Dynamo Equations

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1 Dynamo Equations

Starting with the dimensionless dynamo equations, (5), (6), (7), and (9) as given in Jones [2]:

$$R_1[\frac{\partial \mathbf{u}(r,\theta,\phi,t)}{\partial t} + (\mathbf{u}(r,\theta,\phi,t) \cdot \nabla)\mathbf{u}(r,\theta,\phi,t)] =$$
$$2\mathbf{u}(r,\theta,\phi,t) \times \hat{z} - \nabla p(r,\theta,\phi,t) + (\boldsymbol{\nabla} \times \mathbf{B}(r,\theta,\phi,t)) \times \mathbf{B}(r,\theta,\phi,t)$$
$$+E\nabla^2\mathbf{u}(r,\theta,\phi,t) + R_2(T(r,\theta,\phi,t)\frac{\mathbf{r}}{r_{cmb}}), \tag{1}$$

or we may write

$$R_1[\frac{\partial \mathbf{u}(r,\theta,\phi,t)}{\partial t} + (\mathbf{u}(r,\theta,\phi,t) \cdot \nabla)\mathbf{u}(r,\theta,\phi,t)] = \mathbf{F}(r,\theta,\phi,t) - \nabla p(r,\theta,\phi,t)$$

where

$$\mathbf{F}(r,\theta,\phi,t) = 2\mathbf{u}(r,\theta,\phi,t) \times \hat{z} + (\boldsymbol{\nabla} \times \mathbf{B}(r,\theta,\phi,t))) \times \mathbf{B}(r,\theta,\phi,t)$$
$$+E\nabla^2\mathbf{u}(r,\theta,\phi,t) + R_2(T(r,\theta,\phi,t)\frac{\mathbf{r}}{r_{cmb}}). \tag{2}$$

To the fluid equation (1) we have an equation for $T(r,\theta,\phi,t)$

$$\frac{\partial T(r,\theta,\phi,t)}{\partial t} + [\mathbf{u}(r,\theta,\phi,t) \cdot \nabla)]T(r,\theta,\phi,t) = R_3\nabla^2 T(r,\theta,\phi,t), \tag{3}$$

and the requirements

$$\nabla \cdot \mathbf{u}(r,\theta,\phi,t) = \nabla \cdot \mathbf{B}(r,\theta,\phi,t) = 0. \tag{4}$$

Using the definition and vector identities

$$\boldsymbol{\omega}(r,\theta,\phi,t) = \boldsymbol{\nabla} \times \mathbf{u}(r,\theta,\phi,t), \tag{5}$$

and

$$\frac{1}{2}\nabla(\mathbf{u}(r,\theta,\phi,t)^2) = (\mathbf{u}(r,\theta,\phi,t) \cdot \nabla)\mathbf{u}(r,\theta,\phi,t) + (\mathbf{u}(r,\theta,\phi,t) \times \boldsymbol{\omega}(r,\theta,\phi,t), \tag{6}$$

and

$$\boldsymbol{\nabla} \times (\mathbf{u}(r,\theta,\phi,t) \times \boldsymbol{\omega}(r,\theta,\phi,t)) = \boldsymbol{\omega}(r,\theta,\phi,t) \cdot \nabla \mathbf{u}(r,\theta,\phi,t) - \mathbf{u}(r,\theta,\phi,t) \cdot \nabla \boldsymbol{\omega}(r,\theta,\phi,t). \tag{7}$$

This eliminates the pressure term in Eq. (1) which now reads

$$\frac{\partial \boldsymbol{\omega}(r,\theta,\phi,t)}{\partial t} + \mathbf{u}(r,\theta,\phi,t)\cdot\nabla\boldsymbol{\omega}(r,\theta,\phi,t) =$$
$$[\boldsymbol{\omega}(r,\theta,\phi,t)\cdot\nabla]\mathbf{u}(r,\theta,\phi,t) + \boldsymbol{\nabla}\times\mathbf{F}(r,\theta,\phi,t). \tag{8}$$

Now we have equations for $\mathbf{u}(r,\theta,\phi,t)$ and $\mathbf{B}(r,\theta,\phi,t)$ and $T(r,\theta,\phi,t)$ where the pressure term has been eliminated.

R_1, R_2, and R_3 are dimensionless constants given in [2].

2 Representing a Solenoidal Vector Field

In [1] it is shown that if $\nabla\cdot A(r,\theta,\phi)$ =0, one may write

$$A(r,\theta,\phi) = \boldsymbol{\nabla}\times(\boldsymbol{\Lambda}a(r,\theta,\phi)) + \boldsymbol{\Lambda}b(r,\theta,\phi), \tag{9}$$

where we have the operator

$$\boldsymbol{\Lambda} = \mathbf{r}\times\nabla. \tag{10}$$

This yields

$$\mathbf{r}\cdot A(r,\theta,\phi) = \boldsymbol{\Lambda}^2 a(r,\theta,\phi), \tag{11}$$

and

$$\boldsymbol{\Lambda}\cdot A(r,\theta,\phi) = \boldsymbol{\Lambda}^2 b(r,\theta,\phi) \tag{12}$$

By inverting the operator $\boldsymbol{\Lambda}^2$, we may solve for $a(r,\theta,\phi)$ and $b(r,\theta,\phi)$.
If we have $\boldsymbol{\Lambda}^2 f(\Omega) = g(\Omega)$, then

$$f(\Omega) = \frac{1}{\boldsymbol{\Lambda}^2} g(\Omega)$$
$$f(\Omega) = \frac{1}{4\pi}\int d\Omega' g(\Omega')\log_e(1-\Omega\cdot\Omega'). \tag{13}$$

In this Ω is a vector of length 1 on the surface of a 2D-sphere.

$$a(r,\theta,\phi) = \frac{1}{\boldsymbol{\Lambda}^2}(\mathbf{r}\cdot A(r,\theta,\phi))$$
$$a(r,\theta,\phi) = \frac{1}{4\pi}\int d\Omega'(\mathbf{r}\cdot A(r,\theta',\phi'))\log_e(1-\Omega\cdot\Omega').$$
$$b(r,\theta,\phi) = \frac{1}{4\pi}\int d\Omega'(\boldsymbol{\Lambda}\cdot A(r,\theta',\phi'))\log_e(1-\Omega\cdot\Omega') \tag{14}$$

References

[1] George Backus. A class of self-sustaining dissipative spherical dynamics. *Annals of Physics*, 4:372–447, 1958.

[2] Chris A. Jones. Planetary magnetic fields and fluid dynamos. *Annual Reviews of Fluid Mechanics*, 43:583–614.