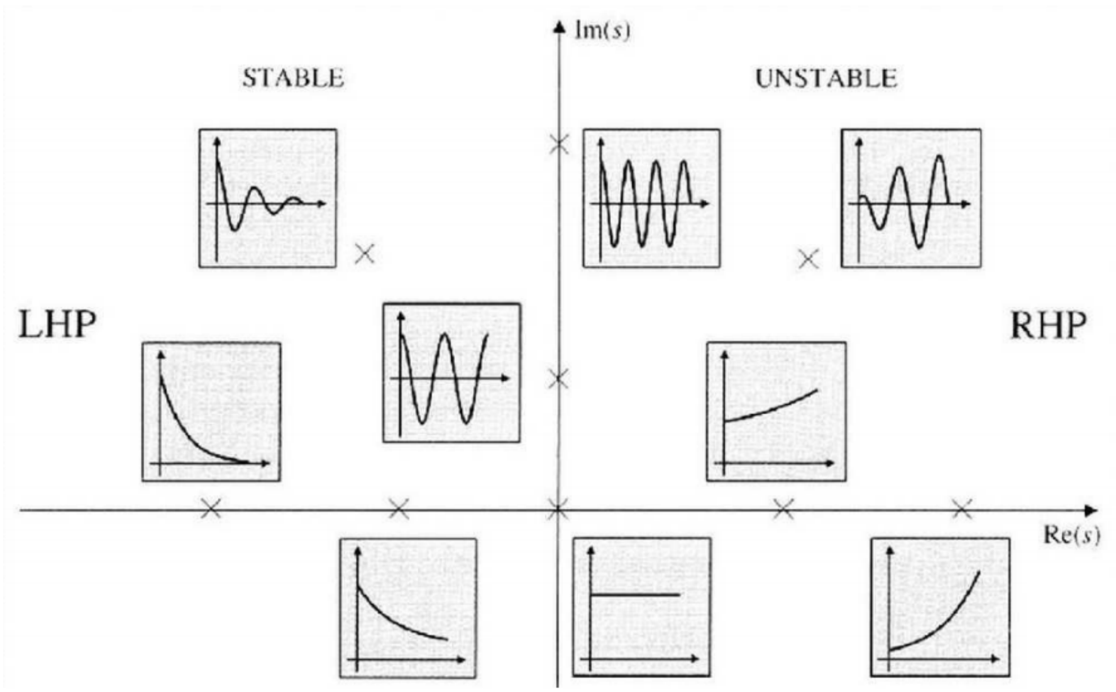


Figure 5.7: System Characteristics based on Pole Location

6 | Transient Response

Systems with energy storage exhibit transient responses to external inputs. Usually, we characterize a system's transient response via its response to unit-step function. Transient response characteristics are based on assumption of zero initial conditions.

6.1 Transient Response

For transient response to unit-step function we specify various important characteristics of the systems output.

6.1.1 Overshoot & Peak Time

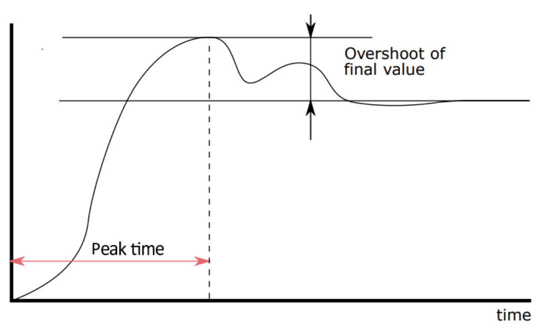


Figure 6.1: Overshoot & Peak Time

The overshoot is typically given as a percentage of the final steady-state value of the system.

6.1.2 Delay Time

Other criteria than the 50% limit can also be used e.g. 10%.

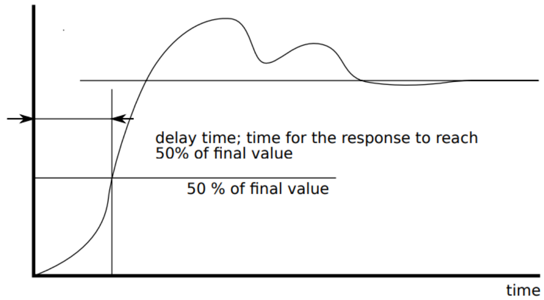


Figure 6.2: Delay Time

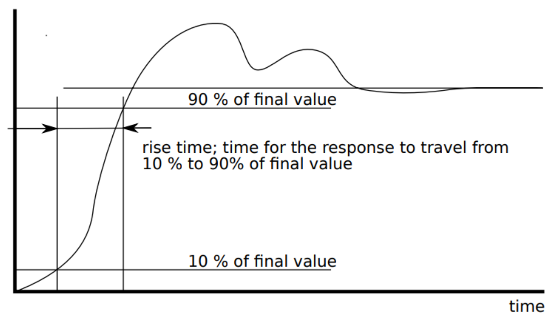


Figure 6.3: Rise Time

6.1.3 Rise Time

Different rise-time criteria are also possible, e.g. the 0% to 100%, or 5% to 95%

6.1.4 Settling Time

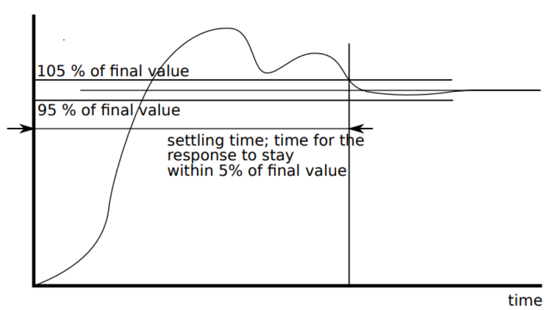


Figure 6.4: Settling Time

Within 10%, 5% and 2% are common settling time criteria.

6.2 Design of First-Order System

The unit-step response of the system: $y(t) = 1 - e^{-t/\tau}$

- Design τ for desired delay time: $y\left(t^{\text{delay}}\right) = 0.5$
$$\tau = -\frac{t^{\text{delay}}}{\ln 0.5}$$
- Design τ for desired rise time (10% to 90%) $\tau = -\frac{t^{\text{rise}}}{\ln 1/9}$
- Design τ for desired 2% settling time $y\left(t^{\text{settle}}\right) = 0.98$
$$\tau = -\frac{t^{\text{settle}}}{\ln 0.02}$$

6.3 Design of Second-Order System

The unit-step response (under=damped) of the system
$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

- Design ω_d for desired peak time
$$t^{\text{peak}} = \pi/\omega_d$$
- Design ζ for desired overshoot
$$e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$
- Design ζ & ω_n for desired rise time (10% to 90%)
$$t^{\text{rise}} = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\arctan\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

7 | Poles & Zeros

7.1 Pole Locations & Dominant Poles

Pole placement design is changing the locations of the closed-loop poles of the system. Control effort is related to how far open-loop poles are moved by feedback. When a zero is near a pole, system may be nearly uncontrollable (moving such poles needs large control gains/effort).

Characteristics of transient response of closed-loop systems is closely related to location of closed-loop poles. When a variable gain is in closed-loop system, location of closed-loop poles depend on gain value.

7.1.1 Dominant Poles

Dominant Poles: Poles closest to imaginary axis in s-plane give rise to longest lasting terms of system's transient response. These poles are called dominant poles.

For example, for $H_1 = 1/(s+1), H_2 = 10/(s+10)$. H_1 has a pole at -1 while H_2 has a pole at -10. Thus, since H_1 has a pole closer to the imaginary axis it is dominant in the systems response.

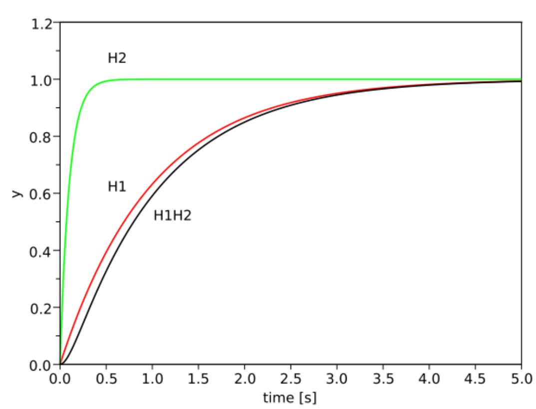


Figure 7.1: Dominant Poles

7.1.2 Zeros

Adding a zero increases the overshoot in unit-step response as it adds a derivative term to the output.

$$\begin{aligned}
&H_0(s) = \frac{1}{s^2+2\zeta s+1} \\
&H_d(s) = \frac{1}{\alpha\zeta}\frac{s}{s^2+2\zeta s+1} \\
&H(s) = H_0(s) + H_d(s) \\
&y(t) = y_0(t) + y_d(t) = y_0(t) + \frac{1}{\alpha\zeta}\dot{y}_0(t)
\end{aligned} \tag{7.1}$$

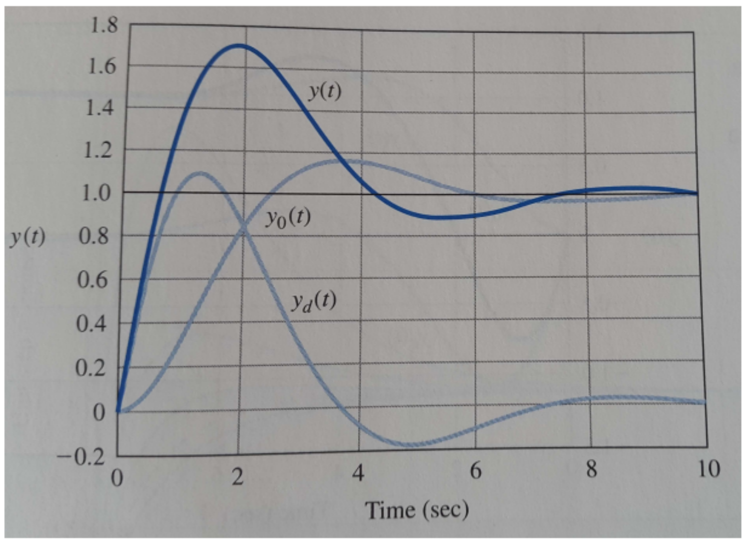


Figure 7.2: Effect of Zeros

8 Root Locus

The design of controllers is important for

- **Stability** (All poles of transfer function must be in LHP)
- **Tracking** (Force output follow reference input as closely as possible)
- **Regulation** (Keep response error small in presence of disturbances)

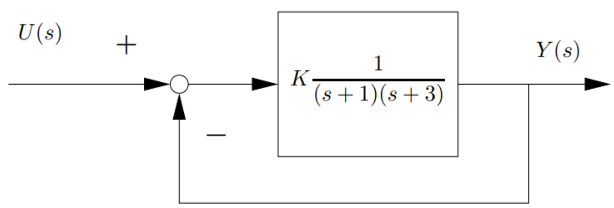


Figure 8.1: Block Diagram of System with Gain

The transfer function of the system with $H_1 = K/(s+1)(s+3)$ and $H_2 = 1$ is given by:

$$H_c(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)} = \frac{K}{(s+1)(s+3)+K} \tag{8.1}$$

With a characteristic equation: $s^2 + 4s + K + 3 = 0$ which has poles $p_{1,2}(K) = -2 \pm \sqrt{1-K}$. The pole locations for various gains have been plotted in the complex plain.

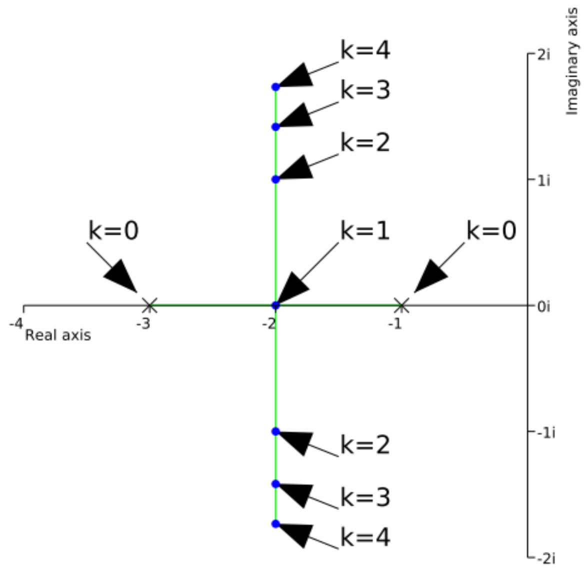


Figure 8.2: Pole Locations for various Gains

8.1 Cartesian Representation

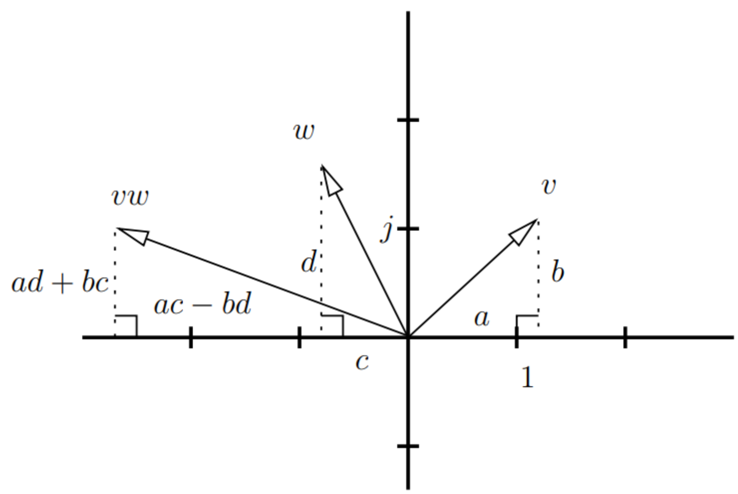


Figure 8.3: Cartesian Representation

Multiplication:

$$vw = (a + jb)(c + jd) = (ac - bd) + j(ad + bc) \tag{8.2}$$

Division:

$$\frac{v}{w} = \frac{a + jb}{c + jd} = \frac{a + jb}{c + jd} \cdot \frac{c - jd}{c - jd} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2} \tag{8.3}$$

8.2 Polar Representation

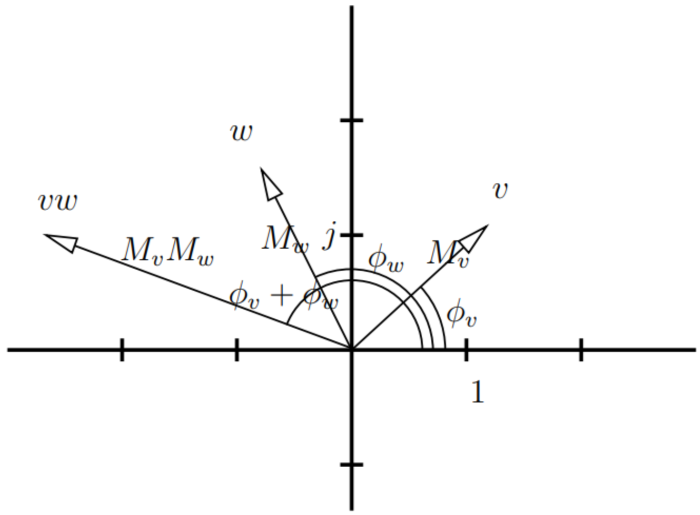


Figure 8.4: Polar Representation

$$\begin{aligned} v &= M_v e^{j\phi_v} \\ w &= M_w e^{j\phi_w} \end{aligned} \tag{8.4}$$

Multiplication:

$$vw = M_v e^{j\phi_v} M_w e^{j\phi_w} = M_v M_w e^{j(\phi_v + \phi_w)} \tag{8.5}$$

Division:

$$\frac{v}{w} = \frac{M_v e^{j\phi_v}}{M_w e^{j\phi_w}} = \frac{M_v}{M_w} e^{j(\phi_v - \phi_w)} \tag{8.6}$$

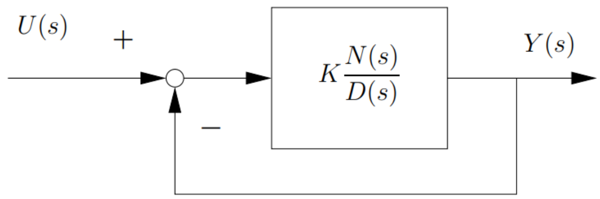
$$e^{j\phi} = \cos\phi + j\sin\phi \tag{8.7}$$

8.3 Root Locus Method

The root locus is the locus of roots of the characteristic equation of the closed-loop system as a specific parameter (usually gain K) varies from 0 to ∞. The plot shows the contributions of each pole and zero of the transfer function of open-loop system to the location of closed-loop poles.

Open-loop complex conjugate zeros and poles (if any) are always located symmetrically about real axis. Hence, root loci are always symmetrical w.r.t. real axis. Therefore, we only need to construct upper half of root loci.

Consider a transfer function $G(s) = K\frac{N(s)}{D(s)}$ in a closed loop:



With an equivalent transfer function

$$H_c(s) = \frac{KN(s)}{D(s) + KN(s)} \tag{8.8}$$

And a characteristic equation:

$$D(s) + KN(s) = 0 \quad \Rightarrow KN(s)/D(s) = -1 \tag{8.9}$$

To sketch the root loci of a system using the root-locus method the poles and zeros of the characteristic equation must be determined. Paths representing the location of roots of the characteristic equation when K (assumed to be positive) varies from 0 to ∞ are drawn in the complex plane.

Assuming that the characteristic equation has the form

$$K\frac{N(s)}{D(s)} = K\frac{(s-z)}{(s-p_1)(s-p_2)} \tag{8.10}$$

If K = 0, the poles of the system are the initial open-loop poles.

$$D(s) + 0N(s) = 0 \Rightarrow \quad (s-p_1)(s-p_2) = 0 \tag{8.11}$$

If $K \to \infty$, s can become z_1 or $|s| \Rightarrow \infty$

$$K\frac{N(s)}{D(s)} = -1 \Rightarrow \quad K\frac{|s-z_1|}{|s-p_1|\,|s-p_2|} = |-1| \tag{8.12}$$

8.3.1 Angle Condition

$$\angle\left(K\frac{(s-z_1)}{(s-p_1)(s-p_2)}\right) = \angle(-1) \tag{8.13}$$

$$\angle(s-z_1) - \angle(s-p_1) - \angle(s-p_2) = \pm 180^\circ(2l+1) \quad l = 0, 1, 2, \ldots \tag{8.14}$$

8.3.2 Magnitude Condition

$$|K|\frac{|s-z_1|}{|s-p_1|\,|s-p_2|} = |-1| \tag{8.15}$$

8.4 Break-Away Points

Use characteristic equation to compute $dK/ds = 0$. The solutions for s that result in a positive K can be considered as break-away points.

8.5 Evans‘ Rules

For a transfer function with n number of poles and m number of zeros

- For $K = 0$ the root-locus starts in open-loop poles
- Determine parts of real axis that are part of root-locus using angle condition
- For $K \Rightarrow \infty$, $n - m$ poles follow the aymptotes to infinity
- Intersection of asymptotes with real axis
  Assymptote Angle
  $$\Phi = \frac{180°(2l+1)}{n-m} for l = 0, 1, 2, .. \tag{8.16}$$
  Assymptote Intersection
  $$\sigma^{\text{int}} = \frac{\sum_{j=1}^{n} Re\,(p_j) - \sum_{i=1}^{m} \mathrm{Re}\,(z_i)}{n-m} \tag{8.17}$$
- Determine break-away points via $dK/ds = 0$
- Angle of departure at poles, arrival at zeroes
- Calculation of gain at each point of root-locus with the magnitude condition

9 | State-Space Method

State-space is the use of differential equations that describe dynamical systems organized as a set of 1st-order differential equations in vector-valued state of system where the solution is a trajectory of state vector in space.

State-space method allows for many inputs and many outputs interrelated in complex manner. Use of matrix-vector representation simplifies mathematical representation of systems of equations. Increase in number of state variables, inputs, or outputs does not increase complexity of equations.

Definitions

- State: Smallest set of variables (called state variables) such that knowledge of these variables at $t = t_0$, and knowledge of input for $t \geq t_0$ completely determines behavior of system for $t \geq t_0$
- State Variables: Variables of a dynamical system that together describe behavior of the system at any time instant
- State vector: Vector consisting of all state variables
- State Space: Space of n dimensions composing of n coordinate axis for n state variables of the dynamical system. Any state of the dynamical system can be represented as a point in state space.

A dynamical state-space system can be modelled using three types of variables: input, output and state variables.

For linear time-invariant systems:

State Equation: $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$
Output Equation: $\dot{\boldsymbol{y}}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$

If $D \neq 0$ the system has "direct feed-through"

For example, the second-order system: $\ddot{y} + 2\dot{y} + y = u$, defining $x_1 = y$ and $x_2 = \dot{y}$ we have

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2x_2 - x_1 + u \end{aligned} \tag{9.1}$$

Rewritten in matrix form gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{9.2}$$

9.1 State-Space to Transfer Functions

State-space representation in time domain for a single-input single-output system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + Bu(t) \\ y(t) &= C\boldsymbol{x}(t) + Du(t) \end{aligned} \tag{9.3}$$

Applying Laplace transform to move to frequency domain

$$\begin{aligned} s\boldsymbol{X}(s) &= A\boldsymbol{X}(s) + B\boldsymbol{U}(s) \\ (sI - A)\boldsymbol{X}(s) &= B\boldsymbol{U}(s) \\ \boldsymbol{X}(s) &= (sI - A)^{-1}B\boldsymbol{U}(s) \end{aligned} \tag{9.4}$$

Determine output in frequency domain

$$\boldsymbol{Y}(s) = \left[C(sI - A)^{-1}B + D\right]\boldsymbol{U}(s) \tag{9.5}$$

Transfer Function: $G(s) = C(sI - A)^{-1}B + D$

With a characteristic polynomial of $G(s)$: $|sI - A| = 0$ where the eigenvalues of A are the poles of $G(S)$.

For a multiple-input multiple-output system (y is m-dimensional, u is r-dimensional, G(s) is an $m \times r$ matrix):

9.2 Transfer Function to State-Space

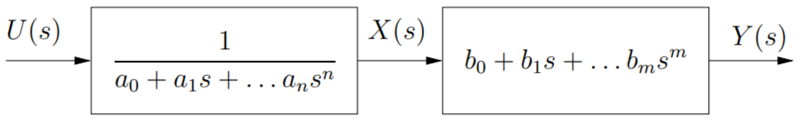
There may be infinitely many possible state-space representations for a transfer function.

9.2.1 Controllable Canonical Form

Consider the following transfer function:

$$\frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s + \dots b_m s^m}{a_0 + a_1 s + \dots a_n s^n} \tag{9.6}$$

Which can be represented with the following block diagram:



Where

$$\begin{aligned} (b_0 + b_1 s + \dots b_m s^m)\, X(s) = Y(s) \\ (a_0 + a_1 s + \dots a_n s^n)\, X(s) = U(s) \end{aligned} \tag{9.7}$$

Transforming into the time domain gives 2 differential equations

$$\begin{aligned} u(t) = a_n \tfrac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} + \dots a_1 \tfrac{\mathrm{d}x(t)}{\mathrm{d}t} + a_0 x(t) \\ y(t) = b_0 x(t) + b_1 \tfrac{\mathrm{d}x(t)}{dt} + \dots + b_m \tfrac{\mathrm{d}^m x(t)}{\mathrm{d}t^m} \end{aligned} \tag{9.8}$$

Suppose we have

$$\begin{aligned} x_1 &= x \\ x_2 &= \frac{\mathrm{d}x}{\mathrm{d}t} \\ \dots \\ x_n &= \frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}} \end{aligned} \tag{9.9}$$

Such that the state-space representation is given by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_3 \\ &\dots \\ \dot{x_n} &= -\frac{a_0}{a_n}x_1 - \frac{a_1}{a_n}x_2 \dots - \frac{a_{n-1}}{a_n}x_n + \frac{1}{a_n}u \end{aligned} \tag{9.10}$$

and

$$y = b_0 x + b_1 \frac{dx}{dt} + \ldots + b_m \frac{d^m x}{dt^m} \\ = b_0 x_1 + b_1 x_2 + \ldots + b_m x_{m+1} \tag{9.11}$$

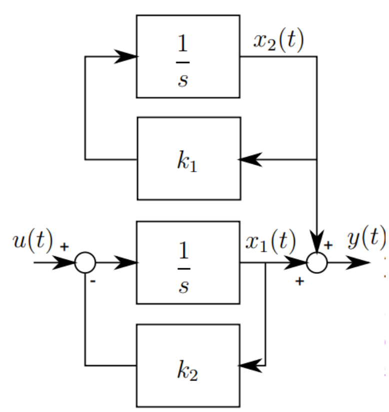
Converting into matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \cdots & \alpha_{nn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta_n \end{pmatrix} u \tag{9.12}$$

$$y = \begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + [d]u \tag{9.13}$$

9.3 Controllability

State variable $x_2(t)$ is not influenced by control input, i.e., the system is not completely state controllable.



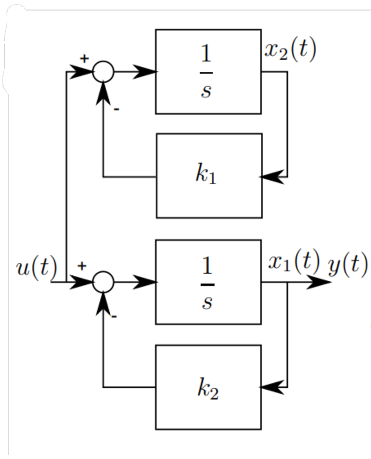
A system is (state) controllable at time t0 if there is a control input u(t) that transfers the system from any initial state $x(t_0)$ to any other state x(t) in a finite interval of time.

Condition for Complete State Controllability: The system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ is completely state controllable if for the controllability matrix C_M if $rank(C_M) = n$.

$$C_M = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \tag{9.14}$$

9.4 Observability

State variable $x_2(t)$ does not end up in the output, so it is not observable.



A system is observable at time t_0 if, knowing the initial state $x(t_0)$ of the system, this state can be determined from the observation of the output y(t) over a finite time interval.

Condition for Complete State Observability: The system $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$ is observable if for the observability matrix C_O if $rank(C_O) = n$.

$$C_O = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} \tag{9.15}$$

Conditions of controllability and observability may govern existence of a complete solution to control system design problem. Solution may not exist if system is not controllable.

System is completely controllable and completely observable if the transfer function has no cancellation. In other words, the canceled transfer function does not carry along all the information characterizing a dynamic system.

9.5 Pole Placement

For open-loop systems $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$, poles are a solution of $|sI - A| = 0$.

If we consider the control signal as a feedback of all state variables, the new location for the poles of the close-loop system is given by

$$\begin{aligned} &u = -Kx \\ &\dot{x} = Ax - BKx = (A - BK)x \\ &|sI - A + BK| = 0 \end{aligned} \tag{9.16}$$

Computing the determinant of the matrix gives

$$\begin{vmatrix} s - a_{11} + b_1k_1 & -a_{12} + b_1k_2 \\ -a_{21} + b_2k_1 & s - a_{22} + b_2k_2 \end{vmatrix} = 0 \tag{9.17}$$

$$(s - a_{11} + b_1k_1)(s - a_{22} + b_2k_2) - (-a_{12} + b_1k_2)(-a_{21} + b_2k_1) = 0 \tag{9.18}$$

$$\begin{aligned} &s^2 + (-a_{11} + b_1k_1 - a_{22} + b_2k_2)\, s + \\ &+ (-a_{11} + b_1k_1)(-a_{22} + b_2k_2) - (-a_{12} + b_1k_2)(-a_{21} + b_{21}k_1) = 0 \end{aligned} \tag{9.19}$$

The poles of the characteristic equation are given

$$\begin{aligned} &(s - p_1)(s - p_2) = 0 \\ &s^2 + (-p_1 - p_2)\, s + p_1p_2 = 0 \end{aligned} \tag{9.20}$$

To determine the poles controller gain the following system of 2 equations is solved

$$\begin{aligned} &(-a_{11} + b_1k_1 - a_{22} + b_2k_2) = (-p_1 - p_2) \\ &(-a_{11} + b_1k_1)(-a_{22} + b_2k_2) - (-a_{12} + b_1k_2)(-a_{21} + b_2k_1) = p_1p_2 \end{aligned} \tag{9.21}$$

9.6 Ackermann‘s Formula

The Ackermann‘s formula is a well-known formula to determine the gain matrix K

- Consider a desired characteristic polynomial for nth-order dynamical system:

$$s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n = 0 \tag{9.22}$$

- Construct a matrix $\alpha_c(A)$:

$$\alpha_c(A) = A^n + \alpha_1 A^{n-1} + \ldots + \alpha_n I \tag{9.23}$$

- Gain matrix K is (Note: you need to feed back all state variables)

$$K = \begin{bmatrix} 0 & \ldots & 1 \end{bmatrix} C_M^{-1} \alpha_c(A) \tag{9.24}$$

9.7 Linear Quadratic Regulator (LQR)

Provides a systematic approach to determine optimal linear control law $u = -Kx$ for the following system:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \quad \boldsymbol{y} = C\boldsymbol{x} \tag{9.25}$$

Such that the following performance index (weighted combination of error and control effort) is minimized. Where matrices Q and R: positive definite, designer's choice, determine the relative importance of error and control effort.

$$J = \int_0^\infty \boldsymbol{x}^\top(t) Q \boldsymbol{x}(t) + \boldsymbol{u}^\top(t) R \boldsymbol{u}(t) \tag{9.26}$$

- Solve the Ricatti equation for P:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{9.27}$$

- Use P to find the feedback gain K:

$$K = -R^{-1}B^T P \tag{9.28}$$

10 Steady-State Response

The steady-state response of a system can be found through the final-value theorem. Final-value theorem is only valid when the $\lim_{t\to\infty} f(t)$ exists, i.e. the final-value theorem is not valid for unstable systems.

Final-Value Theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\{sF(s)\} \tag{10.1}$$

Alternatively, the initial response of the system can also be found with the initial value problem. Initial-value theorem does not give a value of $f(t)$ at exactly $t = 0$, but at a time slightly greater than zero.

Initial-Value Theorem:

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty}\{sF(s)\} \tag{10.2}$$

What is expected from our close-loop system:

- Desired transient response
- Minimum steady state error $e_{ss} = e(t) ast \to \infty$
- Minimum disturbance by $d(t)$ on output $y(t)$
- Stability

There are three types of control system to minimize the steady-state error of a control system:

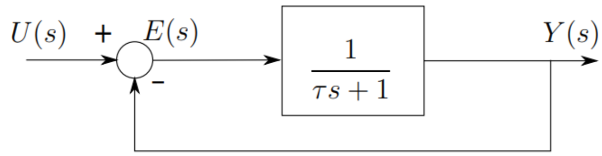
- Proportional Control

- Integral Control
- Derivative Control

10.1 First Order Systems

10.1.1 Uncontrolled System: Unit-Step Response

The steady-state error of the closed-loop system to a unit-step input with $U(S) = 1/s$ is given by



$$
\begin{aligned}
E(S) &= U(S) - Y(S) \\
\frac{E(S)}{U(S)} &= 1 - \frac{Y(S)}{U(S)}
\end{aligned}
\tag{10.3}
$$

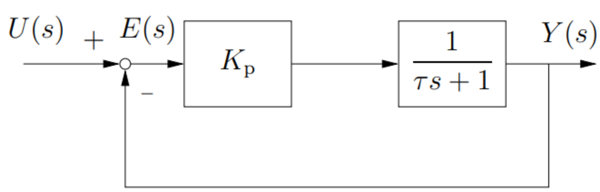
$$
\frac{Y(S)}{U(S)} = G(S) = \frac{\frac{1}{\tau s+1}}{1 + \frac{1}{\tau s+1}} = \frac{1}{\tau s + 2} \tag{10.4}
$$

$$
\frac{E(S)}{U(S)} = 1 - \frac{1}{\tau s + 2} = \frac{\tau s + 1}{\tau s + 2} \tag{10.5}
$$

$$
e_{ss} = \lim_{s \to 0} \{sE(S)\} = \lim_{s \to 0} \left\{ s \cdot U(S) \cdot \frac{\tau s + 1}{\tau s + 2} \right\} = \lim_{s \to 0} \left\{ s \cdot \frac{1}{s} \cdot \frac{\tau s + 1}{\tau s + 2} \right\} = \frac{1}{2} \tag{10.6}
$$

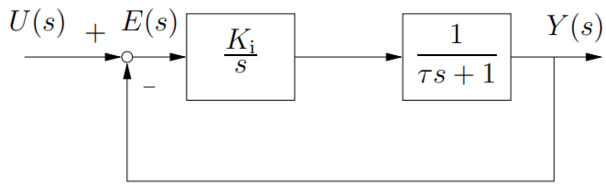
10.1.2 Proportional Control: Unit-step Response

Similarly the steady-state error of a proportional close loop system is given by



$$
e_{ss} = \lim_{s \to 0} \{sE(S)\} = \frac{1}{1 + K_p} \tag{10.7}
$$

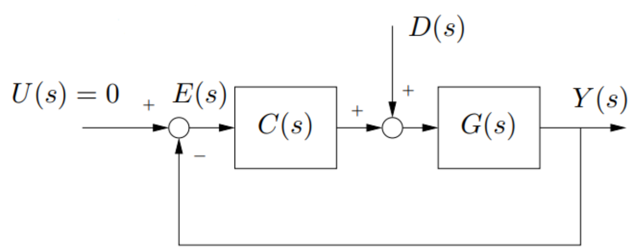
10.1.3 Integral Control: Unit Step Response



$$e_{ss} = \lim_{s \to 0} \{sE(S)\} = 0 \tag{10.8}$$

10.2 Second Order System

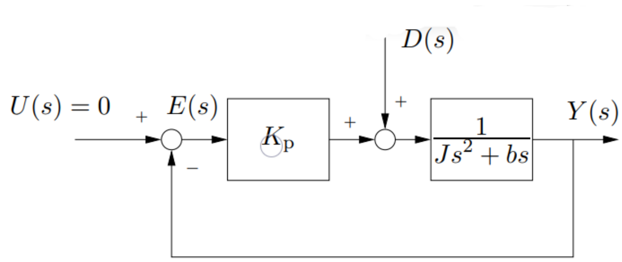
For the second order system relevant transfer functions are given by



$$\frac{Y(s)}{D(s)} = \frac{G(s)}{1 + C(s)G(s)} \tag{10.9}$$

$$\frac{E(s)}{D(s)} = -\frac{G(s)}{1 + C(s)G(s)} \tag{10.10}$$

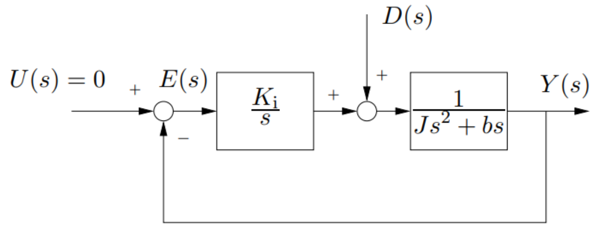
10.2.1 Proportional Control: Unit-Step Response



$$\frac{E(S)}{D(S)} = -\frac{\frac{1}{Js^2+bs}}{1 + \frac{K_p}{Js^2+bs}} = -\frac{1}{Js^2 + bs + K_p} D(S) \tag{10.11}$$

$$e_{ss} = \lim_{s\to 0}\{sE(S)\} = \lim_{s\to 0}\left\{s \cdot \frac{M_d}{s} \cdot -\frac{1}{Js^2+bs+K_p}\right\} = -\frac{M_d}{K_p} \tag{10.12}$$

10.2.2 Integral Control: Unit-Step Response

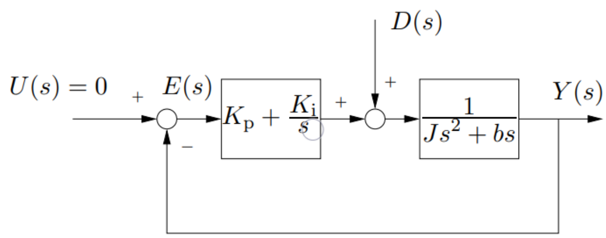


The characteristic equation of $Y(S)/D(S)$ is given by

$$\frac{Y(S)}{D(S)} = \frac{s}{Js^3+bs^2+K_i} \rightarrow Js^3+bs^2+K_i = 0 \tag{10.13}$$

There are roots with positive real parts, therefore the system becomes unstable

10.2.3 Proportional + Integral Control: Unit-Step Response



$$e_{ss} = \lim_{s\to 0}\{sE(S)\} = 0 \tag{10.14}$$

10.3 Recap: Steady State Response

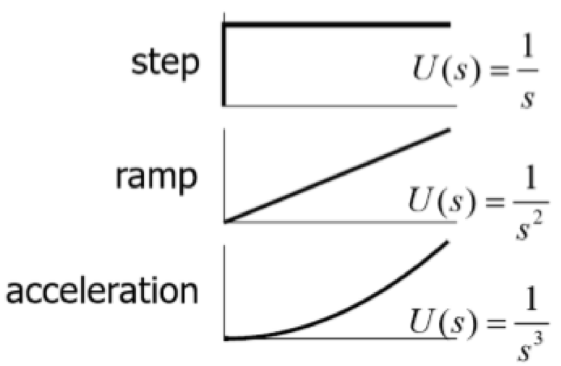
- Any physical control system suffers steady-state error in response to certain inputs.
- A system may have no steady-state error to a step input, but exhibit non-zero steady-state error to ramp input.
- Whether a system exhibits non-zero steady-state error to certain types of inputs depends on the type of the open-loop transfer function.

- Generally type N systems are given by

$$G(s) = \frac{1}{s^N}\left(\frac{b_0 + b_1 s + b_2 s^2 + \dots}{a_0 + a_1 s + a_2 s^2 + \dots}\right) \tag{10.15}$$

- A system of type N has a pole of multiplicity N at the origin
- Classification of systems‘ type is different from systems‘ order
- As the type increase the accuracy improves, but it worsens stability
- A compromise between steady-state accuracy and stability is needed

A type 0 system has a nonzero steady state error.
A type 1 system has a zero steady state position error.
A type 2 system has a zero steady state position and velocity error.



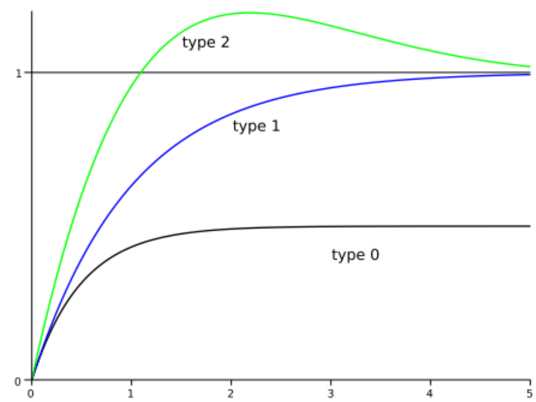


Figure 10.1: Unit-Step Response of Type 0,1,2 systems

10.4 PID Control

- Adding an integral controller can remove steady-state error
- Adding a differential controller can remove oscillations
- A PID controller involves three parameters K_p, K_i, K_d that should be tuned: the process of selecting the controller parameters to meet given performance specifications is called controller tuning

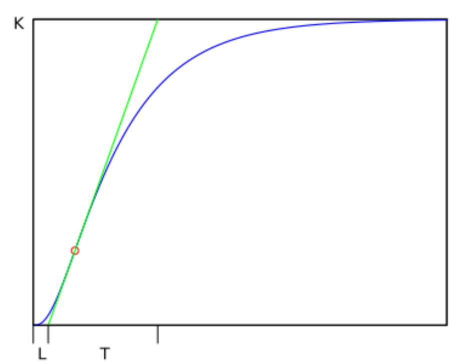
10.4.1 PID tuning: Ziegler-Nichols method

Transfer function corresponding to a PID controller is defined as

$$G_{\mathrm{PID}}(s) = K_{\mathrm{p}} + \frac{K_{\mathrm{i}}}{s} + K_{\mathrm{d}}s = K_{\mathrm{p}}\left(1 + \frac{1}{T_i s} + T_d s\right) \tag{10.16}$$

Ziegler-Nichols tuning includes 2 methods

- **First Method:** Tuning based on experimental unit-step response
  - Obtain experimentally response of system to unit-step input
  - If response is S-Shaped
    - Graphically determine delay time L and time constant T
    - Determine tangent line is drawn at inflection point to intersect with y(t) = 0 and y(t) = k



Controller	K_p	T_i	T_d
P	T/L	NA	NA
PI	0.9T/L	L/0.3	NA
PID	1.2T/L	2L	0.5L

- Second Method: Critical gain method
  1. First consider only a proportional controller
  2. Increase K_p from 0 until the output exhibit sustained oscillations
  3. The corresponding K_p is called critical gain K_{cr}
  4. Corresponding period of oscillations is called critical period P_{cr}

For some systems one or both tuning methods may fail. Tuning via ziegler-Nichols methods provides an initial set-up. Manual fine-tuning while running the controller may be needed.

Controller	K_p	T_i	T_d
P	$0.5K_{cr}$	NA	NA
PI	$0.45K_{cr}$	$P_{cr}/1.2$	NA
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

11 | Frequency-Response Methods

The frequency response is the steady state response of a system to a sinusoidal input. The input‘s frequency is changed over a certain range to study the response. In frequency-response methods data from a system‘s measurements may be used without deriving mathematical models.

Steady-state output of a transfer function system to a sinusoidal input: $u(t) = \overline{u}\sin\omega t$ can be obtained by directly replacing s by jw where w is the frequency input of the sine wave in systems transfer function.

$$y_{\mathrm{ss}} = \overline{u}|G(j\omega)|\sin(\omega t + LG(j\omega)) \tag{11.1}$$

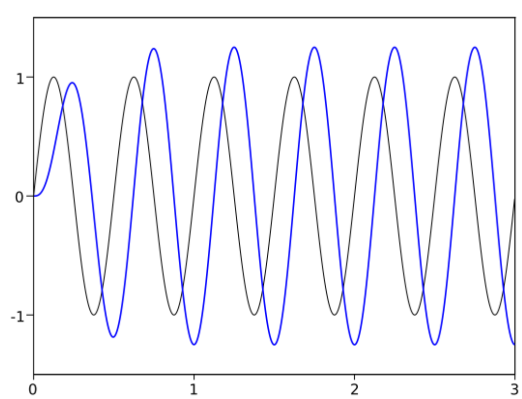
$G(jw)$ is called the sinusoidal transfer function.

The output of a linear time-invariant system to a sinusoidal input: $u(t) = \overline{u}\sin\omega t$ is sinusoidal of same frequency and different amplitude and phase.

Amplitude: $\overline{u} \cdot |G(jw|$

Phase: $\angle G(j\omega)$

Response of $\omega_n^2/\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right), \zeta = 0.4, \omega_n = 4\pi$ to a 2Hz sine wave.



11.1 Proof

Given a sine input in the from of $u(t) = \overline{u}\sin\omega t$. The output of the system is given by:

$$\begin{aligned} Y(s) &= G(s)U(s) = G(s)\frac{\overline{u}\omega}{s^2+w^2} \\ Y(S) &= \frac{\overline{u}\omega\,(s-z_1)\,(s-z_2)\ldots(s-z_m)}{(s+j\omega)(s-j\omega)\,(s-p_1)\,(s-p_1)\ldots(s-p_n)} \end{aligned} \tag{11.2}$$

Using partial fraction expansion if we assume distinct poles

$$Y(s) = \frac{a}{s+j\omega} + \frac{\overline{a}}{s-j\omega} + \frac{b_1}{s-p_1} + \frac{b_2}{s-p_2} + \ldots + \frac{b_n}{s-p_n} \tag{11.3}$$

Performing the inverse Laplace transform to obtain the time-domain response

$$y(t) = ae^{-j\omega t} + \overline{a}e^{j\omega t} + b_1e^{p_1t} + b_2e^{p_2t} + \ldots + b_ne^{p_nt} \tag{11.4}$$

Stability of the system implies negative real parts of the poles $p_1, p_2, p_3, etc.$. such that when $t \to \infty$, $e^{tp_{1,2,\ldots}} \to 0$. The steady state response of the system is given by:

$$y_{\text{ss}} = ae^{-j\omega t} + \overline{a}e^{j\omega t} \tag{11.5}$$

Equating Equation 11.2 to Equation 11.3 and multiplying both sides by $(s + jw)$ and replacing s by $= jw$ gives

$$\frac{a}{s+j\omega} + \frac{\overline{a}}{s-j\omega} + \frac{b_1}{s-p_1} + \frac{b_2}{s-p_2} + \ldots + \frac{b_n}{s-p_n} = G(s)\frac{\overline{u}\omega}{s^2+w^2} \tag{11.6}$$

$$a = G(s)\frac{\overline{u}\omega}{s^2+\omega^2}(s+j\omega)\bigg|_{s=-j\omega} = G(s)\frac{\overline{u}\omega}{s-jw}\bigg|_{s=-j\omega} = -\frac{\overline{u}G(-j\omega)}{2j} \tag{11.7}$$

Similarly

$$\overline{a} = \frac{\overline{u}G(j\omega)}{2j} \tag{11.8}$$

Using the polar representation of $G(j\omega)$

$$G(j\omega) = |G(j\omega)|e^{j\phi} \quad \text{with } \phi = \angle G(j\omega) \tag{11.9}$$

Since $G(j\omega)$ and $G(-j\omega)$ are complex conjugates $|G(j\omega)| = |G(-j\omega)|$

Rewriting the steady state response gives:

$$\begin{aligned}
y_{ss} &= ae^{-j\omega t} + \overline{a}e^{j\omega t} \\
&= \overline{u}G(j\omega)\frac{e^{j\omega t}}{2j} - \overline{u}G(-j\omega)\frac{e^{-j\omega t}}{2j} \\
&= \overline{u}|G(j\omega)|\frac{e^{j\phi}e^{j\omega t}}{2j} - \overline{u}|G(j\omega)|\frac{e^{-j\phi}e^{-j\omega t}}{2j} \\
&= \overline{u}|G(j\omega)|\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} \\
&= \underbrace{\overline{u}|G(j\omega)|}_{\text{amplitude}}\sin(\omega t + \underbrace{\phi}_{\text{phase shift}})
\end{aligned} \tag{11.10}$$

Positive Phase Angle: Phase Lead $\rightarrow$ Lead Network
Negative Phase Angle: Phase Lag $\rightarrow$ Lag Network

11.1.1 Example

For the the transfer function $G(S) = K/(\tau ts + 1)$ find the steady-state response to the sinusoidal input $u(t) = \overline{u}sin(\omega t)$ and determine whether this is a lead or a lag network.

First we determine the transfer function of the system

$$\begin{aligned}
G(j\omega) &= \frac{K}{\tau j\omega + 1} \\
|G(j\omega) &= \frac{K}{\sqrt{1+\tau^2\omega^2}} \\
\angle G(j\omega) &= -\arctan(\tau\omega)
\end{aligned} \tag{11.11}$$

The steady state response of the system is given by

$$y_{ss} = \overline{u}|G(j\omega)|\sin(\omega t + \phi) = \frac{\overline{u}K}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t - \arctan\tau\omega) \tag{11.12}$$

Small ω: Amplitude $\approx \overline{u}K$, Phase Shift is small
Large ω: Amplitude is small, Phase Shift approaches 90 deg

The phase shift is always negative $\rightarrow$ phase lag network

11.2 Why Sine Input?

Any periodic function can be decomposed into a sum of sinusoidal components. If we know the response to each sine frequency and the system is linear, we know the response to any periodic signal.

The Fourier series expansion of an arbitrary function is given by

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\omega_n t\right) + b_n \sin\left(\omega_n t\right)\right] \tag{11.13}$$

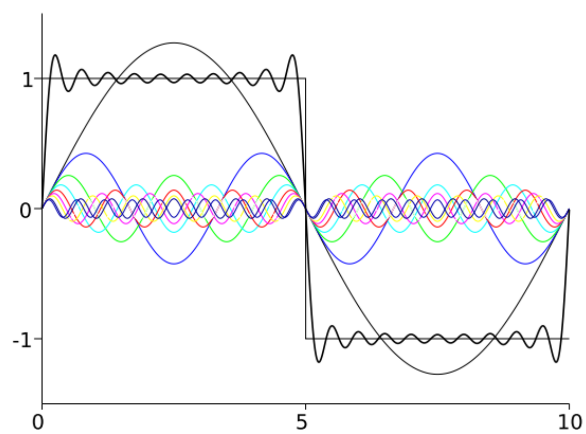


Figure 11.1: Optimal approximation of square wave with 10 sin Functions

11.3 Bode Plots

A sine wave input to an LTI system will always generate a sine wave output of the same frequency

Two Possible Changes:

- Amplitude → corresponds to magnitude of sinusoidal transfer function
- Phase Shift → corresponds to phase angle of sinusoidal transfer function

A Bode diagram consist of 2 graphs

- Magnitude of sinusoidal transfer function in dB, i.e. $20log|G(j\omega)|$ vs $log(\omega)$

- Phase Angle of sinusoidal transfer function in degrees, i.e. $\angle G(j\omega)$ vs $log(\omega)$

Main advantage of Bode plots → multiplication of magnitudes can be converted into addition

11.3.1 Gain K

- A gain K greater than unity has a positive value in dB. A gain K smaller than unity has a negative value in dB.
- The magnitude graph for a constant gain K is a horizontal straight line at $20log(K)dB$
- The phase angle graph for a constant gain K is zero
- Varying the gain K raises or lowers the magnitude curve

11.3.2 Integral & Derivative

- The logarithmic magnitude of $j\omega$ has a slope of $-20dB/decade$ on the magnitude plot (decade is an increase by a factor of 10 of the frequency) it is given by

$$20\log|j\omega| = 20\log\omega \tag{11.14}$$

- The phase angle of $j\omega$ is $-90\,\deg$

- The logarithmic magnitude of $1/(j\omega)$ is given by it has a slope of $20dB/decade$ on the magnitude plot

$$20\log|1/j\omega| = -20\log\omega \tag{11.15}$$

- The phase angle of $1/(j\omega)$ is $90\,\deg$

In Bode plot for reciprocal factors log-magnitude and phase angle curves change in sign only

11.3.3 First-Order Factors

- Magnitude of $1/(1+j\omega\tau)$ in dB is given by

$$20\log 1/(1+j\omega\tau) = -20\log\sqrt{1+\omega^2\tau^2} \quad \text{dB} \tag{11.16}$$

  - $\omega\tau \ll 1 : -20\log\sqrt{1+\omega^2\tau^2} \approx 0$
  - $\omega\tau \gg 1 : -20\log\sqrt{1+\omega^2\tau^2} \approx -20\log(\omega\tau)$

- Starting from $\omega\tau = 0$ the magnitude curve is straight and equal to 0 dB. When $\omega\tau$ is large the magnitude curve decreases by 20 *dB/decade* in a straight line. The critical frequency $\omega_C = 1/\tau \rightarrow \tau$ determines the point when the magnitude curve changes behaviour.
- The phase angle of $1/(1 + j\omega\tau)$ is $\phi = -\arctan(\omega\tau)$
  - $\omega\tau = 0 \Rightarrow \phi = 0\,\text{deg}$
  - $\omega\tau = 1 \Rightarrow \phi = -45\,\text{deg}$
  - $\omega\tau \rightarrow \infty \Rightarrow \phi = -90\,\text{deg}$

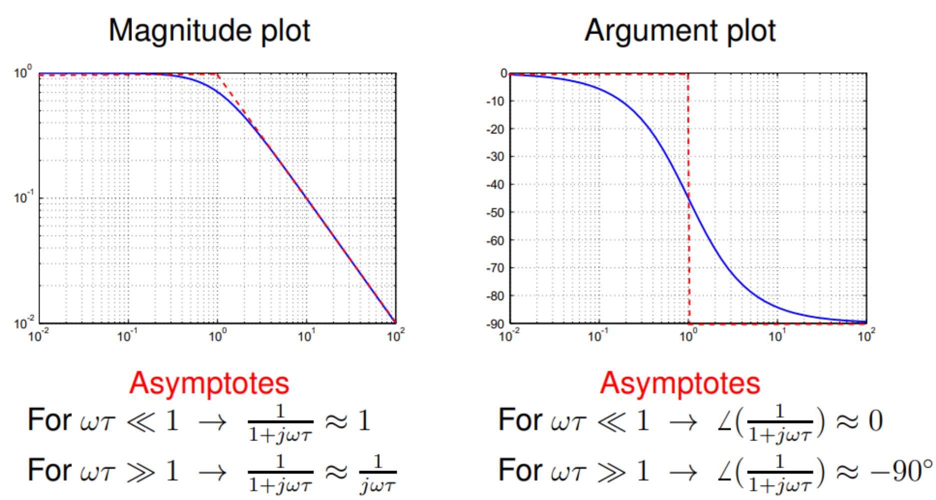


Figure 11.2: Bode Plots of $1/(1 + j\omega\tau)$

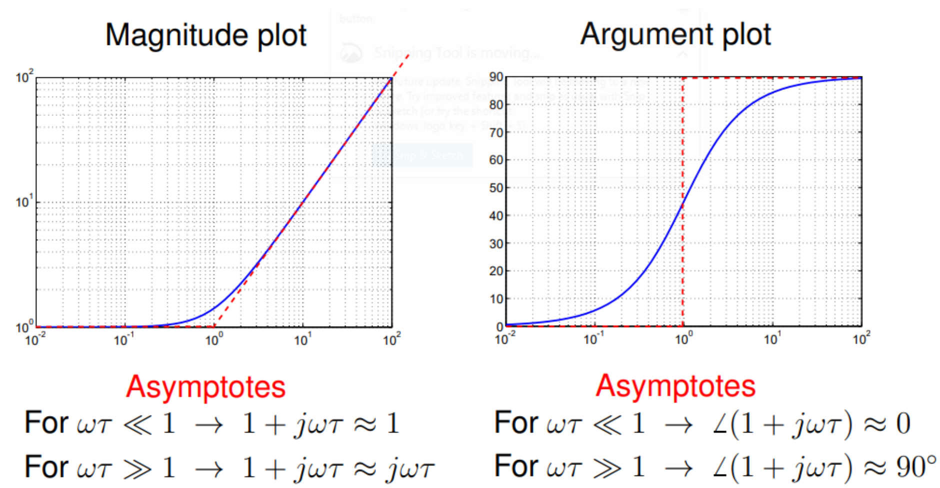


Figure 11.3: Bode Plots of $1 + j\omega\tau$

11.3.4 Second Order Factors

- Magnitude of $G(j\omega) = \frac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2}$ in dB is given by

$$20log(|G(j\omega)|) = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2} \tag{11.17}$$

  - If $\omega/\omega_n \ll 1 \Rightarrow Magnitude \approx 0$, there is a low-frequency asymptote at 0 dB
  - If $\omega/\omega_n \gg 1 \Rightarrow Magnitude \approx -20\log(\omega^2/\omega_n^2) = -40\log(\omega/\omega_n)$, high-frequency asymptote is a straight line of slope -40 $\frac{dB}{decade}$

- Corner Frequency is equal to the natural frequency of the system $\omega_c = \omega_n$. Near Corner Frequency a resonance peak occurs.

  - Damping ratio ζ determines the magnitude of the resonance peak
  - Resonant frequency ω_r (for which $|G(j\omega)|$ is a maximum is given by

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \quad \text{for} \quad 0 \leq \zeta \leq 0.707 \tag{11.18}$$

  - Magnitude of the resonance peak M_r is given by

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \text{for} \quad 0 \leq \zeta \leq 0.707 \tag{11.19}$$

- The phase angle of the second-order factor is given by:

$$\phi = -\arctan\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{11.20}$$

  - $\omega = 0 \Rightarrow \phi = 0\,\text{deg}$
  - $\omega = \omega_c = \omega_n \Rightarrow \phi = -90\,\text{deg}$ (Independent of ζ)
  - $\omega \to \infty \Rightarrow \phi = -180\,\text{deg}$

If ζ increases the resonant peak at ω_n decreases
If ω_n increases the bandwidth increases

11.3.5 Bode Plots from Transfer Functions

Adding logarithms of gains corresponds to multiplying them, sketching curves for individual factors and adding them gives Bode plot of original transfer function

We can use the bode plots of basic factors of transfer functions: gain, integral, derivative, 1st and 2nd order factors to construct a composite logarithmic plot for any general form of the form $G_1(S)G_2(S)G_3(S)...$

$$G(S) = \frac{8s+4}{s(s+3)} = \frac{4}{3} \cdot \frac{1}{s} \cdot \frac{1+2s}{1+(1/3)s} \tag{11.21}$$

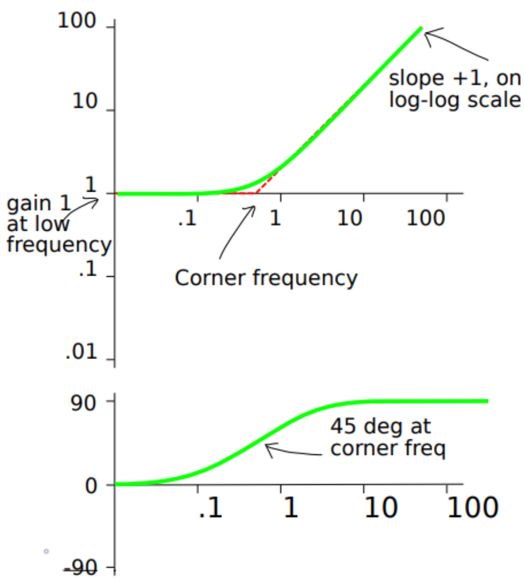


Figure 11.4: Bode Plot of $G(j\omega) = 1 + 2j\omega$

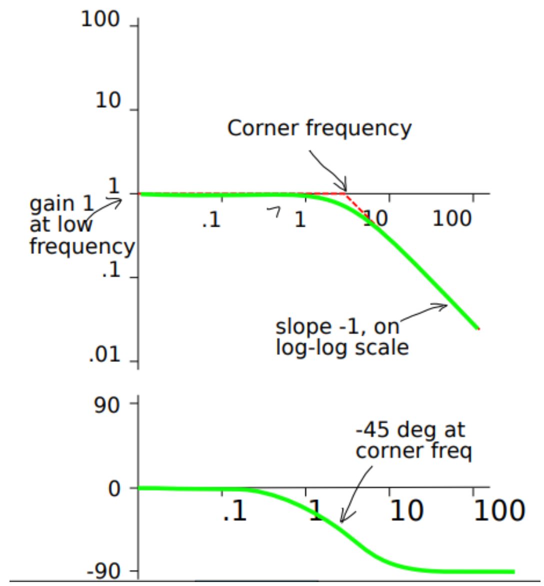


Figure 11.5: Bode Plot of $G(j\omega) = \frac{1}{1+(1/3)j\omega}$

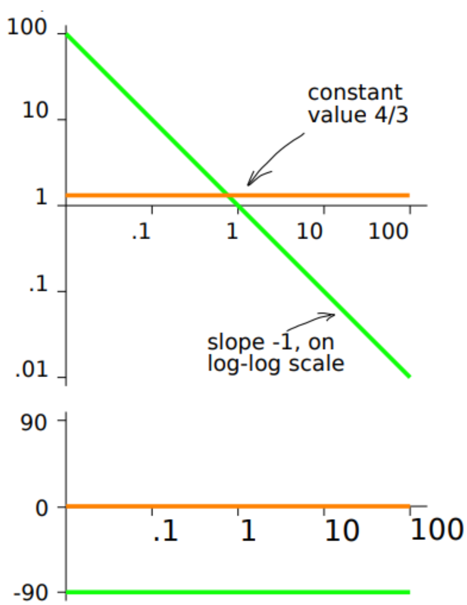


Figure 11.6: Bode Plot of $G(j\omega) = 1/j\omega$ and $G(j\omega) = \frac{4}{3}$

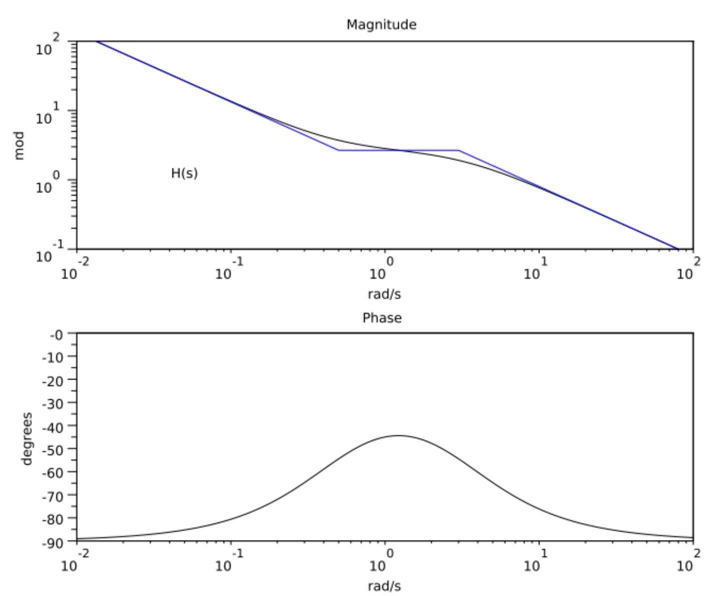


Figure 11.7: Bode Plot of $G(j\omega) = \frac{8s+4}{s(s+3)}$

Terminology

Lead: $1 + j\tau\omega$ (1st order) or $1 + 2\zeta j\omega/\omega_n + (j\omega)^2/\omega_n^2$ (2nd order) term in numerator if $\tau > 0$ or $\zeta > 0$, the phase contribution on output sine is positive, it has a lead over input sine

Lag: $1 + j\tau\omega$ (1st order) or $1 + 2\zeta j\omega/\omega_n + (j\omega)^2/\omega_n^2$ (2nd order) term in denominator if $\tau > 0$ or $\zeta > 0$, the phase contribution on output sine is negative, it has a lag over input sine

Derivative Factor: $j\omega$ in numerator gives a $90\,\mathrm{deg}$ phase lead

Integral Factor: $j\omega$ in denominator gives a $90\,\mathrm{deg}$ phase lag

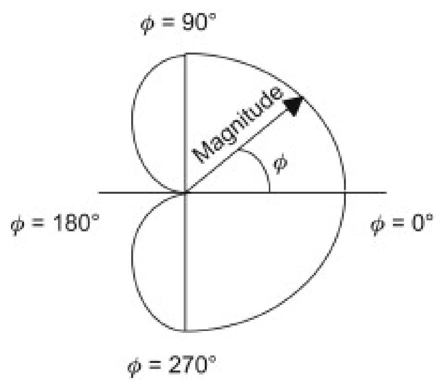
Gain: Constant factor K has no phase effect, unless $K < 0$ then the system has $180\,\mathrm{deg}$ phase change

12 Polar Plots & Root Locus

12.1 Polar/Nyquist Plots

Nyquist/Polar plots are an alternative way of showing the frequency response characteristics (more specifically sinusoidal transfer function $G(j\omega)$) of a system. Polar plot sketches magnitude versus phase angle of $G(j\omega)$) as ω varies from 0 to ∞.

Note: Positive angle is measured counter-clockwise from positive real axis



- **Adv:** Depicts frequency response characteristics of system over entire frequency spectrum in a single plot
- **Dis:** Does not explicitly indicate contributions of each factor of open-loop transfer function

Nyquist/Polar plots for Common Factors:

- **Integral:** Polar plot of integral factor is negative imaginary axis
- **Derivative:** Polar plot of derivative factor is positive imaginary axis
- **1st Order Factors**

- Polar plot of $1/(1+j\omega\tau)$ is a semi-circle under real axis with radius 0.5 centered at (0.5,0)
- Polar plot of $1+j\omega\tau$ is upper half of straight line passing through point (1,0) parallel to imaginary axis

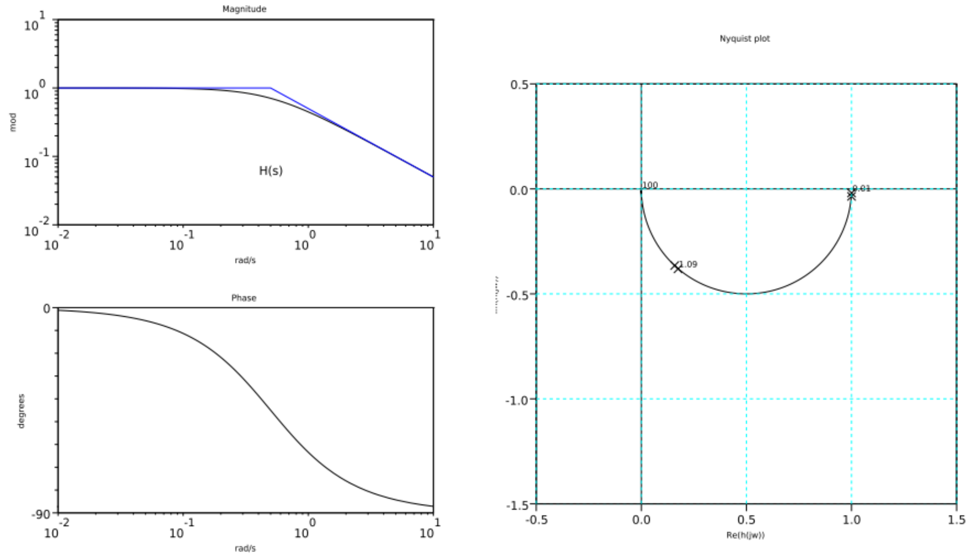


Figure 12.1: Bode Plot vs Polar Plot of $1/(1+j\omega\tau)$

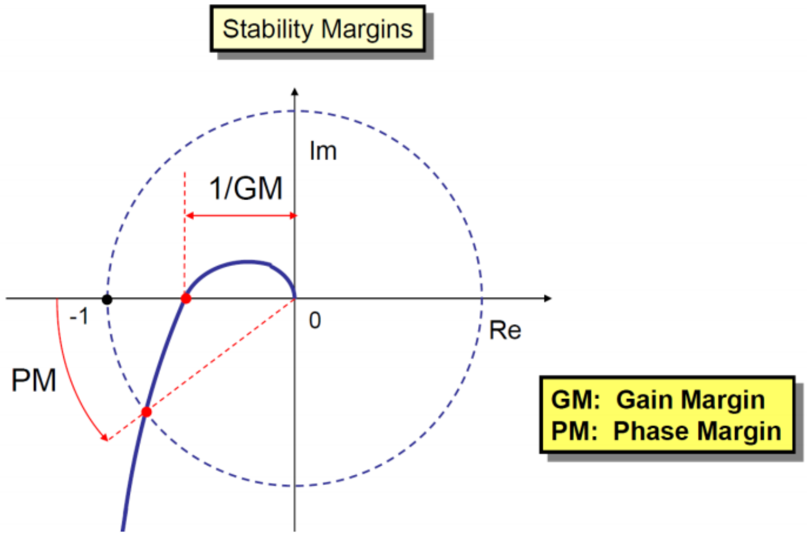
12.1.1 Stability Margin

The point -1 complex plane is the critical point for studying the stability of linear systems in closed loop Corresponds to a neutrally stable system, the roots of the closed loop are in the imaginary axis at $|G(j\omega)| = 1$ and $\angle G(j\omega) = 180\,\text{deg}$.

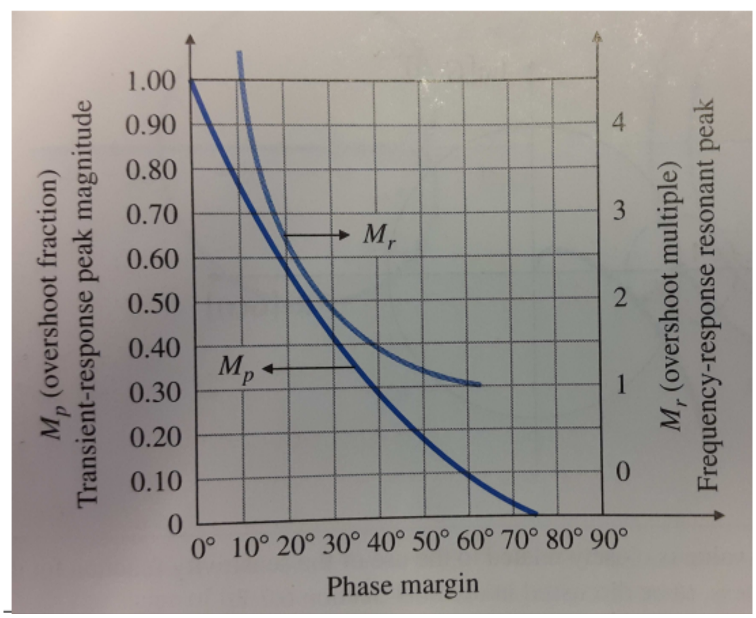
We can measure system's stability margin with

- **Gain Margin (GM)** is the factor by which gain K can be raised before instability occurs
  - From Bode Plot: Vertical distance of $KG(j\omega)$ curve corresponding to $\angle G(j\omega) = \pm 180\,\text{deg}$ with line $KG(j\omega) = 1$. If $GM < 0dB \Rightarrow$ instability
  - From Polar Plot: Reciprocal of the horizontal distance between first intersection with real axis and imaginary axis. If $GM < 1 \Rightarrow$ instability
- **Phase Margin (PM)**
  - From Bode Plot: Amount by which $\angle G(j\omega)$ exceeds $-180\,\text{deg}$ when $|KG(j\omega)| = 1$.
  - From Polar Plot: Angle between negative real axis measure counter-clockwise of intersection of polar plot with a circle with radius 1 and centered at the origin.
- If $PM > 0 \Rightarrow$ system is stable, if $PM = 0 \Rightarrow$ system is neutrally stable and if $PM < 0 \Rightarrow$ system is unstable

- PM is commonly used to specify control system performance, e.g. for 2nd-order systems for $PM < 70\,\text{deg}$ we have $\zeta \approx PM/100$



The overshoot of the closed-loop step response of a system can be inferred from the PM of the system. As the PM increase to 90 deg the overshoot of the system decreases.



12.2 Root Locus

Summary of Rules

- **Rule 1:** Root Locus is symmetrical about the real axis
- **Rule 2:**
  - n (# poles) Branches of root locus start at poles
  - m (# zeros) Branches of root locus end at zeros
- **Rule 3:** Loci on real axis are to the left of an odd number of poles and zeros
- **Rule 4:** $n - m$ # of root loci branches are asymptotic to lines of angles

$$\Phi = \frac{180°(2l+1)}{n-m} \; with \; l = 1, 2, ...$$

that radiate from point

$$\sigma^{\text{int}} = \frac{\sum_{j=1}^{n} \operatorname{Re}(p_j) - \sum_{i=1}^{m} \operatorname{Re}(z_i)}{n-m}$$

Angle Condition

$$\angle\left(\frac{KN(s)}{D(s)}\right) = 180 \deg(2l+1) \; with \; l = 0, 1, 2, ... \tag{12.1}$$

Magnitude Condition

$$\left|\frac{KN(s)}{D(s)}\right| = 1 \tag{12.2}$$

The magnitude condition is used to determine the required gain K for a desired pole

$$Pole \; p_d : K = \frac{|D(p_d)|}{|N(p_d)|} \tag{12.3}$$

12.2.1 Graphical Interpretation of Magnitude Condition

Suppose we have $K\frac{N(s)}{D(s)} = \frac{K}{(s-p_1)(s-p_2)(s-p_3)}$

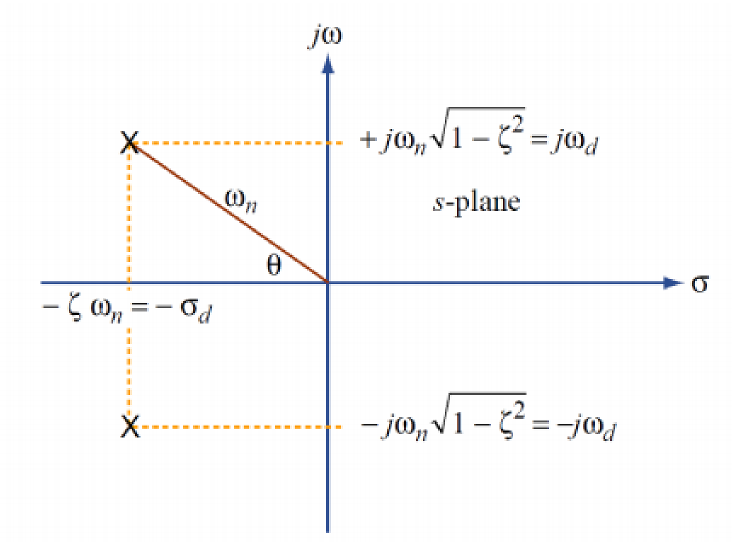
We want our close-loop system to have a pole at $s = p_d$.

The required gain is $K = |p_d - p_1||p_d - p_2||p_d - p_3|$.

Graphically, we can compute gain K to place a pole at pd by measuring lengths of all vectors that connect poles of open-loop system to desired pole pd, and then multiplying them.

12.2.2 Damping Ratio & Natural Frequency

For an under-damped 2nd order system analyzed in chapter 5, the system has poles at: $p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}j$ with a root locus shown below.



Therefore $|p_1| = |p_2| = \omega_n$ and $\theta = arctan(\sqrt{1-\zeta^2/\zeta})$

Equivalently $sin(\theta) = \sqrt{1-\zeta^2}$ and $cos(\theta) = \zeta$

To determine the pole locations of an under-damped second order system in the s-plane:

- Draw straight lines from origin in negative real part for which $cos(\theta) = \zeta$
- Find a point on the line that is a distance ω_n from the origin
- The other pole is the conjugate (reflection along the real-axis)

13 | Compensator Design

In control system design if more than gain adjustment is needed compensators are designed

Effect of Adding Zeros (derivative control)

- Pulling root locus to left: can make system more stable (steady-state effect)
- Speeds up settling time (transient response effect)

Effect of Adding Poles (integral control)

- Pulling root locus to the right: may lower system stability
- Slows down settling time

The basic principles of controller design with frequency response:

- The open loop of the system should have a large gain (preferably infinite) for $\omega \to 0$ such that $G(j\omega)/(1 + G(j\omega) \approx 1$ for $\omega \to 0$
- The open-loop system should have sufficient gain and phase margin to prevent oscillations of the closed-loop system
- Try to make the open-loop system look like $1/s$ (-20 dB/decade) at least in the region around cross-over frequency.

13.1 Compensator Controllers

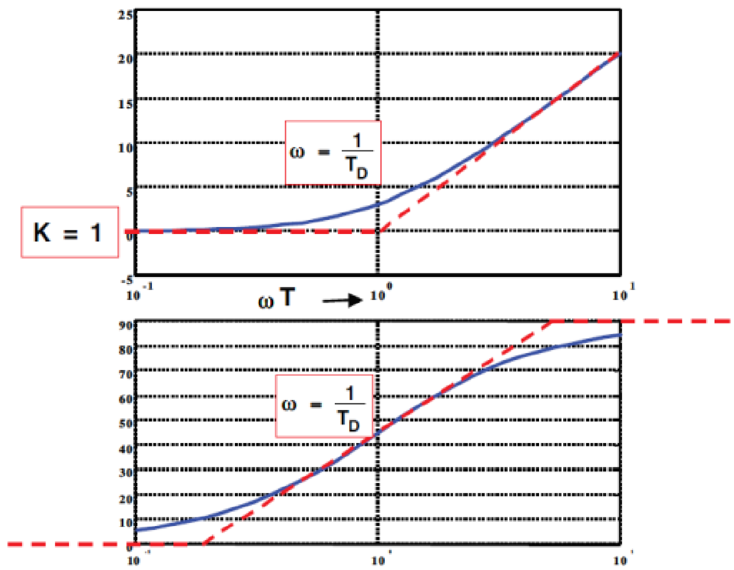
- **Lead filter** – approximated PD function; speeds up response; lowers rise time; decreases overshoot; provides phase lead near cross-over frequency
- **Lag filter** – approximates PI function; improves steady-state accuracy; provides additional gain at low frequencies

- **Lag-Lead** – combination of the above two compensators

$$C(S) = K\frac{s-z}{s-p} \tag{13.1}$$

z < p: lead compensation
z > p: lag compensation

The bode plot for the phase-leading PD controller: $D(j\omega) = K(1 + Tj\omega)$ is shown below. This PD controller has infinite gain at higher frequencies.



13.1.1 Lead-Compensator

Consider a lead-compensator with $z = -1/T$ and $p = -1/(\alpha T)$:

$$C(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1}, \quad 0 < \alpha < 1 \tag{13.2}$$

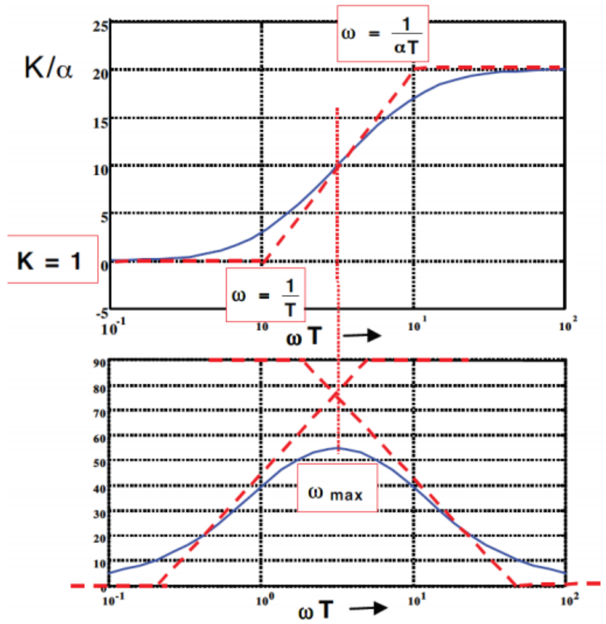
Since $\alpha = 0$ is to the right of pole in the s-plane, for practical purposes $\alpha > 0.05$, implying a maximum phase shift of $\phi \approx 60\,\text{deg}$. The maximum phase shift for a given α from the Nyquist plot is

$$\phi_m = \arcsin\frac{1-\alpha}{1+\alpha} \tag{13.3}$$

When designing a lead-compensator α may be tuned to achieve a specific phase margin

$$\alpha = \frac{1 - \sin\phi_m}{1 + \sin\phi_m} \tag{13.4}$$

The lead-compensator prevents the system from reaching infinite controller gains at high frequencies.



The frequency ω_m corresponding to the maximum phase shift is

$$\log \omega_m = 0.5 \left(\log \frac{1}{T} + \log \frac{1}{\alpha T} \right) \tag{13.5}$$

$$\omega_m = \frac{1}{\sqrt{\alpha} T} \tag{13.6}$$

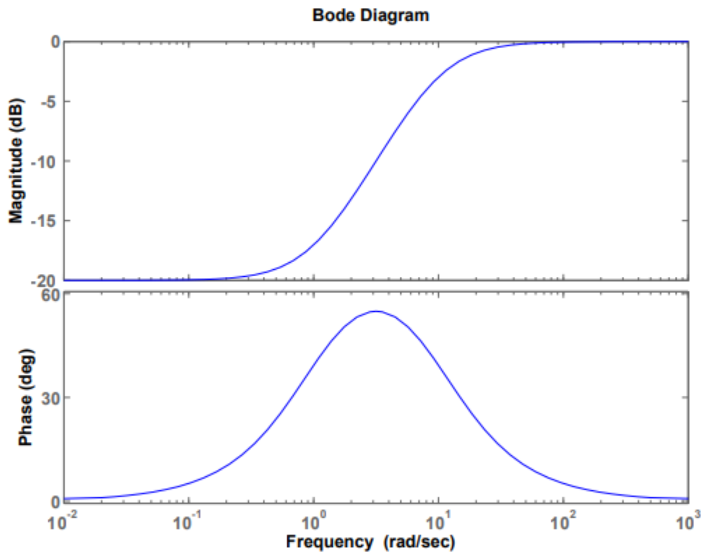
The bode plot for $\alpha = 0.1$ is shown below. A lead compensator is basically a high-pass filter, high frequencies are passed but low frequencies are attenuated.

13.1.2 Lag-Compensator

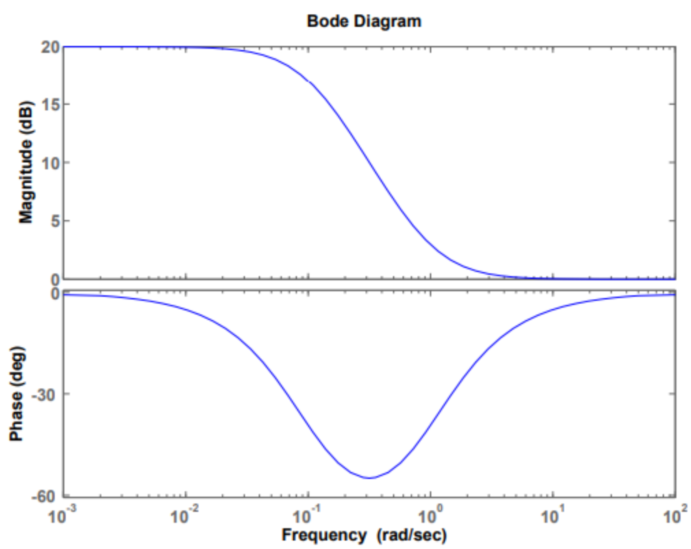
Consider a lag compensator with $z = -1/T$ and $p = -1/(\beta T)$:

$$C(s) = K_c \beta \frac{Ts + 1}{\beta T s + 1}, \quad \beta > 1 \tag{13.7}$$

We require $\beta > 1$ such that the pole is located to the right of zero in s-plane with corner frequencies $1/T$ and $1/(\beta T)$.



The bode plot of the lag compensator with $\beta = 10$ is shown below. Lag compensator is basically a low-pass filter, low frequencies are passed but high frequencies are attenuated.



Lead Compensator	**Lag Compensator**
Achieves desired results through phase lead contribution	Improves steady-state accuracy and achieves desired results through attenuation property at high frequencies
Lead compensator improves transient response, but is sensitive to high-frequency noise signals	Low gain, low bandwidth, slower, while it can attenuate any high-frequency noise

13.2 Special Filters

Notch Filter - Filters out signals at a specific frequency

$$H_{no} = \frac{s^2 + 2\zeta_1 s\omega_n + \omega_n^2}{s^2 + 2\zeta_2 s\omega_n + \omega_n^2} \tag{13.8}$$

Normally, ζ_1 is very small (it can be 0 to totally suppress signals at the specific frequency) and ζ_2 is large.

High Pass Filter - Filters out low-frequency signals:

$$H_{hp} = \frac{\tau s}{1 + \tau s} \tag{13.9}$$

14 Matlab

Relevant Matlab functions for linearizing and trimming systems in Matlab:

- Trimming: trim
- Linearizing: linmod

Defining Transfer Functions in Matlab:

help tf

The best way is simply defining the Laplace variable 's', and starting to type: $s = tf('s')hq = (34 + 24 * s)/(13 + 4 * s + s^2); oruse : hq = tf([2434], [1413])$

Calculating Response of System in Matlab: help lsim help step help impulse