Math 189 HW
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I was aided by the HW Solutions for parts of the HW

1. Let $y = Ax + b$ be a random vector. Show that expectation is linear:

$$E[y] = E[Ax+b] = A\,E[x] + b.$$

Also show;

$$\text{cov}[y] = \text{cov}[Ax+b] = A\,\text{cov}[x]\,A^T = A\Sigma A^T$$

a. Given $y = Ax + b$

$$E[y] = E[Ax+b]$$

$$E[Ax+b] = \int Ax + b \cdot p(x)\,dx$$

$$= A\,E[x] + E[b]$$

given A is a matrix

$$= \overset{\text{matrix}}{A} \int x\, p(x)\,dx + \overset{\text{constant}}{b} \int p(x)\,dx$$

$$= \boxed{A\,E[x] + b}$$

b. Show

$$\text{cov}[y] = \text{cov}[Ax+b] = A\,\text{cov}[x]\,A^T = A\Sigma A^T$$

By def, $\text{cov}[x] = E[(x - E(x))(x - E[x])^T]$

$$\text{cov}[y] = E[(y - E[y])(y - E[y])^T]$$

Substituting $Ax+b$ for y,

$$E[(Ax+b - E[Ax+b])((Ax+b) - E[Ax+b])^T]$$

From part a, we know $E[Ax+b] = A\,E[x] + b$, so

$$\to E[(Ax+b - A\,E[x] - b])((Ax+b) - A\,E[x] - b)^T]$$

$$= E[(Ax - A\,E[x])(Ax - A\,E[x])^T$$

$$= E[A(x - E[x])(A^T(x - E[x])^T$$

$$= A\,E[(x - E[x])\,A^T(x - E[x])^T\,A$$

$$= A\,\text{cov}[x]\,A^T$$

$$= A\Sigma A^T$$

2. Given $D = \{(x,y)\} = \{(0,1), (2,3), (3,6), (4,8)\}$

a. Find least squares estimate $y = \theta^T x$ by hand using Cramer's Rule.

$y = mx + b$

$\vec{y} = \begin{bmatrix} \vdots \end{bmatrix}$

$$\begin{array}{cc} x_i & y_i \end{array}$$
$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 3 & 6 \\ 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} m0 \\ m2 \\ m3 \\ m4 \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \\ b \end{bmatrix}$$

$$x = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} m \\ b \end{bmatrix} = \vec{y}$$

$4 \times 2 \quad 2 \times 1 \to 4 \times 1$

$$X^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$$\underset{X^T}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}} = \underset{X^TX}{\begin{bmatrix} 4 & 9 \\ 9 & 29 \end{bmatrix}}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 \\ 56 \end{bmatrix}$$

$522 - 504 = 18$

$$\theta_0^* = \frac{\begin{vmatrix} 18 & 9 \\ 56 & 29 \end{vmatrix}}{\begin{vmatrix} 4 & 9 \\ 9 & 29 \end{vmatrix}} = \frac{18 \cdot 29 - 9 \cdot 56}{\underbrace{29 \cdot 4 - 9 \cdot 9}_{=35}} = \frac{18}{35}$$

$$\theta_1^* = \frac{\begin{vmatrix} 4 & 18 \\ 9 & 56 \end{vmatrix}}{\begin{vmatrix} 4 & 9 \\ 9 & 29 \end{vmatrix}} = \frac{\overset{62}{4 \cdot 56 - 18 \cdot 9}}{4 \cdot 29 - 9 \cdot 9} = \frac{62}{35}$$

b. $\theta = (X^T X)^{-1} X^T \vec{y}$.

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\frac{1}{35}\begin{bmatrix} 29 & -9 \\ -9 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 9 \\ 9 & 29 \end{bmatrix}^{-1} = \frac{1}{35}\begin{bmatrix} 29 & -9 \\ -9 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix}$$

2×2 2×4 = 2×4

$$\begin{bmatrix} 29 & -9 \\ -9 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 29 & 11 & 2 & -7 \\ -9 & -1 & 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix}$$

2×4 4×1 = 2×1

$$\begin{bmatrix} 29 & 11 & 2 & -7 \\ -9 & -1 & 3 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 18 \\ 62 \end{bmatrix}$$

$$\frac{1}{35}\begin{bmatrix} 18 \\ 62 \end{bmatrix} = \begin{bmatrix} \frac{18}{35} \\ \frac{62}{35} \end{bmatrix}$$