Geodynamo Physical Analysis via sympy

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1 Introduction

To keep things clear, the work within this paper is derivative of the theory as defined in the working document passed between Luke and Henry. We take less space to display everything which he elucidates adequately within his paper since this paper's purpose, and the purpose of the algorithmic system used to produce the results of this paper, is to describe the system of interest, the liquid metal inner core of the Earth and the magnetic field produced by the internal motion within, via a computational mathematics framework. The formulae designed to essentially match the established theory in tandem, meant to be analogous or parallel to the paper theory as written by Henry. The hope being that this computational physics theory for the geodynamo, constructed within python using the 'sympy' package, would serve as a useful tool to analyze the theory in ways which are hard to exhaustively write onto paper, serve as a system to double check results and align the work of multiple group members into a harmonious state, and possibly connect with the other systems the research group utilizes to see if anything interesting or emergent may be found by the coupling of these computational physics frameworks.

1.1 notation and symbology

We take a moment here to lay out some of the notation shorthand utilized to more efficiently print the sometimes large outputs of sympy since the system is explicit to the point of redundancy at times. The following table will seek to encapsulate all forms of shorthand or abstraction the reader might want to reference here moving forward. Note that some of the later items encapsulate differential operators of some form. The important thing to know regarding these abstract operators/transforms, the χ expressions, is that the sympy code processes the system at the raw level of fields, derivatives, and algebraic operations, and these operator abstractions are done with results post-computation for readability purposes ie just a condensed re-expression of the fully-expanded results. Arbitrary function 'f' is used for operators.

Mathematical Expression	Shorthand
Partial Derivative wrt variable 'x'	δ_x
Total Derivative wrt 'x'	δ_x^o
2nd order Partial Derivative wrt 'x' and 'y'	$\delta_x \delta_y$
2nd order Partial wrt 'x'	δ_{xx}
Sin(x)	S_x
Cos(x)	C_x
$(\delta_\theta^o r + \delta_r r\theta)f$	$\chi_1 f$
$(\delta_\phi^o r + \delta_r r S_\theta \phi)f$	$\chi_2 f$
$(-\delta_\phi r\theta + \delta_\theta r S_\theta \phi)f$	$\chi_3 f$
$\frac{-\delta_r(-\chi_1(fS_\theta))+\delta_\phi(-\frac{\chi_3 f}{r^2 S_\theta})}{rS_\theta}$	$\chi_5 f$
$\frac{-\delta_\phi(-\frac{\chi_2 f}{S_\theta})+\delta_\theta(-\chi_1(fS_\theta))}{r^2 S_\theta}$	$\chi_6 f$
$\frac{\delta_r(-\frac{\chi_2 f}{S_\theta})-\delta_\theta(-\frac{\chi_3 f}{r^2 S_\theta})}{r}$	$\chi_7 f$
$\frac{\chi_1 f}{r}$	$\chi_8 f$
$\frac{\chi_3 f}{r^2 S_\theta}$	$\chi_9 f$
$\frac{\chi_2 f}{rS_\theta}$	$\chi_{10} f$
$\frac{\chi_2 f}{rS_\theta}$	$\chi_{11} f$
$\frac{-\delta_r(-\chi_1(fS_\theta))+\delta_\phi(-\frac{\chi_3 f}{r^2 S_\theta})}{rS_\theta}$	$\chi_{12} f$
$\frac{\chi_3 f}{r^2 S_\theta}$	$\chi_{13} f$
$\frac{-\delta_\phi(-\frac{\chi_2 f}{S_\theta})+\delta_\theta(-\chi_1(fS_\theta))}{r^2 S_\theta}$	$\chi_{14} f$
$\frac{\chi_1 f}{r}$	$\chi_{15} f$
$\frac{\delta_r(-\frac{\chi_2 f}{S_\theta})-\delta_\theta(-\chi_{13} f)}{r}$	$\chi_{16} f$

Table 1: Mathematical Shorthands

We have considered further layers of abstraction to condense the formulae; however, the current level seemed sufficient for the time being. Most differential operator expressions have been condensed down to functions acting on the scalar functions of interest; a, b, d, g, and T2. Beyond this point it would seem one's layers of abstraction go beyond those functions and into the fields they constitute, or at least transformations of those fields, given that the next 'layer' of abstraction would involve composite objects of the scalar functions, for example and expression such as $\chi_8 g - \chi_7 d$.

For further context into the application of these shorthands, please reference the lower sections where results are reported ie the expanded pde's in terms a, b, d, g, T2. Also note that χ is being reconsidered for the abstractions since it is also used for the bessel form.

2 Directly Analogous Theoretical Constructions

Before proceeding to show the current results from the computational system, namely the fully expanded set of partial differential equations in terms of the scalar functions associated with the fields U, B, and T, we will display the initial computational constructions which are the basis of the expanded form, and meant to directly reflect the initial theory formulation.

2.1 Basic Definitions

First come a set of 'symbols' within sympy, used to define general global variables or parameters. Also useful for 'stand-in' values, to be determined later.

Then come the objects of primary interest, the fields! The scalar fields temperature and pressure:

$$T_{(t,r,\theta,\phi)}$$

$$P_{(t,r,\theta,\phi)}$$

Item	Symbol
Time	t
momentum equation coefficient, flow evolution term	R_1
momentum equation coefficient, dissipation term	E
momentum equation coefficient, buoyancy term	R_2
Radius of Core-Mantle Boundary	R_{cmb}
Temperature equation 'source'	Q
Temperature equation coefficient, dissipation term	R_3

Table 2: Your table caption

The vector fields, velocity and magnetic, specifically index the coordinate system in their vector form so the coordinate system is defined first as cartesian then a transform into spherical via the 'CoordSys3D' class of sympy. Then the components of the fields are then defined:

$$\mathrm{U}_{\mathrm{r}(t,r,\theta,\phi)}$$

$$\mathrm{U}_{\theta(t,r,\theta,\phi)}$$

$$\mathrm{U}_{\phi(t,r,\theta,\phi)}$$

Where r, th, and ph correspond to components of the field aligned with the spherical coordinate system's bases. The fields themselves follow as so, with the use of basis vectors from the coordinate system 'c' which again is the spherical transformation of the 'parent' coordinate system which happens to be named 'Coord'

The velocity field

$$\left(\mathrm{U}_{\mathrm{r}(t,r,\theta,\phi)}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\mathrm{U}_{\theta(t,r,\theta,\phi)}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\mathrm{U}_{\phi(t,r,\theta,\phi)}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

The magnetic field

$$\left(\mathrm{B}_{\mathrm{r}(t,r,\theta,\phi)}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\mathrm{B}_{\theta(t,r,\theta,\phi)}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\mathrm{B}_{\phi(t,r,\theta,\phi)}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

Also worth noting alternative definitions may be defined such as for vorticity, but we also define vorticity explicitly too

$$\left(\mathrm{W}_{\mathrm{r}(t,r,\theta,\phi)}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\mathrm{W}_{\theta(t,r,\theta,\phi)}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\mathrm{W}_{\phi(t,r,\theta,\phi)}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

The 'alternate' definition for vorticity then being the curl of the velocity field

$$\left(\frac{-\delta_\phi r\,\mathrm{U}_\theta + \delta_\theta r\,\mathrm{U}_\phi\,S_\theta}{r^2 S_\theta}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\frac{-\delta_r r\,\mathrm{U}_\phi\,S_\theta + \delta_\phi\,\mathrm{U}_\mathrm{r}}{r S_\theta}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\frac{\delta_r r\,\mathrm{U}_\theta - \delta_\theta\,\mathrm{U}_\mathrm{r}}{r}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

Finally, useful to define a generalized position field within the coordinate system.

$$(r)\,\hat{\mathbf{i}}_{\mathbf{c}} + (r\theta)\,\hat{\mathbf{j}}_{\mathbf{c}} + (\phi r S_\theta)\,\hat{\mathbf{k}}_{\mathbf{c}}$$

2.2 Dynamo Equations

Then the dynamo equations in their dimensionless form as referenced from Jones (REFERENCE):

The momentum equation consisting of time partial flow

$$(R_1\delta_t\,\mathrm{U}_\mathrm{r})\,\hat{\mathbf{i}}_{\mathbf{c}} + (R_1\delta_t\,\mathrm{U}_\theta)\,\hat{\mathbf{j}}_{\mathbf{c}} + (R_1\delta_t\,\mathrm{U}_\phi)\,\hat{\mathbf{k}}_{\mathbf{c}}$$

advective flow

$$R_1\left(\mathrm{U}_\mathrm{r}\,\delta_r\left((\mathrm{U}_\mathrm{r})\hat{\mathbf{i}}_{\mathbf{c}} + (\mathrm{U}_\theta)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\mathrm{U}_\phi\right)\hat{\mathbf{k}}_{\mathbf{c}}\right) + \frac{\mathrm{U}_\phi\,\delta_\phi\left((\mathrm{U}_\mathrm{r})\hat{\mathbf{i}}_{\mathbf{c}} + (\mathrm{U}_\theta)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\mathrm{U}_\phi\right)\hat{\mathbf{k}}_{\mathbf{c}}\right)}{r S_\theta} + \frac{\mathrm{U}_\theta\,\delta_\theta\left((\mathrm{U}_\mathrm{r})\hat{\mathbf{i}}_{\mathbf{c}} + (\mathrm{U}_\theta)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\mathrm{U}_\phi\right)\hat{\mathbf{k}}_{\mathbf{c}}\right)}{r}\right)$$

coriolis force

$$(-2\,\mathrm{U}_\theta)\,\hat{\mathbf{i}}_{\mathbf{c}} + (2\,\mathrm{U}_\mathrm{r})\,\hat{\mathbf{j}}_{\mathbf{c}}$$

pressure gradient

$$(-\delta_r P)\,\hat{\mathbf{i}}_{\mathbf{c}} + \left(-\frac{\delta_\theta P}{r}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(-\frac{\delta_\phi P}{r S_\theta}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

lorentz force

$$\left(-\frac{(\delta_r r\,\mathrm{B}_\theta - \delta_\theta\,\mathrm{B}_\mathrm{r})\,\mathrm{B}_\theta}{r} + \frac{\left(-\delta_r r\,\mathrm{B}_\phi\,S_\theta + \delta_\phi\,\mathrm{B}_\mathrm{r}\right)\mathrm{B}_\phi}{r S_\theta}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\frac{(\delta_r r\,\mathrm{B}_\theta - \delta_\theta\,\mathrm{B}_\mathrm{r})\,\mathrm{B}_\mathrm{r}}{r} - \frac{\left(-\delta_\phi r\,\mathrm{B}_\theta + \delta_\theta r\,\mathrm{B}_\phi\,S_\theta\right)\mathrm{B}_\phi}{r^2 S_\theta}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(-\frac{\left(-\delta_r r\,\mathrm{B}_\phi\,S_\theta + \delta_\phi\,\mathrm{B}_\mathrm{r}\right)\mathrm{B}_\mathrm{r}}{r S_\theta} + \frac{\left(-\delta_\phi r\,\mathrm{B}_\theta + \delta_\theta r\,\mathrm{B}_\phi\,S_\theta\right)\mathrm{B}_\theta}{r^2 S_\theta}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

dissipation (component along r basis)

$$E\left(\frac{\delta_r\delta_\phi\,\mathrm{U}_\phi}{r S_\theta} - \frac{\left(r\delta_r\,\mathrm{U}_\theta + \mathrm{U}_\theta - \delta_\theta\,\mathrm{U}_\mathrm{r}\right)C_\theta + \left(r\delta_\theta\delta_r\,\mathrm{U}_\theta - \delta_{\theta\theta}\,\mathrm{U}_\mathrm{r} + \delta_\theta\,\mathrm{U}_\theta\right)S_\theta - \frac{-r S_\theta\delta_r\delta_\phi\,\mathrm{U}_\phi - S_\theta\delta_\phi\,\mathrm{U}_\phi + \delta_{\phi\phi}\,\mathrm{U}_\mathrm{r}}{S_\theta}}{r^2 S_\theta}\right.$$

$$+\frac{r^2 S_\theta \delta_{rr}\,\mathrm{U_r} + 4rS_\theta \delta_r\,\mathrm{U_r} + 2\,\mathrm{U_r}\,S_\theta}{r^2 S_\theta} + \frac{rS_\theta \delta_\theta \delta_r\,\mathrm{U}_\theta + rC_\theta \delta_r\,\mathrm{U}_\theta + \mathrm{U}_\theta\,C_\theta + S_\theta \delta_\theta\,\mathrm{U}_\theta}{r^2 S_\theta} - \frac{\delta_\phi\,\mathrm{U}_\phi}{r^2 S_\theta} - \frac{2\left(r\,\mathrm{U}_\theta\,C_\theta + rS_\theta \delta_\theta\,\mathrm{U}_\theta\right)}{r^3 S_\theta} - \frac{2\left(r^2 S_\theta \delta_r\,\mathrm{U_r} + 2r\,\mathrm{U_r}\,S_\theta\right)}{r^3 S_\theta}\Big)$$

dissipation (component along θ basis)

$$E\Bigg(-\frac{-\left(r\delta_{rr}\,\mathrm{U}_\theta + 2\delta_r\,\mathrm{U}_\theta - \delta_\theta \delta_r\,\mathrm{U_r}\right)S_\theta + \frac{rS_\theta \delta_\theta \delta_\phi\,\mathrm{U}_\phi + rC_\theta \delta_\phi\,\mathrm{U}_\phi - r\delta_{\phi\phi}\,\mathrm{U}_\theta}{r^2 S_\theta}}{rS_\theta}$$

$$+\frac{\frac{\delta_\theta \delta_\phi\,\mathrm{U}_\phi}{rS_\theta} - \frac{C_\theta \delta_\phi\,\mathrm{U}_\phi}{r\sin^2\left(\theta\right)} - \frac{\left(r\,\mathrm{U}_\theta\,C_\theta + rS_\theta \delta_\theta\,\mathrm{U}_\theta\right)C_\theta}{r^2\sin^2\left(\theta\right)} - \frac{\left(r^2 S_\theta \delta_r\,\mathrm{U_r} + 2r\,\mathrm{U_r}\,S_\theta\right)C_\theta}{r^2\sin^2\left(\theta\right)} + \frac{-r\,\mathrm{U}_\theta\,S_\theta + rS_\theta \delta_{\theta\theta}\,\mathrm{U}_\theta + 2rC_\theta \delta_\theta\,\mathrm{U}_\theta}{r^2 S_\theta} + \frac{r^2 S_\theta \delta_\theta \delta_r\,\mathrm{U_r} + r^2 C_\theta \delta_r\,\mathrm{U_r} + 2r\,\mathrm{U_r}\,C_\theta + 2rS_\theta \delta_\theta\,\mathrm{U_r}}{r^2 S_\theta}}{r}\Bigg)$$

dissipation (component along ϕ basis)

$$E\Bigg(-\frac{\frac{-rS_\theta \delta_{rr}\,\mathrm{U}_\phi - 2S_\theta \delta_r\,\mathrm{U}_\phi + \delta_r \delta_\phi\,\mathrm{U_r}}{S_\theta} + \frac{\left(r\,\mathrm{U}_\phi\,C_\theta + rS_\theta \delta_\theta\,\mathrm{U}_\phi - r\delta_\phi\,\mathrm{U}_\theta\right)C_\theta}{r^2\sin^2\left(\theta\right)} - \frac{-r\,\mathrm{U}_\phi\,S_\theta + rS_\theta \delta_{\theta\theta}\,\mathrm{U}_\phi + 2rC_\theta \delta_\theta\,\mathrm{U}_\phi - r\delta_\theta \delta_\phi\,\mathrm{U}_\theta}{r^2 S_\theta}}{r}$$

$$+\frac{\frac{\delta_{\phi\phi}\,\mathrm{U}_\phi}{rS_\theta} + \frac{rS_\theta \delta_\theta \delta_\phi\,\mathrm{U}_\theta + rC_\theta \delta_\phi\,\mathrm{U}_\theta}{r^2 S_\theta} + \frac{r^2 S_\theta \delta_r \delta_\phi\,\mathrm{U_r} + 2rS_\theta \delta_\phi\,\mathrm{U_r}}{r^2 S_\theta}}{rS_\theta}\Bigg)$$

and buoyancy

$$\left(\frac{rR_2T}{Rcmb}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\frac{r\theta R_2T}{Rcmb}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\frac{\phi rR_2TS_\theta}{Rcmb}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

Thus formulating a momentum equation

$$TimePartialU + AdvectionU = -Coriolis + GradientPressure + LorentzE\&MForcing + ViscousDissipation + HeatBuoyancy$$

or

$$R_1\delta_t\vec{U} + R_1(\vec{U}\cdot\vec{\nabla})\vec{U} = -2\hat{k}\times\vec{U} - \vec{\nabla}P + (\vec{\nabla}\times\vec{B})\times\vec{B} + E\vec{\nabla}^2U + R_2T\frac{\vec{r}}{R_{cmb}}$$

The magnetic induction equation with terms for time partial evolution

$$\left(\delta_t\,\mathrm{B_r}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\delta_t\,\mathrm{B}_\theta\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\delta_t\,\mathrm{B}_\phi\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

dissipation (component along r basis)

$$\frac{\delta_r \delta_\phi\,\mathrm{B}_\phi}{rS_\theta} - \frac{\left(r\delta_r\,\mathrm{B}_\theta + \mathrm{B}_\theta - \delta_\theta\,\mathrm{B_r}\right)C_\theta + \left(r\delta_\theta \delta_r\,\mathrm{B}_\theta - \delta_{\theta\theta}\,\mathrm{B_r} + \delta_\theta\,\mathrm{B}_\theta\right)S_\theta - \frac{-rS_\theta \delta_r \delta_\phi\,\mathrm{B}_\phi - S_\theta \delta_\phi\,\mathrm{B}_\phi + \delta_{\phi\phi}\,\mathrm{B_r}}{S_\theta}}{r^2 S_\theta} + \frac{r^2 S_\theta \delta_{rr}\,\mathrm{B_r} + 4rS_\theta \delta_r\,\mathrm{B_r} + 2\,\mathrm{B_r}\,S_\theta}{r^2 S_\theta}$$

$$+\frac{rS_\theta \delta_\theta \delta_r\,\mathrm{B}_\theta + rC_\theta \delta_r\,\mathrm{B}_\theta + \mathrm{B}_\theta\,C_\theta + S_\theta \delta_\theta\,\mathrm{B}_\theta}{r^2 S_\theta} - \frac{\delta_\phi\,\mathrm{B}_\phi}{r^2 S_\theta} - \frac{2\left(r\,\mathrm{B}_\theta\,C_\theta + rS_\theta \delta_\theta\,\mathrm{B}_\theta\right)}{r^3 S_\theta} - \frac{2\left(r^2 S_\theta \delta_r\,\mathrm{B_r} + 2r\,\mathrm{B_r}\,S_\theta\right)}{r^3 S_\theta}$$

dissipation (component along theta basis)

$$-\frac{-\left(r\delta_{rr}\,\mathrm{B}_\theta + 2\delta_r\,\mathrm{B}_\theta - \delta_\theta \delta_r\,\mathrm{B_r}\right)S_\theta + \frac{rS_\theta \delta_\theta \delta_\phi\,\mathrm{B}_\phi + rC_\theta \delta_\phi\,\mathrm{B}_\phi - r\delta_{\phi\phi}\,\mathrm{B}_\theta}{r^2 S_\theta}}{rS_\theta}$$

$$+\frac{\frac{\delta_\theta \delta_\phi\,\mathrm{B}_\phi}{rS_\theta} - \frac{C_\theta \delta_\phi\,\mathrm{B}_\phi}{r\sin^2\left(\theta\right)} - \frac{\left(r\,\mathrm{B}_\theta\,C_\theta + rS_\theta \delta_\theta\,\mathrm{B}_\theta\right)C_\theta}{r^2\sin^2\left(\theta\right)} - \frac{\left(r^2 S_\theta \delta_r\,\mathrm{B_r} + 2r\,\mathrm{B_r}\,S_\theta\right)C_\theta}{r^2\sin^2\left(\theta\right)} + \frac{-r\,\mathrm{B}_\theta\,S_\theta + rS_\theta \delta_{\theta\theta}\,\mathrm{B}_\theta + 2rC_\theta \delta_\theta\,\mathrm{B}_\theta}{r^2 S_\theta} + \frac{r^2 S_\theta \delta_\theta \delta_r\,\mathrm{B_r} + r^2 C_\theta \delta_r\,\mathrm{B_r} + 2r\,\mathrm{B_r}\,C_\theta + 2rS_\theta \delta_\theta\,\mathrm{B_r}}{r^2 S_\theta}}{r}$$

dissipation (component along phi basis)

$$-\frac{\frac{-rS_\theta \delta_{rr}\,\mathrm{B}_\phi - 2S_\theta \delta_r\,\mathrm{B}_\phi + \delta_r \delta_\phi\,\mathrm{B_r}}{S_\theta} + \frac{\left(r\,\mathrm{B}_\phi\,C_\theta + rS_\theta \delta_\theta\,\mathrm{B}_\phi - r\delta_\phi\,\mathrm{B}_\theta\right)C_\theta}{r^2\sin^2\left(\theta\right)} - \frac{-r\,\mathrm{B}_\phi\,S_\theta + rS_\theta \delta_{\theta\theta}\,\mathrm{B}_\phi + 2rC_\theta \delta_\theta\,\mathrm{B}_\phi - r\delta_\theta \delta_\phi\,\mathrm{B}_\theta}{r^2 S_\theta}}{r}$$

$$+\frac{\frac{\delta_{\phi\phi}\,\mathrm{B}_\phi}{rS_\theta} + \frac{rS_\theta \delta_\theta \delta_\phi\,\mathrm{B}_\theta + rC_\theta \delta_\phi\,\mathrm{B}_\theta}{r^2 S_\theta} + \frac{r^2 S_\theta \delta_r \delta_\phi\,\mathrm{B_r} + 2rS_\theta \delta_\phi\,\mathrm{B_r}}{r^2 S_\theta}}{rS_\theta}$$

and faraday induction.

$$\left(\frac{-\delta_\phi r\left(-\mathrm{B}_\phi\,\mathrm{U_r} + \mathrm{B_r}\,\mathrm{U}_\phi\right) + \delta_\theta r\left(-\mathrm{B_r}\,\mathrm{U}_\theta + \mathrm{B}_\theta\,\mathrm{U_r}\right)S_\theta}{r^2 S_\theta}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\frac{-\delta_r r\left(-\mathrm{B_r}\,\mathrm{U}_\theta + \mathrm{B}_\theta\,\mathrm{U_r}\right)S_\theta + \delta_\phi\left(\mathrm{B}_\phi\,\mathrm{U}_\theta - \mathrm{B}_\theta\,\mathrm{U}_\phi\right)}{rS_\theta}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\frac{\delta_r r\left(-\mathrm{B}_\phi\,\mathrm{U_r} + \mathrm{B_r}\,\mathrm{U}_\phi\right) - \delta_\theta\left(\mathrm{B}_\phi\,\mathrm{U}_\theta - \mathrm{B}_\theta\,\mathrm{U}_\phi\right)}{r}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

Forming the magnetic induction equation

$$TimePartialB = MagneticDissipation + Induction$$

or

$$\delta_t\vec{B} = \vec{\nabla}^2B + \vec{\nabla}\times(\vec{U}\times\vec{B})$$

The temperature evolution formula involves of course the time partial term

$$\delta_t T$$

the advection term with respect to the temperature field

$$R_3\left(\frac{S_\theta \delta_{\theta\theta} T + C_\theta \delta_\theta T}{r^2 S_\theta} + \frac{r^2 S_\theta \delta_{rr} T + 2r S_\theta \delta_r T}{r^2 S_\theta} + \frac{\delta_{\phi\phi} T}{r^2 \sin^2(\theta)}\right)$$

and the dissipative heat term

$$-\mathrm{U_r}\,\delta_r T - \frac{\mathrm{U}_\phi\,\delta_\phi T}{r S_\theta} - \frac{\mathrm{U}_\theta\,\delta_\theta T}{r}$$

forming the temperature evolution equation

$$TimePartialT + AdvectionT = DissipationT + Heat$$

or

$$\delta_t T + (\vec{U}\cdot\vec{\nabla})T = R_3 \vec{\nabla}^2 T + Q$$

note that heat 'Q' is arbitrary at the current moment

Then finally the vorticity equation, which is simply the curl of the momentum formula. In some ways more complex yet simpler in others such as the capability to ignore the pressure gradient and thus abstain from requiring an equation of state. Also possibly a fitting frame for the system of interest since the flows of particular interest which produce the strong magnetic fields often are characterized by a helical or vortical geometry. Also notable is the connection to magentic helicity whose conservation might provide further insight.

Note that one could technically define vorticity by algorithmically curling the momentum equation, but for double checking purposes we also define the explicit formulation

$$\delta_t \vec{W} + \vec{U}\cdot\vec{\nabla}\vec{W} = (\vec{W}\cdot\vec{\nabla})\vec{U} + \vec{\nabla}\times\vec{F}$$

where F is defined as

$$F = -2\hat{k}\times\vec{U} + (\vec{\nabla}\times\vec{B})\times\vec{B} + E\vec{\nabla}^2 U + R_2 T\frac{\vec{r}}{R_{cmb}}$$

in other words F encapsulates the coriolis, lorentz, viscous dissipation, and buoyancy terms.

in the spirit of the other formulae displayed we'll present the terms starting with the F terms

curl of coriolis

$$\left(\frac{\delta_\theta^{o}0 - \delta_\phi 2r\,\mathrm{U_r}}{r^2 S_\theta}\right)\hat{\mathbf{i}}_{\mathbf{c}} + \left(\frac{-\frac{d}{dr}0 + \delta_\phi\left(-2\,\mathrm{U}_\theta\right)}{r S_\theta}\right)\hat{\mathbf{j}}_{\mathbf{c}} + \left(\frac{\delta_r 2r\,\mathrm{U_r} - \delta_\theta\left(-2\,\mathrm{U}_\theta\right)}{r}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

curl of lorentz (component along r basis)

$$\left(\frac{-\delta_\phi r\left(\frac{\left(\delta_r r\,\mathrm{B}_\theta - \delta_\theta\,\mathrm{B_r}\right)\mathrm{B_r}}{r} - \frac{\left(-\delta_\phi r\,\mathrm{B}_\theta + \delta_\theta r\,\mathrm{B}_\phi\,S_\theta\right)\mathrm{B}_\phi}{r^2 S_\theta}\right) + \delta_\theta r\left(-\frac{\left(-\delta_r r\,\mathrm{B}_\phi\,S_\theta + \delta_\phi\,\mathrm{B_r}\right)\mathrm{B_r}}{r S_\theta} + \frac{\left(-\delta_\phi r\,\mathrm{B}_\theta + \delta_\theta r\,\mathrm{B}_\phi\,S_\theta\right)\mathrm{B}_\theta}{r^2 S_\theta}\right)S_\theta}{r^2 S_\theta}\right)\hat{\mathbf{i}}_{\mathbf{c}}++$$

curl of lorentz (component along th basis)

$$\left(\frac{-\delta_r r\left(-\frac{\left(-\delta_r r\,\mathrm{B}_\phi\,S_\theta + \delta_\phi\,\mathrm{B_r}\right)\mathrm{B_r}}{r S_\theta} + \frac{\left(-\delta_\phi r\,\mathrm{B}_\theta + \delta_\theta r\,\mathrm{B}_\phi\,S_\theta\right)\mathrm{B}_\theta}{r^2 S_\theta}\right)S_\theta + \delta_\phi\left(-\frac{\left(\delta_r r\,\mathrm{B}_\theta - \delta_\theta\,\mathrm{B_r}\right)\mathrm{B}_\theta}{r} + \frac{\left(-\delta_r r\,\mathrm{B}_\phi\,S_\theta + \delta_\phi\,\mathrm{B_r}\right)\mathrm{B}_\phi}{r S_\theta}\right)}{r S_\theta}\right)\hat{\mathbf{j}}_{\mathbf{c}}$$

curl of lorentz (component along ph basis)

$$\left(\frac{\delta_r r\left(\frac{\left(\delta_r r\,\mathrm{B}_\theta - \delta_\theta\,\mathrm{B_r}\right)\mathrm{B_r}}{r} - \frac{\left(-\delta_\phi r\,\mathrm{B}_\theta + \delta_\theta r\,\mathrm{B}_\phi\,S_\theta\right)\mathrm{B}_\phi}{r^2 S_\theta}\right) - \delta_\theta\left(-\frac{\left(\delta_r r\,\mathrm{B}_\theta - \delta_\theta\,\mathrm{B_r}\right)\mathrm{B}_\theta}{r} + \frac{\left(-\delta_r r\,\mathrm{B}_\phi\,S_\theta + \delta_\phi\,\mathrm{B_r}\right)\mathrm{B}_\phi}{r S_\theta}\right)}{r}\right)\hat{\mathbf{k}}_{\mathbf{c}}$$

curl of viscous disspation (component along r basis)

$$\frac{1}{r^2 S_\theta}\Bigg(-\delta_\phi r E\Bigg(-\frac{-\left(r\delta_{rr}\,\mathrm{U}_\theta + 2\delta_r\,\mathrm{U}_\theta - \delta_\theta\delta_r\,\mathrm{U_r}\right)S_\theta + \frac{r S_\theta\delta_\theta\delta_\phi\,\mathrm{U}_\phi + rC_\theta\delta_\phi\,\mathrm{U}_\phi - r\delta_{\phi\phi}\,\mathrm{U}_\theta}{r^2 S_\theta}}{r S_\theta}$$

$$+\frac{\frac{\delta_\theta\delta_\phi\,\mathrm{U}_\phi}{r S_\theta} - \frac{C_\theta\delta_\phi\,\mathrm{U}_\phi}{r\sin^2(\theta)} - \frac{\left(r\,\mathrm{U}_\theta\,C_\theta + rS_\theta\delta_\theta\,\mathrm{U}_\theta\right)C_\theta}{r^2\sin^2(\theta)} - \frac{\left(r^2 S_\theta\delta_r\,\mathrm{U_r} + 2r\,\mathrm{U_r}\,S_\theta\right)C_\theta}{r^2\sin^2(\theta)} + \frac{-r\,\mathrm{U}_\theta\,S_\theta + rS_\theta\delta_{\theta\theta}\,\mathrm{U}_\theta + 2rC_\theta\delta_\theta\,\mathrm{U}_\theta}{r^2 S_\theta} + \frac{r^2 S_\theta\delta_\theta\delta_r\,\mathrm{U_r} + r^2 C_\theta\delta_r\,\mathrm{U_r} + 2r\,\mathrm{U_r}\,C_\theta + 2rS_\theta\delta_\theta\,\mathrm{U_r}}{r^2 S_\theta}}{r}\Bigg)$$

$$+\delta_\theta r E\Bigg(-\frac{\frac{-rS_\theta\delta_{rr}\,\mathrm{U}_\phi - 2S_\theta\delta_r\,\mathrm{U}_\phi + \delta_r\delta_\phi\,\mathrm{U_r}}{S_\theta} + \frac{\left(r\,\mathrm{U}_\phi\,C_\theta + rS_\theta\delta_\theta\,\mathrm{U}_\phi - r\delta_\phi\,\mathrm{U}_\theta\right)C_\theta}{r^2\sin^2(\theta)} - \frac{-r\,\mathrm{U}_\phi\,S_\theta + rS_\theta\delta_{\theta\theta}\,\mathrm{U}_\phi + 2rC_\theta\delta_\theta\,\mathrm{U}_\phi - r\delta_\theta\delta_\phi\,\mathrm{U}_\theta}{r^2 S_\theta}}{r}$$

$$+\frac{\frac{\delta_{\phi\phi}\,\mathrm{U}_\phi}{r S_\theta} + \frac{rS_\theta\delta_\theta\delta_\phi\,\mathrm{U}_\theta + rC_\theta\delta_\phi\,\mathrm{U}_\theta}{r^2 S_\theta} + \frac{r^2 S_\theta\delta_r\delta_\phi\,\mathrm{U_r} + 2rS_\theta\delta_\phi\,\mathrm{U_r}}{r^2 S_\theta}}{r S_\theta}\Bigg)S_\theta\Bigg)\hat{\mathbf{i}}_{\mathbf{c}}$$

curl of viscous disspation (component along th basis)

$$\frac{1}{rS_\theta}\Bigg(\delta_\phi E\Bigg(\frac{\delta_r\delta_\phi\,\mathrm{U}_\phi}{rS_\theta}-\frac{\Big(r\delta_r\,\mathrm{U}_\theta+\mathrm{U}_\theta-\delta_\theta\,\mathrm{U}_\mathrm{r}\Big)C_\theta+\Big(r\delta_\theta\delta_r\,\mathrm{U}_\theta-\delta_{\theta\theta}\,\mathrm{U}_\mathrm{r}+\delta_\theta\,\mathrm{U}_\theta\Big)S_\theta-\frac{-rS_\theta\delta_r\delta_\phi\,\mathrm{U}_\phi-S_\theta\delta_\phi\,\mathrm{U}_\phi+\delta_{\phi\phi}\,\mathrm{U}_\mathrm{r}}{S_\theta}}{r^2S_\theta}$$

$$+\frac{r^2S_\theta\delta_{rr}\,\mathrm{U}_\mathrm{r}+4rS_\theta\delta_r\,\mathrm{U}_\mathrm{r}+2\,\mathrm{U}_\mathrm{r}\,S_\theta}{r^2S_\theta}+\frac{rS_\theta\delta_\theta\delta_r\,\mathrm{U}_\theta+rC_\theta\delta_r\,\mathrm{U}_\theta+\mathrm{U}_\theta\,C_\theta+S_\theta\delta_\theta\,\mathrm{U}_\theta}{r^2S_\theta}-\frac{\delta_\phi\,\mathrm{U}_\phi}{r^2S_\theta}-\frac{2\Big(r\,\mathrm{U}_\theta\,C_\theta+rS_\theta\delta_\theta\,\mathrm{U}_\theta\Big)}{r^3S_\theta}-\frac{2\Big(r^2S_\theta\delta_r\,\mathrm{U}_\mathrm{r}+2r\,\mathrm{U}_\mathrm{r}\,S_\theta\Big)}{r^3S_\theta}\Bigg)$$

$$-\delta_r rE\Bigg(-\frac{\frac{-rS_\theta\delta_{rr}\,\mathrm{U}_\phi-2S_\theta\delta_r\,\mathrm{U}_\phi+\delta_r\delta_\phi\,\mathrm{U}_\mathrm{r}}{S_\theta}+\frac{\Big(r\,\mathrm{U}_\phi\,C_\theta+rS_\theta\delta_\theta\,\mathrm{U}_\phi-r\delta_\phi\,\mathrm{U}_\theta\Big)C_\theta}{r^2\sin^2\Big(\theta\Big)}-\frac{-r\,\mathrm{U}_\phi\,S_\theta+rS_\theta\delta_{\theta\theta}\,\mathrm{U}_\phi+2rC_\theta\delta_\theta\,\mathrm{U}_\phi-r\delta_\theta\delta_\phi\,\mathrm{U}_\theta}{r^2S_\theta}}{r}$$

$$+\frac{\frac{\delta_{\phi\phi}\,\mathrm{U}_\phi}{rS_\theta}+\frac{rS_\theta\delta_\theta\delta_\phi\,\mathrm{U}_\theta+rC_\theta\delta_\phi\,\mathrm{U}_\theta}{r^2S_\theta}+\frac{r^2S_\theta\delta_r\delta_\phi\,\mathrm{U}_\mathrm{r}+2rS_\theta\delta_\phi\,\mathrm{U}_\mathrm{r}}{r^2S_\theta}}{rS_\theta}\Bigg)S_\theta\Bigg)\mathbf{\hat{j}_c}$$

curl of viscous disspation (component along ph basis)

$$\frac{1}{r}\Bigg(-\delta_\theta E\Bigg(\frac{\delta_r\delta_\phi\,\mathrm{U}_\phi}{rS_\theta}-\frac{\Big(r\delta_r\,\mathrm{U}_\theta+\mathrm{U}_\theta-\delta_\theta\,\mathrm{U}_\mathrm{r}\Big)C_\theta+\Big(r\delta_\theta\delta_r\,\mathrm{U}_\theta-\delta_{\theta\theta}\,\mathrm{U}_\mathrm{r}+\delta_\theta\,\mathrm{U}_\theta\Big)S_\theta-\frac{-rS_\theta\delta_r\delta_\phi\,\mathrm{U}_\phi-S_\theta\delta_\phi\,\mathrm{U}_\phi+\delta_{\phi\phi}\,\mathrm{U}_\mathrm{r}}{S_\theta}}{r^2S_\theta}$$

$$+\frac{r^2S_\theta\delta_{rr}\,\mathrm{U}_\mathrm{r}+4rS_\theta\delta_r\,\mathrm{U}_\mathrm{r}+2\,\mathrm{U}_\mathrm{r}\,S_\theta}{r^2S_\theta}+\frac{rS_\theta\delta_\theta\delta_r\,\mathrm{U}_\theta+rC_\theta\delta_r\,\mathrm{U}_\theta+\mathrm{U}_\theta\,C_\theta+S_\theta\delta_\theta\,\mathrm{U}_\theta}{r^2S_\theta}-\frac{\delta_\phi\,\mathrm{U}_\phi}{r^2S_\theta}-\frac{2\Big(r\,\mathrm{U}_\theta\,C_\theta+rS_\theta\delta_\theta\,\mathrm{U}_\theta\Big)}{r^3S_\theta}-\frac{2\Big(r^2S_\theta\delta_r\,\mathrm{U}_\mathrm{r}+2r\,\mathrm{U}_\mathrm{r}\,S_\theta\Big)}{r^3S_\theta}\Bigg)$$

$$+\delta_r rE\Bigg(-\frac{-\Big(r\delta_{rr}\,\mathrm{U}_\theta+2\delta_r\,\mathrm{U}_\theta-\delta_\theta\delta_r\,\mathrm{U}_\mathrm{r}\Big)S_\theta+\frac{rS_\theta\delta_\theta\delta_\phi\,\mathrm{U}_\phi+rC_\theta\delta_\phi\,\mathrm{U}_\phi-r\delta_{\phi\phi}\,\mathrm{U}_\theta}{r^2S_\theta}}{rS_\theta}$$

$$+\frac{\frac{\delta_\theta\delta_\phi\,\mathrm{U}_\phi}{rS_\theta}-\frac{C_\theta\delta_\phi\,\mathrm{U}_\phi}{r\sin^2\Big(\theta\Big)}-\frac{\Big(r\,\mathrm{U}_\theta\,C_\theta+rS_\theta\delta_\theta\,\mathrm{U}_\theta\Big)C_\theta}{r^2\sin^2\Big(\theta\Big)}-\frac{\Big(r^2S_\theta\delta_r\,\mathrm{U}_\mathrm{r}+2r\,\mathrm{U}_\mathrm{r}\,S_\theta\Big)C_\theta}{r^2\sin^2\Big(\theta\Big)}+\frac{-r\,\mathrm{U}_\theta\,S_\theta+rS_\theta\delta_{\theta\theta}\,\mathrm{U}_\theta+2rC_\theta\delta_\theta\,\mathrm{U}_\theta}{r^2S_\theta}+\frac{r^2S_\theta\delta_\theta\delta_r\,\mathrm{U}_\mathrm{r}+r^2C_\theta\delta_r\,\mathrm{U}_\mathrm{r}+2r\,\mathrm{U}_\mathrm{r}\,C_\theta+2rS_\theta\delta_\theta\,\mathrm{U}_\mathrm{r}}{r^2S_\theta}}{r}\Bigg)\Bigg)\mathbf{\hat{k}_c}$$

curl of buoyancy

$$\Bigg(\frac{-\delta_\phi\frac{r^2\theta R_2T}{Rcmb}+\delta_\theta\frac{\phi r^2R_2T\sin^2\Big(\theta\Big)}{Rcmb}}{r^2S_\theta}\Bigg)\mathbf{\hat{i}_c}+\Bigg(\frac{\delta_\phi\frac{rR_2T}{Rcmb}-\delta_r\frac{\phi r^2R_2T\sin^2\Big(\theta\Big)}{Rcmb}}{rS_\theta}\Bigg)\mathbf{\hat{j}_c}+\Bigg(\frac{-\delta_\theta\frac{rR_2T}{Rcmb}+\delta_r\frac{r^2\theta R_2T}{Rcmb}}{r}\Bigg)\mathbf{\hat{k}_c}$$

time partial of vorticity

$$\Big(\delta_t\,\mathrm{W}_\mathrm{r}\Big)\mathbf{\hat{i}_c}+\Big(\delta_t\,\mathrm{W}_\theta\Big)\mathbf{\hat{j}_c}+\Big(\delta_t\,\mathrm{W}_\phi\Big)\mathbf{\hat{k}_c}$$

the first advective term, $\vec{U}\cdot\vec{\nabla}\vec{W}$

$$\mathrm{U}_\mathrm{r}\,\delta_r\Big(\Big(\mathrm{W}_\mathrm{r}\Big)\mathbf{\hat{i}_c}+\Big(\mathrm{W}_\theta\Big)\mathbf{\hat{j}_c}+\Big(\mathrm{W}_\phi\Big)\mathbf{\hat{k}_c}\Big)+\frac{\mathrm{U}_\phi\,\delta_\phi\Big(\Big(\mathrm{W}_\mathrm{r}\Big)\mathbf{\hat{i}_c}+\Big(\mathrm{W}_\theta\Big)\mathbf{\hat{j}_c}+\Big(\mathrm{W}_\phi\Big)\mathbf{\hat{k}_c}\Big)}{rS_\theta}+\frac{\mathrm{U}_\theta\,\delta_\theta\Big(\Big(\mathrm{W}_\mathrm{r}\Big)\mathbf{\hat{i}_c}+\Big(\mathrm{W}_\theta\Big)\mathbf{\hat{j}_c}+\Big(\mathrm{W}_\phi\Big)\mathbf{\hat{k}_c}\Big)}{r}$$

the second advective term, $(\vec{W}\cdot\vec{\nabla})\vec{U}$

$$\mathrm{W}_\mathrm{r}\,\delta_r\Big(\Big(\mathrm{U}_\mathrm{r}\Big)\mathbf{\hat{i}_c}+\Big(\mathrm{U}_\theta\Big)\mathbf{\hat{j}_c}+\Big(\mathrm{U}_\phi\Big)\mathbf{\hat{k}_c}\Big)+\frac{\mathrm{W}_\phi\,\delta_\phi\Big(\Big(\mathrm{U}_\mathrm{r}\Big)\mathbf{\hat{i}_c}+\Big(\mathrm{U}_\theta\Big)\mathbf{\hat{j}_c}+\Big(\mathrm{U}_\phi\Big)\mathbf{\hat{k}_c}\Big)}{rS_\theta}+\frac{\mathrm{W}_\theta\,\delta_\theta\Big(\Big(\mathrm{U}_\mathrm{r}\Big)\mathbf{\hat{i}_c}+\Big(\mathrm{U}_\theta\Big)\mathbf{\hat{j}_c}+\Big(\mathrm{U}_\phi\Big)\mathbf{\hat{k}_c}\Big)}{r}$$

2.3 Solenoidal Vector Field Represenation

We find both our velocity and magnetic fields subject to divergenceless conditions

$$\vec{\nabla}\cdot\vec{U}=\vec{\nabla}\cdot\vec{B}=0$$

Hence we proceed with an alternate definition of these fields via the angular momentum operator Λ

$$\vec{\Lambda}=\vec{r}\times\vec{\nabla}$$

or

$$\Bigg(-\frac{-\delta_\phi r^2\theta+\delta_\theta\phi r^2\sin^2\Big(\theta\Big)}{r^2S_\theta}\Bigg)\mathbf{\hat{i}_c}+\Bigg(-\frac{\delta_\phi^o r-\delta_r\phi r^2\sin^2\Big(\theta\Big)}{rS_\theta}\Bigg)\mathbf{\hat{j}_c}+\Bigg(-\frac{-\delta_\theta^o r+\delta_r r^2\theta}{r}\Bigg)\mathbf{\hat{k}_c}$$

solenoidal fields being defined as so, the velocity field

$$\vec{U}=\vec{\nabla}\times(\vec{\Lambda}a_{(r,\theta,\phi)})+\vec{\Lambda}b_{(r,\theta,\phi)}$$

with the sympy form (component along r basis)

$$\Bigg(-\frac{\Big(-\delta_\phi r^2\theta+\delta_\theta\phi r^2\sin^2\Big(\theta\Big)\Big)b}{r^2S_\theta}+\frac{-\delta_\phi\Bigg(-\frac{\Big(\delta_\phi^o r-\delta_r\phi r^2\sin^2\Big(\theta\Big)\Big)a}{S_\theta}\Bigg)+\delta_\theta\Big(-\Big(-\delta_\theta^o r+\delta_r r^2\theta\Big)aS_\theta\Big)}{r^2S_\theta}\Bigg)\mathbf{\hat{i}_c}$$

(component along th basis)

$$+\Bigg(-\frac{\Big(\delta_\phi^o r-\delta_r\phi r^2\sin^2\Big(\theta\Big)\Big)b}{rS_\theta}+\frac{-\delta_r\Big(-\Big(-\delta_\theta^o r+\delta_r r^2\theta\Big)aS_\theta\Big)+\delta_\phi\Bigg(-\frac{\Big(-\delta_\phi r^2\theta+\delta_\theta\phi r^2\sin^2\Big(\theta\Big)\Big)a}{r^2S_\theta}\Bigg)}{rS_\theta}\Bigg)\mathbf{\hat{j}_c}$$

(component along ph basis)

$$+\left(-\frac{\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) b}{r}+\frac{\delta_{r}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{S_{\theta}}\right)-\delta_{\theta}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{r^{2} S_{\theta}}\right)}{r}\right) \hat{\mathbf{k}}_{\mathbf{c}}$$

the magnetic field

$$\vec{B}=\vec{\nabla} \times (\vec{\Lambda} d_{(r,\theta,\phi)})+\vec{\Lambda} g_{(r,\theta,\phi)}$$

with the sympy form (component along r basis)

$$\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) g}{r^{2} S_{\theta}}+\frac{-\delta_{\phi}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) d}{S_{\theta}}\right)+\delta_{\theta}\left(-\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) d S_{\theta}\right)}{r^{2} S_{\theta}}\right) \hat{\mathbf{i}}_{\mathbf{c}}$$

(component along th basis)

$$+\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) g}{r S_{\theta}}+\frac{-\delta_{r}\left(-\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) d S_{\theta}\right)+\delta_{\phi}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) d}{r^{2} S_{\theta}}\right)}{r S_{\theta}}\right) \hat{\mathbf{j}}_{\mathbf{c}}$$

(component along ph basis)

$$+\left(-\frac{\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) g}{r}+\frac{\delta_{r}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) d}{S_{\theta}}\right)-\delta_{\theta}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) d}{r^{2} S_{\theta}}\right)}{r}\right) \hat{\mathbf{k}}_{\mathbf{c}}$$

also note the definition of an 'alternate' temperature function, $T2$, already being a scalar so simply defined as a placeholder for the eventual substitution of the innermost expansion with respect to all scalar fields; a, b, d, g, T2. The expansion form mentioned will be displayed in the following subsection, 'Innermost Representations of the Scalar Fields'

Again we can technically curl the velocity representation algorithmically, but might as well define the vorticity form explicitly.

$$\vec{W}=-\vec{\Lambda}\vec{\nabla}^{2} a+\vec{\nabla} \times (\vec{\Lambda} b)$$

with the sympy form (component along r basis)

$$\left(\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right)\left(\Delta a\right)}{r^{2} S_{\theta}}+\frac{-\delta_{\phi}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) b}{S_{\theta}}\right)+\delta_{\theta}\left(-\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) b S_{\theta}\right)}{r^{2} S_{\theta}}\right) \hat{\mathbf{i}}_{\mathbf{c}}$$

(component along th basis)

$$+\left(\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right)\left(\Delta a\right)}{r S_{\theta}}+\frac{-\delta_{r}\left(-\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) b S_{\theta}\right)+\delta_{\phi}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) b}{r^{2} S_{\theta}}\right)}{r S_{\theta}}\right) \hat{\mathbf{j}}_{\mathbf{c}}$$

(component along ph basis)

$$+\left(\frac{\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right)\left(\Delta a\right)}{r}+\frac{\delta_{r}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) b}{S_{\theta}}\right)-\delta_{\theta}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) b}{r^{2} S_{\theta}}\right)}{r}\right) \hat{\mathbf{k}}_{\mathbf{c}}$$

then as a sanity check we can also curl the solenoidal velocity definition algorithmically for the form (component along r basis)

$$\frac{1}{r^{2} S_{\theta}}\left(-\delta_{\phi} r\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) b}{r S_{\theta}}+\frac{-\delta_{r}\left(-\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) a S_{\theta}\right)+\delta_{\phi}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{r^{2} S_{\theta}}\right)}{r S_{\theta}}\right)\right.$$

$$\left.+\delta_{\theta} r\left(-\frac{\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) b}{r}+\frac{\delta_{r}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{S_{\theta}}\right)-\delta_{\theta}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{r^{2} S_{\theta}}\right)}{r}\right) S_{\theta}\right) \hat{\mathbf{i}}_{\mathbf{c}}$$

(component along th basis)

$$+\frac{1}{r S_{\theta}}\left(-\delta_{r} r\left(-\frac{\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) b}{r}+\frac{\delta_{r}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{S_{\theta}}\right)-\delta_{\theta}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{r^{2} S_{\theta}}\right)}{r}\right) S_{\theta}\right.$$

$$\left.+\delta_{\phi}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) b}{r^{2} S_{\theta}}+\frac{-\delta_{\phi}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{S_{\theta}}\right)+\delta_{\theta}\left(-\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) a S_{\theta}\right)}{r^{2} S_{\theta}}\right)\right) \hat{\mathbf{j}}_{\mathbf{c}}$$

(component along ph basis)

$$+\frac{1}{r}\left(\delta_{r} r\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) b}{r S_{\theta}}+\frac{-\delta_{r}\left(-\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) a S_{\theta}\right)+\delta_{\phi}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{r^{2} S_{\theta}}\right)}{r S_{\theta}}\right)\right.$$

$$\left.-\delta_{\theta}\left(-\frac{\left(-\delta_{\phi} r^{2} \theta+\delta_{\theta} \phi r^{2} \sin^{2}\left(\theta\right)\right) b}{r^{2} S_{\theta}}+\frac{-\delta_{\phi}\left(-\frac{\left(\delta_{\phi}^{o} r-\delta_{r} \phi r^{2} \sin^{2}\left(\theta\right)\right) a}{S_{\theta}}\right)+\delta_{\theta}\left(-\left(-\delta_{\theta}^{o} r+\delta_{r} r^{2} \theta\right) a S_{\theta}\right)}{r^{2} S_{\theta}}\right)\right) \hat{\mathbf{k}}_{\mathbf{c}}$$

2.4 Innermost Representations of the Scalar Fields

One layer deeper into the theoretical formulation is the expansions which constitute the scalar functions. Then each of the scalar fields associated with the solenoidal field representations: a and b for velocity, d and g for magnetic field, and T2 for the temperature field. The general form of these functions is as follows

$$s_{(r,\theta,\phi,t)} = \sum_{n=1}^{\infty}(\sum_{l=0}^{\infty}(\sum_{m=-l}^{l}(S_{lmn}(t)\chi_n(r)Y_{lm}(\theta,\phi))))$$

where

$$\chi_n(r) = J_c j_n(r) + Y_c y_n(r)$$

with $J_c = J_{coefficient}$ and $Y_c = Y_{coefficient}$.

The special function classes are defined more by their mathematical properties under transformations rather than as some sort of dictionary, so we find it instructive to show the basis of J.

$$j_n = \sqrt{\frac{\pi}{2r}} J_{n+\frac{1}{2}}(r)$$

The Bessel function of the first kind $J_n(z)$ is defined to satisfy

$$z^2\frac{d^2\omega}{dz^2} + z\frac{d\omega}{dz} + (z^2 - n^2)\omega = 0$$

sympy then chooses the representation of a Laurent expansion

$$J_n(z) = z^n(\frac{1}{\Gamma(n+1)2^n} + O(z^2))$$

then for clarity's sake, their definition of Gamma function is as follows

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$$

that being the methods used for j_n now for y_n

$$y_n(r) = \sqrt{\frac{\pi}{2r}} Y_{n+\frac{1}{2}}(r)$$

where the Bessel function of the second kind $Y_n(z)$ is designed to satisfy

$$Y_n(z) = \lim_{\mu \longrightarrow n} \frac{J_\mu(z)\cos\pi\mu - J_{-\mu}(z)}{\sin\pi\mu}$$

Then onto the spherical harmonic functions which are defined as follows

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} e^{im\phi} P_{lm}(\cos\theta)$$

with associated Legendre polynomial definition

$$P_{lm}(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m P_l(x)}{dx^m}$$

again the special function for the Legendre polynomial basis is not defined by its dictionary set, but rather as a class of functions orthogonal on [-1,1] with respect to the constant weight 1. They satisfy $P_n(1) = 1$ for all n, and P_n is odd for odd n and even for even n.

These special functions can be expressed into their particulate forms, given specification, but one may also keep them generalized within their sympy formula manipulation.

3 Dynamical System in terms of Scalar Functions

Now that we have both formulations for the basic fields and equations governing the physical system as well as the solenoidal field representations, we may seemlessly substitute the alternate field representations into place within the terms of the differential equations they belong to, thus forming a set of 7 partial differential equations, 3 each for the vector equations and one for the temperature equation, in terms of the scalar functions a, b, d, g, T2.

4 notes

- my indents aren't all working for some reason, weird. only some

5 Appendices

5.1 1: Sympy code

Note: This code is the working version as of (May 25, 2023)

5.2 2: Sympy-Latex coupling functions