

The goal of Heron's formula is to get the area of a triangle given three sides. We will start with the formula $[ABC] = \frac{1}{2}ab \sin C$. The law of sines isn't useful here, because we would be introducing the circumradius, a value that we don't know. Let's try the law of cosines:

$$\sin C = \sqrt{1 - \cos^2 C}$$

$$\begin{aligned} [ABC] &= \frac{1}{2}ab \sin C = \frac{ab}{2} \sqrt{1 - \cos^2 C} \\ &= \frac{ab}{2} \sqrt{1 - \frac{(c^2 - a^2 - b^2)^2}{(2ab)^2}} = \frac{ab}{2} \sqrt{1 - \frac{(c^2 - a^2 - b^2)^2}{4a^2b^2}} \\ &= \sqrt{\frac{a^2b^2}{4} \left(1 - \frac{(c^2 - a^2 - b^2)^2}{4a^2b^2}\right)} = \sqrt{\frac{a^2b^2}{4} - \frac{(c^2 - a^2 - b^2)^2}{16}} \\ &= \sqrt{\frac{4a^2b^2 - (c^2 - a^2 - b^2)^2}{16}} = \sqrt{\frac{(2ab - c^2 + a^2 + b^2)(2ab + c^2 - a^2 - b^2)}{16}} \\ &= \sqrt{\frac{[(a+b)^2 - c^2][c^2 - (a-b)^2]}{16}} = \sqrt{\frac{(a+b-c)(a+b+c)(a-b+c)(-a+b+c)}{16}} \end{aligned}$$

Using $s = \frac{a+b+c}{2}$ as the semiperimeter, we finally have Heron's formula:

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$$